Part III – Functional Analysis (Incomplete)

Based on lectures by Dr András Zsák

Notes taken by Yaël Dillies

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Contents

0 Introduction

Prerequisites

- some basic functional analysis
- a bit of measure theory
- a bit of complex analysis

Books

Books relevant to the course are:

- Bollobás, *Linear Analysis*
- Murphy, *C^*-algebras*
- Rudin
- Graham-Allan

Notation

We will use $\mathbb{K}$ to mean "either $\mathbb{R}$ or $\mathbb{C}$".

For X a normed space, we define

$$\begin{aligned} B_X &= \{x \in X | \, \|x\| \leq 1\} \\ S_X &= \{x \in X | \, \|x\| = 1\} \\ D_X &= \{x \in X | \, \|x\| < 1\} \end{aligned}$$

For X, Y normed spaces, we write $X \sim Y$ if X, Y are isomorphic, ie there exists a linear bijection $T : X \to Y$ such that T and T^{-1} are continuous. We write $X \cong Y$ if X, Y are isometrically isomorphic, ie there exists a surjective linear map $T : X \to Y$ such that $\|Tx\| = \|x\|$ for all x.

1 Hahn-Banach extension theorems

Lecture 1

Let X be a normed space. The **dual space** of X is the space X^* of bounded linear functionals on X. X^* is always a Banach space in the operator norm: for $f \in X^*$,

$$\|f\| = \sup_{x \in B_X} |f(x)|$$

Example. For $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$, $\ell_p^* \cong \ell_q$.
We also have $\ell_1^* \cong \ell_\infty$, $c_0^* \cong \ell_1$.
If H is a Hilbert space, then $H^* \cong H$ (the isomorphism is conjugate-linear in the complex case).

For $x \in X, f \in X^*$, we write $\langle x, f \rangle = f(x)$. Note that

$$\langle x, f \rangle = |f(x)| \leq \|f\| \, \|x\|$$

Definition. Let X be a *real* vector space. A functional $p : X \to \mathbb{R}$ is

- **positive homogeneous** if $p(tx) = tp(x)$ for all $x \in X$, $t \geq 0$
- **subadditive** if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$

Definition. Let P be a preorder, $A \subseteq P, x \in P$. We say

- x is an **upper bound** for A if $\forall a \in A, a \leq x$.
- A is a **chain** if $\forall a, b \in A, a \leq b \vee b \leq a$.
- x is a **maximal element** if $\forall y \in P, x \not< y$

Fact (Zorn's lemma)**.** A nonempty preorder in which all nonempty chains have an upper bound has a maximal element.

Theorem (Hahn-Banach, positive homogeneous version)**.** Let X be a real vector space and $p : X \to \mathbb{R}$ be positive homogeneous and subadditive. Let Y be a subspace of X and $g : Y \to \mathbb{R}$ be linear such that $\forall y \in Y, g(y) \leq p(y)$. Then there exists $f : X \to \mathbb{R}$ linear such that $f \restriction_Y = g$ and $\forall x \in X, f(x) \leq p(x)$.

Proof. Let P be the set of pairs (Z, h) where Z is a subspace of X with $Y \subseteq Z$ and $h : Z \to \mathbb{R}$ linear, $h \restriction_Y = g$ and $\forall z \in Z, h(z) \leq p(z)$. P is nonempty since $(Y, g) \in P$, and is partially ordered by

$$(Z_1, h_1) \leq (Z_2, h_2) \iff Z_1 \subseteq Z_2 \wedge h_2 \restriction_{Z_1} = h_1$$

If $\{(Z_i, h_i) | i \in I\}$ is a chain with I nonempty, then we can define

$$Z := \bigcup_{i \in I} Z_i, h \restriction_{Z_i} = h_i$$

The definition of h makes sense thanks to the chain assumption. $(Z, h) \in P$ is therefore an upper bound for the chain.
Hence find by Zorn a maximal element (Z, h) of P. If $Z = X$, we won. So assume there is some $x \in X \; Z$. Let $W = \mathrm{Span}(Z \cup \{x\})$ and define $f : W \to \mathbb{R}$ by

$$f(z + \lambda x) = h(z) + \lambda \alpha$$

for some $\alpha \in \mathbb{R}$. Then f is linear and $f\restriction_Z = h$. We now look for α such that $\forall w \in W, f(w) \leq p(w)$. We would then have $(W, f) \in P$ and $(Z, h) < (W, f)$, contradicting maximality of (Z, h).
We need

$$h(z) + \lambda\alpha \leq p(z + \lambda x) \forall z \in Z, \lambda \in \mathbb{R}$$

Since p is positive homogeneous, this becomes

$$h(z) + \alpha \leq p(z + x) h(z) - \alpha \leq p(z - x) \tag{1}$$

ie

$$h(z) - p(z - x) \leq \alpha \leq p(z + x) - h(z) \forall z \in Z$$

The existence of α now amounts to

$$h(z_1) - p(z_1 - x) \leq \alpha \leq p(z_2 + x) - h(z_2) \forall z_1, z_2 \in Z$$

But indeed

$$h(z_1) + h(z_2) = h(z_1 + z_2) \leq p(z_1 + z_2) \leq p(z_1 - x) + p(z_2 + x)$$

□

Definition. Let X be a $\mathbb{K}$-vector space. A **seminorm** on X is a functional $p : X \to \mathbb{R}$ such that

- $\forall x \in X, p(x) \geq 0$
- $\forall x \in X, \lambda \in \mathbb{K}, p(\lambda x) = |\lambda| p(x)$
- $\forall x, y \in X, p(x + y) \leq p(x) + p(y)$

Remark.

$$\text{norm} \implies \text{seminorm} \implies \text{positive homogeneous}$$

Lecture 2

Theorem (Hahn-Banach, absolute homogeneous version)**.** Let X be a real of complex vector space and p a seminorm on X. Let Y be a subspace of X, g a linear functional on Y such that $\forall y \in Y, |g(y)| \leq p(y)$. Then there exists a linear functional f on X such that $f\restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x)$.

Proof.
Real case

$$\forall y \in Y, g(y) \leq |g(y)| \leq p(y)$$

By Theorem 1, there exists $f : X \to \mathbb{R}$ such that $f\restriction_Y = g$ and $\forall x \in X, f(x) \leq p(x)$. We also have

$$\forall x \in X, -f(x) = f(-x) \leq p(-x) = p(x)$$

Hence $|f(x)| \leq p(x)$
Complex case
$\operatorname{Re} g : Y \to \mathbb{R}$ is real-linear.

$$\forall y \in Y, |\operatorname{Re} g(y)| \leq |g(y)| \leq p(y)$$

By the real case, find $h : X \to \mathbb{R}$ real-linear such that $h\restriction_Y = \operatorname{Re} g$

Claim. There exists a unique complex-linear $f : X \to \mathbb{C}$ such that $h = \operatorname{Re} f$.

Proof.
Uniqueness
If we have such f, then

$$\begin{aligned} f(x) &= \operatorname{Re} f(x) + i \operatorname{Im} f(x) \\ &= \operatorname{Re} f(x) - i \operatorname{Re} f(ix) \\ &= h(x) - ih(ix) \end{aligned}$$

Existence
Define $f(x) = h(x) - ih(ix)$. Then f is real-linear and $f(ix) = if(x)$, so f is complex-linear with $\operatorname{Re} f = h$. $\square$

We now have $f : X \to \mathbb{C}$ such that $\operatorname{Re} f = h$.

$$\operatorname{Re} f \restriction_Y = h \restriction_Y = \operatorname{Re} g$$

So, by uniqueness, $f \restriction_Y = g$.
Given $x \in X$, find λ with $|\lambda| = 1$ such that

$$\begin{aligned} |f(x)| &= \lambda f(x) \\ &= f(\lambda x) \\ &= \operatorname{Re} f(\lambda x) \\ &= h(\lambda x) \\ &\leq p(\lambda x) \\ &= p(x) \end{aligned}$$

$\square$

Remark. For a complex vector space X, if we write $X_{\mathbb{R}}$ for X considered as a real vector space, the above proof shows that

$$\operatorname{Re} : (X^*)_{\mathbb{R}} \to X^*_{\mathbb{R}}$$

is an isometric isomorphism.

Corollary. Let X be a $\mathbb{K}$-vector space, p a seminorm on X, $x_0 \in X$. Then there exists a linear functional f on X such that $f(x_0) = p(x_0)$ and $\forall x \in X, |f(x)| \leq p(x)$.

Proof. Let $Y = \operatorname{Span}(x_0)$,

$$\begin{aligned} g : Y &\to \mathbb{K} \\ \lambda x_0 &\mapsto \lambda p(x_0) \end{aligned}$$

We see that $\forall y \in Y, g(y) \leq p(y)$. Hence find by Theorem 1 a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x)$. We check that $f(x_0) = g(x_0) = p(x_0)$. $\square$

Theorem (Hahn-Banach, existence of support functionals)**.** Let X be a real or complex normed space. Then

1. If Y is a subspace of X and $g \in Y^*$, then there exists $f \in X^*$ such that $f \restriction_Y = g$ and $\|f\| = \|g\|$.
2. Given $x_0 \neq 0$, there exists $f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.

Proof.

1. Let $p(x) = \|g\| \|x\|$. Then p is a seminorm on X and
$$\forall y \in Y, |g(y)| \leq \|g\| \|y\| = p(y)$$
Find by Theorem 1 a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x) = \|g\| \|x\|$. So $\|f\| \leq \|g\|$. Since $f \restriction_Y = g$, we also have $\|g\| \leq \|f\|$. Hence $\|f\| = \|g\|$.

2. Apply Corollary 1 with $p(x) = \|x\|$ to get $f \in X^*$ such that
$$\forall x \in X, |f(x)| \leq \|x\| \text{ and } f(x_0) = \|x_0\|$$
It follows that $\|f\| = 1$.

□

Remarks.

- Part 1 is a sort of linear version of Tietze's extension theorem: Given K compact Hausdorff, $L \subseteq K$ closed, $g : L \to \mathbb{K}$ continuous, there exists $f : K \to \mathbb{K}$ such that $f \restriction_L = g$ and $\|f\|_\infty = \|g\|_\infty$.
- Part 2 shows that for all $x \neq y$ in X there exists $f \in X^*$ such that $f(x) \neq f(y)$, namely X^* **separates points** of X. This is a sort of linear version of Urysohn: $C(K)$ separates points of K.
- The f in part 2 is called a **norming functional**, aka **support functional**, for x_0. The existence of support functionals shows that
$$x_0 = \max_{g \in B_{X^*}} \langle x_0, g \rangle$$
Assuming X is a real normed space and $\|x_0\| = 1$, we have $B_X \subseteq \{x \in X | f(x) \leq 1\}$. Visually, TODO: insert tangency diagram

1.1 Bidual

Let X be a normed space. Then X^{**} is called the **bidual** or **second dual** of X.

For $x \in X$, define $\hat{x} : X^* \to \mathbb{K}$, the **evaluation at** x, by $\hat{x}(f) = f(x)$. $\hat{x}$ is linear and $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\|$, so $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leq \|x\|$.

The map $x \mapsto \hat{x} : X \to X^{**}$ is called the **canonical embedding** of X into X^{**}.

Theorem. The canonical embedding is an isometric embedding.

Proof.
Linearity

$$\begin{aligned} \widehat{x+y}(f) &= f(x+y) = f(x) + f(y) = \hat{x}(f) + \hat{y}(f) \\ \widehat{\lambda x}(f) &= f(\lambda x) = \lambda f(x) = \lambda \hat{x}(f) \end{aligned}$$

Isometry
If $x \neq 0$, there exists a support functional f for x. Then

$$\|\hat{x}\| \geq |\hat{x}(f)| = |f(x)| = \|x\|$$

□

Remarks.

- In bracket notation, $\langle f, \hat{x} \rangle = \langle x, f \rangle$
- Let $\hat{X}$ be the image of X in X^{**}. Theorem 1.1 says

  $$X \cong \hat{X} \subseteq X^{**}$$

  We often identify $\hat{X}$ with X and think of X as living isometrically inside X^{**}. Note that

  $$X \text{ complete} \iff \hat{X} \text{ closed in } X^{**}$$

- More generally, $\bar{\hat{X}}$ is a Banach space containing an isometric copy of X as a dense subspace. We proved that normed spaces have completions!

Definition. A normed space X is **reflexive** if the canonical embedding $X \to X^{**}$ is surjective.

Example.

- Some reflexive spaces are Hilbert spaces, finite-dimensional spaces, ℓ_p and $L_p(\mu)$ for $1 < p < \infty$.
- Some non-reflexive spaces are $c_0, \ell_1, \ell_\infty, L_1[0,1]$.

Remarks.

- If X is reflexive, then $X \cong X^{**}$, so X is complete.
- There are Banach spaces X such that $X \cong X^{**}$ but X is not reflexive, eg **James' space**. Any isomorphism to the bidual is then necessarily not the canonical embedding.

2 The dual of $L_p(\mu)$ and $C(K)$

2.1 Dual operators

Lecture 3

Let X, Y be normed spaces. Recall

$$\mathcal{B}(X,Y) = \{T : X \to Y | T \text{ linear, bounded}\}$$

This is a normed space in the operator norm:

$$\|T\| = \sup_{x \in B_X} \|Tx\|$$

If Y is complete, then so is $\mathcal{B}(X,Y)$. For $T \in \mathcal{B}(X,Y)$, the **dual operator** of T is the map $T^* : Y^* \to X^*$ given by $T^* g = g \circ T$. In bracket notation $\langle x, T^* g\rangle = \langle Tx, g\rangle$ for $x \in X, g \in Y^*$.

T^* is linear

$$\begin{aligned}
\langle x, T^*(g+h)\rangle &= \langle Tx, g+h\rangle \\
&= \langle Tx, g\rangle + \langle Tx, h\rangle \\
&= \langle x, T^* g\rangle + xT^* h \\
&= \langle x, T^* g + T^* h\rangle
\end{aligned}$$

$$\begin{aligned}
\langle x, T^*(\lambda g)\rangle &= \langle Tx, \lambda g\rangle \\
&= \lambda \langle Tx, g\rangle \\
&= \lambda \langle x, T^* g\rangle \\
&= \langle x, \lambda T^* g\rangle
\end{aligned}$$

T^* is bounded

$$\begin{aligned}
\|T^*\| &= \sup_{g \in B_{Y^*}} \|T^* g\| \\
&= \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^* g\rangle| \\
&= \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g\rangle| \\
&= \sup_{x \in B_X} \|Tx\| \text{ by Theorem 1 (ii)} \\
&= \|T\|
\end{aligned}$$

Remarks.

- Hahn-Banach is crucial here. Without it, the dual could be 0.
- $\langle \cdot, \cdot \rangle$ is linear in both arguments. This contrasts with the Hilbert space case where $\langle \cdot, \cdot \rangle$ is conjugate-linear in one of the arguments. This comes from the conjugate-linearity of the identification $H^* \cong H$.
- If X, Y are Hilbert spaces and we identify X, Y with X^*, Y^*, respectively, then T^* is the adjoint of T.

Example. Let $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$ and define $R : \ell_p \to \ell_p$ to be the **right shift operator** $(x_0, x_1, \dots) \mapsto (0, x_0, \dots)$. Then $R^* : \ell_q \to \ell_q$ is the **left shift operator** $(x_0, x_1, \dots) \mapsto (x_1, x_2, \dots)$.

Some properties of the dual operator are

1. $\mathrm{id}_X^* = \mathrm{id}_{X^*}$
2. $(S+T)^* + S^* + T^*$, $(\lambda T)^* = \lambda T^*$
3. $(ST)^* = T^* S^*$
4. $T \mapsto T^* : \mathcal{B}(X,Y) \to \mathcal{B}(Y^*, X^*)$ is an *into* isomorphism.
5. The double dual of an operator commutes with the double dual embedding. TODO: Insert commutative diagram For all x,

$$\langle g, T^{**}\hat{x}\rangle = \langle T^* g, \hat{x}\rangle = \langle x, T^* g\rangle = \langle Tx, g\rangle = \left\langle g, \hat{Tx}\right\rangle$$

   So $T^{**}\hat{x} = \widehat{Tx}$.

Remark. From the above properties, if $X \sim Y$, then $X^* \sim Y^*$. Interestingly, if X and Y are reflexive, then we can deduce $X \sim Y$ from $X^* \sim Y^*$.

2.2 Quotient spaces

Let X be a normed space and Y be a *closed* subspace.. Then the quotient space X/Y becomes a normed space in the quotient norm:

$$\|x + Y\| = d(x, Y) = \inf_{y \in Y} \|x + y\|$$

The quotient map $q : X \to X/Y$ is linear and bounded: $\|q(x)\| \leq \|x\|$, so $\|q\| \leq 1$.

q maps the open unit ball D_X onto $D_{X/Y}$. Indeed, if $x \in D_X$, then $\|q(x)\| \leq \|x\| < 1$. Reciprocally, if $q(x) \in D_{X/Y}$, then there exists $y \in Y$ such that $\|x + y\| < 1$. So $x + y \in D_X$ and $q(x+y) = q(x)$. It follows that q is an open map and $\|q\| = 1$.

If Z is another normed space, $T \in \mathcal{B}(X, Z)$ and $Y \subseteq \ker T$, then there exists a unique map $\tilde{T}$ is linear and $\tilde{T}(D_{X/Y}) = \tilde{T}(q(D_X)) = T(D_X)$. It follows that $\left\|\tilde{T}\right\| = \|T\|$.

Theorem. Let X be a normed space. If X^* is separable, then so is X.

Remark. The converse is false, as $X = \ell_1, X^* = \ell_\infty$ shows.

Proof. Since X^* is separable, so is S_{X^*}. Let f_n be a dense subset of S_{X^*}. For every n, find $x_n \in B_X$ such that $f_n(x_n) > \frac{1}{2}$. Let

$$Y = \overline{\mathrm{Span}\{x_n | n \in \mathbb{N}\}}$$

Claim. $Y = X$

Then we're done since Y is separable via $Y = \overline{\mathrm{Span}_{\mathbb{Q}}\{x_n | n \in \mathbb{N}\}}$.

Proof. Assume not. Then we can pick $g \in (X/Y)^*$, $\|g\| = 1$ (by Theorem 1 (ii)). Let $f = g \circ q$. Then $\|f\| = \|g\| = 1$, ie $f \in S_{X^*}$. Thus find n such that $\|f - f_n\| < \frac{1}{4}$, so that

$$\frac{1}{4} > \|f - f_n\| \|x_n\| \geq |(f - f_n)(x_n)| = |f_n(x_n)| > \frac{1}{2}$$

contradiction. □

□

Theorem. Let X be a separable normed space. Then X embeds isometrically into ℓ_∞.

Proof. Let $\{x_n | n \in \mathbb{N}\}$ be dense in X. For every n, find $f_n \in S_{X^*}$, $f_n(x_n) = \|x_n\|$ (assuming $X \neq \{0\}$). Define $T : X \to \ell_\infty$ by $(Tx)_n = f_n(x)$.

Well definition

$$|(Tx)_n| = |f_n(x)| \leq \|f_n\| \|x\| = \|x\|$$

Hence $\|Tx\|_\infty \leq \|x\| < \infty$.

Linearity

$$(T(x+y))_n = f_n(x+y) = f_n(x) + f_n(y) = (Tx + Ty)_n$$

$$(T(\lambda x))_n = f_n(\lambda x) = \lambda f_n(x)) = (\lambda Tx)_n$$

so $T(x+y) = Tx + Ty, T(\lambda x) = \lambda Tx$.

Isometry

We already know $\|Tx\|_\infty \leq \|x\|$. On the other hand, find f a supporting functional for x and f_{n_k} a subsequence converging to f. Then

$$\|Tx\|_\infty \geq \sup_k (Tx)_{n_k} = \sup_k |f_{n_k}(x)| \geq |f(x)| = \|x\|$$

□