POWER ITERATION ALGORITHM

(1)

$v^{(0)}$ = SOME VECTOR WITH $\|v^{(0)}\| = 1$.

for $k = 1, 2, \ldots$

$\quad w = A v^{(k-1)}$

$\quad v^{(k)} = w / \|w\|$

$\quad \lambda^{(k)} = (v^{(k)})^T A v^{(k)}$

END FOR ONCE "CONVERGED"

- POWER ITERATION IS RESTRICTED TO FINDING THE ~~LARGEST~~ ~~E-VA~~ EIGENVECTOR ASSOCIATED WITH THE LARGEST EIGENVALUE.

- POWER ITERATION HAS A LINEAR CONVERGENCE WITH RATE BASED ON $|\lambda_2 / \lambda_1|$

INVERSE ITERATION

(2)

FOR $\mu \in \mathbb{R}$ ~~AN~~ NOT AN E-VAL OF A,

- THE E-VECS OF THE MATRIX $(A - \mu I)^{-1}$ ARE THE SAME AS A.

- THE E-VALS OF $(A - \mu I)^{-1}$ ARE $\{(\lambda_j - \mu)^{-1}\}$ WHERE λ_j ARE THE EVALS OF A.

CAN CHOOSE μ CLOSE TO λ_J, SUCH THAT $(\lambda_J - \mu)^{-1}$ IS MUCH LARGER THAN $(\lambda_j - \mu)^{-1}$ $\forall j \neq J$.

INVERSE ITERATION ALGORITHM ③

$v^{(0)}$ = SOME VECTOR $\|v^{(0)}\| = 1$

for $k = 1, 2, \ldots$

SOLVE $(A - \mu I) w = v^{(k-1)}$

$v^{(k)} = w / \|w\|$

$\lambda^{(k)} = (v^{(k)})^T A (v^{(k)})$

THOUGH CONVERGENCE IS STILL LINEAR, ITS RATE CAN BE CONTROLLED BY CHOOSING μ TO BE CLOSE TO THE E-VAL OF INTEREST.

BASED ON THE CONVERGENCE OF THE POWER ITERATION,

$$\|v^{(k)} - (\pm q_J)\| = O\left(\left| \frac{\mu - \lambda_J}{\mu - \lambda_K} \right|^k \right) ④$$

$$|\lambda^{(k)} - \lambda_J| = O\left(\left| \frac{\mu - \lambda_J}{\mu - \lambda_K} \right|^{2k} \right)$$

~~RQ~~ RAYLEIGH QUOTIENT ITERATION

COMBINE OUTPUT OF THE RAYLEIGH QUOTIENT WITH INVERSE ITERATION, I.E. SET $\mu = \lambda^{(k)}$ AT EACH ITERATION.

ALGORITHM

$v^{(0)}$ = SOME VECTOR $\|v^{(0)}\| = 1$

$\lambda^{(0)} = (\cancel{\lambda} v^{(0)})^T A v^{(0)}$

for $k = 1, 2 \ldots$

SOLVE $(A - \lambda^{(k-1)} I)w = v^{(k-1)}$

$$v^{(k)} = w/\|w\|$$

$$\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$$

THIS SIMPLE MODIFICATION GREATLY ACCELERATES CONVERGENCE

SUPPOSE WE HAVE $v^{(k)}$ AND $\lambda^{(k)}$ SUFFICIENTLY CLOSE TO q_J AND λ_J.

$$\|v^{(k+1)} - q_J\| = O(|\lambda^{(k)} - \lambda_J| \, \|v^{(k)} - q_J\|)$$

IF $\|v^{(k)} - q_J\| \le \delta$ THEN FROM RAYLEIGH QUOTIENT THAT

$$|\lambda^{(k)} - \lambda_J| \le O(\delta^2)$$

$$\|v^{(k+1)} - q_J\| \le O(\delta^3)$$

THUS WE SEE THE FOLLOWING CONVERGENCE PATTERN EMERGE.

$$\begin{tikzcd}
\|v^{(k)} - q_J\| & |\lambda^{(k)} - \lambda_J| \\
\delta \arrow[r, "R.Q."] \arrow[d, "I.I."'] & O(\delta^2) \arrow[dl] \\
O(\delta^3) \arrow[r, "R.Q."] \arrow[d] & O(\delta^6) \arrow[dl] \\
O(\delta^9) \arrow[r] & O(\delta^{18}) \arrow[dl] \\
\vdots &
\end{tikzcd}$$

OPERATION COUNTS

- POWER ITERATION
  - MATRIX-VEC. MULTIPLICATION AT EACH ITERATION
    $\Rightarrow O(m^2)$
- INVERSE ITERATION
  - NEED TO INVERT SAME LINEAR SYSTEM AT EACH STEP.
  - 1 $O(m^3)$ COMPUTATION
  - $N_{ITER} \times O(m^2)$
- ~~RAY~~ RAYLEIGH QUOTIENT

  INVOLVES A NEW LINEAR SYSTEM AT EVERY ~~STE~~ ITERATION.

$\Rightarrow O(m^3)$ AT EACH ITERATION.

NOW WE SEE THE REASON FOR "PRETREATING" A.

IF A IS TRIDIAGONAL EACH METHOD IS $O(m)$.

THESE METHODS GIVE US ONE E-VAL AND ONE E-VEC. HOW ~~DOES~~ DO WE OBTAIN THE SPECTRUM AND EIGEN SPACE OF A?

QR ALGORITHM

"PURE" QR ALGORITHM

$$A^{(0)} = A$$

for $k = 1, 2, \ldots$

$$Q^{(k)} R^{(k)} = A^{(k-1)}$$

$$A^{(k)} = R^{(k)} Q^{(k)}.$$

AS $k \to \infty$ $A^{(k)} \to T$ (SCHUR FORM)

OR IF A IS HERMITIAN

$$A^{(k)} \to \Lambda.$$

$A^{(k)}$ IS ~~A~~ SIMILAR TO $A^{(k-1)}$ SINCE

$$R^{(k)} = \left(Q^{(k)}\right)^T A^{(k-1)}$$

THUS,

$$A^{(k)} = \left(Q^{(k)}\right)^T A^{(k-1)} Q^{(k)}.$$

THIS STILL SEEM IMPOSSIBLY SIMPLE!

HOW / WHY DOES IT WORK?

UNNORMALIZED SIMULTANEOUS ITERATION

- APPLY POWER ITERATION TO SEVERAL VECTORS AT ONCE.