1

WHAT DOES STABILITY MEAN IN THE CONTEXT OF LINEAR ALGEBRA COMPUTATIONS?

PROBLEM: QR FACTORIZATION OF A.

INPUT OR DATA: A

OUTPUT OR SOLUTION: Q, R

WITH $QR = A$ AND Q IS UNITARY AND R UPPER TRIANGULAR

ALGORITHM: HOUSEHOLDER ON A COMPUTER SATISFYING FPA I & II.

INPUT: A, OUTPUT: $\tilde{Q}, \tilde{R}$

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IF BACKWARDS STABLE THEN $\tilde{Q}, \tilde{R}$ ARE SOLUTIONS TO THE PROBLEM WITH INPUT $A + \delta A$ FOR SOME $\frac{\|\delta A\|}{\|A\|} = O(\epsilon)$

SINCE THIS IS A SOLUTION WE KNOW

$$\tilde{Q}\tilde{R} = A + \delta A$$

SO IN OUR EXPERIMENT, WE WERE MEASURING $\frac{\|\delta A\|}{\|A\|}$.

TYPICALLY THIS ERROR IS CALLED <u>BACKWARD ERROR</u> OR THE <u>RESIDUAL</u>.

THM: LET THE QR FACTORIZATION $A = QR$ OF $A \in \mathbb{C}^{m \times n}$ BE COMPUTED BY HOUSEHOLDER ON A COMPUTER SATISFYING FPA I & II. THEN ~~WIT~~ WE HAVE

$$\tilde{Q}\tilde{R} = A + \delta A \qquad \frac{\|\delta A\|}{\|A\|} = O(\epsilon)$$

FOR SOME $\delta A \in \mathbb{C}^{m \times n}$

WHAT IS MEANT BY $\tilde{R}$ AND $\tilde{Q}$?

$\tilde{R}$ IS THE TRIANGULAR MATRIX THAT GETS COMPUTED.

$\tilde{Q}$ IS THE EXACTLY UNITARY MATRIX FORMED USING THE COMPUTED VECTORS $\tilde{v}_k$.

TYPICALLY WE WOULD LIKE TO FACTOR OUR MATRIX TO DO SOMETHING, E.G. SOLVE A LINEAR SYSTEM.

IS BACKWARD STABILITY ENOUGH OR DO WE NEED ACCURATE Q AND R?

SOLVING $Ax = b$ USING QR.

ALGORITHM

1. $QR = A$ DONE USING HOUSEHOLDER.

2. $y = Q^* b$ CONSTRUCT $Q^* b$ USING IMPLICIT MULTIPLICATION.

3. $x = R^{-1} y$ SOLVING THE TRIANGULAR SYSTEM USING BACKWARD SUBSTITUTION.

WE'VE ALREADY DISCUSSED THE BACKWARD STABILITY OF STEP 1. AND THAT IT ~~GI~~ OUTPUTS $\tilde{Q}, \tilde{R}$.

STEP 2 ~~IS~~ IS ALSO BACKWARD STABLE. USING $\tilde{Q}$ GIVEN BY STEP 1. THIS MEANS.

$$\text{(i)} \quad (\tilde{Q} + \delta Q)\tilde{y} = b \qquad \|\delta Q\| = O(\epsilon)$$

STEP 3 IS ALSO BACKWARD STABLE

$$\text{(ii)} \quad (\tilde{R} + \delta R)\tilde{x} = \tilde{y}$$

$$\text{WITH} \quad \frac{\|\delta R\|}{\|R\|} = O(\epsilon).$$

PROVING THESE RESULTS IS VERY TEDIOUS!!! SEE LECTURE 17 IN TREFETHEN & BAU.

<u>THEOREM</u>: THE QR ALGORITHM TO SOLVE $Ax = b$ IS BACKWARD STABLE, SATISFYING

$$(A + \Delta A)\tilde{x} = b$$

~~W~~ FOR SOME $\dfrac{\|\Delta A\|}{\|A\|} = O(\epsilon)$.

Proof: Combining (i) + (ii)

$$b = (\tilde{Q} + \delta Q)(\tilde{R} + \delta R)\,\tilde{x}$$

$$= \left[\tilde{Q}\tilde{R} + (\delta Q)\tilde{R} + \tilde{Q}(\delta R) + (\delta Q)(\delta R)\right]\tilde{x}$$

From step 1. we know

$$\tilde{Q}\tilde{R} = A + \delta A$$

$$b = A\tilde{x} + \underbrace{\left[\delta A + (\delta Q)\tilde{R} + \tilde{Q}(\delta R) + (\delta Q)(\delta R)\right]}_{\Delta A}\tilde{x}$$

Show each term is $O(\epsilon)$ to get result.

Since $\tilde{Q}\tilde{R} = A + \delta A$

$$\Rightarrow \tilde{R} = \tilde{Q}^*(A + \delta A)$$

$$\frac{\|\tilde{R}\|}{\|A\|} \leq \frac{\|\tilde{Q}^*\|\,\|A + \delta A\|}{\|A\|}$$

$$\leq 1 + \frac{\|\delta A\|}{\|A\|} = O(1)$$

Thus

$$\frac{\|\delta Q\,\tilde{R}\|}{\|A\|} \leq \|\delta Q\|\,\frac{\|\tilde{R}\|}{\|A\|} = O(\epsilon)$$

$$\frac{\|\tilde{Q}(\delta R)\|}{\|A\|} \leq \frac{\|\delta R\|}{\|\tilde{R}\|}\,\|\tilde{Q}\|\,\frac{\|\tilde{R}\|}{\|A\|} = O(\epsilon)$$

$$\frac{\|\delta Q\,\delta R\|}{\|A\|} \leq \|\delta Q\| \frac{\|\delta R\|}{\|\tilde{R}\|} \frac{\|\tilde{R}\|}{\|A\|} = \mathcal{O}(\epsilon^2)$$

Thus $$\frac{\|\Delta A\|}{\|A\|} = \mathcal{O}(\epsilon).$$