QR FACTORIZATION

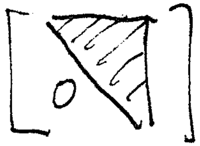
$$A = QR \quad \text{(FULL QR)}$$

A: $m \times n$

Q: $m \times m$ UNITARY MATRIX $(Q^{-1} = Q^*)$

R: $m \times n$ UPPER TRIANGULAR MATRIX

$r_{ij} = 0$
$i > j$

$R =$ 

$$A = \hat{Q}\hat{R} \quad \text{(REDUCED QR)}$$

$\hat{Q}$: $m \times n$ MATRIX USE COLUMNS FORM AN ORTHONORMAL SET OF VECTORS.

$\hat{R}$: $n \times n$ UPPER TRIANGULAR

WHY DO THIS ?

(1) SOLVE LINEAR SYSTEMS

$$Ax = b$$

A: $m \times m$, FULL RANK.

(a) $QRx = b$ (PERFORM QR)

(b) MULTIPLY BY Q^*

$$\underbrace{Q^* Q}_{I} R x = \underbrace{Q^* b}_{y}$$

(c) USE BACKWARDS SUBSTITUTION ~~FOR~~ TO SOLVE $Rx = y$

LAST ROW

$$r_{mm} x_m = y_m$$

$$x_m = y_m / r_{mm}$$

PENULTIMATE ROW

$$r_{m-1\,m-1}\, x_{m-1} + r_{m-1\,m}\, x_m = y_{m-1}$$

(2) LEAST SQUARE PROBLEMS.

$A \in \mathbb{C}^{m \times n}$

FIND x s.t.

$\|Ax - b\|_2$ is MINIMIZED.

THE SOLUTION IS GIVEN BY

$$Ax = \hat{Q}\hat{Q}^* b$$

$$\hat{Q}\hat{R}x = \hat{Q}\hat{Q}^* b$$

MULTIPLY BY $\hat{Q}^*$

$$\Rightarrow \hat{R}x = \hat{Q}^* b$$

SOLVE FOR x USING BACKWARDS SUBSTITUTION.

COMPUTING QR FACTORIZATION.

- CLASSICAL GRAM-SCHMIDT.
- MODIFIED GRAM-SCHMIDT.
- HOUSEHOLDER TRIANGULARIZATION.

USING THE MATRIX A $(m \times n)$ WE CAN ~~S~~ CONSTRUCT SUCCESSIVE SPACES SPANNED BY ITS COLUMNS.

$$\langle a_1 \rangle \subseteq \langle a_1, a_2 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \ldots$$

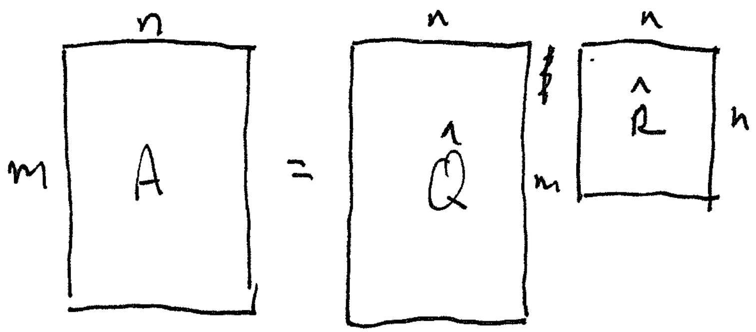
$\langle \ldots \rangle$ SPACE SPANNED BY THE VECTORS IN THE BRACKETS.

IDEA BEHIND QR:
CONSTRUCT SEQUENCE OF ORTHONORMAL VECTORS $q_1, q_2, \ldots$ THAT SPAN THE SUCCESSIVE SPACES.

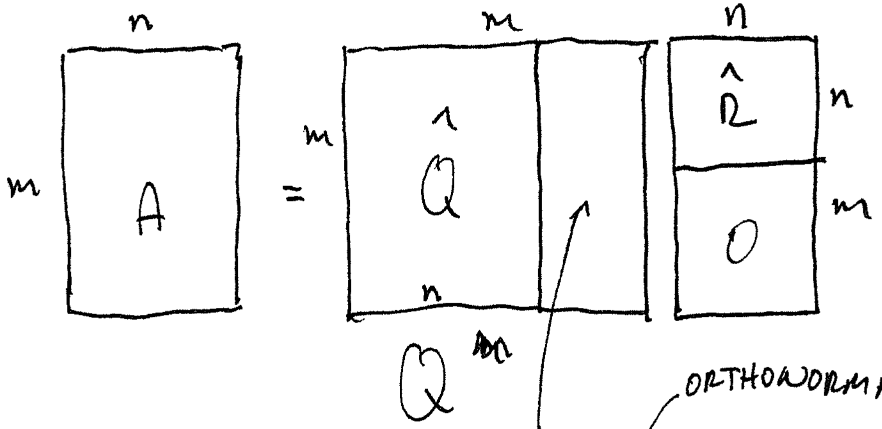
$$\underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}}_{A\ (m \times n)} = \underbrace{\begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}}_{\hat{Q}\ (m \times n)} \times \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & \\ & 0 & \ddots & \\ & & & r_{nn} \end{bmatrix}}_{\hat{R}\ (n \times n)}$$

THE REDUCED QR FACTORIZATION

FOR $m \geq n$



FULL QR ($m \geq n$)



ADDITIONAL ORTHONORMAL VECTORS THAT IF A IS FULL RANK SPAN RANGE$(A)^{\perp}$ $\{$ OR NULL$(A^*)\}$ DONE TO MAKE Q UNITARY.

GRAM-SCHMIDT ORTHOGONALIZATION

GIVEN $a_1, a_2, \cdots, a_n$ FIND $q_1, q_2, \cdots, q_n$ AND r_{ij} BY SUCCESSIVE ORTHOGONALIZATION

FIND $q_j \in \langle a_1, a_2, \dots, a_j \rangle$

THAT IS ORTHOGONAL TO

$q_1, q_2, \cdots, q_{j-1}$

1. $v_j = a_j - (q_1^* a_j) q_1 - (q_2^* a_j) q_2$
$\qquad - \dots - (q_{j-1}^* a_j) q_{j-1}$

2. $q_j = v_j / \|v_j\|_2$

BY THE FACT THAT $A = \hat{Q}\hat{R}$

WE KNOW

$$q_1 = a_1 / r_{11}$$

$$q_2 = \frac{a_2 - r_{12} q_1}{r_{22}}$$

$$\vdots$$

$$q_n = \frac{a_n - \sum_{i=1}^{n-1} r_{in} q_i}{r_{nn}}$$

BY COMPARING THE TWO APPROACHES WE SEE THAT

$$r_{ij} = q_i^* a_j$$

AND $|r_{jj}| = \| a_j - \sum_{i=1}^{j-1} r_{ij} q_i \|_2$

CLASSICAL GRAM-SCHMIDT (CGS) ALGORITHM

for $j = 1$ to n
  $v_j = a_j$
  for $i = 1$ to $j-1$
    $r_{ij} = q_i^* a_j$
    $v_j = v_j - r_{ij} q_i$

END FOR

$r_{jj} = \|v_j\|_2$

$q_j = v_j / r_{jj}$

END FOR

CAN USE CGS TO PROVE EXISTENCE & UNIQUENESS (FULL RANK A)

(TREFETHEN & BAU p. 51)

BUT TURNS OUT TO BE UNSTABLE FOR NUMERICAL PURPOSES.