Principle of Uniform Boundedness

`boundedness, norm-topology`
`principle-uniform-Boundedness`

If $\mathcal{T} \subseteq X^*$ is pointwise bounded ($\forall x, \sup_{T \in \mathcal{T}} \|Tx\| < \infty$), then it is uniformly bounded ($\sup_{T \in \mathcal{T}} \|T\| < \infty$).

If $A \subseteq X$ is weak-bounded, then it is norm-bounded.

`boundedness, weak-topology, norm-topology`
`weak-bounded-implies-norm-bounded`

This is exactly PUB applied to $\hat{A} = \{\hat{x} \mid x \in A\}$.

If $B \subseteq X^*$ is w*-bounded, then it is norm-bounded.

`boundedness, weak-star-topology, norm-topology`
`weak-star-bounded-implies-norm-bounded`

This is exactly PUB applied to B.

Mazur's theorem

`convexity, norm-topology, weak-topology`
`mazur`

Let C be a convex set in a normed space. Then $\overline{C}^{\|\cdot\|} = \overline{C}^{\mathrm{w}}$. In particular,

$$C \text{ norm-closed} \iff C \text{ w-closed}$$

Proof. WLOG C is nonempty. We already know $\overline{C}^{\|\cdot\|} \subseteq \overline{C}^{\mathrm{w}}$ as the weak topology is weaker than the norm-topology. If $x \notin \overline{C}^{\|\cdot\|}$, then Hahn-Banach with $A = \{x\}$ and $B = \overline{C}^{\|\cdot\|}$ gives us $f \in X^*$ such that $f(x) < \inf_B f$. Then $\{z \mid f(z) < \inf_B f\}$ is a w-open neighborhood of x disjoint from B. So $x \notin \overline{C}^{\mathrm{w}}$. $\square$

Definitions of $\mathcal{P}(K), \mathcal{R}(K), \mathcal{O}(K), A(K)$

$$\begin{aligned}
\mathcal{P}(K) &= \overline{\{f \in C(K) \mid f \text{ polynomial}\}} \\
\mathcal{R}(K) &= \overline{\{f \in C(K) \mid f \text{ rational function without poles}\}} \\
\mathcal{O}(K) &= \overline{\{f \in C(K) \mid f \text{ holomorphic on a nhbd of } K\}} \\
A(K) &= \{f \in C(K) \mid f \text{ is holomorphic on } \operatorname{int} K\}
\end{aligned}$$

p-r-o-a-def

Inclusions between $\mathcal{P}(K), \mathcal{R}(K), \mathcal{O}(K), A(K)$

$$\mathcal{P}(K) \subseteq \mathcal{R}(K) \subseteq \mathcal{O}(K) \subseteq A(K) \subseteq C(K)$$

$$\begin{aligned}
&\mathcal{P}(K) = \mathcal{R}(K) \iff K^c \text{ connected} \\
&\mathcal{R}(K) = \mathcal{O}(K) \text{ always} \\
&\mathcal{O}(K) \neq A(K) \text{ in general} \\
&\mathcal{A}(K) = C(K) \iff \operatorname{int} K = \varnothing
\end{aligned}$$

p-r-o-a-inclusions

Any Banach algebra A is a closed subalgebra of $\mathcal{B}(X)$ for some X.

WLOG A is unital. For $a \in A$, consider the map

$$\begin{aligned}
L_a : A &\to A \\
b &\mapsto ab
\end{aligned}$$

$L_a \in \mathcal{B}(A)$ and $\|L_a\| = \|a\|$. Hence

$$L : A \to \mathcal{B}(A)$$

is a unital isometric homomorphism.

closed-subalgebra-b

Let A be a Banach algebra and let $x \in A$. Then $\sigma_A(x)$ is a compact subset of

$$\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|x\|\}$$

First, if $|\lambda| > \|x\|$, then $\left\|\frac{x}{\lambda}\right\| < 1$ and $1 - \frac{x}{\lambda}$ is invertible. So $\lambda 1 - x$ is invertible and $\lambda \notin \sigma_A(x)$.

$\sigma_A(x)$ is the preimage of the closed set $G(A)^c$ under the continuous map $\lambda \mapsto \lambda 1 - x$, hence is closed. Since it is bounded, it is also compact.

spectrum
spectrum-compact

Let A be a normed algebra and let $x \in A$. Then $\sigma_A(x)$ is nonempty.

WLOG A is a Banach algebra. If $\sigma_A(x)$ is empty, then

$$\begin{aligned} f : \mathbb{C} &\to A \\ \lambda &\mapsto (\lambda 1 - x)^{-1} \end{aligned}$$

is holomorphic since it is continuous and $f(\lambda) - f(\mu) = (\mu - \lambda) f(\lambda) f(\mu)$, namely

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} \underset{\lambda \to \mu}{\to} -f(\mu)^2$$

Also, as $|\lambda| \to \infty$,

$$\|f(\lambda)\| \leq \frac{1}{|\lambda| - \|x\|} \to 0$$

meaning that f is bounded. By vector-valued Liouville, f is constant, which is clearly nonsense.

```
spectrum
spectrum-nonempty
```

Gelfand-Mazur theorem

Any complex unital normed division algebra is isomorphic to $\mathbb{C}$.

Proof. Consider

$$\begin{aligned} f : \mathbb{C} &\to A \\ \lambda &\mapsto \lambda 1 \end{aligned}$$

f is an isometric homomorphism. Since $\sigma_A(x)$ is nonempty, there is some λ such that $\lambda 1 - x$ is not invertible, namely $\lambda 1 = x$ and $f(\lambda) = x$. So f is surjective. $\square$

```
division-algebra
gelfand-Mazur
```

Spectral Mapping Theorem for polynomials

Let A be a unital Banach algebra, $x \in A$, p a polynomial. Then

$$\sigma_A(p(x)) = p(\sigma_A(x))$$

Proof. For a fixed $\mu \in \mathbb{C}$, write $\mu - p(z) = c \prod_{i=1}^n (\lambda_i - z)$ for some $c \neq 0$ and some $\lambda_1, \ldots, \lambda_n$. Then

$$\begin{aligned} \mu \notin \sigma_A(p(x)) &\iff \mu 1 - p(x) = c \prod_{i=1}^n (\lambda_i 1 - x) \text{ invertible} \\ &\iff \forall i, \lambda_i 1 - x \text{ invertible} \\ &\iff \forall \lambda \in \sigma_A(x), \forall i, \lambda_i \neq \lambda \\ &\iff \forall \lambda \in \sigma_A(x), \mu - p(\lambda) \neq 0 \end{aligned}$$

$\square$

```
spectrum
spectral-mapping-polynomial
```

Beurling-Gelfand Spectral Radius Formula

$$r_A(x) = \lim_n \|x^n\|^{1/n} = \inf_n \|x^n\|^{1/n}$$

Proof. If $\lambda \in \sigma_A(x)$, then $\lambda^n \in \sigma_A(x^n)$. So $|\lambda| \leq \|x^n\|^{1/n}$.

Let's show $\frac{x^n}{\lambda^n} \overset{w}{\to} 0$ if $|\lambda| > r_A(x)$. This implies $\|x^n\|^{1/n} \leq C^{1/n} |\lambda|$ for some C. Let $\varphi \in A^*$. Define $f : \mathbb{C} \to \mathbb{C}, \lambda \mapsto \varphi((\lambda 1 - x)^{-1})$. Observe that for all $|\lambda| > \|x\|$ we have the Laurent series

$$f(\lambda) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \varphi\left(\frac{x^n}{\lambda^n}\right)$$

By unicity of Laurent series, this also holds for all $|\lambda| > r_A(x)$, meaning that $\varphi\left(\frac{x^n}{\lambda^n}\right) \to 0$, as wanted. $\square$

```
spectrum:spectral-radius
spectral-radius-formula
```