HESSENBERG MATRIX

- PHASE 1 OF THE EIGENVALUE COMPUTATION.

$$H = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ & \times & \times & \times \\ & & \times & \times \end{bmatrix}$$

- FOR HERMITIAN MATRICES THE ASSOCIATED HESSENBERG MATRIX WILL BE TRIDIAGONAL

$$\Delta = \begin{bmatrix} \times & \times & & \\ \times & \times & \times & \\ & \times & \times & \times \\ & & \times & \times \end{bmatrix}$$

TO FORM H FROM A WE'LL NEED TO USE A SIMILARITY TRANSFORMATION TO PRESERVE THE E-VALS.

WE CAN USE HOUSEHOLDER REFLECTORS TO CONSTRUCT H FROM A.

$$\underset{A}{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}} \xrightarrow{Q_1^*} \underset{Q_1^* A}{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix}}$$

$$\xrightarrow{Q_1} \underset{Q_1^* A Q_1}{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix}}$$

③

FOR LARGE m

$\#$ OF FLOPS $\sim 4m(m-k)$

FOR (ii)

~~THUS FOR m TIMES~~

$$\# \text{ OF FLOPS} \sim 4\sum_{k=1}^{m-2} m(m-k)$$

$$\sim 2m^3$$

TOTAL COST $\sim \frac{10}{3}m^3$

IF A IS A HERMITIAN MATRIX, THE WORK IS REDUCED TO $\sim \frac{4}{3}m^3$ DUE TO

(a) SYMMETRY

(b) SPARSITY.

④

THIS ALGORITHM IS BACKWARD STABLE.

RAYLEIGH QUOTIENT AND INVERSE ITERATION

- CLASSICAL E-VAL ALGORITHMS.
- FROM THIS POINT ON, RESTRICT TO REAL, SYMMETRICAL MATRICES.

  AND TAKE $\|\cdot\| = \|\cdot\|_2$

  - EIGENVALUES ARE ALL REAL, $\lambda_1, \ldots, \lambda_m$
  - ORTHONORMAL E-VECS $q_1, \ldots, q_m$.

(5)

RAYLEIGH QUOTIENT

$$r(x) = \frac{x^T A x}{x^T x}$$

NOTICE IF x IS AN E-VEC THEN $r(x) = \lambda$ WHERE λ IS THE E-VAL.

VIEW $x \in \mathbb{R}^m$ AS VARIABLE AND $r : \mathbb{R}^m \longrightarrow \mathbb{R}$.

TAKE PARTIAL DERIVATIVE OF r W.R.T. x_j

$$\frac{\partial r}{\partial x_j} = \frac{1}{x^T x} \frac{\partial}{\partial x_j} (x^T A x)$$

(6)

$$- \frac{(x^T A x)}{(x^T x)^2} \frac{\partial}{\partial x_j} (x^T x)$$

$$= \frac{2}{x^T x} (Ax - r(x)x)_j$$

COLLECT THE PARTIAL DERIVATIVES FOR $j = 1, \dots, m$ INTO ONE m VECTOR

$$\nabla r(x) = \frac{2}{x^T x} (Ax - r(x)x)$$

THUS, $\nabla r = 0$ IF x IS AN E-VEC.

E-VECTORS ARE STATIONARY POINTS OF $r(x)$.

WHEN WE RESTRICT TO $\|x\| = 1$, $r(x)$ IS A CONTINUOUS FUNCTION ON THE UNIT SPHERE.

IF q_J IS AN E-VEC, SINCE $\nabla r(q_J) = 0$ AND r IS SMOOTH, TAYLOR EXPANSION REVEALS

$$r(x) - r(q_J) = O(\|x - q_J\|^2)$$

AS $x \to q_J$

RAYLEIGH QUOTIENT GIVES QUADRATICALLY ACCURATE ESTIMATE OF THE E-VAL.

POWER ITERATION

METHOD TO FIND E-VEC ~~ASSOCIATE~~ ASSOCIATED WITH THE LARGEST E-VAL.

START WITH $v^{(0)}$, $\|v^{(0)}\| = 1$

AND APPLY A AND NORMALIZE AT EACH ITERATION.

$$v^{(1)} = \frac{Av^{(0)}}{\|Av^{(0)}\|}$$

$$v^{(2)} = \frac{Av^{(1)}}{\|Av^{(1)}\|}$$

REPEATING THE ~~RE~~ PROCESS $m-2$ TIMES WE OBTAIN THE HESSENBERG FORM

$$Q_{m-2}^* \cdots Q_2^* Q_1^* A Q_1 Q_2 \cdots Q_{m-2}$$

ALGORITHM: HOUSEHOLDER REDUCTION TO HESSENBERG FORM.

for $k = 1$ to $m-2$

$$x = A_{k+1:m,k}$$

$$v_k = \text{sign}(x_1)\|x\|_2 e_1 + x$$

$$v_k = v_k / \|v_k\|_2$$

(i) $$A_{k+1:m,k:m} = A_{k+1:m,k:m} - 2v_k(v_k^* A_{k+1:m,k:m})$$

(ii) $$A_{1:m,k+1:m} = A_{1:m,k+1:m} - 2(A_{1:m,k+1:m} v_k)v_k^*$$

END FOR

OPERATION ~~COU~~ COUNT

WORK DOMINATED BY (i) AND (ii)

· OPERATION COUNT FOR (i) IS EXACTLY THAT FOR HOUSEHOLDER TRIANGULARIZATION.

$\#$ OF FLOPS $\sim \frac{4}{3} m^3$

· WORK FOR (ii)

AND KEEP GOING! (11)

WRITE $v^{(0)}$ AS A LIN. COMB. OF THE E-VECS OF A.

$$v^{(0)} = a_1 q_1 + a_2 q_2 + \dots + a_m q_m$$

SINCE $v^{(k)}$ IS A MULTIPLE OF $A^k v^{(0)}$, WE HAVE FOR SOME c_k

$$v^{(k)} = c_k A^k v^{(0)}$$

$$= c_k \left(a_1 \lambda_1^k q_1 + a_2 \lambda_2^k q_2 + \dots + a_m \lambda_m^k q_m \right)$$

SUPPOSE OUR E-VALS ARE INDEXED SUCH THAT (12)

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_m|$$

$$v^{(k)} = c_k \lambda_1^k \left(a_1 q_1 + a_2 (\lambda_2/\lambda_1)^k q_2 + \dots + a_m (\lambda_m/\lambda_1)^k q_m \right)$$

THUS,

$$\| v^{(k)} - (\pm q_1) \| = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right)$$

AND USING THE RAYLEIGH QUOTIENT $\lambda^{(k)} = r(v^{(k)})$ WE HAVE

$$| \lambda^{(k)} - \lambda_1 | = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^{2k} \right)$$