# Part III – Combinatorics (Incomplete)

Based on lectures by Prof Béla Bollobás
Notes taken by Yaël Dillies

Michaelmas 2023

## Contents

# 0 Introduction

For a finite set $A$, we write its cardinality $|A|$.

For a graph $G = (V, E)$ and $A, B \subseteq V$, we denote $\Gamma(A) = \{b | \exists a \in A, a \sim b\}$ the set of neighbors of $A$ and $e(A, B)$ the number of edges between $A$ and $B$.

# 1 Basic Results

## 1.1 Chains, Antichains and Scattered Sets of Vectors

*Lecture 1*

During WW2, Littlewood and Offord were interested in roots of polynomials with random coefficients. They came up with the following neat theorem.

**Theorem** (Littlewood-Offord, 1943)**.** If $z_1, \dots, z_n \in \mathbb{C}$ with $|z_i| \geq 1$, then, for any disk $D$ of radius $r$,

$$\#\{\varepsilon \in \{-1,1\}^n | \sum_i \varepsilon_i z_i \in D\} \leq c \log n \frac{2^n}{\sqrt{n}}$$

for some constant $c$ depending only on $r$.

Upon seeing this theorem, Erdős immediately knew he could drastically improve the bound if the $z_i$ were real.

**Theorem** (Erdős, 1945)**.** If $x_1, \dots, x_n \in \mathbb{R}$, $|x_i| \geq 1$, then, for any interval $I$ of length 2,

$$\#\{\varepsilon \in \{-1,1\}^n | \sum_i \varepsilon_i z_i \in I\} \leq \binom{n}{\frac{n}{2}}$$

This is best possible, as we see by taking $x_1 = \dots = x_n = 1$.

Let $G$ be a bipartite graph with parts $U$ and $W$. A **complete matching** from $U$ to $W$ is an injective function $f : U \to W$ such that $\forall u \in U, u \sim f(u)$.
If $G$ has a complete matching, then certainly $|A| \leq |\Gamma(A)|$. Surprisingly, this is enough.

**Theorem** (Kőnig-Egerváry-Hall Theorem, Hall's Marriage Theorem)**.**

$$G \text{ has a complete matching} \iff \forall A \subseteq U, |A| \leq |\Gamma(A)|$$

*Proof.* Exercise □

Let $\mathcal{F} = (F_1, \dots, F_m)$ where the $F_i$ are finite sets. We say $a_1, \dots, a_m$ is a **set of distinct representatives**, aka **SDR** if they are distinct and $\forall i, a_i \in F_i$. Certainly, if $\mathcal{F}$ has SDR, then $|I| \leq \left|\bigcup_{i \in I} F_i\right|$ for all $I \subseteq [m]$.

**Theorem.**

$$\mathcal{F} \text{ is a SDR} \iff \forall I \subseteq [m], |I| \leq \left|\bigcup_{i \in I} F_i\right|$$

*Proof.* Define a bipartite graph $G$ with parts $[m]$ and $\bigcup_i F_i$ by $i \sim a \iff a \in F_i$. For all $I \subseteq [m]$, $|I| \leq \left|\bigcup_{i \in I} F_i\right| = |\Gamma(I)|$, so Theorem 1.1 applies. □

**Theorem.** If $G$ is a bipartite graph with parts $U$, $W$ such that $\deg(u) \geq \deg(w)$ for all $u \in U, w \in W$, then there is a complete matching from $U$ to $W$.

*Proof.* Find $d$ such that $\deg(u) \geq d \geq \deg(w)$ for all $u \in U, w \in W$. For all $A \subseteq U$, we have

$$d\,|A| \leq e(A, \Gamma(A)) \leq d\,|\Gamma(A)|$$

Hence $|A| \leq |\Gamma(A)|$. We're done by Theorem 1.1. □

For $A \subseteq U, B \subseteq W$, define $w(A) = \frac{|A|}{|U|}, w(B) = \frac{|B|}{|W|}$.

Say a bipartite graph $G$ with parts $U, W$ is $(k, \ell)$**-biregular** if $\deg(u) = k, \deg(w) = \ell$ for all $u \in U, w \in W$.

**Lemma.** If $G$ is biregular with parts $U, W$ and $A \subseteq U$, then $w(A) \leq w(\Gamma(A))$.

*Proof.* First, $k|U| = e(G) = \ell|W|$. Second,

$$k|A| = e(A, \Gamma(A)) \leq \ell|\Gamma(A)|$$

Dividing the inequality by the equality gives the result. □

*Lecture 2*

**Corollary.** Let $G$ be a $(k, \ell)$-biregular graph with parts $U, W$. If $k \geq \ell$ (or equivalently $|U| \leq |W|$), then there is a complete matching from $U$ to $W$.

**Corollary.** If $\left|s - \frac{n}{2}\right| \leq \left|r - \frac{n}{2}\right|$, then there exists an injection $f : X^{(r)} \hookrightarrow X^{(s)}$ such that either

- $r \leq s$ and $A \subseteq f(A)$ for all $A \in X^{(r)}$
- $s \leq r$ and $f(A) \subseteq A$ for all $A \in X^{(r)}$

**Theorem** (Sperner, 1928)**.** Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an antichain. Then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

*Proof.* A chain and an antichain can intersect in at most one element. If we manage to partition $\mathcal{P}(X)$ into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains, we win.

But we can repeatedly use Corollary 1.1 to construct matchings $X^{(0)}$ to $X^{(1)}$, $X^{(1)}$ to $X^{(2)}$, ..., $X^{(\lceil \frac{n}{2} \rceil - 1)}$ to $X^{(\lceil \frac{n}{2} \rceil)}$ and $X^{(n)}$ to $X^{(n-1)}$, $X^{(n-1)}$ to $X^{(n-2)}$, ..., $X^{(\lfloor \frac{n}{2} \rfloor + 1)}$ to $X^{(\lfloor \frac{n}{2} \rfloor)}$, then "stack" the matchings together to make chains (if an element of the middle layer). Each chain goes through $X^{(\lfloor fracn2 \rfloor)}$, so we made $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. □

We can now understand the observation of Erdős (1945) about Littlewood-Offord (1943).

**Corollary.** Let $x_1, \dots, x_n \in \mathbb{R}$ be such that $|x_i| \geq 1$. Then at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ of the sums $\sum_i \varepsilon_i x_i$, $\varepsilon_i = \pm 1$ fall into the interior of an interval $I$ of length 2.

*Proof.* WLOG $\forall i, x_i \geq 1$. Set $F_\varepsilon = \{i | \varepsilon_i = 1\}$. $\{F\varepsilon | \sum_i \varepsilon_i x_i \in I\}$ is an antichain (if $F_\varepsilon \subsetneq F_{\varepsilon'}$, then $\sum_i \varepsilon'_i x_i \geq \sum_i \varepsilon_i x_i + 2$, so both sums can't lie in $I$). □

**Definition.** A partial order $P$ is **graded** if it has a partition $P_i$ such that

- if $x < y$, $x \in P_i$, $y \in P_j$, then $i < j$ (in particular each $P_i$ is an antichain)
- if $x < y$, $x \in P_i$, $y \in P_j$, $i + 2 \leq j$, then there exists $z$ such that $x < z < y$.

For $a \in P$, we call the unique $i$ for which $a \in P_i$ the **grade** or **rank** of $a$.
A graded order is **regular** if for every $i$ there exists $p_i$ such that every $x \in P_i$ is less than exactly $p_k$ elemenets of $P_{i+1}$.
For $A \subseteq P$, define $A_i = A \cap P_i$ and $w(A) = \sum_i \frac{|A_i|}{|P_i|}$.

TODO: Insert picture

**Theorem.** Let $A$ be an antichain in a connected regular graded order $P$. Then $w(A) \leq 1$.

*Proof.* Homework. Solution is written in white below:

*Lecture 3*