UNIT II

CHAPTER 2

Design of Infinite Impulse Response Digital Filters

2.1 Concept of Filtering :

Analog filters are designed using analog components like resistors (R), inductors (L) and capacitors (C). While digital filters are implemented using difference equation.

The digital filters described by differential equations can be implemented using software like C or assembly language. We can easily change the algorithm; so we can easily change the filter characteristics according to our requirement.

Basically there are two types of filters as follows :

1. FIR (Finite impulse response) filter.
2. IIR (Infinite impulse response) filter.

We will study each type in detail; later in this chapter. Presently we will compare analog and digital filters by studying advantages and disadvantages of digital filters.

2.1.1 Advantages of Digital Filters :

1. Many input signals can be filtered by one digital filter without replacing the hardware.
2. Digital filters have characteristic like linear phase response. Such characteristic is not possible to obtain in case of analog filters.
3. The performance of digital filters, does not vary with environmental parameters. But the environmental parameters like temperature, humidity etc., change the values of components in case of analog filters. So it is required to calibrate analog filters periodically.
4. In case of digital filters; since the filtering is done with the help of digital computer, both filtered and unfiltered data can be saved for further use.
5. Unlike analog filters; the digital filters are portable.
6. From unit to unit the performance of digital filters is repeatable.
7. The digital filters are highly flexible.

8. *Using VLSI technology;* the hardware of digital filters can be reduced. Similarly the po consumption can be reduced.
9. Digital filters can be used at very low frequencies, for example in Biomedical applications.
10. In case of analog filters; maintenance is frequently required. But for digital filters it is required.

2.1.2 Disadvantages of Digital Filters :

1. **Speed limitation :**

In case of digital filters, ADC and DAC are used. So the speed of digital filter depends on conversion time of ADC and the settling time of DAC. Similarly the speed of operation digital filter depends on the speed of digital processor. Thus the bandwidth of input sig processed is limited by ADC and DAC. In real time applications, the bandwidth of digital fil is much lower than analog filters.

2. **Finite wordlength effect :**

The accuracy of digital filter depends on the wordlength used to encode them in binary for Wordlength should be long enough to obtain the required accuracy.

The digital filters are also affected by the ADC noise, resulting from the quantization continuous signals. Similarly the accuracy of digital filters is also affected by the round-off no occurred during computation.

3. **Long design and development time :**

An initial design and development time for digital hardware is more than analog filters.

Proove that physically realizable IIR filter cannot have

2.1.3 Ideal Filters and/Approximations : linear phase **UPTU - 2010-201**

see page no 2-82

First will study why ideal frequency selective filters are practically not realizable ?

Two important characteristics of ideal filters are :

(1) Ideal filters have constant gain in the passband and zero gain in the stop band.

(2) Ideal filter has linear phase response.

In order to realize (design) digital filter, an important condition is that; the response filter should be causal. We have studied a causal LTI system. Impulse response of causal LTI syst is denoted by h (n). Now system is causal when h (n) has some value for positive values of n h (n) is zero for negative values of 'n'. Thus the condition of causality can be summarized as

$$h(n) = 0 \text{ for } n < 0$$

As an example, let us consider the magnitude response of ideal low pass filter (L.P.F.). It is shown in Fig. 2.1.1(a).

Here, ω_c = Cut-off frequency

and $|H(\omega)|$ = Magnitude of filter

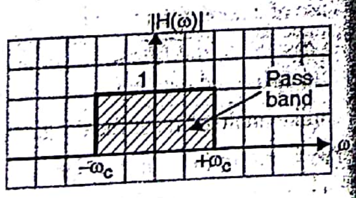


Fig. 2.1.1(a) : Magnitude response of ideal L

As shown in Fig. 2.1.1(a), the magnitude is '1' in the frequency range $-\omega_c$ to $+\omega_c$. Th

$$H(\omega) = \begin{cases} 1 & \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Now the value of h (n) is obtained by taking inverse fourier of H (ω). According to d efinition of inverse fourier we have,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega \quad ...(2.1.3)$$

Using Equation (2.1.2) we get,

$$h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\cdot e^{j\omega n}\, d\omega \quad ...(2.1.4)$$

Now we will consider two conditions as follows :

Condition (i) : When n = 0 :

Putting n = 0 in Equation (2.1.4) we get,

$$h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\cdot e^{0}\, d\omega$$

$$\therefore\ h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega \quad ...(\because e^0 = 1)$$

$$\therefore\ h(n) = \frac{1}{2\pi}[\omega]_{-\omega_c}^{\omega_c} = \frac{2\omega_c}{2\pi}$$

$$\therefore\ h(n) = \frac{\omega_c}{\pi} \text{ ...when } n = 0. \quad ...(2.1.5)$$

Condition (ii) : When n ≠ 0 :

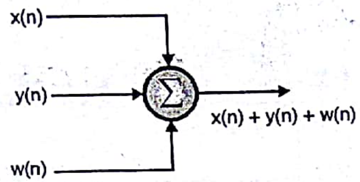
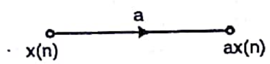
Taking integration of Equation (2.1.4) we get,

$$h(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn}[e^{j\omega_c n} - e^{-j\omega_c n}]$$

$$\therefore\ h(n) = \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] \quad ...(2.1.6)$$

But according to Euler's identity,

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

Sr. No.	Name of the block	Symbol
2.	Summing device	 Fig. 1.1.1(b)
3.	Constant multiplier	 Fig. 1.1.1(c)
4.	Delay by k samples	 Fig. 1.1.1(d)
5.	Advance by k samples	 Fig. 1.1.1(e)

Advantages of representing the digital filter in block diagram form :

1. Just by observing the block diagram ; the computational algorithm can be easily written.
2. From the block diagram, the relationship between input and output can be determined.
3. The memory requirement and the computational complexity can be determined using block diagram structure.
4. From the transfer function; variety of equivalent block diagram representation can be determined.

What is the meaning of canonic and non-canonic structures ?

If the number of delay elements in the block diagram is equal to the order of difference equation of a digital filter, then the realization structure is called as **canonic structure**. Otherwise it is called as **non-canonic structure**.

1.1.2 Transfer Function of FIR Filters :

We know that FIR stands for finite impulse response. The difference equation of FIR system :

$$y(n) = \sum_{k=0}^{M-1} h(k)\,x(n-k) \qquad \ldots(1.1.1)$$

Equation (1.1.1) shows that the system has length M as the limits of summation are from 0 to M – 1. These limits also indicates that the system is causal.

Taking Z-transform of Equation (1.1.1) we get,

$$Y(Z) = \sum_{k=0}^{M-1} h(k)\,Z^{-k}\,X(k) \qquad \ldots(1.1.2)$$

Here we have used time shifting property of Z–transform.

$$\therefore\ Z\{x(n-k)\} = Z^{-k}X(k)$$

Now the system transfer function is given by,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

Thus from Equation (1.1.2) we get,

$$\boxed{H(Z) = \frac{Y(Z)}{X(Z)} = \sum_{k=0}^{M-1} h(k)\,Z^{-k}} \qquad \ldots(1.1.3)$$

Equation (1.1.3) is called as system transfer function of FIR filter.

1.2 Direct Form Structure :

The direct form realization of FIR filter can be obtained by using the equation of linear convolution. It is,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \qquad \ldots(1.2.1)$$

If we consider that there are 'M' samples then Equation (1.2.1) becomes,

$$y(n) = \sum_{k=0}^{M-1} h(k)\,x(n-k) \qquad \ldots(1.2.2)$$

Expanding Equation (1.2.2) we get,

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots + h(M-1)x(n-M+1) \qquad \ldots(1.2.3)$$

□□□

CHAPTER 1

UNIT I

Realization of Digital Systems

1.1 Introduction :

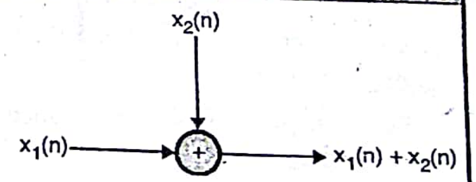
In order to realize (design) a digital filter; the given differential equation is broken into small equations. Then for each small equation; a structure using elementary blocks is drawn. Finally all these blocks are interconnected. The different types of structures used to realize the digital filter are as follows :

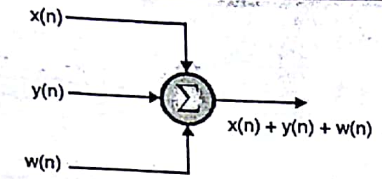
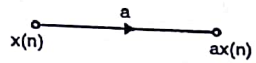
1. Direct form structure
2. Cascade form structure
3. Parallel form structure etc.

1.1.1 Block Diagram Representation of Linear Constant Coefficient Difference Equations (LCCDE) :

Before studying each structure in detail; we will study the different basic blocks used to represent a discrete time system. These blocks are shown in Table 1.1.1.

Table 1.1.1

Sr. No.	Name of the block	Symbol
1.	Adder	 Fig. 1.1.1(a)

Sr. No.	Name of the block	Symbol
2.	Summing device	 Fig. 1.1.1(b)
3.	Constant multiplier	 Fig. 1.1.1(c)
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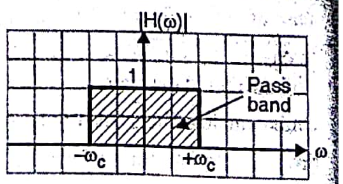

Fig. 2.1.1(a) : Magnitude response of ideal l

2 Filter Design Methods :

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In order to design the digital IIR filter; analog IIR filter is designed first. Then analog filte converted into the digital filter. Here you may ask a question, why to design digital filter from anal filter ?

The reasons are as follows :

(1) The procedure to design analog filter is readily available and it is highly advanced.

(2) When we design digital filter using analog filter then the implementation becomes simple.

The different methods to design IIR filters are :

(1) Impulse invariance (2) Bilinear transformation.

2.2.1 Impulse Invariant Method :

In this method, the design starts from the specifications of analog filter. Here we have to repl analog filter by digital filter. This is achieved if impulse response of digital filter resembles sampled version of impulse response of analog filter. If impulse response of both, analog and dig filter matches then, both filters perform in a similar manner.

Before studying this method we will list out the different notations, we are going to use.

$h(t)$ = Impulse response in time domain

Analog $H_a(s)$ = Transfer function of analog filter; here 's' is Laplace operator

$h(nT_s)$ = Sampled version of $h(t)$, obtained by replacing t by nT_s.

$H(Z)$ = Z transform of $h(nT_s)$. This is response of digital filter.

Ω = Analog frequency

ω = Digital frequency

Transformation of analog system function $H_a(s)$ to digital system function $H(Z)$:

Now let the system transfer function of analog filter be $H_a(s)$. We can express $H_a(s)$ in ter of partial fraction expansion. That means,

$$H_a(s) = \frac{A_1}{s-P_1} + \frac{A_2}{s-P_2} + \frac{A_3}{s-P_3} \dots$$

$$\therefore \quad H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s-P_k} \quad \dots(2$$

Here $A_k = A_1, A_2 \dots A_N$ are the coefficients of partial fraction expansion.

and $P_k = P_1, P_2 \dots P_N$ are the poles.

Here 's' is the laplace operator. So we can obtain impulse response of analog filter, $h(t)$ fro $H_a(s)$ by taking inverse laplace of $H_a(s)$. So using standard relation of inverse laplace we get,

$$h(t) = \sum_{k=1}^{N} A_k e^{P_k t} \quad \dots(2$$

Now unit impulse response for discrete structure is obtained by sampling $h(t)$. That means, $h(n)$ can be obtained from $h(t)$ by replacing 't' by nT_s in Equation (2.2.2).

$$\therefore \quad h(n) = \sum_{k=1}^{N} A_k e^{P_k n T_s} \quad \dots(2.2.3)$$

Here T_s is the sampling time.

The system transfer function of digital filter is denoted by $H(Z)$. It is obtained by taking Z-transform of $h(n)$. According to the definition of Z – transform for causal system,

$$H(Z) = \sum_{n=0}^{\infty} h(n) Z^{-n} \quad \dots(2.2.4)$$

Putting Equation (2.2.3) in Equation (2.2.4) we get,

$$H(Z) = \sum_{n=0}^{\infty} \left[\sum_{k=1}^{N} A_k e^{P_k n T_s} \right] \cdot Z^{-n}$$

$$\therefore \quad H(Z) = \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} e^{P_k T_s n} \cdot Z^{-n}$$

$$\therefore \quad H(Z) = \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} (e^{P_k T_s} \cdot Z^{-1})^n \quad \dots(2.2.5)$$

Using the standard summation formula,

$$\sum_{n=0}^{\infty} a^k = \frac{1}{1-a}, \quad \Rightarrow \frac{1}{1-e^{P_k T_s} Z^{-1}}$$

Equation (2.2.5) becomes,

$$\boxed{H(Z) = \sum_{k=1}^{N} A_k \frac{1}{1-e^{P_k T_s} Z^{-1}}} \quad \dots(2.2.6)$$

This is the required transfer function of digital filter.

Thus comparing Equations (2.2.1) and (2.2.6), we can say that the transfer function of digital filter is obtained from the transfer function of analog filter by doing the transformation.

$$\boxed{\frac{1}{s-P_k} \rightarrow \frac{1}{1-e^{P_k T_s} Z^{-1}}} \quad \dots(2.2.7)$$

Equation (2.2.7) shows, how the poles from analog domain are transferred into the digital domain. This transformation of poles is called as **mapping of poles**.

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2.2.1.1 Relationship of S-plane to Z-plane (Mapping between S-plane and Z-plane) :

We know that the poles of analog filters are located at $s = P_k$. Now from Equation (2.2.7) we can say that the poles of digital filter, H (Z) are located at,

$$Z = e^{P_k T_s} \quad ...(2.2.8)$$

This equation indicates that the poles of analog filter at $s = P_k$ are transformed into the poles of digital filter at $Z = e^{P_k T_s}$. Thus the relationship between laplace ('s' domain) and Z domain is given by,

$$Z = e^{s T_s} \quad ...(2.2.9)$$

Here $s = P_k$ and T_s is the sampling time.

Now 's' is the laplace operator and it is expressed as,

$$s = \sigma + j\Omega \quad ...(2.2.10)$$

Here σ = Attenuation factor

and Ω = Analog frequency

We know the 'Z' can be expressed in polar form as,

$$Z = r e^{j\omega} \quad ...(2.2.11)$$

Here 'r' is magnitude and 'ω' is the digital frequency.

Putting Equations (2.2.10) and (2.2.11) in Equation (2.2.9) we get,

$$r e^{j\omega} = e^{(\sigma + j\Omega)T_s}$$

$$\therefore \quad r e^{j\omega} = e^{\sigma T_s} \cdot e^{j\Omega T_s} \quad ...(2.2.12)$$

Separating real and imaginary parts of Equation (2.2.12) we get,

$$r = e^{\sigma T_s} \quad ...(2.2.13)$$

and $e^{j\omega} = e^{j\Omega T_s}$

$$\therefore \quad \omega = \Omega T_s \quad ...(2.2.14)$$

Now we will find the relationship between s plane and Z plane. Basically plot in 's'–domain means, σ is plotted on X–axis and $j\Omega$) is plotted on Y– axis. And Z-domain representation means real Z is plotted on X– axis and imaginary Z is plotted on Y–axis.

Now consider Equation (2.2.13), it is

$$r = e^{\sigma T_s}$$

We will discuss the following conditions :

(i) If $\sigma < 0$, then r is equal to reciprocal of 'e' raise to some constant. Thus range of r will be 0 to 1.

$$\boxed{\therefore \ \sigma < 0 \Rightarrow 0 < r < 1} \quad ...(2.2.15)$$

Now $\sigma < 0$ means negative values of σ. That is L.H.S. of s plane. We know that 'r' is radius of circle is Z plane.

So '$0 < r < 1$' indicates interior part of unit circle. Thus we can conclude that,

L.H.S. of 's' plane is mapped inside the unit circle.

(ii) If $\sigma = 0$ then $r = e^0 = 1$

$$\boxed{\sigma = 0 \Rightarrow r = 1}$$

Now $\sigma = 0$ indicates $j\Omega$ axis and $r = 1$ indicates unit circle. Thus, **$j\Omega$ axis in 's' plane is mapped on the unit circle.**

(iii) If $\sigma > 0$ then, r is equal to 'e' raise to some constant. That means $r > 1$.

$$\boxed{\sigma > 0 \Rightarrow r > 1}$$

Now $\sigma > 0$ indicates R.H.S. of 's' plane and 'r' > 1 indicates exterior part of unit circle. Thus, **R.H.S. of 's' plane is mapped outside the unit circle.**

Combining all conditions; this mapping is shown in Fig. 2.2.1.

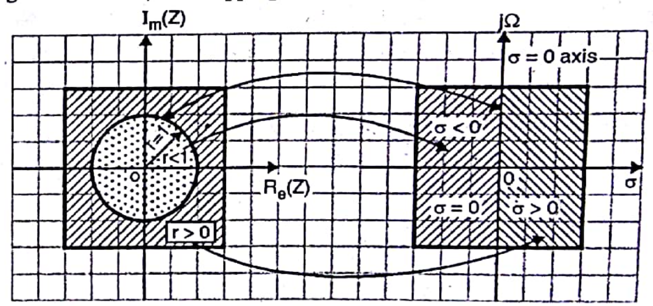

Fig. 2.2.1 : Relationship of s plane to Z-plane

Disadvantages of impulse invariance method :

(1) We know that 'Ω' is analog frequency and its range is from $\frac{\pi}{T_s}$ to $-\frac{\pi}{T_s}$. While the digital frequency 'ω' varies from $-\pi$ to π. That means from $\frac{\pi}{T_s}$ to $-\frac{\pi}{T_s}$ 'ω' maps from $-\pi$ to π. Let k be any integer. Then, we can write the general range of Ω as $(k-1)\frac{\pi}{T_s}$ to $(k+1)\frac{\pi}{T_s}$; but for this range also; 'ω' maps from $-\pi$ to π. Thus mapping from analog frequency 'Ω' to digital frequency 'ω' is **many to one**. This mapping is not one to one.

(2) Analog filters are not band limited so there will be aliasing due to the sampling process. Because of this aliasing, the frequency response of resulting digital filter will not be identical to the original frequency response of analog filter.

(3) The change in the value of sampling time (T_s) has no effect on the amount of aliasing.

Some standard formulae for transformation in impulse invariance method are as follows :

(i) $\dfrac{1}{s-P_k} \longrightarrow \dfrac{1}{1-e^{P_k T_s} \cdot Z^{-1}}$

(ii) $\dfrac{s+a}{(s+a)^2+b^2} \longrightarrow \dfrac{1-e^{-aT_s}[\cos bT_s] Z^{-1}}{1-2e^{-aT_s}[\cos bT_s] Z^{-1}+e^{-2aT_s} \cdot Z^{-2}}$

(iii) $\dfrac{b}{(s+a)^2+b^2} \longrightarrow \dfrac{e^{-aT_s}[\sin bT_s] Z^{-1}}{1-2e^{-aT_s}[\cos bT_s] Z^{-1}+e^{-2aT_s} \cdot Z^{-2}}$

2.2.1.2 Design Steps for Impulse Invariance Method :

Step I : Analog frequency transfer function H (s) will be given. If it is not given then obtain expression of H (s) from the given specifications.

Step II : If required expand H (s) by using partial fraction expansion (PFE).

Step III : Obtain Z-transform of each PFE term using impulse invariance transformation equation.

Step IV : Obtain H (Z), this is required digital IIR filter.

Ex. 2.2.1 : Find out H (Z) using impulse invariance method at 5 Hz sampling frequency from H(s) as given below :

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Soln. :

Step I : Given analog transfer function is,

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Step II : We will expand H (s) using partial fraction expansion as :

$$\therefore \quad H(s) = \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)}$$

Thus poles are at $P_1 = -1$ and $P_2 = -2$.

Now values of A_1 and A_2 are calculated as follows :

$$A_1 = (s-P_1) H(s) \Big|_{s=P_1}$$

$$\therefore \quad A_1 = (s+1) \cdot \frac{2}{(s+1)(s+2)} \Big|_{s=-1}$$

$$\therefore \quad A_1 = \frac{2}{-1+2} = 2$$

and $\quad A_2 = (s-P_2) H(s) \Big|_{s=P_1} = (s+2) \cdot \dfrac{2}{(s+1)(s+2)} \Big|_{s=-2}$

$$\therefore \quad A_2 = \frac{2}{-2+1} = -2$$

$= \dfrac{1}{1-e^{PT} z^{-1}}$

Putting values of A_1 and A_2 in Equation (2) we get,

$$H(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)} \quad ...(3)$$

Step III : Now we will obtain the Z–transform using impulse invariance transformation equation. It is,

$$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{P_k T_s} \cdot Z^{-1}} \quad ...(4)$$

$\dfrac{1}{s-P_k} = \dfrac{1}{1-e^{P}}$

$\dfrac{1}{s-P_1} = \dfrac{1}{1-}$

Here T_s = Sampling time. Now given sampling frequency is $F_s = 5$ Hz.

$$\therefore \quad T_s = \frac{1}{F_s} = \frac{1}{5} = 0.2 \text{ sec.}$$

We have poles at $P_1 = -1$ and $P_2 = -2$

So using Equation (4) we get,

$$\frac{1}{s+1} \longrightarrow \frac{1}{1-e^{-1(0.2)} \cdot Z^{-1}} = \frac{1}{1-e^{-0.2} \cdot Z^{-1}} \quad ...(5)$$

and $\quad \dfrac{1}{s+2} \longrightarrow \dfrac{1}{1-e^{-2(0.2)} \cdot Z^{-1}} = \dfrac{1}{1-e^{-0.4} \cdot Z^{-1}} \quad ...(6)$

Step IV : The transfer function of digital filter is given by,

$$H(Z) = \sum_{k=1}^{N} \frac{A_k}{1-e^{P_k T_s} \cdot Z^{-1}}$$

In this case we get,

$$H(Z) = \frac{A_1}{1-e^{P_1 T_s} \cdot Z^{-1}} + \frac{A_2}{1-e^{P_2 T_s} \cdot Z^{-1}} \quad ...(7)$$

Using Equations (5) and (6) we get,

$$H(Z) = \frac{2}{1-e^{-0.2} \cdot Z^{-1}} - \frac{2}{1-e^{-0.4} \cdot Z^{-1}}$$

$$\therefore \quad H(Z) = \frac{2}{1-0.818\,Z^{-1}} - \frac{2}{1-0.67\,Z^{-1}}$$

To convert each term into positive powers of Z ; multiplying numerator and denominator of each term by Z we get,

$$H(Z) = \frac{2Z}{Z-0.818} - \frac{2Z}{Z-0.67}$$

$$\therefore \quad H(Z) = \frac{2Z(Z-0.67) - 2Z(Z-0.818)}{(Z-0.818)(Z-0.67)}$$

$$H(Z) = \frac{(-j\,0.707)\,[1-e^{(-0.707-j\,0.707)}\,Z^{-1}] + (j\,0.707)\,[1-e^{(-0.707+j\,0.707)}\,Z^{-1}]}{[1-e^{(-0.707+j\,0.707)}\,Z^{-1}]\,[1-e^{(-0.707-j\,0.707)}\,Z^{-1}]}$$

$$H(Z) = \frac{(-j\,0.707)+(j\,0.707).\,e^{-0.707}.e^{-j0.707}Z^{-1} + (j\,0.707) - j\,0.707.e^{-0.707}.e^{j0.707}Z^{-1}}{1-e^{-0.707}.e^{-j0.707}Z^{-1} - e^{-0.707}.e^{j0.707}Z^{-1} + e^{-1.414}.Z^{-2}}$$

$$\therefore H(Z) = \frac{j\,0.707\,e^{-0.707}.Z^{-1}\,[e^{-j0.707} - e^{j0.707}]}{1-e^{-0.707}.Z^{-1}\,[e^{-j0.707}+e^{j0.707}] + e^{-1.414}.Z^{-2}}$$

Now we have trigonometric identities.

$$\left\{\begin{aligned} \sin\theta &= \frac{e^{j\theta}-e^{-j\theta}}{2j} \\ \text{and} \qquad \cos\theta &= \frac{e^{j\theta}+e^{-j\theta}}{2} \end{aligned}\right\}$$

We will arrange equation (2) as follows,

$$H(Z) = \frac{-j\,0.707\cdot e^{-0.707}.Z^{-1}\left[\dfrac{e^{j0.707}-e^{-j0.707}}{2j}\right]\cdot 2j}{1-e^{-0.707}.Z^{-1}\left[\dfrac{e^{j0.707}+e^{-j0.707}}{2}\right]\cdot 2 + e^{-1.414}.Z^{-2}}$$

$$\therefore H(Z) = \frac{1.414\,e^{-0.707}\,Z^{-1}.\sin(0.707)}{1-2e^{-0.707}.Z^{-1}\cos(0.707)+e^{-1.414}.Z^{-2}}$$

$$\therefore H(Z) = \frac{0.4529\,Z^{-1}}{1-0.7498\,Z^{-1}+0.2432\,Z^{-2}}$$

2.2.2 Bilinear Transformation Method (BLT) :

UPTU - 2006-2007, 2007-2

In case of impulse invariance method, we have studied that the mapping is many to one. So method is not suitable to design high-pass filter and band reject filter.

In case of bilinear transformation, the mapping is one to one from s domain to the Z domain there is no aliasing effect. The limitations of impulse invariance method are overcome by using method.

Consider an analog integrator as shown in Fig. 2.2.2(a).

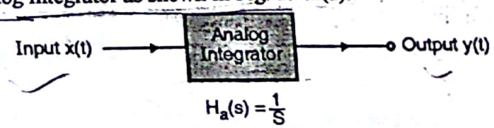

Fig. 2.2.2(a)

The response of analog integrator is,

$$H_a(s) = \frac{1}{s} \qquad ...(2$$

Input in laplace domain is X (s) and output in laplace domain is Y (s).

For time period T; the difference in output is given by,

$$Y(t_2) - Y(t_1) = \int_{t_1}^{t} x(n\,T_s)\,dT \qquad ...(2.2.17)$$

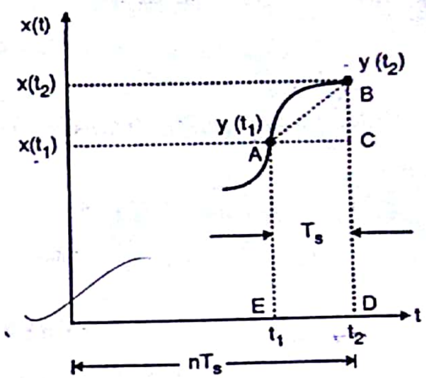

Fig. 2.2.2(b)

Consider input signal as shown in Fig. 2.2.2(b).

Here we have assumed two input positions x (t_1) and x (t_2). Corresponding output is denoted by Y (t_1) and Y (t_2) respectively. Now area under the curve is addition of area of triangle ABC and area of rectangle ACDE.

$$\therefore\; Y(t_2) - Y(t_1) = \frac{1}{2}(t_2-t_1)\,[x(t_2)-x(t_1)] + x(t_1)(t_2-t_1) \qquad ...(2.2.18)$$

As shown in Fig. 2.2.2(b), time period $t_2 = n\,T_s$. Thus time period t_1 is $n\,T_s - T_s$ that means $t_2 - t_1 = T_s$

($t_1 = nT_s - T_s$)

Putting these values in Equation (2.2.18) we get,

$$Y(n\,T_s) - Y(n\,T_s - T_s) = \frac{1}{2}T_s[x(n\,T_s) - x(n\,T_s - T)] + x(n\,T_s - T)\,T_s$$

$$Y(n\,T_s) - Y(n\,T_s - T_s) = \frac{1}{2}T_s\,x(nT) - \frac{1}{2}T_s\,x(n\,T_s - T_s) + T_s\,x(n\,T_s - T_s)$$

$$Y(n\,T_s) - Y(n\,T_s - T_s) = \frac{1}{2}T_s\,x(n\,T_s) + \frac{1}{2}T_s\,x(n\,T_s - T_s)$$

$$Y(nT_s) - Y(n\,T_s) - T_s = \frac{T_s}{2}\,[x(n\,T_s) + x(n\,T_s - T_s)] \qquad ...(2.2.19)$$

$-Y(nT_s - T_s)$

$YZ(n-1)$

Taking Z transform of both sides we get,

$$Y(Z) - Z^{-1}\,Y(Z) = \frac{T_s}{2}\,[X(Z) + Z^{-1}\,X(Z)]$$

$$\therefore Y(Z)\,[1 - Z^{-1}] = \frac{T_s}{2}\,[X(Z)\,(1+Z^{-1})]$$

$$\therefore \frac{Y(Z)}{X(Z)} = H(Z) = \frac{T_s}{2}\,\frac{(1+Z^{-1})}{(1-Z^{-1})} \qquad ...(2.2.20)$$

Now we have transfer function of analog filter,

$$H_a(s) = \frac{1}{s} \qquad ...(2.2.21)$$

Equating (2.2.20) and (2.2.21) we get,

$$\frac{1}{s} = \frac{T_s}{2}\,\frac{(1+Z^{-1})}{(1-Z^{-1})} \qquad ...(2.2.22)$$

Thus relationship between s plane and Z-plane is given by,

$$s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right) \qquad ...(2.2.23)$$

put

Here T_s is the sampling time.

We know that 's' is the Laplace operator and it can be expressed as,

$$s = \sigma + j\Omega \quad ...(2.2.24)$$

Now the equation of Z in polar form is,

$$Z = r e^{j\omega} \quad ...(2.2.25)$$

Putting Equations (2.2.24) and (2.2.25) in Equation (2.2.1) we get,

Multiplying numerator and denominator by $(re^{-j\omega} + 1)$ we get,

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right] \times \frac{re^{-j\omega} + 1}{re^{-j\omega} + 1}$$

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 \cdot e^{j\omega} \cdot e^{-j\omega} + re^{j\omega} - re^{-j\omega} - 1}{r^2 e^{j\omega} \cdot e^{-j\omega} + re^{j\omega} + re^{-j\omega} + 1}\right] \quad ...(2.2.26)$$

But $e^{j\omega} \cdot e^{-j\omega} = e^0 = 1$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 + re^{j\omega} - re^{-j\omega} - 1}{r^2 + re^{j\omega} + re^{-j\omega} + 1}\right] \quad ...(2.2.27)$$

Now we have, $\frac{e^{j\omega} - e^{-j\omega}}{2j} = \sin\omega$ and $\frac{e^{j\omega} + e^{-j\omega}}{2} = \cos\omega$

We will rearrange Equation (2.2.27) as follows :

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 - 1 + j2r\left(\frac{e^{j\omega} - e^{-j\omega}}{2j}\right)}{r^2 + 2r\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 1}\right]$$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 - 1 + j2r\sin\omega}{r^2 + 2r\cos\omega + 1}\right]$$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 - 1}{r^2 + 2r\cos\omega + 1} + j\frac{2r\sin\omega}{r^2 + 2r\cos\omega + 1}\right] \quad ...(2.2.28)$$

Equating real and imaginary parts of Equation (2.2.28) we get,

$$\sigma = \frac{2}{T_s} \times \frac{r^2 - 1}{r^2 + 2r\cos\omega + 1} \quad ...(2.2.29)$$

and

$$\Omega = \frac{2}{T_s} \times \frac{2r\sin\omega}{r^2 + 2r\cos\omega + 1} \quad ...(2.2.30)$$

Now we will discuss the following conditions related to Equation (2.2.29).

(i) **When $r < 1$ then $\sigma < 0$**

Here $r < 1$, means interior part of circle having unit circle and $\sigma < 0$, means σ is negativ... Here $r < 1$, means interior part of circle having unit circle and $\sigma < 0$, means σ is negative... *means σ is negativ[e]*. So this condition indicates that L.H.S. of s plane maps inside th[e]... is L.H.S. of s-plane. So this condition indicates that L.H.S. of s plane maps inside the circle.

(ii) **When $r = 1$ then $\sigma = 0$**

Now $r = 1$ means unit circle and $\sigma = 0$ means $j\Omega$ axis. Thus this condition indicates that the $j\Omega$ axis maps on the unit circle.

(iii) **When $r > 1$ then $\sigma > 0$**

Here $r > 1$, means exterior part of unit circle and $\sigma > 0$ indicates that σ is positive means R.H.S. of s-plane. So this condition indicates that R.H.S. of s-plane maps outside the unit circle.

This mapping is similar to the mapping in impulse invariance method, as shown in Fig. 2.2.3. But in impulse invariance method mapping is valid only for poles ; while in bilinear transformation, mapping is valid for poles as well as zeros.

2.2.3 How Stable Analog Filter is Converted into Stable Digital Filter ?

Analog filter is stable if the poles lie on the L.H.S. of s-plane. While the digital filter is stable if the poles are inside the unit circle in the Z-domain. Now condition (i) indicates that L.H.S. of s-plane maps inside the unit circle. Thus stable analog filter is converted into stable digital filter.

2.2.4 Frequency Warping Concept : 2011-12 (warping)

UPTU - 2007-2008, 2009-2010

Here we will obtain the relationship of $j\Omega$ axis in s-plane to the unit circle in the Z-plane (r = 1) Recall Equation (2.2.30).

$$\boxed{\Omega = \frac{2}{T_s} \times \frac{2r\sin\omega}{r^2 + 2r\cos\omega + 1}}$$

For the unit circle, $r = 1$. Thus putting $r = 1$ in the equation of Ω we get,

$$\Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{1 + 2\cos\omega + 1}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{2 + 2\cos\omega}$$

$$\Omega = \frac{2}{T_s} \times \frac{\sin\omega}{1 + \cos\omega} \quad ...(2.2.31)$$

We have the trignometric identities,

$$\sin\omega = 2\sin\frac{\omega}{2} \cdot \cos\frac{\omega}{2} \text{ and } 2\cos^2\frac{\omega}{2} = 1 + \cos\omega$$

Digital Sig... Thus Equatio... $\Omega =$

Thus Equation (2.2.31) becomes,

$$\Omega = \frac{2}{T_s} \times \frac{2 \sin\frac{\omega}{2} \cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}}$$

$$\therefore \Omega = \frac{2}{T_s} \times \frac{2 \sin\frac{\omega}{2}}{2 \cos \omega / 2}$$

$$\therefore \Omega = \frac{2}{T_s} \tan\frac{\omega}{2}$$

$$\therefore \omega = 2 \tan^{-1}\left(\frac{\Omega T_s}{2}\right)$$

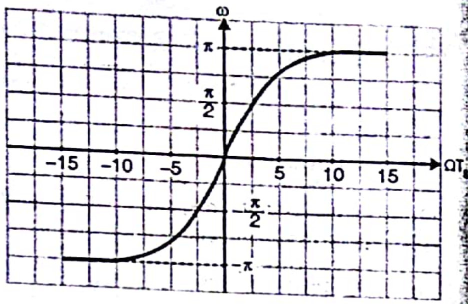

Fig. 2.2.3 : Mapping between ω and Ω

Now for different values of ΩT_s; the graph of ΩT_s versus ω is as shown in Fig. 2.2.3.

Observations :

1. The entire range in Ω is mapped only once into the range $-\pi \le \omega \le \pi$.
2. The mapping is one to one.
3. The mapping is highly non-linear,

Now because of the non-linearity of tangent function $2 \tan^{-1}\left(\frac{\Omega T_s}{2}\right)$; there exists frequency warping or frequency compression.

2.2.5 What is Frequency Warping ?

UPTU - 2007-2008

Because of the non-linear mapping; the amplitude response of digital IIR filter is expanded at lower frequencies and compressed at higher frequencies in comparison to the analog filter. This effect is called as frequency warping.

2.2.5.1 Prewarping Procedure :

UPTU - 2007-2008

We have discussed the warping effect. Because of this effect, there is non-linear compression of Ω to ω values. To compensate this effect ; prewarping or prescaling procedure is used. This procedure is as follows :

(i) The 'Ω' scale is changed to prewarpped scale denoted by Ω^* and $\Omega^* = \frac{2}{T_s} \tan\left(\frac{\omega T_s}{2}\right)$

(ii) Then analog filter transfer function H (s) is obtained using values of Ω^*.

(iii) By applying the bilinear transformation, the desired digital frequency response H (Z) is obtained.

2.2.5.2 Advantages of Bilinear Transformation Method :

UPTU - 2010-2011

1. There is one to one transformation from the s-domain to the Z- domain.
2. The mapping is one to one.
3. There is no aliasing effect.
4. Stable analog filter is transformed into the stable digital filter.

2.2.5.3 Disadvantage of Bilinear Transformation Method :

UPTU - 2010-2011

The mapping is non-linear and because of this; frequency warping effect takes place.

2.2.6 Comparison between Impulse Invariance and Bilinear Transformation Method :

UPTU - 2009-2010, 2010-2011

Sr. No.	Impulse invariance method	Bilinear transformation method
1.	Poles are transferred by using the equation, $\frac{1}{s-P_k} \rightarrow \frac{1}{1-e^{P_k T_s} \cdot Z^{-1}}$	Poles are transferred by using the equation. $s = \frac{2}{T_s} \frac{Z-1}{Z+1}$
2.	Mapping is many to one.	Mapping is one to one.
3.	Aliasing effect is present.	Aliasing effect is not present.
4.	It is not suitable to design high-pass filter and band reject filter.	High pass filter and band reject filter can be designed.
5.	Only poles of the system can be mapped.	Poles as well as zeros can be mapped.
6.	No frequency warping effect.	Frequency warping effect is present

Solved problems using BLT :

Ex. 2.2.6 : An analog filter has the following transfer function $H(s) = \frac{1}{s+1}$. Using bilinear transformation technique, determine the transfer function of digital filter H (Z) and also write the difference equation of digital filter.

Soln. : The given transfer function is,

$$H(s) = \frac{1}{S+1} \quad ...(1)$$

In bilinear transformation H (Z) is obtained by putting,

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right]$$

Here T_S is the sampling time; which is not given. So assume $T_s = 1$ sec.

$$\therefore \quad s = 2\left[\frac{Z-1}{Z+1}\right] \quad ...(2)$$

Putting this value in Equation (1),

$$H(Z) = \frac{1}{1+2\left(\frac{Z-1}{Z+1}\right)} = \frac{1}{\frac{(Z+1)+2(Z-1)}{Z+1}}$$

$$\therefore \quad H(Z) = \frac{Z+1}{Z+1+2Z-2}$$

$$\therefore \quad H(Z) = \frac{Z+1}{3Z-1} \quad ...(3)$$

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$$\therefore\ H(Z) = \frac{1}{10^{-6} \times 1.025 \left\{ \frac{2}{1/1.28 \times 10^{3}} \left[\frac{Z-1}{Z+1}\right]\right\}^{2} + 1.432 \times 10^{-3} \left[\frac{2}{\frac{1}{1.28 \times 10^{3}}} \left[\frac{Z-1}{Z+1}\right]\right] + 1}$$

$$\therefore\ H(Z) = \frac{1}{6.71\left[\frac{Z-1}{Z+1}\right]^{2} + 3.67\left[\frac{Z-1}{Z+1}\right] + 1}$$

$$\therefore\ \boxed{H(Z) = \frac{1}{6.71\left[\frac{1-Z^{-1}}{1+Z^{-1}}\right]^{2} + 3.67\left[\frac{1-Z^{-1}}{1+Z^{-1}}\right] + 1}}$$

2.3 Butterworth Filter Approximation :

UPTU - 2010-2011

A typical characteristic of a butterworth low pass filter is as shown in Fig. 2.3.1.

This type of response is called as butterworth response because its main characteristic is that the passband is maximally flat. That means there are no variations (ripples) in the passband.

Now the magnitude squared response of low pass butterworth filter is given by,

$$|H(\Omega)|^{2} = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} \qquad \ldots(2.3.1)$$

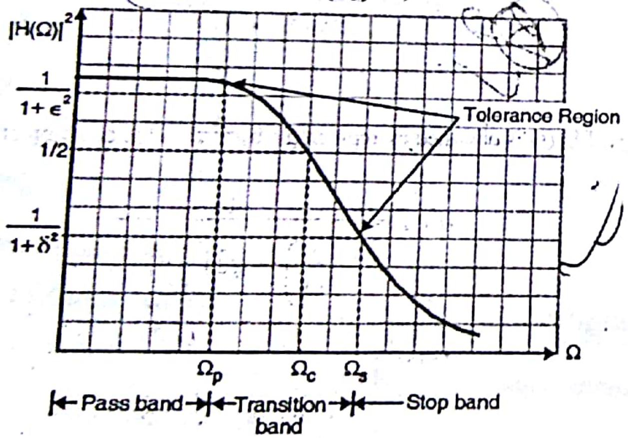

Fig. 2.3.1 : Typical characteristics of analog L.P.F.

This equation is also expressed as,

$$|H(\Omega)|^{2} = \frac{1}{1 + \epsilon^{2}\left(\frac{\Omega}{\Omega_p}\right)^{2N}} \qquad \ldots(2.3.2)$$

Here
- $|H(\Omega)|$ = Magnitude of analog low pass filter.
- Ω_c = Cut-off frequency (– 3B frequency).
- Ω_p = Pass band edge frequency.
- $1 + \epsilon^2$ = Pass band edge value.
- $1 + \delta^2$ = Stop band edge value.
- ϵ = Parameter related to ripples in pass band.
- δ = Parameter related to ripples in stop band.
- N = Order of the filter.

We know that, in case of low pass filter the frequencies will pass upto the value of cut-off frequency (Ω_c). This is called as pass band. After that the frequencies are attenuated. This is called as stop band. Ideal characteristic is shown by dotted line in Fig. 2.3.2. Ideally, at the value of cut-off frequency (Ω_c) the frequencies should be stopped. But in practical cases this is not happening.

Now the order of filter is denoted by 'N'. Roughly we can say order of filter means, the number of stages used in the design of analog filter. As the order of filter 'N' increases, the response of filter is more close to the ideal response as shown in Fig. 2.3.2.

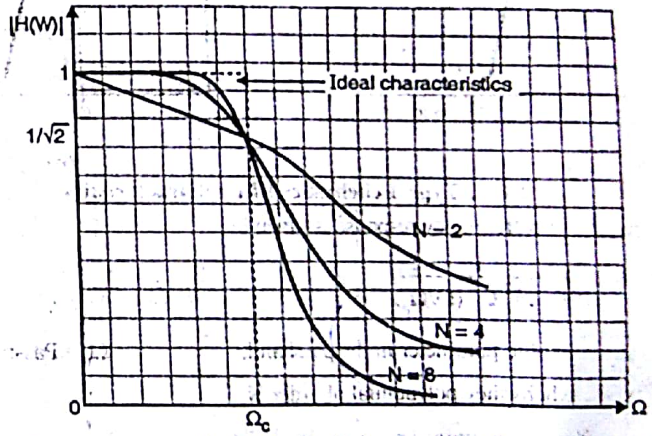

Fig. 2.3.2 : Effect of N on frequency response characteristics

Salient features of low pass butterworth filter :

1. The magnitude response is nearly constant (equal to 1) at lower frequencies. That means pass band is maximally flat.
2. There are no ripples in the pass band and stop band.
3. The maximum gain occurs at $\Omega = 0$ and it is $|H(0)| = 1$.
4. The magnitude response is monotonically decreasing.