9. NPTEL - Dutta Roy Chowdhary
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Unit-4

- Discrete Time Fourier Transform (DTFT):-

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n) e^{-j\omega n}$$ {for causal signal}

$n \longrightarrow \omega$ (discrete time angular freq.)

→ $0 \leq \omega \leq 2\pi$: Range
→ ω is periodic.

{Ω $(-\infty < \Omega < \infty)$ is for continuous freq. which may or may not be periodic depends upon range taken}

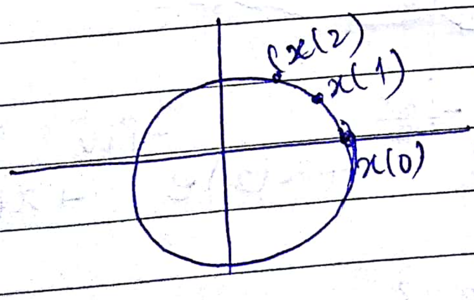
- Inverse DTFT:-

$$x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

- Discrete Fourier Transform :-

$x(n)$ where $n = 0, 1, \ldots, N-1$

$$X(e^{j\omega}) = x(0) + x(1)e^{-j\omega} + x(2)e^{-j\omega} + \cdots + x(N-1)e^{-j\omega}.$$



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$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi nk}{N}}} \longrightarrow \text{DFT}$$

On comparing, $\boxed{\omega = \frac{2\pi k}{N}}$ $k = 0, 1, 2 \ldots, N-1$

If $N=2$, then $X(k)$ is 2-point DFT

Note :- DFT is just frequency sampled version of DTFT.

- Inverse DFT :-

$$\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j\omega_n k}{N}}} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}$$

where $n = 0, 1, \ldots, N-1$

$\omega_n = 2\pi n$

- Methods to solve DFT :-

1. Matrix Method :-

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j\omega_n k}{N}} = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi nk}{N}}$$

If $N=4$:- (4 point DFT):

$$X(k) = \sum_{n=0}^{3} x(n) e^{-\frac{j2\pi nk}{4}}$$

$$X(k) = x(0) + x(1) e^{-\frac{jk\pi}{2}} + x(2) e^{-j\pi k} + x(3) e^{-\frac{j3\pi k}{2}}$$

where $k = 0, 1, 2, 3$

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Order of W_N matrix $= N \times N$

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for $k = 0, 1, 2, 3$:-

$$X(0) = x(0) + x(1) + x(2) + x(3)$$

$$X(1) = x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$$

$$X(2) = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$X(3) = x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$$

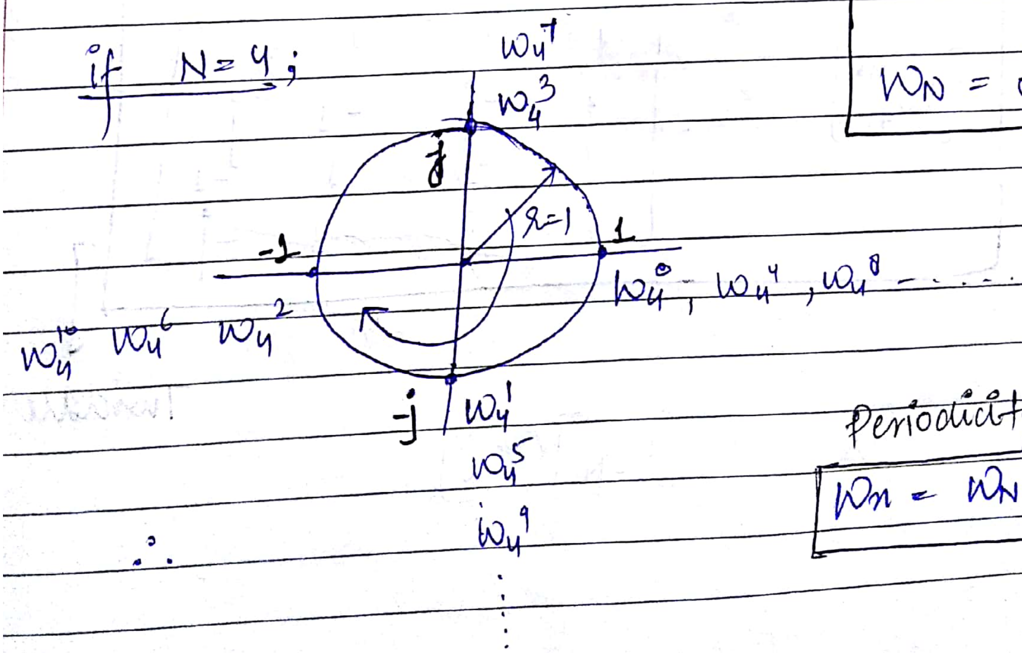
$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/2} & e^{-j\pi} & e^{-j3\pi/2} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} \\ 1 & e^{-j3\pi/2} & e^{-j3\pi} & e^{-j9\pi/2} \end{bmatrix}}_{\text{Twiddle Matrix}} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

In $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}}$

$$W_N = \boxed{e^{-j\frac{2\pi}{N}} = \text{Twiddle factor}}$$

$$\boxed{W_N = e^{-j\frac{2\pi}{N}}}$$

if $N = 4$;



$\therefore$

Periodicity Property:-

$$\boxed{W_n = W_{N+n}}$$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)(W_N^{nk})}$$

- Formation of Twiddle Matrix :-

$$\begin{array}{c|cccc} k \backslash n & 0 & 1 & 2 & 3 \\ \hline 0 & W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ 1 & W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ 2 & W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ 3 & W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{array}$$

$$[W_4^{nk}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\boxed{[W_4^{nk}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}}$$

Twiddle Matrix

$$[X_N] = [W_N][x_N]$$

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Q. Given $x(n) = \{1, 2, 3, 4\}$. find DFT i.e $X(k)$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_4^{nk} \qquad (N = 4)$$

$$X(k) = \underset{4\times 4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}$$

$$X(k) = \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2+3-4 \\ 1+2j-3-4j \end{bmatrix} = \begin{bmatrix} 10 \\ 2j-2 \\ -2 \\ -2-2j \end{bmatrix}$$

$$X(k) = x(0) + x(1) + (x(2) + x(3)$$

$$= 10 + 2j - 2 - 2 - 2 - 2j$$

$$X(k) = 4$$

- IDFT by Matrix Method

$$[x_N] = \frac{1}{N}[W_N^{-1}][X_N]$$

$$[x_N] = \frac{1}{N}[W_N^{*}][X_N] \qquad [W_N^{*}] = [W_N^{-1}]$$

Q. Compute the DFT of 4-point sequence, $x(n) = \{0, 1, \dots$

Properties of DFT

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$X(K) = [W_4][x(n)]$

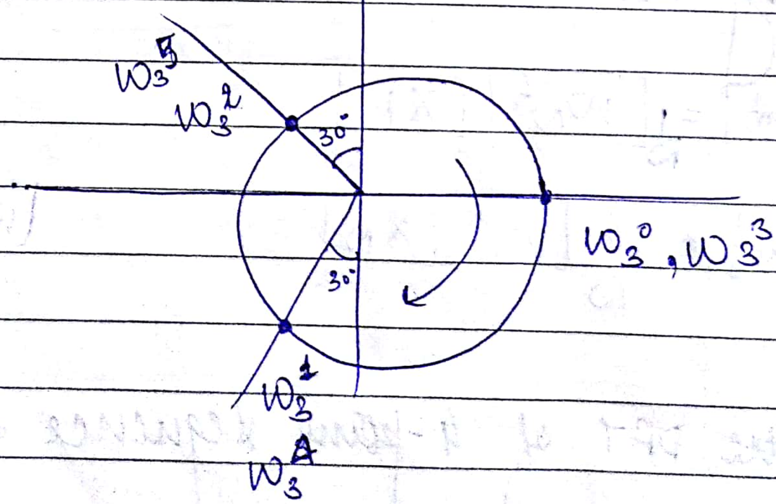
$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$X(k) = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Q.1 Determine the DFT of sequence $x(n) = \begin{cases} 1/4 \ ; \text{ for } 0 \le n \le 2 \\ 0 \quad \text{otherwise} \end{cases}$

Q.2. Derive the DFT of sample data sequence $x(n) = \{1, 1, 2, 2, 3, 3\}$ and compute the corresponding amplitude and phase spectrum.

1.



$$[W_3^{nk}] = \begin{array}{c|c} {}_{k}\backslash^{n} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_2^2 & W_3^4 \end{bmatrix} \end{array}$$

$$W_3^{nk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{-1-j\sqrt{3}}{2} & \frac{-1+j\sqrt{3}}{2} \\ 1 & \frac{-1+j\sqrt{3}}{2} & \frac{-1-j\sqrt{3}}{2} \end{bmatrix}.$$

$$\therefore \quad X = [W_3^{nk}][x(n)]$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{-1-j\sqrt{3}}{2} & \frac{-1+j\sqrt{3}}{2} \\ 1 & \frac{-1+j\sqrt{3}}{2} & \frac{-1-j\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 \\ 0 \\ \cancel{0}\ 8/4 \end{bmatrix}$$

2. $x(n) = \{1, 1, 2, 2, 3, 3\}$

$\therefore N = 6.$ (6 point DFT)

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$$[X(k)] = [W_N^{nk}][x(n)]$$

$$[X(k)] = [W_6^{nk}][x(n)]$$

$$W_6^{nk} = \begin{array}{c|cccccc} k \backslash n & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ 2 & 1 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10} \\ 3 & 1 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15} \\ 4 & 1 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20} \\ 5 & 1 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{array}$$

where $W_6^{nk} = e^{\left(-\frac{2\pi}{6}\right)nk} = e^{-\frac{\pi}{3}nk}$

$$W_6^{nk} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\pi/3} & e^{-2\pi/3} & e^{-\pi} & e^{-\frac{4\pi}{3}} & e^{-5\pi/3} \\ 1 & e^{-2\pi/3} & e^{-4\pi/3} & e^{-2\pi} & e^{-8\pi/3} & e^{-10\pi/3} \\ 1 & e^{-\pi} & e^{-2\pi} & e^{-3\pi} & e^{-4\pi} & e^{-5\pi} \\ 1 & e^{-4\pi/3} & e^{-8\pi/3} & e^{-4\pi} & e^{-16\pi/3} & e^{-20\pi/3} \\ 1 & e^{-5\pi/3} & e^{-\frac{10\pi}{3}} & e^{-5\pi} & e^{-\frac{20\pi}{3}} & e^{-25\pi/3} \end{bmatrix}$$

$$W_6^{nk} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-j\sqrt{3}}{2} & \frac{-1-j\sqrt{3}}{2} & -1 & \frac{-1+j\sqrt{3}}{2} & \frac{1+j\sqrt{3}}{2} \\ 1 & \frac{-1-j\sqrt{3}}{2} & \frac{-1+j\sqrt{3}}{2} & 1 & \frac{-1-j\sqrt{3}}{2} & \frac{-1+j\sqrt{3}}{2} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1+j\sqrt{3}}{2} & \frac{-1-j\sqrt{3}}{2} & 1 & \frac{-1+j\sqrt{3}}{2} & \frac{-1-j\sqrt{3}}{2} \\ 1 & \frac{1+j\sqrt{3}}{2} & \frac{-1+j\sqrt{3}}{2} & -1 & \frac{-1-j\sqrt{3}}{2} & \frac{1-j\sqrt{3}}{2} \end{bmatrix}$$

Q3. Find the DFT of the finite length sequence of length N when
$x(n) = a^n \quad ; 0 < a < 1$.

$x(n) = \{1, a, a^2, a^3 \ldots, a^{N-1}\}$.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} \left(a e^{\frac{-j2\pi k}{N}}\right)^n$$

$$X(k) = 1 + a e^{\frac{-j2\pi k}{N}} + a^2 e^{\frac{-j4\pi k}{N}} + \ldots\ldots + a^{N-1} e^{\frac{-j2\pi k (N-1)}{N}}$$

$$\left[\text{Sum of A.P} = \frac{a(1-r^n)}{1-r}\right]$$

$$\therefore X(k) = \frac{\left[1 - \left(a e^{\frac{-j2\pi k}{N}}\right)^N\right]}{1 - a e^{\frac{-j2\pi k}{N}}}$$

$$\boxed{X(k) = \frac{1 - a^N}{1 - a e^{\frac{-j2\pi k}{N}}}} \text{ — Ans} \qquad \left\{\because e^{-j2\pi k} = \cos(2\pi k) = 1\right\}$$

Q4. Find the 4-point DFT of the sequence
$x(n) = \cos \frac{n\pi}{4}$

$$[X(k)] = [W_N^{nk}][x(n)]$$

for 4 point DFT, $N=4$

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$\therefore \quad X(k) = [W_4^{nk}][x(n)]$

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$X(k) = \begin{bmatrix} 1+\sqrt{2} \\ 1-j\sqrt{2} \\ 1 \\ 1+j\sqrt{2} \end{bmatrix} \quad \text{-Ans}$$

Q5 Find the inverse DFT of $X(k) = \{1, 2, 3, 4\}$.

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{+j2\pi nk}{N}} \qquad n = 0, 1, \ldots, N-1$$

$$[x(n)] = [W_N^{-1}]\,[X(k)]$$

$$x_N = W_N^{-1} X_N$$

$$x_N = \left[\frac{W_N^{*}}{N}\right] X_N$$

$$W_N^{*} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$x_N = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

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$$x_N = \left\{ \frac{5}{2}, \frac{-1-j}{2}, \frac{-1}{2}, \frac{-1+j}{2} \right\}$$

Circular Convolution :–

Circular Convolution can be found by using DFT

Step 1:– Find the DFT of 2 sequences whose convolution is to be calculated

Step 2:– Multiply the DFTs of both the sequences. eg. $x_1(n_1) \cdot x_2(n_1)$, $x_1(n_2) \cdot x_2(n_2)$ etc

Step 3:– Take the IDFT of the obtained multiplication.

Q. Calculate circular convolution of following sequences using DFT. when

$x_1(n) = \{1, 1, 2, 1\}$

$x_2(n) = \{1, 2, 3, 4\}$

$$X_1(K) = [W_4][x_1(n)]$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

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$X_1(k) = \{5, -1, 1, -1\}$

Similarly $X_2(k) = \{10, -2+2j, -2, -2-2j\}$

Step 2: Multiplication :-

$X(k) = X_1(k) \cdot X_2(k) = \{50, 2-2j, -2, 2+2j\}$.

Step 3:- Inverse DFT of $X(k)$:-

$$x(n) = \frac{1}{N}[W_4^*][X(k)]$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 50 \\ 2-2j \\ -2 \\ 2+2j \end{bmatrix}$$

$$x(n) = \begin{bmatrix} 13 \\ 14 \\ 11 \\ 12 \end{bmatrix}$$

$\therefore$ Circular convolution of $x_1(n)$ and $x_2(n)$ is $\{13, 14, 11, 12\}$

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⓪ Fast Fourier Transform Algorithm :-

Q. Why we switched from DFT to FFT?

Ans FFT reduces the complexity i.e, no of adders and multipliers.

DFT :- $X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$, $k = 0, 1, \ldots, N-1$

$$X(0) = \sum_{n=0}^{N-1} x(n) \cdot W_N^0$$

$$= x(0) W_N^0 + x(1) W_N^0 + \ldots \quad x(N-1) W_N^0$$

No. of adders required $= N(N-1)$
No. of multipliers $= N^2$.

for large value of N, the no. of adders and multipliers required becomes very large. Hence complex.

$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$, $n = 0, 1, 2 \ldots, N-1$

$\frac{N}{2}$-point $\qquad$ $\frac{N}{2}$-point

$\frac{N}{2}\left(\frac{N}{2} - 1\right) \simeq \frac{N^2}{2}$ no. of multipliers. addition

$\left(\frac{N}{2} - 1\right)^2$ no. of multipliers

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FFT

Decimation-in-Time | Decimation-in-frequency.

1) Decimation-in-time :-

$x(n) \longrightarrow N = 2^{r}$

$x_e(n)$ even, $x_o(n)$

$x_e(n) = x(2n) \; ; \; n = 0, 1, 2 \ldots, \frac{N}{2}-1$

$x_o(n) = x(2n+1) \; ; \; n = 0, 1, 2 \ldots, \frac{N}{2}-1$

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \; ; \; n = 0, 1, \ldots, N-1$$

$$= \sum_{n=\text{even}} x(n) W_N^{nk} + \sum_{n=\text{odd}} x(n) W_N^{kn}$$

$$= \sum_{l=0}^{\frac{N}{2}-1} x_e(l) W_N^{2lk} + \sum_{l=0}^{\frac{N}{2}-1} x_o(l) W_N^{(2l+1)k}$$

$$= \underbrace{\sum_{l=0}^{\frac{N}{2}-1} x_e(l) W_{N/2}^{lk}}_{\frac{N}{2}\text{-pnt DFT of } x_e(l)} + \left[\sum_{l=0}^{\frac{N}{2}-1} x_o(l) W_{N/2}^{lk}\right] W_N^{k}$$

By using property of Periodicity;

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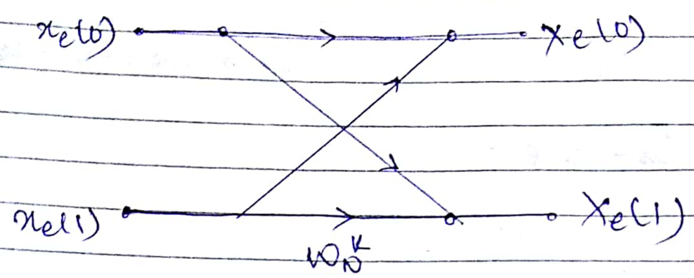
$X(K) = X_e(K) + W_N^K \cdot X_o(K)$; $K = 0, 1 \ldots, N-1$



$$W_N^{K+\frac{N}{2}} = W_N^K \cdot W_N^{N/2}$$

$$W_N^{K+\frac{N}{2}} = -W_N^K \qquad \left\{ W_N^{N/2} = -1 \right\}$$

$$\left\{ W_N^{N/2} = e^{-j\left(\frac{2\pi}{N}\right)\cdot\frac{N}{2}} = -1 \right\}$$

$$\boxed{\gamma = \log_2 N}$$

γ = No. of stage

→ $N \cdot \gamma = N \log_2 N$ = No. of additions

→ $\frac{N}{2} \log_2 N$ = No. of multiplies

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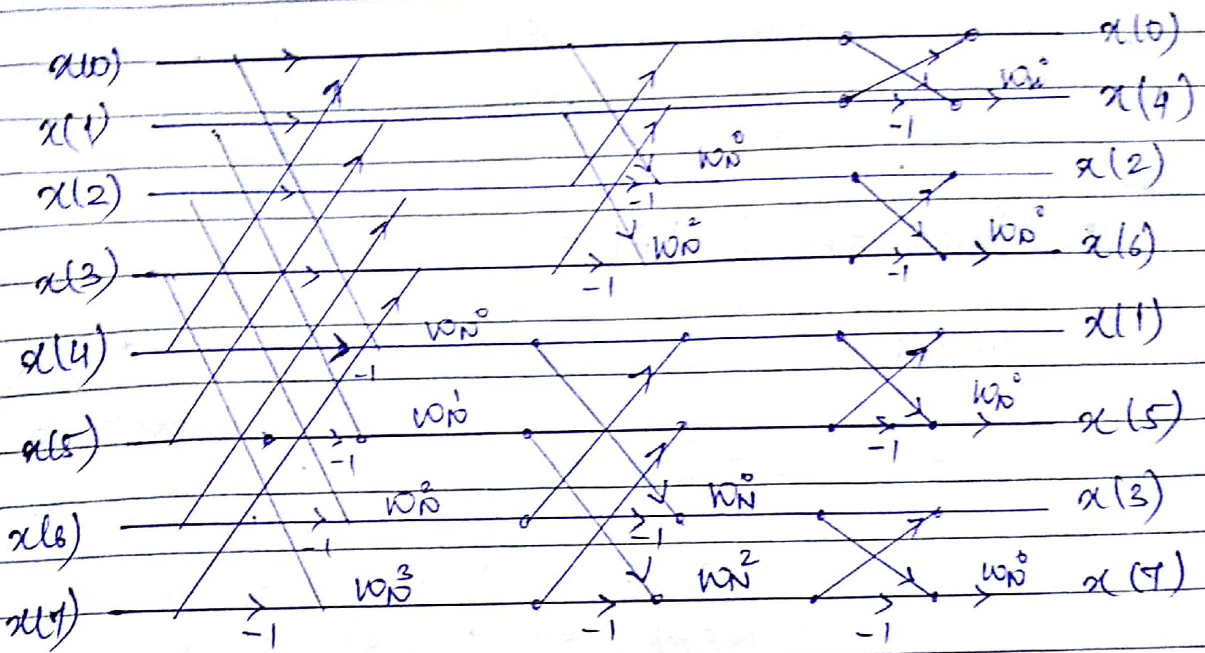
2) Decimation-in-Frequency (FFT-Algorithm):-

We will break (decimate) the freq. term and break the DFT.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$



DIF for N=8

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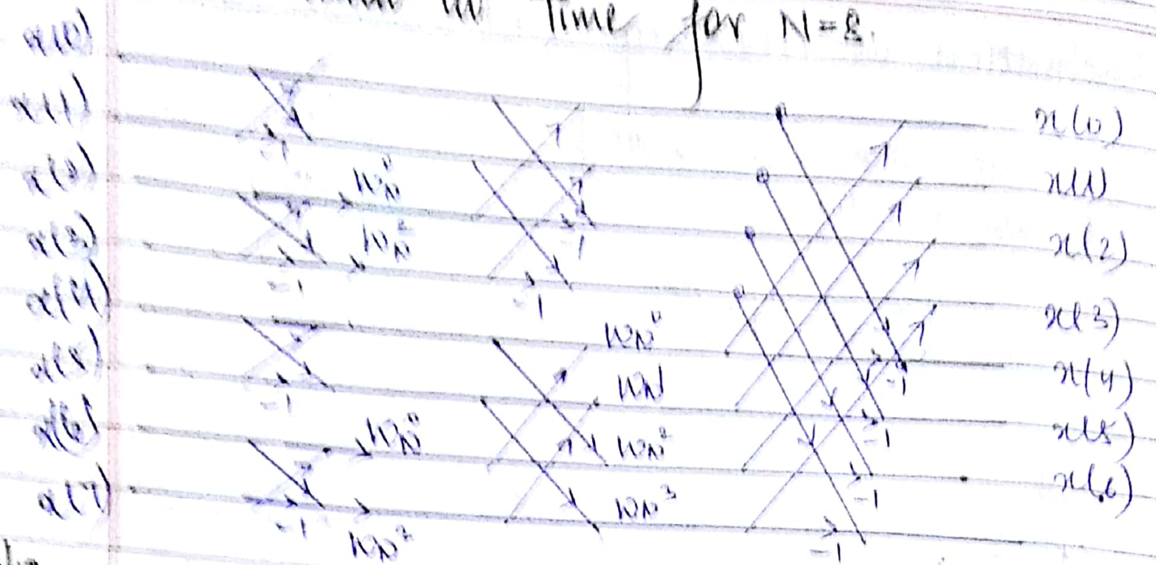
Decimation in Time for N=8.



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• Periodicity of DFT :-

① Periodicity :

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x(n+N) \longleftrightarrow x(n) \quad \forall\ n$

$X(K+N) \longleftrightarrow X(K) \quad \forall\ K$

② Linearity :-

If $x_1(n) \longleftrightarrow X_1(K)$

& $x_2(n) \longleftrightarrow X_2(K)$

then $a_1x_1(n) + a_2x_2(n) \longleftrightarrow a_1X_1(K) + a_2X_2(K)$

③ Circular symmetry :-

If $x(n) = x((n-K))_N$

$= x(n-K, \text{modulo} N)$

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(4) Symmetry Properties :-

$$x(n) = x_R^e(n) + x_R^o(n) + j\left(x_I^e(n) + x_I^o(n)\right)$$

$x_R^e(n)$ = real sequence with even property.
$x_R^o(n)$ = real sequence with odd property.
$x_I^e(n)$ = imaginary sequence with even property.

$$X(K) = X_R^e(K) + X_R^o(K) + j\left[X_I^e(K) + X_I^o(K)\right]$$

where $\boxed{x_R^e(n) \longleftrightarrow X_R^e(K)}$

$\boxed{x_R^o(n) \longleftrightarrow X_I^o(K)}$

$\boxed{x_I^e(n) \longleftrightarrow X_I^e(K)}$

$\boxed{x_I^o(n) \longleftrightarrow X_R^o(K)}$

(5) Time Reversal :-

$$x(N-n) \longleftrightarrow X(N-K)$$

(6) Circular Time Shift :-

$$x((n-l))_N \xleftrightarrow{DFT} X(k) \cdot e^{-j\frac{2\pi k l}{N}}$$

(7) Circular Frequency Shift :-

$$x(n)\, e^{\frac{j2\pi k l}{N}} \xleftrightarrow{DFT} X((K-l))_N$$

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8). Complex Conjugate :-

$$x^*(n) \longleftrightarrow X^*(N-k)$$

(9). Multiplication of two sequences :-

circular convolution

$$x_1(n) \cdot x_2(n) \longleftrightarrow \frac{1}{N}\{X_1(k) \cdot Ⓝ X_2(k)\}$$

(10). Multiplication of two DFTs :-

$$X_3(k) = X_1(k) \cdot X_2(k) \xleftrightarrow{IDFT} \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N = x_3(m)$$

where $m = 0, 1, \ldots N-1$

$$\xleftrightarrow{IDFT} x_1(n) \cdot Ⓝ x_2(n)$$

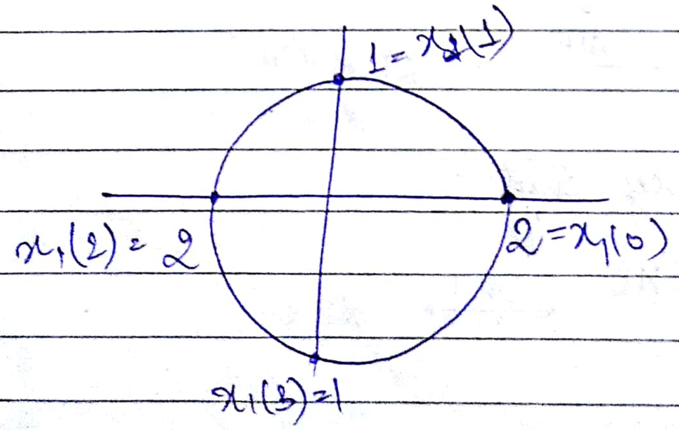
(11) Circular Correlation :-

$$x(n) Ⓝ y^*(-n) \longleftrightarrow X(k) \cdot Y^*(k)$$

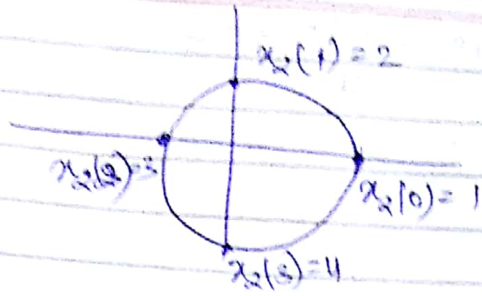
Q. $x_1(n) = \{2, 1, 2, 1\}$ and $x_2(n) = \{1, 2, 3, 4\}$

find $x_1(n) Ⓝ x_2(n)$

Plot $x_1(n)$ in anticlockwise direction ;



$x_2(n) \longrightarrow$



But we need $x_2(-n)$

→ from Periodicity Property :- $x_2(-n) = x_2(N-n)$

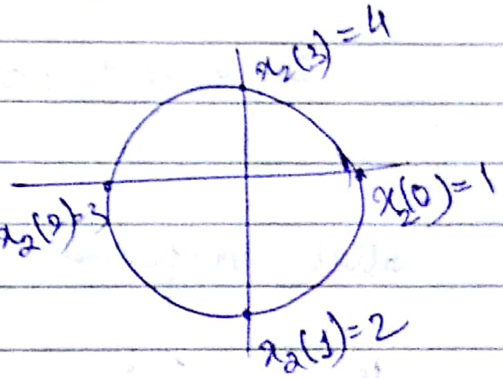
$\therefore \quad x_2(-n) = x_2(4-n)$.

$x_2(4-\cancel{n}) = x_2[-(n-4)]$

for $x(n-p) \Rightarrow$ shift $x(n)$ by 'p' points in anticlockwise direction

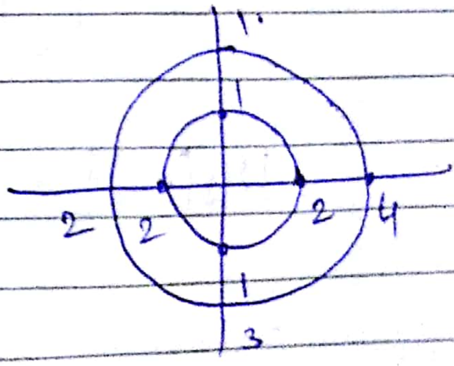
$x[-(n-4)] \Rightarrow$ shift $x(-n)$ by '4' points in anti

$\therefore \; x_2(-n) = x_2(4-n) \longrightarrow$



$x_2(0) = x_2(4)$
$x_2(-1) = x_2(3)$
$x_2(-2) = x_2(2)$
$x_2(-3) = x_2(1)$

$x_3(1) = 8+1+4+3$
$= 16$



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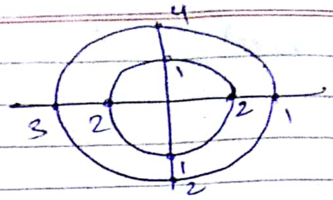
$x_3(0) = 2+4+6+2$
$= 14$



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Q. Design a HPF linear phase s/m using Kaiser window with specifications.

$$|H(e^{j\omega})| \le \delta_2 \quad ; \quad |\omega| \le \omega_s$$
$$1-\delta_1 \le |H(e^{j\omega})| \le 1+\delta_1 \quad ; \quad \omega_p \le |\omega| \le \pi$$

$\delta_1 = 0.024$, $\delta_2 = 0.021$, $\omega_s = 0.35\pi$ rad/sample
$\omega_p = 0.5\pi$ rad/sample.

Ideal response for HPF :-

$$H_d(\omega) = \begin{cases} 0 & ; |\omega| < \omega_c \\ e^{-j\omega\left(\frac{N-1}{2}\right)} & ; \omega_c \le |\omega| \le \pi \end{cases}$$

$$|H_d(\omega)| = \begin{cases} 0 & ; |\omega| < \omega_c \\ 1 & ; \omega_c \le \omega \le \pi \end{cases}$$

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$$h_d(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_d(\omega) e^{j\omega n} \, d\omega .$$

$$= \frac{1}{2\pi}$$

$$h_d(n) = \frac{\sin \pi(n-\alpha)}{\pi(n-\alpha)} - \frac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)}$$

$$h(n) = h_d(n) \cdot \omega(n)$$

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Ex. $x(n) = \{1, 2, 3, 4\}$. Find 4-point DFT

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}} \quad ; K = 0, 1, \dots, N-1$$

$$X(K) = \{10, -2+2j, -2, -2-2j\}$$

Observation :-

Only for Real seq.

$$X(0) = x(0) + x(1) + x(2) + x(3)$$

$$X(2) = x(0) - x(1) + x(2) - x(3)$$

$$X(1) = X^*(3) \rightarrow \text{conjugate symmetry Property}$$

- Linear Convolution by using DFT :-

If $x_1(n) \xrightarrow{DFT} X_1(K)$ {'P' - length}

$x_2(n) \xrightarrow{DFT} X_2(K)$ {'L' - length}

$$\boxed{x_3(n) = IDFT[X_1(K) \cdot X_2(K)]} = x_1(n) \circledast x_2(n)$$

$$\boxed{\text{length of } x_3(n) = P + L - 1}$$

If the length of $x_3(n) < P+L-1$, then it is not linear ~~circul~~ convolution, it will be circular convolution

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⊙ Circular convolution from Linear Convolution :-

Q. $x_1(n) = \{1, 2, 3, 4\}$ (4)
$x_2(n) = \{4, 3, 2, 1\}$ (4-point)

Linear Convolution, $x_3(n) = x_1(n) \circledast x_2(n)$
$= \{4, 11, 20, 30, 20, 11, 4\}$

$\Rightarrow 4 + 4 - 1 = 7$ point in $x_3(n)$

$\{4, 11, 20, 30 \,|\, 20, 11, 4\} \; 0$

24 22 24 30

$\therefore$ Circular Convolution (4 point) $= \{24, 22, 24, 30\}$

Q. $x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\}$ find DFT

$\Rightarrow x(0) = 1+1+1+1+1+1+1+1 = 8.$

$\Rightarrow x\left(\frac{N}{2}\right) = x(4) = x(0) - x(1) + x(2) - x(3) + x(4) - x(5) + x(6) - x(7)$
$= 0$

$$X(k) = \sum_{n=0}^{7} x(n) e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{7} 1 \cdot e^{\left(\frac{-j2\pi k}{8}\right)^n}$$

$$\left\{ \sum_{n=a}^{b} r^n = \frac{r^a - r^{b+1}}{r^a - r^b} \right.$$

$$X(k) = \frac{1 - e^{-j2\pi k}}{1 - e^{\frac{-j2\pi k}{8}}}$$

$$X(k) = \begin{cases} 0 & ; \text{ for } k = 1, 2, 3, 4, 5, 6, 7 \\ 1 & ; \; k = 0 \end{cases}$$

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Unit - III
FIR FILTER DESIGN

I/p Analog signal (Continuous-time signal) → Sampling → ADC → Digital

- Filter Designing is done by the approximation.

- The Filter Design is implemented with Digital computations. It filters signals derived from continuous time signal by periodic ~~signal~~ Sampling followed by ADC. That's why these discrete filters are called digital filter.

Stages of Digital filter :-

1) Specification of desired properties of system
2) The approximation of specifications by using a causal discrete time system.
3) Realization of the system.

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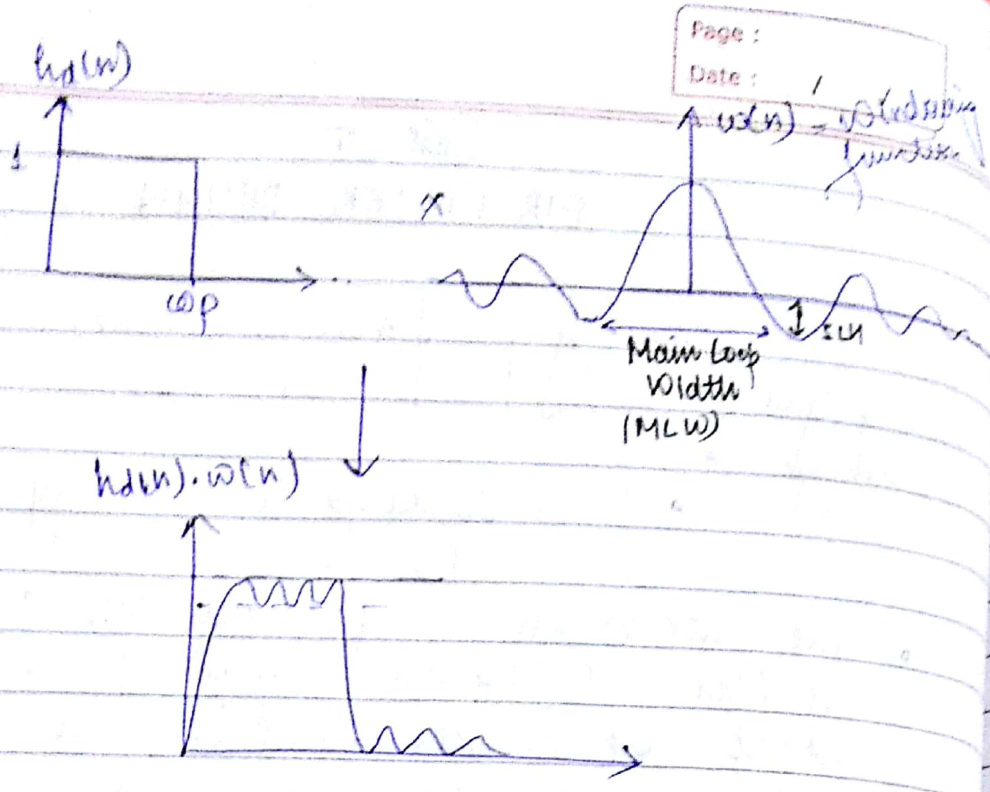
- FIR Filter Design by using Windowing technique :-

$$hd(n) = w(n) \cdot h$$



- Transition Bandwidth of Ideal LPF = 0.
  Transition Bandwidth of Practical LPF ≠ 0.

- In Ideal L.P.F, Pass Band and Stop Band are FLAT.
  In practical L.P.F, PB and S.B are NOT FLAT.

SLH = Side Lobe height (error)

MLW and SLH should be as low as possible.

DTFT of hd(n) :-

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & ; |\omega| < \omega_p < \pi \\ 0 & ; \text{otherwise} \end{cases}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{jn\omega}\, d\omega$$

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$\boxed{N \cdot \Delta f = \text{const}}$ where Δf = MLW = Transition width

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$$h_d(n) = \begin{cases} \dfrac{\sin \omega_p (n-\tau)}{\pi (n-\tau)} & , n \neq \tau \\ \dfrac{\omega_p}{\pi} & , n = \tau \end{cases}$$

Property of $h_d(n)$:- $\boxed{h_d(n) = h_d(N-1-n)}$

$$\frac{\sin \omega_p (-(n-\tau))}{\pi (n-\tau)} = \frac{\sin \omega_p (N-1-n-\tau)}{\pi (N-1-n-\tau)}$$

$$\Rightarrow -(n-\tau) = N-1-n-\tau$$

$$\Rightarrow \boxed{\tau = \frac{N-1}{2}}$$

where N = No. of samples in impulse response.

→ As $N\uparrow$ → MLW ↓ and SLH ↑
so there is a trade-off b/w error and signal B.W

⦿ Different Windows Technique :-

① Rectangular Window :- $w(n) = \begin{cases} 1 & ; 0 \le n \le N-1 \\ 0 & ; \text{otherwise} \end{cases}$

Disadvantage :- Rectangular window is rarely used becoz of Gibb's Phenomenon

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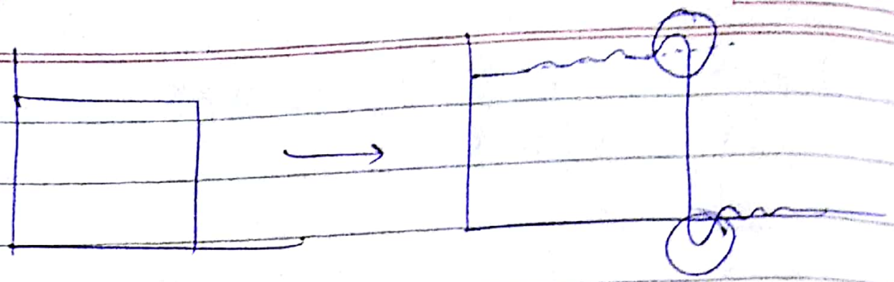
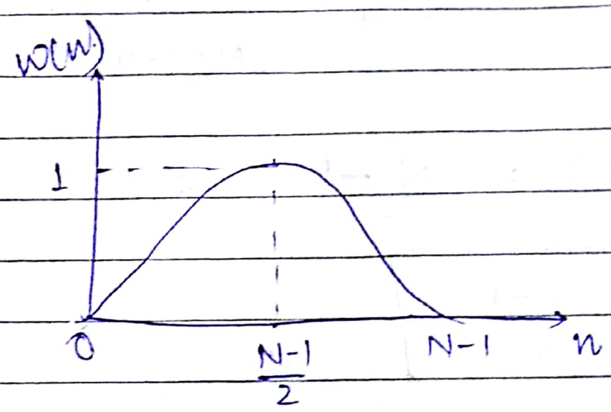
Gibb's Phenomenon

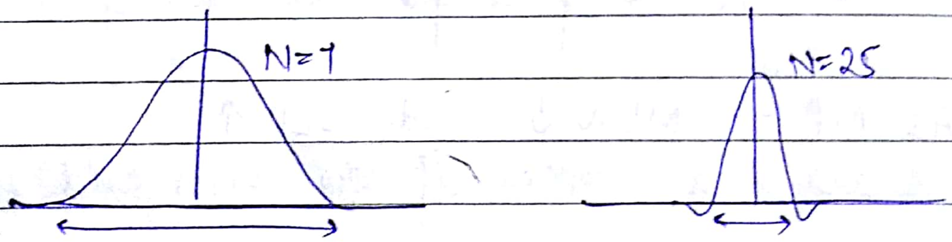


② Hamming Window :-

$$w(n) = \begin{cases} \frac{1}{2}\left(1 - \cos\frac{2\pi n}{N-1}\right) & ; 0 \le n \le N-1 \\ 0 & ; \text{otherwise} \end{cases}$$



N = odd integer



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③ Hamming Window :-

$$w(n) = \begin{cases} 0.54 - 0.46 \cos\frac{2\pi n}{N-1} & ; 0 \le n \le N-1 \\ 0 & ; \text{otherwise} \end{cases}$$

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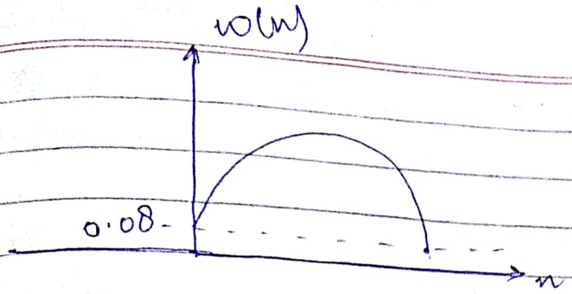


∴ $W(0) = 0.08$ for $n=0$, that's why it is also called raised cosine window.

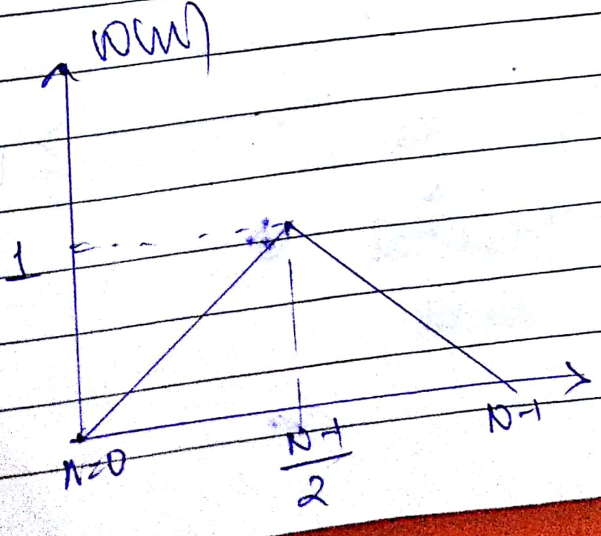
④ Blackmann Window :-

$$w(n) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right) & ; 0 \le n \le N-1 \\ 0 & ; \text{otherwise} \end{cases}$$

~~In this there~~ is 2nd harmonic of cosine funⁿ.

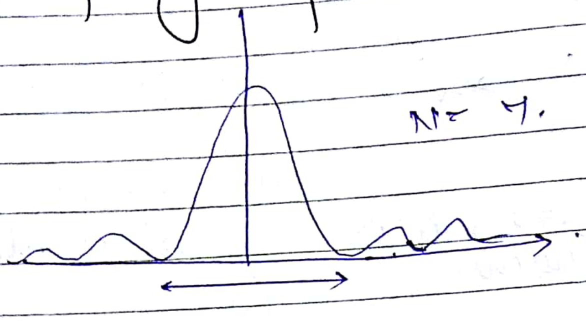
⑤ Barlett Window (Triangular Window) :-

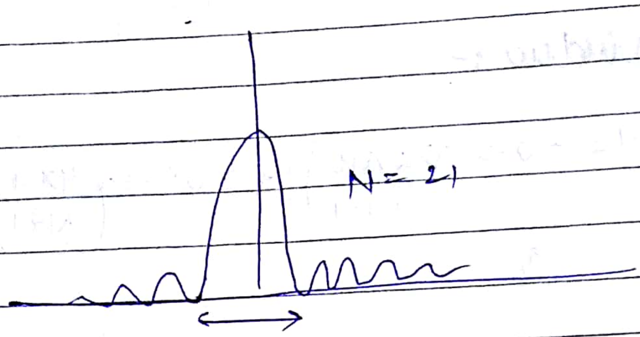
$$w(n) = \begin{cases} \frac{2n}{N-1} & ; 0 \le n \le \frac{N-1}{2} \\ 2 - \frac{2n}{N-1} & ; \frac{N-1}{2} \le n \le N-1 \end{cases}$$



There is no direct relation b/w 'N' and 'SLW'.
SLW depends upon windowing technique

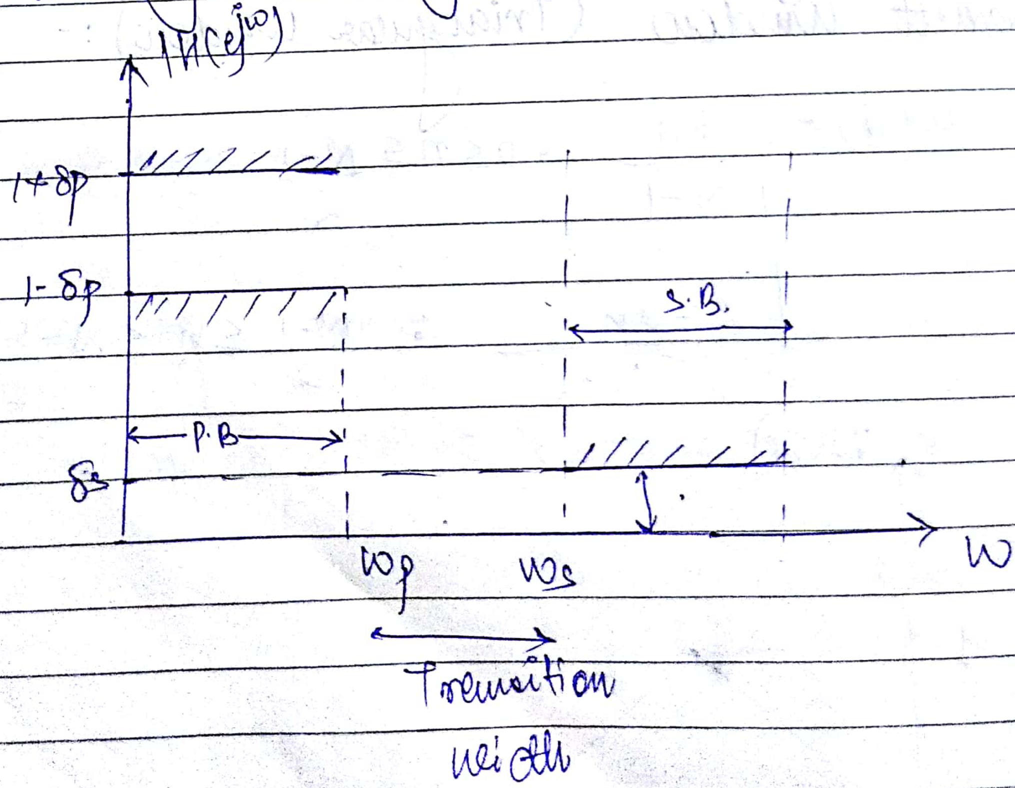
Frequency Response :-





→ Advantage of Triangle window is that its side-lobes are +ve.

Example:-



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Windowing Technique	Side lobe Amplitude (dB)	Transition Width, Δf	Stopband Attenuation (dB)
1) Rectangular	-13	$0.9/N-1$	-21 dB
Bartlett	-25		-25 dB
2) Hanning	-31	$3.1/N-1$	-44 dB
3) Hamming	-41	$3.3/N-1$	-53 dB
4) Blackman	-57	$3.5/N-1$	-74 dB

Ex. Design a LPF with $|H_d(e^{j\omega})| = \begin{cases} 1 & ; |\omega| \le 0.2\pi \\ 0 & ; 0.2\pi < |\omega| < \pi \end{cases}$

and $N=25$

$$h_d(n) = F^{-1}[H_d(e^{j\omega})]$$

$$= \cancel{\phi} \Rightarrow H_d(e^{j\omega}) = \begin{cases} e^{j\omega} & ; -0.2\pi \le \omega \le 0.2\pi \\ 0 & ; \text{otherwise} \end{cases}$$

$$\therefore h_d(n) = \frac{\sin\left(\omega_p\left(\frac{N-1}{2}-n\right)\right)}{\left(\frac{N-1}{2}-n\right)\pi}$$

$$h_d(n) = \sin\left(\frac{0.2\pi(n-12)}{(n-12)\pi}\right)$$

$$h(n) = h_d(n) \times w(n)$$

where $w(n)$ = window func.

Let take (1) Rectangular Windowing technique

1) Rectangular Window, $\Delta\omega = \dfrac{2\pi \times 0.9}{N-1}$

$= \dfrac{2\pi \times 0.9}{25-1}$

$= 0.235$

2) Hanning Window, $\Delta\omega = \dfrac{2\pi \times 3.1}{N-1}$

$= \dfrac{2\pi \times 3.1}{24} = 0.811$

Hamming Window, $\Delta\omega = \dfrac{2\pi \times 3.3}{24} = 0.8635$

Blackmann Window, $\Delta\omega = \dfrac{2\pi \times 3.5}{24} = 0.9150$

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⑥ Kaiser Window :-

$$w(n) = \begin{cases} \dfrac{I_0\left[\beta\left(1-\left(\dfrac{n-\alpha}{\alpha}\right)^2\right)^{1/2}\right]}{I_0(\beta)} & ; \; 0 \le n \le N-1 \\ 0 & ; \; \text{otherwise} \end{cases}$$

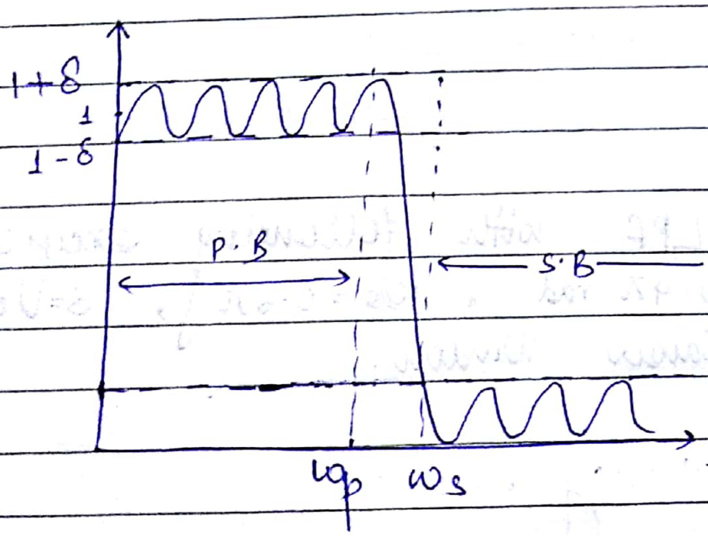
length N

Here, design parameters are -

1) N, length of window {N is always odd integer}

2) β, shape parameter

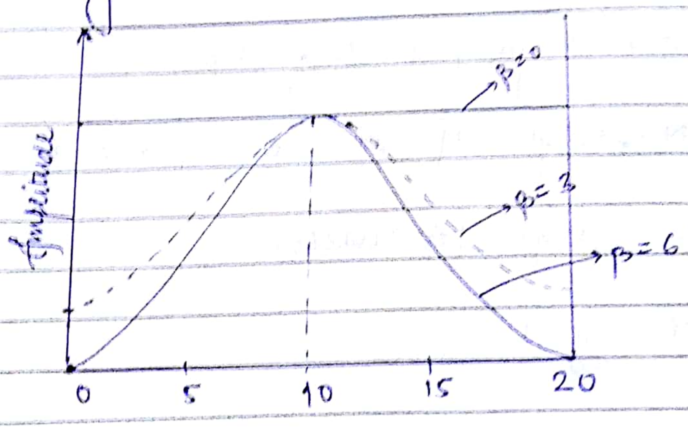
$$\alpha = \frac{N-1}{2}$$



- δ = Peak approximation error
- $w_{p\,max}$: $|H(e^{j\omega})| \ge 1-\delta$
- $\Delta w = \omega_s - \omega_p$ = transition width
- $A = -20\log_{10}\delta$

$$\beta = \begin{cases} 0.1102(A-8.7) & ; A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & ; 21 \le A \le 50 \\ 0.0 & ; A < 21 \end{cases}$$

for $\beta = 0$, Kaiser window will behave as rectangular window.



$$\Delta\omega,\ \delta \longrightarrow N-1 = \frac{A-8}{2.285\,\Delta\omega}$$

Example: Design a LPF with following specifications, $\omega_p = 0.4\pi$ rad., $\omega_s = 0.6\pi$, $\delta = 0.001$ by using Kaiser Window.

Step 1:- Cut-off frequency, ω_c :-

$$\boxed{\omega_c = \frac{\omega_p + \omega_s}{2}}$$

$$\boxed{\omega_c = 0.5\pi}$$

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Step 2: To determine Kaiser Window parameters :- (i.e, N & β)

$\Delta\omega = \omega_s - \omega_p$

$\boxed{\Delta\omega = 0.2\pi}$

$A = -20\log_{10}\delta$

$= -20\log_{10} 10^{-3}$

$A = 60$

$\Rightarrow N-1 = \dfrac{60-8}{2.285 \times 0.2 \times \pi} = 36.2$

$\boxed{N = 37.2} \simeq 39$ (∵ N should be odd)

$\beta = 0.1102(60-8.7)$

$\boxed{\beta = 5.65}$

Step 3:- Now, $h(n) = h_d(n) \cdot w(n)$ = Impulse response.

$$h_d(n) = \begin{cases} \dfrac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} & \text{for LPF} \\ 0 & \end{cases}$$

$$w(n) = \frac{I_0\left[\beta\left(1-\left[\frac{n-\alpha}{\alpha}\right]^2\right)^{1/2}\right]}{I_0(\beta)}$$

$$= \frac{I_0\left[5.65\left(1-\left[\frac{n-19}{19}\right]^2\right)^{1/2}\right]}{I_0(5.65)} \qquad \left[\alpha = \frac{N-1}{2}\right]$$

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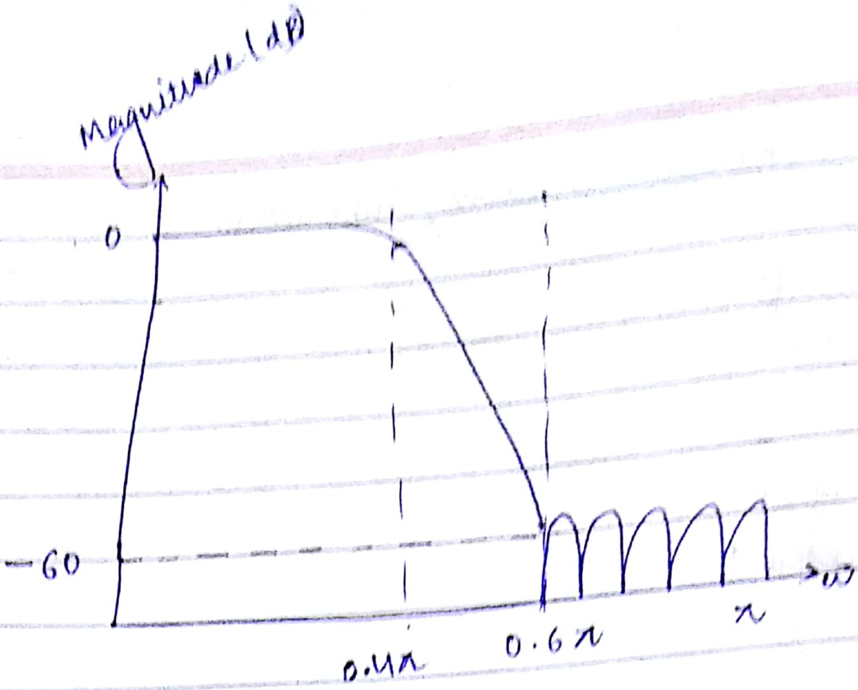

- Magnitude Response of LPF.

→ types of window

↳

↳

↳

→ LPF using above window

→ Kaiser Window