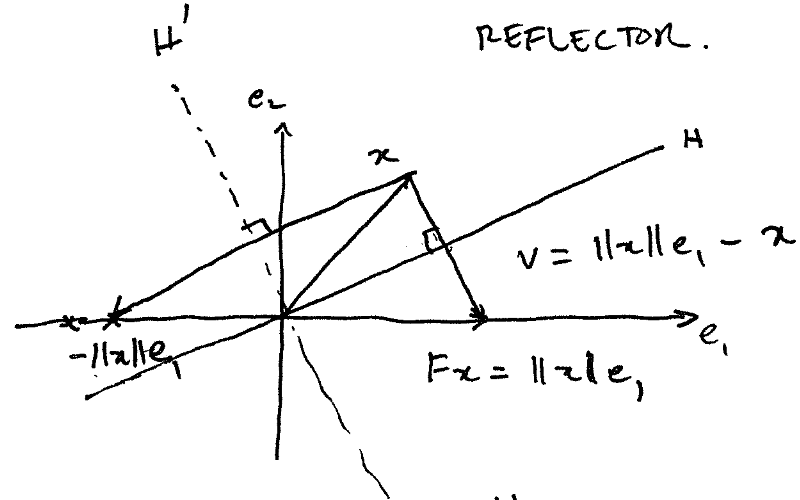
HOUSEHOLDER TRIANGULARIZATION

$$Q_n \cdots Q_2 Q_1 A = R$$

$$Q_k = \begin{bmatrix} I & 0 \\ 0 & F \end{bmatrix}$$

HOUSEHOLDER REFLECTOR.



PROJECTOR ONTO H

$$P = I - qq^* \quad \text{WHERE} \quad q = \frac{v}{\|v\|}$$

TO GO TWICE THE DISTANCE IN q

$$F = I - 2qq^* = I - 2\frac{vv^*}{v^*v}$$

TURNS OUT THERE ARE TWO CHOICES.

WHILE MATHEMATICALLY SPEAKING BOTH CHOICES ARE EQUALLY GOOD, NUMERICALLY WE'D LIKE TO CHOOSE THE ~~REFLECT~~ REFLECTION FURTHEST AWAY FROM x.

$$v = -\operatorname{sign}(x_1)\|x\|e_1 - x$$

OR

$$v = \operatorname{sign}(x_1)\|x\|e_1 + x$$

TAKING ALSO $\operatorname{sign}(0) = 1$

~~Holes~~

HOUSEHOLDER ALGORITHM

for $k = 1$ to n

$x = A_{k:m,k}$ ← VECTOR OF THE k THROUGH m ENTRIES OF THE KTH COLUMN

$$v_k = \text{sign}(x_1)\|x\|_2 e_1 + x \qquad \left[\begin{array}{c} a_k \\ \hline x \end{array}\right]$$

$$v_k = v_k / \|v_k\|_2$$

$$A_{k:m,k:n} = A_{k:m,k:n} - 2v_k(v_k^* A_{k:m,k:n})$$

END. FOR

A IS REDUCED TO UPPER TRIANGULAR FORM WITHOUT EVER EXPLICITLY COMPUTING Q.

HOWEVER, WE KNOW HOW TO APPLY Q (AND Q^*) ONCE THE v_k'S ARE KNOWN

SINCE

$$Q^* = Q_n \cdots Q_2 Q_1$$

$$Q = Q_1 Q_2 \cdots Q_n$$

RATHER THAN COMPUTING Q FROM THE Q_k'S THEN APPLYING Q THE ~~SEQ~~ SEQUENCE CAN BE APPLIED DIRECTLY

IMPLICIT CALCULATION OF $Q^* b$

for $k = 1$ to n

$$b_{k:m} = b_{k:m} - 2v_k(v_k^* b_{k:m})$$

END FOR

IMPLICIT CALCULATION OF ~~Q^*b~~ Qx ⑤

for $k = n$ DOWN TO 1

$$x_{k:m} = x_{k:m} - 2v_k(v_k^* x_{k:m})$$

END FOR

TO FORM Q WE RECOGNIZE THAT

$$QI = Q$$

AND PERFORM THE IMPLICIT CALCULATION

$$Qe_i = q_i \quad \text{FOR ALL } i = 1, \ldots, m$$

EXAMPLE: LEGENDRE POLYNOMIALS

$$P_n(x) \quad \text{ON} \quad [-1, 1]$$

POLYNOMIALS THAT ARE ORTHOGONAL WITH RESPECT TO

⑥

THE INNER PRODUCT

$$(f, g) = \int_{-1}^{1} \overline{f(x)} g(x)\, dx$$

FOR $P_n(x)$ AND $P_m(x)$

$$\int_{-1}^{1} \overline{P_n(x)}\, P_m(x)\, dx = \frac{2}{2n+1} \delta_{nm}$$

THE FIRST FEW ARE

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{3x^2}{2} - \frac{1}{2}$$

$$P_3(x) = \frac{5x^3}{2} - \frac{3}{2}x$$

$$\vdots$$

THEY CAN BE GENERATED BY THE CONTINUOUS VERSION OF QR APPLIED TO THE "MATRIX" WHOSE "COLUMNS" ARE THE MONOMIALS

$$A = \left[1 \,\middle|\, x \,\middle|\, x^2 \,\middle|\, x^3 \,\middle|\, \cdots \,\middle|\, x^{n-1} \right]$$

USING A CONTINUOUS VERSION OF GRAM-SCHMIDT REPLACING

$$q_i^* v_j \quad \text{BY} \quad \underbrace{\int_{-1}^{1} \bar{q}_i(x)\, v_j(x)\, dx}_{r_{ij}}$$

AND

$$\|v\|_2 \quad \text{BY} \quad \underbrace{\left(\int_{-1}^{1} \bar{v}(x) v(x)\, dx \right)^{1/2}}_{r_{ii}}$$

WE OBTAIN

$$A = \left[q_0(x) \,\middle|\, q_1(x) \,\middle|\, \cdots \,\middle|\, q_{n-1}(x) \right] \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ & & r_{nn} \end{bmatrix}$$

$P_n(x)$ IS RELATED TO $q_n(x)$ THROUGH

$$P_n(x) = \frac{q_n(x)}{q_n(1)}$$