Part III – Introduction to Additive Combinatorics (Incomplete)

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Contents

1 Fourier-analytic techniques

Lecture 1

Let $G = \mathbb{F}_p^n$ where p is a small fixed prime and n is large.

Notation. Given a finite set B and any function $f : B \to \mathbb{C}$, write

$$\mathbb{E}_{x \in B} f(x) = \frac{1}{|B|} \sum_{x \in B} f(x)$$

Write $\omega = e^{\frac{\tau i}{p}}$. Note $\sum_{a \in \mathbb{F}_p} \omega^a = 0$.

Definition 1.1. Given $f : \mathbb{F}_p^n \to \mathbb{C}$, define its **Fourier transform** $\hat{f} : \mathbb{F}_p^n \to \mathbb{C}$ by

$$\hat{f}(t) = \mathbb{E}_{x \in \mathbb{F}_p^n} f(x) \omega^{x \cdot t}$$

It is easy to verify the **inversion formula**

$$f(x) = \sum_{t \in \mathbb{F}_p^n} \hat{f}(t) \omega^{-x \cdot t}$$

Indeed,

$$\begin{aligned}
\sum_{t \in \mathbb{F}_p^n} \hat{f}(t) \omega^{-x \cdot t} &= \sum_{t \in \mathbb{F}_p^n} \left(\mathbb{E}_y f(y) \omega^{y \cdot t} \right) \omega^{-x \cdot t} \\
&= \mathbb{E}_y f(y) \sum_t \omega^{(y-x) \cdot t} \\
&= \mathbb{E}_y f(y) 1_{y=x} p^n \\
&= f(x)
\end{aligned}$$

Notation. Given a set A of a finite group G, write

- 1_A the *characteristic function* of A, ie

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

- μ_A the *characteristic measure* of A, ie

$$\mu_A = \alpha^{-1} 1_A$$

  where $\alpha = \frac{|A|}{|G|}$.

- f_A the *balanced function* of A, ie

$$f_A(x) = 1_A(x) - \alpha$$

Note $\mathbb{E}_x f_A(x) = 0, \mathbb{E}_x \mu_A(x) = 1, \widehat{1_A}(0) = \mathbb{E}_x 1_A(x) = \alpha$. Writing $-A = \{-a | a \in A\}$, we have

$$\begin{aligned}
\widehat{1_{-A}}(t) &= \mathbb{E}_x 1_{-A}(x) \omega^{x \cdot t} \\
&= \mathbb{E}_x 1_A(-x) \omega^{x \cdot t} \\
&= \mathbb{E}_x 1_A(x) \omega^{-x \cdot t} \\
&= \overline{\widehat{1_A}(t)}
\end{aligned}$$

Example 1.2. Let $V \le \mathbb{F}_p^n$. Then

$$\widehat{1_V}(t) = \mathbb{E}_x 1_V(x)\omega^{x\cdot t} = \frac{|V|}{|G|} 1_{V^\perp}(t)$$

So

$$\widehat{\mu_V}(t) = 1_{V^\perp}(t)$$

Example 1.3. Let $R \subseteq \mathbb{F}_p^n$ be such that each x is included with probability $\frac{1}{2}$ independently. Then with high probability

$$\sup_{t\neq 0}\left|\widehat{1_R}(t)\right| = O\left(\sqrt{\frac{\log(p^n)}{p^n}}\right)$$

This is on Example Sheet 1 using a **Chernoff-type bound**: Given $\mathbb{C}$-valued independent random variables $X_1, \dots, X_n$ with mean 0 and $\theta \ge 0$, we have

$$\mathbb{P}\left(\left|\sum_i X_i\right| \ge \theta \sqrt{\sum_i \|X_i\|_{L^\infty}^2}\right) \le 4\exp\left(-\frac{\theta^2}{4}\right)$$

Example 1.4. Let $Q = \{x \in \mathbb{F}_p^n \mid x \cdot x = 0\}$. Then $|Q| = \left(\frac{1}{p} + O(p^{-n})\right) p^n$ and $\sup_{t\neq 0}\left|\widehat{1_Q}(t)\right| = O(p^{-\frac{n}{2}})$. See Example Sheet 1.

Notation. Given $f, g : \mathbb{F}_p^n \to \mathbb{C}$, write

$$\langle f, g\rangle = \mathbb{E}_x f(x)\overline{g(x)}$$
$$\left\langle \hat{f}, \hat{g}\right\rangle = \sum_t \hat{f}(t)\overline{\hat{g}(t)}$$

Consequently,

$$\|f\|_2^2 = \mathbb{E}_x |f(x)|^2$$
$$\left\|\hat{f}\right\|_2^2 = \sum_t \left|\hat{f}(t)\right|^2$$

Lemma 1.5. For all $f, g : \mathbb{F}_p^n \to \mathbb{C}$,

$$\langle f, g\rangle = \left\langle \hat{f}, \hat{g}\right\rangle \qquad \text{(Plancherel)}$$

$$\|f\|_2 = \left\|\hat{f}\right\|_2 \qquad \text{(Parseval)}$$

Proof. Exercise. □

Definition 1.6. Let $\rho > 0$ and $f : \mathbb{F}_p^n \to \mathbb{C}$. Define the ρ-large spectrum of f to be

$$\mathrm{Spec}_\rho(f) = \{t \mid |\hat{f}(t)| \ge \rho \|f\|_1\}$$

Example 1.7. By Example 1.2, if $V \le \mathbb{F}_p^n$, then $\mathrm{Spec}_\rho(1_V) = V^\perp$ for all $\rho > 0$.

Lemma 1.8. For all $\rho > 0$, $\left|\mathrm{Spec}_\rho(f)\right| \le \rho^{-2}\frac{\|f\|_2^2}{\|f\|_1^2}$.

Proof.

$$\|f\|_2^2 = \left\|\hat{f}\right\|_2^2 \geq \sum_{t \in \mathrm{Spec}_\rho(f)} \left|\hat{f}(t)\right|^2 \geq \left|\mathrm{Spec}_\rho(f)\right| (\rho \|f\|_1)^2$$

□

Lecture 2

Definition 1.9. Given $f, g : \mathbb{F}_p^n \to \mathbb{C}$, define their **convolution** $f * g : \mathbb{F}_p^n \to \mathbb{C}$ by

$$(f * g)(x) = \mathbb{E}_y f(y) g(x - y)$$

Example 1.10. Given $A, B \subseteq \mathbb{F}_p^n$,

$$\begin{aligned}(1_A * 1_B)(x) &= \mathbb{E}_y 1_A(y) 1_B(x - y) \\ &= \frac{1}{p^n} |A \cap (x - B)| \\ &= \frac{\#\text{ ways to write } x = a + b, a \in A, b \in B}{p^n}\end{aligned}$$

In particular, the support of $1_A * 1_B$ is the **sum set**

$$A + B = \{a + b \mid a \in A, b \in B\}$$

Lemma 1.11. Given $f, g : \mathbb{F}_p^n \to \mathbb{C}$,

$$\widehat{f * g}(t) = \hat{f}(t)\hat{g}(t)$$

Proof.

$$\begin{aligned}\widehat{f * g}(t) &= \mathbb{E}_x \left(\mathbb{E}_y f(y) g(x - y)\right) \omega^{x \cdot t} \\ &= \mathbb{E}_y f(y) \mathbb{E}_u g(u) \omega^{(u+y) \cdot t} \\ &= \hat{f}(t)\hat{g}(t)\end{aligned}$$

□

Example 1.12. $\left\|\hat{f}\right\|_4^4 = \mathbb{E}_{x+y=z+w} f(x) f(y) \overline{f(z) f(w)}$. See Example Sheet 1.

Lemma 1.13 (Bogolyubov)**.** If $A \subseteq \mathbb{F}_p^n$ is of density $\alpha > 0$, then there exists a subspace V of codimension at most $2\alpha^{-2}$ such that $V \subseteq (A + A) - (A + A)$.

Proof. Observe that $(A + A) - (A + A) = \mathrm{supp}(\underbrace{1_A * 1_A * 1_{-A} * 1_{-A}}_{g})$, so we wish to find V such that $g(x) > 0$ for all $x \in V$. Let $K = \mathrm{Spec}_\rho(1_A)$ for some $\rho > 0$ and define $V = \langle K \rangle^\perp$. By Lemma 1.8, $\operatorname{codim} V \leq |K| \leq \rho^{-2}\alpha^{-1}$. We calculate

$$\begin{aligned}g(x) &= \sum_{t \in \mathbb{F}_p^n} 1_A * 1_A \widehat{* 1_{-A}} * 1_{-A}(t) \omega^{-x \cdot t} \\ &= \sum_{t \in \mathbb{F}_p^n} \left|\widehat{1_A}(t)\right|^4 \omega^{-x \cdot t} \\ &= \alpha^4 + \underbrace{\sum_{t \in K \setminus \{0\}} \left|\widehat{1_A}(t)\right|^4 \omega^{-x \cdot t}}_{(1)} + \underbrace{\sum_{t \notin K} \left|\widehat{1_A}(t)\right|^4 \omega^{-x \cdot t}}_{(2)}\end{aligned}$$

We now see that

$$(1) = \sum_{t \in K \setminus \{0\}} \left|\widehat{1_A}(t)\right|^4 \geq 0$$

and

$$|(2)| \leq \sum_{t \notin K} \left|\widehat{1_A}(t)\right|^4 \leq \sup_{t \notin K} \left|\widehat{1_A}(t)\right|^2 \sum_{t \notin K} \left|\widehat{1_A}(t)\right|^2 \leq (\rho\alpha)^2 \|1_A\|_2^2 = \rho^2\alpha^3$$

by Parseval. Picking $\rho = \sqrt{\frac{\alpha}{2}}$, we thus get $\rho^2\alpha^3 \leq \frac{\alpha^4}{2}$ and $g(x) > 0$ whenever $x \in V$. $\square$

Example 1.14. The set $A = \{x \in \mathbb{F}_2^n \mid |x| \geq \frac{n}{2} + \frac{\sqrt{n}}{2}\}$ has density at least $\frac{1}{4}$ but there is no coset C of any subspace of codimension $\sqrt{n}$ such that $C \subseteq A + A$. See Example Sheet 1.

Lemma 1.15. Let $A \subseteq \mathbb{F}_p^n$ of density α be such that $\operatorname{Spec}_\rho(1_A)$ contains some $t \neq 0$. Then there exist $V \leq \mathbb{F}_p^n$ of codimension 1 and $x \in \mathbb{F}_p^n$ such that

$$|A \cap (x + V)| \geq \alpha\left(1 + \frac{\rho}{2}\right)|V|$$

Proof. Let $t \neq 0$ be such that $\left|\widehat{1_A}(t)\right| \geq \rho\alpha$ and let $V = \langle t \rangle^\perp$. For $j = 1, \ldots, p$, write

$$v_j + V = \{x \in \mathbb{F}_p^n \mid x \cdot t = j\}$$

the cosets of V. Then

$$\begin{aligned}
\widehat{1_A}(t) &= \widehat{f_A}(t) \\
&= \mathbb{E}_{x \in \mathbb{F}_p^n}(1_A(x)) - \alpha)\omega^{x \cdot t} \\
&= \mathbb{E}_j \omega^j \mathbb{E}_{x \in v_j + V}(1_A(x) - \alpha) \\
&= \mathbb{E}_j a_j \omega^j
\end{aligned}$$

where $a_j = \frac{|A \cap (v_j + V)|}{|V|} - \alpha$. Since $\sum_j a_j = 0$, we get

$$\rho\alpha \leq \left|\widehat{1_A}(t)\right| \leq \mathbb{E}_j |a_j| = \mathbb{E}_j(|a_j| + a_j)$$

So there is some j such that $|a_j| + a_j \geq \rho\alpha$. In particular, this a_j is positive, so

$$\frac{|A \cap (v_j + V)|}{|V|} \geq \alpha + \frac{\rho\alpha}{2}$$

as wanted. $\square$

Lecture 3

Lemma 1.16. Let $p \geq 3$ and $A \subseteq \mathbb{F}_p^n$ of density $\alpha > 0$ be such that $\sup_{t \neq 0} \left|\widehat{1_A}(t)\right| = o(1)$. Then A contains $(\alpha^3 + o(1))|G|^2$ three terms arithmetic progressions (aka 3AP).

Notation. Given $f, g, h : \mathbb{F}_p^n \to \mathbb{C}$, write

$$T_3(f, g, h) = \mathbb{E}_x f(x) g(x + d) h(x + 2d)$$

Given $A \subseteq \mathbb{F}_p^n$, write $2 \cdot A = \{2a \mid a \in A\}$. This is distinct from $2A = \{a + b \mid a, b \in A\}$.

Proof. The number of 3AP (including the trivial ones of the form a, a, a) in A is $|G|^2$ times

$$\begin{aligned}T_3(1_A, 1_A, 1_A) &= \mathbb{E}_{x,d} 1_A(x) 1_A(x+d) 1_A(x+2d) \\ &= \mathbb{E}_{x,y} 1_A(x) 1_A(y) 1_A(2y-x) \\ &= \mathbb{E}_y (1_A * 1_A)(2y) 1_A(y) \\ &= \langle 1_A * 1_A, 1_{2\cdot A} \rangle \\ &= \left\langle \widehat{1_A}^2, \widehat{1_{2\cdot A}} \right\rangle \\ &= \alpha^3 + \sum_{t \neq 0} \widehat{1_A}(t)^2 \overline{\widehat{1_{2\cdot A}}(t)} \text{ by Plancherel}\end{aligned}$$

In absolute value, the error term is at most

$$\sup_{t \neq 0} \left| \widehat{1_{2\cdot A}}(t) \right| \sum_t \left| \widehat{1_A}(t) \right|^2 = \alpha \sup_{t \neq 0} \left| \widehat{1_A}(t) \right|$$

□

Theorem 1.17 (Meshulam)**.** Let $p \geq 3$ and $A \subseteq \mathbb{F}_p^n$ be a set containing only trivial 3APs. Then

$$|A| = O\left(\frac{p^n}{\log(p^n)}\right)$$

Proof. By assumption, $T_3(1_A, 1_A, 1_A) = \frac{\alpha}{p^n}$. But, as in Lemma 1.16,

$$\left| T_3(1_A, 1_A, 1_A) - \alpha^3 \right| \leq \alpha \sup_{t \neq 0} \left| \widehat{1_A}(t) \right|$$

Hence, provided that $2\alpha^{-2} \leq p^n$, Lemma 1.15 gives us a subspace $V \leq \mathbb{F}_p^n$ of codimension 1 and $x \in \mathbb{F}_p^n$ such that

$$|A \cap (x + V)| \geq \alpha \left(1 + \frac{\alpha^2}{4}\right) |V|$$

We iterate this observation. Let $A_0 = A, V_0 = \mathbb{F}_p^n$. At step i, we are given a set $A_i \subseteq V_i$ of density α_i with only trivial 3APs. Provided that $2\alpha_i^{-2} \leq p^{\dim V_i}$, find $V_{i+1} \leq V_i$ of codimension 1 and $x \in V_i$ such that $|A_i \cap (x + V_i)| \geq \left(\alpha_i + \frac{\alpha_i^2}{4}\right) |V_{i+1}|$ and set $A_{i+1} = (A_i - x) \cap V_i$. Note that $\alpha_{i+1} \geq \alpha_i + \frac{\alpha_i^2}{4}$ and A_{i+1} only contains trivial 3APs (because, very importantly, 3AP are **translation-invariant**).
Through this iteration, the density of A increases from α to 2α in at most $\lceil 4\alpha^{-1} \rceil$ steps, from 2α to 4α in at most $\lceil 2\alpha^{-1} \rceil$ steps, etc... Since density can't increase past 1, it takes at most

$$\underbrace{\lceil 4\alpha^{-1} \rceil + \lceil 2\alpha^{-1} \rceil + \dots}_{\lceil \log \alpha^{-1} \rceil \text{ terms}} \leq (4\alpha^{-1} + 1) + (2\alpha^{-1} + 1) + \dots \leq 8\alpha^{-1} + \log \alpha^{-1} + 1 \leq 9\alpha^{-1}$$

steps to reach a point where the condition $2\alpha_i^{-2} \leq p^{\dim V_i}$ is not respected anymore. Now either $\alpha \leq \sqrt{2} p^{-\frac{n}{4}}$ (in which case the inequality is obvious) or $\alpha \geq \sqrt{2} p^{-\frac{n}{4}}$ and

$$p^{n - 9\alpha^{-1}} \leq p^{\dim V_i} \leq 2\alpha_i^{-2} \leq 2\alpha^{-2} \leq p^{\frac{n}{2}}$$

namely $\alpha \leq \frac{18}{n}$, as wanted. □

We have proved that if $A \subseteq \mathbb{F}_3^n$ only contains trivial 3APs then $|A| = O(\frac{3^n}{n})$. The largest known set in $\mathbb{F}_3^n$ with only trivial 3APs has size $\geq 2.218^n$ (Tyrrell, 2022). We will return to this later.

From now on, let G be a finite abelian group. G comes equipped with a set of **characters**, ie group homomorphisms $\gamma : G \to \mathbb{C}^\times$. Characters themselves form a group denoted $\hat{G}$ and called the **Pontryagin dual** (aka **dual group**) of G. It turns out that if G is finite abelian then $\hat{G} \cong G$ (but *non-canonically*). For instance,

- If $G = \mathbb{F}_p^n$, then $\hat{G} = \{\gamma_t : x \mapsto \omega^{x \cdot t} \mid t \in G\}$
- If $G = \mathbb{Z}/n\mathbb{Z}$, then $\hat{G} = \{\gamma_t : x \mapsto \omega^{xt} \mid t \in G\}$

The latter is a special case of the former, but again n should thought of as an asymptotic variable.

Definition 1.18. Given $f : G \to \mathbb{C}$, define its **Fourier transform** $\hat{f} : \hat{G} \to \mathbb{C}$ by

$$\hat{f}(\gamma) = \mathbb{E}_{x \in G} f(x)\gamma(x)$$

It is easy to verify that $f(x) = \sum_{\gamma \in \hat{G}} \hat{f}(\gamma)\overline{\gamma(x)}$. Similarly, Definitions 1.6, 1.9, Examples 1.3, 1.10 and Lemmas 1.5, 1.8, 1.11 go through in this more general context.

Example 1.19. Let p be a prime, $L < p$ be even and $J = [-\frac{L}{2}, \frac{L}{2}] \subseteq \mathbb{F}_p$. Then for all $t \neq 0$ we have

$$\widehat{1_J}(t) \leq \min\left(\frac{L+1}{p}, \frac{1}{2|t|}\right)$$

See Example Sheet 1.

Theorem 1.20 (Roth)**.** Let $A \subseteq [N]$ be a set containing only trivial 3APs. Then $|A| = O(\frac{N}{\log \log N})$.

Lemma 1.21. Let $A \subseteq [N]$ of density $\alpha > 0$ containing only trivial 3APs and satisfying $N > 50\alpha^{-2}$. Let p be a prime in $[\frac{N}{3}, \frac{2N}{3}]$ and write $A' = A \cap [p] \subseteq \mathbb{F}_p$. Then either

1. $\sup_{t \neq 0} \left|\widehat{1_A}(t)\right| \geq \frac{\alpha^2}{10}$ (where the Fourier coefficients are computed in $\mathbb{F}_p$)
2. or there exists an interval J of length $\geq \frac{N}{3}$ such that
$$|A \cap J| \geq \alpha\left(1 + \frac{\alpha}{400}\right)|J|$$

Proof. If $|A'| \leq \alpha\left(1 - \frac{\alpha}{200}\right)p$, then

$$|A \cap [p+1, N]| \geq \alpha(N-p) + \frac{\alpha^2 p}{200} \geq \alpha\left(1 + \frac{\alpha}{400}\right)(N-p)$$

and we are in Case 2 with $J = [p+1, N]$. Let $A'' = A' \cap [\frac{p}{3}, \frac{2p}{3}]$. Note that all 3APs of the form $(x, x+d, x+2d) \in A' \times A'' \times A''$ are in fact 3APs in $[N]$ (and in particular they are trivial).
If $\left|A' \cap [\frac{p}{3}]\right|$ or $\left|A' \cap [\frac{2p}{3}, p]\right|$ were at least $\frac{2}{5}|A'|$, then we would again be in Case 2. We may therefore assume that $|A''| \geq \frac{|A'|}{5}$.
Now, as in Lemma 1.16 and Theorem 1.17 with $\alpha' = \frac{|A'|}{p}, \alpha'' = \frac{|A''|}{p}$,

$$\frac{\alpha''}{p} = T_3(1_{A'}, 1_{A''}, 1_{A''}) = \alpha'\alpha''^2 + \sum_{t \neq 0} \widehat{1_{A'}}(t)\widehat{1_{A''}}(t)\overline{\widehat{1_{2 \cdot A'}}(t)}$$

So, as before, $\frac{\alpha'\alpha''}{2} \leq \alpha'' \sup_{t \neq 0} \left|\widehat{1_{A'}}(t)\right|$, provided $\frac{\alpha''}{p} \leq \frac{\alpha'\alpha''^2}{2}$. This holds by assumption since $p \geq \frac{N}{3}, N \geq 50\alpha^{-2}, \alpha' \geq \frac{199}{200}\alpha, \alpha'' \geq \frac{\alpha'}{5}$. □

Lecture 5

We now want to convert the large Fourier coefficient into a density increment. This is harder now that the number of values of xt grows as $n \to \infty$. Compare this to the finite field case where $x \cdot t$ only take p different values regardless of n. If we can't find a single big coefficient, then we might instead be able to find an interval of coefficients whose total contribution is big.

TODO: Insert picture

Lemma 1.22. Let $m \in \mathbb{N}$ and $\phi : [m] \to \mathbb{F}_p$ be multiplication by some fixed $t \neq 0$. Given $\varepsilon > 0$, there exists a partition of $[m]$ into progressions P_i of length $\in [\frac{\varepsilon\sqrt{m}}{2}, \varepsilon\sqrt{m}]$ such that $\operatorname{diam}(\phi(P_i)) \leq \varepsilon p$.

Proof. Let $u = \lfloor\sqrt{m}\rfloor$ and consider $0, t, \ldots, ut$. By pigeonhole, find $0 \leq v < w \leq u$ such that $|wt - vt| \leq \frac{p}{u}$. Set $s = w - v \leq u$ so that $|st| \leq \frac{p}{u}$. Divide $[m]$ into residue classes mod s. Each has size at least $\lfloor\frac{m}{s}\rfloor \geq \lfloor\frac{m}{u}\rfloor$ and can be divided into progressions of the form $a, a+s, \ldots, a+ds$ with $\frac{\varepsilon u}{2} < d \leq \varepsilon u$. The diameter of each progression under ϕ is $|dst| \leq \varepsilon p$. □

Lemma 1.23. Let $A \subseteq [N]$ be of density $\alpha > 0$. Let p be a prime in $[\frac{N}{3}, \frac{2N}{3}]$ and write $A' = A \cap [p]$. Suppose there exists $t \neq 0$ such that $\left|\widehat{1_A}(t)\right| \geq \frac{\alpha^2}{10}$. Then there exists a progression p of length at least $\alpha^2\frac{\sqrt{N}}{500}$ such that

$$|A \cap P| \geq \alpha\left(1 + \frac{\alpha}{50}\right)|P|$$

Proof. Let $\varepsilon = \frac{\alpha^2}{40\pi}$ and use Lemma 1.22 to partition $[p]$ into progressions P_i of length at least $\frac{\varepsilon\sqrt{p}}{2} \geq \frac{\alpha^2}{80\pi}\sqrt{\frac{N}{3}} \geq \frac{\alpha^2\sqrt{N}}{500}$ and $\operatorname{diam}\phi(P_i) \leq \varepsilon p$. Fix one x_i inside each P_i.

$$\begin{aligned}
\frac{\alpha^2}{10} &\leq \left|\widehat{f_{A'}}(t)\right| \\
&= \frac{1}{p}\left|\sum_i \sum_{x \in P_i} f_{A'}(x)\omega^{xt}\right| \\
&= \frac{1}{p}\left|\sum_i \sum_{x \in P_i} f_{A'}(x)\omega^{x_i t} + \sum_i \sum_{x \in P_i} f_{A'}(x)(\omega^{xt} - \omega^{x_i t})\right| \\
&\leq \frac{1}{p}\sum_i \left|\sum_{x \in P_i} f_{A'}(x)\omega^{x_i t}\right| + \frac{1}{p}\sum_i \sum_{x \in P_i} |f_{A'}(x)|\, 2\pi\varepsilon \\
&\leq \frac{1}{p}\sum_i \left|\sum_{x \in P_i} f_{A'}(x)\omega^{x_i t}\right| + \frac{\alpha^2}{20}
\end{aligned}$$

So

$$\sum_i \left|\sum_{x \in P_i} f_{A'}(x)\right| \geq \frac{\alpha^2 p}{20}$$

Since $f_{A'}$ has mean zero, there exists i such that $\sum_{x \in P_i} f_{A'}(x) \geq \frac{\alpha^2|P_i|}{40}$. □

Proof of Roth's theorem. Put the ingredients together, Similarly to Meshulam. See Example Sheet 1 for details. $\square$

Example 1.24 (Behrend's construction)**.** There exists a set $A \subseteq [N]$ containing non nontrivial 3APs of size at least $e^{-O(\sqrt{\log n})}$. See Example Sheet 1.

Definition 1.25. Let $\Gamma \subseteq \hat{G}$. The **Bohr set** of **frequencies** Γ and width ρ is

$$B(\Gamma, \rho) = \{x \in G \mid \forall \gamma \in \Gamma, |\gamma(x) - 1| \leq \rho\}$$

$|\Gamma|$ is the **rank** of the Bohr set.

Example 1.26. When $G = \mathbb{F}_p^n$, $B(\Gamma, \rho) = \langle \Gamma \rangle^\perp$ for all small enough ρ (depending only on p, not n).

Lemma 1.27. Let B be a Bohr set of rank d and width ρ. Then $|B| \geq \left(\frac{\rho}{2\pi}\right)^d |G|$.

Proof. See Example Sheet 2. $\square$

Lecture 6

Lemma 1.28 (Bogolyubov)**.** Given $A \subseteq \mathbb{F}_p$ of density $\alpha > 0$, there exists $\Gamma \subseteq \widehat{\mathbb{F}_p}$ of size at most $2\alpha^{-2}$ such that $B(\Gamma, \frac{1}{2}) \subseteq (A + A) - (A + A)$.

Proof. Recall $(1_A * 1_A * 1_{-A} * 1_{-A})(x) = \sum_{t \in \widehat{\mathbb{F}_p}} \left|\widehat{1_A}(t)\right|^4 \omega^{-xt}$. Let $\Gamma = \mathrm{Spec}_{\sqrt{\frac{\alpha}{2}}}(1_A)$ and note that we have $\cos(\frac{2\pi xt}{p}) > 0$ for all $x \in B(\Gamma, \frac{1}{2})$ and $t \in \Gamma$. Hence

$$\begin{aligned} \mathrm{Re} \sum_{t \in \widehat{\mathbb{F}_p}} \left|\widehat{1_A}(t)\right|^4 \omega^{-xt} &= \sum_{t \in \Gamma} \left|\widehat{1_A}(t)\right|^4 \cos\left(\frac{2\pi xt}{p}\right) + \sum_{t \notin \Gamma} \left|\widehat{1_A}(t)\right|^4 \cos\left(\frac{2\pi xt}{p}\right) \\ &\geq \alpha^4 - \frac{\alpha^4}{2} > 0 \end{aligned}$$

$\square$

2 Combinatorial methods

For now, let G be an abelian group. Given $A, B \subseteq G$, we defined

$$A \pm B = \{a \pm b \mid a \in A, b \in B\}$$

If A and B are finite and nonempty, then

$$\max(|A|, |B|) \leq |A \pm B| \leq |A| |B|$$

Better bounds are available in certain settings.

Example 2.1. Let $V \leq \mathbb{F}_p^n$ be a subspace. Then $V + V$, so $|V + V| = |V|$. In fact, if $A \subseteq \mathbb{F}_p^n$ is such that $|A + A| = |A|$, then A is a coset of some subspace.

Example 2.2. Let $A \subseteq \mathbb{F}_p^n$ be such that $|A + A| < \frac{3}{2} |A|$. Then there exists $V \leq \mathbb{F}_p^n$ such that A is contained in a coset of V and $|V| < \frac{3}{2} |A|$. See Example Sheet 2.

Example 2.3. Let $A \subseteq \mathbb{F}_p^n$ be a set of linearly independent vectors. Then $|A+A| = \binom{|A|+1}{2}$. This is big doubling, but $|A| \le n$ is small!
Let $A \subseteq \mathbb{F}_p^n$ be a set where each point is taken randomly with probability $p^{-\theta n}$ where $\theta \in]\frac{1}{2}, 1]$. Then with high probability $|A+A| = (1+o(1))\frac{|A|^2}{2}$.

Definition 2.4. Given finite sets $A, B \subseteq G$, we define the Ruzsa distance between A and B to be

$$d(A,B) = \log \frac{|A-B|}{\sqrt{|A|\,|B|}}$$

$d(A,B)$ is clearly nonnegative and symmetric. However, $d(A,A) \neq 0$ in general.

Lemma 2.5 (Ruzsa's triangle inequality)**.** For $A, B, C \subseteq G$ finite,

$$d(A,C) \le d(A,B) + d(B,C)$$

Proof. The inequality reduces to

$$|B|\,|A-C| \le |A-B|\,|B-C|$$

This is true because

$$\begin{aligned} \phi : B \times (A-C) &\to (A-B) \times (B-C) \\ (b,d) &\mapsto (a_d - b, b - c_d) \end{aligned}$$

is injective, where for each $d \in A-C$ we have chosen $a_d \in A, c_d \in C$ such that $d = a - c$. ☐

Definition 2.6. Given a finite set $A \subseteq G$, we write $\sigma(A) = \frac{|A+A|}{|A|}$ the **doubling constant** and $\delta(A) = \frac{|A-A|}{|A|}$ the **difference constant** of A.

$d(A,A) = \log \sigma(A)$ and $d(A,-A) = \log \delta(A)$, so Lemma 2.5 for $A, -A, -A$ tells us that $\delta(A) \le \sigma(A)^2$.

Lecture 7

Notation. Given $A \subseteq G$ and $\ell, m \in \mathbb{N}$, write $\ell A - mA$ for the set

$$\underbrace{A + \cdots + A}_{\ell \text{ times}} - \underbrace{A + \cdots + A}_{m \text{ times}}$$

Theorem 2.7 (Plünnecke's inequality)**.** Let $A, B \subseteq G$ be finite such that $|A+B| \le K\,|A|$. Then for all ℓ, m,

$$|\ell B - mB| \le K^{\ell+m}\,|A|$$

Idea. A should be thought of as being approximately a subspace. The assumption then says that B is efficiently contained in (a translate of) A and the conclusion now reads that B must itself have small multiples. This makes sense, since we can use multiples of A (which are not much bigger than A) to efficiently contain the multiples of B.

Proof. WLOG $|A+B| = K\,|A|$. Choose $A' \subseteq A$ nonempty such that the ratio $\frac{|A'+B|}{|A'|} = K'$ is minimised. Note $K' \le K$ and $|A''+B| \ge K'\,|A''|$ for all $A'' \subseteq A$.

Claim. For all finite $C \subseteq G$, $|A'+B+C| \le K'\,|A'+C|$.

From the claim, we show that $|A' + mB| \leq K'^m |A'|$ for all m by induction: That's true for $m = 0$. For $m + 1$, the claim with $C = mB$ gives

$$|A' + (m+1)B| = |A' + B + C| \leq K' |A' + C| \leq K'^{m+1} |A'|$$

Now, by the triangle inequality,

$$|A'| |\ell B - mB| \leq |A' + \ell B| |A' + mB| \leq K'^\ell |A'| K'^m |A'|$$

Namely, $|\ell B - mB| \leq K'^{\ell+m} |A'| \leq K^{\ell+m} |A|$.

Proof of the claim. Do induction on C. For $C = \varnothing$, it's true. For $C' = C \cup \{x\}$ with x $\notin C$, observe that

$$\begin{aligned} A' + B + C' &= A' + B + C \cup A' + B + x \\ &= A' + B + C \cup A' + B + x \setminus D + B + x \end{aligned}$$

where $D = \{a \in A' \mid a + B + x \subseteq A' + B + C\}$. By definition of K', $|D + B| \geq K' |D|$, so

$$\begin{aligned} |A' + B + C'| &\leq |A' + B + C| + |A' + B + x \setminus D + B + x| \\ &\leq |A' + B + C| + |A' + B| - |D + B| \\ &\leq K' |A' + C| + K' |A'| - K' |D| \\ &= K'(|A' + C| + |A'| - |D|) \end{aligned}$$

We now apply the same argument again, writing

$$A' + C' = A' + C \cup A' + x \setminus E + x$$

where $E = \{a \in A' \mid a + x \in A' + C\} \subseteq D$. This time, the union is disjoint, so

$$|A' + C'| = |A' + C| + |A'| - |E| \geq |A' + C| + |A| - |D|$$

Hence $|A' + B + C'| \leq K' |A' + C'|$ which proves the claim. □

□

We are now in a position to generalise Example 2.2.

Theorem 2.8 (Freiman-Ruzsa)**.** Let $A \subseteq \mathbb{F}_p^n$ be such that $|A + A| \leq K |A|$ for some $K > 0$. Then A is contained in a subspace $H \leq \mathbb{F}_p^n$ of size $|H| \leq K^2 p^{K^4} |A|$.

Proof. Write $S = A - A$ and choose $X \subseteq A + S$ maximal such that the translates $x + A$ for $x \in X$ are disjoint.
X cannot be too large. Indeed, $\forall x \in X, x + A \subseteq 2A + S$. Hence $\bigcup_{x \in X}(x + A) \subseteq 2A + S$ and $|X| |A| = \left|\bigcup_{x \in X}(x + A)\right| \leq |2A + S| \leq K^4 |A|$ by Plünnecke, namely $|X| \leq K^4$.
Now observe that $A + S \subseteq X + S$. Indeed, if $y \in A + S$, then either $y \in X \subseteq X + S$ (because $0 \in S$) or $y \notin X$, meaning that $x + A$ and $y + A$ are not disjoint (X is maximal), namely $y \in x + A - A \subseteq X + S$.
By induction, $\ell A + S \subseteq \ell X + S$ for all ℓ. Hence, writing

$$H = \langle A \rangle = \bigcup_\ell (\ell A + S) \subseteq \bigcup_\ell (\ell X + S) = \langle X \rangle + S$$

the subgroup generated by A, we see that A is contained in a subgroup of size

$$|H| \leq |\langle X \rangle| |S| \leq p^{|X|} K^2 |A| \leq K^2 p^{K^4} |A|$$

□

Example 2.9. Let $A = H \cup R \subseteq \mathbb{F}_p^n$ where H is a subspace of dimension $K \ll d \ll n-k$ and R consists of $K-1$ linearly independent vectors in $H^\perp$. Then $|A| = |H \cup R| \sim |H|$ and $|A+A| = |H \cup H + R \cup R + R| \sim K|H| \sim K|A|$ but any subspace $V \leq \mathbb{F}_p^n$ containing A must have size $\geq p^{d+(K-1)} = p^{K-1}|H| \sim p^{K-1}|A|$ where the constant is exponential in K.

Conjecture 1 (Polynomial Freiman-Ruzsa)**.** Let $A \subseteq \mathbb{F}_p^n$ be such that $|A+A| \leq K|A|$. Then there is a subspace $H \leq \mathbb{F}_p^n$ of size at most $C_1(K)|A|$ and $x \in \mathbb{F}_p^n$ such that $|A \cap (x+H)| \geq \frac{|A|}{C_2(K)}$ where $C_1(K)$ and $C_2(K)$ are polynomials.

For $p = 2$, this is now a theorem.

Definition 2.10. Given an abelian group G and finite sets $A, B \subseteq G$, define **additive quadruples** to be the tuples $(a, a', b, b') \in A^2 \times B^2$ such that $a + b = a' + b'$ and the **additive energy between** A **and** B to be

$$E(A,B) = \frac{\#\{\text{additive quadruples}\}}{|A|^{\frac{3}{2}}|B|^{\frac{3}{2}}}$$

Write $E(A) = E(A,A)$ the **additive energy of** A.

Observe that, if G is finite, then

$$\begin{aligned} |A|^3 E(A) &= |G|^3 \mathbb{E}_{x+y=z+w} 1_A(x)1_A(y)1_A(z)1_A(w) \\ &= |G|^3 \left\| \widehat{1_A} \right\|_4^4 \end{aligned}$$

Example 2.11. When $H \leq \mathbb{F}_p^n$, we have $E(H) = 1$.

Lemma 2.12. Let G be abelian and $A, B \subseteq G$ be finite. Then $E(A,B) \geq \frac{\sqrt{|A||B|}}{|A \pm B|}$.

Proof. Write $r(x) = \#\{(a,b) \in A \times B \mid a + b = x\}$ so that

$$|A|^{\frac{3}{2}}|B|^{\frac{3}{2}} E(A,B) = \#\{\text{additive quadruples}\} = \sum_x r(x)^2$$

Observe that $\sum_x r(x) = |A||B|$, therefore

$$\begin{aligned} |A|^{\frac{3}{2}}|B|^{\frac{3}{2}} E(A,B) &= \sum_x r(x)^2 \\ &\geq \frac{\sum_x r(x) 1_{A+B}(x)}{\sum_x 1_{A+B}(x)^2} \text{ by Cauchy-Schwarz} \\ &= \frac{(|A||B|)^2}{|A+B|} \end{aligned}$$

and similarly for $A - B$. □

In particular, if $|A+A| \leq K|A|$ then $E(A) \geq \frac{1}{K}$. The mantra is "Small doubling implies big energy". The converse is **not** true.

Example 2.13. Let G be your favorite family of abelian groups. Then there are constants $\eta, \theta > 0$ such that for all sufficiently large n there exists $A \subseteq G$ with $|A| = n$ satisfying $E(A) \gg \eta$ and $|A+A| \geq \theta |A|^2$. See Example Sheet 2.

If we can't hope for a set to have small doubling when its energy is big, we might at least be able to find a big subset with big energy.

Theorem 2.14 (Balog-Szemerédi-Gowers)**.** Let G be an abelian group and let $A \subseteq G$ be finite such that $E(A) \geq \eta$ for some $\eta > 0$. Then there exists $A' \subseteq A$ of size at least $c(\eta)|A|$ such that $|A' + A'| \leq C(\eta)|A|$ where $c(\eta)$ and $C(\eta)$ are polynomials in η.

We first prove a technical lemma using a method known as "dependent random choice".

Lemma 2.15. Let $A_1, \dots, A_m \subseteq [n]$ and suppose that $\sum_{i,j} |A_i \cap A_j| \geq \delta^2 n m^2$. Then there exists $X \subseteq [m]$ of size at least $\frac{\delta^5 m}{\sqrt{2}}$ such that $|A_i \cap A_j| \geq \frac{\delta^2 n}{2}$ for at least 90% of the pairs $(i, j) \in X^2$.

Proof. Let $x_1, \dots, x_5$ be taken uniformly at random from $[n]$ and let

$$X = \{i \in [m] \mid \forall k, x_k \in A_i\}$$

Observe that $\mathbb{P}(i, j \in X) = \left(\frac{|A_i \cap A_j|}{n}\right)^5$. Hence

$$\frac{\mathbb{E}|X|^2}{m^2} = \mathbb{E}_{i,j}\mathbb{P}(i, j \in X) \geq \left(\frac{\mathbb{E}_{i,j}|A_i \cap A_j|}{n}\right)^5 \geq \delta^{10}$$

Call a pair **bad** if $|A_i \cap A_j| < \frac{\delta^2 n}{2}$. Note that

$$\mathbb{P}(i, j \in X \mid (i, j) \text{ bad}) = \mathbb{P}(x_1 \in A_i \cap A_j \mid (i, j) \text{ bad})^5 \leq \frac{\delta^{10}}{2^5}$$

Hence

$$\mathbb{E}[\#\{\text{bad pairs in } X^2\}] \leq \frac{\delta^{10} m^2}{2^5}$$

meaning that

$$\frac{\delta^{10} m^2}{2} + 16\mathbb{E}[\#\{\text{bad pairs in } X^2\}] \leq \mathbb{E}[|X|^2]$$

We can therefore find $x_1, \dots, x_5$ such that $\frac{\delta^{10} m^2}{2} + 16\#\{\text{bad pairs in } X^2\} \leq |X|^2$. This both means that $|X| \geq \frac{\delta^5 m}{\sqrt{2}}$ and that

$$\#\{\text{bad pairs in } X^2\} \leq \frac{|X|^2}{16} \leq 10\% \, |X|^2$$

$\square$