UNNORMALIZED SIMULTANEOUS ITERATION (1)

- APPLY POWER ITERATION TO SEVERAL VECTORS SIMULTANEOUSLY.

- TAKE

$$V^{(0)} = \left[v_1^{(0)} \middle| v_2^{(0)} \middle| v_3^{(0)} \middle| \cdots \middle| v_n^{(0)} \right]_{m \times n}$$

- SET $V^{(k)} = A^k V^{(0)}$

- CAN PROVE THAT THE SPACE

$$\langle A^k v_1^{(0)}, \ldots, A^k v_n^{(0)} \rangle$$

CONVERGES TO THE SPACE SPANNED BY THE DOMINANT E-VECS OF A (2)

$$\langle q_1, \ldots, q_n \rangle$$

THUS IF

$$\hat{Q}^{(k)} \hat{R}^{(k)} = V^{(k)}$$

THEN WE HAVE THAT THE COLUMNS OF $\hat{Q}^{(k)}$ CONVERGE TO $q_1 \cdots q_n$.

SIMULTANEOUS ITERATION

SINCE ALL $v_j^{(k)}$ CONVERGE TO q_1, $V^{(k)}$ IS ILLCONDITIONED

REMEDY: COMPUTE

$\hat{Q}^{(k)} \hat{R}^{(k)} = V^{(k)}$ AT EACH

ITERATION AND USE

$\hat{Q}^{(k)}$ INSTEAD OF $V^{(k)}$ AT

THE NEXT ITERATION.

SIMULTANEOUS ITERATION

ALGORITHM

PICK $\hat{Q}^{(0)} \in \mathbb{R}^{m \times n}$

for $k = 1, 2, \ldots$

$$Z = A \hat{Q}^{(k-1)}$$

$$\hat{Q}^{(k)} \hat{R}^{(k)} = Z$$

IT TURNS OUT SIMULTANEOUS ITERATION AND THE QR ALGORITHM ARE EQUIVALENT.

TAKE $n = m$, REDUCED $\rightarrow$ FULL QR

~~SIE~~ SIMULTANEOUS ITERATION

$$\underline{Q}^{(0)} = I$$

$$Z = A \underline{Q}^{(k-1)}$$

$$\underline{Q}^{(k)} R^{(k)} = Z.$$

$$A^{(k)} = (\underline{Q}^{(k)})^T A \underline{Q}^{(k)}$$

"PURE" QR ALGORITHM

$$A^{(0)} = A$$

$$A^{(k-1)} = Q^{(k)} R^{(k)}$$

$$A^{(k)} = R^{(k)} Q^{(k)}$$

$$\underline{Q}^{(k)} = Q^{(1)} Q^{(2)} \dots Q^{(k)}$$

FOR BOTH,

$$\underline{R}^{(k)} = R^{(k)} R^{(k-1)} \dots R^{(1)}$$

<u>THM</u>: SIMULTANEOUS ITERATION WITH $\underline{Q}^{(0)} = I$ AND QR ALGORITHM GENERATE IDENTICAL ~~SE~~ SEQUENCES $\underline{R}^{(k)}$, $\underline{Q}^{(k)}$ AND $A^{(k)}$

WHERE

$$A^{k} = \underline{Q}^{(k)} \underline{R}^{(k)}$$

AND

$$A^{(k)} = (\underline{Q}^{(k)})^{T} A \underline{Q}^{(k)}$$

WE SEE NOW WHY QR ALGORITHM WORKS!

- $\underline{Q}^{(k)}$ ~~IS~~ CONTAINS OUR APPROXIMATION OF THE EIGENVECTORS.
- THE DIAGONAL ENTRIES OF $A^{(k)}$ ARE THE RAYLEIGH QUOTIENTS.

WE CAN MODIFY THE "PURE" QR ALGORITHM TO BE "~~B~~ PRACTICAL".

"PRACTICAL" QR ALGORITHM

$$(Q^{(0)})^T A^{(0)} Q^{(0)} = A$$

for $k = 1, 2, \ldots$

PICK SHIFT $\mu^{(k)}$

$$Q^{(k)} R^{(k)} = A^{(k-1)} - \mu^{(k)} I$$

$$A^{(k)} = R^{(k)} Q^{(k)} + \mu^{(k)} I$$

IF OFF-DIAGONAL $A^{(k)}_{j,j+1}$ IS CLOSE TO ZERO

SET $A_{j,j+1} = A_{j+1,j} = 0$

TO OBTAIN

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

APPLY SAME ALGORITHM TO A_1, A_2 SEPARATELY.

CONNECTION TO ~~INVERSE~~ INVERSE ITERATION

$$A^k = \underline{Q}^{(k)} \underline{R}^{(k)}$$

TAKE THE INVERSE

$$A^{-k} = (\underline{R}^{(k)})^{-1} (\underline{Q}^{(k)})^T$$

USING THE SYMMETRY OF A^{-1}:

$$A^{-k} = \underline{Q}^{(k)} (\underline{R}^{(k)})^{-T}$$

TAKE

$$P = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

SINCE $P^2 = I$

$$\left\{ A^{-k} = (A^{-1})^k \right\}$$

$$A^{-k} P = \left[\underline{Q}^{(k)} P \right] \left[P (\underline{R}^{(k)})^{-T} P \right]$$

THIS IS THE SIMULTANEOUS ITERATION OF A^{-1} WITH INITIAL MATRIX P.

FIRST COLUMN OF $\underline{Q}^{(k)} P$ IS ~~THAT~~ THE LAST COLUMN OF $\underline{Q}^{(k)}$

THIS CAN BE VIEWED AS INVERSE ITERATION APPLIED TO e_m.

THIS CAN THEREFORE BE ACCELERATED USING SHIFTS! $\mu^{(k)}$.

$$A^{(k-1)} - \mu^{(k)} I = Q^{(k)} R^{(k)}$$

$$A^{(k)} = R^{(k)} Q^{(k)} + \mu^{(k)} I$$

THIS RETAINS

$$A^{(k)} = (Q^{(k)})^T A^{(k-1)} Q^{(k)}$$

AND CONSEQUENTLY

$$A^{(k)} = (\underline{Q}^{(k)})^T A \underline{Q}^{(k)}.$$