STABILITY

AN ALGORITHM $\tilde{f}$ IS STABLE IF FOR EACH $x \in X$

$$\frac{\|\tilde{f}(x) - \tilde{f}(\tilde{x})\|}{\|f(\tilde{x})\|} = O(\epsilon)$$

FOR SOME $\tilde{x} \in X$

$$\frac{\|\tilde{x} - x\|}{\|x\|} = O(\epsilon)$$

ACCURACY OF A BACKWARD STABLE ~~ST~~ ALGORITHM.

THM: SUPPOSE A BACKWARD STABLE ALGORITHM $\tilde{f}$ IS APPLIED TO SOLVE A

PROBLEM $f : X \to Y$ WITH CONDITION NUMBER κ ON A COMPUTER SATISFYING FPA I & II. THEN THE RELATIVE ERROR SATISFIES

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\kappa(x)\,\epsilon)$$

PROOF: SINCE $\tilde{f}$ IS BACKWARD STABLE, $\tilde{f}(x) = f(\tilde{x})$ FOR SOME $\tilde{x}$ WITH $\frac{\|\tilde{x} - x\|}{\|x\|} = O(\epsilon)$.

DEFINE

$$\kappa_\epsilon(x) = \sup_{\substack{\delta x \\ \frac{\|\delta x\|}{\|x\|} \le \epsilon}} \frac{\|\delta f\|}{\|f\|} \Big/ \frac{\|\delta x\|}{\|x\|}$$

THUS,

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = \frac{\|f(\tilde{x}) - f(x)\|}{\|f(x)\|}$$

$$\leq \kappa_{\epsilon}(x) \frac{\|\tilde{x} - x\|}{\|x\|}$$

TAKING THE LIMIT AS $\epsilon \to 0$

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\kappa(x)\epsilon)$$

BACKWARDS ERROR ANALYSIS

INVESTIGATE ACCURACY USING THE CONDITIONING AND STABILITY.

IF OUR ALGORITHM IS BACKWARDS STABLE THEN

THE ERROR IS REFLECTED BY THE CONDITION NUMBER.

FORWARD ERROR ANALYSIS

KEEP A RUNNING TALLY OF THE ~~AT~~ ERROR COMMITTED AT EACH STEP OF THE ALGORITHM.

STABILITY OF FLOATING POINT OPERATIONS

$\oplus$, $\ominus$, $\otimes$, $\oslash$ ARE ALL BACKWARDS STABLE.

FOR SUBTRACTION:

DATA: x_1, x_2

SOLUTION: $f(x_1, x_2) = x_1 - x_2$

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ALGORITHM: $\tilde{f}(x_1, x_2) = fl(x_1) \ominus fl(x_2)$

BY FPA I:

$$fl(x_1) = x_1(1+\epsilon_1)$$

$$fl(x_2) = x_2(1+\epsilon_2)$$

WITH $|\epsilon_1|, |\epsilon_2| \le \epsilon$

BY FPA II:

$$fl(x_1) \ominus fl(x_2)$$

$$= \left(fl(x_1) - fl(x_2)\right)(1+\epsilon_3)$$

WITH $|\epsilon_3| \le \epsilon$

COMBINING THESE RESULTS.

(6)

$$fl(x_1) \ominus fl(x_2)$$

$$= \left[x_1(1+\epsilon_1) - x_2(1+\epsilon_2)\right] * (1+\epsilon_3)$$

$$= x_1(1+\epsilon_1)(1+\epsilon_3) - x_2(1+\epsilon_2)(1+\epsilon_3)$$

$$= \overbrace{x_1(1+\epsilon_4)}^{\tilde{x}_1} - \overbrace{x_2(1+\epsilon_5)}^{\tilde{x}_2}$$

WITH $|\epsilon_4|, |\epsilon_5| \le 2\epsilon + O(\epsilon^2)$

THUS, $\tilde{f}(x_1, x_2) = \tilde{x}_1 - \tilde{x}_2$

$$= f(\tilde{x}_1, \tilde{x}_2)$$

$$\frac{|\tilde{x}_{1,2} - x_{1,2}|}{|x_{1,2}|} = O(\epsilon)$$

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INNER PRODUCT

$$x, y \in \mathbb{R}^m, \quad \alpha = x^T y$$

ALGORITHM:

$$\tilde{\alpha} = \sum_{i=1}^{m} fl(x_i) \otimes fl(y_i)$$

THIS IS BACKWARDS STABLE.

$$= \sum_{i=1}^{m} x_i(1+\epsilon_i) \otimes y_i(1+\delta_i)$$

WITH $|\epsilon_i|, |\delta_i| \le \epsilon$

$$= \sum_{i=1}^{m} \left[x_i(1+\epsilon_i) \times y_i(1+\delta_i)\right](1+\gamma_i)$$

WITH $|\gamma_i| \le \epsilon$

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$$= \sum_{i=1}^{m} \left[x_i(1+\epsilon_i)\, y_i(1+\tilde{\epsilon}_i)\right]$$

WITH $|\tilde{\epsilon}_i| \le 2\epsilon + O(\epsilon^2)$

TAKE THE FIRST TWO TERMS

$$x_1(1+\epsilon_1)y_1(1+\tilde{\epsilon}_1) \oplus x_2(1+\epsilon_2)y_2(1+\tilde{\epsilon}_2)$$

$$= \left[x_1(1+\epsilon_1)y_1(1+\tilde{\epsilon}_1) + x_2(1+\epsilon_2)y_2(1+\tilde{\epsilon}_2)\right](1+\beta_1)$$

WITH $|\beta_1| \le \epsilon$

$$= x_1(1+\epsilon_1)y_1(1+\hat{\epsilon}_1) + x_2(1+\epsilon_2)y_2(1+\hat{\epsilon}_2)$$

WITH $|\hat{\epsilon}_1|, |\hat{\epsilon}_2| \le 3\epsilon + O(\epsilon^2)$

FOLLOWING THE SAME PROCESS WITH THE REMAINING TERMS

$$\tilde{\alpha} = \sum_{i=1}^{m} \underbrace{x_i (1+\epsilon_i)}_{\tilde{x}_i} \underbrace{y_i (1+\hat{\delta}_i)}_{\tilde{y}_i}$$

$$|\hat{\delta}_i| \leq (m+1)\epsilon + O(\epsilon^2)$$

WE HAVE THAT

$$\tilde{\alpha}(x, y) = \alpha(\tilde{x}, \tilde{y})$$

FOR $\frac{\|\tilde{x} - x\|}{\|x\|} = O(\epsilon)$ AND

$$\frac{\|\tilde{y} - y\|}{\|y\|} = O(\epsilon).$$

STABLE BUT NOT B. STABLE

$$f(x) = x + 1$$

$$\tilde{f}(x) = fl(x) \oplus 1$$

CAN SHOW

$$\tilde{f}(x) = x(1+\epsilon_3) + 1 + \underline{\underline{\epsilon_2}}$$

UNSTABLE ALGORITHM

USING $\det(\lambda I - A) = 0$ TO FIND THE EIGENVALUES OF A.