Study of an Annular Model of Planetary Convection

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Objectives

To study the behaviour of thermal convection in an annulus.
This was done by combining the solutions of two sub-problems:

1. Solving the Advection-Diffusion equation for a scalar quantity given the velocity field and Dirichlet BCs.
2. Solving the vorticity equation and consequently deriving the stream function, ψ, to compute the velocity field, again, for Dirichlet BCs.

Introduction

Thermal convection occurs because of fluids undergoing thermal expansion. This leads to changes in buoyancy within a fluid with a non-uniform temperature field that ultimately leads to convection under the influence of gravity. An everyday example of this phenomena is how convection currents develop in water boiling on a pan heated from below (Figure 1).
An annulus is a plane figure consisting of the area between a pair of concentric circles. As such it can be used as a simplified description of the volume between 3D spheres.
Thus thermal convection in an annulus is an interesting problem applicable in many situations. One example would be the flow of molten iron in the Earth's outer core (Figure 1) which in turn affects the magnetic field of the earth [1].

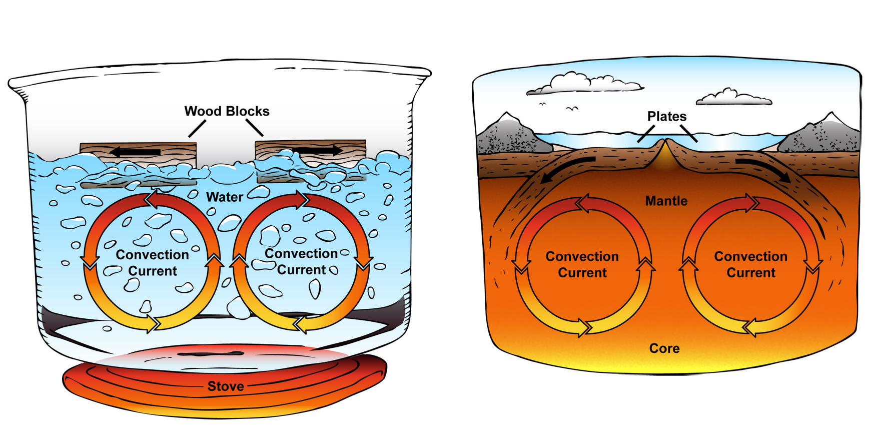

Figure 1: Motion of Liquid iron in the Earth's Outer Core [2]

Useful Definitions

- R_a: The Rayleigh number is a measure of the thermal driving force.
- P_r: The Prandtl number is the ratio of the fluid kinematic viscosity to the thermal diffusivity
- κ: The ratio between the impact of advection to diffusion for the temperature being convected
- BCs: Boundary Conditions

Governing Equations

The non-dimensionalized equations that we are are solving are

1. The advection-diffusion equation for the temperature field
$$T_t + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T \qquad (1)$$
2. The vorticity equation for the velocity field
$$P_r^1(\mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = \nabla p R_a T \hat{g} + \nabla^2 \mathbf{u}$$
3. The incompressibility of the fluid
$$\nabla \mathbf{u} = 0 \qquad (2)$$
This allows $\mathbf{u}$ to be written in the form of a stream function, ψ, such that
$$u_{(r)} = \frac{\psi_\theta}{r} \quad \text{and} \quad u_{(\theta)} = \psi_r \qquad (3)$$

Discretization

The domain is restricted to
$$1 < r < b \text{ and } 0 \le \theta < 2\pi$$
So it is split into a grid of size $(M+1) \times N$ where
$$\delta r = \frac{b-1}{M} \text{ and } \delta\theta = \frac{2\pi}{N}$$

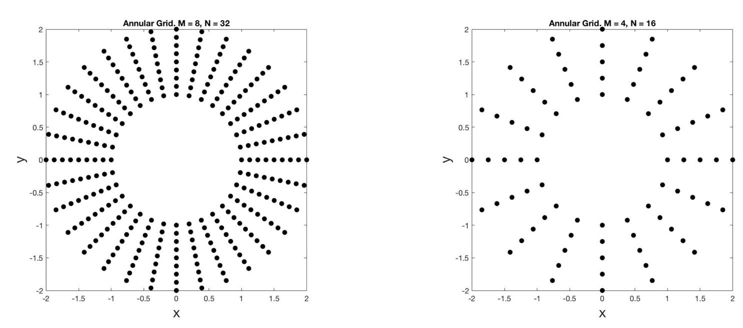

Figure 2: Discrete points on the Annular Grid for $b = 2$. Image on the right has M and N halved compared to the image on the left.

Resultant Plots

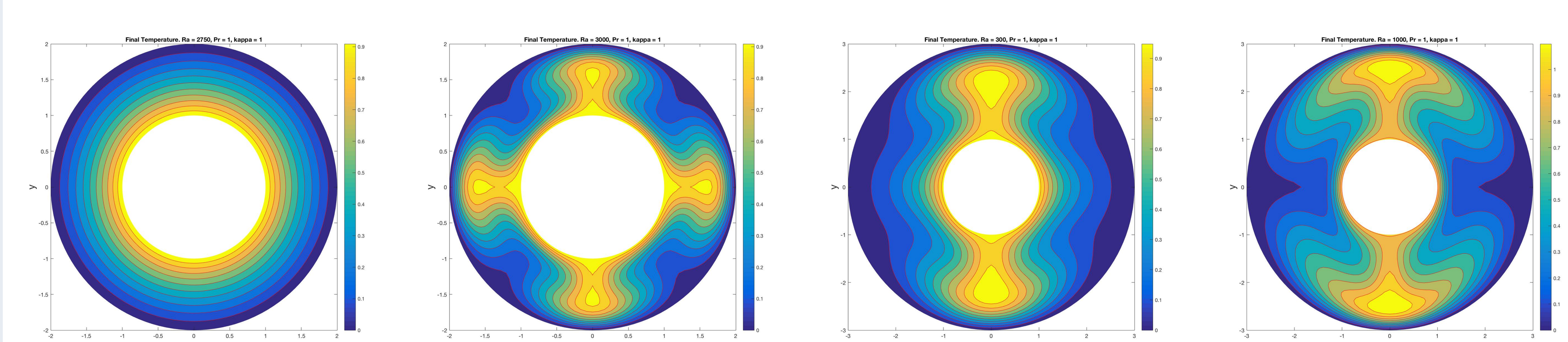

Figure 3: $\boldsymbol{b = 2}$, Left: $R_a = 2750$. Right: $R_a = 3000 R_c \approx 2850$. $\boldsymbol{b = 3}$, Left: $R_a = 300$. Right: $R_a = 1000$. With $P_r = 1 = \kappa$

Advection-Diffusion Equation

In order to solve Equation 1, an operator splitting approach is taken by first solving for $T_t = -\mathbf{u} \cdot \nabla T$ (Advection) and then $T_t = \kappa \nabla^2 T$ (Diffusion) independently.
The Advection part is solved by using a **Richtmyer's second-order Lax-Wendroff routine** that conserves the scalar field, T, being an explicit method. This routine is $\mathcal{O}(\delta r^2, \delta\theta^2, \delta t^2)$ [3].
The Diffusion part is solved using an **implicit Crank-Nicolson scheme**. This done by solving a system of linear of equations ($Ax = b$) using the **MultiGrid method** that helps compute solutions extremely fast. It does do by accelerating the convergence of an iterative method (**Gauss-Seidel** in our case) by computing an approximation on a coarse grid and using it to provide a correction on the finer grid problem.
Figure 4 shows the impact of the changing the value of κ on the advection-diffusion sub-problem.

Vorticity Equations

The vorticity equation is
$$\omega_t + \mathbf{u} \cdot \nabla \omega = P_r \nabla^2 \omega + P_r F \qquad (4)$$
and it is related to the stream function by
$$-\omega = \nabla^2 \psi \qquad (5)$$
Equation 4 was again solved using operator splitting by solving $\omega_t + \mathbf{u} \cdot \nabla \omega = 0$ (Advection) and then $\omega_t = P_r \nabla^2 \omega + P_r F$ (Diffusion) with modified BCs and fundamentally the same schemes as before. F is taken to be 0 in the full problem. Then $-\omega = \nabla^2 \psi$ is solved using a modified MultiGrid solver and $\mathbf{u}$ computed using Equations 3.

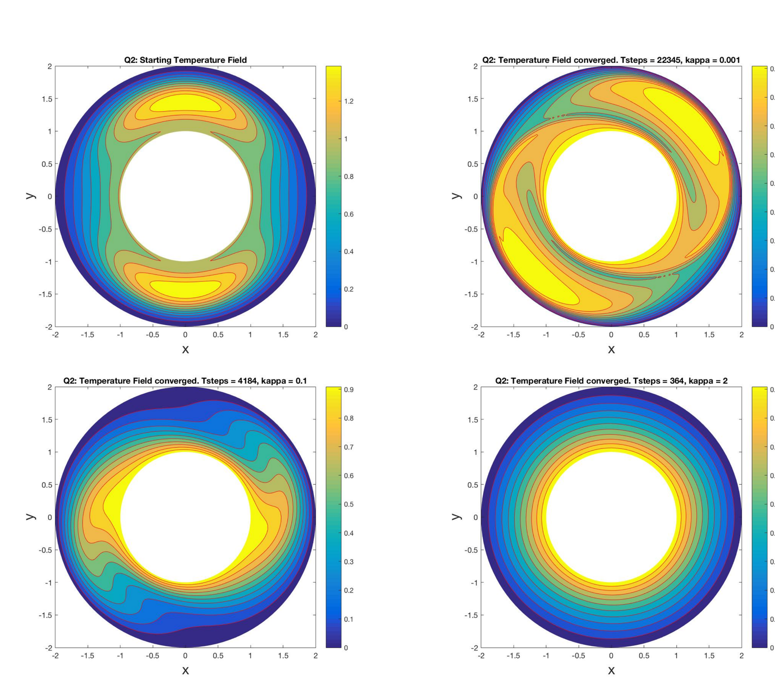

Figure 4: Top Left: Inital T. Top Right: Converged T for $\kappa = 0.001$. BottomLeft: $\kappa = 0.1$. BottomRight: $\kappa = 2$

The Full Problem

The full problem is studied by first solving the Advection-Diffusion equations for T and then computing $\mathbf{u}$ through the vorticity equations for each time step and waiting for the solution to converge.
The main focus of the study was initially seeding a θ-dependent perturbation and checking if it grows or decays when varying R_a. After a critical value, R_c, the behaviour flips. We set $P_r = 1 = \kappa$ and try the compute the value of R_c for various values of b (Figure 3)

Resultant Plots

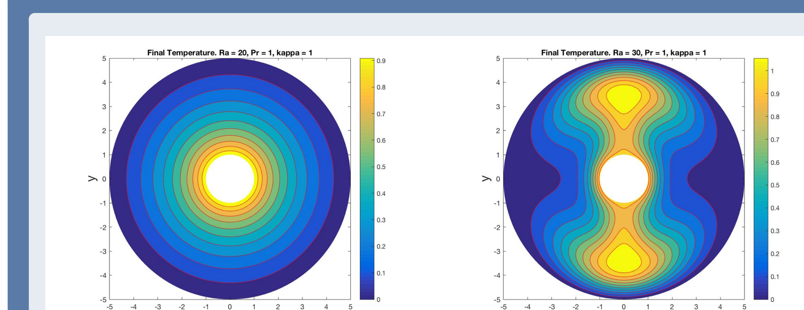

Figure 5: $\boldsymbol{b = 5}$, Left: $R_a = 20$. Right: $R_a = 30$. With $P_r = 1 = \kappa$

Closing Remarks

The Noting that the solution must be studied for various values of b and various values of κ as well this means that the parameter space we are interested in exploring is 4 dimension. This is obviously quite large and thus will not be exhaustively studied. Furthermore, we will stick to assessing situations arising from our specific perturbations.
Finally it is worth noting that a lot of the derived results are specific to the initial conditions and vary massively from situation to the situation. But nevertheless, it seems that the entire routine is function sufficiently and thus can be used, given enough time, to solve any specified problem of this type.

References

[1] R. Monchaux, M. Berhanu, M. Bourgoin, M. Moulin, Ph. Odier, J.-F. Pinton, R. Volk, S. Fauve, N. Mordant, F. Pétrélis, A. Chiffaudel, F. Daviaud, B. Dubrulle, C. Gasquet, L. Marié, and F. Ravelet. Generation of a magnetic field by dynamo action in a turbulent flow of liquid sodium. *Phys. Rev. Lett.*, 98:044502, Jan 2007.

[2] Convection currents - 1. cgc 1d 2016 - 2017, Accessed: April 2017. https://sites.google.com/a/ocsb.ca/cgc-1d/a-unit-2-3-natural-disasters--earthquakes/1-4-convection-currents/2-convection-currents.

[3] Jonathan Mestel. m4n10 lecture notes.

[4] Nathaniel Johnston and John Hammersley. Jacobs portrait poster. From http://www.LaTeXTemplates.com and https://teamwork.jacobs-university.de:8443/confluence/display/CoPandBiG/LaTeX+Poster License: CC BY-NC-SA 3.0 http://creativecommons.org/licenses/by-nc-sa/3.0/.

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