Part III – Ramsey Theory on Graphs (Incomplete)

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0 Introduction

Lecture 1

Notation. We write

- $[n] = \{1, \ldots, n\}$
- K_n for the complete graph on n vertices.
- For X a set, $r \in \mathbb{N}$, $X^{(r)} = \{S \subseteq X | |S| = r\}$
- χ for a k-coloring of the edges of K_n

$$\begin{aligned} &\chi : E(K_n) \to [k] \\ &\chi : E(K_n) \to \{\text{red}, \text{blue}\} && \text{(if } k = 2\text{)} \end{aligned}$$

Ramsey theory is usually concerned with the following question:

Can we find some order in enough disorder?

In this course, we will specialise this question to graphs. We are thus interested in the following:

What can we say about the structure of an arbitrary 2-coloring of the edges of K_n?

Definition. Define the **Ramsey number** $R(\ell, k)$ to be the least n for which every 2-edge coloring contains either a blue K_ℓ or a red K_k, and the **diagonal Ramsey number** $R(k) = R(k, k)$ to be the least n for which every 2-edge coloring contains a monochromatic K_k.

It is unclear that such a n even exists! We shall prove it in due course.
$R(\ell, k) = R(k, \ell)$. By convention, we will usually assume $\ell \leq k$.

Example. $R(3) = 6$ because

- The following coloring shows that $R(3) > 5$. TODO: add picture
- If we have 6 vertices, we can pick a vertex v. By pigeonhole, three of the neighbors of v are connected to v via the same color, say red. Now either two of those neighbors are connected with a red edge, in which case they form a red triangle with v, or they are connected with blue edges to each other, in which case they form a blue triangle. As a way to remember this proof, we encourage you to watch the following music video: Everybody's looking for Ramsey

1 Old bounds on $R(k)$

Theorem (Erdős-Szekeres, 1935)**.**

$$R(\ell, k) \leq \binom{k + \ell - 1}{\ell - 1}$$

In particular, $R(\ell, k)$ is well-defined.

Lemma. For all $k, \ell \geq 3$,

$$R(\ell, k) \leq \underbrace{R(\ell - 1, k)}\, a + \underbrace{R(\ell, k - 1)}\, b$$

Proof. Let $n = a + b$. Pick a vertex v. By pigeonhole, either

- v has at least a red neighbors. Either these neighbors contain a red $K_{\ell-1}$ (in which case we chuck v in), or contain a blue K_k (in which case we already won).
- v has at least b blue neighbors. Either these neighbors contain a blue K_{k-1} (in which case we chuck v in), or contain a red K_ℓ (in which case we already won).

□

Proof of Erdős-Szekeres. Use that $R(\ell, 2) = \ell$ and induct on k and ℓ. □

Corollary.

$$R(k) \leq \binom{2k}{k} \leq C\frac{4^k}{\sqrt{k}}$$

for some constant C.

1.1 Lower bounds

Can we find edge colorings on many vertices without a monochromatic K_k? Certainly, we can at least do so on $(k-1)^2$ vertices.

TODO: Insert figure

This polynomial lower bound is eons away from our exponential upper bound. For quite some time (in the 1930s), people thought that the lower bound was closer to the truth than the upper bound. Surprisingly, it is possible to show an exponential lower bound without actually exhibiting such a coloring!

Theorem (Erdős, 1948)**.**

$$R(k) \geq (1 + o(1))\frac{2^{\frac{k}{2}}}{e\sqrt{2}}$$

Fact.

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

Proof. Let $n = \left\lceil \frac{1-\varepsilon}{e} 2^{\frac{k-1}{2}} \right\rceil$ and χ be a random red/blue edge coloring of K_n (each edge is independently colored red or blue with probability $\frac{1}{2}$). We see that

$$\begin{aligned}
\mathbb{P}(\chi \text{ contains a monochromatic } K_k) &= \mathbb{P}\left(\bigcup_{S \in [n]^{(k)}} \{S \text{ monochromatic}\}\right) \\
&\leq \binom{n}{k}\mathbb{P}([k] \text{ monochromatic}) \\
&= \binom{n}{k} 2^{-\binom{k}{2}+1} \\
&\leq 2\left(\frac{en}{k}\right)^k 2^{-\frac{k(k-1)}{2}} \\
&= 2\left(\frac{en}{k} 2^{-\frac{k-1}{2}}\right)^k \\
&\leq 2(1-\varepsilon)^k \\
&\to 0
\end{aligned}$$

Hence

$$2^{\frac{k}{2}} \leq R(k) \leq 4^k$$

□

This proof is remarkable by the fact that it proves that the probability of some object existing is 1, without actually constructing such an object. In fact it is still an important open problem to explicitly construct a K_k-free edge-coloring of K_n with n exponential in k. In other words, *even though K_k-free edge-colorings are abundant, we don't know how to write down a single one.*

Remark. The use of "constructive" here is quite different to that in other areas of mathematics. We do not mean that the proof requires the Law of Excluded Middle or the Axiom of Choice, nor that we do not provide an algorithm to find a graph without monochromatic K_k.
Since there are only finitely many red/blue edge-colorings of K_n for a fixed n, there trivially is an algorithm to find such a coloring: enumerate them all and try them one by one. Less obviously, there is a procedure to systematically remove any use of the axiom of choice from the proofs of most of the results in this course. Excluded Middle is also redundant since the case splits we consider can be decided in finite time (again, everything is finite).
A more careful definition of "constructive" here is about complexity of the description of the object: Erdős' lower bound does not provide any better algorithm than "Try all edge-colorings", and this has complexity $O\left(2^{\binom{n}{2}}\right)$. In contrast, we would expect a constructive lower bound to yield an edge-coloring in a polynomial number of operations in n.

Question. What's the base of the exponent here? Is there such a base?

Lecture 2