INVERSE

$$A^{-1}A = I$$

INVERSE OF A

THM: FOR $A \in \mathbb{C}^{m \times m}$ THE FOLLOWING ARE EQUIVALENT.

(a) A HAS AN INVERSE A^{-1}

(b) $\operatorname{RANK}(A) = m$

(c) $\operatorname{RANGE}(A) = \mathbb{C}^m$

(d) $\operatorname{NULL}(A) = \{0\}$

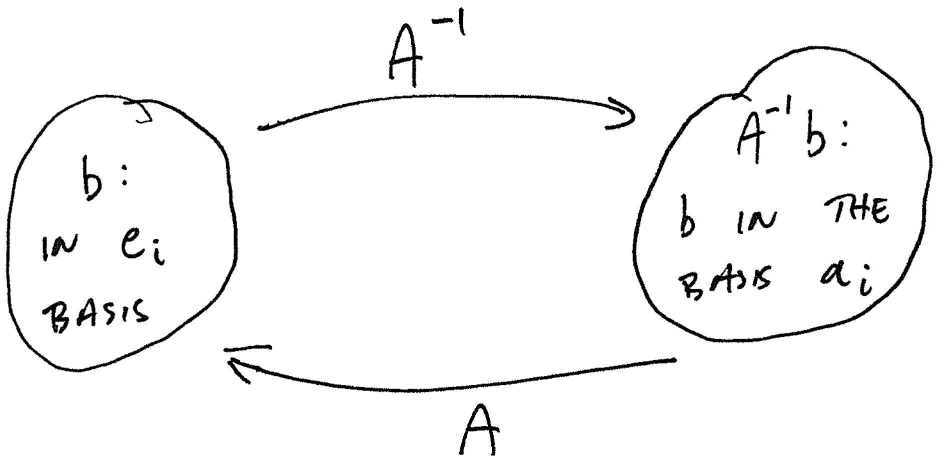
(e) 0 IS NOT AN EIGENVALUE OF A

(f) 0 IS NOT A SINGULAR VALUE OF A

(g) $\det(A) \neq 0$

$(Ax = b)$

SINCE $x = A^{-1}b$ IS THE VECTOR OF COEFFICIENTS OF THE UNIQUE LINEAR EXPANSION OF b IN THE BASIS OF THE COLUMNS OF A, MULTIPLICATION BY A^{-1} CAN BE VIEWED AS A CHANGE OF BASIS.



ORTHOGONAL VECTORS AND MATRICES

DEF: ADJOINT OR HERMETIAN CONJUGATE OF $A \in \mathbb{C}^{m \times n}$

WE WRITE AS $A^* \in \mathbb{C}^{n \times m}$

WHERE
$$a^*_{ij} = \overline{a}_{ji} \quad \left(\text{COMPLEX CONJUGATE OF } a_{ji} \right)$$

EXAMPLE

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad A^* = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{21} & \overline{a}_{31} \\ \overline{a}_{12} & \overline{a}_{22} & \overline{a}_{23} \end{bmatrix}$$

IF $A = A^*$, WE SAY A IS HERMETIAN.

FOR REAL MATRICES.

$$A^* = A^T$$

IF $A = A^T$ THE MATRIX IS SYMMETRIC.

FOR MATRICES ~~WITH~~ A, B WITH COMPATIBLE DIMENSIONS.

$$(AB)^* = B^* A^*$$

(FOR INVERTIBLE MATRICES
$$(AB)^{-1} = B^{-1} A^{-1}$$
)

INNER PRODUCT

$x, y \in \mathbb{C}^m$

$$\longrightarrow x^* y = \sum_{i=1}^{m} \overline{x}_i y_i$$

EUCLIDEAN LENGTH OF A VECTOR

$$\|x\| = \sqrt{x^* x}$$

INNER PRODUCT IS ~~BILEAR~~ BILINEAR.

- $(x_1 + x_2)^* y = x_1^* y + x_2^* y$
- $x^*(y_1 + y_2) = x^* y_1 + x^* y_2$
- $(\alpha x)^* (\beta y) = \bar{\alpha} \beta x^* y$

ORTHOGONAL VECTORS

- $x, y \in \mathbb{C}^m$ ARE ORTHOGONAL IF $x^* y = 0$
- SETS OF VECTORS X AND Y ARE ORTHOGONAL IF

$$x^* y = 0 \quad \forall \; x \in X, \; y \in Y.$$

- A SET OF VECTORS S IS ITSELF ORTHOGONAL IF FOR ALL $x, y \in S$ $x \neq y$ $x^* y = 0$.
- THE ~~A~~ SET ~~OF~~ S IS ~~ORTHOR~~ ORTHONORMAL IF WE ALSO HAVE $\|x\| = 1$ FOR EACH $x \in S$

THM: THE VECTOR IN A ORTHOGONAL SET S ARE LINEARLY INDEPENDENT.

COMPONENTS OF A VECTOR

$\{q_1, q_2, \cdots, q_n\}$ IS AN ~~ORTHON~~ ORTHONORMAL SET AND v IS AN ARBITRARY VECTOR (ALL $\in \mathbb{C}^m$)

DEFINE:

$$r = v - (q_1^* v) q_1 - \cdots - (q_n^* v) q_n$$

r IS ORTHOGONAL TO THE SET $\{q_1, \dots, q_n\}$ SINCE

$$q_i^* r = q_i^* v - (q_1^* v)(q_i^* q_1) - \cdots - (q_n^* v)(q_i^* q_n)$$

SINCE $q_i^* q_j = 0$ FOR $i \neq j$

$$q_i^* q_i = 1$$

$$q_i^* r = q_i^* v - (q_i^* v)\underbrace{(q_i^* q_i)}_{1} = 0$$

THUS,

$$v = r + \sum_{i=1}^{n} \overbrace{(q_i^* v)}^{\text{COEFFICIENT}} q_i$$

VECTOR

$$= r + \sum_{i=1}^{n} \underbrace{(q_i q_i^*)}_{\text{RANK-1}} v$$

IF $\{q_i\}$ IS A BASIS FOR ~~$\mathbb{C}$~~ $\mathbb{C}^n$, THEN $n = m$ AND $r = 0$.

THUS,

$$v = \sum_{i=1}^{m} \overbrace{(q_i q_i^*)}^{\text{PROJECTOR}} v$$

UNITARY MATRICES

$Q \in \mathbb{C}^{m\times m}$ IS UNITARY

IF $Q^* = Q^{-1}$.

FOR REAL MATRICES, A MATRIX IS ORTHOGONAL IF $Q^T = Q^{-1}$

$(Q \in \mathbb{R}^{m\times m})$

FOR A UNITARY MATRIX,

$$Q^* Q = I$$

$$\begin{bmatrix} q_1^* \\ \hline q_2^* \\ \hline \vdots \\ \hline q_m^* \end{bmatrix} \left[\begin{array}{c|c|c|c} q_1 & q_2 & \cdots & q_m \end{array}\right] = I$$

THIS HOLDS IF

$$q_i^* q_j = \delta_{ij} \quad \text{(KRONECKER DELTA)}$$

MULTIPLICATION BY Q^*, Q

$$\begin{tikzcd} \begin{array}{c} b\text{: BASIS} \\ e_i \end{array} \arrow[r, shift left, "Q^*"] & \begin{array}{c} Q^*b\text{:} \\ b \text{ IN THE} \\ \text{BASIS } q_i \end{array} \arrow[l, shift left, "Q"] \end{tikzcd}$$

OTHER PROPERTIES

- $\|Qx\| = \|x\|$
- $(Qx)^*(Qy) = x^* y$