CGS PROJECTORS

GS CAN ALSO BE WRITTEN INTERMS OF PROJECTORS.

$$q_1 = \frac{P_1 a_1}{\|P_1 a_1\|}, \quad q_2 = \frac{P_2 a_2}{\|P_2 a_2\|}$$

$$q_n = \frac{P_n a_n}{\|P_n a_n\|}$$

P_j IS AN ORTHOGONAL PROJECTOR AN $m \times m$ MATRIX THAT PROJECTS $\mathbb{C}^m$ ORTHOGONALLY ONTO $\langle q_1, q_2, \ldots, q_{j-1} \rangle$

LET $\hat{Q}_{j-1} = \left[q_1 \middle| q_2 \middle| \cdots \middle| q_{j-1} \right]$

THEN

$$P_j = I - \hat{Q}_{j-1} \hat{Q}_{j-1}^*$$

AT EACH STEP, CGS COMPUTES

$$v_j = P_j a_j \qquad (*)$$

MODIFIED GS (MGS)

DECOMPOSE P_j INTO $j-1$ PROJECTORS OF RANK $m-1$

$$P_j = P_{\perp q_{j-1}} \cdots P_{\perp q_2} P_{\perp q_1}$$

WHERE

$$P_{\perp q_j} = I - q_j q_j^*$$

THUS, (*) CAN BE WRITTEN AS

$$v_j = P_{\perp q_{j-1}} \cdots P_{\perp q_2} P_{\perp q_1} a_j$$

AT EACH STEP MGS APPLIES $P_{\perp q_i}$ TO ALL COLUMNS.

MGS ALGORITHM

for $i = 1$ to n

  $v_i = a_i$

END FOR

for $i = 1$ to n

  $r_{ii} = \|v_i\|_2$

  $q_i = v_i / r_{ii}$

  for $j = i+1$ to n

    $r_{ij} = q_i^* v_j$

    $v_j = v_j - r_{ij} q_i$

  END FOR

END FOR

MGS CAN BE THOUGHT OF AS TRIANGULAR ORTHOGONALIZATION.

- FIRST STEP CAN BE VIEWED AS

$$\left[v_1 \mid v_2 \mid \cdots \mid v_n \right] \underbrace{\begin{bmatrix} \frac{1}{r_{11}} & -\frac{r_{12}}{r_{11}} & \cdots & -\frac{r_{1n}}{r_{11}} \\ & 1 & & 0 \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_{R_1} = \left[q_1 \mid v_2^{(2)} \mid \cdots \mid v_n^{(2)} \right]$$

RIGHT MULTIPLY BY R_1

SECOND STEP: RIGHT MULTIPLY BY R_2

$$R_2 = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ & 1/r_{22} & -\frac{r_{23}}{r_{22}} & \cdots & -\frac{r_{2n}}{r_{22}} \\ & 0 & 1 & & 0 \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

DOING THIS n TIMES

$$\underbrace{A R_1 R_2 \cdots R_n}_{\hat{R}^{-1}} = \hat{Q}$$

TRIANGULAR ORTHOGONALIZATION USING TRIANGULAR MATRICES TO REDUCE A TO A MATRIX WITH ORTHONORMAL COLUMNS.

- R_i'S ARE NEVER FORMED EXPLICITLY BUT THEY GIVE US A CLEAR WAY OF THINKING ABOUT WHAT IS HAPPENING AT EACH STEP.

3. HOUSEHOLDER TRIANGULARIZATION

· ORTHOGONAL TRIANGULARIZATION.

TRIANGULARIZE A USING ORTHOGONAL MATRICES.

$$\underbrace{Q_n \cdots Q_2 Q_1}_{Q^*} A = R$$

FORM __FULL__ QR OF A.

TRIANGULARIZE BY INTRODUCING ZEROS.

$$\underset{A}{\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_1 A}{\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}} \xrightarrow{Q_2} \underset{Q_2 Q_1 A}{\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & \times \end{bmatrix}}$$

$$\xrightarrow{Q_3} \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q_3 Q_2 Q_1 A$$

FIND Q_k (UNITARY) TO INTRODUCE ZEROS BELOW THE kTH DIAGONAL WHILE ~~PRES~~ PRESERVING THE ZEROS PREVIOUSLY INTRODUCED.

__APPROACH__

$$Q_k = \begin{bmatrix} I & 0 \\ 0 & F \end{bmatrix}$$

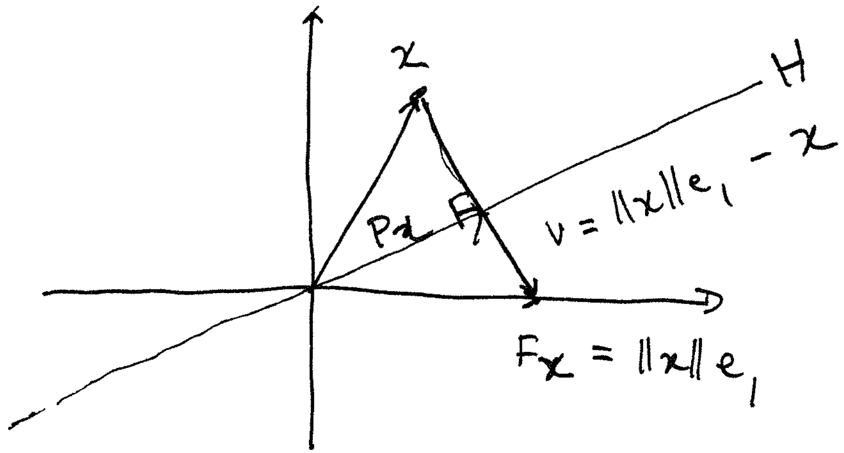
I IS $(k-1)\times(k-1)$ IDENTITY

F IS $(m-k+1)\times(m-k+1)$ UNITARY MATRIX CALL HOUSEHOLDER REFLECTOR.

FOR THE KTH STEP THE $k,\dots,m$ ENTRIES OF THE KTH COLUMN WE WRITE AS $x \in \mathbb{C}^{m-k+1}$

$$x = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix} \xrightarrow{F} Fx = \begin{bmatrix} \|x\| \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \|x\| e_1$$



F MAPS A VECTOR ON ONE SIDE OF H TO ITS MIRROR IMAGE.

THE PROJECTOR OF x ONTO H IS

$$P = I - \frac{vv^*}{v^*v}$$

TO REFLECT ABOUT H WE NEED TO GO TWICE AS FAR IN THE DIRECTION OF v.

$$F = I - 2\,\frac{VV^*}{V^*V}$$

FULL RANK AND UNITARY.