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MATRIX NORMS

- CAN THINK OF A MATRIX AS AN mn LENGTH VECTOR
- APPLY USUAL VECTOR NORMS TO THIS VECTOR.
- MUST SATISFY

(1) $\|A\| \geq 0$ AND $\|A\| = 0$ ONLY IF $A = 0$

(2) $\|A+B\| \leq \|A\| + \|B\|$

(3) $\|\alpha A\| = |\alpha| \|A\|$

FROBENIUS NORM

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

CAN SHOW:

$$\|AB\|_F \leq \|A\|_F \|B\|_F$$

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- INDUCED MATRIX NORM.

FOR AN $m \times n$ MATRIX A AND VECTOR NORMS $\|\cdot\|_{(n)}$ AND $\|\cdot\|_{(m)}$ ON THE DOMAIN AND RANGE

$$\|A\|_{(m,n)} = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}}$$

$$= \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_{(n)} = 1}} \|Ax\|_{(m)}$$

- MEASURES THE MAXIMUM FACTOR BY WHICH A WILL "STRETCH" A VECTOR.

EXAMPLES

(i) $\|A\|_2$, $A \in \mathbb{R}^{2\times 2}$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

ALL UNIT VECTOR IN $\mathbb{R}^2$ CAN BE WRITTEN AS

$$x = (\cos\theta, \sin\theta) \qquad \theta \in [0, 2\pi)$$

$$Ax = \begin{bmatrix} \cos\theta + 2\sin\theta \\ 2\sin\theta \end{bmatrix}$$

$$\|Ax\|_2^2 = \cos^2\theta + 4\sin\theta\cos\theta + 8\sin^2\theta$$

TO FIND MAX. VALUE

$$0 = \frac{\partial}{\partial\theta}\|Ax\|_2^2 = 14\sin\theta\cos\theta + 4(\cos^2\theta - \sin^2\theta)$$

$$0 = 7\sin 2\theta + 4\cos 2\theta$$

$$\Rightarrow \tan 2\theta = -\frac{4}{7}$$

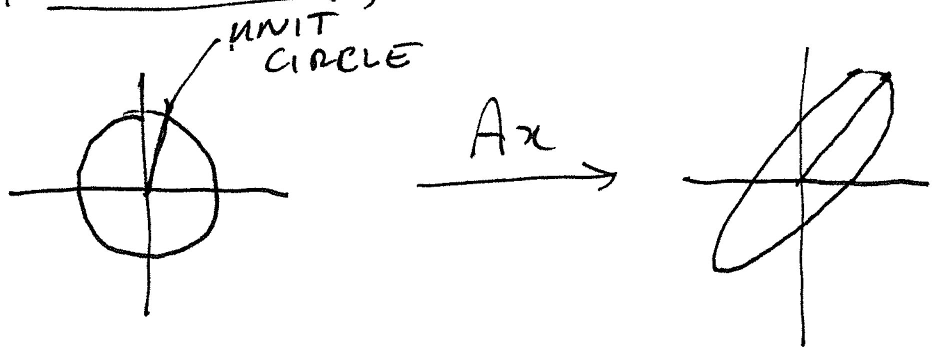
$$2\theta = \tan^{-1}\left(-\frac{4}{7}\right) + m\pi \quad (n \in \mathbb{Z})$$

$$\theta = \frac{1}{2}\tan^{-1}\left(-\frac{4}{7}\right) + m\frac{\pi}{2}$$

MAXIMUM ~~AT~~ FOR $m=1$ FOR EXAMPLE, WHICH GIVES.

$$\|A\|_2 = 2.9208\ldots$$

GRAPHICALLY,



(iii) 2-NORM OF A DIAGONAL MATRIX

$$D = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_m \end{bmatrix}$$

· STRETCHES OR COMPRESSES ALONG THE DIRECTIONS e_i

THUS, $\|D\|_2 = \max_i |d_i|$.

OFTEN DIFFICULT TO FIND EXACT VALUES, SO WE LOOK FOR BOUNDS.

HÖLDER INEQUALITY

$$|x^* y| \le \|x\|_p \|y\|_q$$

FOR $1 = \frac{1}{p} + \frac{1}{q}$

CAUCHY-SCHWARZ

$$p = q = 2$$

$$|x^* y| \le \|x\|_2 \|y\|_2$$

EXAMPLE: 2-NORM OF AN OUTER PRODUCT.

$$A = uv^*$$

$$\|Ax\|_2 = \|u \overbrace{v^* x}^{\text{SCALAR}}\|_2$$

$$= |v^* x| \, \|u\|_2$$

$$\le \|u\|_2 \|v\|_2 \|x\|_2$$

$$\Rightarrow \|A\|_2 \le \|u\|_2 \|v\|_2$$

BOUNDS FOR $\|AB\|$

$A: l \times m$

$B: m \times n$

$$\|ABx\|_{(l)} \le \|A\|_{(l,m)} \|Bx\|_{(m)}$$

$$\le \|A\|_{(l,m)} \|B\|_{(m,n)} \|x\|_{(n)}$$

$$\|AB\|_{(l,n)} \le \|A\|_{(l,m)} \|B\|_{(m,n)}$$

CONDITIONING AND CONDITION NUMBER

- CONDITIONING PERTAINS TO THE PERTURBATION BEHAVIOUR OF THE MATHEMATICAL PROBLEM.

- A "PROBLEM" AS A FUNCTION

$$f : X \longrightarrow Y$$

NORMED VECTOR SPACES.

f CAN BE NONLINEAR.

- USUALLY CONCERNED WITH THE BEHAVIOUR AT A PARTICULAR VALUE OF $x \in X$.

PROBLEM (INSTANCE)

- WELL-CONDITIONED PROBLEM
  ALL SMALL PERTURBATIONS OF x LEAD TO SMALL CHANGES IN $f(x)$.

- ILL-CONDITIONED PROBLEM

  SMALL CHANGES IN x GIVE LARGE CHANGES IN $f(x)$.

MEASURED VIA THE CONDITION NUMBER.