DFT = Discrete Fourier Transform

To perform frequency analysis on a descreate-time signal {x(n)}. we convert the time-domain sequence to an equivalent frequency domain representation by using Fourier transform X(ω) of the sequence {x(n)}

→ X(ω) continuous function of frequency

→ so this is not a computationally convenient representation of the sequence {x(n)}

→ we consider the representation of a sequence {x(n)} by samples of its spectrum X(ω).

→ such a frequency-domain representation leads to the discreate Fourier transform (DFT)

→ Powerful computational tool for performing frequency analysis of discreate-time signal

DFT

& Di

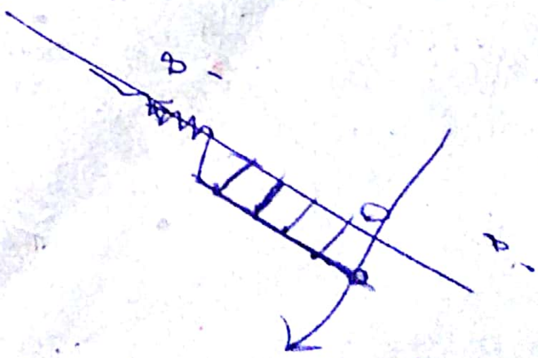
aperiodic discreate-time signal x(n)

Fourier transform

$$\boxed{X(\omega) = \sum_{x=-\infty}^{\infty} x(n)\, e^{-j\omega n}}$$

Example Compute DFT four-point sequence

x(n)=(0 1 2 3)



DTFT

Date/ 20/11/018

By definition

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n)\, e^{-j\omega n}$$

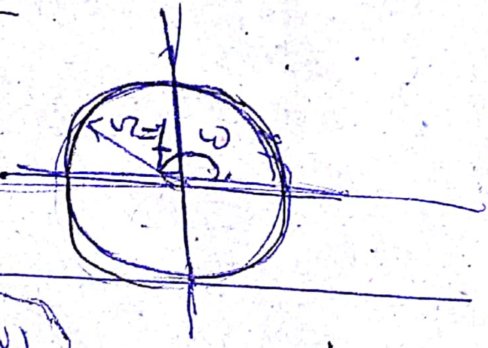
$n \rightarrow \omega$ - frequency

discrete time frequency → discrete time frequency (angular)

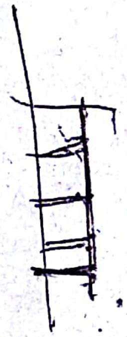
$(\omega = 0 - 2\pi)$ → digital frequency

Ω → Anlog frequency

$-\infty$ to $+\infty$



$$\begin{tikzcd} x(n) \arrow[r, shift left, "IDFT"'] \arrow[d] & X(e^{j\omega}) \arrow[l, shift left, "D.F.T."'] \arrow[d] \\ \text{discrete signal} & \text{continuous } \omega(0, 2\pi) \end{tikzcd}$$

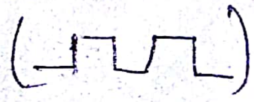


IDTFT→

$$x(n) = \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

DTFT→

$$X(\omega) = \sum_{n=0}^{\infty} x(n)\, e^{-j\omega n}$$



DFT→

discrete signal

digital

$$\begin{tikzcd} x(n) \arrow[r, shift left, "DFT"] & X(k) \arrow[l, shift left, "IDFT"] \arrow[r] & \text{digital signal} \end{tikzcd}$$

By def→

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

$(k = 0, 1, 2, \ldots N-1)$

IDFT

$$x(n) = \sum_{k=0}^{N-1} X(k)\, e^{\frac{+j2\pi nk}{N}}$$

DFT properties

1) Linearity →

DFT obeys the law of linearity.

If $x_1(n) \xleftrightarrow[N]{DFT} x_1(k)$ & $x_2(n) \xleftrightarrow[N]{DFT} x_2(k)$ then for any two constants, a & b

~~$a x_1(n) + b$~~

$$a x_1(n) + b x_2(n) \xleftrightarrow[N]{DFT} a x_1(k) + b x_2(k)$$

~~Proof → $x(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$~~

2) Periodicity →

If $x(n) \xleftrightarrow[N]{DFT} x(k)$ then

$x(n+N) = x(n)$ for all n

then $\boxed{X(k+N) = x(k)}$ for all k

3) multiplication of two DFT's & circular ~~convolution~~ convolution.

$$x_1(n) \textcircled{N} x_2(n) \xleftrightarrow[N]{DFT} x_1(k) \times x_2(k)$$

$$\boxed{x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2[(n-m)]N)}$$

4) Time reversal of a sequence →

If $x(n) \xleftrightarrow[N]{DFT} x(k)$ then

$x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$

(or)

$X(N-n) \xleftrightarrow[N]{DFT} X(N-k)$

when N-point sequence is reverse in time, it is equivalent to reversing the DFT values.

5) circular frequency shift

If $x(n) \xleftrightarrow[N]{DFT} x(k)$:

$x(n) e^{\frac{j2\pi ln}{N}} \xleftrightarrow[N]{DFT} X((k-l))N$

6) circular time shift →

If $x(n) \xleftrightarrow[N]{DFT} x(k)$

then, $x((n-l))_N \xleftrightarrow[N]{DFT} X(k) e^{-j2\pi kl/N}$

7) multiplication of two sequences →

If $x(n) \xleftrightarrow[N]{DFT} x_1(k)$

$x_2(n) \xleftrightarrow[n]{DFT} X_2(k)$

then $x_1(n) x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} X_1(k) \textcircled{N} X_2(k)$

8) circular correlation →

for complex valued sequence $x(n)$ & $y(n)$

$x(n) \xleftrightarrow[N]{DFT} X(k)$

$y(n) \xleftrightarrow[N]{DFT} Y(k)$

then, $r_{xy}(l) \xleftrightarrow[N]{DFT} R_{xy}(k) = x(k) Y^*(k)$

where $r_{xy}(l)$ is the circular cross correlation sequence

$$r_{xy}(l) = \sum_{n=0}^{N-1} x(n) y^*((n-l))_N$$

(10) Parseval's theorem →

for complex valued sequence x(n) & y(n)

$x(n) \xleftrightarrow[N]{DFT} x(k)$

$y(n) \xleftrightarrow[N]{DFT} y(k)$

then → $\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(K)\, y^*(k)$

DFT of even & odd sequence

+ for even sequence → Purely real

$x(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi nk}{N}\right)$

+ for odd sequence → Purely imaginary

$x(k) = \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi nk}{N}\right)$

→ periodic ⓞⓡ circular convolution

linear convolution ✓ circular ✓

sample (m) & (n) → (m) & (n)

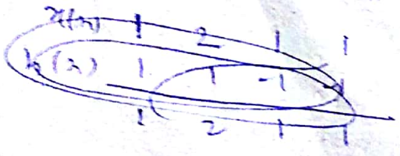
⇒ (m+n-1) max (m, n)

4 sample

$X(n) = \{1, 2, 1, 1\}$ $h(n) = \{1, 1, -1, -1\}$

4 sample

$y(n) = x(n) \circledast h(n)$



x(n) 1 2 1 1
h(n) 1 1 -1 -1

1 2 1 1
1 2 1 1
-1 -2 -1 -1
-1 -2 -1 -1

1 3 2 -1 -2 -2 -1
-2 -2

max (4, 4) ⇒ 4 sample

1 3 2 -1
-2 -2 -1

-1 +1 1 -1

so circular convolution is {-1 1 1 -1}

Eg $x(n) = \{2, 2\}$ 2 sample $h(n) = \{1, 1, -1\}$ 3

	1	1	-1
2	2	2	-2
2	2	2	-2

$y(n) = \{2, 4, 0, -2\}$

linear convolution - sample = (n+m-1)
= (2+3-1)
= (4)

result ⇒ {2, 4, 0, -1}

circular convolution sample max(2, 3) = 3

2, 4, 0, -2

⇒ so result ⇒ {0, 4, 0}

Eg.- $x(n) = \{1\ 1\ 2\ 2\}$ 15,17,15,13

$h(n) = \{1\ 2\ 3\ 4\}$

h(n) → x(n) ↓	1	2	3	4
1	1	2	3	4
1	1	2	3	4
2	2	4	6	8
2	2	4	6	8

linear convolution = $n+m-1 = 4+4-1 = 7$ sample

$= \{1, 3, 7, 13, 14, 14, 8\}$

circular convolution - max (4,4) = 4

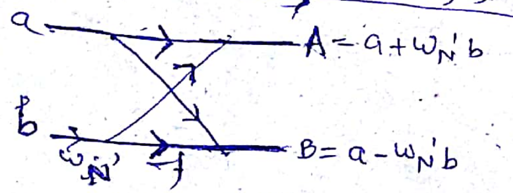
1, 3, 7, 13, 14, 14, 8

1+14, 3+14, 7+8, 13

result ⇒ $\boxed{15, 17, 15, 13}$

DIT - FFT → by using Butterfly structure → convert time domain → frequency domain

Basic Butterfly (DIT - FFT) →



$A = a + W_N^r b$

$B = a - W_N^r b$

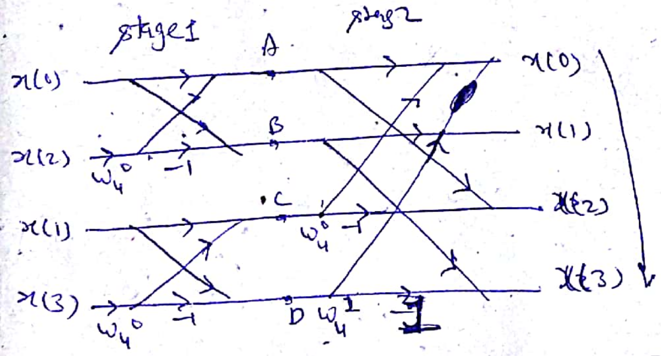
index	Bit representation	Bit reverse representation	bit. reversed index
0	00	00	0
1	01	10	2
2	10	01	1
3	11	11	3

↓ decimated.

Decimation in time fast fourier Transformation (DIT - FFT) Algorithm →

DIT means rearranging the time signal. give input in ~~[illegible]~~ order / decimated order then output will be in proper order

see let us see 4 point dFT →



$a = x(0)$

$b = x(2)$

$c = x(1)$

$d = X(3)$

$W_4^0 = 1,\ W_4^1 = -j$

x(0), x(1), x(2), x(3)

0 2   1 3

stage 1

$a+b = A$

$a-b = B$

$c+d = C$

$c-d = D$

stage 2

$A+C$

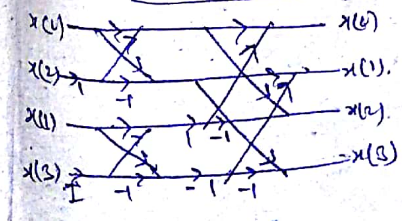
$B-Dj$

$A-C$

$B+Dj$

Que → obtain the. 2D-DFT of the given image using DIT - FFT algorithm.

$$X = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$



Row 1 [0, 1, 2, 1]

0 1 2 3

x(0) = 0

x(2) = 2

x(1) = 1

x(3) = 1

Stage 1.

2

-2

2

0

Stage 2

4

-2 - 0j

0

-2

Row 2 [1 2 3 2] → Row 4

x(0) = 1

x(2) = 3

x(1) = 2

x(3) = 2

Stage 1

4

-2

4

0

Stage 2

8

-2

0

-2

Row 3 ⇒ [2, 3, 4, 3]

	stage 1	stage 2
$x(0) = 2$	6	12
$x(2) = 4$	-2	-2
$x(1) = 3$	6	0
$x(3) = 3$	0	-2

Intermediate matrix ⇒

$$\begin{bmatrix} 4 & -2 & 0 & -2 \\ 8 & -2 & 0 & -2 \\ 12 & -2 & 0 & -2 \\ 8 & -2 & 0 & -2 \end{bmatrix}$$

column 1 [4, 8, 12, 8]

	stage 1	stage 2
$x(0) = 4$	16	32
$x(2) = 12$	-8	-8
$x(1) = 8$	16	0
$x(3) = 8$	0	-8

column 2 [-2, -2, -2, -2]

	stage 1	stage 2
$x(0) = -2$	-4	-8
$x(2) = -2$	0	0
$x(1) = -2$	-4	0
$x(3) = -2$	0	0

column = 4

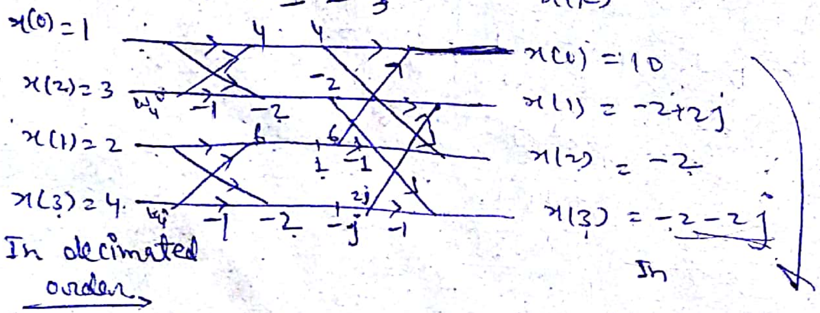
column 3 [0, 0, 0, 0]

stage 2: 0, 0, 0, 0

80) final values =

$$\begin{bmatrix} 32 & -8 & 0 & -8 \\ -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \end{bmatrix}$$

eg ⇒ $x(n) = \{1, 2, 3, 4\}$
$\quad\quad\quad\quad\quad\;\; 0\;\; 1\;\; 2\;\; 3$



In decimated order

In

$x(k) = \{10, -2+2j, -2, -2-2j\}$

DFT and IDFT using matrix short form method

compare the 4-point sequences.

① $x_1(n) = \{1, 1, 2, 1\}$

② $x_2(n) = \{1, 2, 3, 4\}$.

solve using DFT & IDFT methods

Sol:

DFT $x(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}$

IDFT $x_2(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(k) e^{\frac{-j2\pi nk}{N}}$

DFT ⇒

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 46 \\ -2+2j \\ 2 \\ -2-2j \end{bmatrix}$$

IDFT ⇒

$$\frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 10 \\ -2-2j \\ -2 \\ -2+2j \end{bmatrix} \Rightarrow \begin{bmatrix} 10/4 \\ -2-2j/4 \\ -2/4 \\ \frac{-2+2j}{4} \end{bmatrix}$$