Toric

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April 20, 2025

Chapter 0

Prerequisites

0.1 Affine Monoids

Lemma 0.1.1 (Multivariate Laurent polynomials are an integral domain)**.** *Multivariate Laurent polynomials over an integral domain are an integral domain.*

Proof. Come on. $\square$

Definition 0.1.2 (Affine monoid)**.** An *affine monoid* is a finitely generated commutative monoid which is:

- cancellative: if $a + c = b + c$ then $a = b$, and
- torsion-free: if $na = nb$ then $a = b$ (for $n \geq 1$).

Proposition 0.1.3 (Embedding an affine monoid inside a lattice)**.**
If M is an affine monoid, then M can be embedded inside $\mathbb{Z}^n$ for some n.

Proof. Embed M inside its Grothendieck group G. Prove that G is finitely generated free. $\square$

Proposition 0.1.4 (Affine monoid algebras are domains)**.**
If R is an integral domain M is an affine monoid, then $R[M]$ is an integral domain and is a finitely generated R-algebra.

Proof.
$i : R[M] \hookrightarrow R[\mathbb{Z}M]$ injects into an integral domain so is an integral domain. It's finitely generated by χ^{a_i} where $\mathcal{A} = \{a_1, \dots, a_s\}$ is a finite generating set for M. $\square$

Definition 0.1.5 (Irreducible element)**.** An element x of a monoid M is *irreducible* if $x = y + z$ implies $y = 0$ or $z = 0$.

Proposition 0.1.6 (Irreducible elements lie in all sets generating a salient monoid)**.**
If M is a monoid with a single unit, and S is a set generating M, then S contains all irreducible elements of M.

Proof. Assume p is an irreducible element. Since S generates M, write

$$p = \sum_i a_i$$

where the a_i are finitely many elements (not necessarily distinct) elements of S. Since p is irreducible, we must have

$$p = a_i \in S$$

for some i. $\square$

Proposition 0.1.7 (A salient finitely generated monoid has finitely many irreducible elements)**.**

If M is a finitely generated monoid with a single unit, then only finitely many elements of M are irreducible.

Proof.
Let S be a finite set generating M. Write I the set of irreducible elements. By Proposition 0.1.6, $I \subseteq S$. Hence I is finite. $\square$

Proposition 0.1.8 (A salient finitely generated cancellative monoid is generated by its irreducible elements)**.**
If M is a finitely generated cancellative monoid with a single unit, then M is generated by its irreducible elements.

Proof. We do not follow the proof from [1].
Let S be a finite minimal generating set and assume for contradiction that $r \in S$ is reducible, say $r = a + b$ where a, b are non-units. Write

$$a = \sum_{s \in S} m_s s, b = \sum_{s \in S} n_s s$$

for some $m_s, n_s \in \mathbb{N}$, so that

$$r = \sum_{s \in S} (m_s + n_s) s.$$

We distinguish three cases

- $m_r + n_r = 0$. Then
$$r = \sum_{s \in S \setminus \{r\}} (m_s + n_s) s \in \langle S \setminus \{r\} \rangle$$
contradicting the minimality of S.

- $m_r + n_r = 1$. Then
$$0 = \sum_{s \in S \setminus \{r\}} (m_s + n_s) s \qquad \Longrightarrow \forall s \in S \setminus \{r\}, m_s s = n_s s = 0$$
Furthermore, either $m_r = 0$ or $n_r = 0$, so $a = 0$ or $b = 0$, contradicting the fact that a and b are non-units.

- $m_r + n_r \geq 2$. Then
$$0 = r + \sum_{s \in S \setminus \{r\}} (m_s + n_s) s$$
and $r = 0$, contradicting the minimality of S once again.

$\square$

0.2 Tensor Product

Lemma 0.2.1 (The tensor product of linearly independent families)**.** *If f and g are linearly independent families of points in semimodules M and N, then $ij \mapsto fi \otimes gj$ is a linearly independent family of points in $M \otimes N$.*

Proof. Assume
$$\sum_{i,j} c_{i,j} fi \otimes gj = \sum_{i,j} d_{i,j} fi \otimes gj$$
Then
$$\sum_i fi \otimes \left(\sum_j c_{i,j} gj \right) = \sum_i fi \otimes \left(\sum_j d_{i,j} gj \right)$$
Since f is linearly independent,
$$\sum_j c_{i,j} gj = \sum_j d_{i,j} gj$$
for every i. Since g is linearly independent, $c_{i,j} = d_{i,j}$ for all i, j, as wanted. $\square$

0.3 Hopf algebras

0.3.1 Group-like elements

Definition 0.3.1 (Group-like elements)**.** An element a of a bi-algebra A is *group-like* if it is a unit and $\Delta(a) = a \otimes a$, where Δ is the comultiplication map.

Lemma 0.3.2 (The image of a group-like element by the counit is 1)**.**
If A is a bialgebra, then the counit of A sends every group-like element of A to 1.

Proof. Let Δ be the comultiplication map and ϵ be the counit map. Let a be a group-like element of A. Using the coalgebra axioms and the fact that $\Delta(x) = x \otimes x$, we get:
$$x \otimes 1 = (\mathbf{1}_A \otimes \epsilon)(\Delta(x)) = x \otimes \epsilon(x).$$
As x is invertible, this implies that $1 \otimes 1 = 1 \otimes \epsilon(x)$ in $R \otimes_R A$. But the algebra map from R to A is injective because A is a bialgebra (hence the counit is a left inverse of this map), so $1 = \epsilon(x)$. $\square$

Proposition 0.3.3 (Group-like elements form a group)**.**
Group-like elements of a bi-algebra A form a group under multiplication.

Proof. Check that group-like elements are closed under unit, multiplication and inverses. $\square$

Lemma 0.3.4 (Bialgebra homs preserve group-like elements)**.**
Let $f : A \to B$ be a bi-algebra hom. If $a \in A$ is group-like, then $f(a)$ is group-like too.

Proof. a is a unit, so $f(a)$ is a unit too. Then
$$f(a) \otimes f(a) = (f \otimes f)(\Delta_A(a)) = \Delta_B(f(a))$$
so $f(a)$ is group-like. $\square$

Lemma 0.3.5.

If R is a commutative semiring, A is a Hopf algebra over R and G is a group, then every element of the image of G in $A[G]$ is group-like.

Proof. This is an easy check. □

Lemma 0.3.6.

If R is a commutative semiring, A is a Hopf algebra over R and G is a group, then the group-like elements in $A[G]$ span $A[G]$ as an A-module.

Proof.

This follows immediately from 0.3.5. □

Lemma 0.3.7 (Independence of group-like elements)**.**

The group-like elements in a bialgebra A over a field are linearly independent.

Proof.

See Lemma 4.23 in [2]. □

Lemma 0.3.8 (Group-like elements in a group algebra)**.**

Let k be a field. The group-like elements of $k[M]$ are exactly the image of M.

Proof.

See Lemma 12.4 in [2]. □

0.3.2 Diagonalizable bialgebras

Definition 0.3.9 (Diagonalizable bialgebras)**.** A bialgebra is called diagonalizable if it is isomorphic to a group algebra.

Lemma 0.3.10.

A diagonalizable bialgebra is spanned by its group-like elements.

Proof.

This is true for a group algebra by 0.3.6, and the property of being spanned by its group-like elements is preserves by isomorphisms of bialgebras. □

Proposition 0.3.11.

Let A be a bialgebra over a field k, and let G be the set of group-like elements of A (which is a group by 0.3.3). If A is generated by G, then the unique bialgebra morphism from $k[G]$ to A sending each element of G to iself is bijective.

Proof.

This morphism is injective by the linear independence of group-like elements (0.3.7), and surjective because the group-like elements of A span A by assumption. □

Corollary 0.3.12.

A bialgebra over a field is diagonalizable if and only if it is spanned by its group-like elements.

Proof.

We know that a diagonalizable bialgebra is spanned by its group-like elements by 0.3.10, and that a bialgebra over a field that is spanned by its group-like elements is diagonalizable by 0.3.11 (and by the fact that a bijective morphism of bialgebras is an isomorphism).

□

Proposition 0.3.11 and Corollary 0.4.15 are false over a general commutative ring. Indeed, let R be a commutative ring and let G be a group. Then the group-like elements of $R[G]$ correspond to locally constant maps from $Spec(R)$ to G (with the discrete topology), hence they are of the form $e_1 g_1 + \cdots + e_r g_r$, with the g_i in G and $e_1, \ldots, e_r$ a family of pairwise orthogonal idempotent elements of R that sum to 1. So $R[G]$ is not isomorphic to the group algebra over its group-like elements unless $Spec(R)$ is connected. As for the corollary, a bialgebra of the form $R_1[G_1] \times \cdots \times R_n[G_n]$, seen as a bialgebra over $R_1 \times \cdots \times R_n$, is generated by its group-like elements but not diagonalizable.

It is likely that both results are still true if the base ring has a connected spectrum.

0.3.3 The group algebra functor

Proposition 0.3.13 (The antipode is a antihomomorphism)**.** *If A is a R-Hopf algebra, then the antipode map $s : A \to A$ is anti-commutative, ie $s(a * b) = s(b) * s(a)$. If further A is commutative, then $s(a * b) = s(a) * s(b)$.*

Proof. Any standard reference will have a proof. □

Proposition 0.3.14 (Hopf algebras are cogroup objects in the category of algebras)**.** *From a R-Hopf algebra, one can build a cogroup object in the category of R-algebras.*

From a cogroup object in the category of R-algebras, one can build a R-Hopf algebra.

Proof.

Turn the arrows around. □

Definition 0.3.15 (The group algebra functor)**.**

For a commutative ring R, we have a functor $G \rightsquigarrow R[G] : \mathrm{Grp} \to \mathrm{Hopf}_R$.

Proposition 0.3.16 (The group algebra functor is fully faithful)**.**

For a field K, the functor $G \rightsquigarrow K[G]$ from the category of groups to the category of Hopf algebras over K is fully faithful.

Proof.

It is clearly faithful. Now for the full part, if $f : K[G] \to K[H]$ is a Hopf algebra hom, then we get a series of maps

$$G \simeq \text{ group-like elements of } R[G] \to \text{ group-like elements of } R[H] \simeq H$$

and each map separately is clearly multiplicative. □

0.4 Group Schemes

0.4.1 Correspondence between Hopf algebras and affine group schemes

We want to show that Hopf algebras correspond to affine group schemes. This can easily be done categorically assuming both categories on either side are defined thoughtfully. However, the categorical version will not be workable with if we do not also have links to the non-categorical notions. Therefore, one solution would be to build the left, top and right edges of the following diagram so that the bottom edge can be obtained by composing the three:

Bundling/unbundling Hopf algebras

We have already done the left edge in the previous section.

Spec of a Hopf algebra

Now let's do the top edge.

Proposition 0.4.1 (Sliced adjoint functors)**.** *If $a : F \vdash G$ is an adjunction between $F : C \to D$ and $G : D \to C$ and $X : C$, then there is an adjunction between $F/X : C/X \to D/F(X)$ and $G/X : D/F(X) \to C/X$.*

Proof. See https://ncatlab.org/nlab/show/sliced+adjoint+functors+-+section. □

Proposition 0.4.2 (Limit-preserving functors lift to over categories)**.** *Let J be a shape (i.e. a category). Let $\widetilde{J}$ denote the category which is the same as J, but has an extra object $*$ which is terminal. If $F : C \to D$ is a functor preserving limits of shape $\widetilde{J}$, then the obvious functor $C/X \to D/F(X)$ preserves limits of shape J.*

Proof. Extend a functor $K : J \to C/X$ to a functor $\widetilde{K} : \widetilde{J} \to C$, by letting $\widetilde{K}(*) = X$. □

Proposition 0.4.3 (Fully faithful product-preserving functors lift to group objects)**.** *If a finite-products-preserving functor $F : C \to D$ is fully faithful, then so is $\mathrm{Grp}(F) : \mathrm{Grp}\, C \to \mathrm{Grp}\, D$.*

Proof. Faithfulness is immediate.

For fullness, assume $f : F(G) \to F(H)$ is a morphism. By fullness of F, find $g : G \to H$ such that $F(g) = f$. g is a morphism because we can pull back each diagram from D to C along F which is faithful. □

Definition 0.4.4 (Spec as a functor on algebras)**.** Spec is a contravariant functor from the category of R-algebras to the category of schemes over Spec_R.

Proposition 0.4.5 (Spec as a functor on algebras is fully faithful)**.**

Spec is a fully faithful contravariant functor from the category of R-algebras to the category of schemes over Spec_R*, preserving all limits.*

Proof.

$\mathrm{Spec} : \mathrm{Ring} \to \mathrm{Sch}$ is a fully faithful contravariant functor which preserves all limits, hence so is $\mathrm{Spec} : \mathrm{Ring}_R \to \mathrm{AffSch}_{\mathrm{Spec}\, R}$ by Proposition 0.4.2 (alternatively, by Proposition 0.4.1). □

Definition 0.4.6 (Spec as a functor on Hopf algebras)**.**

Spec is a fully faithful contravariant functor from the category of R-algebras to the category of group schemes over Spec_R.

Proposition 0.4.7 (Spec as a functor on Hopf algebras is fully faithful)**.**

Spec is a fully faithful contravariant functor from the category of R-Hopf algebras to the category of group schemes over Spec_R.

Proof.

$\mathrm{Spec} : \mathrm{Ring}_R \to \mathrm{Sch}_{\mathrm{Spec}\, R}$ is a fully faithful contravariant functor preserving all limits according to Proposition 0.4.4, therefore $\mathrm{Spec} : \mathrm{Hopf}_R \to \mathrm{GrpSch}_{\mathrm{Spec}\, R}$ too is fully faithful according to 0.4.3. □

0.4.2 Essential image of Spec on Hopf algebras

Finally, let's do the right edge.

Proposition 0.4.8 (Essential image of a sliced functor). *If $F : C \to D$ is a full functor between cartesian-monoidal categories, then $F/X : C/X \hom D/F(X)$ has the same essential image as F.*

Proof. Transfer all diagrams. □

Proposition 0.4.9 (Equivalences lift to group object categories). *If $e : C \backsimeq D$ is an equivalence of cartesian-monoidal categories, then $\mathrm{Grp}(e) : \mathrm{Grp}(C) \backsimeq \mathrm{Grp}(D)$ too is an equivalence of categories.*

Proof. Transfer all diagrams. □

Proposition 0.4.10 (Essential image of a functor on group objects). *If $F : C \to D$ is a fully faithful functor between cartesian-monoidal categories, then $\mathrm{Grp}(F) : \mathrm{Grp}(C) \hom \mathrm{Grp}(D)$ has the same essential image as F.*

Proof.
Transfer all diagrams. □

Proposition 0.4.11 (Essential image of Spec on algebras).
The essential image of $\mathrm{Spec} : \mathrm{Ring}_R \to \mathrm{Sch}_{\mathrm{Spec}\, R}$ is precisely affine schemes over $\mathrm{Spec}\, R$.

Proof.
Direct consequence of Proposition 0.4.8. □

Proposition 0.4.12 (Essential image of Spec on Hopf algebras).
The essential image of $\mathrm{Spec} : \mathrm{Hopf}_R \to \mathrm{GrpSch}_{\mathrm{Spec}\, R}$ is precisely affine group schemes over $\mathrm{Spec}\, R$.

Proof.
Direct consequence of Propositions 0.4.10 and 0.4.11. □

0.4.3 Diagonalisable groups

Definition 0.4.13.
For a commutative group G we define $D_R(G)$ as the spectrum $\mathrm{Spec}\, R[G]$ of the group algebra $R[G]$.

Definition 0.4.14.
An algebraic group G over $\mathrm{Spec}\, R$ is **diagonalisable** if it is isomorphic to $D_R(G)$ for some commutative group G.

Theorem 0.4.15.
An algebraic group G over a field k is diagonalizable if and only if group-like elements span $\Gamma(G)$.

Proof.
See Theorem 12.8 in [2]. □

Theorem 0.4.16.
For a field k, D_k is a fully faithful contravariant functor from the category of commutative groups to the category of group schemes over $\mathrm{Spec}\, k$.

Proof.

Compose Propositions 0.4.7 and 0.3.16.

Also see Theorem 12.9(a) in [2]. See SGA III Exposé VIII for a proof that works for R an arbitrary commutative ring in place of k. $\square$

Chapter 1

Affine Toric Varieties

1.1 Introduction to Affine Toric Varieties

1.1.1 The Torus

Definition 1.1.1 (The split torus)**.** The split torus ${\mathbb{G}_m}^n$ over a scheme S is the pullback of $\operatorname{Spec}\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ along the unique map $S \to \operatorname{Spec}\mathbb{Z}$.

Lemma 1.1.2 (The split torus over $\operatorname{Spec} R$)**.**
The split torus over $\operatorname{Spec} R$ *is isomorphic to* $\operatorname{Spec}(R[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$.

Proof. Ask any toddler on the street. □

Definition 1.1.3 (Characters of a group scheme)**.**
For a group scheme G over S, the **character lattice** of G is

$$X(G) := \operatorname{Hom}_{\mathsf{GrpSch}_S}(G, \mathbb{G}_m).$$

An element $X(G)$ is (unsurprisingly) called a **character**.

Proposition 1.1.4 (Character lattice of the torus)**.**
Characters of the torus over a field k are isomorphic to $\mathbb{Z}^n$. $X({\mathbb{G}_m}^n) = \mathbb{Z}^n$.

Proof.
By Propositions 1.1.2 and 0.4.16 in turn, we have

$$X({\mathbb{G}_m}^n) = \operatorname{Hom}_{\mathsf{GrpSch}}({\mathbb{G}_m}^n, \mathbb{G}_m) = \operatorname{Hom}(k[\mathbb{Z}], k[\mathbb{Z}^n]) = \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}^n) = \mathbb{Z}^n.$$

□

Proposition 1.1.5 (The image of a torus is a torus)**.**
Let T_1 and T_2 be split tori over a field k and let $\Phi : T_1 \to T_2$ be a homomorphism, then Φ factors as

$$T_1 \xrightarrow{\Phi} T_2 = T_1 \xrightarrow{\phi} T \xrightarrow{\iota} T_2,$$

where T is a split torus, ι is a closed subgroup embedding and ϕ is an fpqc homomorphism.

Proof. Let $M_1 = X(T_1), M_2 = X(T_2)$. Define M to be the image of the homomorphism $M_2 \to M_1$ corresponding to Φ and take $T = D_k(M)$. The homomorphisms ι, ϕ correspond to the canonical quotient map $M_2 \to M$ and the canonical inclusion $M \to M_1$ respectively. Hence $\Phi = \iota \circ \phi$.

M is a subgroup of a finitely-generated free abelian group M_1, hence itself a finitely-generated free abelian group. Thus T is a split torus.

ι is a closed embedding since the corresponding ring map is a quotient map with kernel generated by the kernel of $M_2 \to M_1$.

ϕ is affine, hence quasi-compact. A collection of coset representatives for M/M_1 gives a basis for $k[M]$ as a $k[M_1]$ module, hence ϕ is faithfully flat. □

Proposition 1.1.6 (A subgroup of a torus is a torus)**.**
Let T be a split torus. If $H \subseteq T$ is an irreducible subgroup, then H is a split torus.

Proof.
Let $M = X(T), N = X(H)$. Since H is a closed subscheme $M \to N$ is surjective, so N is a finitely-generated abelian group. Since H is irreducible it is connected, so N is torsion-free, hence free. Thus H is a split torus. □

Definition 1.1.7 (The character eigenspace)**.**
For a finite dimensional representation of a torus T on W, the **character eigenspace** of a character $\chi \in X(T)$ is

$$W_m = \{w \in W : t \cdot w = \chi(t) \text{ for all } t \in T\}.$$

Proposition 1.1.8 (Decomposition into character eigenspaces)**.**
The space decomposes into the direct sum of the character eigenspaces.

Proof. TODO □

Definition 1.1.9.
For a group scheme G, the **cocharacter lattice** of G is $\mathrm{Hom}_{\mathsf{GrpSch}_S}(\mathbb{G}_m, G)$. An element is called a **cocharacter** or **one-parameter subgroup**.

Definition 1.1.10 (The character-cocharacter pairing)**.**
Character lattice and one-parameter subgroup pairing.

Proposition 1.1.11 (Cocharacter lattice of the torus)**.**
$N = \mathrm{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^n$*. For $u \in N$ we write λ^u for the corresponding cocharacter.*

Proof.
By Propositions 1.1.2 and 0.4.16 in turn, we have

$$\mathrm{cochar}({\mathbb{G}_m}^n) = \mathrm{Hom}_{\mathsf{GrpSch}}(\mathbb{G}_m, {\mathbb{G}_m}^n) = \mathrm{Hom}(k[\mathbb{Z}^n], k[\mathbb{Z}]) = \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n.$$

□

1.1.2 The Definition of Affine Toric Variety

Definition 1.1.12.
A **toric variety** is a variety X with

- an open embedding $T := (\mathbb{C}^\times)^n \hookrightarrow X$ with dense image
- such that the natural action $T \times T \to T$ of the torus on itself extends to an (algebraic) action $T \times X \to X$.

1.1.3 Lattice Points

Definition 1.1.13.
Given a finite set $\mathcal{A} = \{a_1, \dots, a_s\} \subseteq M$, define $\Phi_{\mathcal{A}} : T \to \mathbb{A}^s$ given by $\Phi_{\mathcal{A}}(t) = (\chi^{a_1}(t), \dots, \chi^{a_s}(t))$.

Definition 1.1.14.
$Y_{\mathcal{A}}$ is the (Zariski) closure of $\operatorname{im} \Phi_{\mathcal{A}}$ in $\mathbb{A}^s$.

Proposition 1.1.15.
Proposition 1.1.8

Proof.
TODO ☐

1.1.4 Toric Ideals

Proposition 1.1.16.
The ideal of the affine toric variety $Y_{\mathcal{A}}$ is

$$I(Y_{\mathcal{A}}) = \langle x^{\ell_+} - x^{\ell_-} | \ell \in L \rangle$$

Proof. See [1]. ☐

Definition 1.1.17. The ideal $I_L = \langle x^\alpha - x^\beta | \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L \rangle$ is called the **lattice ideal** of the lattice $L \subseteq \mathbb{Z}^s$.
A toric ideal is a prime lattice ideal.

Definition 1.1.18. A **toric ideal** is a prime lattice ideal.

Proposition 1.1.19.
Proposition 1.1.11: an ideal is toric if and only if it's prime and generated by binomials $x^\alpha - x^\beta$.

Proof.
☐

Proposition 1.1.20 (The spectrum of an affine monoid algebra is an affine toric variety)**.**
If S is an affine monoid, then $\operatorname{Spec}(\Bbbk[S])$ *is an affine toric variety.*

Proof.
Identify the torus with $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ using Lemma 1.1.2. i induces a morphism $T \to \operatorname{Spec}(\Bbbk[S])$. It's an open embedding as i gives the localization of $\Bbbk[S]$ at χ^{a_i}, so $\operatorname{im} i$ is an affine open. It's dominant as $\operatorname{Spec}(\Bbbk[S])$ is integral and so is irreducible, and $\operatorname{im} i$ is open and nonempty, so dense. The torus action is given by the natural restriction of comultiplication on $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ using Proposition 0.3.16. ☐

Proposition 1.1.21 (The character lattice of the spectrum of an affine monoid algebra)**.**
If S is an affine monoid, then the character lattice of $\operatorname{Spec}(\Bbbk[S])$ *is $\mathbb{Z}S$.*

Proof. It is what it is. ☐

Proposition 1.1.22.
If S is an affine monoid and $\mathcal{A}$ is a finite set generating S as a monoid, then $\operatorname{Spec}(\Bbbk[S]) = Y_{\mathcal{A}}$.

Proof.

We get a $\Bbbk$-algebra homomorphism $\pi : \Bbbk[x_1, \dots, x_s] \to \Bbbk[\mathbb{Z}S]$ given by $\mathcal{A}$; this induces a morphism $\Phi_{\mathcal{A}} : T \to \Bbbk^s$. The kernel of π is the toric ideal of $Y_{\mathcal{A}}$ and π is clearly surjective, so $Y_{\mathcal{A}} = \mathbb{V}(\ker(\pi)) = \operatorname{Spec}(\Bbbk[x_1, \dots, x_s]/\ker(\pi)) = \operatorname{Spec}(\mathbb{C}[S])$. □

Definition 1.1.23.

Torus action on semigroup algebra

1.1.5 Equivalence of Constructions

Lemma 1.1.24.

Proof. □

Theorem 1.1.25. *TFAE:*

1. *V is an affine toric variety.*
2. *$V = Y_{\mathcal{A}}$ for some finite $\mathcal{A}$.*
3. *V is an affine variety defined by a toric ideal.*
4. *$V = \operatorname{Spec} \Bbbk[S]$ for an affine monoid S.*

Proof. □

1.2 Cones and Affine Toric Varieties

1.2.1 Convex Polyhedral Cones

Fix a pair of dual real vector spaces M and N.

Definition 1.2.1 (Convex cone generated by a set)**.** For a set $S \subseteq N$, the **cone generated by** S, aka **cone hull of** S, is

$$\operatorname{Cone}(S) := \left\{ \sum_{u \in S} \lambda_u u | \lambda_u \geq 0 \right\}$$

Definition 1.2.2 (Convex polyhedral cone)**.**

A **polyhedral cone** is a set that can be written as $\operatorname{Cone}(S)$ for some finite set S.

Definition 1.2.3 (Convex hull)**.** For a set $S \subseteq N$, the **convex hull of** S is

$$\operatorname{Conv}(S) := \left\{ \sum_{u \in S} \lambda_u | \lambda_u \geq 0, \sum_{u} \lambda_u = 1 \right\}$$

Definition 1.2.4 (Polytope)**.**

A **polytope** is a set that can be written as $\operatorname{Conv}(S)$ for some finite set S.

1.2.2 Dual Cones and Faces

Definition 1.2.5 (Dual cone)**.**
Given a polyhedral cone $\sigma \subseteq N$, its **dual cone** is defined by

$$\sigma^\vee = \{m \in M | \forall u \in \sigma, \langle m, u \rangle \geq 0\}$$

.

Proposition 1.2.6 (Dual of a polyhedral cone)**.**
If σ is polyhedral, then its dual $\sigma^\vee$ is polyhedral too.

Proof. Classic, use Fourier-Motzkin eliminiation. □

Proposition 1.2.7 (Dual cone of a sumset)**.**
If σ_1, σ_2 are two cones, then

$$(\sigma_1 + \sigma_2)^\vee = \sigma_1^\vee \cap \sigma_2^\vee.$$

Proof. Classic. See [3] maybe. □

Proposition 1.2.8 (Double dual of a polyhedral cone)**.**
If σ is polyhedral, then $\sigma^{\vee\vee} = \sigma$.

Proof. Classic. See [3] maybe. □

Given $m \neq 0$ in M, we get the hyperplane

$$H_m = \{u \in N | \langle m, u \rangle = 0\} \subseteq N$$

and the closed half-space

$$H_m^+ = \{u \in N | \langle m, u \rangle \geq 0\} \subseteq N.$$

Definition 1.2.9 (Face of a cone)**.** If σ is a cone, then a subset of σ is a **face** iff it is the intersection of σ with some halfspace. We write this $\tau \preceq \sigma$. If furthermore $\tau \neq \sigma$, we call τ a proper face and write $\tau \prec \sigma$.

Definition 1.2.10 (Edge of a cone)**.**
A dimension 1 face of a cone is called an *edge*.

Definition 1.2.11 (Facet of a cone)**.**
A codimension 1 face of a cone is called a *facet*.

Lemma 1.2.12 (Face of a polyhedral cone)**.**
If σ is a polyhedral cone, then every face of σ is a polyhedral cone.

Lemma 1.2.13 (Intersection of faces)**.**
If σ is a polyhedral cone, then the intersection of two faces of σ is a face of σ.

Proof. Classic. See [3] maybe. □

Lemma 1.2.14 (Face of a face)**.**
A face of a face of a polyhedral cone σ is again a face of σ.

Proof. Classic. See [3] maybe. □

Lemma 1.2.15.
Let τ be a face of a polyhedral cone σ. If $v, w \in \sigma$ and $v + w \in \tau$, then $v, w \in \tau$.

Proof. Classic. See [3] maybe. ☐

Proposition 1.2.16 (Dual cone of the intersection of halfspaces)**.**
If $\sigma = H^+_{m_1} \cap \cdots \cap H^+_{m_s}$*, then*

$$\sigma^\vee = \mathrm{Cone}(m_1, \ldots, m_s).$$

Proof. Classic. See [3] maybe. ☐

Proposition 1.2.17 (Facets of a full dimensional cone)**.**
If σ *is a full dimensional cone, then facets of* σ *are of the form* $H_m \cap \sigma$*.*

Proof. Classic. See [3] maybe. ☐

Proposition 1.2.18 (Intersection of facets containing a face)**.**
Every proper face $\tau \prec \sigma$ *of a polyhedral cone* σ *is the intersection of the facets of* σ *containing* τ*.*

Proof. Classic. See [3] maybe. ☐

Definition 1.2.19 (Dual face)**.**
Given a cone σ and a face $\tau \preceq \sigma$, the **dual face** to τ is

$$\tau^* := \sigma^\vee \cap \tau^\perp$$

Proposition 1.2.20 (The dual face is a face of the dual)**.**
If $\tau \preceq \sigma$*, then* $\tau^* \preceq \sigma^\vee$*.*

Proof. Classic. See [3] maybe. ☐

Proposition 1.2.21 (The double dual of a face)**.**
If $\tau \preceq \sigma$*, then* $\tau^{**} = \tau$*.*

Proof.
Classic. See [3] maybe. ☐

Proposition 1.2.22 (The dual of a face is antitone)**.**
If $\tau' \preceq \tau \preceq \sigma$*, then* $\tau' \preceq \tau$*.*

Proof. Classic. See [3] maybe. ☐

Proposition 1.2.23 (The dimension of the dual of a face)**.**
If $\tau \preceq \sigma$*, then*

$$\dim \tau + \dim \tau^* = \dim N.$$

Proof. Classic. See [3] maybe. ☐

1.2.3 Relative Interiors

Definition 1.2.24 (Relative interior)**.** The **relative interior**, aka **intrinsic interior**, of a cone σ is the interior of σ as a subset of its span.

Lemma 1.2.25 (The relative interior in terms of the inner product)**.**
For a cone σ,

$$u \in \operatorname{Relint}(\sigma) \iff \forall m \in \sigma^\vee \setminus \sigma^\perp, \langle m, u \rangle > 0$$

Proof. Classic. See [3] maybe. □

Lemma 1.2.26 (Relative interior of a dual face)**.**
If $\tau \preceq \sigma$ and $m \in \sigma^\vee$, then

$$m \in \operatorname{Relint}(\tau^*) \iff \tau = H_m \cap \sigma$$

Proof. Classic. See [3] maybe. □

Lemma 1.2.27 (Minimal face of a cone)**.**
If σ is a cone, then $W := \sigma \cap (-\sigma)$ is a subspace. Furthermore, $W = H_m \cap \sigma$ whenever $m \in \operatorname{Relint}(\sigma^\vee)$.

Proof. Classic. See [3] maybe. □

1.2.4 Strong Convexity

Definition 1.2.28 (Salient cones)**.** A cone σ is **salient**, aka **pointed** or **strongly convex**, if $\sigma \cap (-\sigma) = \{0\}$.

Proposition 1.2.29 (Alternative definitions of salient cones)**.**
The following are equivalent

1. *σ is salient*
2. *$\{0\} \preceq \sigma$*
3. *σ contains no positive dimensional subspace*
4. *$\dim \sigma^\vee = \dim N$*

Proof. Classic. See [3] maybe. □

1.2.5 Separation

Lemma 1.2.30 (Separation lemma)**.**
Let σ_1, σ_2 be polyhedral cones meeting along a common face τ. Then

$$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2$$

for any $m \in \operatorname{Relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$.

Proof.
See [1]. □

1.2.6 Rational Polyhedral Cones

Let M and N be dual lattices with associated vector spaces $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}, N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 1.2.31 (Rational cone)**.**
A cone $\sigma \subseteq N_{\mathbb{R}}$ is **rational** if $\sigma = \mathrm{Cone}(S)$ for some finite set $S \subseteq N$.

Lemma 1.2.32 (Faces of a rational cone)**.**
If $\tau \preceq \sigma$ is a face of a rational cone, then τ itself is rational.

Proof. Classic. See [3] maybe. □

Lemma 1.2.33 (The dual of a rational cone)**.**
$\sigma^\vee$ is a rational cone iff σ is.

Proof. Classic. See [3] maybe. □

Definition 1.2.34 (Ray generator)**.**
If ρ is an edge of a rational cone σ, then the monoid $\rho \cap N$ is generated by a unique element $u_\rho \in \rho \cap N$, which we call the **ray generator** of ρ.

Definition 1.2.35 (Minimal generators)**.**
The **minimal generators** of a rational cone σ are the ray generators of its edges.

Lemma 1.2.36 (A rational cone is generated by its minimal generators)**.**
A salient convex rational polyhedral cone is generated by its minimal generators.

Proof. Classic. See [3] maybe. □

Definition 1.2.37 (Regular cone)**.**
A salient rational polyhedral cone σ is **regular**, aka **smooth**, if its minimal generators form part of a $\mathbb{Z}$-basis of N.

Definition 1.2.38 (Simplicial cone)**.**
A salient rational polyhedral cone σ is **simplicial** if its minimal generators are $\mathbb{R}$-linearly independent.

1.2.7 Semigroup Algebras and Affine Toric Varieties

Definition 1.2.39 (Dual lattice of a cone)**.**
If $\sigma \subseteq N_{\mathbb{R}}$ is a polyhedral cone, then the lattice points

$$S_\sigma := \sigma^\vee \cap M$$

form a monoid.

Proposition 1.2.40 (Gordan's lemma)**.**
S_σ is finitely generated as a monoid.

Proof.
See [1]. □

Definition 1.2.41 (Affine toric variety of a rational polyhedral cone)**.**
$U_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma]$ is an affine toric variety.

Theorem 1.2.42 (Dimension of the affine toric variety of a rational polyhedral cone)**.**

$$\dim U_\sigma = \dim N \iff \textit{the torus of } U_\sigma \textit{ is } T_N = N \otimes_[\mathbb{Z}]\mathbb{C}^* \iff \sigma \textit{ is salient.}$$

Proof.
See [1]. □

Proposition 1.2.43 (The irreducible elements of the dual lattice of a cone)**.**
If $\sigma \subseteq N_\mathbb{R}$ is salient of maximal dimension, then the irreducible elements of S_σ are precisely the minimal generators of $\sigma^\vee$.

Proof.
See [1]. □

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