LU DECOMPOSITION

FOR $A \in \mathbb{C}^{m \times m}$ WE HAVE WITH PARTIAL PIVOTING.

$$L_{m-1} P_{m-1} \cdots L_2 P_2 L_1 P_1 A = U$$

THIS CAN ALSO BE WRITTEN AS

$$\underbrace{(L'_{m-1} \cdots L'_2 L'_1)}_{L^{-1}} \overbrace{(P_{m-1} \cdots P_2 P_1)}^{P} A = U$$

$$L'_k = P_{m-1} \cdots P_{k+1} L_k P_{k+1}^{-1} \cdots P_{m-1}^{-1}$$

THUS

$$PA = LU$$

ALGORITHM FOR LU

$U = A$

$L = I$

$P = I$

for $k = 1$ to $m-1$

  SELECT $i \geq k$ to MAXIMIZE $|u_{ik}|$ } SELECT PIVOT

  $u_{k,k:m} \longleftrightarrow u_{i,k:m}$

  $l_{k,1:k-1} \longleftrightarrow l_{i,1:k-1}$ } EXCHANGE OF ROWS.

  $p_{k,:} \longleftrightarrow p_{i,:}$

  for $j = k+1$ to m

    $l_{jk} = u_{jk} / u_{kk}$

$$u_{j,k:m} = u_{j,k:m} - l_{jk}\, u_{k,k:m}$$

END FOR

END FOR.

OPERATION COUNT

$\#$ OF FLOPS $\sim \frac{2}{3} m^3$

STABILITY OF LU

CONSIDER AGAIN

$$A = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix}$$

STANDARD GAUSSIAN ~~ELIMA~~ ELIMINATION LEADS TO LARGE ENTRIES OF L AND U.

THIS AMPLIFICATION IS AT THE HEART OF THE INSTABILITY

THM: $A = LU$ FOR NONSINGULAR $A \in \mathbb{C}^{m \times m}$ BE COMPUTED BY STANDARD GAUSSIAN ELIMINATION ON A COMPUTER SATISFYING FPA I AND FPA II. & IF NO ZERO-PIVOTS ARE ENCOUNTERED THEN COMPUTED $\tilde{L}$ AND $\tilde{U}$ SATISFY

$$\tilde{L}\tilde{U} = A + \delta A \qquad \frac{\|\delta A\|}{\|L\|\|U\|} = O(\epsilon)$$

FOR SOME $\delta A \in \mathbb{C}^{m \times m}$

BACKWARD STABILITY IF

$$\|L\|\,\|U\| = O(\|A\|)$$

BUT IF $\|L\|\|U\| \neq O(\|A\|)$
CAN EXPECT BACKWARD INSTABILITY.

PIVOTING

THM APPLIES REPLACING
A BY PA.

WITH PARTIAL PIVOTING
WE KNOW

$$\|L\| = O(1)$$

THEN FOR BACKWARD
STABILITY, WE MUST HAVE

$$\|U\| = O(\|A\|).$$

THIS ~~&~~ CAN BE MEASURED
BY A GROWTH FACTOR.

$$\rho = \frac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|}$$

SO. THEN ~~WHE~~ WE HAVE

$$\|U\| = O(\rho\|A\|)$$

THM: $PA = LU$ BE COMPUTED
USING GAUSSIAN ELIMINATION
WITH PIVOTING ON A COMPUTER
SATISFYING FPA I AND FPA II.
THEN $\tilde{L}$, $\tilde{U}$, AND $\tilde{P}$ SATISFY

$$\tilde{L}\tilde{U} = \tilde{P}A + \delta A$$

$$\frac{\|\delta A\|}{\|A\|} = O(\rho\epsilon)$$

FOR SOME $\delta A \in \mathbb{C}^{m \times m}$ WHERE ρ
IS THE GROWTH FACTOR.

EXAMPLE: (7)

$$A = \begin{bmatrix} 1 & & & & 1 \\ -1 & 1 & & & 1 \\ -1 & -1 & 1 & & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

AFTER EACH STEP k, THE $k+1$ THROUGH m ENTRIES IN THE LAST COLUMN WILL BE 2^k.

$$U = \begin{bmatrix} 1 & & & & 1 \\ & 1 & & & 2 \\ & & 1 & & 4 \\ & & & 1 & 8 \\ & & & & 16 \end{bmatrix}$$

ENTRY u_{mm} IS 2^{m-1}

THUS, $\rho = 2^{m-1}$

(8)

THIS IS HUGE FOR LARGE MATRIC(E)S!