EIGENVALUES AND EIGENVECTORS

$A \in \mathbb{C}^{m \times m}$ (SQUARE)

$x \in \mathbb{C}^{m}$ EIGENVECTOR AND

$\lambda \in \mathbb{C}$ CORRESPONDING EIGENVALUE

IF

$$Ax = \lambda x$$

EIGENSPACE: SET OF THE EIGENVECTORS

SPECTRUM: SET OF ALL EIGENVALUES.

WHY COMPUTE THIS

1. E-VALS AND E-VECS ENCODE INFORMATION ABOUT A.
2. ARISE IN SAY, STABILITY CALCULATIONS.
3. USE E-VECS AS A BASIS TO SOLVE A LINEAR SYSTEM.

EIGENVALUE DECOMPOSITION

$$A = X \Lambda X^{-1}$$

X: NON-SINGULAR $m \times m$ WITH x_k (EIGENVECTOR k) IN THE kTH COLUMN.

Λ: DIAGONAL MATRIX WITH

$\Lambda_{kk} = \lambda_k$, THE KTH

EIGENVALUE.

THIS DECOMPOSITION IS NOT

ALWAYS POSSIBLE!

GEOMETRIC MULTIPLICITY OF λ

NUMBER OF E-VECS

CORRESPONDING TO

EIGENVALUE λ.

ALGEBRAIC MULTIPLICITY

CHARACTERISTIC POLYNOMIAL OF

A

$$P_A(z) = \det(zI - A)$$

THE ROOTS OF $P_A(z)$

ARE THE EIGENVALUES.

$$P_A(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_m)$$

THE ALGEBRAIC MULTIPLICITY

IS THE NUMBER OF TIMES

λ IS A ROOT OF $P_A(z)$.

A E-VAL IS SIMPLE IF IT

HAS ALGEBRAIC MULTIPLICITY 1.

THM: THE ALGEBRAIC MULTIPLICITY

OF λ IS AT LEAST AS

GREAT AS ITS GEOMETRIC

MULTIPLICITY.

DEFECTIVE MATRICES AND EIGENVALUES

- AN EIGENVALUE IS DEFECTIVE IF ITS ALGEBRAIC MULTIPLICITY EXCEEDS ITS GEOMETRIC MULTIPLICITY.
- A MATRIX IS DEFECTIVE IF IT HAS AT LEAST ONE DEFECTIVE E-VAL.

<u>THM</u>: AN $m \times m$ MATRIX A IS NON-DEFECTIVE IFF IT HAS AN EIGENVALUE DECOMPOSITION

$$A = X \Lambda X^{-1}$$

SIMILARITY TRANSFORMATION

FOR $X \in \mathbb{C}^{m \times m}$ NONSINGULAR THEN THE MAP $A \mapsto X^{-1}AX$ IS A SIMILARITY TRANSFORMATION OF A.

MATRICES A & B ARE SIMILAR IF $\exists\ X$ S.T. $B = X^{-1}AX$.

<u>THM</u>: IF X IS NON-SINGULAR THEN A AND $X^{-1}AX$ HAVE SAME CHARACTERISTIC POLYNOMIAL, EIGENVALUES, AND MULTIPLICITIES.

SCHUR FACTORIZATION

$$A = \underset{\text{UNITARY MATRIX}}{Q}\ \underset{\text{UPPER-TRIANGULAR.}}{T}\ Q^*$$

THM: EVERY SQUARE MATRIX HAS A SCHUR FACTORIZATION.

SINCE THIS IS A SIMILARITY TRANSFORMATION, A & T HAVE THE SAME E-VALS.

SINCE T IS TRIANGULAR THE E-VALS ARE THE DIAGONAL ENTRIES!

UNITARY DIAGONALIZATION

MATRIX A IS UNITARY DIAGONALIZABLE IF $\exists$ UNITARY Q S.T.

$$A = Q \Lambda Q^*$$

THM: A HEMITIAN MATRIX $(A = A^*)$ IS UNITARY DIAGONALIZABLE AND ITS E-VALS ARE REAL.

EIGENVALUE ALGORITHMS

~~ALREADY~~
ALREADY DISCUSSED THAT A ROOT FINDING ALGORITHM FOR $P_A(z)$ WOULD BE UNSTABLE.

THIS CONNECTION WITH ROOT FINDING, HOWEVER, TELLS US THAT WE SHOULD NOT BE ABLE TO FIND THE EIGENVALUES IN A FINITE NUMBER OF STEPS.

ANY EIGENVALUE SOLVER MUST BE ITERATIVE!

COMPUTE SCHUR FACTORIZATION ITERATIVELY.

GENERATE A SEQUENCE OF UNITARY Q_j SUCH THAT

$$Q_j^* \cdots Q_2^* Q_1^* A Q_1 Q_2 \cdots Q_j$$

CONVERGES TO T AS $j \to \infty$.

IF A IS HERMITIAN, T WILL BE DIAGONAL.

TWO ~~PA~~ PHASES OF E-VAL COMPUTATION

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \xrightarrow[1]{\text{DIRECT}} \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ & x & x & x \\ & & x & x \end{bmatrix}$$

H (HESSENBERG MATRIX)

2 ITERATIVE

$$\begin{bmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{bmatrix}$$

T