GAUSSIAN ELIMINATION

EXAMPLE

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

$$L_1 A = \begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ -4 & & 1 & \\ -3 & & & 1 \end{bmatrix} \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & 3 & 5 & 5 \\ & 4 & 6 & 8 \end{bmatrix}$$

$$L_2 L_1 A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -3 & 1 & \\ & -4 & & 1 \end{bmatrix} \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 2 \\ & & 2 & 4 \end{bmatrix}$$

$$L_3 L_2 L_1 A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 2 \\ & & & 2 \end{bmatrix} = U$$

WHAT ABOUT L? IF $A = LU$

$$L^{-1} = L_3 L_2 L_1$$

$$\Rightarrow L = L_1^{-1} L_2^{-1} L_3^{-1}$$

$$L_1^{-1} = \begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ -\cancel{3}4 & & 1 & \\ -\cancel{4}3 & & & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & & 1 & \\ 3 & & & 1 \end{bmatrix}$$

FOR L_k^{-1} JUST REPLACE ~~VALUE~~ VALUES BELOW l_{kk} BY -1 TIMES THE VALUE.

$$L = L_1^{-1} L_2^{-1} L_3^{-\cancel{3}1}$$

$$= \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 3 & 1 & \\ 3 & 4 & 1 & 1 \end{bmatrix}$$

TO GET L, SIMPLY NEED TO COMPILE ALL THE SUBDIAGONAL ENTRIES OF L_k^{-1} IN ONE MATRIX!

NON-ZERO

RECALL:

MGS WE SAID WAS TRIANGULAR ORTHOGONALIZATION

HOUSEHOLDER WE SAID WAS ORTHOGONAL TRIANGULARIZATION.

GAUSSIAN ELIMINATION IS TRIANGULAR TRIANGULARIZATION.

GENERAL FORMULA

LET x_k BE THE KTH COLUMN OF THE MATRIX AT THE KTH STEP.

$$x_k = \begin{bmatrix} x_{1k} \\ \vdots \\ x_{kk} \\ x_{k+1,k} \\ \vdots \\ x_{mk} \end{bmatrix}$$

WOULD LIKE L_k SUCH THAT

$$L_k x_k = \begin{bmatrix} x_{1k} \\ \vdots \\ x_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

TO DO THIS, WE MUST HAVE

$$l_{jk} = \frac{x_{jk}}{x_{kk}} \quad \text{for } j = k+1, \ldots, m$$

SO THAT

$$L_k = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -l_{k+1,k} & 1 & \\ & & \vdots & & \ddots \\ & & -l_{mk} & & & 1 \end{bmatrix}$$

THEN CAN WRITE

$$L_k = I - l_k e_k^*$$

WHERE

$$l_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{mk} \end{bmatrix}$$

AND e_k IS THE VECTOR WITH KTH ENTRY 1 AND 0 ELSEWHERE.

SINCE, $e_k^* l_k = 0$

$$(I - l_k e_k^*)(I + l_k e_k^*) = I$$

THUS,

$$L_k^{-1} = I + l_k e_k^*$$

$$L_k^{-1} L_{k+1}^{-1} = (I + l_k e_k^*)(I + l_{k+1} e_{k+1}^*)$$

$$= I + l_k e_k^* + l_{k+1} e_{k+1}^*$$

SINCE $e_k^* l_{k+1} = 0$.

THUS,

$$L = L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1}$$

$$= \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & l_{32} & \ddots & \\ \vdots & \vdots & & \ddots \\ l_{m1} & l_{m2} & \cdots\ l_{m,m-1} & 1 \end{bmatrix}$$

GAUSSIAN ELIMINATION ALGORITHM

$A \in \mathbb{C}^{m \times m}$ (SQUARE)

$U = A$

$L = I$

for $k = 1$ to $m-1$

  for $j = k+1$ to m

    $l_{jk} = u_{jk} / u_{kk}$

    (*) $u_{j,k:m} = u_{j,k:m} - l_{jk}\, u_{k,k:m}$

  END FOR

END FOR

OPERATION COUNT

WORK DOMINATED BY LINE (*)

CONSISTS OF:

(i) $m-k+1$: $\times$

(ii) $m-k+1$: $-$

$2(m-k+1)$ FLOPS EVERY TIME (*) IS REACHED.

$$\#\text{ OF FLOPS} \sim \sum_{k=1}^{m-1} \sum_{j=k+1}^{m} 2(m-k+1)$$

$$= \sum_{k=1}^{m-1} 2(m-k+1) \underbrace{\sum_{j=k+1}^{m} 1}_{m-k}$$

$$= \sum_{k=1}^{m-1} 2(m-k+1)(m-k)$$

$$\sim \sum_{k=1}^{m-1} 2m^2 - 4mk + 2k^2$$

$$\sim 2m^3 - \frac{4m^3}{2} + \frac{2}{3}m^3$$

$$\#\text{ OF FLOPS} \sim \frac{2}{3}m^3$$