Site percolation

definition
percolation

percolation-site-def

For each vertex x of a graph G, we draw an independent Bernoulli random variable $w(x)$ with probability p. We say x is an *open site* if $w(x) = 1$. An edge is *open* if both of its sites are open.

Basic results about the existence of an infinite cluster

infinite-cluster
percolation

percolation-infinite-cluster-basic

- Every site has the same probability of being in an infinite cluster. Proof: Translation invariance.
- $P_p(\text{infinite cluster}) > 0 \iff P_p(0 \leftrightarrow \infty) > 0$. Proof: Translation invariance + countability of $\mathbb{Z}^d$.
- Probability is monotone in p. Proof: Coupling.
- Equivalent to the existence of an infinite open path. Proof: Build the path inductively.

The probability of an infinite cluster is either 0 or 1.

infinite-cluster
percolation

percolation-infinite-cluster-zero-one

Existence of an infinite cluster is a tail event, so done by Kolmogorov's 0-1 law.

Prove $0 < p_c$ in $\mathbb{Z}^d$.

infinite-cluster
percolation::zd

percolation-zd-critical-probability-pos

If $p < \frac{1}{2d}$, then, for all n,

$$\begin{aligned} P_p(0 \leftrightarrow \infty) &\leq P_p(\exists \text{ path of length } n \text{ starting at } 0) \\ &\leq (2d)^n p^{n+1} \\ &\to 0 \end{aligned}$$

Prove $p_c < 1$ in $\mathbb{Z}^d, d \geq 2$.

infinite-cluster
percolation::zd

percolation-zd-critical-probability-lt-one

It's enough to show it for $d = 2$. If $p > \frac{7}{8}$, then

$$\begin{aligned}
&P_p(\text{no infinite cluster}) \\
&= P_p(\exists \text{ closed loop around } [-n,n]^2) \\
&\leq \sum_m P_p(\exists \text{ closed loop around } [-n,n]^2 \text{ through } (m,0)) \\
&\leq \sum_{m \geq n} P_p(\exists \text{ closed path of length } m) \\
&\leq \sum_{m \geq n} 8^m (1-p)^{m+1} \\
&\to 0
\end{aligned}$$

Translation-invariant percolation events have probability 0 or 1.

percolation::zd

percolation-translation-invariant

Any percolation event can be approximated by a cylindrical event (by Dynkin). Hence if A is a translation-invariant event, find B a cylindrical event such that $P_p(A \triangle B) \leq \varepsilon$. Shift the event B enough so that the resulting cylindrical event B' is independent from B. Then

$$\begin{aligned}
\left|P_p(A) - P_p(B)^2\right| &= |P_p(A) - P_p(B \cap B')| \\
&\leq P_p(A \triangle (B \cap B')) \\
&\leq P_p(A \triangle B) + P_p(A \triangle B') \\
&\leq 2\varepsilon
\end{aligned}$$

Taking $\varepsilon \to 0$, we get $P_p(A)^2 = P_p(A)$, as wanted.

The number of infinite clusters is ae constant for supercritical percolation in $\mathbb{Z}^d$.

infinite-cluster supercritical-percolation
percolation::zd

percolation-zd-supercritical-clusters-ae-constant

For each k, $N = k$ is a translation-invariant event, hence has probability 0 or 1.

The number of infinite clusters is 1 or ∞ for supercritical percolation in $\mathbb{Z}^d$.

infinite-cluster supercritical
percolation::zd

percolation-zd-supercritical-clusters-one-or-infty

If $N = k \in]1, \infty[$, there's a nonzero probability to connect two clusters, hence $P(N = k) < 1$. So $P(N = k) = 0$.

The number of infinite clusters is not ∞ for supercritical percolation in $\mathbb{Z}^d$.

infinite-cluster supercritical-percolation
percolation::zd

percolation-zd-supercritical-clusters-not-infty

Assume $P(N = \infty) = 1$.

- For all k, there exists n such that the probability of k disjoint clusters intersecting $[-n, n]^d$ is strictly positive. Proof: Union over n of these events is $N \geq k$.
- Find a box intersecting three disjoint clusters that are far enough apart. Resample that box.
- The probability of a point being a trifurcation is translation invariant and strictly positive.
- $Cm^d = \mathbb{E}[\#\text{ trifurcations in } [1, m]^d] \leq \#\partial[1, m]^d = O(m^{d-1})$. Contradiction.

Pivotals

pivotal
percolation::zd

percolation-pivotal-def

For an increasing event A, a site z is *A-pivotal* for a configuration v if $v^{z,0} \notin A$ but $v^{z,1} \in A$.

Russo's formula

pivotal
percolation::zd

percolation-russo-formula

If A is an increasing cylindrical event, then

$$\frac{dP_p(A)}{dp} = \mathbb{E}_p[\#\text{ pivotals of } A]$$

Proof. Write S the finite set of states that A depends on. Couple percolations by $w_p(x) := 1_{X(x) \leq p}$ where $X(x) \sim \mathrm{Unif}[0, 1]$ are independent. This shows $p \mapsto P_p(A)$ is monotone. Hence, for $\varepsilon > 0$ (treat $\varepsilon < 0$ similarly),

$$\begin{aligned} &P_{p+\varepsilon}(A) - P_p(A) = P(w_{p+\varepsilon} \in A, w_p \notin A) \\ &= \sum_{x \in S} P(X(x) \in [p, p+\varepsilon[, x \text{ pivotal for } w_p) + O(\varepsilon^2) \\ &= \varepsilon \mathbb{E}_p[\#\text{ pivotals of } A] \end{aligned}$$

as $X(x) \in [p, p+\varepsilon[$ for several $x \in S$ with probability $O(\varepsilon^2)$. $\square$

Definitions and setup for exponential decay

infinite-cluster subcritical
percolation::zd

percolation-exponential-decay-setup

- $\mathcal{S}$ the set of finite connected sets in $\mathbb{Z}^d$ containing the origin and whose complement is connected.
- If $S \in \mathcal{S}$, $O(S)$ is the set of neighbors of S and $\tilde{S} = O(S) \cup S$.
- For $S \in \mathcal{S}$, C_S is the connected component of 0 for percolation inside S.
- $u_n(p) = P_p(0 \leftrightarrow \lambda_n)$.
- $\varphi_p(S) = \mathbb{E}[|O(S) \cap O(C_S)|]$ is the expected number of sites of S^c that are neighbors of C_S.

If $\varphi_p(S) < 1$ for some $S \in \mathcal{S}$, $P_p(0 \leftrightarrow \lambda_n)$ decays exponentially.

infinite-cluster subcritical
percolation::zd

percolation-exponential-decay-subcritical

Find n_0 such that $S \subseteq \Lambda_{n_0}$. If $0 \leftrightarrow \lambda_n$, then there is a site $x \in C_S$ adjacent to a site $y \in O(S)$ such that $y \leftrightarrow \lambda_n$ outside of $\tilde{C}_S$. Therefore, if $0 \in D \subseteq S$,

$$\begin{aligned}
&P_p(C_S = D, 0 \leftrightarrow \lambda_n)\\
&\leq \sum_{y \in O(S) \cap O(D)} P_p(C_S = D, y \leftrightarrow \lambda_n \text{ outside of } \tilde{D})\\
&= P_p(C_S = D) \sum_{y \in O(S) \cap O(D)} P_p(y \leftrightarrow \lambda_n \text{ outside of } \tilde{D})\\
&\leq P_p(C_S = D)\, |O(S) \cap O(D)|\, u_{n-n_0}
\end{aligned}$$

Summing over D, we get $u_n \leq \varphi_p(S) u_{n-n_0}$, namely exponential decay.

If $p > p_c$, then $\inf_{S \in \mathcal{S}} \varphi_p(S) > 0$.

infinite-cluster
percolation::zd

percolation-exponential-decay-supercritical

For all $S \in \mathcal{S}$,

$$\varphi_p(S) \geq P_p(0 \leftrightarrow \infty) > 0$$

If $\inf_{S \in \mathcal{S}} \varphi_{p_0}(S) > 0$, then $p_0 \geq p_c$.

infinite-cluster
percolation::zd

percolation-exponential-decay-not-subcritical

Percolate in Λ_n. Call U the set of points connected to λ_n. The expected number of closed $0 \leftrightarrow \lambda_n$-pivotals is $(1-p)\frac{du_n(p)}{dp}$ by Russo. A pivotal y is closed iff there is an open path from 0 to a neighbor of y in $S(U)$ (the component of 0 in $\tilde{U}^c$), and in particular $y \in O(S(U))$. Hence, if $p > p_0$,

$$\begin{aligned}
\frac{du_n(p)}{dp} &= \frac{1}{1-p} \sum_{V \not\ni 0} \mathbb{E}\left[1_{U=V} \varphi_p(S(V))\right]\\
&\geq \frac{\alpha}{1-p} P_p(0 \notin U) \geq \alpha
\end{aligned}$$

Integrating,

$$P_p(0 \leftrightarrow \infty) \lim_{n \to \infty} u_n(p) \geq (p - p_0)\alpha > 0$$

The probability of a left-right crossing in a rhombus is $\frac{1}{2}$.

crossing
percolation::triangular

percolation-triangular-top-bottom-left-right

Look at the set of sites connected to the top boundary. Either it reaches the bottom (and we have a top-bottom open crossing) or it doesn't (and the "lower boundary" of the set is a left-right closed crossing). Hence the probabilities of a top-bottom open crossing and of a left-right closed crossing add up to 1. But they are equal by symmetry, hence they must be $\frac{1}{2}$.

For triangular percolation, $p_c \leq \frac{1}{2}$.

At $p = \frac{1}{2}$, the probability of a left-right crossing is $\frac{1}{2}$. In particular, the probability of a point belonging to a cluster of diameter at least N is at least $\frac{1}{2(N+1)}$. Hence we do not have exponential decay and $p_c \leq \frac{1}{2}$.

Glauber dynamic

Update the configuration one state at a time. Forget a random state and pick between the two possible configurations c and d with probabilities

$$\frac{P(c)}{P(c)+P(d)}, \frac{P(d)}{P(c)+P(d)}$$

The state needn't be chosen with the same probability, but they must each have positive probability of being chosen.

The Glauber dynamic gives rise to a unique stationary measure because...

...the Markov chain is

- aperiodic
- irreducible
- reversible

Indeed it is a random walk on the space of configurations (which is connected and finite).

Harris inequality

If A, B, are two increasing cylindrical events, then $P_\beta(A \cap B) \geq P_\beta(A)P_\beta(B)$

Proof. Construct two Markov chains X_n and Y_n coupled through a Glauber dynamic such that $X_n \leq Y_n$ and Y_n is constrained to B (possible because B increasing and cylindrical). So $X_n \in A \implies Y_n \in A$ (A is increasing). This proves $P_\beta(A \mid B) \geq P_\beta(A)$. $\square$

Triangular percolation has no infinite cluster at $p = \frac{1}{2}$.

Tags
percolation::triangular

Label

Assume there is an infinite cluster with probability 1. Consider the $(2N+1) \times (2N+1)$ rhombus R_N centered at the origin and its sides L_1, E_2, L_3, L_4. Define E_i the event that $L_i \leftrightarrow \infty$. By Harris, these events are positively correlated, so

$$P(E_1^c)^4 = \prod_i P(E_i^c) \leq P(E_1^c, \dots, E_4^c) \leq P(R_N \not\leftrightarrow \infty) \to 0$$

Hence $P(E_i) \to 1$ for each i and the following happens with strictly positive probability: There are infinite open paths from L_1 and L_3 and infinite closed paths from L_2 and L_4. But in that case it is impossible to have a single infinite cluster. Contradiction.

Ising distribution

definition
ising-model

ising-distrib-def

$$\begin{aligned} P_\beta(\sigma) &= \frac{1}{Z_\beta} \exp\left(-\beta \sum_{x \sim y} 1_{\sigma_x \neq \sigma_y}\right) \\ &= \frac{1}{Z'_\beta} \exp\left(-\frac{\beta}{2} \sum_{x \sim y} \sigma_x \sigma_y\right) \end{aligned}$$

How to extend the Ising measure to an infinite graph?

ising-model

extend-ising-infinite

Consider a cobounded sequence of sets of states S_n, define P_n^+ the Ising model conditioned on the spins being $+1$ outside S_n. For every increasing cylindrical event A, $P_n^+(A)$ decreases, so it has a limit $P^+(A)$. This defines P^+ for increasing cylindrical events. Now extend by Carathéodory. Define P^- similarly.

How to couple the σ_n^+ together?

ising-model

ising-coupling-sigma-n

Pick a measure μ on $\mathbb{Z}^d$ such that $\mu\{x\} > 0$ for all x. Create a Markov chain on $\{(\sigma_0^+, \sigma_1^+, \dots) \mid \sigma_0^+ \leq \sigma_1^+ \leq \dots\}$, by starting at 1 everywhere and each time resampling x with probability $\mu\{x\}$. Each truncated Markov chain $(\sigma_0^+, \dots, \sigma_n^+)$ is irreducible, aperiodic, reversible and has a finite state space, so converges to a unique stable measure. Piece these measures together by Kolmogorov extension.