

# Part III – Functional Analysis (Incomplete)

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## 0 Introduction

### Prerequisites

- some basic functional analysis
- a bit of measure theory
- a bit of complex analysis

### Books

Books relevant to the course are:

- Bollobás, *Linear Analysis*
- Murphy,  *$C^*$ -algebras*
- Rudin
- Graham-Allan

### Notation

We will use  $\mathbb{K}$  to mean "either  $\mathbb{R}$  or  $\mathbb{C}$ ".

For  $X$  a normed space, we define

$$B_X = \{x \in X \mid \|x\| \leq 1\}$$

$$S_X = \{x \in X \mid \|x\| = 1\}$$

$$D_X = \{x \in X \mid \|x\| < 1\}$$

For  $X, Y$  normed spaces, we write  $X \sim Y$  if  $X, Y$  are isomorphic, ie there exists a linear bijection  $T : X \rightarrow Y$  such that  $T$  and  $T^{-1}$  are continuous. We write  $X \cong Y$  if  $X, Y$  are isometrically isomorphic, ie there exists a surjective linear map  $T : X \rightarrow Y$  such that  $\|Tx\| = \|x\|$  for all  $x$ .

# 1 Hahn-Banach extension theorems

## Lecture 1

Let  $X$  be a normed space. The **dual space** of  $X$  is the space  $X^*$  of bounded linear functionals on  $X$ .  $X^*$  is always a Banach space in the operator norm: for  $f \in X^*$ ,

$$\|f\| = \sup_{x \in B_X} |f(x)|$$

**Example.** For  $1 < p, q < \infty$ ,  $p^{-1} + q^{-1} = 1$ ,  $\ell_p^* \cong \ell_q$ .

We also have  $\ell_1^* \cong \ell_\infty$ ,  $c_0^* \cong \ell_1$ .

If  $H$  is a Hilbert space, then  $H^* \cong H$  (the isomorphism is conjugate-linear in the complex case).

For  $x \in X$ ,  $f \in X^*$ , we write  $\langle x, f \rangle = f(x)$ . Note that

$$\langle x, f \rangle = |f(x)| \leq \|f\| \|x\|$$

**Definition.** Let  $X$  be a *real* vector space. A functional  $p : X \rightarrow \mathbb{R}$  is

- **positive homogeneous** if  $p(tx) = tp(x)$  for all  $x \in X$ ,  $t \geq 0$
- **subadditive** if  $p(x + y) \leq p(x) + p(y)$  for all  $x, y \in X$

**Definition.** Let  $P$  be a preorder,  $A \subseteq P$ ,  $x \in P$ . We say

- $x$  is an **upper bound** for  $A$  if  $\forall a \in A$ ,  $a \leq x$ .
- $A$  is a **chain** if  $\forall a, b \in A$ ,  $a \leq b \vee b \leq a$ .
- $x$  is a **maximal element** if  $\forall y \in P$ ,  $x \not\prec y$

**Fact** (Zorn's lemma). A nonempty preorder in which all nonempty chains have an upper bound has a maximal element.

**Theorem 1.1** (Hahn-Banach, positive homogeneous version). Let  $X$  be a real vector space and  $p : X \rightarrow \mathbb{R}$  be positive homogeneous and subadditive. Let  $Y$  be a subspace of  $X$  and  $g : Y \rightarrow \mathbb{R}$  be linear such that  $\forall y \in Y$ ,  $g(y) \leq p(y)$ . Then there exists  $f : X \rightarrow \mathbb{R}$  linear such that  $f \upharpoonright_Y = g$  and  $\forall x \in X$ ,  $f(x) \leq p(x)$ .

*Proof.* Let  $P$  be the set of pairs  $(Z, h)$  where  $Z$  is a subspace of  $X$  with  $Y \subseteq Z$  and  $h : Z \rightarrow \mathbb{R}$  linear,  $h \upharpoonright_Y = g$  and  $\forall z \in Z$ ,  $h(z) \leq p(z)$ .  $P$  is nonempty since  $(Y, g) \in P$ , and is partially ordered by

$$(Z_1, h_1) \leq (Z_2, h_2) \iff Z_1 \subseteq Z_2 \wedge h_2 \upharpoonright_{Z_1} = h_1$$

If  $\{(Z_i, h_i) \mid i \in I\}$  is a chain with  $I$  nonempty, then we can define

$$Z := \bigcup_{i \in I} Z_i, h \upharpoonright_{Z_i} = h_i$$

The definition of  $h$  makes sense thanks to the chain assumption.  $(Z, h) \in P$  is therefore an upper bound for the chain.

Hence find by Zorn a maximal element  $(Z, h)$  of  $P$ . If  $Z = X$ , we won. So assume there is some  $x \in X \setminus Z$ . Let  $W = \text{Span}(Z \cup \{x\})$  and define  $f : W \rightarrow \mathbb{R}$  by

$$f(z + \lambda x) = h(z) + \lambda \alpha$$

for some  $\alpha \in \mathbb{R}$ . Then  $f$  is linear and  $f|_Z = h$ . We now look for  $\alpha$  such that  $\forall w \in W, f(w) \leq p(w)$ . We would then have  $(W, f) \in P$  and  $(Z, h) < (W, f)$ , contradicting maximality of  $(Z, h)$ .

We need

$$h(z) + \lambda\alpha \leq p(z + \lambda x) \forall z \in Z, \lambda \in \mathbb{R}$$

Since  $p$  is positive homogeneous, this becomes

$$h(z) + \alpha \leq p(z + x)h(z) - \alpha \leq p(z - x) \quad (1)$$

ie

$$h(z) - p(z - x) \leq \alpha \leq p(z + x) - h(z) \forall z \in Z$$

The existence of  $\alpha$  now amounts to

$$h(z_1) - p(z_1 - x) \leq \alpha \leq p(z_2 + x) - h(z_2) \forall z_1, z_2 \in Z$$

But indeed

$$h(z_1) + h(z_2) = h(z_1 + z_2) \leq p(z_1 + z_2) \leq p(z_1 - x) + p(z_2 + x)$$

□

**Definition.** Let  $X$  be a  $\mathbb{K}$ -vector space. A **seminorm** on  $X$  is a functional  $p : X \rightarrow \mathbb{R}$  such that

- $\forall x \in X, p(x) \geq 0$
- $\forall x \in X, \lambda \in \mathbb{K}, p(\lambda x) = |\lambda|p(x)$
- $\forall x, y \in X, p(x + y) \leq p(x) + p(y)$

**Remark.**

$$\text{norm} \implies \text{seminorm} \implies \text{positive homogeneous}$$

## Lecture 2

**Theorem 1.2** (Hahn-Banach, absolute homogeneous version). Let  $X$  be a real or complex vector space and  $p$  a seminorm on  $X$ . Let  $Y$  be a subspace of  $X$ ,  $g$  a linear functional on  $Y$  such that  $\forall y \in Y, |g(y)| \leq p(y)$ . Then there exists a linear functional  $f$  on  $X$  such that  $f|_Y = g$  and  $\forall x \in X, |f(x)| \leq p(x)$ .

*Proof.*

**Real case**

$$\forall y \in Y, g(y) \leq |g(y)| \leq p(y)$$

By Theorem 1.1, there exists  $f : X \rightarrow \mathbb{R}$  such that  $f|_Y = g$  and  $\forall x \in X, f(x) \leq p(x)$ . We also have

$$\forall x \in X, -f(x) = f(-x) \leq p(-x) = p(x)$$

Hence  $|f(x)| \leq p(x)$

**Complex case**

$\text{Re } g : Y \rightarrow \mathbb{R}$  is real-linear.

$$\forall y \in Y, |\text{Re } g(y)| \leq |g(y)| \leq p(y)$$

By the real case, find  $h : X \rightarrow \mathbb{R}$  real-linear such that  $h|_Y = \text{Re } g$

**Claim.** There exists a unique complex-linear  $f : X \rightarrow \mathbb{C}$  such that  $h = \text{Re } f$ .

*Proof.*

**Uniqueness**

If we have such  $f$ , then

$$\begin{aligned} f(x) &= \operatorname{Re} f(x) + i \operatorname{Im} f(x) \\ &= \operatorname{Re} f(x) - i \operatorname{Re} f(ix) \\ &= h(x) - ih(ix) \end{aligned}$$

**Existence**

Define  $f(x) = h(x) - ih(ix)$ . Then  $f$  is real-linear and  $f(ix) = if(x)$ , so  $f$  is complex-linear with  $\operatorname{Re} f = h$ .  $\square$

We now have  $f : X \rightarrow \mathbb{C}$  such that  $\operatorname{Re} f = h$ .

$$\operatorname{Re} f \upharpoonright_Y = h \upharpoonright_Y = \operatorname{Re} g$$

So, by uniqueness,  $f \upharpoonright_Y = g$ .

Given  $x \in X$ , find  $\lambda$  with  $|\lambda| = 1$  such that

$$\begin{aligned} |f(x)| &= \lambda f(x) \\ &= f(\lambda x) \\ &= \operatorname{Re} f(\lambda x) \\ &= h(\lambda x) \\ &\leq p(\lambda x) \\ &= p(x) \end{aligned}$$

$\square$

**Remark.** For a complex vector space  $X$ , if we write  $X_{\mathbb{R}}$  for  $X$  considered as a real vector space, the above proof shows that

$$\operatorname{Re} : (X^*)_{\mathbb{R}} \rightarrow X_{\mathbb{R}}^*$$

is an isometric isomorphism.

**Corollary 1.3.** Let  $X$  be a  $\mathbb{K}$ -vector space,  $p$  a seminorm on  $X$ ,  $x_0 \in X$ . Then there exists a linear functional  $f$  on  $X$  such that  $f(x_0) = p(x_0)$  and  $\forall x \in X, |f(x)| \leq p(x)$ .

*Proof.* Let  $Y = \operatorname{Span}(x_0)$ ,

$$\begin{aligned} g : Y &\rightarrow \mathbb{K} \\ \lambda x_0 &\mapsto \lambda p(x_0) \end{aligned}$$

We see that  $\forall y \in Y, g(y) \leq p(y)$ . Hence find by Theorem 1.2 a linear functional  $f$  on  $X$  such that  $f \upharpoonright_Y = g$  and  $\forall x \in X, |f(x)| \leq p(x)$ . We check that  $f(x_0) = g(x_0) = p(x_0)$ .  $\square$

**Theorem 1.4** (Hahn-Banach, existence of support functionals). Let  $X$  be a real or complex normed space. Then

1. If  $Y$  is a subspace of  $X$  and  $g \in Y^*$ , then there exists  $f \in X^*$  such that  $f \upharpoonright_Y = g$  and  $\|f\| = \|g\|$ .
2. Given  $x_0 \neq 0$ , there exists  $f \in S_{X^*}$  such that  $f(x_0) = \|x_0\|$ .

*Proof.*

1. Let  $p(x) = \|g\| \|x\|$ . Then  $p$  is a seminorm on  $X$  and

$$\forall y \in Y, |g(y)| \leq \|g\| \|y\| = p(y)$$

Find by Theorem 1.1 a linear functional  $f$  on  $X$  such that  $f \upharpoonright_Y = g$  and  $\forall x \in X, |f(x)| \leq p(x) = \|g\| \|x\|$ . So  $\|f\| \leq \|g\|$ . Since  $f \upharpoonright_Y = g$ , we also have  $\|g\| \leq \|f\|$ . Hence  $\|f\| = \|g\|$ .

2. Apply Corollary 1.3 with  $p(x) = \|x\|$  to get  $f \in X^*$  such that

$$\forall x \in X, |f(x)| \leq \|x\| \text{ and } f(x_0) = \|x_0\|$$

It follows that  $\|f\| = 1$ .

□

### Remarks.

- Part 1 is a sort of linear version of Tietze's extension theorem: Given  $K$  compact Hausdorff,  $L \subseteq K$  closed,  $g : L \rightarrow \mathbb{K}$  continuous, there exists  $f : K \rightarrow \mathbb{K}$  such that  $f \upharpoonright_L = g$  and  $\|f\|_\infty = \|g\|_\infty$ .
- Part 2 shows that for all  $x \neq y$  in  $X$  there exists  $f \in X^*$  such that  $f(x) \neq f(y)$ , namely  $X^*$  **separates points** of  $X$ . This is a sort of linear version of Urysohn:  $C(K)$  separates points of  $K$ .
- The  $f$  in part 2 is called a **norming functional**, aka **support functional**, for  $x_0$ . The existence of support functionals shows that

$$x_0 = \max_{g \in B_{X^*}} \langle x_0, g \rangle$$

Assuming  $X$  is a real normed space and  $\|x_0\| = 1$ , we have  $B_X \subseteq \{x \in X | f(x) \leq 1\}$ . Visually, TODO: insert tangency diagram

## 1.1 Bidual

Let  $X$  be a normed space. Then  $X^{**}$  is called the **bidual** or **second dual** of  $X$ .

For  $x \in X$ , define  $\hat{x} : X^* \rightarrow \mathbb{K}$ , the **evaluation at  $x$** , by  $\hat{x}(f) = f(x)$ .  $\hat{x}$  is linear and  $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\|$ , so  $\hat{x} \in X^{**}$  and  $\|\hat{x}\| \leq \|x\|$ .

The map  $x \mapsto \hat{x} : X \rightarrow X^{**}$  is called the **canonical embedding** of  $X$  into  $X^{**}$ .

**Theorem 1.5.** The canonical embedding is an isometric embedding.

*Proof.*

### Linearity

$$\begin{aligned} \widehat{x+y}(f) &= f(x+y) = f(x) + f(y) = \hat{x}(f) + \hat{y}(f) \\ \widehat{\lambda x}(f) &= f(\lambda x) = \lambda f(x) = \lambda \hat{x}(f) \end{aligned}$$

### Isometry

If  $x \neq 0$ , there exists a support functional  $f$  for  $x$ . Then

$$\|\hat{x}\| \geq |\hat{x}(f)| = |f(x)| = \|x\|$$

□

**Remarks.**

- In bracket notation,  $\langle f, \hat{x} \rangle = \langle x, f \rangle$
- Let  $\hat{X}$  be the image of  $X$  in  $X^{**}$ . Theorem 1.5 says

$$X \cong \hat{X} \subseteq X^{**}$$

We often identify  $\hat{X}$  with  $X$  and think of  $X$  as living isometrically inside  $X^{**}$ . Note that

$$X \text{ complete} \iff \hat{X} \text{ closed in } X^{**}$$

- More generally,  $\overline{\hat{X}}$  is a Banach space containing an isometric copy of  $X$  as a dense subspace. We proved that normed spaces have completions!

**Definition.** A normed space  $X$  is **reflexive** if the canonical embedding  $X \rightarrow X^{**}$  is surjective.

**Example.**

- Some reflexive spaces are Hilbert spaces, finite-dimensional spaces,  $\ell_p$  and  $L_p(\mu)$  for  $1 < p < \infty$ .
- Some non-reflexive spaces are  $c_0, \ell_1, \ell_\infty, L_1[0, 1]$ .

**Remarks.**

- If  $X$  is reflexive, then  $X \cong X^{**}$ , so  $X$  is complete.
- There are Banach spaces  $X$  such that  $X \cong X^{**}$  but  $X$  is not reflexive, eg **James' space**. Any isomorphism to the bidual is then necessarily not the canonical embedding.

## 1.2 Dual operators

*Lecture 3*

Let  $X, Y$  be normed spaces. Recall

$$\mathcal{B}(X, Y) = \{T : X \rightarrow Y \mid T \text{ linear, bounded}\}$$

This is a normed space in the operator norm:

$$\|T\| = \sup_{x \in B_X} \|Tx\|$$

If  $Y$  is complete, then so is  $\mathcal{B}(X, Y)$ . For  $T \in \mathcal{B}(X, Y)$ , the **dual operator** of  $T$  is the map  $T^* : Y^* \rightarrow X^*$  given by  $T^*g = g \circ T$ . In bracket notation  $\langle x, T^*g \rangle = \langle Tx, g \rangle$  for  $x \in X, g \in Y^*$ .

**$T^*$  is linear**

$$\begin{aligned} \langle x, T^*(g + h) \rangle &= \langle Tx, g + h \rangle \\ &= \langle Tx, g \rangle + \langle Tx, h \rangle \\ &= \langle x, T^*g \rangle + \langle x, T^*h \rangle \\ &= \langle x, T^*g + T^*h \rangle \end{aligned}$$

$$\begin{aligned}
\langle x, T^*(\lambda g) \rangle &= \langle Tx, \lambda g \rangle \\
&= \lambda \langle Tx, g \rangle \\
&= \lambda \langle x, T^*g \rangle \\
&= \langle x, \lambda T^*g \rangle
\end{aligned}$$

$T^*$  is bounded

$$\begin{aligned}
\|T^*\| &= \sup_{g \in B_{Y^*}} \|T^*g\| \\
&= \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^*g \rangle| \\
&= \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g \rangle| \\
&= \sup_{x \in B_X} \|Tx\| \text{ by Theorem 1.4 (ii)} \\
&= \|T\|
\end{aligned}$$

**Remarks.**

- Hahn-Banach is crucial here. Without it, the dual could be 0.
- $\langle \cdot, \cdot \rangle$  is linear in both arguments. This contrasts with the Hilbert space case where  $\langle \cdot, \cdot \rangle$  is conjugate-linear in one of the arguments. This comes from the conjugate-linearity of the identification  $H^* \cong H$ .
- If  $X, Y$  are Hilbert spaces and we identify  $X, Y$  with  $X^*, Y^*$ , respectively, then  $T^*$  is the adjoint of  $T$ .

**Example.** Let  $1 < p, q < \infty$ ,  $p^{-1} + q^{-1} = 1$  and define  $R : \ell_p \rightarrow \ell_p$  to be the **right shift operator**  $(x_0, x_1, \dots) \mapsto (0, x_0, \dots)$ . Then  $R^* : \ell_q \rightarrow \ell_q$  is the **left shift operator**  $(x_0, x_1, \dots) \mapsto (x_1, x_2, \dots)$ .

Some properties of the dual operator are

1.  $\text{id}_X^* = \text{id}_{X^*}$
2.  $(S + T)^* = S^* + T^*$ ,  $(\lambda T)^* = \lambda T^*$
3.  $(ST)^* = T^*S^*$
4.  $T \mapsto T^* : \mathcal{B}(X, Y) \rightarrow \mathcal{B}(Y^*, X^*)$  is an *into* isomorphism.
5. The double dual of an operator commutes with the double dual embedding.  
TODO: Insert commutative diagram For all  $x$ ,

$$\langle g, T^{**}\hat{x} \rangle = \langle T^*g, \hat{x} \rangle = \langle x, T^*g \rangle = \langle Tx, g \rangle = \langle g, \hat{Tx} \rangle$$

$$\text{So } T^{**}\hat{x} = \widehat{Tx}.$$

**Remark.** From the above properties, if  $X \sim Y$ , then  $X^* \sim Y^*$ . Interestingly, if  $X$  and  $Y$  are reflexive, then we can deduce  $X \sim Y$  from  $X^* \sim Y^*$ .

### 1.3 Quotient spaces

Let  $X$  be a normed space and  $Y$  be a *closed* subspace.. Then the quotient space  $X/Y$  becomes a normed space in the quotient norm:

$$\|x + Y\| = d(x, Y) = \inf_{y \in Y} \|x + y\|$$



The quotient map  $q : X \rightarrow X/Y$  is linear and bounded:  $\|q(x)\| \leq \|x\|$ , so  $\|q\| \leq 1$ .

$q$  maps the open unit ball  $D_X$  onto  $D_{X/Y}$ . Indeed, if  $x \in D_X$ , then  $\|q(x)\| \leq \|x\| < 1$ . Reciprocally, if  $q(x) \in D_{X/Y}$ , then there exists  $y \in Y$  such that  $\|x + y\| < 1$ . So  $x + y \in D_X$  and  $q(x + y) = q(x)$ . It follows that  $q$  is an open map and  $\|q\| = 1$ .

If  $Z$  is another normed space,  $T \in \mathcal{B}(X, Z)$  and  $Y \subseteq \ker T$ , then there exists a unique map  $\tilde{T}$  is linear and  $\tilde{T}(D_{X/Y}) = \tilde{T}(q(D_X)) = T(D_X)$ . It follows that  $\|\tilde{T}\| = \|T\|$ .

**Theorem 1.6.** Let  $X$  be a normed space. If  $X^*$  is separable, then so is  $X$ .

**Remark.** The converse is false, as  $X = \ell_1, X^* = \ell_\infty$  shows.

*Proof.* Since  $X^*$  is separable, so is  $S_{X^*}$ . Let  $f_n$  be a dense subset of  $S_{X^*}$ . For every  $n$ , find  $x_n \in B_X$  such that  $f_n(x_n) > \frac{1}{2}$ . Let

$$Y = \overline{\text{Span}\{x_n \mid n \in \mathbb{N}\}}$$

**Claim.**  $Y = X$

Then we're done since  $Y$  is separable via  $Y = \overline{\text{Span}_{\mathbb{Q}}\{x_n \mid n \in \mathbb{N}\}}$ .

*Proof.* Assume not. Then we can pick  $g \in (X/Y)^*$ ,  $\|g\| = 1$  (by Theorem 1.4 (ii)). Let  $f = g \circ q$ . Then  $\|f\| = \|g\| = 1$ , ie  $f \in S_{X^*}$ . Thus find  $n$  such that  $\|f - f_n\| < \frac{1}{4}$ , so that

$$\frac{1}{4} > \|f - f_n\| \|x_n\| \geq |(f - f_n)(x_n)| = |f_n(x_n)| > \frac{1}{2}$$

contradiction. □

□

**Theorem 1.7.** Let  $X$  be a separable normed space. Then  $X$  embeds isometrically into  $\ell_\infty$ .

*Proof.* Let  $\{x_n \mid n \in \mathbb{N}\}$  be dense in  $X$ . For every  $n$ , find  $f_n \in S_{X^*}$ ,  $f_n(x_n) = \|x_n\|$  (assuming  $X \neq \{0\}$ ). Define  $T : X \rightarrow \ell_\infty$  by  $(Tx)_n = f_n(x)$ .

**Well definition**

$$|(Tx)_n| = |f_n(x)| \leq \|f_n\| \|x\| = \|x\|$$

Hence  $\|Tx\|_\infty \leq \|x\| < \infty$ .

**Linearity**

$$(T(x + y))_n = f_n(x + y) = f_n(x) + f_n(y) = (Tx + Ty)_n$$

$$(T(\lambda x))_n = f_n(\lambda x) = \lambda f_n(x) = (\lambda Tx)_n$$

so  $T(x + y) = Tx + Ty, T(\lambda x) = \lambda Tx$ .

**Isometry**

We already know  $\|Tx\|_\infty \leq \|x\|$ . On the other hand, find  $f$  a supporting functional for  $x$  and  $f_{n_k}$  a subsequence converging to  $f$ . Then

$$\|Tx\|_\infty \geq \sup_k (Tx)_{n_k} = \sup_k |f_{n_k}(x)| \geq |f(x)| = \|x\|$$

□

**Remarks.**

- The result says that  $\ell_\infty$  is isometrically universal for the class  $\mathcal{SB}$  of separable Banach spaces.
- There is a dual result: Every separable Banach space is a quotient of  $\ell_1$ .

**Theorem 1.8** (Vector-valued Liouville). Let  $X$  be a complex Banach space,  $f : \mathbb{C} \rightarrow X$  holomorphic and bounded. Then  $f$  is constant.

*Proof.* Find  $M \geq 0$  such that  $\forall z \in \mathbb{C}, |f(z)| \leq M$ . Fix  $\phi \in X^*$ .  $\phi \circ f : \mathbb{C} \rightarrow \mathbb{C}$  is

**bounded**

$$|\phi(f(z))| \leq \|\phi\| \|f(z)\| \leq M \|\phi\|$$

**holomorphic**

$$\frac{\phi(f(z)) - \phi(f(w))}{z - w} = \phi \left( \frac{f(z) - f(w)}{z - w} \right) \rightarrow \phi(f'(z))$$

By scalar Liouville,  $\phi \circ f$  is constant. For every  $z \in \mathbb{C}$ ,  $\phi \in X^*$ ,  $\phi(f(z)) = \phi(f(0))$ . Since  $X^*$  separates points of  $X$ ,  $f(z) = f(0)$ .  $\square$

**Remark.** This is a typical example of how to transfer a scalar result to a vector-valued one: Prove the result once composed with any functional, then go back using Hahn-Banach.

## 1.4 Locally convex spaces

**Definition.** A **locally convex space** is a  $\mathbb{K}$ -vector space such that there exists a family  $\mathcal{P}$  of seminorms on  $X$  that separate points of  $X$  in the sense that for all  $x \neq 0$  there exists  $p \in \mathcal{P}$  such that  $p(x) \neq 0$ .

The family  $\mathcal{P}$  defines a topology on  $X$ :

$$U \subseteq X \text{ open} \iff \forall x \in U, \exists s \subseteq \mathcal{P} \text{ finite}, \varepsilon > 0, \{y \in X \mid \forall p \in s, p(x) < \varepsilon\} \subseteq U$$

**Remarks.**

1. Addition and scalar multiplication are continuous.
2. The topology is Hausdorff as  $\mathcal{P}$  separates points.
3.  $x_n \rightarrow x \iff \forall p \in \mathcal{P}, p(x_n - x) \rightarrow 0$
4. Let  $Y$  be a subspace of  $X$  and  $\mathcal{P}_Y = \{p|_Y \mid p \in \mathcal{P}\}$ . Then  $(Y, \mathcal{P}_Y)$  is a LCS and its topology is the subspace topology.
5. Let  $\mathcal{P}, \mathcal{Q}$  be two families of seminorms on  $X$  both separating points of  $X$ . We say  $\mathcal{P}, \mathcal{Q}$  are **equivalent**, write  $\mathcal{P} \sim \mathcal{Q}$ , if they induce the same topology on  $X$ . One interesting result is that

$$(X, \mathcal{P}) \text{ metrisable} \iff \mathcal{P} \text{ equivalent to some countable family}$$

6. We make  $\mathcal{P}$  part of the data here out of simplicity, but in grown up mathematics we instead assume that  $X$  already comes with a topology and that this topology coincides with the one induced by  $\mathcal{P}$ .

**Definition.** A **Fréchet space** is a complete metrisable LCS.

**Example.**

1. A normed space is a LCS with  $\mathcal{P} = \{\|\cdot\|\}$ .
2. Let  $U \subseteq \mathbb{C}$  nonempty open. Let  $\mathcal{O}(U) = \{f : U \rightarrow \mathbb{C} \mid f \text{ holomorphic}\}$ . For compact  $K \subseteq U$ , define  $p_K(f) = \sup_{z \in K} |f(z)|$ . Let  $\mathcal{P} = \{p_K \mid K \subseteq U \text{ compact}\}$ . Then  $(\mathcal{O}(U), \mathcal{P})$  is a LCS. If we replace  $\{K \subseteq U \text{ compact}\}$  by a compact exhaustion of  $U$ , then we get a countable separating family equivalent to  $\mathcal{P}$ . So  $(\mathcal{O}(U), \mathcal{P})$  is metrisable. However it is not normable: no norm on  $\mathcal{O}(U)$  induces the topology of  $(\mathcal{O}(U), \mathcal{P})$ , which is the topology of uniform convergence. This is a consequence of Montel's theorem.
3. Fix  $d \in \mathbb{N}$ ,  $\Omega \subseteq \mathbb{R}^d$  a nonempty open set. Let

$$C^\infty(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ infinitely differentiable}\}$$

Given a multi-index  $\alpha \in \mathbb{Z}^d$ ,  $\alpha$  defines a differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$$

For a compact  $K \subseteq \Omega$ ,  $\alpha \in \mathbb{Z}^d$ , define

$$p_{K,\alpha}(f) = \sup_{z \in K} |D^\alpha f(z)|$$

Let

$$\mathcal{P} = \{p_{K,\alpha} \mid K \subseteq U \text{ compact}, \alpha \in \mathbb{Z}^d\}$$

Then  $(C^\infty, \mathcal{P})$  is a LCS. It is in fact a non-normable Fréchet space.

**Lemma 1.9.** Let  $(X, \mathcal{P}), (Y, \mathcal{Q})$  be LCS,  $T : X \rightarrow Y$  linear. TFAE

1.  $T$  is continuous
2.  $T$  is continuous at 0
3.  $\forall q \in \mathcal{Q}, \exists s \subseteq \mathcal{P} \text{ finite}, C \geq 0, \forall x \in X, q(Tx) \leq C \max_{p \in s} p(x)$

*Proof.*

(i)  $\iff$  (ii)

Translation is continuous.

(ii)  $\implies$  (iii)

Given  $q \in \mathcal{Q}$ , let  $V = \{y \in Y \mid q(y) \leq 1\}$ . Then  $V$  is a neighborhood of 0 in  $Y$ . So there exists  $U$  neighborhood of 0 in  $X$  such that  $T(U) \subseteq V$ . WLOG

$$U = \{x \in X \mid \forall p_K \in s, p_K(x) \leq \varepsilon\}$$

Let  $p = \max_{p_K \in s} p_K(x)$ . If  $p(x) = 1$ , then  $p(\varepsilon x) = \varepsilon$ , so  $\varepsilon x \in U$  and

$$q(T(\varepsilon x)) \leq 1$$

By homogeneity,  $q(Tx) \leq \frac{1}{\varepsilon} p(x)$  for all  $x$  such that  $p(x) > 0$ . If  $p(x) = 0$ , then  $p(\lambda x) = 0$  for all scalar  $\lambda$ . So  $q(T(\lambda x)) \leq 1$  for all  $\lambda$ . Hence  $q(Tx) = 0 \leq \frac{1}{\varepsilon} p(x)$ .

(iii)  $\implies$  (ii)

Assume  $t \subseteq \mathcal{Q}$  is finite,  $\varepsilon > 0$ , and let  $V = \{y \in Y \mid \forall q \in t, q(y) \leq \varepsilon\}$  the corresponding

neighborhood of 0. For each  $q \in t$ , find  $s_q \subseteq \mathcal{P}$  finite and  $C_q$  so that  $\forall x \in X, q(Tx) \leq C_q \max_{p \in s_q} p(x)$ . Let

$$U = \left\{ x \in X \mid \forall q \in \mathcal{Q}, p \in s_q, p(x) \leq \frac{\varepsilon}{C_q} \right\}$$

Then  $U$  is a neighborhood of 0 and  $T(U) \subseteq V$ .  $\square$

**Definition.** Let  $(X, \mathcal{P})$  be a LCS. The **dual space** of  $X$  is the space of continuous linear functionals  $X \rightarrow \mathbb{K}$ .

## Lecture 5

**Lemma 1.10.** Let  $f$  be a linear functional on a LCS  $(X, \mathcal{P})$ . Then

$$f \in X^* \iff \ker f \text{ closed}$$

*Proof.*

$\implies$

$\ker f = f^{-1}(0)$  is closed since  $f$  is continuous.

$\impliedby$

If  $\ker f = 0$ , then  $f = 0$  is continuous. Else fix some  $x_0 \notin \ker f$ . Since  $(\ker f)^c$  is open, find  $s \subseteq \mathcal{P}$  finite,  $\varepsilon > 0$  such that

$$\underbrace{\{x \in X \mid \forall p \in s, p(x - x_0) < \varepsilon\}}_U \subseteq (\ker f)^c$$

Then  $U$  is a neighborhood of 0 and  $(x_0 + U) \cap \ker f = \emptyset$ . Note that  $U$  is convex and **balanced** ( $x \in U, |\lambda| \leq 1 \implies \lambda x \in U$ ), hence so is  $f(U)$  as  $f$  is linear.

If  $f(U)$  is unbounded, then it is the whole scalar field, hence so is  $f(x_0 + U) = f(x_0) + f(U)$ . But  $0 \in \ker f$ , contradicting disjointness.

So find  $M$  such that  $|f(x)| < M$  for all  $x \in U$ . For all  $\delta > 0$ ,  $\frac{\delta}{M}U$  is a neighborhood of 0 and  $f(\frac{\delta}{M}U) \subseteq \{\lambda \in \mathbb{K} \mid |\lambda| < \delta\}$ . Thus  $f$  is continuous.  $\square$

**Theorem 1.11** (Hahn-Banach). Let  $(X, \mathcal{P})$  be a LCS.

1. Given a subspace  $Y$  of  $X$  and  $g \in Y^*$ , there exists  $f \in X^*$  such that  $f|_Y = g$ .
2. Given a closed subspace  $Y$  of  $X$  and  $x_0 \notin Y$ , there exists  $f \in X^*$  such that  $f|_Y = 0$ ,  $f(x_0) \neq 0$ .

**Remark.** This means that  $X^*$  separates points of  $X$ .

*Proof.*

1. By Lemma 1.9, find  $s \subseteq \mathcal{P}$  finite,  $C \geq 0$  such that

$$\forall y \in Y, |g(y)| \leq C \max_{p \in s} p(y)$$

Let  $p(x) = C \max_{p \in s} p(x)$ . Then  $p$  is a seminorm on  $X$  and  $\forall y \in Y, |g(y)| \leq p(y)$ . By Theorem 1.2, find a linear functional  $f$  on  $X$  such that  $f|_Y = g, \forall x \in X, |f(x)| \leq p(x)$ . By Lemma 1.9,  $f \in X^*$ .

2. Let  $Z = \text{Span}(Y \cup \{x_0\})$  and define a linear functional  $g$  on  $Z$  by

$$g(y + \lambda x_0) = \lambda, y \in Y, \lambda \in \mathbb{K}$$

Then  $g|_Y = 0, g(x_0) = 1 \neq 0$  and  $\ker g = Y$  is closed, so  $g \in Z^*$  by Lemma 1.10. By part (i), find  $f \in X^*$  such that  $f|_Z = g$ . This works.

□

## 2 The dual of $L_p(\mu)$ and $C(K)$

Let  $(\Omega, \mathcal{F}, \mu)$  be measure space.

$1 \leq p < \infty$

$$L_p(\mu) = \{f : \Omega \rightarrow \mathbb{K} \mid f \text{ measurable and } \int_{\Omega} |f|^p d\mu < \infty\}$$

This is a normed space in the  $L_p$ -norm:

$$\|f\|_p = \left( \int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}}$$

$p = \infty$

A measurable function  $f : \Omega \rightarrow \mathbb{K}$  is **essentially bounded** if there exists  $N \in \mathcal{F}$  such that  $\mu(N) = 0$  and  $f|_{N^c}$  is bounded.

$$L_p(\mu) = \{f : \Omega \rightarrow \mathbb{K} \mid f \text{ measurable and essentially bounded}\}$$

This is a normed space in the  $L_{\infty}$ -norm:

$$\|f\|_{\infty} = \text{esssup } |f| = \inf_{|f| \leq k \text{ ae}} k$$

The inf is attained: there exists some  $N \in \mathcal{F}, \mu(N) = 0$  such that  $\|f\|_{\infty} = \sup_{N^c} |f|$ .

In all cases, we identify functions up to almost everywhere equality.

**Theorem 2.1.**  $L_p(\mu)$  is complete for  $1 \leq p \leq \infty$ .

**Definition** (Complex measures). A **complex measure** on  $\mathcal{F}$  is a countably additive set function  $\nu : \mathcal{F} \rightarrow \mathbb{C}$ .

The **total variation measure**  $|\nu|$  is defined by

$$|\nu|(A) = \sup_{\substack{A_1, \dots, A_n \text{ measurable} \\ \text{partition of } A}} \sum_k |\nu(A_k)|$$

$|\nu| : \mathcal{F} \rightarrow [0, \infty]$  is a positive measure. Later we'll see that  $|\nu|$  is a finite measure.

The **total variation** of  $\nu$  is  $\|\nu\|_1 = |\nu|(\Omega)$ .

**Proposition.** If  $\nu$  is a complex measure on  $\mathcal{F}$  and  $A_n \in \mathcal{F}$  for all  $n$ , then

- If  $A$  is monotone, then  $\nu(\bigcup_n A_n) = \lim_{n \rightarrow \infty} \nu(A_n)$ .
- If  $A$  is antitone, then  $\nu(\bigcap_n A_n) = \lim_{n \rightarrow \infty} \nu(A_n)$ .

**Definition** (Signed measures). A **signed measure** on  $\mathcal{F}$  is a countably additive set function  $\nu : \mathcal{F} \rightarrow \mathbb{R}$ .

**Theorem 2.2.** If  $\nu$  is a signed measure, then there exists a measurable partition  $\Omega = P \cup N$  such that for all  $A \in \mathcal{F}$

$$\begin{aligned} A \subseteq P &\implies \nu(A) \geq 0 \\ A \subseteq N &\implies \nu(A) \leq 0 \end{aligned}$$

**Remarks.**

1. This decomposition is called the **Hahn decomposition** of  $\nu$ .

2. Define  $\nu^+(A) = \nu(A \cap P)$ ,  $\nu^-(A) = -\nu(A \cap N)$ . Then  $\nu^+, \nu^-$  are finite positive measures such that  $\nu = \nu^+ - \nu^-$ . This determines  $\nu^+, \nu^-$  uniquely and the decomposition  $\nu = \nu^+ - \nu^-$  is called the **Jordan decomposition** of  $\nu$ .
3. If  $\nu$  is a complex measure on  $\mathcal{F}$ , then  $\operatorname{Re} \nu, \operatorname{Im} \nu$  are signed measures with Jordan decomposition  $\nu_1 - \nu_2, \nu_3 - \nu_4$  respectively. Hence  $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$  is the Jordan decomposition of  $\nu$ .

$$\nu_1, \nu_2, \nu_3, \nu_4 \leq |\nu| \leq \nu_1 + \nu_2 + \nu_3 + \nu_4$$

So  $|\nu|$  is a finite measure.

*Sketch.* Define  $\nu^+(A) = \sup_{\substack{B \in \mathcal{F} \\ B \subseteq A}} \nu(B)$ .  $\nu^+$  is nonnegative and finitely additive.

**Key step:**  $\nu^+(\Omega) < \infty$

By contradiction, construct inductively sequences  $A_n, B_n$  such that

$$\nu^+(A_n) = \infty, \nu(B_n) > n$$

by taking  $A_0 = \Omega, B_{n+1} \subseteq A_n$  such that  $\nu(B_n) > n$  (exists by continuity) and  $A_{n+1} = B_{n+1}$  or  $A_n \setminus B_{n+1}$ . This contradicts countable additivity.

Now find a sequence  $A_n$  such that  $\nu(A_n) > \nu^+(\Omega) - 2^{-n}$  and set  $P = \liminf_n A_n, N = P^c$ . Check that this works.  $\square$

## Lecture 6

**Definition** (Absolute continuity). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $\nu : \mathcal{F} \rightarrow \mathbb{C}$  a complex measure.  $\nu$  is **absolutely continuous** with respect to  $\mu$ , written  $\nu \ll \mu$ , if  $\forall A \in \mathcal{F}, \mu(A) = 0 \implies \nu(A) = 0$ .

**Remarks.**

- $\nu \ll \mu \implies |\nu| \ll \mu$ , so if  $\nu$  has Jordan decomposition  $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$  then  $\nu_1, \nu_2, \nu_3, \nu_4 \ll \mu$ .
- If  $\nu \ll \mu$ , then  $\forall \varepsilon > 0, \exists \delta > 0, \forall A \in \mathcal{F}, \mu(A) < \delta \implies |\nu(A)| < \varepsilon$ .

**Example.** Let  $f \in L_1(\mu)$ . Define  $\nu(A) = \int_A f d\mu$  for  $A \in \mathcal{F}$ . By Dominated Convergence,  $\nu$  is a complex measure and  $\mu(A) = 0 \implies \nu(A) = 0$ . So  $\nu \ll \mu$ .

**Definition.**  $A \in \mathcal{F}$  is  **$\sigma$ -finite** if there exists  $A_n$  with  $\mu(A_n) < \infty$  such that  $A = \bigcup_n A_n$ . Say  $\mu$  is  **$\sigma$ -finite** if  $\Omega$  is  $\sigma$ -finite.

**Theorem 2.3** (Radon-Nikodym). Let  $\mu$  be a  $\sigma$ -finite measure and  $\nu$  a complex measure such that  $\nu \ll \mu$ . Then there exists a unique  $f \in L_1(\mu)$  such that, for all  $A \in \mathcal{F}$ ,  $\nu(A) = \int_A f d\mu$ . Moreover,  $f$  takes values in  $\mathbb{C}/\mathbb{R}/\mathbb{R}^+$  depending on where  $\nu$  is valued.

*Proof.*

**Uniqueness**

standard

**Existence**

$\nu$  is a finite measure (by the Jordan decomposition). WLOG  $\mu$  is a finite measure (by  $\sigma$ -finiteness). Let

$$\mathcal{H} = \left\{ h : \Omega \rightarrow \mathbb{R}^+ \mid h \text{ integrable}, \forall A \in \mathcal{F}, \int_A h d\mu \ll \nu(A) \right\}$$

$\mathcal{H} \neq \emptyset$  (eg  $0 \in \mathcal{H}$ ). Let  $\alpha = \sup_{h \in \mathcal{H}} \int_{\Omega} h d\mu$ . We see  $0 \leq \alpha \leq \nu(\Omega)$ .

**Claim**

There exists  $f \in \mathcal{H}$  such that  $\alpha = \int_{\Omega} f d\mu$ .

**Idea**

If  $\int_A f d\mu < \nu(A)$ , then  $f + \frac{1}{n}1_A \in \mathcal{H}$  (morally, not literally), contradicting the definition of  $\alpha$ .

Pick that  $f$ . Define  $\nu_n(A) = \nu(A) - \int_A f d\mu - \frac{1}{n}\mu(A)$ .  $\nu_n$  has Hahn decomposition  $\Omega = P_n \cup N_n$ . Then  $f + \frac{1}{n}1_{P_n} \in \mathcal{H}$ . By definition of  $\alpha$ ,  $\mu(P_n) = 0$ . Since  $\nu \ll \mu$ ,  $\nu(P_n) = 0$ . Let  $P = \bigcup_n P_n, N = \bigcap_n N_n$ . Then  $\Omega = P \cup N, \mu(P) = \nu(P) = 0$ . For  $A \in \mathcal{F}$ ,

$$\begin{aligned} A \subseteq P &\implies \int_A f d\mu = \nu(A) = 0 \\ A \subseteq N &\implies \forall n, \nu_n \leq 0 \implies \int_A f d\mu \geq \nu(A) \end{aligned}$$

□

**Remarks.**

- Without assuming  $\nu \ll \mu$ , the proof shows there is a decomposition  $\nu = \nu_1 + \nu_2$  where  $\nu_1(A) = \int_A f d\mu$  and  $\nu_2 \perp \mu$  (**orthogonal**, ie there exists a measurable decomposition  $\Omega = P \cup N$  such that  $\mu(P) = 0, |\nu_2|(N) = 0$ ).  $\nu = \nu_1 + \nu_2$  is the **Lebesgue decomposition** of  $\nu$ .
- The unique  $f$  in Theorem 2.3 is the **Radon-Nikodym derivative** of  $\nu$  with respect to  $\mu$ , denoted  $\frac{d\nu}{d\mu}$ . The result says

$$\nu(A) = \int_{\Omega} 1_A d\nu = \int_A \frac{d\nu}{d\mu} d\mu = \int_{\Omega} 1_A \frac{d\nu}{d\mu} d\mu$$

Hence a measurable function  $g$  is  $\nu$ -integrable iff  $g \frac{d\nu}{d\mu}$  is  $\mu$ -integrable and then

$$\int_{\Omega} f d\nu = \int_{\Omega} g \frac{d\nu}{d\mu} d\mu$$

## 2.1 Dual space of $L_p(\mu)$

Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space,  $1 \leq p < \infty, 1 < q \leq \infty$  such that  $p^{-1} + q^{-1} = 1$ . For  $g \in L_q$ , define  $\phi_g : L_p \rightarrow \mathbb{K}$  by  $\phi_g(f) = \int_{\Omega} f g d\mu$ . By Hölder,  $f g \in L_1$ , and  $|\phi_g(f)| \leq \|f\|_p \|g\|_q$ . So  $\phi_g$  is well-defined, linear and bounded with  $\|\phi_g\| \leq \|g\|_q$ . Hence  $\phi_g \in L_p^*$  and  $\phi : L_q \rightarrow L_p^*$  is linear and bounded with  $\|\phi\| \leq 1$ .

**Theorem 2.4.**

1. If  $1 < p < \infty$ , then  $\phi$  is an isometric isomorphism. So  $L_p^* \cong L_q$ .
2. If  $p = 1$  and  $\mu$  is  $\sigma$ -finite, then  $\phi$  is an isometric isomorphism. So  $L_1^* \cong L_{\infty}$ .

*Proof.*



1.  $\phi$  is isometric

Let  $g \in L_1$ . We know  $\|\phi_g\| \leq \|g\|_q$ . Let  $\lambda$  be a measurable function with  $|\lambda| = 1$ ,  $\lambda g = |g|$ . let  $f = \lambda |g|^{q-1}$ . Then

$$\|f\|_p^p = \int_{\Omega} |g|^{pq-p} d\mu = \int_{\Omega} |g|^q d\mu = \|g\|_q^q$$

So  $f \in L_p$  and  $\|f\|_p = \|g\|_q^{\frac{q}{p}}$ . Then

$$\|g\|_q^{\frac{q}{p}} \|\phi_g\| \geq |\phi_g(f)| = \int_{\Omega} |g|^q d\mu = \|g\|_q^q$$

So  $\|\phi_g\| \geq \|g\|_q^{q-\frac{q}{p}} = \|g\|_q$ .

$\phi$  is onto

Fix  $\psi \in L_p^*$ . We seek  $g \in L_q$  such that  $\psi = \phi_g$ . Idea: We want  $\psi(1_A) = \int_A g d\mu$ .

**Case 1:  $\mu$  is finite**

For  $A \in \mathcal{F}$ ,  $1_A \in L_p$ , so define  $\nu(A) = \psi(1_A)$ .  $\nu(\emptyset) = 0$  and, if  $A = \bigcup_p A_n \in \mathcal{F}$ , then  $\sum_k 1_{A_k} = 1_A$  in  $L_p$ , so

$$\sum_k \nu(A_k) = \sum_k \psi(1_{A_k}) = \psi(1_A)$$

Hence  $\nu$  is a complex measure.

If  $A \in \mathcal{F}$ ,  $\mu(A) = 0$ , then  $1_A = 0$  ae in  $L_p$ , so  $\nu(A) = \psi(1_A) = 0$ . Hence  $\nu \ll \mu$ .

By Theorem 2.3, find  $g \in L_1$  such that  $\forall A \in \mathcal{F}, \nu(A) = \int_A g d\mu$ . Hence

$$\begin{aligned} \psi(1_A) &= \int_{\Omega} 1_A g d\mu \text{ for all } A \in \mathcal{F} \\ \psi(f) &= \int_{\Omega} f g d\mu \text{ for all simple function } f \end{aligned}$$

Given  $f \in L_{\infty}$ , find simple functions  $f_n$  tending to  $f$  in  $L_{\infty}$ . So  $\psi(f_n) \rightarrow \psi(f)$  and  $f_n g \rightarrow f g$  (by Hölder for  $\infty, 1$ ), meaning that

$$\psi(f) = \int_{\Omega} f g d\mu \text{ for all } f \in L_{\infty}$$

For  $n \in \mathbb{N}$ , let  $A = \{|g| \leq n\}$  and  $f_n = \lambda 1_{A_n} |g|^{q-1}$  where  $|\lambda| = 1, \lambda g = |g|$ . As  $f_n \in L_{\infty}$ ,

$$\int_{\Omega} f_n g d\mu = \int_{A_n} |g|^q d\mu = \psi(f_n)$$

So  $(\int_A |g|^q d\mu)^{q^{-1}} \leq \|\psi\|$ . By Monotone Convergence,  $g \in L_q$ .

Given  $f \in L_p$ , find simple functions  $f_n$  tending to  $f$  in  $L_p$ . So  $\psi(f_n) \rightarrow \psi(f)$  and  $f_n g \rightarrow f g$  in  $L_1$  (by Hölder for  $p, q$ ). Hence  $\psi(f) = \int_{\Omega} f g d\mu$ , as wanted.

Before going onto Case 2, for  $A \in \mathcal{F}$ , let  $\mathcal{F}_A = \{B \in \mathcal{F} \mid B \subseteq A\}$  and  $\mu_A = \mu \upharpoonright_{\mathcal{F}_A}$  so that  $(A, \mathcal{F}_A, \mu_A)$  is a measure space. Then  $L_p(\mu_A) \subseteq L_p(\mu)$  (by extending  $f \in L_p(\mu_A)$  by 0 outside  $A$ ). Let  $\psi_A = \psi \upharpoonright_{L_p(\mu_A)}$ .

Lecture 7

**Claim.** If  $A, B \in \mathcal{F}$  are disjoint, then

$$\|\psi_{A \cup B}\| = (\|\psi_A\|^q + \|\psi_B\|^q)^{\frac{1}{q}}$$

*Proof.*

$$\begin{aligned}
(\|\psi_A\|^q + \|\psi_B\|^q)^{\frac{1}{q}} &= \sup_{\substack{a, b \geq 0 \\ a^p + b^p \leq 1}} a \|\psi_A\| + b \|\psi_B\| \\
&= \sup_{\substack{a, b \geq 0 \\ a^p + b^p \leq 1 \\ f \in B_{L_p(\mu_A)} \\ g \in B_{L_p(\mu_B)}}} a |\psi_A(f)| + b |\psi_B(g)| \\
&= \sup_{\substack{|a|^p + |b|^p \leq 1 \\ f \in B_{L_p(\mu_A)} \\ g \in B_{L_p(\mu_B)}}} \underbrace{|a\psi_A(f) + b\psi_B(g)|}_{\psi_{A \cup B}(af + bg)} \\
&= \sup_{h \in L_p(\mu_{A \cup B})} |\psi_{A \cup B}(h)| \\
&= \|\psi_{A \cup B}\|
\end{aligned}$$

□

**Case 2:  $\mu$  is  $\sigma$ -finite**

Find a measurable partition  $\Omega = \bigcup_n A_n$  such that  $\mu(A_n) < \infty$ . By Case 1, find, for each  $n$ ,  $g_n \in L_q(A_n)$  such that  $\psi_{A_n} = \phi_{g_n}$ , ie

$$\psi(f) = \int_{A_n} f g_n d\mu \text{ for all } f \in L_q(\mu_{A_n})$$

If we define  $g$  on  $\Omega$  by  $g = g_n$  on  $A_n$ , then  $g \in L_q$  and

$$\psi(f) = \phi_g(f) \text{ for all } f \in L_p(\mu_{A_n})$$

Hence  $\psi = \phi_g$  on  $\overline{\text{Span}} \bigcup_n L_p(\mu_{A_n}) = L_p(\mu)$ .

**Case 3: General  $n$**

First observe that, for  $f \in L_p(\mu)$ ,  $\{f \neq 0\}$  is  $\sigma$ -finite. Indeed,

$$\{f \neq 0\} = \bigcup_n \left\{ \frac{1}{n} < |f| \right\}$$

and

$$\mu \left\{ \frac{1}{n} < |f| \right\} \leq |n^p| \|f\|_p^p < \infty \text{ by Markov}$$

Choose  $f_n \in B_{L_p}$  such that  $\psi(f_n) \rightarrow \|\psi\|$ . Then  $A = \bigcup_n \{f_n \neq 0\}$  is  $\sigma$ -finite and  $\|\psi_A\| = \|\psi\|$ . By the claim,

$$\|\psi\| = (\|\psi_A\|^q + \|\psi_{A^c}\|^q)^{\frac{1}{q}}$$

So  $\Psi_{A^c} = 0$ . By Case 2, find  $g \in L_q(\mu_A) \subseteq L_q(\mu)$  such that  $\psi_A = \phi_g$ , so that

$$\psi(f) = \psi_A f \upharpoonright_A + \psi_{A^c}(f \upharpoonright_{A^c}) = \int_A f g d\mu + 0 = \int_\Omega f g d\mu$$

**2.  $p = 1, \mu$  is  $\sigma$ -finite**

**$\phi$  is isometric**

Let  $g \in L_\infty$ . We know  $\|\phi_g\| \leq \|g\|_\infty$  (by Hölder) Fix  $s < \|g\|_\infty$ . Then  $\mu\{s < |g|\} > 0$ . Since  $\mu$  is  $\sigma$ -finite, find  $A \subseteq \{s < |g|\}$  such that  $0 < \mu(A) < \infty$ . Choose a

measurable function  $\lambda$  such that  $|\lambda| = 1$ ,  $\lambda g = |g|$ . Then  $\lambda 1_A \in L_1$ ,  $\|\lambda 1_A\|_1 = \mu(A)$ . Now,

$$\mu(A) \|\phi_g\| \geq |\phi_g(\lambda 1_A)| = \int_A |g| d\mu \geq s\mu(A)$$

So  $\|\phi_g\| \geq s$ . Taking the sup,  $\|\phi_g\| \geq \|g\|_\infty$ .

**$\phi$  is onto**

Fix  $\psi \in L_q^*$ . We seek  $g \in L_\infty$  such that  $\psi = \phi_g$ .

**Case 1:  $\mu$  is finite**

Define  $\nu(A) = \psi(1_A)$  for all  $A \in \mathcal{F}$ . Follow the same steps as for  $1 < p < \infty$ .

**Case 2:  $\mu$  is  $\sigma$ -finite**

This time, prove that

$$\|\psi_{A \cup B}\| = \max(\|\psi_A\|, \|\psi_B\|)$$

for all  $A, B \in \mathcal{F}$  disjoint and proceed as before.

□

**Corollary 2.5.** For  $1 < p < \infty$ ,  $L_p(\mu)$  is reflexive.

*Proof.* Let  $\psi \in L_p^{**}$ . Then  $g \mapsto \langle \phi_g, \psi \rangle : L_q \rightarrow \mathbb{K}$  is in  $L_q^*$ . By Theorem 2.4.i, find  $f \in L_p$  such that

$$\langle \phi_g, \psi \rangle = \int_\Omega f g d\mu \quad \langle f, \psi_g \rangle = \langle \phi_g, \hat{f} \rangle$$

Since  $L_p^* = \{\phi_g \mid g \in L_q\}$ , this proves  $\psi = \hat{f}$ .

□

## 2.2 Dual space of $C(K)$

Throughout,  $K$  will be a compact Hausdorff topological space. Define

$$\begin{aligned} C(K) &= \{f : K \rightarrow \mathbb{C} \mid f \text{ continuous}\} \\ C^{\mathbb{R}}(K) &= \{f : K \rightarrow \mathbb{R} \mid f \text{ continuous}\} \\ C^+(K) &= \{f : K \rightarrow \mathbb{R}^+ \mid f \text{ continuous}\} \\ M(K) &= C(K)^* \\ M^{\mathbb{R}}(K) &= \{\phi \in M(K) \mid \forall f \in C^{\mathbb{R}}(K), \phi(f) \in \mathbb{R}\} \\ M^+(K) &= \{\phi : C(K) \rightarrow \mathbb{C} \mid \phi \text{ is } \mathbb{R}\text{-linear}, \forall f \in C^+(K), 0 \leq \phi(f) \in \mathbb{R}\} \end{aligned}$$

$C(K), C^{\mathbb{R}}(K)$  are complex/real Banach spaces in the sup norm:  $\|f\|_{\infty} = \sup_K |f|$ .  $M(K)$  is a complex Banach space in the operator norm.  $M^{\mathbb{R}}(K)$  is a closed real-linear subspace of  $M(K)$ . Elements of  $M^+(K)$  are called **positive linear functionals**.

**Aim.** Identify  $M(K), M^{\mathbb{R}}(K)$ .

Lecture 8

The next lemma tells us that it's enough to understand  $M^+(K)$ .

**Lemma 2.6.**

1. For all  $\phi \in M(K)$ , there are unique  $\phi_1, \phi_2 \in M^{\mathbb{R}}(K)$  such that  $\phi = \phi_1 + i\phi_2$ .
2.  $\phi \mapsto \phi \upharpoonright_{C^{\mathbb{R}}(K)} : M^{\mathbb{R}}(K) \rightarrow C^{\mathbb{R}}(K)^*$  is an isometric isomorphism.
3.  $M^+(K) \subseteq M(K)$  and  $M^+(K) = \{\phi \in M(K) \mid \|\phi\| = \phi(1)\}$
4. For all  $\phi \in M^{\mathbb{R}}(K)$ , there are unique  $\phi^+, \phi^- \in M^+(K)$  such that  $\phi = \phi^+ - \phi^-$  and  $\|\phi\| = \|\phi^+\| + \|\phi^-\|$ .

*Proof.*

1. Let  $\phi \in M(K)$ . Then  $\bar{\phi}$  sending  $f \mapsto \phi(\bar{f})$  is in  $M(K)$  as well and  $\phi \in M^{\mathbb{R}}(K) \iff \bar{\phi} = \phi$ .

**Uniqueness**

Assume  $\phi = \phi_1 + i\phi_2$  where  $\phi_1, \phi_2 \in M^{\mathbb{R}}(K)$ . Then  $\bar{\phi} = \phi_1 - i\phi_2$ , so

$$\phi_1 = \frac{\phi + \bar{\phi}}{2}, \phi_2 = \frac{\phi - \bar{\phi}}{2i}$$

**Existence**

Check that the above works

2. Let  $\phi \in M^{\mathbb{R}}(K)$ . We show  $\|\phi \upharpoonright_{C^{\mathbb{R}}(K)}\| = \|\phi\|$ . Clearly,  $\|\phi \upharpoonright_{C^{\mathbb{R}}(K)}\| \leq \|\phi\|$ . Let  $f \in B_{C(K)}$ . Choose  $\lambda \in \mathbb{C}, |\lambda| = 1, \lambda\phi(f) = |\phi(f)|$ , so that

$$\begin{aligned} |\phi(f)| &= \lambda\phi(f) \\ &= \phi(\lambda f) \\ &= \phi(\operatorname{Re}(\lambda f)) + \phi(\operatorname{Im}(\lambda f)) \xrightarrow{0} \\ &\leq \|\phi \upharpoonright_{C^{\mathbb{R}}(K)}\| \|\operatorname{Re}(\lambda f)\|_{\infty} \\ &\leq \|\phi \upharpoonright_{C^{\mathbb{R}}(K)}\| \end{aligned}$$

Hence  $\|\phi\| \leq \|\phi \upharpoonright_{C^\mathbb{R}(K)}\|$ .

Finally, given  $\psi \in C^\mathbb{R}(K)$ , define  $\phi(f) = \psi(\operatorname{Re} f) + i\psi(\operatorname{Im} f)$ . Then  $\phi \in M(K)$  and  $\phi \upharpoonright_{C^\mathbb{R}(K)} = \psi$ .

3.  $M^+(K) \subseteq M(K)$

Let  $\phi \in M^+(K)$ . For  $f \in B_{C^\mathbb{R}(K)}$ , we have  $1 \pm f \geq 0$ , so  $\phi(1 \pm f) \geq 0$ . Hence  $\phi(f) \in \mathbb{R}$  and  $|\phi(f)| \leq \phi(1)$ . So  $\phi \upharpoonright_{C^\mathbb{R}(K)} \in C^\mathbb{R}(K)^*$  and  $\|\phi \upharpoonright_{C^\mathbb{R}(K)}\| = \phi(1)$ . By (ii),  $\phi \in M(K)$ ,  $\|\phi\| = \phi(1)$ .

$M^+(K) = \{\phi \in M(K) \mid \|\phi\| = \phi(1)\}$

We have already checked one inclusion. Let  $\phi \in M(K)$  with  $\|\phi\| = \phi(1)$ . WLOG  $\|\phi\| = \phi(1) = 1$ . Let  $f \in B_{C^\mathbb{R}(K)}$  and write  $\phi(f) = a + ib$  where  $a, b \in \mathbb{R}$ . We want  $b = 0$ . For  $t \in \mathbb{R}$ ,

$$\begin{aligned} |\phi(f + it)|^2 &= a^2 + (b + t)^2 = a^2 + b^2 + t^2 + 2bt \\ &\leq \|f + it\|_\infty \leq 1 + t^2 \end{aligned}$$

So  $b = 0$ .

Given  $f \in C^+(K)$  with  $0 \leq f \leq 1$ , we have  $-1 \leq 2f - 1 \leq 1$ , so  $|\phi(2f - 1)| \leq \|2f - 1\|_\infty \leq 1$ , ie  $-1 \leq 2\phi(f) - 1 \leq 1$ . So  $\phi(f) \geq 0$ .

4. Let  $\phi \in M^\mathbb{R}(K)$ . Assume for a moment that  $\phi = \psi_1 - \psi_2$  where  $\psi_1, \psi_2 \in M^+(K)$ . For  $f, g \in C^+(K)$  with  $0 \leq g \leq f$ , we have  $\psi_1(f) \geq \psi_1(g) = \phi(g) + \psi_2(g) \geq \phi(g)$ . So

$$\psi_1(f) \geq \sup_{0 \leq g \leq f} \phi(g)$$

For  $f \in C^+(K)$ , define

$$\phi^+(f) = \sup_{0 \leq g \leq f} \phi(g)$$

Observe that  $\phi^+ \geq 0$ ,  $\phi^+(f) \leq \|\phi\| \|f\|_\infty$ ,  $\phi^+(f) \geq \phi(f)$ ,  $\phi^+$  is linear.

Next, for  $f \in C^\mathbb{R}(K)$ , write  $f = f_1 - f_2$  where  $f_1, f_2 \in C^+(K)$  and define  $\phi^+(f) = \phi^+(f_1) - \phi^+(f_2)$ . This is well-defined and  $\mathbb{R}$ -linear. Then  $\phi$  is  $\mathbb{C}$ -linear since  $\phi^+(f) \geq 0$ . For all  $f \in C^+(K)$  and  $\phi^+ \in M^+(K)$ .

Define  $\phi^- = \phi^+ - \phi$ . For  $f \in C^+(K)$ ,  $\phi^+(f) \geq \phi(f)$ , so  $\phi^-(f) \geq 0$ , namely  $\phi^- \in M^+(K)$ .

We now see that  $\|\phi\| \leq \|\phi^+\| + \|\phi^-\|$ . Given  $f \in C^+(K)$ ,  $0 \leq f \leq 1$ , we have  $-1 \leq 2f - 1 \leq 1$ , so

$$2\phi(f) - \phi(1) = \phi(2f - 1) \leq \|\phi\|$$

Taking the sup over  $f$ , we thus check that

$$\|\phi^+\| + \|\phi^-\| = \phi^+(1) + \phi^-(1) = 2\phi^+(1) - \phi(1) \leq \|\phi\|$$

**Uniqueness**

Assume  $\phi = \psi_1 - \psi_2$ ,  $\psi_1, \psi_2 \in M^+(K)$ ,  $\|\phi\| = \|\psi_1\| + \|\psi_2\|$ . From the initial observation,  $\psi_1 \geq \phi^+$ , hence  $\psi_2 = \psi_1 - \phi \geq \phi^+ - \phi = \phi^-$ . Therefore  $\psi_1 - \phi^+, \psi_2 - \phi^- \in M^+(K)$ . By (iii),

$$\|\psi_1 - \phi^+\| + \|\psi_2 - \phi^-\| = \psi_1(1) - \phi^+(1) + \psi_2(1) - \phi^-(1) = \|\phi\| - \|\phi\| = 0$$

Hence  $\psi_1 = \phi^+, \psi_2 = \phi^-$ .

□

**Topological preliminaries**

1.  $K$  being compact Hausdorff, it is **normal**: given disjoint closed sets  $E, F$  in  $K$ , there are disjoint open sets  $U, V$  such that  $E \subseteq U, F \subseteq V$ . Equivalently, given  $E \subseteq U \subseteq K$ ,  $E$ , closed,  $U$  open, there exists  $V$  open such that  $E \subseteq V \subseteq \overline{V} \subseteq U$ .
2. Urysohn says: given disjoint closed sets  $E, F$ , there is a continuous function  $f : K \rightarrow [0, 1]$  such that  $f = 0$  on  $E$ ,  $f = 1$  on  $F$ .
3. Write  $f \prec U$  to mean that  $U$  is an open set,  $f$  is continuous and  $\text{supp } f \subseteq U$ . Write  $E \prec f$  to mean that  $E$  is closed,  $f$  is continuous and  $f = 1$  on  $E$ .
4. Urysohn then becomes: Given  $E \subseteq U$ , there exists  $f$  such that  $E \prec f \prec U$ .

**Lemma 2.7.** Let  $E$  closed,  $U_1, \dots, U_n$  open such that  $E \subseteq \bigcup_n U_n$ . Then

1. There exist open sets  $V_j$  such that  $\overline{V_j} \subseteq U_j$  and  $E \subseteq \bigcup_j V_j$ .
2. There exist  $f_j \prec U_j$  such that  $0 \leq \sum_j f_j \leq 1$  and  $\sum_j f_j = 1$  on  $E$ .

*Proof.*

1. Induction on  $n$ :  $n = 0$   
Obvious.

$n > 0$

$E \setminus U_n \subseteq \bigcup_{j < n} U_j$  so, by induction, find open sets  $V_j$  such that  $\overline{V_j} \subseteq U_j$  for all  $j < n$  and  $E \setminus U_n \subseteq \bigcup_{j < n} U_j$ . So  $E \setminus \underbrace{\bigcup_{j < n} V_j}_{\text{closed}} \subseteq \underbrace{U_n}_{\text{open}}$ . By Urysohn, find an open  $V_n$

such that

$$E \setminus \bigcup_{j < n} V_j \subseteq V_n \subseteq \overline{V_n} \subseteq U_n$$

2. Find the  $V_j$  as in (i) for  $1 \leq j \leq n$  and by Urysohn find  $h_j$  such that  $\overline{V_j} \prec h_j \prec U_j$ . By Urysohn again, find  $h_0$  such that  $\left(\bigcup_j U_j\right)^c \prec h_0 \prec E^c$ . Let  $h = \sum_{j=0}^n h_j \geq 1$  and  $f_j = \frac{h_j}{h}$  for  $1 \leq j \leq n$ . Then  $0 \leq \sum_{j=1}^n f_j \leq 1$ ,  $f_j \prec U_j$  and  $\sum_{j=1}^n f_j = 1$  on  $E$ .

□

**Definition** (Borel measures). Let  $X$  be a Hausdorff space and  $\mathcal{G}$  its family of open sets. The **Borel  $\sigma$ -algebra** is  $\mathcal{B} := \sigma(\mathcal{G})$ , the  $\sigma$ -algebra generated by open sets. Elements of  $\mathcal{B}$  are called **Borel sets**. A **Borel measure** on  $X$  is a measure  $\mu$  on  $\mathcal{B}$ . We say  $\mu$  is **regular** if

1.  $\mu(E) < \infty$  for all compact  $E \subseteq X$
2.  $\mu(A) = \inf_{\substack{U \text{ open} \\ A \subseteq U}} \mu(U)$  for all Borel set  $A$
3.  $\mu(U) = \sup_{\substack{E \text{ compact} \\ E \subseteq U}} \mu(E)$  for all open  $U$

A complex Borel measure  $\nu$  is **regular** if  $|\nu|$  is regular.

If  $X$  is compact and  $\mu$  is a Borel measure on  $X$ , then

$$\begin{aligned} \mu \text{ regular} &\iff \mu(X) < \infty \text{ and } \forall A \in \mathcal{B}, \mu(A) = \inf_{\substack{U \text{ open} \\ A \subseteq U}} \mu(U) \\ &\iff \mu(X) < \infty \text{ and } \forall A \in \mathcal{B}, \mu(A) = \sup_{\substack{E \text{ closed} \\ E \subseteq A}} \mu(E) \end{aligned}$$

**Definition** (Integration with respect to a complex measure). Let  $\Omega$  be a set,  $\mathcal{F}$  a  $\sigma$ -algebra on  $\Omega$ ,  $\nu$  a complex measure on  $\mathcal{F}$ . Write  $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$  the Jordan decomposition. Say a measurable function is  $\nu$ -**integrable** if  $f$  is  $|\nu|$ -integrable, or equivalently if  $f$  is  $\nu_1, \nu_2, \nu_3, \nu_4$ -integrable. Define

$$\int_{\Omega} f d\nu = \int_{\Omega} f d\nu_1 - \int_{\Omega} f d\nu_2 + i \int_{\Omega} f d\nu_3 - i \int_{\Omega} f d\nu_4$$

Lecture 9

**Proposition.**

1.  $\int_{\Omega} d\nu = \nu(A)$  for all  $A \in \mathcal{F}$ .
2. Linearity: If  $f, g : \Omega \rightarrow \mathbb{C}$  are  $\nu$ -integrable and  $\lambda \in \mathbb{C}$ , then

$$\int_{\Omega} f + g d\nu = \int_{\Omega} f d\nu + \int_{\Omega} g d\nu, \int_{\Omega} \lambda f d\nu = \lambda \int_{\Omega} f d\nu$$

3. Dominated Convergence: Let  $f_n, f, g$  be measurable functions  $\Omega \rightarrow \mathbb{C}$  such that  $f_n \rightarrow f$  ae (with respect to  $|\nu|$ ),  $g \in L_1$  and  $\forall n, f_n \leq g$  ae. Then  $f$  is  $\nu$ -integrable and  $\int_{\Omega} f_n d\nu \rightarrow \int_{\Omega} f d\nu$ .
4.  $|\int_{\Omega} f d\nu| \leq \int_{\Omega} |f| d|\nu|$  for all  $\nu$ -integrable  $f$ . This is true for simple functions by properties 1 and 2. For general  $f$ , use Dominated Convergence.

Let  $\nu$  be a complex Borel measure on  $K$ . Then for  $f \in C(K)$  we have

$$\int_K |f| d|\nu| \leq \|f\|_{\infty} |\nu|(K) = \|f\|_{\infty} \|\nu\|_1$$

So  $f$  is  $\nu$ -integrable. Define  $\phi : C(K) \rightarrow \mathbb{C}$  by  $\phi(f) = \int_{\Omega} f d\nu$ . Then  $\phi \in M(K)$  and  $\|\phi\| \leq \|\nu\|_1$ . If  $\nu$  is a signed measure, then  $\phi \in M^{\mathbb{R}}(K)$ . If  $\nu$  is a positive measure, then  $\phi \in M^+(K)$ .

**Theorem 2.8** (Riesz Representation Theorem). For every  $\phi \in M^+(K)$ , there exists a unique regular Borel measure  $\mu$  on  $K$  that represents  $\phi$ :  $\phi(f) = \int_K f d\mu$  for all  $f \in C(K)$ . Moreover,  $\|\phi\| = \mu(K) = \|\mu\|_1$ .

*Proof.*

**Uniqueness**

Assume  $\mu_1, \mu_2$  both represent  $\phi$ . Let  $E \subseteq U \subseteq K$  where  $E$  closed,  $U$  open. By Urysohn, find  $f$  such that  $E \prec f \prec U$ . Now,

$$\mu_1(E) \leq \int_K f d\mu_1 = \phi(f) = \int_K f d\mu_2 \leq \mu_2(U)$$

Taking the inf over  $U$ , we get  $\mu_1(E) \leq \mu_2(E)$ . By symmetry,  $\mu_1(E) = \mu_2(E)$ . By regularity,  $\mu_1 = \mu_2$ .

**Existence**

For  $U$  open, define  $\mu^*(U) = \sup_{f \prec U} \phi(f)$ . Note that

$$\mu^*(U) \geq 0, \mu \text{ monotone}, \mu^*(K) = \phi(1)$$

It follows that, for  $V$  open,  $\mu^*(V) = \inf_{U \supseteq V} \mu^*(U)$ . Hence extend the definition of  $\mu^*$  to

$$\mu^*(A) = \inf_{U \supseteq A} \mu^*(U)$$

We will show that  $\mu^*$  is an outer measure.

- $\mu(\emptyset) = 0$
- If  $A \subseteq B$ , then  $\mu^*(A) \leq \mu^*(B)$ .
- Do we have  $\mu^*(\bigcup_n A_n) = \sum_n \mu^*(A_n)$ ?  
 First assume that the  $A_n = U_n$  are open. Let  $U = \bigcup_n U_n$ . Assume  $f \prec U$  and let  $E = \text{supp } f$ .  $E \subseteq \bigcup_n U_n$ , so by compactness find  $N$  such that  $E \subseteq \bigcup_{n=1}^N U_n$ . By Lemma 2.7, find  $h_n \prec U_n$  with  $\sum_{n=1}^N h_n \leq 1$  and  $\sum_{n=1}^N h_n = 1$  on  $E$ . So  $f = \sum_{n=1}^N fh_n$  and

$$\begin{aligned} \phi(f) &= \sum_{n=1}^N \phi(fh_n) \\ &\leq \sum_{n=1}^N \mu^*(U_n) \text{ as } fh_n \prec U_n \\ &\leq \sum_n \mu^*(U_n) \end{aligned}$$

Taking the sup over  $f$ , we get  $\mu^*(U) \leq \sum_n \mu^*(U_n)$ . It follows that

$$\mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n)$$

We now let  $\mathcal{M}$  be the set of  $\mu^*$ -measurable sets. Then  $\mathcal{M}$  is a  $\sigma$ -algebra and  $\mu^* \upharpoonright_{\mathcal{M}}$  is a measure on  $\mathcal{M}$ .

To restrict it further to a Borel, we now show that  $\mathcal{B} \subseteq \mathcal{M}$ . It's enough to show that  $\mathcal{G} \subseteq \mathcal{M}$ .

Let  $U$  open. We need

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U) \text{ for all } A$$

First, let  $A = V \in \mathcal{G}$ . Fix  $f \prec V \cap U$  and  $g \prec V \setminus \text{supp } f$ . Then  $f + g \prec V$ , thus

$$\mu^*(V) \geq \phi(f + g) = \phi(f) + \phi(g)$$

Taking the sup over  $g$ ,

$$\mu^*(V) \geq \phi(f + g) = \phi(f) + \mu^*(V \setminus \text{supp } f) \geq \phi(f) + \mu^*(V \setminus U)$$

Taking the sup over  $f$ ,

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$$

Now let  $A$  be arbitrary. Fix  $V$  open such that  $A \subseteq V$ . then

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$$

Taking the inf over  $V$ ,

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$$

Now,  $\mu := \mu^* \upharpoonright_{\mathcal{B}}$  is a Borel measure on  $K$ . We have

$$\mu(K) = \phi(1) = \|\phi\| < \infty$$

and by definition  $\mu$  is regular. It remains to show that  $\phi(f) = \int_K f d\mu$  for all  $f \in C(K)$ . It is enough to check that for  $f \in C^{\mathbb{R}}(K)$  and enough to check that



$\phi(f) \leq \int_K f d\mu$  (apply this to  $-f$ ).

Fix  $0 < a < b$  in  $\mathbb{R}$  such that  $\phi(1) \in [a, b]$ . Let  $\varepsilon > 0$ . Choose  $0 \leq y_0 < a \leq y_1 < \dots < y_n = b$  such that

$$y_j < y_{j-1} + \varepsilon$$

Let  $A_j = f^{-1}[y_{j-1}, y_j]$ . Those sets form a measurable partition of  $K$ . Choose closed sets  $E_j$  and open sets  $U_j$  such that  $E_j \subseteq A_j \subseteq U_j$  and  $\mu(U_j \setminus E_j) < \frac{\varepsilon}{n}$  (by regularity) and  $f(U_j) \subseteq [y_{j-1}, y_j]$ . By Lemma 2.7, find  $h_j \prec U_j$  for each  $j$  such that  $\sum_j h_j = 1$ . Now,

$$\begin{aligned} \phi(f) &= \sum_j \phi(f_j) \\ &\leq \sum_j (y_j + \varepsilon) \phi(h_j) \\ &\leq \sum_j (y_{j-1} + 2\varepsilon) \left( \mu(E_j) + \frac{\varepsilon}{n} \right) \\ &= \sum_j y_{j-1} \mu(E_j) + \underbrace{\sum_j (b + \varepsilon) + 2\varepsilon \mu(K) + 2\varepsilon^2}_{o(1)} \\ &= \int_K \sum_j y_{j-1} 1_{E_j} d\mu + o(1) \leq \int_K f d\mu + o(1) \end{aligned}$$

since  $f \leq y_j + \varepsilon$  on  $U_j$ ,  $h_j \prec U_j$  and  $\phi \in M^+(K)$ . So  $\phi(f) \leq \int_K f d\mu$ .

□

## Lecture 10

**Corollary 2.9.** For every  $\phi \in M(K)$ , there exists a unique regular complex Borel measure  $\nu$  on  $K$  that represents  $\phi$ :  $\phi(f) = \int_K f d\nu$  for all  $f \in C(K)$ . Moreover,  $\|\phi\| = \|\nu\|_1$  and if  $\phi \in M^{\mathbb{R}}(K)$  then  $\nu$  is a signed measure.

*Proof.*

### Existence

Apply Lemma 2.6 and Theorem 2.8 to obtain a regular complex Borel measure representing  $\phi$ . We now want  $\|\phi\| = \|\nu\|_1$ .

We already know  $\|\phi\| \leq \|\nu\|_1$ . Take a measurable partition  $K = \bigcup_{j=1}^n A_j$ . Fix  $\varepsilon > 0$  and closed sets  $E_j$ , open sets  $U_j$  such that  $E_j \subseteq A_j \subseteq U_j$ ,  $|\nu|(U_j \setminus E_j) < \frac{\varepsilon}{n}$  ( $\nu$  is regular). We can also assume  $U_i \subseteq \bigcap_{j \neq i} E_j^c$ . Fix  $\lambda_j \in \mathbb{C}$  such that  $|\lambda_j| = 1$ ,  $\lambda_j \nu(E_j) = |\nu(E_j)|$ . By Lemma 2.7, find  $h_j \prec U_j$  such that  $\sum_{j=1}^n h_j = 1$ . Then  $E_j \prec h_j$ , hence

$$\begin{aligned} \left| \int_K \left( \sum_{j=1}^n \lambda_j 1_{E_j} - \sum_{j=1}^n \lambda_j h_j \right) d\nu \right| &\leq \sum_{j=1}^n \int_K |1_{E_j} - h_j| d|\nu| \\ &\leq \sum_{j=1}^n |\nu|(U_j \setminus E_j) < \varepsilon \end{aligned}$$

Now,

$$\begin{aligned}
\sum_{j=1}^n |\nu(A_j)| &\leq \sum_{j=1}^n |\nu(E_j)| + \varepsilon \\
&= \sum_{j=1}^n \lambda_j \nu(E_j) + \varepsilon \\
&= \int_K \sum_{j=1}^n \lambda_j 1_{E_j} d\nu + \varepsilon \\
&\leq \left| \int_K \sum_{j=1}^n \lambda_j h_j d\nu \right| + 2\varepsilon \\
&\leq \left| \phi \left( \sum_{j=1}^n \lambda_j h_j \right) \right| + 2\varepsilon \\
&\leq \|\phi\| \left\| \sum_{j=1}^n \lambda_j h_j \right\|_{\infty} + 2\varepsilon \\
&\leq \|\phi\| + 2\varepsilon
\end{aligned}$$

It follows that  $\|\nu\|_1 \leq \|\phi\|$ . □

**Corollary 2.10.** The space of regular real (resp. complex) Borel measures on  $K$  is a real (resp. complex) Banach space in  $\|\cdot\|_1$  isomorphic to  $M^{\mathbb{R}}(K)$  (resp.  $M(K)$ ).

### 3 Weak topologies

Let  $X$  be a set and  $\mathcal{F}$  a set of functions on  $X$  such that each  $f \in \mathcal{F}$  is a function  $X \rightarrow Y_f$  where  $Y_f$  is a topological space. The **weak topology**  $\sigma(X, \mathcal{F})$  on  $X$  **generated by**  $\mathcal{F}$  is the smallest topology on  $X$  that makes each  $f \in \mathcal{F}$  continuous.

**Remarks.**

1.  $S = \{f^{-1}(U) \mid f \in \mathcal{F}, U \subseteq Y_f \text{ open}\}$  is a subbase of  $\sigma(X, \mathcal{F})$ . So

$$\begin{aligned} V \subseteq X \text{ open} &\iff \forall x \in V, \exists f_1, \dots, f_n \in \mathcal{F}, U_i \subseteq Y_{f_i}, x \in \bigcap_i f_i^{-1}(U_i) \subseteq V \\ &\iff \forall x \in V, \exists f_1, \dots, f_n \in \mathcal{F}, \\ &\quad \text{open neighborhoods } U_i \text{ of } f_i(x), \bigcap_i U_i \subseteq V \end{aligned}$$

2. More generally, if  $S_f$  is a subbase in  $Y_f$ , then  $\{f^{-1}(U) \mid f \in \mathcal{F}, U \in S_f\}$  is a subbase of  $\sigma(X, \mathcal{F})$ .
3. If  $Y_f$  is Hausdorff for all  $f \in \mathcal{F}$  and  $\mathcal{F}$  **separates points of**  $X$  ( $\forall x \neq y, \exists f \in \mathcal{F}, f(x) \neq f(y)$ ), then  $\sigma(X, \mathcal{F})$  is Hausdorff.
4. Let  $Y \subseteq X, \mathcal{F}_Y = \{f|_Y \mid f \in \mathcal{F}\}$ . Then  $\sigma(Y, \mathcal{F}_Y) = \sigma(X, \mathcal{F})|_Y$ .
5. **Universal property:** Let  $Z$  be a topological space and  $g : Z \rightarrow X$ . then

$$g \text{ continuous} \iff \forall f \in \mathcal{F}, f \circ g : Z \rightarrow Y_f \text{ continuous}$$

**Example.**

1. Let  $X$  be a topological space,  $Y \subseteq X$  and  $\iota : Y \rightarrow X$  the inclusion map. Then  $\sigma(Y, \iota)$  is the subspace topology on  $Y$ .
2. Let  $\Gamma$  be a set,  $X_\gamma$  a topological space for each  $\gamma \in \Gamma$ ,  $X = \prod_{\gamma \in \Gamma} X_\gamma$ . For each  $\gamma$ , we have  $\pi_\gamma : X \rightarrow X_\gamma$  sending  $x \mapsto x_\gamma$ , the **evaluation map at**  $\gamma$ , or **projection onto**  $X_\gamma$ . The weak topology  $\sigma(X, \{\pi_\gamma \mid \gamma \in \Gamma\})$  is called the **product topology** on  $X$ .

$$V \subseteq X \text{ open} \iff \forall x \in V, \exists s \subseteq \Gamma \text{ finite, } U_\gamma \text{ neighborhood of } x_\gamma, \{y \mid \forall \gamma \in s, y_\gamma \in U_\gamma\} \subseteq V$$

**Proposition 3.1.** Let  $X$  be a set. For each  $n$ , let  $(Y_n, d_n)$  be a metric space and  $f_n : X \rightarrow Y_n$  be a separating family of functions. Then  $\sigma(X, \{f_n \mid n \in \mathbb{N}\})$  is metrisable.

*Proof.* Call  $\sigma := \sigma(X, \{f_n \mid n \in \mathbb{N}\})$ . Define

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n} \min(d(f_n(x), f_n(y)), 1)$$

This is a metric on  $X$ . Given  $0 < \varepsilon < 1$ , if  $d(x, y) < 2^{-n}\varepsilon$ , then  $d(f_n(x), f_n(y)) < \varepsilon$ . So each  $f_n$  is continuous with respect to the topology  $\tau$  induced by that metric. Hence  $\sigma \subseteq \tau$ .

Reciprocally,  $y \mapsto d(x, y)$  is  $\sigma$ -continuous for each  $x$  by the Weierstrass M-test since

$$y \mapsto 2^{-n} \min(d(f_n(x), f_n(y)), 1)$$

is  $\sigma$ -continuous for each  $n$ . □

**Theorem 3.2** (Tychonoff). The product of compact topological spaces is compact.

*Proof.* Assume each  $X_\gamma$  is compact. Let  $\mathcal{E}$  be a family of closed subsets with the FIP (finite intersection property). We want  $\bigcap_{E \in \mathcal{E}} E \neq \emptyset$ .

By Zorn, find a maximal family  $\mathcal{A}$  of sets in  $X$  such that  $\mathcal{E} \subseteq \mathcal{A}$  and  $\mathcal{A}$  has the FIP. We will show that  $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$ . Maximality of  $\mathcal{A}$  means that

- $\mathcal{A}$  is closed under finite intersections.
- If  $B$  intersects every  $A \in \mathcal{A}$ , then  $B \in \mathcal{A}$ .

For each  $\gamma \in \Gamma$ ,  $\{\pi_\gamma(A) \mid A \in \mathcal{A}\}$  has the FIP, hence find by compactness of  $X_\gamma$  some  $x_\gamma \in \bigcap_{A \in \mathcal{A}} \overline{\pi_\gamma(A)}$ .

We show that all neighborhoods of  $x$  are in  $\mathcal{A}$ . Then  $\forall A \in \mathcal{A}, x \in \overline{A}$ .

It's enough to show it for neighborhoods of the form  $U = \bigcap_{\gamma \in s} \pi_\gamma^{-1}(U_\gamma)$  for some  $s \subseteq \Gamma$  finite where each  $U_\gamma$  is a neighborhood of  $x_\gamma$ . For such  $U$ , we see that  $\pi_\gamma^{-1}(U_\gamma)$  intersects every  $A \in \mathcal{A}$ , so  $\pi_\gamma^{-1}(U_\gamma) \in \mathcal{A}$  by the second remark. Hence  $U \in \mathcal{A}$  by the first remark.  $\square$

### 3.1 Weak topologies on vector spaces

Lecture 11

Let  $E$  be a real or complex vector space. Let  $F$  be a subspace of the space of all linear functionals on  $E$  that separates points of  $E$ , ie  $\forall x \in E, x \neq 0 \implies \exists f \in F, f(x) \neq 0$ .

Consider the weak topology  $\sigma(E, F)$

$$U \text{ open} \iff \forall x \in U, \exists f_1, \dots, f_n \in F, \varepsilon > 0, \{y \mid \forall i, |f_i(x - y)| < \varepsilon\} \subseteq U$$

For  $f \in F, x \in E$ , let  $p_f(x) = |f(x)|$ . Let  $\mathcal{P} = \{p_f \mid f \in F\}$ . Then  $(E, \mathcal{P})$  is a LCS whose topology is  $\sigma(E, F)$ . So  $\sigma(E, F)$  is Hausdorff and vector addition and scalar multiplication are continuous.

**Lemma 3.3.** Let  $E$  be as above,  $f, g_1, \dots, g_n$  linear functionals on  $E$  such that

$$\bigcap_i \ker g_i \subseteq \ker f$$

Then  $f \in \text{Span}\{g_1, \dots, g_n\}$ .

*Proof.* Reinterpret the  $g_i$  as a single linear map  $g : E \rightarrow \mathbb{K}^n$ . Then

$$\ker g = \bigcap_i \ker g_i \subseteq \ker f$$

Hence we have a factorisation  $f = h \circ g$ . Find  $a_1, \dots, a_n$  such that  $h(y) = \sum_i a_i y_i$  for all  $y \in \mathbb{K}^n$ . Then

$$f(x) = h(g(x)) = \sum_i a_i g_i(x)$$

for all  $x$ , so  $f = \sum_i a_i g_i \in \text{Span}\{g_1, \dots, g_n\}$ .  $\square$

**Proposition.** Let  $E, F$  be as above and  $f$  a linear functional on  $E$ . Then

$$f \text{ is } \sigma(E, F)\text{-continuous} \iff f \in F$$

Namely,

$$(E, \sigma(E, F))^* = F$$

*Proof.*

$\Leftarrow$

True by definition.

$\Rightarrow$

Find an open neighborhood  $U$  of 0 in  $E$  such that  $\forall x \in U, |f(x)| < 1$ . WLOG  $U = \{x \mid \forall i, |g_i(x)| < \varepsilon\}$  for some  $\varepsilon > 0, g_1, \dots, g_n \in F$ .

If  $x \in \bigcap_i \ker g_i$ , then  $\lambda x \in U$  for all  $\lambda$ , hence

$$|\lambda| |f(x)| = |f(\lambda x)| < 1$$

for all  $\lambda$ , so  $f(x) = 0$ . By Lemma 3.3,  $f \in \text{Span}\{g_1, \dots, g_n\} \subseteq F$ .  $\square$

**Example.**

1. Let  $X$  be a normed space. The **weak topology** on  $X$  is the topology  $\sigma(X, X^*)$  on  $X$  ( $X^*$  separates points of  $X$  by Hahn-Banach). We sometimes write  $(X, w)$  for  $(X, \sigma(X, X^*))$ . Open sets in  $\sigma(X, X^*)$  are called **weak open** or **w-open**.

$U \subseteq X$  is w-open

$\iff$

$$\forall x \in U, \exists \varepsilon > 0, f_1, \dots, f_n \in X^*, \{y \in X \mid \forall i, |f_i(y - x)| < \varepsilon\} \subseteq U$$

2. Let  $X$  be a normed space. The **weak star topology** or **w\*-topology** on  $X^*$  is the topology  $\sigma(X^*, X)$ . Here we identify  $X$  with its image  $\hat{X}$  in  $X^{**}$  under the canonical embedding. Open sets in  $\sigma(X^*, X)$  are called **w\*-open**.

$U \subseteq X^*$  is w\*-open

$\iff$

$$\forall f \in U, \exists \varepsilon > 0, x_1, \dots, x_n \in X, \{g \in X^* \mid \forall i, |g(x_i) - f(x_i)| < \varepsilon\} \subseteq U$$

**Properties.**

1.  $(X, w)$  and  $(X^*, w^*)$  are LCS, hence Hausdorff with continuous vector space operations.
2.  $\sigma(X, X^*)$  is a subtopology of the norm topology, with equality iff  $X$  is finite dimensional.
3.  $\sigma(X^*, X)$  is a subtopology of  $\sigma(X^*, X^{**})$ , with equality iff  $X$  is reflexive.
4. Let  $Y$  be a subspace of  $X$ . Then

$$\sigma(X, X^*) \upharpoonright_Y = \sigma(Y, \{f \upharpoonright_Y \mid f \in X^*\}) \stackrel{\text{Hahn-Banach}}{=} \sigma(Y, Y^*)$$

Similarly,

$$\sigma(X^{**}, X^*) \upharpoonright_X = \sigma(X, X^*) = \sigma(X, \{\hat{f} \upharpoonright_X \mid f \in X^*\})$$

So the canonical embedding is a homeomorphism  $\sigma(X, X^*) \rightarrow \sigma(\hat{X}, X^*)$ .

**Proposition 3.4.** Let  $X$  be a normed space.

1. A linear functional  $f$  on  $X$  is w-continuous iff  $f \in X^*$ . So  $(X, w)^* = X^*$ .
2. A linear functional  $\Lambda$  on  $X^*$  is w\*-continuous iff  $\Lambda \in \hat{X}$ . So  $(X^*, w^*)^* = X$ .

It follows that  $\sigma(X^*, X) = \sigma(X^*, X^{**})$  iff  $X$  is reflexive.

**Definition.** Let  $X$  be a normed space.

1. A set  $A$  in  $X$  is **weakly bounded** if  $\{f(x) \mid x \in A\}$  is bounded for all  $f \in X^*$ , or equivalently if for all  $w$ -neighborhood  $U$  there exists  $\lambda$  such that  $A \subseteq \lambda U$ .
2. A set  $B$  in  $X^*$  is  **$w^*$ -bounded** if  $\{f(x) \mid f \in B\}$  is bounded for all  $x \in X$ , or equivalently if for all  $w^*$ -neighborhood  $U$  there exists  $\lambda$  such that  $B \subseteq \lambda U$ .

**Theorem** (Principle of uniform Bounded, PUB). Let  $X$  be a Banach space,  $Y$  a normed space  $\mathcal{T} \subseteq \mathcal{B}(X, Y)$ . If  $\mathcal{T}$  is **pointwise bounded** ( $\forall x \in X, \sup_{T \in \mathcal{T}} \|Tx\| < \infty$ ), then  $\mathcal{T}$  is **uniformly bounded** ( $\sup_{T \in \mathcal{T}} \|T\| < \infty$ ).

**Proposition 3.5.** Let  $X$  be a normed space.

1. If  $A \subseteq X$  is weakly bounded, then  $A$  is norm-bounded.
2. If  $X$  is complete and  $B \subseteq X^*$  is  $w^*$ -bounded, then  $B$  is norm-bounded.

*Proof.*

1.  $A$  being weak bounded means that  $\hat{A} = \{\hat{x} \mid x \in A\}$  is pointwise bounded. So we're done by PUB.
2.  $B$  being  $w^*$ -bounded means that  $B$  is pointwise bounded. So we're done by PUB.

□

**Notation.** We write  $x_n \xrightarrow{w} x$  if  $x_n$  converges to  $x$  in the weak topology. Note that

$$x_n \xrightarrow{w} x \iff \forall f \in X^*, \langle x_n, f \rangle \rightarrow \langle x, f \rangle$$

We write  $f_n \xrightarrow{w^*} f$  if  $f_n$  converges to  $f$  in the  $w^*$ -topology. Note that

$$f_n \xrightarrow{w^*} f \iff \forall x \in X, \langle x, f_n \rangle \rightarrow \langle x, f \rangle$$

**Theorem** (Consequence of PUB). Let  $X$  be a Banach space,  $Y$  a normed space,  $T_n$  a sequence in  $\mathcal{B}(X, Y)$ . If  $T_n$  converges pointwise to some function  $T : X \rightarrow Y$ , then  $T \in \mathcal{B}(X, Y)$ ,  $\sup_n \|T_n\| < \infty$  and  $\|T\| \leq \liminf_n \|T_n\|$ .

**Proposition 3.6.** Let  $X$  be a normed space.

1. If  $x_n \xrightarrow{w} x$  in  $X$ , then  $\sup_n \|x_n\| < \infty$  and  $\|x\| \leq \liminf \|x_n\|$ .
2. If  $f_n \xrightarrow{w^*} f$  in  $X^*$ , then  $\sup_n \|f_n\| < \infty$  and  $\|f\| \leq \liminf \|f_n\|$ .

*Proof.*

1.  $\widehat{x_n} \rightarrow \hat{x}$  pointwise in  $X^{**}$ . Result follows by PUB.
2.  $f_n \rightarrow f$  pointwise in  $X^*$ . Result follows by PUB.

□