CONDITION NUMBER

ABSOLUTE CONDITION NUMBER

INTRODUCE AN INFINITESIMAL PERTURBATION δx TO x.

$$\delta f = f(x+\delta x) - f(x)$$

ABSOLUTE CONDITION NUMBER

$$\hat{\kappa} = \sup_{\delta x} \frac{\|\delta f\|}{\|\delta x\|}$$

FOR DIFFERENTIABLE $f(x)$ CAN FIND

$$J(x), \quad J_{ij}(x) = \frac{\partial f_i}{\partial x_j}$$

JACOBIAN

FOR INFINITESIMAL QUANTITIES

$$\delta f = J(x)\,\delta x$$

$$\hat{\kappa} = \sup_{\delta x} \frac{\|J(x)\,\delta x\|}{\|\delta x\|}$$

$$= \|J(x)\|$$

RELATIVE CONDITION NUMBER

$$\kappa = \sup_{\delta x} \left(\frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta x\|}{\|x\|} \right)$$

FOR DIFFERENTIABLE f

$$\kappa = \cancel{\sup} \|J(x)\| \Big/ \frac{\|f(x)\|}{\|x\|}$$

FOR FLOATING POINT NUMBERS INTRODUCE RELATIVE ERRORS, SO κ IS USED IN NUMERICAL ANALYSIS. (3)

$\kappa = 1 - 10^2$ WELL-CONDITIONED

$\kappa > 10^6$ ILL-CONDITIONED.

EXAMPLES (i) $f : x \longmapsto \sqrt{x}$

$$J = \frac{df}{dx} = \frac{1}{2} x^{-1/2}$$

$$\kappa = \frac{\left|\frac{1}{2} x^{-1/2}\right|}{\frac{|x^{1/2}|}{|x|}} = \frac{1}{2}$$

(ii) FINDING THE ROOTS OF A POLYNOMIAL GIVEN THE COEFFICIENTS.

CONSIDER (4)

$$x^2 - 2x + 1 = (x-1)^2$$

ROOTS ARE $x = 1$ TWICE.

NOW TAKE

$$x^2 - 2x + 0.9999$$

$$= (x - 0.99)(x - 1.01)$$

ROOTS ARE $x = 0.99$ AND $x = 1.01$

RELATIVE CHANGE OF 10^{-4} IN COEFF GIVES ~~CH~~ RELATIVE CHANGE OF 10^{-2} IN THE ROOTS.

IN FACT, $\delta r = C\sqrt{\delta c}$

CONST.

CHANGE IN ROOTS

CHANGE IN COEFFICIENTS

THIS GIVES

$$\kappa = \infty$$

THIS PROBLEM IS ILL-CONDITIONED.

~~EX~~ A MORE COMPLEX EXAMPLE ⑤

~~WITH~~ WILKENSON POLYNOMIAL

$$p(x) = \prod_{i=1}^{20} (x-i)$$

CONDITION NUMBER OF MATRIX VECTOR MULTIPLICATION

Fix $A \in \mathbb{C}^{m \times n}$, $f: x \to Ax$

$$\kappa = \sup_{\delta x} \frac{\|A(x+\delta x) - Ax\|}{\|Ax\|} \Big/ \frac{\|\delta x\|}{\|x\|}$$

$$= \sup_{\delta x} \frac{\|A\,\delta x\|}{\|\delta x\|} \Big/ \frac{\|Ax\|}{\|x\|}$$

$$= \|A\| \Big/ \frac{\|Ax\|}{\|x\|}$$

IF $A \in \mathbb{C}^{m \times m}$ AND NONSINGULAR ⑥

$$\|x\| = \|A^{-1}Ax\|$$

$$\leq \|A^{-1}\|\,\|Ax\|$$

$$\Rightarrow \frac{\|x\|}{\|Ax\|} \leq \|A^{-1}\|$$

$$\Rightarrow \kappa \leq \|A\|\,\|A^{-1}\|$$

THEOREM: LET $A \in \mathbb{C}^{m \times m}$ AND NONSINGULAR AND CONSIDER $Ax = b$. THE PROBLEM OF COMPUTING b GIVEN x HAS

$$\kappa = \|A\| \frac{\|x\|}{\|b\|} \leq \|A\|\,\|A^{-1}\|$$

THE PROBLEM OF COMPUTING x GIVEN b HAS

$$\kappa = \|A^{-1}\| \frac{\|b\|}{\|x\|} \leq \|A\|\,\|A^{-1}\|.$$

THE CONDITION NUMBER OF A

$$\kappa(A) = \|A\| \|A^{-1}\|$$

CONDITION NUMBER OF A SYSTEM, $Ax=b$

- b IS FIXED
- $f: A \to x$

HOW DO PERTURBATIONS IN A, δA, CHANGE x?

$$(A + \delta A)(x + \delta x) = b$$

USE THE FACT THAT $Ax=b$ AND INGNORING $\delta A \, \delta x$, WE HAVE

$$\delta A x + A \delta x = 0$$

$$\delta x = - A^{-1} \delta A x$$

$$\|\delta x\| \leq \|A^{-1}\| \|\delta A\| \|x\|$$

CONDITION NUMBER

$$\kappa = \sup_{\delta A} \frac{\|\delta x\|}{\|x\|} \Big/ \frac{\|\delta A\|}{\|A\|}$$

$$\leq \|A^{-1}\| \|A\| = \kappa(A)$$

EQUALITY IF

$$\|A^{-1} \delta A x\| = \|A^{-1}\| \|\delta A\| \|x\|$$

CAN ALWAYS BE FOUND.

$$\kappa = \kappa(A).$$

<u>THEOREM</u>: LET b BE FIXED AND CONSIDER $x = A^{-1} b$, WHERE A IS SQUARE AND NON-SINGULAR. THE CONDITION NUMBER OF THIS PROBLEM W.R.T. ~~B~~ PERTURBATIONS IN A IS

$$\kappa = \kappa(A) = \|A\| \|A^{-1}\|.$$