Part III – advanced Probability (Incomplete)

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Contents

0 Introduction

Lecture 1

This course is concerned with advanced topics in modern probability theory. Two examples are

Martingales
Martingales are processes indexed by discrete time such that

$$M_{n+1} = M_n + \text{ extra randomness}$$

where

$$\mathbb{E}[\text{extra randomness}|M_n] = 0$$

A typical example is Markov chains.

Brownian motion
Brownian motion is a continuous version of discrete random walks. It also arises naturally as the scaling limit of such. If $X_1, \ldots$ are iid with mean μ and variance σ^2 and set $S_n = X_1 + \cdots + X_n$, we have several theorems about

$$\frac{S_n}{n} \to \mu$$

namely the Law of Large Numbers, the Central Limit Theorem, and Large Deviation results.

If we now set $B_t^{(n)} = \frac{S_{\lfloor nt \rfloor} - \mu nt}{\sigma\sqrt{n}}$, we have that $B_t^{(n)}$ tends to Brownian motion as $n \to \infty$.

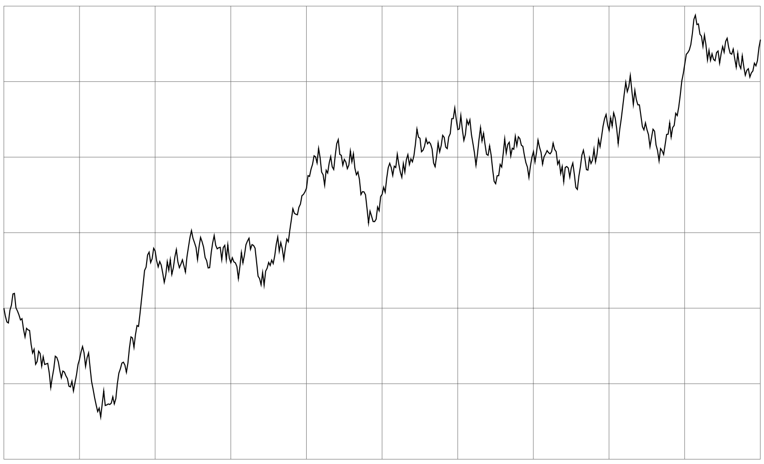

Figure 1: Standard Brownian motion

TODO: Label Gaussian in figure

Recall **Dirichlet's problem**: If $\mathcal{D} \subseteq \mathbb{C}$ is a simply connected domain and $f : \partial\mathcal{D} \to \mathbb{C}$, can we find a harmonic function $u : \mathcal{D} \to \mathbb{C}$ equal to f on $\mathcal{D}$?

Brownian motion lets us define such a u as follows:
Start a Brownian motion at $x \in \mathcal{D}$. Say it first hits the boundary of $\mathcal{D}$ in B_T. Evaluate

f at B_T.
Now take the expectation of the result and define

$$u(x) = \mathbb{E}[f(B_T)]$$

The resulting u is harmonic and clearly equals f on $\mathcal{D}$.

One can easily see that the corresponding construction in the discrete setting works by conditioning on the first move of the random walk.

TODO: Insert figure

1 Conditional Expectation

1.1 Basic measure theory recap

Definition. A collection $\mathcal{F}$ of sets in Ω is a **σ-algebra** if

- $\in \mathcal{F}$
- If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
- If $A_n \in \mathcal{F}$, then $\bigcup_n \in \mathcal{F}$

Definition. $\mathbb{P} : \mathcal{P}(\mathcal{P}(\Omega))$ is a **probability measure** if

- $\mathbb{P}() = 0$
- $\mathbb{P}(\Omega) = 1$
- When the A_n are disjoint, $\mathbb{P}\left(\bigcup_n\right) = \sum_n \mathbb{P}(A_n)$

From now on, Ω will be a set equipped with a σ-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$.

Definition. For $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, define

$$\sigma(\mathcal{A}) = \bigcap \{\mathcal{F} | \mathcal{F} \subseteq A \text{ is a } \sigma\text{-algebra}\}$$

the smallest σ-algebra containing $\mathcal{A}$, aka **σ-algebra generated by** $\mathcal{A}$.
The **Borel σ-algebra** $\mathcal{B}$ is the σ-algebra generated by the open sets in $\mathbb{R}$.

Definition. $X : \Omega \to \mathbb{R}$ is a **random variable** if X is measurable with respect to $\mathcal{B}$, namely if $X^{-1}(U) \in \mathcal{F}$ for all opens $U \subseteq \mathbb{R}$.

If the X_i, $i \in I$ are functions $\Omega \to \mathbb{R}$, we write $\sigma(X_i | i \in I)$ for $\sigma(\{X_i^{-1}(U) | i \in I, U \subseteq \mathbb{R} \text{ open}\})$, the smallest σ-algebra making all the X_i measurable.

1.2 Expectiation

Definition. A **simple function** is a function that can be written as a weighted sum of finitely many indicator functions.

Definition. For a simple function $f = \sum_i a_i 1_{A_i}$, we define

$$\mathbb{E}[f] = \sum_i a_i \mathbb{P}(A_i)$$

For a nonnegative function f, we define

$$\mathbb{E}[f] = \sup_{g \leq f \text{ simple}} \mathbb{E}[g]$$

For an arbitrary function f, write $f = f^+ - f^-$ with $f^+, f^- \geq 0$, and define

$$\mathbb{E}[f] = \mathbb{E}[f^+] - \mathbb{E}[f^-]$$

assuming at least one of $\mathbb{E}[f^+], \mathbb{E}[f^-]$ is finite.

Definition (Expectation conditional to an event)**.** For $A \in \mathcal{F}$, define

$$\mathbb{E}[X|A] = \frac{\mathbb{E}[1_A X]}{\mathbb{P}(A)}$$

Lecture 2

Lecture 3

Lecture 4

Lecture 5

Lecture 6

Lecture 7

Lecture 8

Lecture 9

Lecture 10

Lecture 11

Proposition 1.1. Let X be a continuous process and A an open set. Then

$$T_A = \inf\{t \mid X_t \in A\}$$

is a stopping time with respect to $\mathcal{F}_{t^+}$.

Proof. We need to show

$$\{T_A \leq t\} \in \mathcal{F}_{t^+}$$

But, for every s,

$$\{T_A < s\} = \bigcup_{\substack{q \in \mathbb{Q} \\ q < s}} \{X_q \in A\} \in \mathcal{F}_s$$

So

$$\{T_A \leq t\} \in \mathcal{F}_{t^+} = \bigcap_n \left\{T_A < t + \frac{1}{n}\right\} \in \bigcap_n \mathcal{F}_{t+\frac{1}{n}} = \mathcal{F}_{t^+}$$

□

Let X be a càdlàg stochastic process taking values in $\mathbb{R}^+ \to E$. Write

$$\begin{aligned} &C(\mathbb{R}^+, E) \text{ the space of continuous functions} \\ &D(\mathbb{R}^+, E) \text{ the space of càdlàg functions} \end{aligned}$$

Endow the spaces C, D with the product σ-algebra that makes all evaluations measurable. This σ-algebra is generated by the cylinder sets

$$\left\{\bigcap_{s \in J} \{f(s) \in A_s\} \mid J \text{ finite}, A_s \in B\right\}$$

For A in the product σ-algebra, we write $\mu(A) = \mathbb{P}(X \in A)$ and call μ the **law** of A. For every $J \subseteq \mathbb{R}^+$ finite, write μ_J for the law of $\prod_{r \in J} X_r$. The measures μ_J are called the **finite dimensional marginals** of X.
The μ_j completely characterise the law μ This follows from the uniqueness of extension theorem because cylinder sets generate the product σ-algebra.

Example. Let $X_t = 0$ for all t. Let $U \sim \mathcal{U}[0,1]$ and $X'_t = 1_{U=t}$ for all t. Both of them have the same finite dimensional marginals, namely Dirac measures at 0, but the two processes are not equal:

$$\mathbb{P}(\forall t, X_t = 0) = 1, \mathbb{P}(\forall t, X'_t = 0) = 0$$

However,

$$\mathbb{P}(X_t = X'_t) = 1$$

Definition. For two processes X and X', we say X **is a version of** X' if $X_t = X'_t$ ae for all t.

Definition. Say a set A is a **null set** if $\mu(A) = 0$. Write $\mathcal{N}$ for the collection of all null sets. Define $\tilde{\mathcal{F}}_t = \sigma(\mathcal{F}_{t^+}, \mathcal{N})$. If $\tilde{\mathcal{F}}_t = \mathcal{F}_t$ for all t, we say that $\mathcal{F}$ **satisfies the usual conditions**.

Theorem (Martingale regularisation theorem)**.** Let X be a martingale with respect to $\mathcal{F}_t$. There exists a càdlàg process $\tilde{X}$ such that

$$X_t \overset{\text{ae}}{=} \mathbb{E}[\tilde{X}_t \mid \mathcal{F}_t]$$

for all t and $\tilde{X}$ is a martingale with respect to $\tilde{\mathcal{F}}_t$. If $\mathcal{F}$ satisfies the usual conditions, then $\tilde{X}$ is a version of X.

Lemma 1.2. Let $f : \mathbb{Q}^+ \to \mathbb{R}$ be such that f is bounded on bounded sets and

$$\mathcal{N}(a, b, f \upharpoonright_I) < \infty$$

for all bounded I. Then the limits $\lim_{\substack{s \to t^- \\ s \in \mathbb{Q}^+}} f(s), \lim_{\substack{s \to t^+ \\ s \in \mathbb{Q}^+}} f(s)$ exist and are finite.

Proof. For any $s_n \to t^+$, $f(s_n)$ converges by the finite upcrossing proof. Further, for any other $t_n \to t^+$, we can combine the two sequences together to get that $\lim_n f(s_n) = \lim_n f(t_n)$. Since f is bounded on bounded sets, the limits are finite. □

Lecture 12

Proof of the martingale regularisation theorem.

Goal. Define $\tilde{X}_t = \lim_{\substack{s \to t^- \\ s \in \mathbb{Q}}} X_s$ on a set of measure 1 and 0 elsewhere.

Steps.

1. Show that the limit exists and is finite on a set of measure 1.
2. Show that $\tilde{X}$ is $\tilde{F}$-measurable and is finite on a set of measure 1.
3. Martingale property of $\tilde{X}$.
4. Càdlàg property of $\tilde{X}$.

Construct the set of measure 1

Let $I \subseteq \mathbb{Q}_+$ be bounded. We want $\mathbb{P}(\sup_{t \in I} |X_t| < \infty) = 1$. Write

$$\sup_{t \in I} |X_t| = \sup_{\substack{J \subseteq I \\ \text{finite}}} \sup_{t \in J} |X_t|$$

Let $J = \{j_1, \ldots, j_n\} \subseteq I, j_1 < \cdots < j_n$ and $K > \sup I$. $X \restriction_J$ is a discrete-time martingale. By Doob's maximum inequality,

$$\lambda \mathbb{P}\left(\sup_{t \in J} |X_t| \geq \lambda\right) \leq \mathbb{E}[|X_{j_n}|] \leq \mathbb{E}[|X_K|]$$

Taking the limit as $J \to I$,

$$\lambda \mathbb{P}\left(\sup_{t \in I} |X_t| \geq \lambda\right) \leq \mathbb{E}[|X_K|]$$

So

$$\mathbb{P}\left(\sup_{t \in I} |X_t| < \infty\right) = 1$$

Set $I_M = \mathbb{Q} \cap [0, M]$. Then

$$\mathbb{P}\left(\bigcap_{M \in \mathbb{N}} \left\{\sup_{t \in I_M} |X_t| < \infty\right\}\right) = 1$$

Let $a, b \in \mathbb{Q}, a < b, I \subseteq \mathbb{Q}_+$ bounded. Write

$$N(a, b, I, x) = \sup_{\substack{J \subseteq I \\ \text{finite}}} N(a, b, J, x)$$

Let $J = \{j_1, \ldots, j_n\} \subseteq I, j_1 < \cdots < j_n$ and $K > \sup I$. $X \restriction_J$ is a discrete-time martingale. By Doob's upcrossing inequality,

$$(b - a)\mathbb{E}[N(a, b, J, X)] \leq \mathbb{E}[(X_{j_n} - a)^-] \leq \mathbb{E}[(X_k - a)^-]$$

By monotone convergence, we get

$$(b - a)\mathbb{E}[N(a, b, I, X)] \leq \mathbb{E}[(X_k - a)^-]$$

Define

$$\Omega_0 = \bigcap_{M \in \mathbb{N}} \bigcap_{\substack{a, b \in \mathbb{Q} \\ a < b}} \left\{\sup_{t \in I_M} |X_t| < \infty\right\} \cap \{N(a, b, I_m, X) < \infty\}$$

On Ω_0, the lemma tells us that $\lim_{\substack{s \to t^- \\ s \in \mathbb{Q}_+}} X_s$ exists, and we have $\mathbb{P}(\Omega_0) = 1$. Define

$$\tilde{X}_t = \begin{cases} \lim_{\substack{s \to t^- \\ s \in \mathbb{Q}_+}} X_s & \text{on } \Omega_0 \\ 0 & \text{elsewhere} \end{cases}, \tilde{\mathcal{F}}_t = \sigma(\mathcal{F}_{t^+}, \mathcal{N})$$

From the definition, we see that $\tilde{X}$ is $\tilde{\mathcal{F}}$-adapted.

$X_t = \mathbb{E}[\tilde{X}_t \mid \mathcal{F}_t]$ ae

Let $t_n \to t^-, t_n \in \mathbb{Q}_+$. Then by definition $\tilde{X}_t = \lim_n X_{t_n}$ ae. X_{t_n} is a backwards martingale with respect to $\mathcal{F}_{t_n}$. So X_{t_n} converges ae and in L_1. But then $\mathbb{E}[X_{t_n} \mid \mathcal{F}_t]$ converges to both X_t and $\mathbb{E}[\tilde{X}_t \mid \mathcal{F}_t]$, so $X_t = \mathbb{E}[\tilde{X}_t \mid \mathcal{F}_t]$ ae.

$\tilde{X}$ is a martingale

Let $s < t$. We want $\mathbb{E}[\tilde{X}_t]$ ae.

Claim. For all random variables X and σ-algebra $\mathcal{G}$,

$$\mathbb{E}[X \mid \sigma(\mathcal{G}, \mathcal{N})] = \mathbb{E}[X \mid \mathcal{G}]$$

In our case, this means

$$\mathbb{E}[X_t \mid \tilde{\mathcal{F}}_s] \stackrel{\text{ae}}{=} \tilde{X}_s$$

and now the