Part III – Combinatorics (Incomplete)

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0 Introduction

For a finite set A, we write its cardinality $|A|$.

For a graph $G = (V, E)$ and $A, B \subseteq V$, we denote $\Gamma(A) = \{b | \exists a \in A, a \sim b\}$ the set of neighbors of A and $e(A, B)$ the number of edges between A and B.

1 Basic Results

1.1 Chains, Antichains and Scattered Sets of Vectors

Lecture 1

During WW2, Littlewood and Offord were interested in roots of polynomials with random coefficients. They came up with the following neat theorem.

Theorem (Littlewood-Offord, 1943)**.** If $z_1, \dots, z_n \in \mathbb{C}$ with $|z_i| \geq 1$, then, for any disk D of radius r,

$$\#\{\varepsilon \in \{-1,1\}^n | \sum_i \varepsilon_i z_i \in D\} \leq c \log n \frac{2^n}{\sqrt{n}}$$

for some constant c depending only on r.

Upon seeing this theorem, Erdős immediately knew he could drastically improve the bound if the z_i were real.

Theorem (Erdős, 1945)**.** If $x_1, \dots, x_n \in \mathbb{R}$, $|x_i| \geq 1$, then, for any interval I of length 2,

$$\#\{\varepsilon \in \{-1,1\}^n | \sum_i \varepsilon_i z_i \in I\} \leq \binom{n}{\frac{n}{2}}$$

This is best possible, as we see by taking $x_1 = \dots = x_n = 1$.

Let G be a bipartite graph with parts U and W. A **complete matching** from U to W is an injective function $f : U \to W$ such that $\forall u \in U, u \sim f(u)$.
If G has a complete matching, then certainly $|A| \leq |\Gamma(A)|$. Surprisingly, this is enough.

Theorem (Kőnig-Egerváry-Hall Theorem, Hall's Marriage Theorem)**.**

$$G \text{ has a complete matching} \iff \forall A \subseteq U, |A| \leq |\Gamma(A)|$$

Proof. Exercise □

Let $\mathcal{F} = (F_1, \dots, F_m)$ where the F_i are finite sets. We say $a_1, \dots, a_m$ is a **set of distinct representatives**, aka **SDR** if they are distinct and $\forall i, a_i \in F_i$. Certainly, if $\mathcal{F}$ has SDR, then $|I| \leq \left|\bigcup_{i \in I} F_i\right|$ for all $I \subseteq [m]$.

Theorem.

$$\mathcal{F} \text{ is a SDR} \iff \forall I \subseteq [m], |I| \leq \left|\bigcup_{i \in I} F_i\right|$$

Proof. Define a bipartite graph G with parts $[m]$ and $\bigcup_i F_i$ by $i \sim a \iff a \in F_i$. For all $I \subseteq [m]$, $|I| \leq \left|\bigcup_{i \in I} F_i\right| = |\Gamma(I)|$, so Theorem 1.1 applies. □

Theorem. If G is a bipartite graph with parts U, W such that $\deg(u) \geq \deg(w)$ for all $u \in U, w \in W$, then there is a complete matching from U to W.

Proof. Find d such that $\deg(u) \geq d \geq \deg(w)$ for all $u \in U, w \in W$. For all $A \subseteq U$, we have

$$d\,|A| \leq e(A, \Gamma(A)) \leq d\,|\Gamma(A)|$$

Hence $|A| \leq |\Gamma(A)|$. We're done by Theorem 1.1. □

For $A \subseteq U, B \subseteq W$, define $w(A) = \frac{|A|}{|U|}, w(B) = \frac{|B|}{|W|}$.

Say a bipartite graph G with parts U, W is (k, ℓ)**-biregular** if $\deg(u) = k, \deg(w) = \ell$ for all $u \in U, w \in W$.

Lemma. If G is biregular with parts U, W and $A \subseteq U$, then $w(A) \leq w(\Gamma(A))$.

Proof. First, $k\,|U| = e(G) = \ell\,|W|$. Second,

$$k\,|A| = e(A, \Gamma(A)) \leq \ell\,|\Gamma(A)|$$

Dividing the inequality by the equality gives the result. □

Lecture 2