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Radiative Transfer

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Abstract. Chandrasekhar's work in radiative transfer theory began in 1944 and culminated with the publication of his influential treatise *Radiative Transfer* in 1950. In this review his major contributions to radiative transfer will be recounted and evaluated. These include his development of the discrete ordinates method, the invariance principles, and his formulation and solution of the transfer equation for polarized light.

1. Introduction

Chandrasekhar's work in radiative transfer was typical of his lifelong pattern of working intensively in some particular field of physics for a few years, publishing a treatise on his work, and then essentially leaving the field.

The main bulk of the work was published during the period 1944-48 in a series of twenty-four articles in the *Astrophysical Journal* under the general title "On the Radiative Equilibrium of a Stellar Atmosphere.' '[1] This series represented a remarkable period of creativity for Chandrasekhar, bringing a wealth of new results and insights, which redirected and invigorated research in radiative transfer for many years afterwards.

In accordance with his custom, Chandrasekhar capped his achievements with the publication in 1950 of his treatise, *Radiative Transfer*[2] [37], after which he published only a half-dozen more papers in the field.[3] As was often the case with his treatises, RT is noteworthy for its clarity, elegance, and style. Even forty-five years later it remains an important reference and guide to many of the fundamental concepts and methods of radiative transfer. This is more than just a volume of collected papers: his previous work is reorganized, with new insights added, and often with a simpler, more compact formulation.

Chandrasekhar never intended RT to be considered the last word on the subject, or his departure from the field to indicate that there was no more to be done.

[1] Specified here by the symbol RE followed by a roman numeral, e.g., reference [23] is denoted RE-XIII. Many of these papers are reprinted in the collection [49]; this is indicated in the reference list by the notation, e.g., [SP2-41], for paper 41.

[2] Hereinafter RT.

[3] [38], [39], [40], [41], [43], & [50].

While RT had brought radiative transfer theory to a new high level, it left many clear challenges for the succeeding generation of workers in the field. As to his departure, his time to move on simply had come.

The purpose of this review is to give an overall feeling for the scope of Chandrasekhar's work and some appreciation of its significance. It is directed mainly at those who have some familiarity with radiative transfer theory, but who are not necessarily experts in it.

Because of the dominance of his treatise RT, the original papers of the ". . . Radiative Equilibrium . . ." series are seldom referenced today; RT has become the reference of choice for Chandrasekhar's work. Accordingly, the original papers will be cited here only when this seems necessary for understanding the historical and motivational aspects of Chandrasekhar's work. For convenience and consistency, the notation of RT will be adopted throughout, even if this was not the notation of the original papers. Some discussion of alternate notations will be given below, where appropriate.

In §2 a brief presentation of some relevant radiative transfer theory is given. This is to provide a convenient reference point for the subsequent discussion and is not intended as a full introduction to the subject. Those desiring fuller explanations or more complete derivations should consult RT.

The major part of this review in §§3 – 4 will focus on what are generally regarded as Chandrasekhar's three areas of greatest contribution, his work on: the discrete ordinate method, the invariance principles, and polarization. This division into separate areas is done for convenience and should not be taken too literally, since the developments often proceeded in parallel, and results in one area often influenced another. For example, even after having developed his invariance techniques rather fully, he still needed considerable guidance in choosing appropriate forms for the solutions, and for this he often depended on results he had previously obtained using the discrete ordinate method. In §6 a brief discussion is given on some of the influences of Chandrasekhar's work on the succeeding development of the field.

Chandrasekhar's work on radiative transfer theory represents a formidable achievement, one in which he took particular pride. He spoke fondly of this period,

> My research on radiative transfer gave me the most satisfaction. I worked on it for five years, and the subject, I felt, developed on its own initiative and momentum. Problems arose one by one, each more complex and difficult than the previous one, and they were solved. The whole subject attained an elegance and a beauty which I do not find to the same degree in any of my other work. [4].

It is hoped that this review can convey some of the "elegance and beauty" Chandraskhar brought to the field of radiative transfer.

[4] Quoted by Wali [51, p. 190].

2. Background

Problems in radiative transfer come in a bewildering number of forms, reflecting the various physical processes that may operate in different circumstances. In some cases, where the source function can be specified *a priori,* the problem is almost trivially solved using the so-called *formal solution* (see, e.g., RT, §7). Apart from such special simple cases, radiative transfer problems typically involve *scattering*, which implies a source function that itself depends on the radiation field. This leads mathematically to an *integro-differential* equation of transfer. For these cases of scattering, the formal solution does not provide an explicit solution, although it may be used to re-formulate the problem as an integral equation.

The great complexity of radiative transfer problems led early workers to concentrate on simple prototypical problems, for which some analytic progress might be made. Perhaps the simplest nontrivial geometry is *plane parallel,* in which all physical variables depend on only one cartesian coordinate, say z (0 at the surface and measured inwards); this dependence is often specified instead by the normal optical depth τ. The direction of the ray can then be specified by the two spherical polar variables, θ, the angle of the ray with respect to the outward normal, and the corresponding azimuthal angle φ. It turns out to be very convenient to use the variable $\mu = \cos\theta$, $-1 \le \mu \le 1$, instead of θ itself. The monochromatic specific intensity at frequency ν then depends on just three variables $I_\nu(\tau, \mu, \varphi)$.

Another common simplification is that the frequency of the radiation is unchanged upon scattering, the case of *elastic* or *monochromatic scattering.* In this case the frequency can be treated as a parameter, and the specific intensity can be written $I(\tau, \mu, \varphi)$.

Much of the character of general radiative transfer problems already appears in what is perhaps the simplest example, the case of unpolarized radiation with isotropic scattering in plane-parallel geometry with axial symmetry. In this case the radiative transfer equation is,

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} = I(\tau,\mu) - \frac{\varpi_0}{2} \int_{-1}^{1} I(\tau,\mu')\, d\mu' \ , \tag{1}$$

where ϖ_0 has the important meaning of the fraction of radiation that is scattered (as opposed to absorbed), and is called the *single scattering albedo.* Accordingly, the case $\varpi_0 = 1$ refers to *conservative scattering.* Because of the axial symmetry, the azimuthal angle φ does not appear here.

For the more general case of anisotropic scattering, the transfer equation takes the form

$$\mu \frac{\partial I(\tau,\mu,\varphi)}{\partial \tau} = I(\tau,\mu,\varphi) - \frac{1}{4\pi} \int_{-1}^{+1} \int_{0}^{2\pi} p(\mu,\varphi;\mu',\varphi') I(\tau,\mu',\varphi')\, d\mu' d\varphi', \tag{2}$$

(RT, p. 13, Eq. 71). Here the *phase function* $p(\mu, \varphi; \mu', \varphi)$ describes the scattering from direction (μ', φ') into direction (μ, φ). Much of Chandrasekhar's work was concerned with the simple transfer equation (2). This equation shows the typical influence of scattering in that the specific intensity in one direction depends on the specific intensity in all others. It is this implicit coupling that gives transfer problems their special difficulty.

For randomly oriented (non-aligned) scatterers, the phase function is a function of the cosine of the scattering angle Θ alone, that is,

$$\cos\Theta = \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\varphi-\varphi'). \tag{3}$$

Then the general phase function can be conveniently expressed as a series in Legendre polynomials

$$p(\cos\Theta) = \sum_{l=0}^{\infty} \varpi_l P_l(\cos\Theta), \tag{4}$$

(RT, p. 7, Eq. 33). The coefficient ϖ_0 again has the meaning of the single scattering albedo, and $\varpi_0 = 1$ refers to conservative scattering. A simplification in all of Chandrasekhar's work is that the coefficients ϖ_l are all independent of depth, the *homogeneous* case.

Among the cases treated by Chandrasekhar were the *isotropic scattering phase function,*

$$p(\cos\Theta) = \varpi_0, \tag{5}$$

the *linear phase function*

$$p(\cos\Theta) = \varpi_0(1 + x\cos\Theta), \tag{6}$$

and *Rayleigh 's phase function*

$$p(\cos\Theta) = \frac{3}{4}(1+\cos^2\Theta). \tag{7}$$

It is necessary to distinguish between problems involving scattering of unpolarized radiation in accordance with the Rayleigh's phase function (7) and full *Rayleigh scattering* which includes the treatment of polarization (see §5).

The geometries considered by Chandrasekhar were either *semi-infinite,* extending in optical depth from $\tau = 0$ to $\tau = \infty$, or *finite,* extending from $\tau = 0$ to $\tau = \tau_1$.

The various classes of transfer problems are now defined by their particular boundary conditions on the intensities. One important example is the *radiative equilibrium problem* or *Milne problem,* which is defined as a semi-infinite, conservative atmosphere with no incident intensity at $\tau = 0$, and with a condition at infinity that the intensities should not grow exponentially,

$$\begin{aligned} I(0,\mu,\varphi) &= 0, & -1 \le \mu \le 0 \\ I(\tau,\mu,\varphi)e^{\epsilon\tau} &\to 0, & \tau \to \infty \end{aligned} \tag{8}$$

for any $\varepsilon > 0$. Because the problem is homogeneous, one can also specify the (constant) net flux F carried in the medium. The radiative equilibrium problem provides one of the simplest models for a stellar atmosphere.

Another important class of problems involves the *diffuse transmission and reflection* of radiation for a finite medium, or just the diffuse reflection for the semi-infinite medium. With radiation incident on $\tau = 0$ in direction $(-\mu_0, \varphi_0)$, and with net flux πF normal to the direction of the beam, the problem is to find the emergent diffuse intensities at the boundaries.

The term *diffuse* in this context refers to the separation of the radiation field into the unscattered part travelling in direction $(-\mu_0, \varphi_0)$, and the remaining so-called diffuse part, which has scattered at least once. The diffuse radiation is much smoother in its angular dependence, and so is more suitable to be treated by the discrete ordinate method, for example. It is easily shown that the diffuse radiation field satisfies a modified integro-differential equation,

$$\mu \frac{\partial I(\tau, \mu, \varphi)}{\partial \tau} = I(\tau, \mu, \varphi) - \frac{1}{4\pi} \int_{-1}^{+1} \int_0^{2\pi} p(\mu, \varphi; \mu', \varphi') I(\tau, \mu', \varphi')\, d\mu' d\varphi' - \frac{1}{4} F e^{-\tau/\mu_0} p(\mu, \varphi; -\mu_0, \varphi_0), \tag{9}$$

(RT, p. 22, Eq. 126). The incident radiation is now fully accounted for in the inhomogeneous term, and the boundary conditions are for zero incident diffuse radiation at the boundaries.

The *diffuse scattering and transmission functions, S* and *T,* are now defined in terms of the emergent intensities at each boundary

$$\begin{aligned} I(0, \mu, \varphi) &= \frac{F}{4\pi} S(\tau_1; \mu, \varphi; \mu_0, \varphi_0), \\ I(\tau_1, -\mu, \varphi) &= \frac{F}{4\pi} T(\tau_1; \mu, \varphi; \mu_0, \varphi_0). \end{aligned} \tag{10}$$

For the semi-infinite case no transmission function is defined. If the incident radiation has axial symmetry, only simplified versions of the scattering and transmission functions are required that do not depend on the azimuthal variables, which are then denoted[5] $S(\mu, \mu_0)$ and $T(\mu, \mu_0)$.

Using these scattering and transmission functions, along with the linear superposition principle, it is possible to express the reflected and transmitted intensities corresponding to any arbitrary incident radiation. As will be seen below, the scattering and transmission functions are not merely of interest in special circumstances, but define the fundamental structure of the radiative transfer properties of the medium, and play a crucial role in the invariance principles.

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[5] The argument τ_1 is often omitted if it is clear from context.

3. The discrete ordinate method

An integro-differential equation of transfer with two-point boundary conditions (e.g., Eq. [1] or [2]) presents a difficult mathematical problem. By 1944, fully analytic solutions had been presented for only a few problems (e.g., the solution to the semi-infinite Milne problem by Wiener and Hopf [3]), but these methods did not extend to all of the problems that were of interest to Chandrasekhar. This led him to adopt a scheme, introduced earlier by Wick [6], that reduced an intero-differential equation to an approximate, finite set of ordinary differential equations by the introduction of a quadrature scheme into the integral term. Because of Chandrasekhar's subsequent extensive development of this method, it is now often known as the *Wick-Chandrasekhar discrete ordinate method* or simply the *discrete ordinate method.*

The quadrature formula used in the Wick-Chandrasekhar method approximates the integral of an arbitrary function $f(\mu)$ from –1 to +1 by a sum over discrete values of the function:

$$\int_{-1}^{1} f(\mu)\, d\mu = \sum_j a_j f(\mu_j). \tag{11}$$

The quadrature constants a_j and μ_j can be chosen in a variety of ways. Wick [6] suggested that the best choice for the constants are those of the Gaussian quadrature formula, for which the formula (11) is *exact* for any polynomial in μ of degree less than $(2n -1)$. Because of the symmetry between the positive and negative values of μ, by restricting the order of quadrature to be even, say $2n$, these constants can be numbered in such a way that j takes the values $\pm 1, \pm 2, \ldots, \pm n$ where $a_{-j} = a_j$ and $\mu_{-j} = \mu_j$.

Applying the above quadrature formula to the integral in equation (1), and setting $\mu = \mu_i$, $i = \pm 1, \pm 2, \ldots, \pm n$, one obtains the equations of the discrete ordinate method,

$$\mu_i \frac{dI_i}{d\tau} = I_i - \frac{\varpi_0}{2} \sum_j a_j I_j, \tag{12}$$

where $I_j = I(\tau, \mu_j)$. These represent a system of $2n$ coupled, ordinary differential equations for the components of specific intensity at the discrete angles μ_i. Solution of these equations are expected to yield an approximation to the true solution of the full integro-differential equation (1).

The discrete ordinate method gave highly accurate solutions in a surprisingly compact form. Consider, for example, the radiative equilibrium problem (Milne problem) for a constant flux F, where $\varpi_0 = 1$. Of particular interest is the angular distribution of the emergent intensity for this problem, which can be expressed

$$I(0, \mu) = \frac{\sqrt{3}}{4} F H(\mu) \tag{13}$$

which defines $H(\mu)$, the H-function.

By use of the discrete ordinate equations of order n, Wick showed that the H-function could be expressed simply. As a preliminary step, one finds the n positive roots[6] k_a for k of the *characteristic equation*

$$1 = \frac{\varpi_0}{2} \sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 k^2}. \tag{14}$$

In principle, this is equivalent to solving the n-order polynomial equation in k^2 that results upon multiplying both sides by the product of denominators. The H-function is then given by the simple formula,

$$H(\mu) = \frac{\prod_{i=1}^{n} (1 + \mu/\mu_i)}{\prod_{\alpha=1}^{n} (1 + \mu k_\alpha)}. \tag{15}$$

It is interesting to compare this approximate expression for $H(\mu)$ to the earlier exact analytic formula for the Milne problem, due to Hopf [4, p. 105]:

$$H(\mu) = (1+\mu) \exp\left\{ -\frac{\mu}{\pi} \int_0^{\pi/2} \frac{\log[(1 - \phi \cot\phi)/ \sin^2\phi]}{\cos^2\phi + \mu^2 \sin^2\phi} \, d\phi \right\}. \tag{16}$$

While this exact formula is beautifully elegant in its own right, it does require a separate numerical quadrature to determine the value of $H(\mu)$ for each value of μ. By contrast, the discrete ordinate method requires the solution to the characteristic equation (14), but once this is done the formula (15) is a simple closed expression for $H(\mu.)$ for any value of μ.

Chandrasekhar's numerical comparison of low order results with the exact analytic result (16) convinced him that the approximate results of the discrete ordinate method would converge to the exact results in the limit $n \to \infty$. Only much later was this convergence proved mathematically [44].

Chandrasekhar also applied the method of discrete ordinates to the problem of diffuse reflection, in which radiation is incident on the medium at angle μ_0, and one is required to find the radiation emergent at angle μ. This relationship is given in terms of a scattering function $S(\mu,\mu_0)$-It had previously been shown by Hopf [4, Eq. 191] that the scattering function is related to the above H-function (16) by means of the relation

$$\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) S(\mu, \mu_0) = \varpi_0 H(\mu) H(\mu_0). \tag{17}$$

Thus, the function $S(\mu,\mu_0)$ of *two* variables, can be simply expressed in terms of a function of a *single* variable, the same H-function that appears in the solution for the radiative equilibrium problem.

[6] For $\varpi_0 = 1$ there are only $n - 1$ such roots.

When Chandrasekhar applied the discrete ordinate method to the semi-infinite diffuse reflection problem (cf. RT, §26), he found a result of the same form as Eq.(17), where the H-function (15) were precisely the same as that for the discrete ordinate solution (15) to the radiative equilibrium problem.

For the case of a finite medium, besides the diffuse scattering function $S(\mu,\mu_0)$ there is also a diffuse transmission function $T(\mu, \mu_0)$ to be determined. These functions satisfy the extended relations, given by Ambartsumian [5],

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right) S(\mu,\mu_0) = \varpi_0\,[X(\mu)X(\mu_0) - Y(\mu)Y(\mu_0)]\,,$$
$$\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right) T(\mu,\mu_0) = \varpi_0\,[X(\mu)Y(\mu_0) - Y(\mu)X(\mu_0)]\,, \tag{18}$$

where the functions[7] X and Y were solutions to certain functional equations. After seeing these forms in Ambartsumian's paper, Chandrasekhar (RE-XXI [31]) was able to put the discrete ordinate solution for this problem into the same form, where the X-and Y-functions were also expressible in closed form in terms of the roots of the characteristic function. This solution required almost fifteen pages of algebra in RE-XXI (shortened somewhat in RT, §59). Anyone who has tried to reproduce this result, starting from scratch and properly arranging the vast arrays of equations into intelligible form, will gain tremendous respect for Chandrasekhar's feat of complex algebraic manipulation.

It may seem strange, even paradoxical, that Chandrasekhar's work in radiative transfer, rightly regarded as a *tour de force* in mathematical physics, should have depended so strongly and intimately on a numerical approximation scheme such as the discrete ordinate method. But one should remember that the problems confronted by mathematical physicists, whatever the field, are usually highly idealized. The transfer problems Chandrasekhar considered had already been simplified by making a number of physical assumptions and approximations: e.g., plane-parallel geometry, coherent scattering, and single-scattering albedo independent of depth. In a sense, choosing discrete angles is just one more simplifying approximation, on a par with the others made. Then the crucial questions to ask are whether these simplified equations are of practical use, can increase our mathematical or physical understanding, or satisfy some criterion of mathematical beauty. I believe for the discrete ordinates method the answer is "yes" to each of these questions.

As to the practicality of the method, remember that Chandrasekhar was not acting solely as a mathematical physicist, but as an astrophysicist attempting to find answers to practical problems in stellar atmospheres and planetary atmospheres. Much of his effort was devoted to the construction of detailed tables of functions to be used for solving real problems. Martin Schwarzschild said of him,[8]

[7] Ambartsumian used the notations ϕ and ψ for these functions.

[8] Quoted by Wali [51, p. 188].

> Chandra had no snobbishness in regard to his mathematical work. He did not shy away from numerical, computational solutions. He mixed rigorous analysis with numerical calculations, as the problem required.

The discrete ordinate method gave him highly accurate solutions in a completely straightforward way.

As to the method's relation to mathematical and physical understanding, Chandrasekhar was obviously delighted when he continually found many results of this method that were perfectly consistent with exact requirements of the theory. Many known analytically exact results are obeyed precisely to all orders of approximation in the discrete ordinate method, for example, the Hopf-Bronstein relation $J(0) = \sqrt{3}F/4$ in the Milne problem, the structure of the diffuse scattering and transmission functions as given in Eqs. (17) and (18), and their reciprocity relations. Often he was able to determine the form of the exact solutions only after he had solved the discrete ordinate equations first. These circumstances convinced Chandrasekhar that the discrete ordinate method was more than just a convenient numerical method; it also preserved essential mathematical and physical characterstics of the problem being investigated.

Mathematical beauty is a highy personal matter, and many would not apply such a term to the discrete ordinate method. However, in its defense one notes true that many of the basic results of the method are expressible in forms that are surprisingly compact and elegant, much more so than one might have imagined when first tackling a complicated set of coupled differential equations. It is clear that a great proportion of Chandrasekhar's day-to-day work in radiative transfer must have been involved with manipulations of quite complex sets of discrete ordinate equations. One can only assume that he found this perfectly consistent with his view of the field as one of elegance and beauty.

4. Invariance principles

An important component of Chandrasekhar's work in radiative transfer was devoted to the development of the *invariance principles.* These principles were first introduced by Ambartsumian in 1943 [5] and 1944 [7], but, because of the difficulties in communications during World War II, Chandrasekhar did not immediately learn of this work. In [50] he relates how he became aware of [5] during the summer of 1945, and became aware only "very much later" of [7].

After seeing Ambartsumian's work [5] in 1945, Chandrasekhar quickly realized the importance of the invariance principles. Besides their intrinsic mathematical elegance, the invariance principles greatly simplified radiative transfer problems by showing from the outset the underlying structure of the solutions, e.g., the forms of the scattering and transmission functions in Eqs. (17) and (18). Subsequently, Chandrasekhar began a substantial redirection of his efforts in radiative transfer. He spent much of the next years in presenting expanded derivations of the invariance principles and generalizing them to more complex cases, ultimately to include polarization.

In the context of plane-parallel radiative transfer problems, the invariance principles result from consideration of what happens upon addition (or subtraction) of layers of material to the surfaces of the medium. In the simplest case, the medium has constant properties, such as phase function and single scattering albedo, and the added layer shares these same properties.

Ambartsumian [5] considered the problem of diffuse reflection from an isotropically scattering semi-infinite medium. He observed the obvious physical fact that the scattering function would be unchanged if a small layer of material were added at the surface, since the resulting medium would still be semi-infinite. By simply tracking through the changes in intensities due to the added layer, and setting the net change to zero, he found that the scattering function become immediately expressible in the form given in Eq. (17). Thus the remarkable factorization of the scattering function in the semi-infinite case is a direct result of an invariance principle.

In the same paper [5], Ambartsumian also considered the appropriate generalization to the invariance principle for a finite medium. In this case, the reflection and transmission functions are unchanged by adding an infinitesimal layer to one surface while simultaneously subtracting a layer of the same thickness from the other surface. Without giving details, Ambartsumian stated that, as a result of this invariance principle, the reflection and transmission functions for the finite case were expressible in the simple forms given in Eq. (18). Using the Ambartsumian invariance results for guidance, Chandrasekhar was able in RE-XXI [31] to solve the discrete ordinate solution for isotropic scattering in a finite medium. He then extended the theory to include various cases of anisotropic scattering (RT, §§64-65).

At this time Chandrasekhar began his own investigations into the invariance principles, which extended the groundbreaking work of Ambartsumian. The foundations of his work were four invarince principles, which were presented first in RE-XVII [27], and later, in their most elegant form, in RT, (pp. 161-166, §50). Because of their importance and beauty, it is worthwhile here to write Chandrasekhar's four principles out in full. They are expressed first in words, and then mathematically.

I. *The intensity,* $I(\tau, +\mu, \varphi)$ *in the outward direction at any level* τ *results from the reflection of the reduced incident flux* $\pi F e^{-\tau/\mu_0}$ *and the diffuse radiation* $I(\tau, -\mu', \varphi')$ $(0 < \mu' \leq 1)$ *incident on the surface* τ *, by the atmosphere of optical thickness* $(\tau_1 - \tau)$*, below* τ.

$$I(\tau, +\mu, \varphi) = \frac{F}{4\pi} e^{-\tau/\mu_0} S(\tau_1 - \tau; \mu, \varphi; \mu_0, \varphi_0) + \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} S(\tau_1 - \tau; \mu, \varphi; \mu', \varphi') I(\tau, -\mu', \varphi')\, d\mu' d\varphi' \quad (19)$$

II. *The intensity, $I(\tau, -\mu, \varphi)$, in the inward direction at any level τ results from the transmission of the incident flux by the atmosphere of optical thickness τ, above the surface τ, and the reflection by this same surface of the diffuse radiation $I(\tau, +\mu', \varphi')$* $(0 \le \mu' \le 1)$ *incident on it from below.*

$$I(\tau, -\mu, \varphi) = \frac{F}{4\pi} T(\tau; \mu, \varphi; \mu_0, \varphi_0) + \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} S(\tau; \mu, \varphi; \mu', \varphi') I(\tau, +\mu', \varphi')\, d\mu' d\varphi'. \quad (20)$$

III. *The diffuse reflection of the incident light by the entire atmosphere is equivalent to the reflection by the part of the atmosphere of optical thickness τ, above the level τ, and the transmission by this same atmosphere of the diffuse radiation $I(\tau, +\mu', \varphi')$* $(0 \le \mu' \le 1)$ *incident on the surface τ from below.*

$$\frac{F}{4\pi} S(\tau_1; \mu, \varphi; \mu_0, \varphi_0) = \frac{F}{4\pi} S(\tau_1 - \tau; \mu, \varphi; \mu_0, \varphi_0) + e^{-\tau/\mu} I(\tau, +\mu, \varphi) + \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T(\tau; \mu, \varphi; \mu', \varphi') I(\tau, +\mu', \varphi')\, d\mu' d\varphi'. \quad (21)$$

IV. *The diffuse transmission of the incident light by the entire atmosphere is equivalent to the transmission of the reduced incident flux $\pi F e^{-\tau/\mu_0}$ and the diffuse radiation $I(\tau - \mu', \varphi')$ $(0 < \mu' \le 1)$ incident on the surface τ by the atmosphere of optical thickness $(\tau_1 - \tau)$ below τ.*

$$\frac{F}{4\pi} T(\tau_1; \mu, \varphi; \mu_0, \varphi_0) = \frac{F}{4\pi} e^{-\tau/\mu_0} T(\tau_1 - \tau; \mu, \varphi; \mu_0, \varphi_0) + e^{-(\tau_1 - \tau)/\mu} I(\tau, -\mu, \varphi) + \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T(\tau_1 - \tau; \mu, \varphi; \mu', \varphi') I(\tau, -\mu', \varphi')\, d\mu' d\varphi'. \quad (22)$$

These invariance principles were introduced by Chandrasekhar purely on intuitive and physical grounds; he made no attempt to prove them, starting, for example, with the transfer equation.

By differentiating Eqs. (19)–(22) with respect to τ and passing to an appropriate limit, either $\tau = 0$ or $\tau = \tau_1$, and with some further manipulations, Chandrasekhar found a set of four integral relations involving the scattering and transmission functions and their derivatives with respect to τ_1 (RT, §51). By eliminating the derivatives between two pairs of these equations, the two invariance principles of Ambartsumian were recovered. But this left two relationships involving the derivatives, so Chandrasekhar's procedure had in essence doubled the number of known invariance principles.

For example, in the case of isotropic scattering in an axially symmetric, finite atmosphere two of these four relationships are just Ambartsumian's results previously given in Eq. (18). In addition, there are two new results,

$$\begin{aligned} \frac{\partial S(\tau_1;\mu,\varphi)}{\partial \tau_1} &= \varpi_0 Y(\mu) Y(\mu_0), \\ \left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)\frac{\partial T(\tau_1;\mu,\varphi)}{\partial \tau_1} &= \varpi_0\left[\frac{1}{\mu_0}X(\mu)Y(\mu_0)-Y(\mu)X(\mu_0)\right]. \end{aligned} \quad \cdot \quad (23)$$

It became apparent to Chandrasekhar that the invariance principles were a powerful tool for attacking complex problems, in part, simply by showing the structure of the solutions. It also became clear that X-and Y-functions were important, fundamental quantities, and he spent much effort in developing a theory for their properties. This theory is concerned with more general functions than the particular ones appropriate to the isotropic scattering problem. What Chandrasekhar studied were the class of functions that satisfied the pair of equations,

$$\begin{aligned} X(\mu) &= 1+\mu\int_0^1 \frac{\Psi(\mu')}{\mu+\mu'}\left[X(\mu)X(\mu')-Y(\mu)Y(\mu')\right], \\ Y(\mu) &= e^{-\tau_1/\mu}+\mu\int_0^1 \frac{\Psi(\mu')}{\mu+\mu'}\left[Y(\mu)X(\mu')-X(\mu)Y(\mu')\right]. \end{aligned} \quad (24)$$

The *characteristic function* $\psi(\mu)$ is an even function of μ (Chandrasekhar took this to be a polynomial, although much of his development is not dependent on that assumption). For the particular case $\psi(\mu) = 1/2$, the X-and Y –functions defined by these equations are just the ones previously defined for the isotropic scattering problem.

The theory of the X-and Y -functions is very elegantly presented in RT, chapter VIII. A great number of results are given there concerning these functions and their moments, which play a crucial role in Chandrasekhar's subsequent work. One problem that Chandrasekhar left unresolved in his theory of the X-and Y –functions was their non-uniqueness in the conservative case (RT, §58). This was not settled until many years later by Mullikin [46].

The general invariance equations imply that many radiative transfer problems of great complexity, such as those involving anisotropic scattering or polarization, can be drastically simplified. Although the algebra was very difficult, he showed how the scattering and transmission functions could be reduced to expressions involving a number of trigonometric functions of $(\varphi - \varphi_0)$, plus a finite set of pairs of X- and Y-functions, each one with a different characteristic function $\psi(\mu)$. Thus, once the X-and Y -functions are determined, these radiative transfer problems are *completely* solved.

The magnitude of this reduction is impressive. For fixed τ_1, it is equivalent to reducing the problem of solving for functions S and T of *three* variables, μ, μ_0, and

$(\varphi - \varphi_0)$,to one of solving for functions of only *one* variable. In the case of Rayleigh scattering, Chandrasekhar achieved this reduction for each element of the scattering and transmission matrices for the Stokes parameters. Since the discrete ordinate method yields explicit formulas for the X-and Y -functions, Chandrasekhar could legitimately claim to have given complete and exact solutions to these very difficult problems.

Even with the insights provided by the invariance principles, these problems were still remarkably difficult. In order to solve them, Chandrasekhar needed to pick the correct form of the solution, and this he did by a combination of intuitive guessing and by comparison with similar solutions found with the discrete ordinate method. Looking at one of his papers such as the sixty-four page RE-XXI [31], one will also conclude that he possessed a heroic capacity for doing algebra.

5. Polarization

One of the motivations for Chandrasekhar's investigations into radiative transfer was the problem of the polarization of the sunlit sky. This problem had been treated by Lord Raleigh, who first derived and explained the basic molecular scattering mechanisms of the atmosphere. However, Raleigh's detailed predictions of sky polarization were based on the approximation of a single scattering of radiation and, while these matched the observations in very broad outline, substantial unexplained deviations still remained. For example, Rayleigh's simple theory predicted a nonvanishing amount of polarization in all directions (except directly towards or away from the sun). However, it was known that, depending on the sun's location, there exist two (sometimes three) neutral points of zero polarization on the sun's meridian circle, called the *Babinet*, *Brewster*, and *Arago* points. It was generally believed that the discrepancy lay in the assumption of single scattering, and that second (and higher) order scattering needed to be taken into account.

In [50] Chandrasekhar recalls L.V. King's ([2]) opinion on the inclusion of multiple scattering in the Rayleigh problem in 1913,

> The complete solution of the problem from this aspect would require us to split up the incident radiation into two components one of which is polarized in the principal plane and the other at right angles to it: the effect of self-illumination would lead to simultaneous integral equations in three variables the solution of which would be much too complicated to be useful.

Chandrasekhar took this as a direct challenge. The solution to the Rayleigh problem became one of the central goals of his theoretical investigations, and his ultimate success with it gave him considerable personal pride.

In order to attack the Rayleigh problem (and equivalent problems involving Thomson scattering) Chandrasekhar first needed a proper framework for the description of polarized radiation in radiative transfer theory. For the special case of scattering with axial symmetry, it was easy to see that the relevant polarization variables could be chosen as the intensities in the two planes parallel and perpendicular to the meridian plane through the direction of the ray, I_l and I_r.[9] In RE-X [19] Chandrasekhar used these variables to formulate the radiative equilibrium problem for Thomson scattering and was able to give a complete solution using the discrete ordinate method. One simple, noteworthy result of his numerical calculations was that the degree of polarization of the emergent intensity varied from zero for the normal direction $\mu = 1$, to a maximum of 11.7% for grazing emergence $\mu = 0$. This result is widely quoted as a bound for the degree of polarization that can result from Thomson scattering in such situations.

Chandrasekhar realized that the symmetry arguments that led to a choice of polarization variables in the axially symmetric case were insufficient to deal with the full Raleigh problem, where the generally oblique angle of the incident sunlight would spoil axial symmetry, so that the principal planes of polarization would not generally be aligned with the meridian plane. He needed a more general formulation for polarization, but searched in vain through the then standard texts. In [50] he recounts his state of mind at this point,

> Nevertheless, it did not seem to me that the basic question could have been overlooked by the great masters of the nineteenth century.

He set his attention on the old papers of Rayleigh and Stokes and quickly found that the required formalism had been developed by Stokes [1] in 1852, almost a century before. This work had fallen into virtual obscurity, but Chandrasekhar seized upon it and resurrected it into the modem form we now know.

Stokes had considered what general description of the polarization of a beam of radiation was required to account for all the various possibilities of linear, circular, or elliptical polarization *and* the possibility that the beam might be completely or partially unpolarized. His conclusion was that such a description required four parameters, which he called *A*, *B*, *C*, and *D*. Chandrasekhar called these the *Stokes parameters* and gave them the names, *I*, *Q*, *U*, and *V*, which are now widely used in astronomy. Roughly speaking, I measures the total intensity, *Q* and *U* measure the orientation and degree of linear polarization, and *V* measures the degree of circular polarization.

In a series of papers ([20], [23], & [12]) Chandrasekhar formulated the full scattering problem with polarization. His development of the Stokes parameters is separated into two stages: in RE-XI [20] Chandrasekhar introduced only the theory necessary for the strict Rayleigh problem, without the circularity parameter *V*; in RE-XV [25] he gave the full theory of all four Stokes parameters. His clear account of the Stokes parameters in RT (§15) remains one of the best standard treatments of this theory.

[9] The subscripts here are apparently mnemonics using the last letters of “parallel” and “perpendicular.”

In RT (§ 17, Eq. 226) Chandrasekhar shows that the appropriate transfer equation for polarized transfer is exactly analogous to Eq. (2) for unpolarized radiation if one regards I as a vector of Stokes parameters and p as an appropriate phase matrix. With the proper description of polarization in hand, Chandrasekhar was able to attack more complex problems of Rayleigh scattering for polarized radiation (RT, chapter X). In this case the fourth Stokes parameter V plays only a limited role; Chandrasekhar shows that it obeys its own transfer equation uncoupled from the others. One consequence of this is that if the sources of radiation (internal or through boundary conditions) do not introduce any circularity, then the solution will also be completely free of circularity. Thus, both the radiative equilibrium problem and the reflection and transmission problems with incident unpolarized light do not involve V at all.

In RT, §§69–70, Chandrasekhar gives the solution for the diffuse reflection of a semi-infinite Rayleigh scattering atmosphere, which requires the full non-axially symmetry theory. This solution depends on five H-functions, called H_l, H_r, $H^{(1)}$, $H^{(2)}$, and $Hv(\mu)$, defined by their respective characteristic functions

$$\begin{aligned}
\Psi_l(\mu) &= \frac{3}{4}(1-\mu^2),\\
\Psi_r(\mu) &= \frac{3}{8}(1-\mu^2),\\
\Psi^{(1)}(\mu) &= \frac{3}{8}(1-\mu^2)(1+2\mu^2),\\
\Psi^{(2)}(\mu) &= \frac{3}{16}(1+\mu^2)^2,\\
\Psi_v(\mu) &= \frac{3}{4}\mu^2.
\end{aligned} \tag{25}$$

For the finite problem there is a corresponding solution for the diffuse reflection and transmission functions in terms of the analogous five X-and Y -functions defined with the same characteristic functions.

In order to make detailed predictions for the polarization of the sunlit sky, Chandrasekhar also realized that there was possibly an additional physical effect to be taken into account, namely, that some of the radiation striking the earth's surface would be scattered or reflected back into the atmosphere, which Chandrasekhar called the *planetary problem*, to distinguish it from the usual case of no incident radiation, called the *standard problem*. With the assumption of ground reflection obeying Lambert's law (reflected intensities independent of angle), Chandrasekhar found a clever way of reducing the planetary problem to the standard one (RT, §72.1). Thus he was able to give full solutions for the polarization of the sky as a function of the incident sunlight angle.

The solutions for the sunlit sky showed precisely the character of the observations, in particular the various neutral points described above. In 1954, Chandrasekhar & Elbert published a paper [41] with complete polarization sky maps based on his theory, which showed not only the neutral points on the sun's meridian, but also the complete neutral lines.

All this was attained by completely analytic means, except for the final numerical evaluation of the relevant X-and Y -functions. For Chandrasekhar this represented a tremendous triumph for his methods.

6. Perspective

Up to this point, attention has been confined primarily to the bulk of Chandrasekhar's work on radiative transfer theory during 1944–50. A full account of its influence on the field in the succeeding forty-five years would be an inappropriate task for a short review such as this. However, it seems appropriate to end with at least a few examples of Chandrasekhar's continuing influence.

In different ways, the discrete ordinate method was both Chandrasekhar's most transient and his most permanent contribution to the field. After RT, the development of analytical radiative transfer rapidly moved toward full treatment of the angular dependence of the solutions, rather than discrete versions. This could already be seen in Chandrasekhar's own work, where he gradually (but not completely) shifted from the discrete ordinate method to the invariance principles to give him the structure of the solutions. Perhaps the most interesting development in this context was the *singular eigenfunction method*, used in plasma physics by Van Kampen [42], and later applied to transfer theory by Case and others (see, e.g., Case & Zweifel [47]). In a sense, this is the true descendent of the discrete ordinate method, since it also starts by asking for solutions of exponential form, but now confronts the true nature of the continuous angular dependence in the scattering integral.

But if the discrete ordinate method has fallen out of favor for analytical radiative transfer, it remains to this day a very strong component of numerical work in stellar atmospheres and other astrophysical applications, because of its simplicity, accuracy, and adaptability to complex physical situations. In this way, the method continues to serve those seeking practical solutions to real physical problems, as Chandrasekhar himself was.

One of the most surprising long-term implications of the invariance principles has been their generalization and development into an entire mathematical field known as "invariant imbedding" by Bellman and others (see, e.g., [48]). This development was based on the recognition that the invariance principles had converted what was a *boundary value problem* (involving boundary conditions at two or more points) into an *initial value problem* (involving boundary conditions at a single point) through the introduction of the reflection and transmission functions and the integro-differential equations they satisfy. The invariant imbedding methods generalize this idea of transforming from a boundary value problem to an initial value problem to a much wider class of problems than just radiative transfer, including wave propagation and control theory, among others. This was an important practical advance, since initial value problems are generally much more numerically tractable than boundary value problems.

In the introduction to their work [48], Bellman and Wing, after crediting earlier workers, clearly express the special influence of Chandrasekhar in the development of their methods,

> However, it was not until the gifted mathematical physicist S. Chandrasekhar appeared and published his famous book on radiative transfer [RT] that the authors and others began their intensive and extensive studies of the imbedding methods we shall describe in this book. Chandrasekhar developed an elegant theory of principles of invariance, thus completing and considerably extending the ingenious methods of Ambartsumian.

This quotation speaks of the widespread and lasting influence of Chandrasekhar's work and the awe in which it is held. It also shows the wisdom of his custom of distilling his work in the form of a well-written treatise that is much more than a collection of results and facts. In this way he has been able to keep alive the perspective on the field he worked so hard to gain himself, while communicating to others the "elegance and beauty" he felt about it.

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