Part III – Functional Analysis (Incomplete)

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Michaelmas 2023

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0 Introduction

Prerequisites

- some basic functional analysis
- a bit of measure theory
- a bit of complex analysis

Books

Books relevant to the course are:

- Bollobás, *Linear Analysis*
- Murphy, C^**-algebras*
- Rudin
- Graham-Allan

Notation

We will use $\mathbb{K}$ to mean "either $\mathbb{R}$ or $\mathbb{C}$".

For X a normed space, we define

$$\begin{aligned} B_X &= \{x \in X | \, \|x\| \leq 1\} \\ S_X &= \{x \in X | \, \|x\| = 1\} \\ D_X &= \{x \in X | \, \|x\| < 1\} \end{aligned}$$

For X, Y normed spaces, we write $X \sim Y$ if X, Y are isomorphic, ie there exists a linear bijection $T : X \to Y$ such that T and T^{-1} are continuous. We write $X \cong Y$ if X, Y are isometrically isomorphic, ie there exists a surjective linear map $T : X \to Y$ such that $\|Tx\| = \|x\|$ for all x.

1 Hahn-Banach extension theorems

Lecture 1

Let X be a normed space. The **dual space** of X is the space X^* of bounded linear functionals on X. X^* is always a Banach space in the operator norm: for $f \in X^*$,

$$\|f\| = \sup_{x \in B_X} |f(x)|$$

Example. For $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$, $\ell_p^* \cong \ell_q$.
We also have $\ell_1^* \cong \ell_\infty$, $c_0^* \cong \ell_1$.
If H is a Hilbert space, then $H^* \cong H$ (the isomorphism is conjugate-linear in the complex case).

For $x \in X, f \in X^*$, we write $\langle x, f \rangle = f(x)$. Note that

$$\langle x, f \rangle = |f(x)| \leq \|f\| \, \|x\|$$

Definition. Let X be a *real* vector space. A functional $p : X \to \mathbb{R}$ is

- **positive homogeneous** if $p(tx) = tp(x)$ for all $x \in X$, $t \geq 0$
- **subadditive** if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$

Definition. Let P be a preorder, $A \subseteq P, x \in P$. We say

- x is an **upper bound** for A if $\forall a \in A, a \leq x$.
- A is a **chain** if $\forall a, b \in A, a \leq b \vee b \leq a$.
- x is a **maximal element** if $\forall y \in P, x \not< y$

Fact (Zorn's lemma)**.** A nonempty preorder in which all nonempty chains have an upper bound has a maximal element.

Theorem 1.1 (Hahn-Banach, positive homogeneous version)**.** Let X be a real vector space and $p : X \to \mathbb{R}$ be positive homogeneous and subadditive. Let Y be a subspace of X and $g : Y \to \mathbb{R}$ be linear such that $\forall y \in Y, g(y) \leq p(y)$. Then there exists $f : X \to \mathbb{R}$ linear such that $f \restriction_Y = g$ and $\forall x \in X, f(x) \leq p(x)$.

Proof. Let P be the set of pairs (Z, h) where Z is a subspace of X with $Y \subseteq Z$ and $h : Z \to \mathbb{R}$ linear, $h \restriction_Y = g$ and $\forall z \in Z, h(z) \leq p(z)$. P is nonempty since $(Y, g) \in P$, and is partially ordered by

$$(Z_1, h_1) \leq (Z_2, h_2) \iff Z_1 \subseteq Z_2 \wedge h_2 \restriction_{Z_1} = h_1$$

If $\{(Z_i, h_i) | i \in I\}$ is a chain with I nonempty, then we can define

$$Z := \bigcup_{i \in I} Z_i, h \restriction_{Z_i} = h_i$$

The definition of h makes sense thanks to the chain assumption. $(Z, h) \in P$ is therefore an upper bound for the chain.
Hence find by Zorn a maximal element (Z, h) of P. If $Z = X$, we won. So assume there is some $x \in X\ Z$. Let $W = \mathrm{Span}(Z \cup \{x\})$ and define $f : W \to \mathbb{R}$ by

$$f(z + \lambda x) = h(z) + \lambda \alpha$$

for some $\alpha \in \mathbb{R}$. Then f is linear and $f \restriction_Z = h$. We now look for α such that $\forall w \in W, f(w) \leq p(w)$. We would then have $(W, f) \in P$ and $(Z, h) < (W, f)$, contradicting maximality of (Z, h).
We need

$$h(z) + \lambda\alpha \leq p(z + \lambda x) \forall z \in Z, \lambda \in \mathbb{R}$$

Since p is positive homogeneous, this becomes

$$h(z) + \alpha \leq p(z + x) h(z) - \alpha \leq p(z - x) \tag{1}$$

ie

$$h(z) - p(z - x) \leq \alpha \leq p(z + x) - h(z) \forall z \in Z$$

The existence of α now amounts to

$$h(z_1) - p(z_1 - x) \leq \alpha \leq p(z_2 + x) - h(z_2) \forall z_1, z_2 \in Z$$

But indeed

$$h(z_1) + h(z_2) = h(z_1 + z_2) \leq p(z_1 + z_2) \leq p(z_1 - x) + p(z_2 + x)$$

□

Definition. Let X be a $\mathbb{K}$-vector space. A **seminorm** on X is a functional $p : X \to \mathbb{R}$ such that

- $\forall x \in X, p(x) \geq 0$
- $\forall x \in X, \lambda \in \mathbb{K}, p(\lambda x) = |\lambda| p(x)$
- $\forall x, y \in X, p(x + y) \leq p(x) + p(y)$

Remark.

$$\text{norm} \implies \text{seminorm} \implies \text{positive homogeneous}$$

Lecture 2

Theorem 1.2 (Hahn-Banach, absolute homogeneous version)**.** Let X be a real of complex vector space and p a seminorm on X. Let Y be a subspace of X, g a linear functional on Y such that $\forall y \in Y, |g(y)| \leq p(y)$. Then there exists a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x)$.

Proof.
Real case

$$\forall y \in Y, g(y) \leq |g(y)| \leq p(y)$$

By Theorem 1.1, there exists $f : X \to \mathbb{R}$ such that $f \restriction_Y = g$ and $\forall x \in X, f(x) \leq p(x)$.
We also have

$$\forall x \in X, -f(x) = f(-x) \leq p(-x) = p(x)$$

Hence $|f(x)| \leq p(x)$
Complex case
$\operatorname{Re} g : Y \to \mathbb{R}$ is real-linear.

$$\forall y \in Y, |\operatorname{Re} g(y)| \leq |g(y)| \leq p(y)$$

By the real case, find $h : X \to \mathbb{R}$ real-linear such that $h \restriction_Y = \operatorname{Re} g$

Claim. There exists a unique complex-linear $f : X \to \mathbb{C}$ such that $h = \operatorname{Re} f$.

Proof.
Uniqueness
If we have such f, then

$$\begin{aligned} f(x) &= \operatorname{Re} f(x) + i \operatorname{Im} f(x) \\ &= \operatorname{Re} f(x) - i \operatorname{Re} f(ix) \\ &= h(x) - ih(ix) \end{aligned}$$

Existence
Define $f(x) = h(x) - ih(ix)$. Then f is real-linear and $f(ix) = if(x)$, so f is complex-linear with $\operatorname{Re} f = h$. □

We now have $f : X \to \mathbb{C}$ such that $\operatorname{Re} f = h$.

$$\operatorname{Re} f \restriction_Y = h \restriction_Y = \operatorname{Re} g$$

So, by uniqueness, $f \restriction_Y = g$.
Given $x \in X$, find λ with $|\lambda| = 1$ such that

$$\begin{aligned} |f(x)| &= \lambda f(x) \\ &= f(\lambda x) \\ &= \operatorname{Re} f(\lambda x) \\ &= h(\lambda x) \\ &\leq p(\lambda x) \\ &= p(x) \end{aligned}$$

□

Remark. For a complex vector space X, if we write $X_{\mathbb{R}}$ for X considered as a real vector space, the above proof shows that

$$\operatorname{Re} : (X^*)_{\mathbb{R}} \to X^*_{\mathbb{R}}$$

is an isometric isomorphism.

Corollary 1.3. Let X be a $\mathbb{K}$-vector space, p a seminorm on X, $x_0 \in X$. Then there exists a linear functional f on X such that $f(x_0) = p(x_0)$ and $\forall x \in X, |f(x)| \leq p(x)$.

Proof. Let $Y = \operatorname{Span}(x_0)$,

$$\begin{aligned} g : Y &\to \mathbb{K} \\ \lambda x_0 &\mapsto \lambda p(x_0) \end{aligned}$$

We see that $\forall y \in Y, g(y) \leq p(y)$. Hence find by Theorem 1.2 a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x)$. We check that $f(x_0) = g(x_0) = p(x_0)$. □

Theorem 1.4 (Hahn-Banach, existence of support functionals)**.** Let X be a real or complex normed space. Then

1. If Y is a subspace of X and $g \in Y^*$, then there exists $f \in X^*$ such that $f \restriction_Y = g$ and $\|f\| = \|g\|$.
2. Given $x_0 \neq 0$, there exists $f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.

Proof.

1. Let $p(x) = \|g\| \|x\|$. Then p is a seminorm on X and

$$\forall y \in Y, |g(y)| \leq \|g\| \|y\| = p(y)$$

Find by Theorem 1.1 a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x) = \|g\| \|x\|$. So $\|f\| \leq \|g\|$. Since $f \restriction_Y = g$, we also have $\|g\| \leq \|f\|$. Hence $\|f\| = \|g\|$.

2. Apply Corollary 1.3 with $p(x) = \|x\|$ to get $f \in X^*$ such that

$$\forall x \in X, |f(x)| \leq \|x\| \text{ and } f(x_0) = \|x_0\|$$

It follows that $\|f\| = 1$.

□

Remarks.

- Part 1 is a sort of linear version of Tietze's extension theorem: Given K compact Hausdorff, $L \subseteq K$ closed, $g : L \to \mathbb{K}$ continuous, there exists $f : K \to \mathbb{K}$ such that $f \restriction_L = g$ and $\|f\|_\infty = \|g\|_\infty$.
- Part 2 shows that for all $x \neq y$ in X there exists $f \in X^*$ such that $f(x) \neq f(y)$, namely X^* **separates points** of X. This is a sort of linear version of Urysohn: $C(K)$ separates points of K.
- The f in part 2 is called a **norming functional**, aka **support functional**, for x_0. The existence of support functionals shows that

$$x_0 = \max_{g \in B_{X^*}} \langle x_0, g \rangle$$

Assuming X is a real normed space and $\|x_0\| = 1$, we have $B_X \subseteq \{x \in X | f(x) \leq 1\}$. Visually, TODO: insert tangency diagram

1.1 Bidual

Let X be a normed space. Then X^{**} is called the **bidual** or **second dual** of X.

For $x \in X$, define $\hat{x} : X^* \to \mathbb{K}$, the **evaluation at** x, by $\hat{x}(f) = f(x)$. $\hat{x}$ is linear and $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\|$, so $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leq \|x\|$.

The map $x \mapsto \hat{x} : X \to X^{**}$ is called the **canonical embedding** of X into X^{**}.

Theorem 1.5. The canonical embedding is an isometric embedding.

Proof.
Linearity

$$\begin{aligned} \widehat{x+y}(f) &= f(x+y) = f(x) + f(y) = \hat{x}(f) + \hat{y}(f) \\ \widehat{\lambda x}(f) &= f(\lambda x) = \lambda f(x) = \lambda \hat{x}(f) \end{aligned}$$

Isometry
If $x \neq 0$, there exists a support functional f for x. Then

$$\|\hat{x}\| \geq |\hat{x}(f)| = |f(x)| = \|x\|$$

□

Remarks.

- In bracket notation, $\langle f, \hat{x} \rangle = \langle x, f \rangle$
- Let $\hat{X}$ be the image of X in X^{**}. Theorem 1.5 says

$$X \cong \hat{X} \subseteq X^{**}$$

  We often identify $\hat{X}$ with X and think of X as living isometrically inside X^{**}. Note that

$$X \text{ complete} \iff \hat{X} \text{ closed in } X^{**}$$

- More generally, $\bar{\hat{X}}$ is a Banach space containing an isometric copy of X as a dense subspace. We proved that normed spaces have completions!

Definition. A normed space X is **reflexive** if the canonical embedding $X \to X^{**}$ is surjective.

Example.

- Some reflexive spaces are Hilbert spaces, finite-dimensional spaces, ℓ_p and $L_p(\mu)$ for $1 < p < \infty$.
- Some non-reflexive spaces are $c_0, \ell_1, \ell_\infty, L_1[0,1]$.

Remarks.

- If X is reflexive, then $X \cong X^{**}$, so X is complete.
- There are Banach spaces X such that $X \cong X^{**}$ but X is not reflexive, eg **James' space**. Any isomorphism to the bidual is then necessarily not the canonical embedding.

1.2 Dual operators

Lecture 3

Let X, Y be normed spaces. Recall

$$\mathcal{B}(X,Y) = \{T : X \to Y | T \text{ linear, bounded}\}$$

This is a normed space in the operator norm:

$$\|T\| = \sup_{x \in B_X} \|Tx\|$$

If Y is complete, then so is $\mathcal{B}(X,Y)$. For $T \in \mathcal{B}(X,Y)$, the **dual operator** of T is the map $T^* : Y^* \to X^*$ given by $T^* g = g \circ T$. In bracket notation $\langle x, T^* g \rangle = \langle Tx, g \rangle$ for $x \in X, g \in Y^*$.

T^* is linear

$$\begin{aligned} \langle x, T^*(g+h) \rangle &= \langle Tx, g+h \rangle \\ &= \langle Tx, g \rangle + \langle Tx, h \rangle \\ &= \langle x, T^* g \rangle + x T^* h \\ &= \langle x, T^* g + T^* h \rangle \end{aligned}$$

$$\begin{aligned}\langle x, T^*(\lambda g)\rangle &= \langle Tx, \lambda g\rangle \\ &= \lambda \langle Tx, g\rangle \\ &= \lambda \langle x, T^* g\rangle \\ &= \langle x, \lambda T^* g\rangle\end{aligned}$$

T^* is bounded

$$\begin{aligned}\|T^*\| &= \sup_{g\in B_{Y^*}} \|T^* g\| \\ &= \sup_{g\in B_{Y^*}} \sup_{x\in B_X} |\langle x, T^* g\rangle| \\ &= \sup_{x\in B_X} \sup_{g\in B_{Y^*}} |\langle Tx, g\rangle| \\ &= \sup_{x\in B_X} \|Tx\| \text{ by Theorem 1.4 (ii)} \\ &= \|T\|\end{aligned}$$

Remarks.

- Hahn-Banach is crucial here. Without it, the dual could be 0.
- $\langle\cdot,\cdot\rangle$ is linear in both arguments. This contrasts with the Hilbert space case where $\langle\cdot,\cdot\rangle$ is conjugate-linear in one of the arguments. This comes from the conjugate-linearity of the identification $H^* \cong H$.
- If X, Y are Hilbert spaces and we identify X, Y with X^*, Y^*, respectively, then T^* is the adjoint of T.

Example. Let $1 < p, q < \infty$, $p^{-1}+q^{-1} = 1$ and define $R : \ell_p \to \ell_p$ to be the **right shift operator** $(x_0, x_1, \dots) \mapsto (0, x_0, \dots)$. Then $R^* : \ell_q \to \ell_q$ is the **left shift operator** $(x_0, x_1, \dots) \mapsto (x_1, x_2, \dots)$.

Some properties of the dual operator are

1. $\mathrm{id}_X^* = \mathrm{id}_{X^*}$
2. $(S + T)^* + S^* + T^*$, $(\lambda T)^* = \lambda T^*$
3. $(ST)^* = T^* S^*$
4. $T \mapsto T^* : \mathcal{B}(X, Y) \to \mathcal{B}(Y^*, X^*)$ is an *into* isomorphism.
5. The double dual of an operator commutes with the double dual embedding. TODO: Insert commutative diagram For all x,

$$\langle g, T^{**}\hat{x}\rangle = \langle T^* g, \hat{x}\rangle = \langle x, T^* g\rangle = \langle Tx, g\rangle = \left\langle g, \hat{Tx}\right\rangle$$

So $T^{**}\hat{x} = \widehat{Tx}$.

Remark. From the above properties, if $X \sim Y$, then $X^* \sim Y^*$. Interestingly, if X and Y are reflexive, then we can deduce $X \sim Y$ from $X^* \sim Y^*$.

1.3 Quotient spaces

Let X be a normed space and Y be a *closed* subspace.. Then the quotient space X/Y becomes a normed space in the quotient norm:

$$\|x + Y\| = d(x, Y) = \inf_{y\in Y} \|x + y\|$$

The quotient map $q : X \to X/Y$ is linear and bounded: $\|q(x)\| \leq \|x\|$, so $\|q\| \leq 1$.

q maps the open unit ball D_X onto $D_{X/Y}$. Indeed, if $x \in D_X$, then $\|q(x)\| \leq \|x\| < 1$. Reciprocally, if $q(x) \in D_{X/Y}$, then there exists $y \in Y$ such that $\|x + y\| < 1$. So $x + y \in D_X$ and $q(x + y) = q(x)$. It follows that q is an open map and $\|q\| = 1$.

If Z is another normed space, $T \in \mathcal{B}(X, Z)$ and $Y \subseteq \ker T$, then there exists a unique map $\tilde{T}$ is linear and $\tilde{T}(D_{X/Y}) = \tilde{T}(q(D_X)) = T(D_X)$. It follows that $\left\|\tilde{T}\right\| = \|T\|$.

Theorem 1.6. Let X be a normed space. If X^* is separable, then so is X.

Remark. The converse is false, as $X = \ell_1, X^* = \ell_\infty$ shows.

Proof. Since X^* is separable, so is S_{X^*}. Let f_n be a dense subset of S_{X^*}. For every n, find $x_n \in B_X$ such that $f_n(x_n) > \frac{1}{2}$. Let

$$Y = \overline{\text{Span}\{x_n | n \in \mathbb{N}\}}$$

Claim. $Y = X$

Then we're done since Y is separable via $Y = \overline{\text{Span}_{\mathbb{Q}}\{x_n | n \in \mathbb{N}\}}$.

Proof. Assume not. Then we can pick $g \in (X/Y)^*$, $\|g\| = 1$ (by Theorem 1.4 (ii)). Let $f = g \circ q$. Then $\|f\| = \|g\| = 1$, ie $f \in S_{X^*}$. Thus find n such that $\|f - f_n\| < \frac{1}{4}$, so that

$$\frac{1}{4} > \|f - f_n\| \|x_n\| \geq |(f - f_n)(x_n)| = |f_n(x_n)| > \frac{1}{2}$$

contradiction. □

□

Theorem 1.7. Let X be a separable normed space. Then X embeds isometrically into ℓ_∞.

Proof. Let $\{x_n | n \in \mathbb{N}\}$ be dense in X. For every n, find $f_n \in S_{X^*}$, $f_n(x_n) = \|x_n\|$ (assuming $X \neq \{0\}$). Define $T : X \to \ell_\infty$ by $(Tx)_n = f_n(x)$.

Well definition

$$|(Tx)_n| = |f_n(x)| \leq \|f_n\| \|x\| = \|x\|$$

Hence $\|Tx\|_\infty \leq \|x\| < \infty$.

Linearity

$$(T(x + y))_n = f_n(x + y) = f_n(x) + f_n(y) = (Tx + Ty)_n$$
$$(T(\lambda x))_n = f_n(\lambda x) = \lambda f_n(x)) = (\lambda Tx)_n$$

so $T(x + y) = Tx + Ty, T(\lambda x) = \lambda Tx$.

Isometry

We already know $\|Tx\|_\infty \leq \|x\|$. On the other hand, find f a supporting functional for x and f_{n_k} a subsequence converging to f. Then

$$\|Tx\|_\infty \geq \sup_k (Tx)_{n_k} = \sup_k |f_{n_k}(x)| \geq |f(x)| = \|x\|$$

□

Remarks.

- The result says that ℓ_∞ is isometrically universal for the class $\mathcal{SB}$ of separable Banach spaces.
- There is a dual result: Every separable Banach space is a quotient of ℓ_1.

Theorem 1.8 (Vector-valued Liouville)**.** Lex X be a complex Banach space, $f : \mathbb{C} \to X$ holomorphic and bounded. Then f is constant.

Proof. Find $M \geq 0$ such that $\forall z \in \mathbb{C}, |f(z)| \leq M$. Fix $\phi \in X^*$. $\phi \circ f : \mathbb{C} \to \mathbb{C}$ is

bounded

$$|\phi(f(z))| \leq \|\phi\| \, \|f(z)\| \leq M \, \|\phi\|$$

holomorphic

$$\frac{\phi(f(z)) - \phi(f(w))}{z - w} = \phi\left(\frac{f(z) - f(w)}{z - w}\right) \to \phi(f'(z))$$

By scalar Liouville, $\phi \circ f$ is constant. For every $z \in \mathbb{C}, \phi \in X^*$, $\phi(f(z)) = \phi(f(0))$. Since X^* separates points of X, $f(z) = f(0)$. $\square$

Remark. This is a typical example of how to transfer a scalar result to a vector-valued one: Prove the result once composed with any functional, then go back using Hahn-Banach.

1.4 Locally convex spaces

Definition. A **locally convex space** is a $\mathbb{K}$-vector space such that there exists a family $\mathcal{P}$ of seminorms on X that separate points of X in the sense that for all $x \neq 0$ there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$.

The family $\mathcal{P}$ defines a topology on X:

$$U \subseteq X \text{ open} \iff \forall x \in U, \exists s \subseteq \mathcal{P} \text{ finite}, \varepsilon > 0, \{y \in X | \forall p \in s, p(x) < \varepsilon\} \subseteq U$$

Remarks.

1. Addition and scalar multiplication are continuous.
2. The topology is Hausdorff as $\mathcal{P}$ separates points.
3. $x_n \to x \iff \forall p \in \mathcal{P}, p(x_n - x) \to 0$
4. Let Y be a subspace of X and $\mathcal{P}_Y = \{p \restriction_Y | p \in \mathcal{P}\}$. Then $(Y, \mathcal{P}_Y)$ is a LCS and its topology is the subspace topology.
5. Let $\mathcal{P}, \mathcal{Q}$ be two families of seminorms on X both separating points of X. We say $\mathcal{P}, \mathcal{Q}$ are **equivalent**, write $\mathcal{P} \sim \mathcal{Q}$, if they induce the same topology on X. One interesting result is that

$$(X, \mathcal{P}) \text{ metrisable} \iff \mathcal{P} \text{ equivalent to some countable family}$$

6. We make $\mathcal{P}$ part of the data here out of simplicity, but in grown up mathematics we instead assume that X already comes with a topology and that this topology coincides with the one induced by $\mathcal{P}$.

Definition. A **Fréchet space** is a complete metrisable LCS.

Example.

1. A normed space is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.
2. Let $U \subseteq \mathbb{C}$ nonempty open. Let $\mathcal{O}(U) = \{f : U \to \mathbb{C} | f \text{ holomorphic}\}$. For compact $K \subseteq U$, define $p_K(f) = \sup_{z\in K} |f(z)|$. Let $\mathcal{P} = \{p_K | K \subseteq U \text{ compact}\}$ Then $(\mathcal{O}(U), \mathcal{P})$ is a LCS. If we replace $\{K \subseteq U \text{ compact}\}$ by a compact exhaustion of U, then we get a countable separating family equivalent to $\mathcal{P}$. So $(\mathcal{O}(U), \mathcal{P})$ is metrisable. However it is not normable: no norm on $\mathcal{O}(U)$ induces the topology of $(\mathcal{O}(U), \mathcal{P})$, which is the topology of uniform convergence. This is a consequence of Montel's theorem.
3. Fix $d \in \mathbb{N}, \Omega \subseteq \mathbb{R}^d$ a nonempty open set. Let

$$C^\infty(\Omega) = \{f : \Omega \to \mathbb{R} | f \text{ infinitely differentiable}\}$$

Given a multi-index $\alpha \in \mathbb{Z}^d$, α defines a differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$$

For a compact $K \subseteq \Omega, \alpha \in \mathbb{Z}^d$, define

$$p_{K,\alpha}(f) = \sup_{z\in K} |D^\alpha f(z)|$$

Let

$$\mathcal{P} = \{p_{K,\alpha} | K \subseteq U \text{ compact}, \alpha \in \mathbb{Z}^d\}$$

Then $(C^\infty, \mathcal{P})$ is a LCS. It is in fact a non-normable Fréchet space.

Lemma 1.9. Let $(X, \mathcal{P}), (Y, \mathcal{Q})$ be LCS, $T : X \to Y$ linear. TFAE

1. T is continuous
2. T is continuous at 0
3. $\forall q \in \mathcal{Q}, \exists s \subseteq \mathcal{P}$ finite, $C \geq 0, \forall x \in X, q(Tx) \leq C \max_{p\in s} p(x)$

Proof.
(i) $\iff$ (ii)
Translation is continuous.

(ii) $\implies$ (iii)
Given $q \in \mathcal{Q}$, let $V = \{y \in Y | q(y) \leq 1\}$. Then V is a neighborhood of 0 in Y. So there exists U neighborhood of 0 in X such that $T(U) \subseteq V$. WLOG

$$U = \{x \in X | \forall p_K \in s, p_K(x) \leq \varepsilon\}$$

Let $p = \max_{p_K\in s} p_K(x)$. If $p(x) = 1$, then $p(\varepsilon x) = \varepsilon$, so $\varepsilon x \in U$ and

$$q(T(\varepsilon x)) \leq 1$$

By homogeneity, $q(Tx) \leq \frac{1}{\varepsilon} p(x)$ for all x such that $p(x) > 0$. If $p(x) = 0$, then $p(\lambda x) = 0$ for all scalar λ. So $q(T(\lambda x)) \leq 1$ for all λ. Hence $q(Tx) = 0 \leq \frac{1}{\varepsilon} p(x)$.

(iii) $\implies$ (ii)
Assume $t \subseteq \mathcal{Q}$ is finite, $\varepsilon > 0$, and let $V = \{y \in Y | \forall q \in t, q(y) \leq \varepsilon$ the corresponding

neighborhood of 0. For each $q \in t$, find $s_q \subseteq \mathcal{P}$ finite and C_q so that $\forall x \in X, q(Tx) \leq C_q \max_{p \in s_q} p(x)$. Let

$$U = \left\{ x \in X | \forall q \in \mathcal{Q}, p \in s_q, p(x) \leq \frac{\varepsilon}{C_q} \right\}$$

Then U is a neighborhood of 0 and $T(U) \subseteq V$. □

Definition. Let $(X, \mathcal{P})$ be a LCS. The **dual space** of X is the space of continuous linear functionals $X \to \mathbb{K}$.

Lecture 5

Lemma 1.10. Let f be a linear functional on a LCS $(X, \mathcal{P})$. Then

$$f \in X^* \iff \ker f \text{ closed}$$

Proof.
$\implies$
$\ker f = f^{-1}(0)$ is closed since f is continuous.

$\impliedby$
If $\ker f = 0$, then $f = 0$ is continuous. Else fix some $x_0 \notin \ker f$. Since $(\ker f)^c$ is open, find $s \subseteq \mathcal{P}$ finite, $\varepsilon > 0$ such that

$$\underbrace{\{x \in X | \forall p \in s, p(x - x_0) < \varepsilon\}}_{U} \subseteq (\ker f)^c$$

Then U is a neighborhood of 0 and $(x_0 + U) \cap \ker f =$. Note that U is convex and **balanced** ($x \in U, |\lambda| \leq 1 \implies \lambda x \in U$), hence so is $f(U)$ as f is linear.
If $f(U)$ is unbounded, then it is the whole scalar field, hence so is $f(x_0 + U) = f(x_0) + f(U)$. But $0 \in \ker f$, contradicting disjointness.
So find M such that $|f(x)| < M$ for all $x \in U$. For all $\delta > 0$, $\frac{\delta}{M} U$ is a neighborhood of 0 and $f\left(\frac{\delta}{M} U\right) \subseteq \{\lambda \in \mathbb{K} | |\lambda| < \delta\}$. Thus f is continuous. □

Theorem 1.11 (Hahn-Banach)**.** Let $(X, \mathcal{P})$ be a LCS.

1. Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ such that $f \restriction_Y = g$.
2. Given a closed subspace Y of X and $x_0 \notin Y$, there exists $f \in X^*$ such that $f \restriction_Y = 0$, $f(x_0) \neq 0$.

Remark. This means that X^* separates points of X.

Proof.

1. By Lemma 1.9, find $s \subseteq \mathcal{P}$ finite, $C \geq 0$ such that

$$\forall y \in Y, |g(y)| \leq C \max_{p \in s} p(y)$$

   Let $p(x) = C \max_{p \in s} p(x)$. Then p is a seminorm on X and $\forall y \in Y, |g(y)| \leq p(y)$. By Theorem 1.2, find a linear functional f on X such that $f \restriction_Y = g, \forall x \in X, |f(x)| \leq p(x)$. By Lemma 1.9, $f \in X^*$.

2. Let $Z = \operatorname{Span}(Y \cup \{x_0\})$ and define a linear functional g on Z by

$$g(y + \lambda x_0) = \lambda, y \in Y, \lambda \in \mathbb{K}$$

Then $g \restriction_Y = 0, g(x_0) = 1 \neq 0$ and $\ker g = Y$ is closed, so $g \in Z^*$ by Lemma 1.10. By part (i), find $f \in X^*$ such that $f \restriction_Z = g$. This works.

□

2 The dual of $L_p(\mu)$ and $C(K)$

Let $(\Omega, \mathcal{F}, \mu)$ be measure space.

$1 \leq p < \infty$

$$L_p(\mu) = \{f : \Omega \to \mathbb{K} | f \text{ measurable and } \int_\Omega |f|^p \, d\mu < \infty\}$$

This is a normed space in the L_p-norm:

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}}$$

$p = \infty$
A measurable function $f : \Omega \to \mathbb{K}$ is **essentially bounded** if there exists $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $f \restriction_{N^c}$ is bounded.

$$L_p(\mu) = \{f : \Omega \to \mathbb{K} | f \text{ measurable and essentially bounded}\}$$

This is a normed space in the L_∞-norm:

$$\|f\|_\infty = \operatorname{esssup} |f| = \inf_{|f| \leq k \text{ ae}} k$$

The inf is attained: there exists some $N \in \mathcal{F}, \mu(N) = 0$ such that $\|f\|_\infty = \sup_{N^c} |f|$.

In all cases, we identify functions up to almost everywhere equality.

Theorem 2.1. *$L_p(\mu)$ is complete for $1 \leq p \leq infty$.*

Definition (Complex measures)**.** A **complex measure** on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{C}$.
The **total variation measure** $|\nu|$ is defined by

$$|\nu| (A) = \sup_{\substack{A_1, \ldots, A_n \text{ measurable} \\ \text{partition of } A}} \sum_k |\nu(A_k)|$$

$|\nu| : \mathcal{F} \to [0, \infty]$ is a positive measure. Later we'll see that $|\nu|$ is a finite measure.
The **total variation** of ν is $\|\nu\|_1 = |\nu| (\Omega)$.

Proposition. If ν is a complex measure on $\mathcal{F}$ and $A_n \in \mathcal{F}$ for all n, then

- If A is monotone, then $\nu(\bigcup_n A_n) = \lim_{n\to\infty} \nu(A_n)$.
- If A is antitone, then $\nu(\bigcap_n A_n) = \lim_{n\to\infty} \nu(A_n)$.

Definition (Signed measures)**.** A **signed measure** on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{R}$.

Theorem 2.2. *If ν is a signed measure, then there exists a measurable partition $\Omega = P \cup N$ such that for all $A \in \mathcal{F}$*

$$\begin{aligned} A \subseteq P &\implies \nu(A) \geq 0 \\ A \subseteq N &\implies \nu(A) \leq 0 \end{aligned}$$

Remarks.

1. This decomposition is called the **Hahn decomposition** of ν.

2. Define $\nu^+(A) = \nu(A \cap P), \nu^-(A) = -\nu(A \cap N)$. Then ν^+, ν^- are finite positive measures such that $\nu = \nu^+ - \nu^-$. This determines ν^+, ν^- uniquely and the decomposition composition $\nu = \nu^+ - \nu^-$ is called the **Jordan decomposition** of ν.

3. If ν is a complex measure on $\mathcal{F}$, then $\operatorname{Re}\nu, \operatorname{Im}\nu$ are signed measures with Jordan decomposition $\nu_1 - \nu_2, \nu_3 - \nu_4$ respectively. Hence $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ is the Jordan decomposition of ν.

$$\nu_1, \nu_2, \nu_3, \nu_4 \leq |\nu| \leq \nu_1 + \nu_2 + \nu_3 + \nu_4$$

   So $|\nu|$ is a finite measure.

Sketch. Define $\nu^+(A) = \sup_{\substack{B \in \mathcal{F} \\ B \subseteq A}} \nu(B)$. ν^+ is nonnegative and finitely additive.

Key step: $\nu^+(\Omega) < \infty$
By contradiction, construct inductively sequences A_n, B_n such that

$$\nu^+(A_n) = \infty, \nu(B_n) > n$$

by taking $A_0 = \Omega, B_{n+1} \subseteq A_n$ such that $\nu(B_n) > n$ (exists by continuity) and $A_{n+1} = B_{n+1}$ or $A_n \setminus B_{n+1}$. This contradicts countable additivity.

Now find a sequence A_n such that $\nu(A_n) > \nu^+(\Omega) - 2^{-n}$ and set $P = \liminf_n A_n, N = P^c$. Check that this works. $\square$

Lecture 6