DISCRETE LEGENDRE POLYNOMIALS

THE CONTINUOUS FORMULATION CAN BE MADE DISCRETE BY EVALUATING THE MONOMIALS AT m EQUALLY SPACED POINTS.

$$x_i = \frac{2}{m}(i-1) - 1 \,, \quad i = 1, \ldots, m$$

THIS TURNS A INTO AN $m \times n$ VANDERMONDE MATRIX

$$A = \left[\begin{array}{c|c|c|c|c} 1 & x_1 & x_1^2 & & x_1^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_m & x_m^2 & & x_m^{n-1} \end{array}\right]$$

BY PERFORMING THE STANDARD QR, THE RESULTING COLUMNS OF Q, q_k, WILL ~~APP~~ APPROXIMATE $g_k(x_i)$ SINCE THE DISCRETE INNER PRODUCT IS PROPORTIONAL TO AN APPROXIMATION OF THE CONTINUOUS ONE

$$f^* g = \sum_{i=1}^{m} \bar{f}_i g_i$$

$$\approx \sum_{i=1}^{m} \bar{f}(x_i)\, g(x_i)$$

$$\approx \frac{m}{2} \int_{-1}^{1} \bar{f}(x)\, g(x)\, dx$$

THE APPROXIMATE VALUES OF

$P_k(x_i)$ IS

$$P_k(i) = \frac{q_k(i)}{q_k(m)} \quad \text{(MATLAB NOTATION)}$$

LEAST SQUARES PROBLEMS

$A \in \mathbb{C}^{m \times n} \quad m \geq n$

$b \in \mathbb{C}^m$

FIND $x \in \mathbb{C}^n$ SUCH THAT

$\| \underbrace{b - Ax}_{r} \|_2$ IS MINIMIZED

r, RESIDUAL.

THEOREM

GIVEN THE LEAST SQUARES PROBLEM, A VECTOR

x MINIMIZES $\|r\|_2$

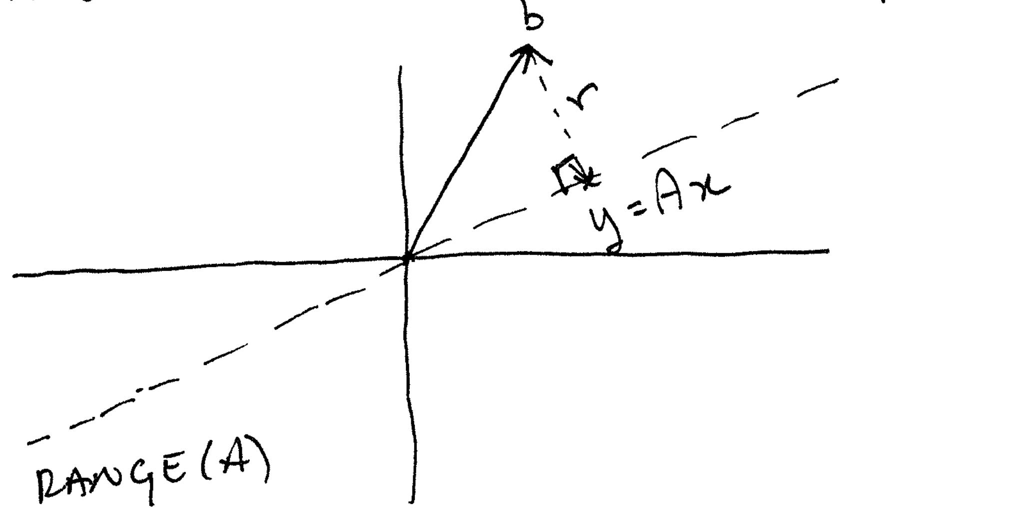
IFF $r \perp \text{RANGE}(A)$, THAT IS

(i) $A^* r = 0$

(ii) $A^* A x = A^* b$ (NORMAL EQUATIONS)

(iii) $Pb = Ax$

WHERE P IS THE ORTHOGONAL PROJECTOR ONTO THE RANGE(A)



IF A IS FULL RANK THEN x IS UNIQUE.

USING REDUCED QR, WE CAN CONSTRUCT THE PROJECTOR P AS

$$P = \hat{Q}\hat{Q}^*$$

SINCE THE COLUMNS OF $\hat{Q}$ SPAN THE RANGE OF A.

THUS, FROM (iii)

$$Pb = \hat{Q}\hat{Q}^* b = Ax = \hat{Q}\hat{R}x$$

LEFT-MULTIPLYING BY $\hat{Q}^*$ WE HAVE

$$\hat{R}x = \hat{Q}^* b$$

THIS CAN BE SOLVED USING BACKWARDS SUBSTITUTION.

EXAMPLE: POLYNOMIAL FITTING.

GIVEN $x_1, \ldots, x_m$ AND $y_1, \ldots, y_m$. FIND

$$p(x) = c_0 + c_1 x + c_2 x^2 + \ldots c_{n-1} x^{n-1}$$

SUCH THAT

$$\sum_{i=1}^{m} \left| p(x_i) - y_i \right|^2$$

IS MINIMIZED.

WE CAN SET THIS UP AS A LEAST SQUARES PROBLEM.

$$b = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad \text{"}x\text{"} = \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

(UNKNOWNS)

$$A = \left[\begin{array}{c|c|c|c} 1 & x_1 & & x_1^{n-1} \\ \vdots & \vdots & \cdots & \\ 1 & x_m & & x_m^{n-1} \end{array}\right]$$

IF x_i ARE EQUALLY SPACED, THEN $\hat{Q}$ ARE RELATED TO THE DISCRETE LEGENDRE POLYNOMIALS ON THE INTERVAL $[x_1, x_m]$.

THUS $\hat{Q}\hat{Q}^* b$ IS THE PROJECTION ONTO THE SPACE

SPANNED BY THE DISCRETE LEGENDRE POLYNOMIALS.