Part III – Functional Analysis (Incomplete)

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Contents

0 Introduction

Prerequisites

- some basic functional analysis
- a bit of measure theory
- a bit of complex analysis

Books

Books relevant to the course are:

- Bollobás, *Linear Analysis*
- Murphy, C^**-algebras*
- Rudin
- Graham-Allan

Notation

We will use $\mathbb{K}$ to mean "either $\mathbb{R}$ or $\mathbb{C}$".

For X a normed space, we define

$$B_X = \{x \in X | \, \|x\| \leq 1\}$$

$$S_X = \{x \in X | \, \|x\| = 1\}$$

For X, Y normed spaces, we write $X \sim Y$ if X, Y are isomorphic, ie there exists a linear bijection $T : X \to Y$ such that T and T^{-1} are continuous. We write $X \cong Y$ if X, Y are isometrically isomorphic, ie there exists a surjective linear map $T : X \to Y$ such that $\|Tx\| = \|x\|$ for all x.

1 Hahn-Banach extension theorems

Lecture 1

Let X be a normed space. The **dual space** of X is the space X^* of bounded linear functionals on X. X^* is always a Banach space in the operator norm: for $f \in X^*$,

$$\|f\| = \sup_{x \in B_X} |f(x)|$$

Example. For $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$, $\ell_p^* \cong \ell_q$.
We also have $\ell_1^* \cong \ell_\infty$, $c_0^* \cong \ell_1$.
If H is a Hilbert space, then $H^* \cong H$ (the isomorphism is conjugate-linear in the complex case).

For $x \in X, f \in X^*$, we write $\langle x, f \rangle = f(x)$. Note that

$$\langle x, f \rangle = |f(x)| \leq \|f\| \, \|x\|$$

Definition. Let X be a *real* vector space. A functional $p : X \to \mathbb{R}$ is

- **positive homogeneous** if $p(tx) = tp(x)$ for all $x \in X$, $t \geq 0$
- **subadditive** if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$

Definition. Let P be a preorder, $A \subseteq P, x \in P$. We say

- x is an **upper bound** for A if $\forall a \in A, a \leq x$.
- A is a **chain** if $\forall a, b \in A, a \leq b \vee b \leq a$.
- x is a **maximal element** if $\forall y \in P, x \not< y$

Fact (Zorn's lemma)**.** A nonempty preorder in which all nonempty chains have an upper bound has a maximal element.

Theorem (Hahn-Banach, positive homogeneous version)**.** Let X be a real vector space and $p : X \to \mathbb{R}$ be positive homogeneous and subadditive. Let Y be a subspace of X and $g : Y \to \mathbb{R}$ be linear such that $\forall y \in Y, g(y) \leq p(y)$. Then there exists $f : X \to \mathbb{R}$ linear such that $f \restriction_Y = g$ and $\forall x \in X, f(x) \leq p(x)$.

Proof. Let P be the set of pairs (Z, h) where Z is a subspace of X with $Y \subseteq Z$ and $h : Z \to \mathbb{R}$ linear, $h \restriction_Y = g$ and $\forall z \in Z, h(z) \leq p(z)$. P is nonempty since $(Y, g) \in P$, and is partially ordered by

$$(Z_1, h_1) \leq (Z_2, h_2) \iff Z_1 \subseteq Z_2 \wedge h_2 \restriction_{Z_1} = h_1$$

If $\{(Z_i, h_i) | i \in I\}$ is a chain with I nonempty, then we can define

$$Z := \bigcup_{i \in I} Z_i, h \restriction_{Z_i} = h_i$$

The definition of h makes sense thanks to the chain assumption. $(Z, h) \in P$ is therefore an upper bound for the chain.
Hence find by Zorn a maximal element (Z, h) of P. If $Z = X$, we won. So assume there is some $x \in X\ Z$. Let $W = \mathrm{Span}(Z \cup \{x\})$ and define $f : W \to \mathbb{R}$ by

$$f(z + \lambda x) = h(z) + \lambda \alpha$$

for some $\alpha \in \mathbb{R}$. Then f is linear and $f \restriction_Z = h$. We now look for α such that $\forall w \in W, f(w) \leq p(w)$. We would then have $(W, f) \in P$ and $(Z, h) < (W, f)$, contradicting maximality of (Z, h).
We need

$$h(z) + \lambda\alpha \leq p(z + \lambda x) \forall z \in Z, \lambda \in \mathbb{R}$$

Since p is positive homogeneous, this becomes

$$h(z) + \alpha \leq p(z + x) h(z) - \alpha \leq p(z - x) \tag{1}$$

ie

$$h(z) - p(z - x) \leq \alpha \leq p(z + x) - h(z) \forall z \in Z$$

The existence of α now amounts to

$$h(z_1) - p(z_1 - x) \leq \alpha \leq p(z_2 + x) - h(z_2) \forall z_1, z_2 \in Z$$

But indeed

$$h(z_1) + h(z_2) = h(z_1 + z_2) \leq p(z_1 + z_2) \leq p(z_1 - x) + p(z_2 + x)$$

□

Definition. Let X be a $\mathbb{K}$-vector space. A **seminorm** on X is a functional $p : X \to \mathbb{R}$ such that

- $\forall x \in X, p(x) \geq 0$
- $\forall x \in X, \lambda \in \mathbb{K}, p(\lambda x) = |\lambda| p(x)$
- $\forall x, y \in X, p(x + y) \leq p(x) + p(y)$

Remark.

$$\text{norm} \implies \text{seminorm} \implies \text{positive homogeneous}$$

Lecture 2

Theorem (Hahn-Banach, absolute homogeneous version)**.** Let X be a real of complex vector space and p a seminorm on X. Let Y be a subspace of X, g a linear functional on Y such that $\forall y \in Y, |g(y)| \leq p(y)$. Then there exists a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x)$.

Proof.
Real case

$$\forall y \in Y, g(y) \leq |g(y)| \leq p(y)$$

By Theorem 1, there exists $f : X \to \mathbb{R}$ such that $f \restriction_Y = g$ and $\forall x \in X, f(x) \leq p(x)$. We also have

$$\forall x \in X, -f(x) = f(-x) \leq p(-x) = p(x)$$

Hence $|f(x)| \leq p(x)$
Complex case
$\operatorname{Re} g : Y \to \mathbb{R}$ is real-linear.

$$\forall y \in Y, |\operatorname{Re} g(y)| \leq |g(y)| \leq p(y)$$

By the real case, find $h : X \to \mathbb{R}$ real-linear such that $h \restriction_Y = \operatorname{Re} g$

Claim. There exists a unique complex-linear $f : X \to \mathbb{C}$ such that $h = \operatorname{Re} f$.

Proof.
Uniqueness
If we have such f, then

$$\begin{aligned} f(x) &= \operatorname{Re} f(x) + i \operatorname{Im} f(x) \\ &= \operatorname{Re} f(x) - i \operatorname{Re} f(ix) \\ &= h(x) - ih(ix) \end{aligned}$$

Existence
Define $f(x) = h(x) - ih(ix)$. Then f is real-linear and $f(ix) = if(x)$, so f is complex-linear with $\operatorname{Re} f = h$. □

We now have $f : X \to \mathbb{C}$ such that $\operatorname{Re} f = h$.

$$\operatorname{Re} f \restriction_Y = h \restriction_Y = \operatorname{Re} g$$

So, by uniqueness, $f \restriction_Y = g$.
Given $x \in X$, find λ with $|\lambda| = 1$ such that

$$\begin{aligned} |f(x)| &= \lambda f(x) \\ &= f(\lambda x) \\ &= \operatorname{Re} f(\lambda x) \\ &= h(\lambda x) \\ &\leq p(\lambda x) \\ &= p(x) \end{aligned}$$

□

Remark. For a complex vector space X, if we write $X_{\mathbb{R}}$ for X considered as a real vector space, the above proof shows that

$$\operatorname{Re} : (X^*)_{\mathbb{R}} \to X^*_{\mathbb{R}}$$

is an isometric isomorphism.

Corollary. Let X be a $\mathbb{K}$-vector space, p a seminorm on X, $x_0 \in X$. Then there exists a linear functional f on X such that $f(x_0) = p(x_0)$ and $\forall x \in X, |f(x)| \leq p(x)$.

Proof. Let $Y = \operatorname{Span}(x_0)$,

$$\begin{aligned} g : Y &\to \mathbb{K} \\ \lambda x_0 &\mapsto \lambda p(x_0) \end{aligned}$$

We see that $\forall y \in Y, g(y) \leq p(y)$. Hence find by Theorem 1 a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x)$. We check that $f(x_0) = g(x_0) = p(x_0)$. □

Theorem (Hahn-Banach, existence of support functionals)**.** Let X be a real or complex normed space. Then

1. If Y is a subspace of X and $g \in Y^*$, then there exists $f \in X^*$ such that $f \restriction_Y = g$ and $\|f\| = \|g\|$.
2. Given $x_0 \neq 0$, there exists $f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.

Proof.

1. Let $p(x) = \|g\| \|x\|$. Then p is a seminorm on X and
$$\forall y \in Y, |g(y)| \leq \|g\| \|y\| = p(y)$$
Find by Theorem 1 a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x) = \|g\| \|x\|$. So $\|f\| \leq \|g\|$. Since $f \restriction_Y = g$, we also have $\|g\| \leq \|f\|$. Hence $\|f\| = \|g\|$.

2. Apply Corollary 1 with $p(x) = \|x\|$ to get $f \in X^*$ such that
$$\forall x \in X, |f(x)| \leq \|x\| \text{ and } f(x_0) = \|x_0\|$$
It follows that $\|f\| = 1$.

□

Remarks.

- Part 1 is a sort of linear version of Tietze's extension theorem: Given K compact Hausdorff, $L \subseteq K$ closed, $g : L \to \mathbb{K}$ continuous, there exists $f : K \to \mathbb{K}$ such that $f \restriction_L = g$ and $\|f\|_\infty = \|g\|_\infty$.
- Part 2 shows that for all $x \neq y$ in X there exists $f \in X^*$ such that $f(x) \neq f(y)$, namely X^* **separates points** of X. This is a sort of linear version of Urysohn: $C(K)$ separates points of K.
- The f in part 2 is called a **norming functional**, aka **support functional**, for x_0. The existence of support functionals shows that
$$x_0 = \max_{g \in B_{X^*}} \langle x_0, g \rangle$$
Assuming X is a real normed space and $\|x_0\| = 1$, we have $B_X \subseteq \{x \in X | f(x) \leq 1\}$. Visually, TODO: insert tangency diagram

1.1 Bidual

Let X be a normed space. Then X^{**} is called the **bidual** or **second dual** of X.

For $x \in X$, define $\hat{x} : X^* \to \mathbb{K}$, the **evaluation at** x, by $\hat{x}(f) = f(x)$. $\hat{x}$ is linear and $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\|$, so $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leq \|x\|$.

The map $x \mapsto \hat{x} : X \to X^{**}$ is called the **canonical embedding** of X into X^{**}.

Theorem. The canonical embedding is an isometric embedding.

Proof.
Linearity
$$\begin{aligned} \widehat{x+y}(f) &= f(x+y) = f(x) + f(y) = \hat{x}(f) + \hat{y}(f) \\ \widehat{\lambda x}(f) &= f(\lambda x) = \lambda f(x) = \lambda \hat{x}(f) \end{aligned}$$
Isometry
If $x \neq 0$, there exists a support functional f for x. Then
$$\|\hat{x}\| \geq |\hat{x}(f)| = |f(x)| = \|x\|$$

□

Remarks.

- In bracket notation, $\langle f, \hat{x} \rangle = \langle x, f \rangle$
- Let $\hat{X}$ be the image of X in X^{**}. Theorem 1.1 says

  $$X \cong \hat{X} \subseteq X^{**}$$

  We often identify $\hat{X}$ with X and think of X as living isometrically inside X^{**}. Note that

  $$X \text{ complete} \iff \hat{X} \text{ closed in } X^{**}$$

- More generally, $\bar{\hat{X}}$ is a Banach space containing an isometric copy of X as a dense subspace. We proved that normed spaces have completions!

Definition. A normed space X is **reflexive** if the canonical embedding $X \to X^{**}$ is surjective.

Example.

- Some reflexive spaces are Hilbert spaces, finite-dimensional spaces, ℓ_p and $L_p(\mu)$ for $1 < p < \infty$.
- Some non-reflexive spaces are $c_0, \ell_1, \ell_\infty, L_1[0,1]$.

Remarks.

- If X is reflexive, then $X \cong X^{**}$, so X is complete.
- There are Banach spaces X such that $X \cong X^{**}$ but X is not reflexive, eg **James' space**. Any isomorphism to the bidual is then necessarily not the canonical embedding.

Lecture 3