FUNDAMENTALS

MATRIX - VECTOR MULTIPLICATION

x: n COLUMN VECTOR $\in \mathbb{C}^n$

A: $m \times n$ MATRIX $\in \mathbb{C}^{m \times n}$

m: # OF ROWS

n: # OF COLUMNS.

$$b = Ax$$

b: m COLUMN VECTOR $\in \mathbb{C}^m$

$$b_i = \sum_{j=1}^{n} a_{ij} x_j \qquad i = 1, \ldots, m$$

THIS IS A LINEAR MAP

$$x \longmapsto Ax$$

SINCE

$$A(x+y) = Ax + Ay$$

$$A(\alpha x) = \alpha A x$$

$$x, y \in \mathbb{C}^n, \quad \alpha \in \mathbb{C}$$

WE CAN WRITE M-V MULTIPLICATION SLIGHTLY DIFFERENTLY.

$$b = \sum_{j=1}^{n} x_j \underbrace{a_j}_{\text{JTH COLUMN OF } A.}$$

$$\begin{bmatrix} b \end{bmatrix} = x_1 \begin{bmatrix} a_1 \end{bmatrix} + x_2 \begin{bmatrix} a_2 \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_n \end{bmatrix}$$

b IS A LINEAR COMBINATION OF THE COLUMNS OF A.

MATRIX TIMES A MATRIX.

$A: l \times m$, $C: m \times n$, $B: l \times n$

(1) $b_{ij} = \sum_{k=1}^{m} a_{ik} c_{kj} \quad (\cancel{A}B = AC)$

(2) $b_j = Ac_j = \sum_{k=1}^{m} c_{kj} a_k$

jTH COLUMN OF B

b_j IS A LIN. COMBINATION OF THE COLUMNS OF A WITH COEFFICIENTS c_{kj}

EXAMPLE: OUTER PRODUCT.

$u \in \mathbb{C}^m$

$v \in \mathbb{C}^n$

$$uv^T = \begin{bmatrix} \\ u \\ \\ \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c|c} v_1 u & v_2 u & \cdots & v_n u \end{array} \right]$$

DEF: RANGE OF A, RANGE(A) IS THE SET OF VECTORS THAT CAN BE EXPRESSED AS Ax FOR SOME x.

THM: RANGE(A) IS THE SPACE SPANNED BY THE COLUMNS OF A.

DEF: NULL SPACE OF $A \in \mathbb{C}^{m \times n}$, NULL(A), IS THE SET OF VECTORS x THAT SATISFY $Ax = 0$.

DEF: RANK OF A IS THE DIMENSION OF THE COLUMN SPACE OF A.

A MATRIX A $(m \times n)$ IS FULL RANK IF IT HAS THE MAXIMUM POSSIBLE RANK.

(THE LESSER OF m AND n)

IF $m \geq n$ ~~THN~~ ~~IE~~ THEN A ~~must~~ MUST HAVE n LINEARLY INDEPENDENT COLUMNS TO BE FULL RANK.

THM: A MATRIX $A \in \mathbb{C}^{m \times n}$ WITH $m \geq n$ HAS FULL RANK IFF IT MAPS NO TWO DISTINCT VECTORS TO THE SAME VECTOR.

INVERSE

A NONSINGULAR OR INVERTIBLE MATRIX IS A SQUARE MATRIX $(m \times m)$ OF FULL RANK.

$\Rightarrow$ COLUMNS PROVIDE A BASIS FOR $\mathbb{C}^m$.

CAN WRITE

$$\underbrace{e_j}_{\substack{\text{UNIT VECTOR}\\ \text{WITH 1 IN}\\ \text{THE } j\text{TH ENTRY}}} = \sum_{i=1}^{m} z_{ij} \underbrace{a_i}_{\substack{i\text{TH COLUMN}\\ \text{OF } A.}}$$

$$\left[e_1 \middle| e_2 \middle| \cdots \middle| e_m \right] = \underbrace{I}_{\substack{m \times m\\ \text{IDENTITY}}} = A \underbrace{Z}_{A^{-1}.}$$