Part III – Functional Analysis (Incomplete)

Based on lectures by Dr András Zsák

Notes taken by Yaël Dillies

Michaelmas 2023

Contents

0 Introduction

Prerequisites

- some basic functional analysis
- a bit of measure theory
- a bit of complex analysis

Books

Books relevant to the course are:

- Bollobás, *Linear Analysis*
- Murphy, C^**-algebras*
- Rudin
- Graham-Allan

Notation

We will use $\mathbb{K}$ to mean "either $\mathbb{R}$ or $\mathbb{C}$".

For X a normed space, we define

$$\begin{aligned} B_X &= \{x \in X \mid \|x\| \le 1\} \\ S_X &= \{x \in X \mid \|x\| = 1\} \\ D_X &= \{x \in X \mid \|x\| < 1\} \end{aligned}$$

For X, Y normed spaces, we write $X \sim Y$ if X, Y are isomorphic, ie there exists a linear bijection $T : X \to Y$ such that T and T^{-1} are continuous. We write $X \cong Y$ if X, Y are isometrically isomorphic, ie there exists a surjective linear map $T : X \to Y$ such that $\|Tx\| = \|x\|$ for all x.

1 Hahn-Banach extension theorems

Lecture 1

Let X be a normed space. The **dual space** of X is the space X^* of bounded linear functionals on X. X^* is always a Banach space in the operator norm: for $f \in X^*$,

$$\|f\| = \sup_{x \in B_X} |f(x)|$$

Example. For $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$, $\ell_p^* \cong \ell_q$.
We also have $\ell_1^* \cong \ell_\infty$, $c_0^* \cong \ell_1$.
If H is a Hilbert space, then $H^* \cong H$ (the isomorphism is conjugate-linear in the complex case).

For $x \in X, f \in X^*$, we write $\langle x, f \rangle = f(x)$. Note that

$$\langle x, f \rangle = |f(x)| \leq \|f\| \, \|x\|$$

Definition. Let X be a *real* vector space. A functional $p : X \to \mathbb{R}$ is

- **positive homogeneous** if $p(tx) = tp(x)$ for all $x \in X$, $t \geq 0$
- **subadditive** if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$

Definition. Let P be a preorder, $A \subseteq P, x \in P$. We say

- x is an **upper bound** for A if $\forall a \in A, a \leq x$.
- A is a **chain** if $\forall a, b \in A, a \leq b \vee b \leq a$.
- x is a **maximal element** if $\forall y \in P, x \not< y$

Fact (Zorn's lemma)**.** A nonempty preorder in which all nonempty chains have an upper bound has a maximal element.

Theorem 1.1 (Hahn-Banach, positive homogeneous version)**.** Let X be a real vector space and $p : X \to \mathbb{R}$ be positive homogeneous and subadditive. Let Y be a subspace of X and $g : Y \to \mathbb{R}$ be linear such that $\forall y \in Y, g(y) \leq p(y)$. Then there exists $f : X \to \mathbb{R}$ linear such that $f \restriction_Y = g$ and $\forall x \in X, f(x) \leq p(x)$.

Proof. Let P be the set of pairs (Z, h) where Z is a subspace of X with $Y \subseteq Z$ and $h : Z \to \mathbb{R}$ linear, $h \restriction_Y = g$ and $\forall z \in Z, h(z) \leq p(z)$. P is nonempty since $(Y, g) \in P$, and is partially ordered by

$$(Z_1, h_1) \leq (Z_2, h_2) \iff Z_1 \subseteq Z_2 \wedge h_2 \restriction_{Z_1} = h_1$$

If $\{(Z_i, h_i) \mid i \in I\}$ is a chain with I nonempty, then we can define

$$Z := \bigcup_{i \in I} Z_i, h \restriction_{Z_i} = h_i$$

The definition of h makes sense thanks to the chain assumption. $(Z, h) \in P$ is therefore an upper bound for the chain.
Hence find by Zorn a maximal element (Z, h) of P. If $Z = X$, we won. So assume there is some $x \in X\ Z$. Let $W = \mathrm{Span}(Z \cup \{x\})$ and define $f : W \to \mathbb{R}$ by

$$f(z + \lambda x) = h(z) + \lambda\alpha$$

for some $\alpha \in \mathbb{R}$. Then f is linear and $f \restriction_Z = h$. We now look for α such that $\forall w \in W, f(w) \leq p(w)$. We would then have $(W, f) \in P$ and $(Z, h) < (W, f)$, contradicting maximality of (Z, h).
We need
$$h(z) + \lambda\alpha \leq p(z + \lambda x) \forall z \in Z, \lambda \in \mathbb{R}$$
Since p is positive homogeneous, this becomes
$$h(z) + \alpha \leq p(z + x) h(z) - \alpha \leq p(z - x) \tag{1}$$
ie
$$h(z) - p(z - x) \leq \alpha \leq p(z + x) - h(z) \forall z \in Z$$
The existence of α now amounts to
$$h(z_1) - p(z_1 - x) \leq \alpha \leq p(z_2 + x) - h(z_2) \forall z_1, z_2 \in Z$$
But indeed
$$h(z_1) + h(z_2) = h(z_1 + z_2) \leq p(z_1 + z_2) \leq p(z_1 - x) + p(z_2 + x)$$
□

Definition. Let X be a $\mathbb{K}$-vector space. A **seminorm** on X is a functional $p : X \to \mathbb{R}$ such that

- $\forall x \in X, p(x) \geq 0$
- $\forall x \in X, \lambda \in \mathbb{K}, p(\lambda x) = |\lambda| p(x)$
- $\forall x, y \in X, p(x + y) \leq p(x) + p(y)$

Remark.
$$\text{norm} \implies \text{seminorm} \implies \text{positive homogeneous}$$

Lecture 2

Theorem 1.2 (Hahn-Banach, absolute homogeneous version)**.** Let X be a real of complex vector space and p a seminorm on X. Let Y be a subspace of X, g a linear functional on Y such that $\forall y \in Y, |g(y)| \leq p(y)$. Then there exists a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x)$.

Proof.
Real case
$$\forall y \in Y, g(y) \leq |g(y)| \leq p(y)$$
By Theorem 1.1, there exists $f : X \to \mathbb{R}$ such that $f \restriction_Y = g$ and $\forall x \in X, f(x) \leq p(x)$.
We also have
$$\forall x \in X, -f(x) = f(-x) \leq p(-x) = p(x)$$
Hence $|f(x)| \leq p(x)$
Complex case
$\operatorname{Re} g : Y \to \mathbb{R}$ is real-linear.
$$\forall y \in Y, |\operatorname{Re} g(y)| \leq |g(y)| \leq p(y)$$
By the real case, find $h : X \to \mathbb{R}$ real-linear such that $h \restriction_Y = \operatorname{Re} g$

Claim. There exists a unique complex-linear $f : X \to \mathbb{C}$ such that $h = \operatorname{Re} f$.

Proof.
Uniqueness
If we have such f, then

$$\begin{aligned} f(x) &= \operatorname{Re} f(x) + i \operatorname{Im} f(x) \\ &= \operatorname{Re} f(x) - i \operatorname{Re} f(ix) \\ &= h(x) - ih(ix) \end{aligned}$$

Existence
Define $f(x) = h(x) - ih(ix)$. Then f is real-linear and $f(ix) = if(x)$, so f is complex-linear with $\operatorname{Re} f = h$. $\square$

We now have $f : X \to \mathbb{C}$ such that $\operatorname{Re} f = h$.

$$\operatorname{Re} f \restriction_Y = h \restriction_Y = \operatorname{Re} g$$

So, by uniqueness, $f \restriction_Y = g$.
Given $x \in X$, find λ with $|\lambda| = 1$ such that

$$\begin{aligned} |f(x)| &= \lambda f(x) \\ &= f(\lambda x) \\ &= \operatorname{Re} f(\lambda x) \\ &= h(\lambda x) \\ &\le p(\lambda x) \\ &= p(x) \end{aligned}$$

$\square$

Remark. For a complex vector space X, if we write $X_{\mathbb{R}}$ for X considered as a real vector space, the above proof shows that

$$\operatorname{Re} : (X^*)_{\mathbb{R}} \to X_{\mathbb{R}}^*$$

is an isometric isomorphism.

Corollary 1.3. Let X be a $\mathbb{K}$-vector space, p a seminorm on X, $x_0 \in X$. Then there exists a linear functional f on X such that $f(x_0) = p(x_0)$ and $\forall x \in X, |f(x)| \le p(x)$.

Proof. Let $Y = \operatorname{Span}(x_0)$,

$$\begin{aligned} g : Y &\to \mathbb{K} \\ \lambda x_0 &\mapsto \lambda p(x_0) \end{aligned}$$

We see that $\forall y \in Y, g(y) \le p(y)$. Hence find by Theorem 1.2 a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \le p(x)$. We check that $f(x_0) = g(x_0) = p(x_0)$. $\square$

Theorem 1.4 (Hahn-Banach, existence of support functionals)**.** Let X be a real or complex normed space. Then

1. If Y is a subspace of X and $g \in Y^*$, then there exists $f \in X^*$ such that $f \restriction_Y = g$ and $\|f\| = \|g\|$.
2. Given $x_0 \neq 0$, there exists $f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.

Proof.

1. Let $p(x) = \|g\| \, \|x\|$. Then p is a seminorm on X and

$$\forall y \in Y, |g(y)| \leq \|g\| \, \|y\| = p(y)$$

Find by Theorem 1.1 a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x) = \|g\| \, \|x\|$. So $\|f\| \leq \|g\|$. Since $f \restriction_Y = g$, we also have $\|g\| \leq \|f\|$. Hence $\|f\| = \|g\|$.

2. Apply Corollary 1.3 with $p(x) = \|x\|$ to get $f \in X^*$ such that

$$\forall x \in X, |f(x)| \leq \|x\| \text{ and } f(x_0) = \|x_0\|$$

It follows that $\|f\| = 1$.

$\square$

Remarks.

- Part 1 is a sort of linear version of Tietze's extension theorem: Given K compact Hausdorff, $L \subseteq K$ closed, $g : L \to \mathbb{K}$ continuous, there exists $f : K \to \mathbb{K}$ such that $f \restriction_L = g$ and $\|f\|_\infty = \|g\|_\infty$.
- Part 2 shows that for all $x \neq y$ in X there exists $f \in X^*$ such that $f(x) \neq f(y)$, namely X^* **separates points** of X. This is a sort of linear version of Urysohn: $C(K)$ separates points of K.
- The f in part 2 is called a **norming functional**, aka **support functional**, for x_0. The existence of support functionals shows that

$$x_0 = \max_{g \in B_{X^*}} \langle x_0, g \rangle$$

Assuming X is a real normed space and $\|x_0\| = 1$, we have $B_X \subseteq \{x \in X | f(x) \leq 1\}$. Visually, TODO: insert tangency diagram

1.1 Bidual

Let X be a normed space. Then X^{**} is called the **bidual** or **second dual** of X.

For $x \in X$, define $\hat{x} : X^* \to \mathbb{K}$, the **evaluation at** x, by $\hat{x}(f) = f(x)$. $\hat{x}$ is linear and $|\hat{x}(f)| = |f(x)| \leq \|f\| \, \|x\|$, so $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leq \|x\|$.

The map $x \mapsto \hat{x} : X \to X^{**}$ is called the **canonical embedding** of X into X^{**}.

Theorem 1.5. The canonical embedding is an isometric embedding.

Proof.
Linearity

$$\widehat{x+y}(f) = f(x+y) = f(x) + f(y) = \hat{x}(f) + \hat{y}(f)$$
$$\widehat{\lambda x}(f) = f(\lambda x) = \lambda f(x) = \lambda \hat{x}(f)$$

Isometry
If $x \neq 0$, there exists a support functional f for x. Then

$$\|\hat{x}\| \geq |\hat{x}(f)| = |f(x)| = \|x\|$$

$\square$

Remarks.

- In bracket notation, $\langle f, \hat{x} \rangle = \langle x, f \rangle$
- Let $\hat{X}$ be the image of X in X^{**}. Theorem 1.5 says

$$X \cong \hat{X} \subseteq X^{**}$$

  We often identify $\hat{X}$ with X and think of X as living isometrically inside X^{**}. Note that

$$X \text{ complete} \iff \hat{X} \text{ closed in } X^{**}$$

- More generally, $\overline{\hat{X}}$ is a Banach space containing an isometric copy of X as a dense subspace. We proved that normed spaces have completions!

Definition. A normed space X is **reflexive** if the canonical embedding $X \to X^{**}$ is surjective.

Example.

- Some reflexive spaces are Hilbert spaces, finite-dimensional spaces, ℓ_p and $L_p(\mu)$ for $1 < p < \infty$.
- Some non-reflexive spaces are $c_0, \ell_1, \ell_\infty, L_1[0,1]$.

Remarks.

- If X is reflexive, then $X \cong X^{**}$, so X is complete.
- There are Banach spaces X such that $X \cong X^{**}$ but X is not reflexive, eg **James' space**. Any isomorphism to the bidual is then necessarily not the canonical embedding.

1.2 Dual operators

Lecture 3

Let X, Y be normed spaces. Recall

$$\mathcal{B}(X,Y) = \{T : X \to Y \mid T \text{ linear, bounded}\}$$

This is a normed space in the operator norm:

$$\|T\| = \sup_{x \in B_X} \|Tx\|$$

If Y is complete, then so is $\mathcal{B}(X,Y)$. For $T \in \mathcal{B}(X,Y)$, the **dual operator** of T is the map $T^* : Y^* \to X^*$ given by $T^* g = g \circ T$. In bracket notation $\langle x, T^* g \rangle = \langle Tx, g \rangle$ for $x \in X, g \in Y^*$.

T^* is linear

$$\begin{aligned}
\langle x, T^*(g+h) \rangle &= \langle Tx, g+h \rangle \\
&= \langle Tx, g \rangle + \langle Tx, h \rangle \\
&= \langle x, T^* g \rangle + x T^* h \\
&= \langle x, T^* g + T^* h \rangle
\end{aligned}$$

$$
\begin{aligned}
\langle x, T^*(\lambda g)\rangle &= \langle Tx, \lambda g\rangle \\
&= \lambda \langle Tx, g\rangle \\
&= \lambda \langle x, T^* g\rangle \\
&= \langle x, \lambda T^* g\rangle
\end{aligned}
$$

T^* is bounded

$$
\begin{aligned}
\|T^*\| &= \sup_{g \in B_{Y^*}} \|T^* g\| \\
&= \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^* g\rangle| \\
&= \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g\rangle| \\
&= \sup_{x \in B_X} \|Tx\| \text{ by Theorem 1.4 (ii)} \\
&= \|T\|
\end{aligned}
$$

Remarks.

- Hahn-Banach is crucial here. Without it, the dual could be 0.
- $\langle \cdot, \cdot \rangle$ is linear in both arguments. This contrasts with the Hilbert space case where $\langle \cdot, \cdot \rangle$ is conjugate-linear in one of the arguments. This comes from the conjugate-linearity of the identification $H^* \cong H$.
- If X, Y are Hilbert spaces and we identify X, Y with X^*, Y^*, respectively, then T^* is the adjoint of T.

Example. Let $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$ and define $R : \ell_p \to \ell_p$ to be the **right shift operator** $(x_0, x_1, \dots) \mapsto (0, x_0, \dots)$. Then $R^* : \ell_q \to \ell_q$ is the **left shift operator** $(x_0, x_1, \dots) \mapsto (x_1, x_2, \dots)$.

Some properties of the dual operator are

1. $\mathrm{id}_X^* = \mathrm{id}_{X^*}$
2. $(S+T)^* + S^* + T^*$, $(\lambda T)^* = \lambda T^*$
3. $(ST)^* = T^* S^*$
4. $T \mapsto T^* : \mathcal{B}(X, Y) \to \mathcal{B}(Y^*, X^*)$ is an *into* isomorphism.
5. The double dual of an operator commutes with the double dual embedding. TODO: Insert commutative diagram For all x,

   $$\langle g, T^{**}\hat{x}\rangle = \langle T^* g, \hat{x}\rangle = \langle x, T^* g\rangle = \langle Tx, g\rangle = \left\langle g, \hat{Tx}\right\rangle$$

   So $T^{**}\hat{x} = \widehat{Tx}$.

Remark. From the above properties, if $X \sim Y$, then $X^* \sim Y^*$. Interestingly, if X and Y are reflexive, then we can deduce $X \sim Y$ from $X^* \sim Y^*$.

1.3 Quotient spaces

Let X be a normed space and Y be a *closed* subspace.. Then the quotient space X/Y becomes a normed space in the quotient norm:

$$\|x + Y\| = d(x, Y) = \inf_{y \in Y} \|x + y\|$$

The quotient map $q : X \to X/Y$ is linear and bounded: $\|q(x)\| \le \|x\|$, so $\|q\| \le 1$.

q maps the open unit ball D_X onto $D_{X/Y}$. Indeed, if $x \in D_X$, then $\|q(x)\| \le \|x\| < 1$. Reciprocally, if $q(x) \in D_{X/Y}$, then there exists $y \in Y$ such that $\|x+y\| < 1$. So $x + y \in D_X$ and $q(x+y) = q(x)$. It follows that q is an open map and $\|q\| = 1$.

If Z is another normed space, $T \in \mathcal{B}(X, Z)$ and $Y \subseteq \ker T$, then there exists a unique map $\tilde{T}$ is linear and $\tilde{T}(D_{X/Y}) = \tilde{T}(q(D_X)) = T(D_X)$. It follows that $\left\|\tilde{T}\right\| = \|T\|$.

Theorem 1.6. Let X be a normed space. If X^* is separable, then so is X.

Remark. The converse is false, as $X = \ell_1, X^* = \ell_\infty$ shows.

Proof. Since X^* is separable, so is S_{X^*}. Let f_n be a dense subset of S_{X^*}. For every n, find $x_n \in B_X$ such that $f_n(x_n) > \frac{1}{2}$. Let

$$Y = \overline{\operatorname{Span}\{x_n \mid n \in \mathbb{N}\}}$$

Claim. $Y = X$

Then we're done since Y is separable via $Y = \overline{\operatorname{Span}_{\mathbb{Q}}\{x_n \mid n \in \mathbb{N}\}}$.

Proof. Assume not. Then we can pick $g \in (X/Y)^*$, $\|g\| = 1$ (by Theorem 1.4 (ii)). Let $f = g \circ q$. Then $\|f\| = \|g\| = 1$, ie $f \in S_{X^*}$. Thus find n such that $\|f - f_n\| < \frac{1}{4}$, so that

$$\frac{1}{4} > \|f - f_n\| \|x_n\| \ge |(f - f_n)(x_n)| = |f_n(x_n)| > \frac{1}{2}$$

contradiction. ◻

◻

Theorem 1.7. Let X be a separable normed space. Then X embeds isometrically into ℓ_∞.

Proof. Let $\{x_n \mid n \in \mathbb{N}\}$ be dense in X. For every n, find $f_n \in S_{X^*}$, $f_n(x_n) = \|x_n\|$ (assuming $X \ne \{0\}$). Define $T : X \to \ell_\infty$ by $(Tx)_n = f_n(x)$.

Well definition

$$|(Tx)_n| = |f_n(x)| \le \|f_n\| \|x\| = \|x\|$$

Hence $\|Tx\|_\infty \le \|x\| < \infty$.

Linearity

$$(T(x+y))_n = f_n(x+y) = f_n(x) + f_n(y) = (Tx + Ty)_n$$
$$(T(\lambda x))_n = f_n(\lambda x) = \lambda f_n(x)) = (\lambda Tx)_n$$

so $T(x+y) = Tx + Ty, T(\lambda x) = \lambda Tx$.

Isometry

We already know $\|Tx\|_\infty \le \|x\|$. On the other hand, find f a supporting functional for x and f_{n_k} a subsequence converging to f. Then

$$\|Tx\|_\infty \ge \sup_k (Tx)_{n_k} = \sup_k |f_{n_k}(x)| \ge |f(x)| = \|x\|$$

◻

Remarks.

- The result says that ℓ_∞ is isometrically universal for the class $\mathcal{SB}$ of separable Banach spaces.
- There is a dual result: Every separable Banach space is a quotient of ℓ_1.

Theorem 1.8 (Vector-valued Liouville)**.** Lex X be a complex Banach space, $f : \mathbb{C} \to X$ holomorphic and bounded. Then f is constant.

Proof. Find $M \geq 0$ such that $\forall z \in \mathbb{C}, |f(z)| \leq M$. Fix $\phi \in X^*$. $\phi \circ f : \mathbb{C} \to \mathbb{C}$ is

bounded

$$|\phi(f(z))| \leq \|\phi\| \, \|f(z)\| \leq M \, \|\phi\|$$

holomorphic

$$\frac{\phi(f(z)) - \phi(f(w))}{z - w} = \phi\left(\frac{f(z) - f(w)}{z - w}\right) \to \phi(f'(z))$$

By scalar Liouville, $\phi \circ f$ is constant. For every $z \in \mathbb{C}, \phi \in X^*$, $\phi(f(z)) = \phi(f(0))$. Since X^* separates points of X, $f(z) = f(0)$. □

Remark. This is a typical example of how to transfer a scalar result to a vector-valued one: Prove the result once composed with any functional, then go back using Hahn-Banach.

1.4 Locally convex spaces

Definition. A **locally convex space** is a $\mathbb{K}$-vector space such that there exists a family $\mathcal{P}$ of seminorms on X that separate points of X in the sense that for all $x \neq 0$ there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$.

The family $\mathcal{P}$ defines a topology on X:

$$U \subseteq X \text{ open} \iff \forall x \in U, \exists s \subseteq \mathcal{P} \text{ finite}, \varepsilon > 0, \{y \in X \mid \forall p \in s, p(x) < \varepsilon\} \subseteq U$$

Remarks.

1. Addition and scalar multiplication are continuous.
2. The topology is Hausdorff as $\mathcal{P}$ separates points.
3. $x_n \to x \iff \forall p \in \mathcal{P}, p(x_n - x) \to 0$
4. Let Y be a subspace of X and $\mathcal{P}_Y = \{p \restriction_Y \mid p \in \mathcal{P}\}$. Then $(Y, \mathcal{P}_Y)$ is a LCS and its topology is the subspace topology.
5. Let $\mathcal{P}, \mathcal{Q}$ be two families of seminorms on X both separating points of X. We say $\mathcal{P}, \mathcal{Q}$ are **equivalent**, write $\mathcal{P} \sim \mathcal{Q}$, if they induce the same topology on X. One interesting result is that
$$(X, \mathcal{P}) \text{ metrisable} \iff \mathcal{P} \text{ equivalent to some countable family}$$
6. We make $\mathcal{P}$ part of the data here out of simplicity, but in grown up mathematics we instead assume that X already comes with a topology and that this topology coincides with the one induced by $\mathcal{P}$.

Definition. A **Fréchet space** is a complete metrisable LCS.

Example.

1. A normed space is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.
2. Let $U \subseteq \mathbb{C}$ nonempty open. Let $\mathcal{O}(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}$. For compact $K \subseteq U$, define $p_K(f) = \sup_{z \in K} |f(z)|$. Let $\mathcal{P} = \{p_K \mid K \subseteq U \text{ compact}\}$ Then $(\mathcal{O}(U), \mathcal{P})$ is a LCS. If we replace $\{K \subseteq U \text{ compact}\}$ by a compact exhaustion of U, then we get a countable separating family equivalent to $\mathcal{P}$. So $(\mathcal{O}(U), \mathcal{P})$ is metrisable. However it is not normable: no norm on $\mathcal{O}(U)$ induces the topology of $(\mathcal{O}(U), \mathcal{P})$, which is the topology of uniform convergence. This is a consequence of Montel's theorem.
3. Fix $d \in \mathbb{N}, \Omega \subseteq \mathbb{R}^d$ a nonempty open set. Let

   $$C^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ infinitely differentiable}\}$$

   Given a multi-index $\alpha \in \mathbb{Z}^d$, α defines a differential operator

   $$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$$

   For a compact $K \subseteq \Omega, \alpha \in \mathbb{Z}^d$, define

   $$p_{K,\alpha}(f) = \sup_{z \in K} |D^\alpha f(z)|$$

   Let

   $$\mathcal{P} = \{p_{K,\alpha} \mid K \subseteq U \text{ compact}, \alpha \in \mathbb{Z}^d\}$$

   Then $(C^\infty, \mathcal{P})$ is a LCS. It is in fact a non-normable Fréchet space.

Lemma 1.9. Let $(X, \mathcal{P}), (Y, \mathcal{Q})$ be LCS, $T : X \to Y$ linear. TFAE

1. T is continuous
2. T is continuous at 0
3. $\forall q \in \mathcal{Q}, \exists s \subseteq \mathcal{P}$ finite, $C \geq 0, \forall x \in X, q(Tx) \leq C \max_{p \in s} p(x)$

Proof.
(i) $\iff$ (ii)
Translation is continuous.

(ii) $\implies$ (iii)
Given $q \in \mathcal{Q}$, let $V = \{y \in Y \mid q(y) \leq 1\}$. Then V is a neighborhood of 0 in Y. So there exists U neighborhood of 0 in X such that $T(U) \subseteq V$. WLOG

$$U = \{x \in X \mid \forall p_K \in s, p_K(x) \leq \varepsilon\}$$

Let $p = \max_{p_K \in s} p_K(x)$. If $p(x) = 1$, then $p(\varepsilon x) = \varepsilon$, so $\varepsilon x \in U$ and

$$q(T(\varepsilon x)) \leq 1$$

By homogeneity, $q(Tx) \leq \frac{1}{\varepsilon} p(x)$ for all x such that $p(x) > 0$. If $p(x) = 0$, then $p(\lambda x) = 0$ for all scalar λ. So $q(T(\lambda x)) \leq 1$ for all λ. Hence $q(Tx) = 0 \leq \frac{1}{\varepsilon} p(x)$.

(iii) $\implies$ (ii)
Assume $t \subseteq \mathcal{Q}$ is finite, $\varepsilon > 0$, and let $V = \{y \in Y \mid \forall q \in t, q(y) \leq \varepsilon$ the corresponding

neighborhood of 0. For each $q \in t$, find $s_q \subseteq \mathcal{P}$ finite and C_q so that $\forall x \in X, q(Tx) \leq C_q \max_{p \in s_q} p(x)$. Let

$$U = \left\{ x \in X \mid \forall q \in \mathcal{Q}, p \in s_q, p(x) \leq \frac{\varepsilon}{C_q} \right\}$$

Then U is a neighborhood of 0 and $T(U) \subseteq V$. □

Definition. Let $(X, \mathcal{P})$ be a LCS. The **dual space** of X is the space of continuous linear functionals $X \to \mathbb{K}$.

Lecture 5

Lemma 1.10. Let f be a linear functional on a LCS $(X, \mathcal{P})$. Then

$$f \in X^* \iff \ker f \text{ closed}$$

Proof.
$\Longrightarrow$
$\ker f = f^{-1}(0)$ is closed since f is continuous.

$\Longleftarrow$
If $\ker f = 0$, then $f = 0$ is continuous. Else fix some $x_0 \notin \ker f$. Since $(\ker f)^c$ is open, find $s \subseteq \mathcal{P}$ finite, $\varepsilon > 0$ such that

$$\underbrace{\{x \in X \mid \forall p \in s, p(x - x_0) < \varepsilon\}}_{U} \subseteq (\ker f)^c$$

Then U is a neighborhood of 0 and $(x_0 + U) \cap \ker f =$. Note that U is convex and **balanced** ($x \in U, |\lambda| \leq 1 \implies \lambda x \in U$), hence so is $f(U)$ as f is linear.
If $f(U)$ is unbounded, then it is the whole scalar field, hence so is $f(x_0 + U) = f(x_0) + f(U)$. But $0 \in \ker f$, contradicting disjointness.
So find M such that $|f(x)| < M$ for all $x \in U$. For all $\delta > 0$, $\frac{\delta}{M}U$ is a neighborhood of 0 and $f\left(\frac{\delta}{M}U\right) \subseteq \{\lambda \in \mathbb{K} \mid |\lambda| < \delta\}$. Thus f is continuous. □

Theorem 1.11 (Hahn-Banach)**.** Let $(X, \mathcal{P})$ be a LCS.

1. Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ such that $f \restriction_Y = g$.
2. Given a closed subspace Y of X and $x_0 \notin Y$, there exists $f \in X^*$ such that $f \restriction_Y = 0$, $f(x_0) \neq 0$.

Remark. This means that X^* separates points of X.

Proof.

1. By Lemma 1.9, find $s \subseteq \mathcal{P}$ finite, $C \geq 0$ such that

$$\forall y \in Y, |g(y)| \leq C \max_{p \in s} p(y)$$

Let $p(x) = C \max_{p \in s} p(x)$. Then p is a seminorm on X and $\forall y \in Y, |g(y)| \leq p(y)$. By Theorem 1.2, find a linear functional f on X such that $f \restriction_Y = g, \forall x \in X, |f(x)| \leq p(x)$. By Lemma 1.9, $f \in X^*$.

2. Let $Z = \mathrm{Span}(Y \cup \{x_0\})$ and define a linear functional g on Z by

$$g(y + \lambda x_0) = \lambda, y \in Y, \lambda \in \mathbb{K}$$

Then $g \restriction_Y = 0, g(x_0) = 1 \neq 0$ and $\ker g = Y$ is closed, so $g \in Z^*$ by Lemma 1.10. By part (i), find $f \in X^*$ such that $f \restriction_Z = g$. This works.

□

2 The dual of $L_p(\mu)$ and $C(K)$

Let $(\Omega, \mathcal{F}, \mu)$ be measure space.

$1 \leq p < \infty$

$$L_p(\mu) = \{f : \Omega \to \mathbb{K} \mid f \text{ measurable and } \int_\Omega |f|^p \, d\mu < \infty\}$$

This is a normed space in the L_p-norm:

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}}$$

$p = \infty$
A measurable function $f : \Omega \to \mathbb{K}$ is **essentially bounded** if there exists $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $f \restriction_{N^c}$ is bounded.

$$L_p(\mu) = \{f : \Omega \to \mathbb{K} \mid f \text{ measurable and essentially bounded}\}$$

This is a normed space in the L_∞-norm:

$$\|f\|_\infty = \operatorname{esssup} |f| = \inf_{|f| \leq k \text{ ae}} k$$

The inf is attained: there exists some $N \in \mathcal{F}, \mu(N) = 0$ such that $\|f\|_\infty = \sup_{N^c} |f|$.

In all cases, we identify functions up to almost everywhere equality.

Theorem 2.1. $L_p(\mu)$ *is complete for* $1 \leq p \leq infty$.

Definition (Complex measures)**.** A **complex measure** on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{C}$.
The **total variation measure** $|\nu|$ is defined by

$$|\nu|(A) = \sup_{\substack{A_1, \ldots, A_n \text{ measurable} \\ \text{partition of } A}} \sum_k |\nu(A_k)|$$

$|\nu| : \mathcal{F} \to [0, \infty]$ is a positive measure. Later we'll see that $|\nu|$ is a finite measure.
The **total variation** of ν is $\|\nu\|_1 = |\nu|(\Omega)$.

Proposition. If ν is a complex measure on $\mathcal{F}$ and $A_n \in \mathcal{F}$ for all n, then

- If A is monotone, then $\nu(\bigcup_n A_n) = \lim_{n\to\infty} \nu(A_n)$.
- If A is antitone, then $\nu(\bigcap_n A_n) = \lim_{n\to\infty} \nu(A_n)$.

Definition (Signed measures)**.** A **signed measure** on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{R}$.

Theorem 2.2. *If ν is a signed measure, then there exists a measurable partition $\Omega = P \cup N$ such that for all $A \in \mathcal{F}$*

$$\begin{aligned} A \subseteq P &\implies \nu(A) \geq 0 \\ A \subseteq N &\implies \nu(A) \leq 0 \end{aligned}$$

Remarks.

1. This decomposition is called the **Hahn decomposition** of ν.

2. Define $\nu^+(A) = \nu(A \cap P), \nu^-(A) = -\nu(A \cap N)$. Then ν^+, ν^- are finite positive measures such that $\nu = \nu^+ - \nu^-$. This determines ν^+, ν^- uniquely and the decomposition composition $\nu = \nu^+ - \nu^-$ is called the **Jordan decomposition** of ν.

3. If ν is a complex measure on $\mathcal{F}$, then $\operatorname{Re}\nu, \operatorname{Im}\nu$ are signed measures with Jordan decomposition $\nu_1 - \nu_2, \nu_3 - \nu_4$ respectively. Hence $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ is the Jordan decomposition of ν.

$$\nu_1, \nu_2, \nu_3, \nu_4 \leq |\nu| \leq \nu_1 + \nu_2 + \nu_3 + \nu_4$$

So $|\nu|$ is a finite measure.

Sketch. Define $\nu^+(A) = \sup_{\substack{B \in \mathcal{F} \\ B \subseteq A}} \nu(B)$. ν^+ is nonnegative and finitely additive.

Key step: $\nu^+(\Omega) < \infty$
By contradiction, construct inductively sequences A_n, B_n such that

$$\nu^+(A_n) = \infty, \nu(B_n) > n$$

by taking $A_0 = \Omega, B_{n+1} \subseteq A_n$ such that $\nu(B_n) > n$ (exists by continuity) and $A_{n+1} = B_{n+1}$ or $A_n \setminus B_{n+1}$. This contradicts countable additivity.

Now find a sequence A_n such that $\nu(A_n) > \nu^+(\Omega) - 2^{-n}$ and set $P = \liminf_n A_n, N = P^c$. Check that this works. □

Lecture 6

Definition (Absolute continuity)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\nu : \mathcal{F} \to \mathbb{C}$ a complex measure. ν is **absolutely continuous** with respect to μ, written $\nu \ll \mu$, if $\forall A \in \mathcal{F}, \mu(A) = 0 \implies \nu(A) = 0$.

Remarks.

- $\nu \ll \mu \implies |\nu| \ll \mu$, so if ν has Jordan decomposition $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ then $\nu_1, \nu_2, \nu_3, \nu_4 \ll \mu$.
- If $\nu \ll \mu$, then $\forall \varepsilon > 0, \exists \delta > 0, \forall A \in \mathcal{F}, \mu(A) < \delta \implies |\nu(A)| < \varepsilon$.

Example. Let $f \in L_1(\mu)$. Define $\nu(A) = \int_A f d\mu$ for $A \in \mathcal{F}$. By Dominated Convergence, ν is a complex measure and $\mu(A) = 0 \implies \nu(A) = 0$. So $\nu \ll \mu$.

Definition. $A \in \mathcal{F}$ is **σ-finite** if there exists A_n with $\mu(A_n) < \infty$ such that $A = \bigcup_n A_n$. Say μ is **σ-finite** if Ω is σ-finite.

Theorem 2.3 (Radon-Nikodym)**.** Let μ be a σ-finite measure and ν a complex measure such that $\nu \ll \mu$. Then there exists a unique $f \in L_1(\mu)$ such that, for all $A \in \mathcal{F}$, $\nu(A) = \int_A f d\mu$. Moreover, f takes values in $\mathbb{C}/\mathbb{R}/\mathbb{R}^+$ depending on where ν is valued.

Proof.
Uniqueness
standard

Existence
ν is a finite measure (by the Jordan decomposition). WLOG μ is a finite measure (by σ-finiteness). Let

$$\mathcal{H} = \left\{ h : \Omega \to \mathbb{R}^+ \,\middle|\, h \text{ integrable}, \forall A \in \mathcal{F}, \int_A h d\mu \ll \nu(A) \right\}$$

$\mathcal{H} \neq \varnothing$ (eg $0 \in \mathcal{H}$). Let $\alpha = \sup_{h \in \mathcal{H}} \int_\Omega h d\mu$. We see $0 \leq \alpha \leq \nu(\Omega)$.

Claim
There exists $f \in \mathcal{H}$ such that $\alpha = \int_\Omega f d\mu$.

Idea
If $\int_A f d\mu < \nu(A)$, then $f + \frac{1}{n} 1_A \in \mathcal{H}$ (morally, not literally), contradicting the definition of α.

Pick that f. Define $\nu_n(A) = \nu(A) - \int_A f d\mu - \frac{1}{n}\mu(A)$. ν_n has Hahn decomposition $\Omega = P_n \cup N_n$. Then $f + \frac{1}{n} P_n \in \mathcal{H}$. By definition of α, $\mu(P_n) = 0$. Since $\nu \ll \mu$, $\nu(P_n) = 0$. Let $P = \bigcup_n P_n, N = \bigcap_n N_n$. Then $\Omega = P \cup N, \mu(P) = \nu(P) = 0$. For $A \in \mathcal{F}$,

$$A \subseteq P \implies \int_A f d\mu = \nu(A) = 0$$
$$A \subseteq N \implies \forall n, \nu_n \leq 0 \implies \int_A f d\mu \geq \nu(A)$$

□

Remarks.

- Without assuming $\nu \ll \mu$, the proof shows there is a decomposition $\nu = \nu_1 + \nu_2$ where $\nu_1(A) = \int_A f d\mu$ and $\nu_2 \perp \mu$ (**orthogonal**, ie there exists a measurable decomposition $\Omega = P \cup N$ such that $\mu(P) = 0, |\nu_2|(N) = 0$). $\nu = \nu_1 + \nu_2$ is the **Lebesgue decomposition** of ν.
- The unique f in Theorem 2.3 is the **Radon-Nikodym derivative** of ν with respect to μ, denoted $\frac{d\nu}{d\mu}$. The result says

$$\nu(A) = \int_\Omega 1_A d\nu = \int_A \frac{d\nu}{d\mu} d\mu = \int_\Omega 1_A \frac{d\nu}{d\mu} d\mu$$

  Hence a measurable function g is ν-integrable iff $g\frac{d\nu}{d\mu}$ is μ-integrable and then

$$\int_\Omega f d\nu = \int_\Omega g \frac{d\nu}{d\mu} d\mu$$

2.1 Dual space of $L_p(\mu)$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $1 \leq p < \infty, 1 < q \leq \infty$ such that $p^{-1} + q^{-1} = 1$. For $g \in L_q$, define $\phi_g : L_p \to \mathbb{K}$ by $\phi_g(f) = \int_\Omega fg d\mu$. By Hölder, $fg \in L_1$, and $|\phi_g(f)| \leq \|f\|_p \|g\|_q$. So ϕ_g is well-defined, linear and bounded with $\|\phi_g\| \leq \|g\|_q$. Hence $\phi_g \in L_p^*$ and $\phi : L_q \to L_p^*$ is linear and bounded with $\|\phi\| \leq 1$.

Theorem 2.4.

1. If $1 < p < \infty$, then ϕ is an isometric isomorphism. So $L_p^* \cong L_q$.
2. If $p = 1$ and μ is σ-finite, then ϕ is an isometric isomorphism. So $L_1^* \cong L_\infty$.

Proof.

1. ϕ **is isometric**
   Let $g \in L_1$. We know $\|\phi_g\| \leq \|g\|_q$. Let λ be a measurable function with $|\lambda| = 1, \lambda g = |g|$. let $f = \lambda |g|^{q-1}$. Then

$$\|f\|_p^p = \int_\Omega |g|^{pq-p} \, d\mu = \int_\Omega |g|^q \, d\mu = \|g\|_q^q$$

   So $f \in L_p$ and $\|f\|_p = \|g\|_q^{\frac{q}{p}}$. Then

$$\|q\|_q^{\frac{q}{p}} \|\phi_g\| \geq |\phi_g(f)| = \int_\Omega |g|^q \, d\mu = \|g\|_q^q$$

   So $\|\phi_g\| \geq \|g\|_q^{q-\frac{q}{p}} = \|g\|_q$.

   ϕ **is onto**
   Fix $\psi \in L_p^*$. We seek $g \in L_q$ such that $\psi = \phi_g$. Idea: We want $\psi(1_A) = \int_A g d\mu$.

   Case 1: μ is finite
   For $A \in \mathcal{F}$, $1_A \in L_p$, so define $\nu(A) = \psi(1_A)$. $\nu() = 0$ and, if $A = \bigcup_p A_n \in \mathcal{F}$, then $\sum_k 1_{A_k} = 1_A$ in L_p, so

$$\sum_k \nu(A_k) = \sum_k \psi(1_{A_k}) = \psi(1_A)$$

   Hence ν is a complex measure.
   If $A \in \mathcal{F}, \mu(A) = 0$, then $1_A = 0$ ae in L_p, so $\nu(A) = \psi(1_A) = 0$. Hence $\nu \ll \mu$.
   By Theorem 2.3, find $g \in L_1$ such that $\forall A \in \mathcal{F}, \nu(A) = \int_A g d\mu$. Hence

$$\psi(1_A) = \int_\Omega 1_A g d\mu \text{ for all } A \in \mathcal{F}$$
$$\psi(f) = \int_\Omega f g d\mu \text{ for all simple function } f$$

   Given $f \in L_\infty$, find simple functions f_n tending to f in L_∞. So $\psi(f_n) \to \psi(f)$ and $f_n g \to fg$ (by Hölder for $\infty, 1$), meaning that

$$\psi(f) = \int_\Omega f g d\mu \text{ for all } f \in L_\infty$$

   For $n \in \mathbb{N}$, let $A = \{|g| \leq n\}$ and $f_n = \lambda 1_{A_n} |g|^{q-1}$ where $|\lambda| = 1, \lambda g = |g|$. As $f_n \in L_\infty$,

$$\int_\Omega f_n g d\mu = \int_{A_n} |g|^q \, d\mu = \psi(f_n)$$

   So $\left(\int_A |g|^q \, d\mu\right)^{q^{-1}} \leq \|\psi\|$. By Monotone Convergence, $g \in L_q$.
   Given $f \in L_p$, find simple functions f_n tending to f in L_p. So $\psi(f_n) \to \psi(f)$ and $f_n g \to fg$ in L_1 (by Hölder for p, q). Hence $\psi(f) = \int_\Omega f g d\mu$, as wanted.

   Before going onto Case 2, for $A \in \mathcal{F}$, let $\mathcal{F}_A = \{B \in \mathcal{F} \mid B \subseteq A\}$ and $\mu_A = \mu \restriction_{\mathcal{F}_A}$ so that $(A, \mathcal{F}_A, \mu_A)$ is a measure space. Then $L_p(\mu_A) \subseteq L_p(\mu)$ (by extending $f \in L_p(\mu_A)$ by 0 outside A). Let $\psi_A = \psi \restriction_{L_p(\mu_A)}$.

Lecture 7

   Claim. If $A, B \in \mathcal{F}$ are disjoint, then

$$\|\psi_{A \cup B}\| = (\|\psi_A\|^q + \|\psi_B\|^q)^{\frac{1}{q}}$$

Proof.

$$\begin{aligned}
(\|\psi_A\|^q + \|\psi_B\|^q)^{\frac{1}{q}} &= \sup_{\substack{a,b\geq 0\\ a^p+b^p\leq 1}} a\|\psi_A\| + b\|\psi_B\| \\
&= \sup_{\substack{a,b\geq 0\\ a^p+b^p\leq 1\\ f\in B_{L_p(\mu_A)}\\ g\in B_{L_p(\mu_B)}}} a|\psi_A(f)| + b|\psi_B(g)| \\
&= \sup_{\substack{|a|^p+|b|^p\leq 1\\ f\in B_{L_p(\mu_A)}\\ g\in B_{L_p(\mu_B)}}} \underbrace{|a\psi_A(f) + b\psi_B(g)|}_{\psi_{A\cup B}(af+bg)} \\
&= \sup_{h\in L_p(\mu_{A\cup B})} |\psi_{A\cup B}(h)| \\
&= \|\psi_{A\cup B}\|
\end{aligned}$$

□

Case 2: μ is σ-finite
Find a measurable partition $\Omega = \bigcup_n A_n$ such that $\mu(A_n) < \infty$. By Case 1, find, for each n, $g_n \in L_q(A_n)$ such that $\psi_{A_n} = \phi_{g_n}$, ie

$$\psi(f) = \int_{A_n} fg_n d\mu \text{ for all } f \in L_q(\mu_{A_n})$$

If we define g on Ω by $g = g_n$ on A_n, then $g \in L_q$ and

$$\psi(f) = \phi_g(f) \text{ for all } f \in L_p(\mu_{A_n})$$

Hence $\psi = \phi_g$ on $\overline{\text{Span}} \bigcup_n L_p(\mu_{A_n}) = L_p(\mu)$.

Case 3: General n
First observe that, for $f \in L_p(\mu)$, $\{f \neq 0\}$ is σ-finite. Indeed,

$$\{f \neq 0\} = \bigcup_n \left\{\frac{1}{n} < |f|\right\}$$

and

$$\mu\left\{\frac{1}{n} < |f|\right\} \leq |n^p| \|f\|_p^p < \infty \text{ by Markov}$$

Choose $f_n \in B_{L_p}$ such that $\psi(f_n) \to \|\psi\|$. Then $A = \bigcup_n \{f_n \neq 0\}$ is σ-finite and $\|\psi_A\| = \|\psi\|$. By the claim,

$$\|\psi\| = (\|\psi_A\|^q + \|\psi_{A^c}\|^q)^{\frac{1}{q}}$$

So $\Psi_{A^c} = 0$. By Case 2, find $g \in L_q(\mu_A) \subseteq L_q(\mu)$ such that $\psi_A = \phi_g$, so that

$$\psi(f) = \psi_A f\restriction_A + \psi A^c(f \restriction A^c) = \int_A fgd\mu + 0 = \int_\Omega fgd\mu$$

2. $p = 1, \mu$ **is σ-finite**
ϕ **is isometric**
Let $g \in L_\infty$. We know $\|\phi_g\| \leq \|g\|_\infty$ (by Hölder) Fix $s < \|g\|_\infty$. Then $\mu\{s < |g|\} > 0$. Since μ is σ-finite, find $A \subseteq \{s < |g|\}$ such that $0 < \mu(A) < \infty$. Choose a

measurable function λ such that $|\lambda| = 1, \lambda g = |g|$. Then $\lambda 1_A \in L_1, \|\lambda 1_A\|_1 = \mu(A)$. Now,

$$\mu(A) \|\phi_g\| \geq |\phi_g(\lambda 1_A)| = \int_A |g| \, d\mu \geq s\mu(A)$$

So $\|\phi_g\| \geq s$. Taking the sup, $\|\phi_g\| \geq \|g\|_\infty$.

ϕ is onto
Fix $\psi \in L_q^*$. We seek $g \in L_\infty$ such that $\psi = \phi_g$.

Case 1: μ is finite
Define $\nu(A) = \psi(1_A)$ for all $A \in \mathcal{F}$. Follow the same steps as for $1 < p < \infty$.

Case 2: μ is σ-finite
This time, prove that

$$\|\psi_{A \cup B}\| = \max(\|\psi_A\|, \|\psi_B\|)$$

for all $A, B \in \mathcal{F}$ disjoint and proceed as before.

□

Corollary 2.5. For $1 < p < \infty$, $L_p(\mu)$ is reflexive.

Proof. Let $\psi \in L_p^{**}$. Then $g \mapsto \langle \phi_g, \psi \rangle : L_q \to \mathbb{K}$ is in L_q^*. By Theorem 2.4.i, find $f \in L_p$ such that

$$\langle \phi_g, \psi \rangle = \int_\Omega fg d\mu \, \langle f, \psi_g \rangle = \left\langle \phi_g, \hat{f} \right\rangle$$

Since $L_p^* = \{\phi_g \mid g \in L_q\}$, this proves $\psi = \hat{f}$. □

2.2 Dual space of $C(K)$

Throughout, K will be a compact Hausdorff topological space. Define

$$\begin{aligned} C(K) &= \{f : K \to \mathbb{C} \mid f \text{ continuous}\} \\ C^{\mathbb{R}}(K) &= \{f : K \to \mathbb{R} \mid f \text{ continuous}\} \\ C^{+}(K) &= \{f : K \to \mathbb{R}^{+} \mid f \text{ continuous}\} \\ M(K) &= C(K)^* \\ M^{\mathbb{R}}(K) &= \{\phi \in M(K) \mid \forall f \in C^{\mathbb{R}}(K), \phi(f) \in \mathbb{R}\} \\ M^{+}(K) &= \{\phi : C(K) \to \mathbb{C} \mid \phi \text{ is } \mathbb{R}\text{-linear}, \forall f \in C^{+}(K), 0 \le \phi(f) \in \mathbb{R}\} \end{aligned}$$

$C(K), C^{\mathbb{R}}(K)$ are complex/real Banach spaces in the sup norm: $\|f\|_\infty = \sup_K |f|$. $M(K)$ is a complex Banach space in the operator norm. $M^{\mathbb{R}}(K)$ is a closed real-linear subspace of $M(k)$. Elements of $M^+(K)$ are called **positive linear functionals**.

Aim. Identify $M(K), M^{\mathbb{R}}(K)$.

Lecture 8

The next lemma tells us that it's enough to understand $M^+(K)$.

Lemma 2.6.

1. For all $\phi \in M(K)$, there are unique $\phi_1, \phi_2 \in M^{\mathbb{R}}(K)$ such that $\phi = \phi_1 + i\phi_2$.
2. $\phi \mapsto \phi \restriction_{C^{\mathbb{R}}(K)}: M^{\mathbb{R}}(K) \to C^{\mathbb{R}}(K)^*$ is an isometric isomorphism.
3. $M^+(K) \subseteq M(K)$ and $M^+(K) = \{\phi \in M(K) \mid \|\phi\| = \phi(1)\}$
4. For all $\phi \in M^{\mathbb{R}}(K)$, there are unique $\phi^+, \phi^- \in M^+(K)$ such that $\phi = \phi^+ - \phi^-$ and $\|\phi\| = \|\phi^+\| + \|\phi^-\|$.

Proof.

1. Let $\phi \in M(K)$. Then $\overline{\phi}$ sending $f \mapsto \phi(\overline{f})$ is in $M(K)$ as well and $\phi \in M^{\mathbb{R}}(K) \iff \overline{\phi} = \phi$.

   Uniqueness
   Assume $\phi = \phi_1 + i\phi_2$ where $\phi_1, \phi_2 \in M^{\mathbb{R}}(K)$. Then $\overline{\phi} = \phi_1 - i\phi_2$, so

   $$\phi_1 = \frac{\phi + \overline{\phi}}{2}, \phi_2 = \frac{\phi - \overline{\phi}}{2i}$$

   Existence
   Check that the above works

2. Let $\phi \in M^{\mathbb{R}}(K)$. We show $\left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\| = \|\phi\|$. Clearly, $\left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\| \le \|\phi\|$. Let $f \in B_{C(K)}$. Choose $\lambda \in \mathbb{C}, |\lambda| = 1, \lambda\phi(f) = |\phi(f)|$, so that

   $$\begin{aligned} |\phi(f)| &= \lambda\phi(f) \\ &= \phi(\lambda f) \\ &= \phi(\mathrm{Re}(\lambda f)) + \cancelto{0}{\phi(\mathrm{Im}(\lambda f))} \\ &\le \left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\| \|\mathrm{Re}(\lambda f)\|_\infty \\ &\le \left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\| \end{aligned}$$

Hence $\|\phi\| \leq \left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\|$.
Finally, given $\psi \in C^{\mathbb{R}}(K)$, define $\phi(f) = \psi(\operatorname{Re} f) + i\psi(\operatorname{Im} f)$. Then $\phi \in M(K)$ and $\phi \restriction_{C^{\mathbb{R}}(K)} = \psi$.

3. $M^+(K) \subseteq M(K)$
Let $\phi \in M^+(K)$. For $f \in B_{C^{\mathbb{R}}(K)}$, we have $1 \pm f \geq 0$, so $\phi(1 \pm f \geq 0$. Hence $\phi(f) \in \mathbb{R}$ and $|\phi(f)| \leq \phi(1)$. So $\phi \restriction_{C^{\mathbb{R}}(K)} \in C^{\mathbb{R}}(K)^*$ and $\left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\| = \phi(1)$. By (ii), $\phi \in M(K), \|\phi\| = \phi(1)$.

   $M^+(K) = \{\phi \in M(K) \mid \|\phi\| = \phi(1)$
   We have already checked one inclusion. Let $\phi \in M(K)$ with $\|\phi\| = \phi(1)$. WLOG $\|\phi\| = \phi(1) = 1$. Let $f \in B_{C^{\mathbb{R}}(K)}$ and write $\phi(f) = a + ib$ where $a, b \in \mathbb{R}$. We want $b = 0$. For $t \in \mathbb{R}$,

$$\begin{aligned} |\phi(f + it)|^2 &= a^2 + (b+t)^2 = a^2 + b^2 + t^2 + 2bt \\ &\leq \|f + it\|_\infty \leq 1 + t^2 \end{aligned}$$

   So $b = 0$.
   Given $f \in C^+(K)$ with $0 \leq f \leq 1$, we have $-1 \leq 2f - 1 \leq 1$, so $|\phi(2f-1)| \leq \|2f - 1\|_\infty \leq 1$, ie $-1 \leq 2\phi(f) - 1 \leq 1$. So $\phi(f) \geq 0$.

4. Let $\phi \in M^{\mathbb{R}}(K)$. Assume for a moment that $\phi = \psi_1 - \psi_2$ where $\psi_1, \psi_2 \in M^+(K)$. For $f, g \in C^+(K)$ with $0 \leq g \leq f$, we have $\psi_1(f) \geq \psi_1(g) = \phi(g) + \psi_2(g) \geq \phi(g)$. So

$$\psi_1(f) \geq \sup_{0 \leq g \leq f} \phi(g)$$

   For $f \in C^+(K)$, define

$$\phi^+(f) = \sup_{0 \leq g \leq f} \phi(g)$$

   Observe that $\phi^+ \geq 0, \phi^+(f) \leq \|\phi\| \|f\|_\infty, \phi^+(f) \geq \phi(f), \phi^+$ is linear.
   Next, for $f \in C^{\mathbb{R}}(K)$, write $f = f_1 - f_2$ where $f_1, f_2 \in C^+(K)$ and define $\phi^+(f) = \phi^+(f_1) - \phi^+(f_2)$. This is well-defined and $\mathbb{R}$-linear. Then ϕ is $\mathbb{C}$-linear since $\phi^+(f) \geq 0$. For all $f \in C^+(K)$ and $\phi^+ \in M^+(K)$.
   Define $\phi^- = \phi^+ - \phi$. For $f \in C^+(K)$, $\phi^+(f) \geq \phi(f)$, so $\phi^-(f) \geq 0$, namely $\phi^- \in M^+(K)$.
   We now see that $\|\phi\| \leq \|\phi^+\| + \|\phi^-\|$. Given $f \in C^+(K), 0 \leq f \leq 1$, we have $-1 \leq 2f - 1 \leq 1$, so

$$2\phi(f) - \phi(1) = \phi(2f - 1) \leq \|\phi\|$$

   Taking the sup over f, we thus check that

$$\left\|\phi^+\right\| + \left\|\phi^-\right\| = \phi^+(1) + \phi^-(1) = 2\phi^+(1) - \phi(1) \leq \|\phi\|$$

   Uniqueness
   Assume $\phi = \psi_1 - \psi_2, \psi_1, \psi_2 \in M^+(K), \|\phi\| = \|\psi_1\| + \|\psi_2\|$. From the initial observation, $\psi_1 \geq \phi^+$, hence $\psi_2 = \psi_1 - \phi \geq \phi^+ - \phi = \phi^-$. Therefore $\psi_1 - \phi^+, \psi_2 - \phi^- \in M^+(K)$. By (iii),

$$\left\|\psi_1 - \phi^+\right\| + \left\|\psi_2 - \phi^-\right\| = \psi_1(1) - \phi^+(1) + \psi_2(1) - \phi^-(1) = \|\phi\| - \|\phi\| = 0$$

   Hence $\psi_1 = \phi^+, \psi_2 = \phi^-$.

$\square$

Topological preliminaries

1. K being compact Hausdorff, it is **normal**: given disjoint closed sets E, F in K, there are disjoint open sets U, V such that $E \subseteq U, F \subseteq V$. Equivalently, given $E \subseteq U \subseteq K$, E, closed, U open, there exists V open such that $E \subseteq V \subseteq \overline{V} \subseteq U$.
2. Urysohn says: given disjoint closed sets E, F, there is a continuous function $f : K \to [0, 1]$ such that $f = 0$ on E, $f = 1$ on F.
3. Write $f \prec U$ to mean that U is an open set, f is continuous and $\operatorname{supp} f \subseteq U$. Write $E \prec f$ to mean that E is closed, f is continuous and $f = 1$ on E.
4. Urysohn then becomes: Given $E \subseteq U$, there exists f such that $E \prec f \prec U$.

Lemma 2.7. Let E closed, $U_1, \dots, U_n$ open such that $E \subseteq \bigcup_n U_n$. Then

1. There exist open sets V_j such that $\overline{V_j} \subseteq U_j$ and $E \subseteq \bigcup_j V_j$.
2. There exist $f_j \prec U_j$ such that $0 \leq \sum_j f_j \leq 1$ and $\sum_j f_j = 1$ on E.

Proof.

1. Induction on n: $n = 0$
   Obvious.

   $n > 0$
   $E \setminus U_n \subseteq \bigcup_{j<n} U_j$ so, by induction, find open sets V_j such that $\overline{V_j} \subseteq U_j$ for all $j < n$ and $E \setminus U_n \subseteq \bigcup_{j<n} U_j$. So $\underbrace{E \setminus \bigcup_{j<n} V_j}_{\text{closed}} \subseteq \underbrace{U_n}_{\text{open}}$. By Urysohn, find an open V_n such that
   $$E \setminus \bigcup_{j<n} V_j \subseteq V_n \subseteq \overline{V_n} \subseteq U_n$$

2. Find the V_j as in (i) for $1 \leq j \leq n$ and by Urysohn find h_j such that $\overline{V_j} \prec h_j \prec U_j$. By Urysohn again, find h_0 such that $\left(\bigcup_j U_j\right)^c \prec h_0 \prec E^c$. Let $h = \sum_{j=0}^n h_j \geq 1$ and $f_j = \frac{h_j}{h}$ for $1 \leq j \leq n$. Then $0 \leq \sum_{j=1}^n \leq 1$, $f_j \prec U_j$ and $\sum_{j=1}^n f_j = 1$ on E.

□

Definition (Borel measures)**.** Let X be a Hausdorff space and $\mathcal{G}$ its family of open sets. The **Borel σ-algebra** is $\mathcal{B} := \sigma(\mathcal{G})$, the σ-algebra generated by open sets. Elements of $\mathcal{B}$ are called **Borel sets**. A **Borel measure** on X is a measure μ on $\mathcal{B}$. We say μ is **regular** if

1. $\mu(E) < \infty$ for all compact $E \subseteq X$
2. $\mu(A) = \inf_{\substack{U \text{ open} \\ A \subseteq U}} \mu(A)$ for all Borel set A
3. $\mu(U) = \sup_{\substack{E \text{ compact} \\ E \subseteq U}} \mu(E)$ for all open U

A complex Borel measure ν is **regular** if $|\nu|$ is regular.

If X is compact and μ is a Borel measure on X, then

$$\begin{aligned} \mu \text{ regular} &\iff \mu(X) < \infty \text{ and } \forall A \in \mathcal{B}, \mu(A) = \inf_{\substack{U \text{ open} \\ A \subseteq U}} \mu(U) \\ &\iff \mu(X) < \infty \text{ and } \forall A \in \mathcal{B}, \mu(A) = \sup_{\substack{E \text{ closed} \\ E \subseteq A}} \mu(E) \end{aligned}$$

Definition (Integration with respect to a complex measure)**.** Let Ω be a set, $\mathcal{F}$ a σ-algebra on Ω, ν a complex measure on $\mathcal{F}$. Write $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ the Jordan decomposition. Say a measurable function is **ν-integrable** if f is $|\nu|$-integrable, or equivalently if f is $\nu_1, \nu_2, \nu_3, \nu_4$-integrable. Define

$$\int_\Omega f d\nu = \int_\Omega f d\nu_1 - \int_\Omega f d\nu_2 + i \int_\Omega f d\nu_3 - i \int_\Omega f d\nu_4$$

Lecture 9

Proposition.

1. $\int_\Omega d\nu = \nu(A)$ for all $A \in \mathcal{F}$.
2. Linearity: If $f, g : \Omega \to \mathbb{C}$ are ν-integrable and $\lambda \in \mathbb{C}$, then
$$\int_\Omega f + g d\nu = \int_\Omega f d\nu + \int_\Omega g d\nu, \int_\Omega \lambda f d\nu = \lambda \int_\Omega f d\nu$$
3. Dominated Convergence: Let f_n, f, g be measurable functions $\Omega \to \mathbb{C}$ such that $f_n \to f$ ae (with respect to $|\nu|$), $g \in L_1$ and $\forall n, f_n \leq g$ ae. Then f is ν-integrable and $\int_\Omega f_n d\nu \to \int_\Omega f d\nu$
4. $\left|\int_\Omega f d\nu\right| \leq \int_\Omega |f| \, d\, |\nu|$ for all ν-integrable f. This is true for simple functions by properties 1 and 2. For general f, use Dominated Convergence.

Let ν be a complex Borel measure on K. Then for $f \in C(K)$ we have

$$\int_K |f| \, d\, |\nu| \leq \|f\|_\infty |\nu| (K) = \|f\|_\infty \|\nu\|_1$$

So f is ν-integrable. Define $\phi : C(K) \to \mathbb{C}$ by $\phi(f) = \int_\Omega f d\nu$. Then $\phi \in M(K)$ and $\|\phi\| \leq \|\nu\|_1$. If ν is a signed measure, then $\phi \in M^{\mathbb{R}}(K)$. If ν is a positive measure, then $\phi \in M^+(K)$.

Theorem 2.8 (Riesz Representation Theorem)**.** For every $\phi \in M^+(K)$, there exists a unique regular Borel measure μ on K that represents ϕ: $\phi(f) = \int_K f d\mu$ for all $f \in C(K)$. Moreover, $\|\phi\| = \mu(K) = \|\mu\|_1$.

Proof.
Uniqueness
Assume μ_1, μ_2 both represent ϕ. Let $E \subseteq U \subseteq K$ where E closed, U open. By Urysohn, find f such that $E \prec f \prec U$. Now,

$$\mu_1(E) \leq \int_K f d\mu_1 = \phi(f) = \int_K f d\mu_2 \leq \mu_2(U)$$

Taking the inf over U, we get $\mu_1(E) \leq \mu_2(E)$. By symmetry, $\mu_1(E) = \mu_2(E)$. By regularity, $\mu_1 = \mu_2$.

Existence
For U open, define $\mu^*(U) = \sup_{f \prec U} \phi(f)$. Note that

$$\mu^*(U) \geq 0, \mu \text{ monotone}, \mu^*(K) = \phi(1)$$

It follows that, for V open, $\mu^*(V) = \inf_{U \supseteq V} \mu^*(U)$. Hence extend the definition of μ^* to

$$\mu^*(A) = \inf_{U \supseteq A} \mu^*(U)$$

We will show that μ^* is an outer measure.

- $\mu(\varnothing) = 0$
- If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$.
- Do we have $\mu^*\left(\bigcup_n A_n\right) = \sum_n \mu^*(A_n)$?
First assume that the $A_n = U_n$ are open. Let $U = \bigcup_n U_n$. Assume $f \prec U$ and let $E = \operatorname{supp} f$. $E \subseteq \bigcup_n U_n$, so by compactness find N such that $E \subseteq \bigcup_{n=1}^N U_n$. By Lemma 2.7, find $h_n \prec U_n$ with $\sum_{n=1}^N h_n \leq 1$ and $\sum_{n=1}^N h_n = 1$ on E. So $f = \sum_{n=1}^N f h_n$ and

$$\begin{aligned}\phi(f) &= \sum_{n=1}^{N} \phi(f h_n) \\ &\leq \sum_{n=1}^{N} \mu^*(U_j) \text{ as } f h_n \prec U_n \\ &\leq \sum_{n} \mu^*(U_n)\end{aligned}$$

Taking the sup over f, we get $\mu^*(U) \leq \sum_n \mu^*(U_n)$. It follows that

$$\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$$

We now let $\mathcal{M}$ be the set of μ^*-measurable sets. Then $\mathcal{M}$ is a σ-algebra and $\mu^* \restriction_{\mathcal{M}}$ is a measure on $\mathcal{M}$.
To restrict it further to a Borel, we now show that $\mathcal{B} \subseteq \mathcal{M}$. It's enough to show that $\mathcal{G} \subseteq \mathcal{M}$.
Let U open. We need

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U) \text{ for all } A$$

First, let $A = V \in \mathcal{G}$. Fix $f \prec V \cap U$ and $g \prec V \setminus \operatorname{supp} f$. Then $f + g \prec V$, thus

$$\mu^*(V) \geq \phi(f + g) = \phi(f) + \phi(g)$$

Taking the sup over g,

$$\mu^*(V) \geq \phi(f + g) = \phi(f) + \mu^*(V \setminus \operatorname{supp} f) \geq \phi(f) + \mu^*(V \setminus U)$$

Taking the sup over f,

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$$

Now let A be arbitrary. Fix V open such that $A \subseteq V$. then

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$$

Taking the inf over V,

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$$

Now, $\mu := \mu^* \restriction_{\mathcal{B}}$ is a Borel measure on K. We have

$$\mu(K) = \phi(1) = \|\phi\| < \infty$$

and by definition μ is regular. It remains to show that $\phi(f) = \int_K f d\mu$ for all $f \in C(K)$. It is enough to check that for $f \in C^{\mathbb{R}}(K)$ and enough to check that

$\phi(f) \leq \int_K f d\mu$ (apply this to $-f$).
Fix $0 < a < b$ in $\mathbb{R}$ such that $\phi(1) \in [a, b]$. Let $\varepsilon > 0$. Choose $0 \leq y_0 < a \leq y_1 < \cdots < y_n = b$ such that

$$y_j < y_{j-1} + \varepsilon$$

Let $A_j = f^{-1}]y_{j-1}, y_j]$. Those sets form a measurable partition of K. Choose closed sets E_j and open sets U_j such that $E_j \subseteq A_j \subseteq U_j$ and $\mu(U_j \setminus E_j) < \frac{\varepsilon}{n}$ (by regularity) and $f(U_j) \subseteq]y_{j-1}, y_j]$. By Lemma 2.7, find $h_j \prec U_j$ for each j such that $\sum_j h_j = 1$. Now,

$$\begin{aligned}
\phi(f) &= \sum_j \phi(f_j) \\
&\leq \sum_j (y_j + \varepsilon)\phi(h_j) \\
&\leq \sum_j (y_{j-1} + 2\varepsilon)\left(\mu(E_j) + \frac{\varepsilon}{n}\right) \\
&= \sum_j y_{j-1}\mu(E_j) + \underbrace{\sum_j (b + \varepsilon) + 2\varepsilon\mu(K) + 2\varepsilon^2}_{o(1)} \\
&= \int_K \sum_j y_{j-1} 1_{E_j} d\mu + o(1) \qquad \leq \int_K f d\mu + o(1)
\end{aligned}$$

since $f \leq y_j + \varepsilon$ on U_j, $h_j \prec U_j$ and $\phi \in M^+(K)$. So $\phi(f) \leq \int_K f d\mu$.

□

Lecture 10

Corollary 2.9. For every $\phi \in M(K)$, there exists a unique regular complex Borel measure ν on K that represents ϕ: $\phi(f) = \int_K f d\nu$ for all $f \in C(K)$. Moreover, $\|\phi\| = \|\nu\|_1$ and if $\phi \in M^{\mathbb{R}}(K)$ then ν is a signed measure.

Proof.
Existence
Apply Lemma 2.6 and Theorem 2.8 to obtain a regular complex Borel measure representing ϕ. We now want $\|\phi\| = \|\nu\|_1$.
We already know $\|\phi\| \leq \|\nu\|_1$. Take a measurable partition $K = \bigcup_{j=1}^n A_j$. Fix $\varepsilon > 0$ and closed sets E_j, open sets U_j such that $E_j \subseteq A_j \subseteq U_j$, $|\nu|(U_j \setminus E_j) < \frac{\varepsilon}{n}$ (ν is regular). We can also assume $U_i \subseteq \bigcap_{j \neq i} E_j^c$. Fix $\lambda_j \in \mathbb{C}$ such that $|\lambda_j| = 1, \lambda_j \nu(E_j) = |\nu(E_j)|$. By Lemma 2.7, find $h_j \prec U_j$ such that $\sum_{j=1}^n h_j = 1$. Then $E_j \prec h_j$, hence

$$\begin{aligned}
\left| \int_K \left(\sum_{j=1}^n \lambda_j 1_{E_j} - \sum_{j=1}^n \lambda_j h_j \right) d\nu \right| &\leq \sum_{j=1}^n \int_K \left| 1_{E_j} - h_j \right| d|\nu| \\
&\leq \sum_{j=1}^n |\nu| (U_j \setminus E_j) < \varepsilon
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{j=1}^{n} |\nu(A_j)| &\leq \sum_{j=1}^{n} |\nu(E_j)| + \varepsilon \\
&= \sum_{j=1}^{n} \lambda_j \nu(E_j) + \varepsilon \\
&= \int_K \sum_{j=1}^{n} \lambda_j 1_{E_j} d\nu + \varepsilon \\
&\leq \left| \int_K \sum_{j=1}^{n} \lambda_j h_j d\nu \right| + 2\varepsilon \\
&\leq \left| \phi \left(\sum_{j=1}^{n} \lambda_j h_j \right) \right| + 2\varepsilon \\
&\leq \|\phi\| \left\| \sum_{j=1}^{n} \lambda_j h_j \right\|_\infty + 2\varepsilon \\
&\leq \|\phi\| + 2\varepsilon
\end{aligned}$$

It follows that $\|\nu\|_1 \leq \|\phi\|$. □

Corollary 2.10. The space of regular real (resp. complex) Borel measures on K is a real (resp. complex) Banach space in $\|\cdot\|_1$ isomorphic to $M^{\mathbb{R}}(K)$ (resp. $M(K)$).

3 Weak topologies

Let X be a set and $\mathcal{F}$ a set of functions on X such that each $f \in \mathcal{F}$ is a function $X \to Y_f$ where Y_f is a topological space. The **weak topology** $\sigma(X, \mathcal{F})$ **on** X **generated by** $\mathcal{F}$ is the smallest topology on X that makes each $f \in \mathcal{F}$ continuous.

Remarks.

1. $S = \{f^{-1}(U) \mid f \in \mathcal{F}, U \subseteq Y_f \text{ open}\}$ is a subbase of $\sigma(X, \mathcal{F})$. So

$$\begin{aligned} V \subseteq X \text{ open} &\iff \forall x \in V, \exists F \subseteq \mathcal{F} \text{ finite}, \forall f \in F, U_f \subseteq Y_f \text{ and } x \in \bigcap_{f \in F} f^{-1}(U_f) \subseteq V \\ &\iff \forall x \in V, \exists F \subseteq \mathcal{F} \text{ finite, open neighborhoods } U_f \text{ of } f(x), \bigcap_{f \in F} U_f \subseteq V \end{aligned}$$

2. More generally, if S_f is a subbase in Y_f, then $\{f^{-1}(U) \mid f \in \mathcal{F}, U \in S_f\}$ is a subbase of $\sigma(X, \mathcal{F})$.
3. If Y_f is Hausdorff for all $f \in \mathcal{F}$ and $\mathcal{F}$ **separates points of** X ($\forall x \neq y, \exists f \in \mathcal{F}, f(x) \neq f(y)$), then $\sigma(X, \mathcal{F})$ is Hausdorff.
4. Let $Y \subseteq X, \mathcal{F}_Y = f \restriction_Y | f \in \mathcal{F}$. Then $\sigma(Y, \mathcal{F}_Y) = \sigma(X, \mathcal{F}) \restriction_Y$.
5. **Universal property**: Let Z be a topological space and $g : Z \to X$. then

$$g \text{ continuous} \iff \forall f \in \mathcal{F}, f \circ g : Z \to Y_f \text{ continuous}$$

Example.

1. Let X be a topological space, $Y \subseteq X$ and $\iota : Y \to X$ the inclusion map. Then $\sigma(Y, \iota)$ is the subspace topology on Y.
2. Let Γ be a set, X_γ a topological space for each $\gamma \in \Gamma$, $X = \prod_{\gamma \in \Gamma} X_\gamma$. For each γ, we have $\pi_\gamma : X \to X_\gamma$ sending $x \mapsto x_\gamma$, the **evaluation map at** γ, or **projection onto** X_γ. The weak topology $\sigma(X, \{\pi_\gamma \mid \gamma \in \Gamma\})$ is called the **product topology** on X.

$$V \subseteq X \text{ open} \iff \underset{\{y \mid \forall \gamma \in s, y_\gamma \in U_\gamma\} \subseteq V}{\forall x \in V, \exists s \subseteq \Gamma \text{ finite}, U_\gamma \text{ neighborhood of } x_\gamma,}$$

Proposition 3.1. Let X be a set. For each n, let (Y_n, d_n) be a metric space and $f_n : X \to Y_n$ be a separating family of functions. Then $\sigma(X, \{f_n \mid n \in \mathbb{N}\})$ is metrisable.

Proof. Call $\sigma := \sigma(X, \{f_n \mid n \in \mathbb{N}\})$. Define

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n} \min(d(f_n(x), f_n(y)), 1)$$

This is a metric on X. Given $0 < \varepsilon < 1$, if $d(x, y) < 2^{-n}\varepsilon$, then $d(f_n(x), f_n(y)) < \varepsilon$. So each f_n is continuous with respect to the topology τ induced by that metric. Hence $\sigma \subseteq \tau$.
Reciprocally, $y \mapsto d(x, y)$ is σ-continuous for each x by the Weierstrass M-test since

$$y \mapsto 2^{-n} \min(d(f_n(x), f_n(y)), 1)$$

is σ-continuous for each n. □

Theorem 3.2 (Tychonoff)**.** The product of compact topological spaces is compact.

Proof. Assume each X_γ is compact. Let $\mathcal{E}$ be a family of closed subsets with the FIP (finite intersection property). We want $\bigcap_{E \in \mathcal{E}} E \neq \varnothing$.
By Zorn, find a maximal family $\mathcal{A}$ of sets in X such that $\mathcal{E} \subseteq \mathcal{A}$ and $\mathcal{A}$ has the FIP. We will show that $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \varnothing$. Maximality of $\mathcal{A}$ means that

- $\mathcal{A}$ is closed under finite intersections.
- If B intersects every $A \in \mathcal{A}$, then $B \in \mathcal{A}$.

For each $\gamma \in \Gamma$, $\{\pi_\gamma(A) \mid A \in \mathcal{A}\}$ has the FIP, hence find by compactness of X_γ some $x_\gamma \in \bigcap_{A \in \mathcal{A}} \overline{\pi_\gamma(A)}$.
We show that all neighborhoods of x are in $\mathcal{A}$. Then $\forall A \in \mathcal{A}, x \in \overline{A}$.
It's enough to show it for neighborhoods of the form $U = \bigcap_{\gamma \in s} \pi_\gamma^{-1}(U_\gamma)$ for some $s \subseteq \Gamma$ finite where each U_γ is a neighborhood of x_γ. For such U, we see that $\pi_\gamma^{-1}(U_\gamma)$ intersects every $A \in \mathcal{A}$, so $\pi_\gamma^{-1}(U_\gamma)$ by the second remark. Hence $U \in \mathcal{A}$ by the first remark. $\square$

Lecture 11