M3/4/5N9 – Project 3
Due 13 Jan 2016

Note: Use the MATLAB format `shorte` when reporting any numerical values. Your report should be a **single** pdf file that contains the responses/explanations, figures, tables, and data. You also must submit your .m files so your codes can be run.

PageRank as a sparse linear system.
In addition to being viewed as an eigenvalue problem, PageRank can also be determined by solving the sparse linear system

$$(\boldsymbol{I} - \alpha \boldsymbol{G}^T)\boldsymbol{x} = \mathbf{1} \tag{1}$$

and then normalizing $\boldsymbol{x}$ such that $\mathbf{1}^T\boldsymbol{x} = 1$.

1. Implement GMRES to solve the linear system and find the PageRank vector for $\alpha = 0.85$ and $\boldsymbol{G}^T$ from Project 2. Set the tolerance for the relative residual, $\|\mathbf{1} - (\boldsymbol{I} - \alpha\boldsymbol{G}^T)\boldsymbol{x}\|/\|\mathbf{1}\|$, to 10^{-8} and provide in your report the number of iterations needed to converge, as well as the top 50 sites. You will receive:

   (a) *Some* marks for a working code that uses MATLAB to take advantage of sparsity in the matrix-vector multiplication and requires an additional $O(n^3)$ work at iteration n.

   (b) *Most* marks for a working code that uses MATLAB to take advantage of sparsity in the matrix-vector multiplication and requires an additional $O(n^2)$ work at iteration n.

   (c) *Full* marks for a working code that uses MATLAB to take advantage of sparsity in the matrix-vector multiplication and requires an additional $O(n)$ work at iteration n.

   If you have implemented an $O(n^2)$ or $O(n)$ approach, provide the detailed algorithm indicating how the reduction from $O(n^3)$ at each iteration was achieved. Do not use backslash or other built-in MATLAB functions for solving linear systems. For additional help, see attached excerpt from *Iterative methods for sparse linear systems* by Yousef Saad.

2. Study and discuss how the convergence of GMRES is affected by the value of α. In particular, provide the number of GMRES iterations required for $\alpha = 0.5, 0.7, 0.9,\ 0.99$, and 0.9999 with tolerances of 10^{-8} and 10^{-10} for the relative residual. Compare your results with the Power method from Project 2 and discuss which method provides a better way of finding the PageRank for different values of α.

3. As discussed in lectures, one way of accelerating the convergence of GMRES is preconditioning. Augment your GMRES implementation to allow for a preconditioner based on the incomplete LU decomposition of the matrix $\boldsymbol{I} - \alpha\boldsymbol{G}^T$. This is most easily done using the MATLAB function `ilu` whose outputs are a sparse lower triangular matrix, $\boldsymbol{L}$, and a sparse upper triangular matrix, $\boldsymbol{U}$ where $\boldsymbol{LU} \approx \boldsymbol{I} - \alpha\boldsymbol{G}^T$. If the input to `ilu` is a sparse data structure, then both $\boldsymbol{L}$ and $\boldsymbol{U}$ will be as well and the forward and backward substitutions involving these matrices can be performed rapidly using backslash (that's right, you can use backslash for the part of the code involving the factored preconditioner).

   Using your implementation of preconditioned GMRES, study and discuss how well preconditioning improves the convergence for different values of α and the tolerance. Provide the number of iterations needed for $\alpha = 0.5, 0.7, 0.9,\ 0.99$, and 0.9999 for tolerances 10^{-8} and 10^{-10} for the relative residual. Discuss if and when the preconditioning should be used to find the PageRank for this data.