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ANALYSING SCHEMES

FOR QR, WE ENCOUNTERED THREE DIFFERENT ALGORITHMS FOR ~~REAM~~ PERFORMING THE SAME COMPUTATION.

WHAT MAKES ONE APPROACH BETTER THAN THE OTHER?

(1) ONE COULD BE FASTER THAN THE OTHERS.

(2) ONE METHOD COULD BE MORE ROBUST THAN THAN THE OTHERS.

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THE FIRST PROPERTY IS CHARACTERIZED BY THE OPERATION COUNT.

THE SECOND IS RELATED TO THE STABILITY OF THE ALGORITHM, OR HOW SENSITIVE IT IS TO PERTURBATIONS.

OPERATION COUNT

- COUNT THE NUMBER OF "FLOPS" (FLOATING OPERATIONS) THAT THE ALGORITHM REQUIRES IN THE LIMIT OF VERY LARGE MATRICES.

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· THE OPERATIONS

$$+, -, \times, \div, \sqrt{}$$

ALL COST ONE FLOP

FOR EACH REAL NUMBER

THE OPERATION COUNT IS A CLASSICAL MEASURE OF THE COST. IN PRACTICE, THERE ARE ALSO OTHER IMPORTANT FACTORS.

(i) MOVEMENT OF DATA IN MEMORY.

(ii) PARALLEL VS. SERIAL COMPUTATION.

④

(iii) OTHER PROGRAMS RUNNING ON THE COMPUTER

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OPERATION COUNT FOR MGS.

THE GRAM-SCHMIDT ALGORITHMS REQUIRE $\sim 2mn^2$ FLOPS TO COMPUTE A QR FACTORIZATION OF AN $m \times n$ MATRIX OF REAL NUMBERS.

"$\sim$" MEANS

$$\lim_{m,n \to \infty} \frac{\#\text{ OF FLOPS}}{2mn^2} = 1$$

FOR MGS, WHEN m & n ARE LARGE THE WORK IS DOMINATED BY WHAT IS DONE IN THE INNER LOOP

(i) $r_{ij} = q_i^* v_j$

INNER PRODUCT OF TWO VECTORS IN $\mathbb{R}^m$

$m : \times$

$m-1 : +$

WHICH GIVE $2m-1$ FLOPS.

(ii) $v_j = v_j - \underbrace{r_{ij} q_i}$

SUBTRACTION OF TWO VECTORS IN $\mathbb{R}^m$

$m : -$

SCALAR TIMES A VECTOR IN $\mathbb{R}^m$

$m : \times$

THIS REQUIRES $2m$ FLOPS.

THUS EACH ITERATION OF THE INNER LOOP REQUIRE $\sim 4m$ FLOPS.

$$\# \text{ OF FLOPS} \sim \sum_{i=1}^{n} \sum_{j=i+1}^{n} 4m$$

$$= 4m \sum_{i=1}^{n} \sum_{j=i+1}^{n} 1$$

$$\approx 4m \sum_{i=1}^{n} i \qquad \left(\int_1^n x \, dx \right)$$

$$\sim 4m \frac{n^2}{2}$$

$$\sim 2mn^2$$

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OPERATION COUNT FOR HOUSEHOLDER.

WORK IS DOMINATED BY

$$A_{k:m,k:n} = A_{k:m,k:n} - \underbrace{\underbrace{(2v_k)\underbrace{(v_k^* A_{k:m,k:n})}_{(1)}}_{(2)}}_{(3)}$$

(1) $v_k^* A_{k:m,k:n}$

$(n-k)$ INNER PRODUCTS OF VECTORS IN $\mathbb{R}^{m-k}$

OF FLOPS $\sim 2(n-k)(m-k)$

⑧

(2) $(2v_k)(\quad)$

· OUTER PRODUCT OF TWO VECTORS IN $\mathbb{R}^{m-k}$ AND $\mathbb{R}^{n-k}$

OF FLOPS $\sim (m-k)(n-k)$

(3) SUBTRACTION OF TWO $(m-k)\times(n-k)$ MATRICES

OF FLOPS $\sim (m-k)(n-k)$

TOTAL COST

$$\sim \sum_{k=1}^{n} 4(n-k)(m-k)$$

$$= 4\left(\sum_{k=1}^{n} nm - k(n+m) + k^2\right)$$

9

$$= 4 \sum_{k=1}^{n} nm - 4 \sum_{k=1}^{n} k(n+m) + 4 \sum_{k=1}^{n} k^2$$

$$\approx 4mn^2 - 4(n+m)\frac{\cancel{k}n^2}{2} + \frac{4n^3}{3}$$

$$= 4mn^2 - 2n^3 - 2mn^2 + \frac{4n^3}{3}$$

$$= 2mn^2 - \frac{2n^3}{3}$$

HOUSEHOLDER REQUIRES FEWER FLOPS TO COMPUTE QR.