Part III – advanced Probability (Incomplete)

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Contents

0 Introduction

Lecture 1

This course is concerned with advanced topics in modern probability theory. Two examples are

Martingales
Martingales are processes indexed by discrete time such that

$$M_{n+1} = M_n + \text{ extra randomness}$$

where

$$\mathbb{E}[\text{extra randomness}|M_n] = 0$$

A typical example is Markov chains.

Brownian motion
Brownian motion is a continuous version of discrete random walks. It also arises naturally as the scaling limit of such. If $X_1, \dots$ are iid with mean μ and variance σ^2 and set $S_n = X_1 + \cdots + X_n$, we have several theorems about

$$\frac{S_n}{n} \to \mu$$

namely the Law of Large Numbers, the Central Limit Theorem, and Large Deviation results.

If we now set $B_t^{(n)} = \frac{S_{\lfloor nt \rfloor} - \mu n t}{\sigma \sqrt{n}}$, we have that $B_t^{(n)}$ tends to Brownian motion as $n \to \infty$.

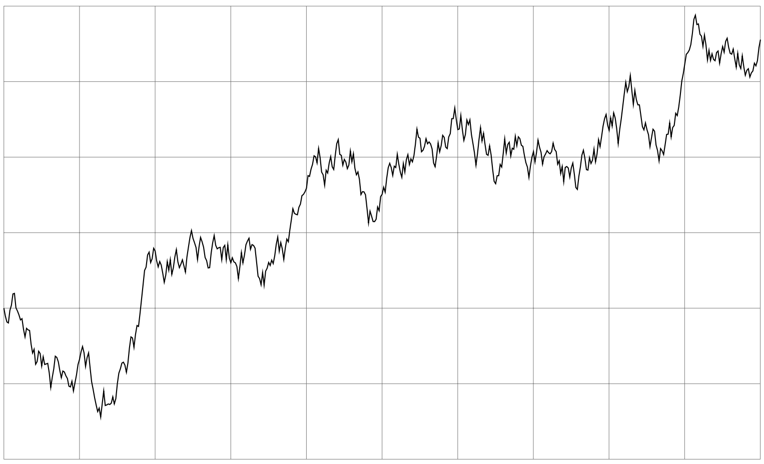

Figure 1: Standard Brownian motion

TODO: Label Gaussian in figure

Recall **Dirichlet's problem**: If $\mathcal{D} \subseteq \mathbb{C}$ is a simply connected domain and $f : \partial\mathcal{D} \to \mathbb{C}$, can we find a harmonic function $u : \mathcal{D} \to \mathbb{C}$ equal to f on $\mathcal{D}$?

Brownian motion lets us define such a u as follows:
Start a Brownian motion at $x \in \mathcal{D}$. Say it first hits the boundary of $\mathcal{D}$ in B_T. Evaluate

f at B_T.
Now take the expectation of the result and define

$$u(x) = \mathbb{E}[f(B_T)]$$

The resulting u is harmonic and clearly equals f on $\mathcal{D}$.

One can easily see that the corresponding construction in the discrete setting works by conditioning on the first move of the random walk.

TODO: Insert figure

1 Conditional Expectation

1.1 Basic measure theory recap

Definition. A collection $\mathcal{F}$ of sets in Ω is a **σ-algebra** if

- $\in \mathcal{F}$
- If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
- If $A_n \in \mathcal{F}$, then $\bigcup_n \in \mathcal{F}$

Definition. $\mathbb{P} : \mathcal{P}(\mathcal{P}(\Omega))$ is a **probability measure** if

- $\mathbb{P}() = 0$
- $\mathbb{P}(\Omega) = 1$
- When the A_n are disjoint, $\mathbb{P}\left(\bigcup_n\right) = \sum_n \mathbb{P}(A_n)$

From now on, Ω will be a set equipped with a σ-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$.

Definition. For $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, define

$$\sigma(\mathcal{A}) = \bigcap \{\mathcal{F} | \mathcal{F} \subseteq A \text{ is a } \sigma\text{-algebra}\}$$

the smallest σ-algebra containing $\mathcal{A}$, aka **σ-algebra generated by** $\mathcal{A}$.
The **Borel σ-algebra** $\mathcal{B}$ is the σ-algebra generated by the open sets in $\mathbb{R}$.

Definition. $X : \Omega \to \mathbb{R}$ is a **random variable** if X is measurable with respect to $\mathcal{B}$, namely if $X^{-1}(U) \in \mathcal{F}$ for all opens $U \subseteq \mathbb{R}$.

If the X_i, $i \in I$ are functions $\Omega \to \mathbb{R}$, we write $\sigma(X_i | i \in I)$ for $\sigma(\{X_i^{-1}(U) | i \in I, U \subseteq \mathbb{R} \text{ open}\})$, the smallest σ-algebra making all the X_i measurable.

1.2 Expectiation

Definition. A **simple function** is a function that can be written as a weighted sum of finitely many indicator functions.

Definition. For a simple function $f = \sum_i a_i 1_{A_i}$, we define

$$\mathbb{E}[f] = \sum_i a_i \mathbb{P}(A_i)$$

For a nonnegative function f, we define

$$\mathbb{E}[f] = \sup_{g \leq f \text{ simple}} \mathbb{E}[g]$$

For an arbitrary function f, write $f = f^+ - f^-$ with $f^+, f^- \geq 0$, and define

$$\mathbb{E}[f] = \mathbb{E}[f^+] - \mathbb{E}[f^-]$$

assuming at least one of $\mathbb{E}[f^+], \mathbb{E}[f^-]$ is finite.

Definition (Expectation conditional to an event)**.** For $A \in \mathcal{F}$, define

$$\mathbb{E}[X|A] = \frac{\mathbb{E}[1_A X]}{\mathbb{P}(A)}$$

Lecture 2