GAUSSIAN ELIMINATION WITH PIVOTING

SUPPOSE $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

STANDARD GAUSSIAN ELIMINATION WILL FAIL AT THE FIRST STEP.

ALSO, PROBLEMATIC IF

$$A = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix}$$

SINCE DIVIDING BY 10^{-20} COULD LEAD TO NUMERICAL INSTABILITIES.

PIVOTS

AT STEP k HAVE MATRIX X

$$\begin{bmatrix} x & x & x & x \\ & x_{kk} & x & x \\ & x & x & x \\ & x & x & x \end{bmatrix} \xrightarrow{L_k} \begin{bmatrix} x & x & x & x \\ & x_{kk} & x & x \\ & 0 & x & x \\ & 0 & x & x \end{bmatrix}$$

ENTRY x_{kk} IS CALLED THE PIVOT.

EVERY ENTRY OF $X_{k+1:m,\,k:m}$ IS SUBTRACTED BY THE PRODUCT OF AN ENTRY IN ROW k AND AN ENTRY IN COLUMN k DIVIDED BY x_{kk}.

IF THE GOAL IS TO INTRODUCE ZEROS IN A COLUMN, THE CHOICE OF x_{kk} IS NOT UNIQUE.

TAKE x_{ik} AS PIVOT:

$$\begin{bmatrix} x & x & x & x \\ & x & x & x \\ & x_{ik} & x & x \\ & x & x & x \end{bmatrix} \longrightarrow \begin{bmatrix} x & x & x & x \\ & 0 & x & x \\ & x_{ik} & x & x \\ & 0 & x & x \end{bmatrix}$$

CAN EVEN TAKE x_{ij}

$$\begin{bmatrix} x & x & x & x \\ & x & x & x \\ & x & x & x_{ij} \\ & x & x & x \end{bmatrix} \longrightarrow \begin{bmatrix} x & x & x & x \\ & x & x & 0 \\ & x & x & x_{ij} \\ & x & x & 0 \end{bmatrix}$$

- FREE TO CHOOSE ANY ENTRY IN $X_{k:m,k:m}$ AS THE PIVOT

- FOR STABILITY, CHOOSE THE LARGEST ENTRY IN $X_{k:m,k:m}$.

- CHOOSING ENTRY OTHER ~~THE~~ THAN x_{kk} WILL DESTROY THE INDUCED TRIANGULAR STRUCTURE

- ADD ADDITIONAL STEP OF PERMUTING THE ROWS AND COLUMNS TO MOVE CHOSEN PIVOT INTO x_{kk}'S POSITION.

COMPLETE PIVOTING

- ALLOW ALL ENTRIES OF $X_{k:m,k:m}$ TO BE A PIVOT.

  # OF FLOPS = $O(m^3)$

PARTIAL PIVOTING

- CHOOSE PIVOT FROM THE KTH ~~k~~ COLUMN
- ONLY ROW EXCHANGES ARE ~~two~~ INVOLVED.
- # OF FLOPS = $O(m^2)$

IDEA:

$$\begin{bmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & x_{ik} & \times & \times \\ & \times & \times & \times \end{bmatrix} \xrightarrow{P_k} \begin{bmatrix} \times & \times & \times & \times \\ & x_{ik} & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \end{bmatrix}$$

SELECT THE PIVOT. EXCHANGE ROWS.

$$\xrightarrow{L_k} \begin{bmatrix} \times & \times & \times & \times \\ & x_{ik} & \times & \times \\ & 0 & \times & \times \\ & 0 & \times & \times \end{bmatrix}$$

ELIMINATION STEP.

EXAMPLE

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

STEP 1: $P_1 A$

$$\begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix} A = \begin{bmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

STEP 2: $L_1 P_1 A$

$$\begin{bmatrix} 1 & & & \\ -1/2 & 1 & & \\ -1/4 & & 1 & \\ -3/4 & & & 1 \end{bmatrix} P_1 A = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & -1/2 & -3/2 & -3/2 \\ & -3/4 & -5/4 & -5/4 \\ & 7/4 & 9/4 & 17/4 \end{bmatrix}$$

STEP 3: $P_2 L_1 P_1 A$

$$\begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{bmatrix} L_1 P_1 A =$$

$\vdots$

EVENTUALLY OBTAIN DECOMPOSITION THAT MAY BE WRITTEN AS

$$\underbrace{\begin{bmatrix} & & 1 & \\ & & & 1 \\ & 1 & & \\ 1 & & & \end{bmatrix}}_{P} A = \underbrace{\begin{bmatrix} 1 & & & \\ 3/4 & 1 & & \\ 1/2 & -2/7 & 1 & \\ 1/4 & -3/7 & 1/3 & 1 \end{bmatrix}}_{L}$$

$$\times \underbrace{\begin{bmatrix} 8 & 7 & 9 & 5 \\ & 7/4 & 9/4 & 17/4 \\ & & -6/7 & -2/7 \\ & & & 2/3 \end{bmatrix}}_{U}$$

$PA = LU$ IS THE TRUE LU DECOMPOSITION.

INTERPRETATION:

1. PERMUTE ROWS OF A ACCORDING P.

2. APPLY GAUSSIAN ELIMINATION WITHOUT PIVOTING TO PA.

WHY IS THIS THE CASE?

WE DID:

$$L_3 P_3 L_2 P_2 L_1 P_1 A = U$$

THESE OPERATIONS CAN BE REORDERED

$$L_3 P_3 L_2 P_2 L_1 P_1 = \underbrace{(L_3' L_2' L_1')}_{L^{-1}} \underbrace{(P_3 P_2 P_1)}_{P}$$

WHERE

$$L_3' = L_3 \qquad L_2' = P_3 L_2 P_3^{-1}$$

$$L_1' = P_3 P_2 L_1 P_2^{-1} P_3^{-1}$$

PERMUTATION MATRICES PRESERVE LOWER TRIANGULAR FORM.