- HERMITIAN NOT HERMETIAN

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- LEC. 3 4 LEC. 6 (TREFETHEN AND BAU)

VECTOR NORMS

- MEASURE THE "LENGTH" OF A VECTOR.

A NORM IS A FUNCTION $\|\cdot\| : \mathbb{C}^m \longrightarrow \mathbb{R}$ THAT SATISFIES

(1) $\|x\| \geq 0$ AND $\|x\| = 0$ ONLY IF $x = 0$

(2) $\|x+y\| \leq \|x\| + \|y\|$ (Δ - INEQUALITY)

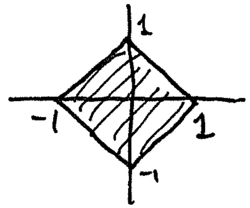
(3) $\|\alpha x\| = |\alpha| \|x\|$

$\forall\ x, y \in \mathbb{C}^m$ AND $\alpha \in \mathbb{C}$

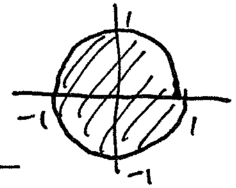
ALREADY SAW THE EUCLIDEAN NORM, BUT THIS IS PART OF A LARGER CLASS OF P- NORMS.

FOR $x \in \mathbb{R}^2$ $\{x \mid \|x\| \leq 1\}$

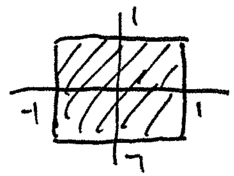
- $\|x\|_1 = \sum_{i=1}^{m} |x_i|$



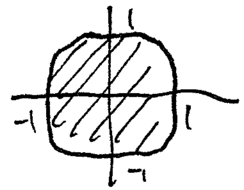
- $$\boxed{\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2}}$$



- $\|x\|_\infty = \max_{1 \leq i \leq m} |x_i|$



- $\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p}$



WEIGHTED NORMS

$$\|x\|_W = \|Wx\|$$

W IS A DIAGONAL MATRIX WITH $w_{ii} \neq 0$ $\forall i$.

WEIGHTED 2-NORM

$$\|x\|_W = \left(\sum_{i=1}^{m} |w_{ii} x_i|^2 \right)^{1/2}$$

PROJECTORS

A PROJECTOR IS A SQUARE MATRIX THAT SATISFIES

$$P^2 = P$$

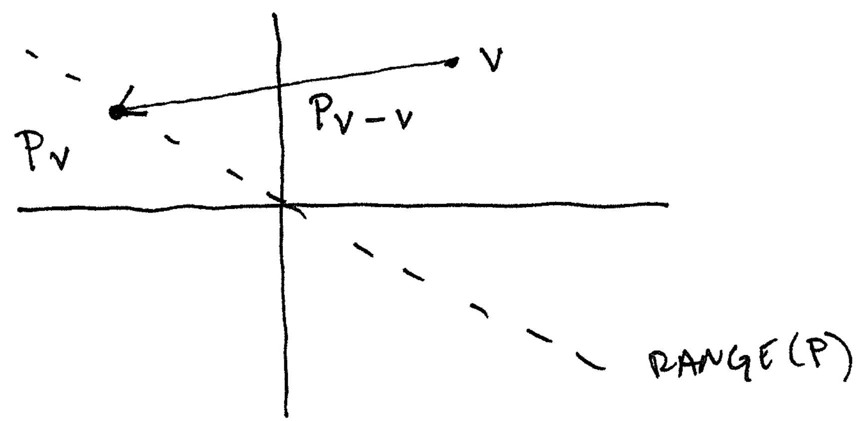
(ALSO SAID TO BE IDEMPOTENT)

- IF $v \in$ RANGE (P)

$\Rightarrow$ $v = Px$ FOR SOME x

$$Pv = P^2 x = Px = v$$

- SUPPOSE THAT $Pv \neq v$



$$P(Pv - v) = P^2 v - Pv = 0$$

$Pv - v \in$ NULL(P).

COMPLEMENTARY PROTECTOR.

5

IF P IS A PROJECTOR, $I-P$ IS ALSO A PROJECTOR.

$$(I-P)^2 = I^2 - 2P + P^2 = I - P$$

$I-P$ IS THE COMPLEMENTARY PROJECTOR OF P.

- IF $Pu = 0$ WE HAVE

$$(I-P)u = u$$

$$\Rightarrow \text{RANGE}(I-P) \supseteq \text{NULL}(P)$$

- ALSO HAVE $\text{RANGE}(I-P) \subseteq \text{NULL}(P)$ SINCE

$$(I-P)u = u - Pu \in \text{NULL}(P)$$

$$\Rightarrow \text{RANGE}(I-P) = \text{NULL}(P)$$

6

SINCE WE ~~CAN~~ ~~WRITE~~ ~~INSTEAD~~ ALSO HAVE $P = I - (I-P)$,

$$\text{NULL}(I-P) = \text{RANGE}(P)$$

THUS, $\text{NULL}(I-P) \cap \text{NULL}(P) = \{0\}$

AND, $\text{RANGE}(P) \cap \text{RANGE}(I-P) = \{0\}$

A PROJECTOR SEPARATES $\mathbb{C}^m$ INTO TWO SPACES.

IN FACT, SUPPOSE SUBSPACES, $S_1, S_2 \subseteq \mathbb{C}^m$ s.t. $S_1 \cap S_2 = \{0\}$ AND $S_1 + S_2 = \mathbb{C}^m$, THEN

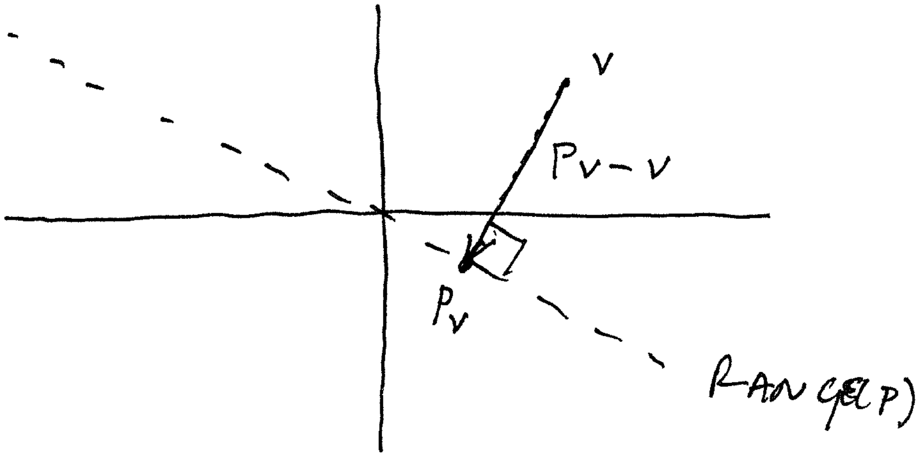
$\exists$ PROJECTOR P s.t.

$\text{RANGE}(P) = S_1$ AND $\text{NULL}(P) = S_2$

P PROJECTS ONTO S_1 ALONG S_2.

ORTHOGONAL PROJECTORS

AN ORTHOGONAL PROJECTOR IS ONE THAT PROJECTS ONTO S_1 ALONG S_2 WHERE S_1 AND S_2 ARE ORTHOGONAL.



CAN CONSTRUCT ORTHOGONAL PROJECTORS FROM SETS OF ORTHONORMAL VECTORS.

LET $\{q_1, \dots, q_n\}$ A SET OF n ORTHONORMAL VECTORS IN $\mathbb{C}^m$

LET

$$\hat{Q} = \left[\, q_1 \,\middle|\, q_2 \,\middle|\, \cdots \,\middle|\, q_n \,\right]$$

HAVE SEEN THAT $v \in \mathbb{C}^m$

$$v = \underbrace{r}_{\text{COMPONENT OF } v \text{ ORTHOGONAL TO THE COLUMN SPACE OF } \hat{Q}} + \underbrace{\sum_{i=1}^{n} (q_i q_i^*) v}_{\text{COMPONENT OF } v \text{ IN THE COLUMN SPACE OF } \hat{Q}}$$

THUS THE map

$$v \longmapsto \underbrace{\sum_{i=1}^{n} (q_i q_i^*) v}_{y}$$

IS AN ORTHOGONAL PROJECTION ONTO RANGE $(\hat{Q})$.

$$y = \sum_{i=1}^{n} (q_i q_i^*) v = \underbrace{\hat{Q}\hat{Q}^*}_{P} v$$

THE COMPLEMENTARY PROJECTOR IS $I - \hat{Q}\hat{Q}^*$

AN IMPORTANT CASE $n = 1$

$P = qq^*$ (RANK-1 ORTHOGONAL PROJECTOR)

$P_\perp = I - qq^*$ (RANK $m-1$)