Part III – Ramsey Theory on Graphs (Incomplete)

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Michaelmas 2023

Contents

0   Introduction

Notation. We write

- $[n] = \{1, \ldots, n\}$

- K_n for the complete graph on n vertices.

- For X a set, $r \in \mathbb{N}$, $X^{(r)} = \{S \subseteq X \,||\, |S| = r\}$

- χ for a k-coloring of the edges of K_n

$$\chi : E(K_n) \to [k]$$
$$\chi : E(K_n) \to \{\text{red}, \text{blue}\} \qquad \text{(if } k = 2\text{)}$$

Ramsey theory is usually concerned with the following question:

> *Can we find some order in enough disorder?*

In this course, we will specialise this question to graphs. We are thus interested in the following:

> *What can we say about the structure of an arbitrary 2-coloring of the edges of K_n?*

Definition. Define the **Ramsey number** $R(\ell, k)$ to be the least n for which every 2-edge coloring contains either a blue K_ℓ or a red K_k, and the **diagonal Ramsey number** $R(k) = R(k, k)$ to be the least n for which every 2-edge coloring contains a monochromatic K_k.

It is unclear that such a n even exists! We shall prove it in due course.

$R(\ell, k) = R(k, \ell)$. By convention, we will usually assume $\ell \leq k$.

Example. $R(3) = 6$ because

- The following coloring shows that $R(3) > 5$. TODO: add picture

- If we have 6 vertices, we can pick a vertex v. By pigeonhole, three of the neighbors of v are connected to v via the same color, say red. Now either two of those neighbors are connected with a red edge, in which case they form a red triangle with v, or they are connected with blue edges to each other, in which case they form a blue triangle. As a way to remember this proof, we encourage you to watch the following music video: Everybody's looking for Ramsey

1   Old bounds on $R(k)$

Theorem (Erdős-Szekeres, 1935)**.**

$$R(\ell, k) \leq \binom{k + \ell - 2}{\ell - 1}$$

In particular, $R(\ell, k)$ is well-defined.

Lemma. For all $k, \ell \geq 3$,

$$R(\ell, k) \leq \underbrace{R(\ell - 1, k)}_{a} + \underbrace{R(\ell, k - 1)}_{b}$$

Proof. Let $n = a + b$. Pick a vertex v. By pigeonhole, either

- v has at least a red neighbors. Either these neighbors contain a red $K_{\ell-1}$ (in which case we chuck v in), or contain a blue K_k (in which case we already won).

- v has at least b blue neighbors. Either these neighbors contain a blue K_{k-1} (in which case we chuck v in), or contain a red K_ℓ (in which case we already won).

$\square$

Proof of Erdős-Szekeres. Use that $R(\ell, 2) = \ell$ and induct on k and ℓ. $\square$

Corollary.
$$R(k) \leq \binom{2k}{k} \leq C \frac{4^k}{\sqrt{k}}$$

for some constant C.

1.1   Lower bounds

Can we find edge colorings on many vertices without a monochromatic K_k? Certainly, we can at least do so on $(k-1)^2$ vertices.

TODO: Insert figure

This polynomial lower bound is eons away from our exponential upper bound. For quite some time (in the 1930s), people thought that the lower bound was closer to the truth than the upper bound. Surprisingly, it is possible to show an exponential lower bound without actually exhibiting such a coloring!

Theorem (Erdős, 1948)**.**
$$R(k) \geq \frac{k-1}{e\sqrt{2}} 2^{\frac{k}{2}}$$

Fact.
$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

Proof. Let $n = \left\lceil \frac{k-1}{e\sqrt{2}} 2^{\frac{k}{2}} \right\rceil$ and χ be a random red/blue edge coloring of K_n (each edge

is independently colored red or blue with probability $\frac{1}{2}$). We see that

$$\mathbb{P}(\chi \text{ contains a monochromatic } K_k) = \mathbb{P}\left(\bigcup_{S \in [n]^{(k)}} \{S \text{ monochromatic}\}\right)$$

$$\leq \binom{n}{k} \mathbb{P}([k] \text{ monochromatic})$$

$$= \binom{n}{k} 2^{-\binom{k}{2}+1}$$

$$\leq 2 \left(\frac{en}{k}\right)^k 2^{-\frac{k(k-1)}{2}}$$

$$= 2 \left(\frac{en}{k} 2^{-\frac{k-1}{2}}\right)^k$$

$$\leq 2 \left(1 - \frac{1}{k}\right)^k$$

$$< 1$$

Hence

$$2^{\frac{k}{2}} \leq R(k) \leq 4^k$$

$\square$

This proof is remarkable by the fact that it proves that the probability of some object existing is high, without actually constructing such an object. In fact it is still an important open problem to explicitly construct a K_k-free edge-coloring of K_n with n exponential in k. In other words, *even though K_k-free edge-colorings are abundant, we don't know how to write down a single one.*

Remark. The use of "constructive" here is quite different to that in other areas of mathematics. We do not mean that the proof requires the Law of Excluded Middle or the Axiom of Choice, nor that we do not provide an algorithm to find a graph without monochromatic K_k.

Since there are only finitely many red/blue edge-colorings of K_n for a fixed n, there trivially is an algorithm to find such a coloring: enumerate them all and try them one by one. Less obviously, there is a procedure to systematically remove any use of the axiom of choice from the proofs of most of the results in this course. Excluded Middle is also redundant since the case splits we consider can be decided in finite time (again, everything is finite).

A more careful definition of "constructive" here is about complexity of the description of the object: Erdős' lower bound does not provide any better *deterministic* algorithm than "Try all edge-colorings", and this has complexity $\Omega\left(2^{\binom{n}{2}}\right)$ (without even accounting for the time it takes to check whether a coloring contains a monochromatic K_k). In contrast, we would expect a constructive lower bound to yield an edge-coloring in a polynomial number of operations in n.

Question. What's the base of the exponent here? Is there even such a base?

Lecture 2

We know

$$R(3, k) \leq \binom{k+1}{2} \leq (k+1)^2$$

Definition. An **independent set** in a graph is a set of vertices that does not contain an edge. The **independence number** $\alpha(G)$ is the maximum size of an independent set of G.

Definition (Binomial Random Graph)**.** For $n \in \mathbb{N}, 0 \leq p \leq 1$, we define $G(n,p)$ the probability space of graphs where each edge is independently present with probability p.

Theorem (Erdős)**.**

$$R(3,k) \geq c \left(\frac{k}{\log k} \right)^{\frac{3}{2}}$$

for some constant $c > 0$.

Proof. Change the language. Discuss the blue graph. We are now looking for the maximum number of edges of a graph with no triangles and no independent set of size k.

Take $n = \left(\frac{k}{\log k} \right)^{\frac{3}{2}}, p = n^{-\frac{2}{3}} = \frac{\log k}{k}$. Now sample $G \sim G(n,p)$ and define $\tilde{G}$ to be G with one vertex removed from each triangle and independent set of size k. By construction, $K_3 \nsubseteq \tilde{G}$ and $\alpha(\tilde{G}) < k$. We will show $\mathbb{E}[|\tilde{G}|] \geq \frac{n}{2}$ using

$$\left| \tilde{G} \right| \geq n - \#\text{triangles in } G - \#\text{independent sets of size } k \text{ in } G$$

First,

$$\mathbb{E}[\#\text{triangles in } G] = \sum_{T \in [n]^{(3)}} \mathbb{P}(T \text{ triangle in } G)$$
$$= \binom{n}{3} p^3 \leq \frac{(np)^3}{6} = \frac{n}{6}$$

Second,

$$\mathbb{E}[\#\text{independent sets of size } k \text{ in } G] = \binom{n}{k} (1-p)^{\binom{k}{2}}$$
$$\leq \left(\frac{en}{k} \right)^k e^{-p\binom{k}{2}}$$
$$\sim \left(\frac{en}{k} e^{-\frac{pk}{2}} \right)^k$$
$$= \left(\frac{ek^{\frac{3}{2}}}{k \log^{\frac{3}{2}} k} e^{-\frac{\log k}{2}} \right)^k$$
$$= \left(\frac{e}{\log^{\frac{3}{2}} k} \right)^k \longrightarrow 0$$

Hence, for large enough k,

$$\mathbb{E}[|\tilde{G}|] \geq n - \frac{n}{6} - 1 \geq \frac{n}{2} = \frac{1}{2} \left(\frac{k}{\log k} \right)^{\frac{3}{2}}$$

By adjusting $c > 0$, we have proved the theorem. $\qquad \square$

We are being wasteful here. Why throw an entire vertex away when we could get away with removing a single edge? Because we might accidentally create an independent set of size k. But we can be smarter...

Idea. Take a maximal collection of edge-disjoint triangles in $G \sim G(n,p)$ and remove all edges from these triangles.

Theorem (Erdős)**.**

$$R(3,k) \geq c \left(\frac{k}{\log k} \right)^2$$

for some constant $c > 0$.

Lemma. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of events in a probability space. Let $\mathcal{E}_t$ be the even that t *independent* events from $\mathcal{F}$ occur. then

$$\mathbb{P}(\mathcal{E}_t) \leq \frac{1}{t!} \left(\sum_{i=1}^{m} \mathbb{P}(A_i) \right)^t$$

Proof. Note that

$$1_{\mathcal{E}_t} \leq \frac{1}{t!} \sum_{\substack{i \in [m]^t \\ A_{i_1}, \ldots, A_{i_t} \text{ independent}}} 1_{A_{i_1}} \ldots 1_{A_{i_t}}$$

So

$$\mathbb{P}(\mathcal{E}_t) \leq \frac{1}{t!} \sum_{\substack{i \in [m]^t \\ A_{i_1}, \ldots, A_{i_t} \text{ independent}}} \mathbb{P}(A_{i_1}) \ldots \mathbb{P}(A_{i_t})$$

$$\leq \frac{1}{t!} \sum_{i \in [m]^t} \mathbb{P}(A_{i_1}) \ldots \mathbb{P}(A_{i_t})$$

$$= \frac{1}{t!} \left(\sum_{i=1}^{m} \mathbb{P}(A_i) \right)^t$$

$\square$

Lecture 3

Lemma. Let $n, k \in \mathbb{N}, p \in [0,1]$ be such that $pk \geq 16 \log n$. Then with high probability every subset of size k of $G \sim G(n,p)$ contains at least $\frac{pk^2}{8}$ edges.

Proof of Erdős' bound. We fix $n = \left(\frac{c_1 k}{\log k} \right)^2, p = c_2 n^{-\frac{1}{2}} = \frac{c_2}{c_1} \frac{\log k}{k}$. Let $G \sim G(n,p), \mathcal{T}$ a maximal collection of edge-disjoint triangles in G, $\tilde{G}$ be G with all edges of $\mathcal{T}$ removed. Note, $\tilde{G}$ contains no triangle. We show

$$\mathbb{P}(\alpha(\tilde{G}) \geq k) < 1$$

Let Q be the event that every set of k vertices of G contains $\geq \frac{pk^2}{8}$ edges. Setting $\frac{c_2}{c_1} = 48$, we get

$$pk = \frac{c_2}{c_1} \frac{\log k}{k} k = 48 \log k > 16 \log n$$

By the lemma, we know so that $\mathbb{P}(Q) = 1 + o(1)$ by the lemma. Now note that

$$\mathbb{P}(\alpha(\tilde{G}) \geq k) \leq \mathbb{P}(\alpha(\tilde{G}) \geq k, Q) + \underbrace{\mathbb{P}(Q^c)}_{0}$$

So we focus on $\mathbb{P}(\alpha(\tilde{G}) \geq k, Q)$. Set $t = \frac{pk^2}{24}$.

$$\mathbb{P}(\alpha(\tilde{G}) \geq k, Q) \leq \mathbb{P}\left(\exists S \in [n]^{(k)}, \mathcal{T} \text{ meets } S \text{ in } \geq \frac{pk^2}{8}\right)$$

$$\leq \binom{n}{k} \mathbb{P}\left(\underbrace{\underset{\text{meet } [k] \text{ in at least two vertices}}{\text{at least } t \text{ triangles of } \mathcal{T}}}_{B}\right)$$

Let $\{T_i\}$ be the collection of triangles in K_n that meet $[k]$ in at least two vertices. Let $A_i = \{T_i \subseteq G\}$. Note that if $T_{i_1}, \ldots, T_{i_k}$ are edge-disjoint, then $A_{i_1}, \ldots, A_{i_k}$ are independent. So

$$\mathbb{P}(B) \leq \mathbb{P}(\mathcal{E}_t)$$

$$\leq \frac{1}{t!}\left(\sum_{\substack{T_i \subseteq K_n \text{ intersects } [k] \\ \text{in at least two vertices}}} \mathbb{P}(T_i \subseteq G)\right)^t$$

$$\leq \frac{1}{t!}(k^2 n p^3)^t$$

$$\leq \left(\frac{ek^2 n p^3}{t}\right)^t$$

$$= (24 e n p^2)^t = (24 e c_2^2)^t = e^{-t}$$

by choosing $c_2 = \frac{1}{\sqrt{24e}}$. To finish, observe that

$$t = \frac{pk^2}{24} = 2k \log k \geq k \log n$$

Hence

$$\binom{n}{k}\mathbb{P}(B) \leq \binom{n}{k} e^{-t} \leq \left(\frac{en}{k} e^{-\log n}\right)^k = \left(\frac{e}{k}\right)^k \to 0$$

$\square$

1.2 Large deviation inequalities

Let Z be a gaussian random variable.

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq t) \leq e^{\frac{-t}{2 \text{Var } Z}}$$

Let $X_1, \ldots, X_n$ be iid Bernoulli random variables. We denote this $X_i \sim \text{Ber}(p)$. Write $S_n = X_1 + \cdots + X_n$. Note $\mathbb{E}[S_n] = np$, $\text{Var}(S_n) = np(1-p)$.

Idea. Often, the tail of S_n looks like a gaussian tail.

Theorem (Chernoff inequality)**.** Let $X_1, \ldots, X_n \sim \text{Ber}(p)$. Then

$$\mathbb{P}\left(|S_n - pn| \geq t\right) \leq 2 \exp\left(\underbrace{-\frac{t}{2pn}}_{\text{meat}} + \underbrace{\frac{t^3}{(pn)^2}}_{\text{error term}}\right)$$

Lecture 4

Proof of the $\frac{pk^8}{8}$ lemma.

$$\mathbb{P}(G \text{ fails the statement}) = \mathbb{P}\left(\exists S \in [n]^{(k)}, e(G[s]) < \frac{pk^2}{8}\right)$$

$$\leq \binom{n}{k}\mathbb{P}\left(e(G[[k]]) < \frac{pk^2}{8}\right)$$

$$\leq \binom{n}{k}\mathbb{P}\left(\frac{pk^2}{4} < \left|e(G[[k]]) - p\binom{k}{2}\right|\right)$$

We are now done by Chernoff. TODO: Are we really? The numbers don't seem to work out. $\square$

1.3 The Local Lemma

The probabilistic method is like finding the hay in the hay stack. What if we want to find the needle?

Definition. Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a family of events in a probability space. A **dependency graph** Γ is a graph with vertices $\mathcal{F}$ such that the event A_i is independent of $\sigma(A_j \mid j \not\sim i)$ for all $i \in [n]$.

Remarks.

- A dependency graph is not unique.

- The complete graph is always a dependency graph.

- The empty graph is a dependency graph iff the A_i are globally independent.

Theorem (The Local Lemma, symmetric version)**.** Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a family of events in a probability space, let Γ be a dependency graph for $\mathcal{F}$ with maximum degree Δ. If $\mathbb{P}(A_i) \leq \frac{1}{e(\Delta+1)}$ for all i, then

$$\mathbb{P}\left(\bigcap_i A_i^c\right) > 0$$

Theorem (Spencer)**.**

$$R(k) \geq (1 + o(1))\frac{\sqrt{2}k}{e}2^{\frac{k}{2}}$$

Proof. Let $n = (1 - \varepsilon)\frac{\sqrt{2}k}{e}2^{\frac{k}{2}}$ for some $\varepsilon > 0$. Let χ be a random edge-coloring of K_n uniformly over all colorings. Define, for $S \in [n]^{(k)}$, the event

$$A_S = \{S \text{ is monochromatic in } \mathbb{Z}\}$$

Note we want $\mathbb{P}\left(\bigcap_{S \in [n]^{(k)}} A_S^c\right) > 0$. Define the dependency graph Γ by

$$S \sim T \iff 1 < |S \cap T| < k$$

The maximum degree of Γ Is

$$\Delta = \sum_{t=2}^{k-1}\binom{k}{t}\binom{n-k}{k-t} = \binom{n}{k} - k\binom{n-k}{k-1} - \binom{n-k}{k} - 1$$

To apply the Local Lemma, we just check

$$\mathbb{P}(A_S) = 2^{-\frac{k}{2}+1} \leq \frac{1}{e(\Delta+1)}$$

$\square$

Theorem (Lopsided Local Lemma). Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a family of events on a probability space, Γ a dependency graph for $\mathcal{F}$, $0 \leq x_1, \ldots, x_n < 1$ satisfying

$$\mathbb{P}(A_i) \leq x_i \prod_{j \sim i}(1 - x_j)$$

Then

$$\mathbb{P}\left(\bigcap_i A_i^c\right) \geq \prod_i (1 - x_i) > 0$$

Theorem (Erdős, 1961).
$$R(3,k) \geq c\left(\frac{k}{\log k}\right)^2$$

for some $c > 0$.

Proof. Let $n = \varepsilon^4 \left(\frac{k}{\log k}\right)^2, p = \frac{\varepsilon}{\sqrt{n}} = \frac{\log k}{\varepsilon k}, G \sim G(n,p)$. For all $T \in [n]^{(3)}$ and $I \in [n]^{(k)}$, define $A_T = \{T \subseteq G\}$ and $B_I = \{I \text{ independent in } B\}$. We want

$$\mathbb{P}\left(\bigcap_{T \in [n]^{(3)}} A_T^c \cap \bigcap_{I \in [n]^{(k)}} B_I^c\right) > 0$$

Let $\mathcal{A} = \{A_T\}, \mathcal{B} = \{B_I\}$. Define Γ so that

$$T \sim T' \iff |T \cap T'| = 2$$
$$T \sim I \iff 2 \leq |I \cap T|$$
$$I \sim I' \iff 2 \leq |I \cap I'| < k$$

A_T has at most $3n$ neighbors in $\mathcal{A}$ and $3n^{k-2}$ neighbors in $\mathcal{B}$. B_I has at most $k^2 n$ neighbors in $\mathcal{A}$ and $k^2 n^{k-2}$ neighbors in $\mathcal{B}$. We now define $x_T = 3p^3, x_I = n^{-k}$. We have check

1. $p^3 = \mathbb{P}(A_T) \leq x_T \prod_{|T \cap T'|=2}(1 - x_T) \prod_{|I \cap T| \geq 2}(1 - x_I) = 3p^3(1 - 3p^3)^{3n}(1 - n^{-k})^{3n^{k-2}}$ Indeed,

$$(1 - 3p^3)^{3n}(1 - n^{-k})^{3n^{k-2}} \geq \exp(-18p^3 n - 6n^{-2}) = \exp(-18\varepsilon^2 p - 6n^{-2}) \to 1$$

Lecture 5

2. $(1 - p)^{\binom{k}{2}} = \mathbb{P}(B_I) \leq x_I \prod_{|T \cap I| \geq 2}(1 - x_T) \prod_{2 \leq |I \cap I'| < k}(1 - x_{I'}) = n^{-k}(1 - n^{-k})^{k^2 k^{k-2}}(1 - 3p^3)^{k^2 n}$

$$
\begin{aligned}
n^{-k}(1 - n^{-k})^{k^2 n^{k-2}}(1 - 3p^3)^{k^2 n} &\geq \exp(-k \log n - 2k^2 n^{-2} - 6k^2 p^3 n) \\
&\geq \exp(-2k \log k - 6k^2 p\varepsilon^2 + o(1)) \\
&\geq \exp\left(-\frac{k \log k}{4\varepsilon} + o(1)\right) \\
&\geq \exp\left(-p\binom{k}{2}\right) \\
&\geq (1 - p)^{\binom{k}{2}}
\end{aligned}
$$

$\square$

Proof of the Local Lemma. Applying $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B \mid A)$ repeatedly, write

$$
\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) &= \prod_{i=1}^{n} \mathbb{P}(A_i^c \mid A_1^c \cap \ldots \cap A_{i-1}^c) \\
&= \prod_{i=1}^{n}(1 - \mathbb{P}(A_i \mid A_1^c \cap \ldots \cap A_{i-1}^c))
\end{aligned}
$$

It is enough to show that $\mathbb{P}(A_i \mid A_1^c \cap \ldots \cap A_{i-1}^c) \leq x_i$. We prove

$$
\mathbb{P}(A_i \mid \bigcap_{j \in S} A_j^c)
$$

for all $S \subseteq [n]$ by induction:

- $S = \varnothing$. Done by assumption.

- Write

$$
I = \bigcap_{\substack{j \in S \\ j \not\sim i}} A_j^c, \quad D = \bigcap_{\substack{j \in S \\ j \sim i}} A_j^c
$$

So

$$
\begin{aligned}
\mathbb{P}(A_i \mid I \cap D) &= \frac{\mathbb{P}(A_i \cap I \cap D)}{\mathbb{P}(I \cap D)} \\
&\leq \frac{\mathbb{P}(A_i \cap I)}{\mathbb{P}(I \cap D)} \\
&= \frac{\mathbb{P}(A_i)\mathbb{P}(I)}{\mathbb{P}(I \cap D)} \\
&= \frac{\mathbb{P}(A_i)}{\mathbb{P}(D \mid I)}
\end{aligned}
$$

Now, write $D = A_{i_1}^c \cap \ldots \cap A_{i_m}^c$ and

$$
\begin{aligned}
\mathbb{P}(D \mid I) &= \prod_{j=1}^{m}(1 - \mathbb{P}(A_{i_j} \mid I \cap A_{i_1}^c \cap \ldots \cap A_{i_{j-1}}^c)) \\
&\geq \prod_{j=1}^{m}(1 - x_{i_j}) \\
&\geq \prod_{j \sim i}(1 - x_j)
\end{aligned}
$$

$\square$

Theorem (Lovasz Local Lemma, symmetric version)**.** For $\mathcal{F} = \{A_1, \ldots, A_n\}$, Γ a dependency graph with maximum degree Δ, if $\mathbb{P}(A_i) \leq \frac{1}{\Delta+1}$ for all i, then

$$\mathbb{P}\left(\bigcap_i A_i^c\right) \geq \left(1 - \frac{1}{e(\Delta+1)}\right)^n > 0$$

Proof. We use the Lopsided Local Lemma with $x_i = \frac{1}{e(\Delta+1)}$. Note

$$x_i \prod_{j \sim i} = \frac{1}{\Delta+1}\left(1 - \frac{1}{\Delta+1}\right)^\Delta \geq \frac{1}{e(\Delta+1)} \geq \mathbb{P}(A_i)$$

So Lopsided Local Lemma applies. $\square$

We now know

$$\frac{ck^2}{(\log k)^2} \leq R(3, k) \leq (k+1)^2$$

Theorem (State of the art on $R(3, k)$)**.**

$$\underbrace{\left(\frac{1}{4} + o(1)\right)\frac{k^2}{\log k}}_{\substack{\text{Fiz Pontiveros} \\ \text{Griffiths} \\ \text{Morris}} + \text{Bohman} \atop \text{Keevash}} \leq R(3, k) \leq \underbrace{(1 + o(1))\frac{k^2}{\log k}}_{\substack{\text{Ajtai} \\ \text{Komlós} \\ \text{Szemerédi}} + \text{Shearer}}$$

1.4 Upper bounds on $R(3, k)$

Theorem (Ajtai, Komlós, Szemerédi)**.**

$$R(3, k) \leq c \frac{k^2}{\log k}$$

for some $c > 0$. In fact, we will see $c = 1 + o(1)$.

Theorem (Ajtai, Komlós, Szemerédi)**.** Let G be a triangle-free graph on n vertices with maximum degree Δ. Then

$$\alpha(G) \geq c \frac{n}{\Delta} \log \Delta$$

for some absolute constant $c > 0$.

Remark. For general graphs, we know $\alpha(G) \geq \frac{n}{\chi(G)} \geq \frac{n}{d+1}$ by the naïve greedy algorithm and this is basically best possible. The extra $\log d$ factor will come from tracking how sparse our graph is becoming as we remove vertices from it.

We apply a random greedy algorithm to prove the following theorem due to Shearer.

Define

$$f(x) = \frac{x \log x - x + 1}{(x-1)^2}$$

extending continuously to $[0, 1]$ by $f(0) = 1$, $f(1) = \frac{1}{2}$. We remark that

- f is continuous and differentiable.
- f is antitone and convex.
- $0 < f(x) < 1$
- $(x+1)f(x) = 1 + (x - x^2)f(x)$

Theorem (Shearer)**.** Let G be a triangle-free graph on n vertices with average degree $d > 0$. Then

$$\alpha(G) \geq nf(d) \geq (1 + o(1)) \frac{n}{\Delta} \log \Delta$$

Proof. Induction on n.

Let x a vertex to be chosen later and $G' := G - x - N(x)$. Writing d' the average degree of G' and applying induction to G', we get

$$\begin{aligned}
\alpha(G) &\geq 1 + \alpha(G') \\
&\geq 1 + (n - \deg x - 1)f(d') \\
&\geq 1 + (n - \deg x - 1)(f(d) + (d - d')f'(d)) \\
&= nf(d) + 1 - (\deg x + 1)f(d) + (d \deg x + d + n(d - d') - d' \deg x - d')f'(d) \\
&= nf(d) + 1 - (\deg x + 1)f(d) + \left(d \deg x + d - 2 \sum_{y \sim x} \deg y \right) f'(d)
\end{aligned}$$

where we used that $N(x)$ is an independent set to get that

$$n(d - d') = n\left(\frac{2e(G)}{n} - \frac{2e(G')}{n - \deg x - 1}\right)$$
$$= 2(e(G) - e(G')) - (\deg x + 1)d'$$
$$= 2\sum_{y \sim x} \deg y - (\deg x + 1)d'$$

So we want to choose x such that

$$(\deg x + 1)f(d) \leq 1 + (d\deg x + d - 2\sum_{y \sim x} \deg y)f'(d)$$

We average over x:

$$\mathbb{E}_x \text{ LHS} = (d+1)f(d)$$
$$\mathbb{E}_x \text{ RHS} = 1 + (d^2 + d - 2\mathbb{E}_x\sum_{y \sim x} \deg y)f'(d)$$

Notice that

$$\mathbb{E}_x \sum_{y \sim x} \deg y = \frac{1}{n}\sum x \sum_{y \sim x} \deg y = \frac{1}{n}\sum_x \deg x^2 \geq \left(\frac{1}{n}\sum_x \deg x\right)^2$$

Since $f'(d) \leq 0$, we get

$$\mathbb{E}_x \text{ RHS} \geq 1 + (d^2 + d - 2d^2)f'(d) = 1 + (d - d^2)f'(d) = (d+1)f(d) = \mathbb{E}_x \text{ LHS}$$

So such x exists. $\qquad\square$

Proof of the AKS bound using Shearer. Let G be a graph on n vertices with neither a triangle nor an independent set of size k. By triangle freeness, the maximum degree Δ is strictly less than k. By AKS, there is an independent set of size $(1 + o(1))\frac{n}{k}\log k$. So

$$(1 + o(1))\frac{n}{k}\log k \leq \alpha(G) < k$$

So $n \leq (1 + o(1))\frac{k^2}{\log k}$ $\qquad\square$

Lecture 7