Part III – Functional Analysis (Incomplete)

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Contents

0 Introduction

Prerequisites

- some basic functional analysis
- a bit of measure theory
- a bit of complex analysis

Books

Books relevant to the course are:

- Bollobás, *Linear Analysis*
- Murphy, *C^*-algebras*
- Rudin
- Graham-Allan

Notation

We will use $\mathbb{K}$ to mean "either $\mathbb{R}$ or $\mathbb{C}$".

For X a normed space, we define

$$\begin{aligned} B_X &= \{x \in X \mid \|x\| \leq 1\} \\ S_X &= \{x \in X \mid \|x\| = 1\} \\ D_X &= \{x \in X \mid \|x\| < 1\} \end{aligned}$$

For X, Y normed spaces, we write $X \sim Y$ if X, Y are isomorphic, ie there exists a linear bijection $T : X \to Y$ such that T and T^{-1} are continuous. We write $X \cong Y$ if X, Y are isometrically isomorphic, ie there exists a surjective linear map $T : X \to Y$ such that $\|Tx\| = \|x\|$ for all x.

1 Hahn-Banach extension theorems

Lecture 1

Let X be a normed space. The **dual space** of X is the space X^* of bounded linear functionals on X. X^* is always a Banach space in the operator norm: for $f \in X^*$,

$$\|f\| = \sup_{x \in B_X} |f(x)|$$

Examples.

- For $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$, $\ell_p^* \cong \ell_q$
- $\ell_1^* \cong \ell_\infty$, $c_0^* \cong \ell_1$
- If H is a Hilbert space, then $H^* \cong H$ (the isomorphism is conjugate-linear in the complex case).

For $x \in X, f \in X^*$, we write $\langle x, f \rangle = f(x)$. Note that

$$\langle x, f \rangle = |f(x)| \leq \|f\| \, \|x\|$$

Definition. Let X be a *real* vector space. A functional $p : X \to \mathbb{R}$ is

- **positive homogeneous** if $p(tx) = tp(x)$ for all $x \in X$, $t \geq 0$
- **subadditive** if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$

Definition. Let P be a preorder, $A \subseteq P, x \in P$. We say

- x is an **upper bound** for A if $\forall a \in A, a \leq x$.
- A is a **chain** if $\forall a, b \in A, a \leq b \vee b \leq a$.
- x is a **maximal element** if $\forall y \in P, x \not< y$

Fact (Zorn's lemma)**.** A nonempty preorder in which all nonempty chains have an upper bound has a maximal element.

Theorem 1.1 (Hahn-Banach, positive homogeneous version)**.** Let X be a real vector space and $p : X \to \mathbb{R}$ be positive homogeneous and subadditive. Let Y be a subspace of X and $g : Y \to \mathbb{R}$ be linear such that $\forall y \in Y, g(y) \leq p(y)$. Then there exists $f : X \to \mathbb{R}$ linear such that $f \restriction_Y = g$ and $\forall x \in X, f(x) \leq p(x)$.

Proof. Let P be the set of pairs (Z, h) where Z is a subspace of X with $Y \subseteq Z$ and $h : Z \to \mathbb{R}$ linear, $h \restriction_Y = g$ and $\forall z \in Z, h(z) \leq p(z)$. P is nonempty since $(Y, g) \in P$, and is partially ordered by

$$(Z_1, h_1) \leq (Z_2, h_2) \iff Z_1 \subseteq Z_2 \wedge h_2 \restriction_{Z_1} = h_1$$

If $\{(Z_i, h_i) \mid i \in I\}$ is a chain with I nonempty, then we can define

$$Z := \bigcup_{i \in I} Z_i, h \restriction_{Z_i} = h_i$$

The definition of h makes sense thanks to the chain assumption. $(Z, h) \in P$ is therefore an upper bound for the chain.

Hence find by Zorn a maximal element (Z, h) of P. If $Z = X$, we won. So assume there is some $x \in X\ Z$. Let $W = \operatorname{Span}(Z \cup \{x\})$ and define $f : W \to \mathbb{R}$ by

$$f(z + \lambda x) = h(z) + \lambda\alpha$$

for some $\alpha \in \mathbb{R}$. Then f is linear and $f \restriction_Z = h$. We now look for α such that $\forall w \in W, f(w) \leq p(w)$. We would then have $(W, f) \in P$ and $(Z, h) < (W, f)$, contradicting maximality of (Z, h).
We need

$$h(z) + \lambda\alpha \leq p(z + \lambda x) \forall z \in Z, \lambda \in \mathbb{R}$$

Since p is positive homogeneous, this becomes

$$h(z) + \alpha \leq p(z + x) h(z) - \alpha \leq p(z - x) \tag{1}$$

ie

$$h(z) - p(z - x) \leq \alpha \leq p(z + x) - h(z) \forall z \in Z$$

The existence of α now amounts to

$$h(z_1) - p(z_1 - x) \leq \alpha \leq p(z_2 + x) - h(z_2) \forall z_1, z_2 \in Z$$

But indeed

$$h(z_1) + h(z_2) = h(z_1 + z_2) \leq p(z_1 + z_2) \leq p(z_1 - x) + p(z_2 + x)$$

□

Definition. Let X be a $\mathbb{K}$-vector space. A **seminorm** on X is a functional $p : X \to \mathbb{R}$ such that

- $\forall x \in X, p(x) \geq 0$
- $\forall x \in X, \lambda \in \mathbb{K}, p(\lambda x) = |\lambda| p(x)$
- $\forall x, y \in X, p(x + y) \leq p(x) + p(y)$

Remark.

$$\text{norm} \implies \text{seminorm} \implies \text{positive homogeneous}$$

Lecture 2

Theorem 1.2 (Hahn-Banach, absolute homogeneous version)**.** Let X be a real of complex vector space and p a seminorm on X. Let Y be a subspace of X, g a linear functional on Y such that $\forall y \in Y, |g(y)| \leq p(y)$. Then there exists a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x)$.

Proof.
Real case

$$\forall y \in Y, g(y) \leq |g(y)| \leq p(y)$$

By Theorem 1.1, there exists $f : X \to \mathbb{R}$ such that $f \restriction_Y = g$ and $\forall x \in X, f(x) \leq p(x)$. We also have

$$\forall x \in X, -f(x) = f(-x) \leq p(-x) = p(x)$$

Hence $|f(x)| \leq p(x)$
Complex case
$\operatorname{Re} g : Y \to \mathbb{R}$ is real-linear.

$$\forall y \in Y, |\operatorname{Re} g(y)| \leq |g(y)| \leq p(y)$$

By the real case, find $h : X \to \mathbb{R}$ real-linear such that $h \restriction_Y = \operatorname{Re} g$

Claim. There exists a unique complex-linear $f : X \to \mathbb{C}$ such that $h = \operatorname{Re} f$.

Proof.
Uniqueness
If we have such f, then

$$\begin{aligned} f(x) &= \operatorname{Re} f(x) + i \operatorname{Im} f(x) \\ &= \operatorname{Re} f(x) - i \operatorname{Re} f(ix) \\ &= h(x) - ih(ix) \end{aligned}$$

Existence
Define $f(x) = h(x) - ih(ix)$. Then f is real-linear and $f(ix) = if(x)$, so f is complex-linear with $\operatorname{Re} f = h$. ☐

We now have $f : X \to \mathbb{C}$ such that $\operatorname{Re} f = h$.

$$\operatorname{Re} f \restriction_Y = h \restriction_Y = \operatorname{Re} g$$

So, by uniqueness, $f \restriction_Y = g$.
Given $x \in X$, find λ with $|\lambda| = 1$ such that

$$\begin{aligned} |f(x)| &= \lambda f(x) \\ &= f(\lambda x) \\ &= \operatorname{Re} f(\lambda x) \\ &= h(\lambda x) \\ &\leq p(\lambda x) \\ &= p(x) \end{aligned}$$

☐

Remark. For a complex vector space X, if we write $X_{\mathbb{R}}$ for X considered as a real vector space, the above proof shows that

$$\operatorname{Re} : (X^*)_{\mathbb{R}} \to X_{\mathbb{R}}^*$$

is an isometric isomorphism.

Corollary 1.3. Let X be a $\mathbb{K}$-vector space, p a seminorm on X, $x_0 \in X$. Then there exists a linear functional f on X such that $f(x_0) = p(x_0)$ and $\forall x \in X, |f(x)| \leq p(x)$.

Proof. Let $Y = \operatorname{Span}(x_0)$,

$$\begin{aligned} g : Y &\to \mathbb{K} \\ \lambda x_0 &\mapsto \lambda p(x_0) \end{aligned}$$

We see that $\forall y \in Y, g(y) \leq p(y)$. Hence find by Theorem 1.2 a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x)$. We check that $f(x_0) = g(x_0) = p(x_0)$. ☐

Theorem 1.4 (Hahn-Banach, existence of support functionals)**.** Let X be a real or complex normed space. Then

1. If Y is a subspace of X and $g \in Y^*$, then there exists $f \in X^*$ such that $f \restriction_Y = g$ and $\|f\| = \|g\|$.

2. Given $x_0 \neq 0$, there exists $f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.

Proof.

1. Let $p(x) = \|g\| \|x\|$. Then p is a seminorm on X and

$$\forall y \in Y, |g(y)| \leq \|g\| \|y\| = p(y)$$

Find by Theorem 1.1 a linear functional f on X such that $f \restriction_Y = g$ and $\forall x \in X, |f(x)| \leq p(x) = \|g\| \|x\|$. So $\|f\| \leq \|g\|$. Since $f \restriction_Y = g$, we also have $\|g\| \leq \|f\|$. Hence $\|f\| = \|g\|$.

2. Apply Corollary 1.3 with $p(x) = \|x\|$ to get $f \in X^*$ such that

$$\forall x \in X, |f(x)| \leq \|x\| \text{ and } f(x_0) = \|x_0\|$$

It follows that $\|f\| = 1$.

□

Remarks.

- Part 1 is a sort of linear version of Tietze's extension theorem: Given K compact Hausdorff, $L \subseteq K$ closed, $g : L \to \mathbb{K}$ continuous, there exists $f : K \to \mathbb{K}$ such that $f \restriction_L = g$ and $\|f\|_\infty = \|g\|_\infty$.
- Part 2 shows that for all $x \neq y$ in X there exists $f \in X^*$ such that $f(x) \neq f(y)$, namely X^* **separates points** of X. This is a sort of linear version of Urysohn: $C(K)$ separates points of K.
- The f in part 2 is called a **norming functional**, aka **support functional**, for x_0. The existence of support functionals shows that

$$x_0 = \max_{g \in B_{X^*}} \langle x_0, g \rangle$$

Assuming X is a real normed space and $\|x_0\| = 1$, we have $B_X \subseteq \{x \in X | f(x) \leq 1\}$. Visually, TODO: insert tangency diagram

1.1 Bidual

Let X be a normed space. Then X^{**} is called the **bidual** or **second dual** of X.

For $x \in X$, define $\hat{x} : X^* \to \mathbb{K}$, the **evaluation at** x, by $\hat{x}(f) = f(x)$. $\hat{x}$ is linear and $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\|$, so $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leq \|x\|$.

The map $x \mapsto \hat{x} : X \to X^{**}$ is called the **canonical embedding** of X into X^{**}.

Theorem 1.5. The canonical embedding is an isometric embedding.

Proof.
Linearity

$$\widehat{x+y}(f) = f(x+y) = f(x) + f(y) = \hat{x}(f) + \hat{y}(f)$$
$$\widehat{\lambda x}(f) = f(\lambda x) = \lambda f(x) = \lambda \hat{x}(f)$$

Isometry
If $x \neq 0$, there exists a support functional f for x. Then

$$\|\hat{x}\| \geq |\hat{x}(f)| = |f(x)| = \|x\|$$

□

Remarks.

- In bracket notation, $\langle f, \hat{x} \rangle = \langle x, f \rangle$
- Let $\hat{X}$ be the image of X in X^{**}. Theorem 1.5 says

  $$X \cong \hat{X} \subseteq X^{**}$$

  We often identify $\hat{X}$ with X and think of X as living isometrically inside X^{**}. Note that

  $$X \text{ complete} \iff \hat{X} \text{ closed in } X^{**}$$

- More generally, $\overline{\hat{X}}$ is a Banach space containing an isometric copy of X as a dense subspace. We proved that normed spaces have completions!

Definition. A normed space X is **reflexive** if the canonical embedding $X \to X^{**}$ is surjective.

Examples.

- Some reflexive spaces are Hilbert spaces, finite-dimensional spaces, ℓ_p and $L_p(\mu)$ for $1 < p < \infty$.
- Some non-reflexive spaces are $c_0, \ell_1, \ell_\infty, L_1[0,1]$.

Remarks.

- If X is reflexive, then $X \cong X^{**}$, so X is complete.
- There are Banach spaces X such that $X \cong X^{**}$ but X is not reflexive, eg **James' space**. Any isomorphism to the bidual is then necessarily not the canonical embedding.

1.2 Dual operators

Lecture 3

Let X, Y be normed spaces. Recall

$$\mathcal{B}(X,Y) = \{T : X \to Y \mid T \text{ linear, bounded}\}$$

This is a normed space in the operator norm:

$$\|T\| = \sup_{x \in B_X} \|Tx\|$$

If Y is complete, then so is $\mathcal{B}(X,Y)$. For $T \in \mathcal{B}(X,Y)$, the **dual operator** of T is the map $T^* : Y^* \to X^*$ given by $T^*g = g \circ T$. In bracket notation $\langle x, T^*g \rangle = \langle Tx, g \rangle$ for $x \in X, g \in Y^*$.

T^* is linear

$$\begin{aligned}
\langle x, T^*(g+h) \rangle &= \langle Tx, g+h \rangle \\
&= \langle Tx, g \rangle + \langle Tx, h \rangle \\
&= \langle x, T^*g \rangle + xT^*h \\
&= \langle x, T^*g + T^*h \rangle
\end{aligned}$$

$$\begin{aligned}\langle x, T^*(\lambda g)\rangle &= \langle Tx, \lambda g\rangle \\ &= \lambda \langle Tx, g\rangle \\ &= \lambda \langle x, T^* g\rangle \\ &= \langle x, \lambda T^* g\rangle\end{aligned}$$

T^* is bounded

$$\begin{aligned}\|T^*\| &= \sup_{g \in B_{Y^*}} \|T^* g\| \\ &= \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^* g\rangle| \\ &= \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g\rangle| \\ &= \sup_{x \in B_X} \|Tx\| \text{ by Theorem 1.4 (ii)} \\ &= \|T\|\end{aligned}$$

Remarks.

- Hahn-Banach is crucial here. Without it, the dual could be 0.
- $\langle \cdot, \cdot \rangle$ is linear in both arguments. This contrasts with the Hilbert space case where $\langle \cdot, \cdot \rangle$ is conjugate-linear in one of the arguments. This comes from the conjugate-linearity of the identification $H^* \cong H$.
- If X, Y are Hilbert spaces and we identify X, Y with X^*, Y^*, respectively, then T^* is the adjoint of T.

Example. Let $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$ and define $R : \ell_p \to \ell_p$ to be the **right shift operator** $(x_0, x_1, \dots) \mapsto (0, x_0, \dots)$. Then $R^* : \ell_q \to \ell_q$ is the **left shift operator** $(x_0, x_1, \dots) \mapsto (x_1, x_2, \dots)$.

Some properties of the dual operator are

1. $\mathrm{id}_X^* = \mathrm{id}_{X^*}$
2. $(S + T)^* + S^* + T^*$, $(\lambda T)^* = \lambda T^*$
3. $(ST)^* = T^* S^*$
4. $T \mapsto T^* : \mathcal{B}(X, Y) \to \mathcal{B}(Y^*, X^*)$ is an *into* isomorphism.
5. The double dual of an operator commutes with the double dual embedding. TODO: Insert commutative diagram For all x,

   $$\langle g, T^{**}\hat{x}\rangle = \langle T^* g, \hat{x}\rangle = \langle x, T^* g\rangle = \langle Tx, g\rangle = \left\langle g, \hat{Tx}\right\rangle$$

   So $T^{**}\hat{x} = \widehat{Tx}$.

Remark. From the above properties, if $X \sim Y$, then $X^* \sim Y^*$. Interestingly, if X and Y are reflexive, then we can deduce $X \sim Y$ from $X^* \sim Y^*$.

1.3 Quotient spaces

Let X be a normed space and Y be a *closed* subspace. Then the quotient space X/Y becomes a normed space in the quotient norm:

$$\|x + Y\| = d(x, Y) = \inf_{y \in Y} \|x + y\|$$

The quotient map $q : X \to X/Y$ is linear and bounded: $\|q(x)\| \le \|x\|$, so $\|q\| \le 1$.

q maps the open unit ball D_X onto $D_{X/Y}$. Indeed, if $x \in D_X$, then $\|q(x)\| \le \|x\| < 1$. Reciprocally, if $q(x) \in D_{X/Y}$, then there exists $y \in Y$ such that $\|x + y\| < 1$. So $x + y \in D_X$ and $q(x + y) = q(x)$. It follows that q is an open map and $\|q\| = 1$.

If Z is another normed space, $T \in \mathcal{B}(X, Z)$ and $Y \subseteq \ker T$, then there exists a unique map $\tilde{T}$ is linear and $\tilde{T}(D_{X/Y}) = \tilde{T}(q(D_X)) = T(D_X)$. It follows that $\left\|\tilde{T}\right\| = \|T\|$.

Theorem 1.6. Let X be a normed space. If X^* is separable, then so is X.

Remark. The converse is false, as $X = \ell_1, X^* = \ell_\infty$ shows.

Proof. Since X^* is separable, so is S_{X^*}. Let f_n be a dense subset of S_{X^*}. For every n, find $x_n \in B_X$ such that $f_n(x_n) > \frac{1}{2}$. Let

$$Y = \overline{\mathrm{Span}\{x_n \mid n \in \mathbb{N}\}}$$

Claim. $Y = X$

Then we're done since Y is separable via $Y = \overline{\mathrm{Span}_{\mathbb{Q}}\{x_n \mid n \in \mathbb{N}\}}$.

Proof. Assume not. Then we can pick $g \in (X/Y)^*$, $\|g\| = 1$ (by Theorem 1.4 (ii)). Let $f = g \circ q$. Then $\|f\| = \|g\| = 1$, ie $f \in S_{X^*}$. Thus find n such that $\|f - f_n\| < \frac{1}{4}$, so that

$$\frac{1}{4} > \|f - f_n\| \|x_n\| \ge |(f - f_n)(x_n)| = |f_n(x_n)| > \frac{1}{2}$$

contradiction. □

□

Theorem 1.7. Let X be a separable normed space. Then X embeds isometrically into ℓ_∞.

Proof. Let $\{x_n \mid n \in \mathbb{N}\}$ be dense in X. For every n, find $f_n \in S_{X^*}$, $f_n(x_n) = \|x_n\|$ (assuming $X \ne \{0\}$). Define $T : X \to \ell_\infty$ by $(Tx)_n = f_n(x)$.

Well definition

$$|(Tx)_n| = |f_n(x)| \le \|f_n\| \|x\| = \|x\|$$

Hence $\|Tx\|_\infty \le \|x\| < \infty$.

Linearity

$$(T(x + y))_n = f_n(x + y) = f_n(x) + f_n(y) = (Tx + Ty)_n$$

$$(T(\lambda x))_n = f_n(\lambda x) = \lambda f_n(x)) = (\lambda Tx)_n$$

so $T(x + y) = Tx + Ty, T(\lambda x) = \lambda Tx$.

Isometry

We already know $\|Tx\|_\infty \le \|x\|$. On the other hand, find f a supporting functional for x and f_{n_k} a subsequence converging to f. Then

$$\|Tx\|_\infty \ge \sup_k (Tx)_{n_k} = \sup_k |f_{n_k}(x)| \ge |f(x)| = \|x\|$$

□

Remarks.

- The result says that ℓ_∞ is isometrically universal for the class $\mathcal{SB}$ of separable Banach spaces.
- There is a dual result: Every separable Banach space is a quotient of ℓ_1.

Theorem 1.8 (Vector-valued Liouville)**.** Lex X be a complex Banach space, $f : \mathbb{C} \to X$ holomorphic and bounded. Then f is constant.

Proof. Find $M \geq 0$ such that $\forall z \in \mathbb{C}, |f(z)| \leq M$. Fix $\phi \in X^*$. $\phi \circ f : \mathbb{C} \to \mathbb{C}$ is

bounded

$$|\phi(f(z))| \leq \|\phi\| \, \|f(z)\| \leq M \, \|\phi\|$$

holomorphic

$$\frac{\phi(f(z)) - \phi(f(w))}{z - w} = \phi\left(\frac{f(z) - f(w)}{z - w}\right) \to \phi(f'(z))$$

By scalar Liouville, $\phi \circ f$ is constant. For every $z \in \mathbb{C}, \phi \in X^*$, $\phi(f(z)) = \phi(f(0))$. Since X^* separates points of X, $f(z) = f(0)$. □

Remark. This is a typical example of how to transfer a scalar result to a vector-valued one: Prove the result once composed with any functional, then go back using Hahn-Banach.

1.4 Locally convex spaces

Definition. A **locally convex space** is a $\mathbb{K}$-vector space such that there exists a family $\mathcal{P}$ of seminorms on X that separate points of X in the sense that for all $x \neq 0$ there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$.

The family $\mathcal{P}$ defines a topology on X:

$$U \subseteq X \text{ open } \iff \forall x \in U, \exists s \subseteq \mathcal{P} \text{ finite}, \varepsilon > 0, \{y \in X \mid \forall p \in s, p(x) < \varepsilon\} \subseteq U$$

Remarks.

1. Addition and scalar multiplication are continuous.
2. The topology is Hausdorff as $\mathcal{P}$ separates points.
3. $x_n \to x \iff \forall p \in \mathcal{P}, p(x_n - x) \to 0$
4. Let Y be a subspace of X and $\mathcal{P}_Y = \{p\restriction_Y \mid p \in \mathcal{P}\}$. Then $(Y, \mathcal{P}_Y)$ is a LCS and its topology is the subspace topology.
5. Let $\mathcal{P}, \mathcal{Q}$ be two families of seminorms on X both separating points of X. We say $\mathcal{P}, \mathcal{Q}$ are **equivalent**, write $\mathcal{P} \sim \mathcal{Q}$, if they induce the same topology on X. One interesting result is that
$$(X, \mathcal{P}) \text{ metrisable} \iff \mathcal{P} \text{ equivalent to some countable family}$$
6. We make $\mathcal{P}$ part of the data here out of simplicity, but in grown up mathematics we instead assume that X already comes with a topology and that this topology coincides with the one induced by $\mathcal{P}$.

Definition. A **Fréchet space** is a complete metrisable LCS.

Examples.

1. A normed space is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.
2. Let $U \subseteq \mathbb{C}$ nonempty open. Let $\mathcal{O}(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}$. For compact $K \subseteq U$, define $p_K(f) = \sup_{z \in K} |f(z)|$. Let $\mathcal{P} = \{p_K \mid K \subseteq U \text{ compact}\}$ Then $(\mathcal{O}(U), \mathcal{P})$ is a LCS. If we replace $\{K \subseteq U \text{ compact}\}$ by a compact exhaustion of U, then we get a countable separating family equivalent to $\mathcal{P}$. So $(\mathcal{O}(U), \mathcal{P})$ is metrisable. However it is not normable: no norm on $\mathcal{O}(U)$ induces the topology of $(\mathcal{O}(U), \mathcal{P})$, which is the topology of uniform convergence. This is a consequence of Montel's theorem.
3. Fix $d \in \mathbb{N}, \Omega \subseteq \mathbb{R}^d$ a nonempty open set. Let

$$C^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ infinitely differentiable}\}$$

   Given a multi-index $\alpha \in \mathbb{Z}^d$, α defines a differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$$

   For a compact $K \subseteq \Omega, \alpha \in \mathbb{Z}^d$, define

$$p_{K,\alpha}(f) = \sup_{z \in K} |D^\alpha f(z)|$$

   Let

$$\mathcal{P} = \{p_{K,\alpha} \mid K \subseteq U \text{ compact}, \alpha \in \mathbb{Z}^d\}$$

   Then $(C^\infty, \mathcal{P})$ is a LCS. It is in fact a non-normable Fréchet space.

Lemma 1.9. Let $(X, \mathcal{P}), (Y, \mathcal{Q})$ be LCS, $T : X \to Y$ linear. TFAE

1. T is continuous
2. T is continuous at 0
3. $\forall q \in \mathcal{Q}, \exists s \subseteq \mathcal{P}$ finite, $C \geq 0, \forall x \in X, q(Tx) \leq C \max_{p \in s} p(x)$

Proof.
(i) $\iff$ (ii)
Translation is continuous.

(ii) $\implies$ (iii)
Given $q \in \mathcal{Q}$, let $V = \{y \in Y \mid q(y) \leq 1\}$. Then V is a neighborhood of 0 in Y. So there exists U neighborhood of 0 in X such that $T(U) \subseteq V$. WLOG

$$U = \{x \in X \mid \forall p_K \in s, p_K(x) \leq \varepsilon\}$$

Let $p = \max_{p_K \in s} p_K(x)$. If $p(x) = 1$, then $p(\varepsilon x) = \varepsilon$, so $\varepsilon x \in U$ and

$$q(T(\varepsilon x)) \leq 1$$

By homogeneity, $q(Tx) \leq \frac{1}{\varepsilon} p(x)$ for all x such that $p(x) > 0$. If $p(x) = 0$, then $p(\lambda x) = 0$ for all scalar λ. So $q(T(\lambda x)) \leq 1$ for all λ. Hence $q(Tx) = 0 \leq \frac{1}{\varepsilon} p(x)$.

(iii) $\implies$ (ii)
Assume $t \subseteq \mathcal{Q}$ is finite, $\varepsilon > 0$, and let $V = \{y \in Y \mid \forall q \in t, q(y) \leq \varepsilon$ the corresponding

neighborhood of 0. For each $q \in t$, find $s_q \subseteq \mathcal{P}$ finite and C_q so that $\forall x \in X, q(Tx) \leq C_q \max_{p \in s_q} p(x)$. Let

$$U = \left\{ x \in X \mid \forall q \in \mathcal{Q}, p \in s_q, p(x) \leq \frac{\varepsilon}{C_q} \right\}$$

Then U is a neighborhood of 0 and $T(U) \subseteq V$. $\square$

Definition. Let $(X, \mathcal{P})$ be a LCS. The **dual space** of X is the space of continuous linear functionals $X \to \mathbb{K}$.

Lecture 5

Lemma 1.10. Let f be a linear functional on a LCS $(X, \mathcal{P})$. Then

$$f \in X^* \iff \ker f \text{ closed}$$

Proof.
$\implies$
$\ker f = f^{-1}(0)$ is closed since f is continuous.

$\impliedby$
If $\ker f = 0$, then $f = 0$ is continuous. Else fix some $x_0 \notin \ker f$. Since $(\ker f)^c$ is open, find $s \subseteq \mathcal{P}$ finite, $\varepsilon > 0$ such that

$$\underbrace{\{x \in X \mid \forall p \in s, p(x - x_0) < \varepsilon\}}_{U} \subseteq (\ker f)^c$$

Then U is a neighborhood of 0 and $(x_0 + U) \cap \ker f =$. Note that U is convex and **balanced** ($x \in U, |\lambda| \leq 1 \implies \lambda x \in U$), hence so is $f(U)$ as f is linear.
If $f(U)$ is unbounded, then it is the whole scalar field, hence so is $f(x_0 + U) = f(x_0) + f(U)$. But $0 \in \ker f$, contradicting disjointness.
So find M such that $|f(x)| < M$ for all $x \in U$. For all $\delta > 0$, $\frac{\delta}{M} U$ is a neighborhood of 0 and $f\left(\frac{\delta}{M} U\right) \subseteq \{\lambda \in \mathbb{K} \mid |\lambda| < \delta\}$. Thus f is continuous. $\square$

Theorem 1.11 (Hahn-Banach)**.** Let $(X, \mathcal{P})$ be a LCS.

1. Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ such that $f \restriction_Y = g$.
2. Given a closed subspace Y of X and $x_0 \notin Y$, there exists $f \in X^*$ such that $f \restriction_Y = 0$, $f(x_0) \neq 0$.

Remark. This means that X^* separates points of X.

Proof.

1. By Lemma 1.9, find $s \subseteq \mathcal{P}$ finite, $C \geq 0$ such that

$$\forall y \in Y, |g(y)| \leq C \max_{p \in s} p(y)$$

Let $p(x) = C \max_{p \in s} p(x)$. Then p is a seminorm on X and $\forall y \in Y, |g(y)| \leq p(y)$. By Theorem 1.2, find a linear functional f on X such that $f \restriction_Y = g, \forall x \in X, |f(x)| \leq p(x)$. By Lemma 1.9, $f \in X^*$.

2. Let $Z = \mathrm{Span}(Y \cup \{x_0\})$ and define a linear functional g on Z by

$$g(y + \lambda x_0) = \lambda, y \in Y, \lambda \in \mathbb{K}$$

Then $g \restriction_Y = 0, g(x_0) = 1 \neq 0$ and $\ker g = Y$ is closed, so $g \in Z^*$ by Lemma 1.10. By part (i), find $f \in X^*$ such that $f \restriction_Z = g$. This works.

□

2 The dual of $L_p(\mu)$ and $C(K)$

Let $(\Omega, \mathcal{F}, \mu)$ be measure space.

$1 \le p < \infty$

$$L_p(\mu) = \{f : \Omega \to \mathbb{K} \mid f \text{ measurable and } \int_\Omega |f|^p \, d\mu < \infty\}$$

This is a normed space in the L_p-norm:

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}}$$

$p = \infty$
A measurable function $f : \Omega \to \mathbb{K}$ is **essentially bounded** if there exists $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $f \restriction_{N^c}$ is bounded.

$$L_p(\mu) = \{f : \Omega \to \mathbb{K} \mid f \text{ measurable and essentially bounded}\}$$

This is a normed space in the L_∞-norm:

$$\|f\|_\infty = \operatorname{esssup} |f| = \inf_{|f| \le k \text{ ae}} k$$

The inf is attained: there exists some $N \in \mathcal{F}, \mu(N) = 0$ such that $\|f\|_\infty = \sup_{N^c} |f|$.

In all cases, we identify functions up to almost everywhere equality.

Theorem 2.1. *$L_p(\mu)$ is complete for $1 \le p \le infty$.*

Definition (Complex measures)**.** A **complex measure** on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{C}$.
The **total variation measure** $|\nu|$ is defined by

$$|\nu|(A) = \sup_{\substack{A_1, \ldots, A_n \text{ measurable} \\ \text{partition of } A}} \sum_k |\nu(A_k)|$$

$|\nu| : \mathcal{F} \to [0, \infty]$ is a positive measure. Later we'll see that $|\nu|$ is a finite measure.
The **total variation** of ν is $\|\nu\|_1 = |\nu|(\Omega)$.

Proposition 2.2. *If ν is a complex measure on $\mathcal{F}$ and $A_n \in \mathcal{F}$ for all n, then*

- *If A is monotone, then $\nu(\bigcup_n A_n) = \lim_{n\to\infty} \nu(A_n)$.*
- *If A is antitone, then $\nu(\bigcap_n A_n) = \lim_{n\to\infty} \nu(A_n)$.*

Definition (Signed measures)**.** A **signed measure** on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{R}$.

Theorem 2.3. *If ν is a signed measure, then there exists a measurable partition $\Omega = P \cup N$ such that for all $A \in \mathcal{F}$*

$$A \subseteq P \implies \nu(A) \ge 0$$
$$A \subseteq N \implies \nu(A) \le 0$$

Remarks.

1. This decomposition is called the **Hahn decomposition** of ν.

2. Define $\nu^+(A) = \nu(A \cap P), \nu^-(A) = -\nu(A \cap N)$. Then ν^+, ν^- are finite positive measures such that $\nu = \nu^+ - \nu^-$. This determines ν^+, ν^- uniquely and the decomposition composition $\nu = \nu^+ - \nu^-$ is called the **Jordan decomposition** of ν.

3. If ν is a complex measure on $\mathcal{F}$, then $\operatorname{Re}\nu, \operatorname{Im}\nu$ are signed measures with Jordan decomposition $\nu_1 - \nu_2, \nu_3 - \nu_4$ respectively. Hence $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ is the Jordan decomposition of ν.

$$\nu_1, \nu_2, \nu_3, \nu_4 \leq |\nu| \leq \nu_1 + \nu_2 + \nu_3 + \nu_4$$

So $|\nu|$ is a finite measure.

Sketch. Define $\nu^+(A) = \sup_{\substack{B \in \mathcal{F} \\ B \subseteq A}} \nu(B)$. ν^+ is nonnegative and finitely additive.

Key step: $\nu^+(\Omega) < \infty$
By contradiction, construct inductively sequences A_n, B_n such that

$$\nu^+(A_n) = \infty, \nu(B_n) > n$$

by taking $A_0 = \Omega, B_{n+1} \subseteq A_n$ such that $\nu(B_n) > n$ (exists by continuity) and $A_{n+1} = B_{n+1}$ or $A_n \setminus B_{n+1}$. This contradicts countable additivity.

Now find a sequence A_n such that $\nu(A_n) > \nu^+(\Omega) - 2^{-n}$ and set $P = \liminf_n A_n, N = P^c$. Check that this works. $\square$

Lecture 6

Definition (Absolute continuity)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\nu : \mathcal{F} \to \mathbb{C}$ a complex measure. ν is **absolutely continuous** with respect to μ, written $\nu \ll \mu$, if $\forall A \in \mathcal{F}, \mu(A) = 0 \implies \nu(A) = 0$.

Remarks.

- $\nu \ll \mu \implies |\nu| \ll \mu$, so if ν has Jordan decomposition $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ then $\nu_1, \nu_2, \nu_3, \nu_4 \ll \mu$.
- If $\nu \ll \mu$, then $\forall \varepsilon > 0, \exists \delta > 0, \forall A \in \mathcal{F}, \mu(A) < \delta \implies |\nu(A)| < \varepsilon$.

Example. Let $f \in L_1(\mu)$. Define $\nu(A) = \int_A f d\mu$ for $A \in \mathcal{F}$. By Dominated Convergence, ν is a complex measure and $\mu(A) = 0 \implies \nu(A) = 0$. So $\nu \ll \mu$.

Definition. $A \in \mathcal{F}$ is **σ-finite** if there exists A_n with $\mu(A_n) < \infty$ such that $A = \bigcup_n A_n$. Say μ is **σ-finite** if Ω is σ-finite.

Theorem 2.4 (Radon-Nikodym)**.** Let μ be a σ-finite measure and ν a complex measure such that $\nu \ll \mu$. Then there exists a unique $f \in L_1(\mu)$ such that, for all $A \in \mathcal{F}$, $\nu(A) = \int_A f d\mu$. Moreover, f takes values in $\mathbb{C}/\mathbb{R}/\mathbb{R}^+$ depending on where ν is valued.

Proof.
Uniqueness
standard

Existence
ν is a finite measure (by the Jordan decomposition). WLOG μ is a finite measure (by σ-finiteness). Let

$$\mathcal{H} = \left\{ h : \Omega \to \mathbb{R}^+ \,\middle|\, h \text{ integrable}, \forall A \in \mathcal{F}, \int_A h d\mu \ll \nu(A) \right\}$$

$\mathcal{H} \neq \varnothing$ (eg $0 \in \mathcal{H}$). Let $\alpha = \sup_{h \in \mathcal{H}} \int_\Omega h d\mu$. We see $0 \leq \alpha \leq \nu(\Omega)$.

Claim
There exists $f \in \mathcal{H}$ such that $\alpha = \int_\Omega f d\mu$.

Idea
If $\int_A f d\mu < \nu(A)$, then $f + \frac{1}{n} 1_A \in \mathcal{H}$ (morally, not literally), contradicting the definition of α.

Pick that f. Define $\nu_n(A) = \nu(A) - \int_A f d\mu - \frac{1}{n}\mu(A)$. ν_n has Hahn decomposition $\Omega = P_n \cup N_n$. Then $f + \frac{1}{n} P_n \in \mathcal{H}$. By definition of α, $\mu(P_n) = 0$. Since $\nu \ll \mu$, $\nu(P_n) = 0$. Let $P = \bigcup_n P_n, N = \bigcap_n N_n$. Then $\Omega = P \cup N, \mu(P) = \nu(P) = 0$. For $A \in \mathcal{F}$,

$$A \subseteq P \implies \int_A f d\mu = \nu(A) = 0$$
$$A \subseteq N \implies \forall n, \nu_n \leq 0 \implies \int_A f d\mu \geq \nu(A)$$

□

Remarks.

- Without assuming $\nu \ll \mu$, the proof shows there is a decomposition $\nu = \nu_1 + \nu_2$ where $\nu_1(A) = \int_A f d\mu$ and $\nu_2 \perp \mu$ (**orthogonal**, ie there exists a measurable decomposition $\Omega = P \cup N$ such that $\mu(P) = 0, |\nu_2|(N) = 0$). $\nu = \nu_1 + \nu_2$ is the **Lebesgue decomposition** of ν.
- The unique f in Theorem 2.4 is the **Radon-Nikodym derivative** of ν with respect to μ, denoted $\frac{d\nu}{d\mu}$. The result says

  $$\nu(A) = \int_\Omega 1_A d\nu = \int_A \frac{d\nu}{d\mu} d\mu = \int_\Omega 1_A \frac{d\nu}{d\mu} d\mu$$

  Hence a measurable function g is ν-integrable iff $g\frac{d\nu}{d\mu}$ is μ-integrable and then

  $$\int_\Omega f d\nu = \int_\Omega g \frac{d\nu}{d\mu} d\mu$$

2.1 Dual space of $L_p(\mu)$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $1 \leq p < \infty, 1 < q \leq \infty$ such that $p^{-1} + q^{-1} = 1$. For $g \in L_q$, define $\phi_g : L_p \to \mathbb{K}$ by $\phi_g(f) = \int_\Omega f g d\mu$. By Hölder, $fg \in L_1$, and $|\phi_g(f)| \leq \|f\|_p \|g\|_q$. So ϕ_g is well-defined, linear and bounded with $\|\phi_g\| \leq \|g\|_q$. Hence $\phi_g \in L_p^*$ and $\phi : L_q \to L_p^*$ is linear and bounded with $\|\phi\| \leq 1$.

Theorem 2.5.

1. If $1 < p < \infty$, then ϕ is an isometric isomorphism. So $L_p^* \cong L_q$.
2. If $p = 1$ and μ is σ-finite, then ϕ is an isometric isomorphism. So $L_1^* \cong L_\infty$.

Proof.

1. **ϕ is isometric**
   Let $g \in L_1$. We know $\|\phi_g\| \leq \|g\|_q$. Let λ be a measurable function with $|\lambda| = 1, \lambda g = |g|$. let $f = \lambda |g|^{q-1}$. Then

$$\|f\|_p^p = \int_\Omega |g|^{pq-p} \, d\mu = \int_\Omega |g|^q \, d\mu = \|g\|_q^q$$

   So $f \in L_p$ and $\|f\|_p = \|g\|_q^{\frac{q}{p}}$. Then

$$\|q\|_q^{\frac{q}{p}} \|\phi_g\| \geq |\phi_g(f)| = \int_\Omega |g|^q \, d\mu = \|g\|_q^q$$

   So $\|\phi_g\| \geq \|g\|_q^{q-\frac{q}{p}} = \|g\|_q$.

   ϕ is onto
   Fix $\psi \in L_p^*$. We seek $g \in L_q$ such that $\psi = \phi_g$. Idea: We want $\psi(1_A) = \int_A g d\mu$.

   Case 1: μ is finite
   For $A \in \mathcal{F}$, $1_A \in L_p$, so define $\nu(A) = \psi(1_A)$. $\nu() = 0$ and, if $A = \bigcup_p A_n \in \mathcal{F}$, then $\sum_k 1_{A_k} = 1_A$ in L_p, so

$$\sum_k \nu(A_k) = \sum_k \psi(1_{A_k}) = \psi(1_A)$$

   Hence ν is a complex measure.
   If $A \in \mathcal{F}, \mu(A) = 0$, then $1_A = 0$ ae in L_p, so $\nu(A) = \psi(1_A) = 0$. Hence $\nu \ll \mu$.
   By Theorem 2.4, find $g \in L_1$ such that $\forall A \in \mathcal{F}, \nu(A) = \int_A g d\mu$. Hence

$$\begin{aligned}
\psi(1_A) &= \int_\Omega 1_A g d\mu \text{ for all } A \in \mathcal{F} \\
\psi(f) &= \int_\Omega f g d\mu \text{ for all simple function } f
\end{aligned}$$

   Given $f \in L_\infty$, find simple functions f_n tending to f in L_∞. So $\psi(f_n) \to \psi(f)$ and $f_n g \to fg$ (by Hölder for $\infty, 1$), meaning that

$$\psi(f) = \int_\Omega f g d\mu \text{ for all } f \in L_\infty$$

   For $n \in \mathbb{N}$, let $A = \{|g| \leq n\}$ and $f_n = \lambda 1_{A_n} |g|^{q-1}$ where $|\lambda| = 1, \lambda g = |g|$. As $f_n \in L_\infty$,

$$\int_\Omega f_n g d\mu = \int_{A_n} |g|^q \, d\mu = \psi(f_n)$$

   So $\left(\int_A |g|^q \, d\mu\right)^{q^{-1}} \leq \|\psi\|$. By Monotone Convergence, $g \in L_q$.
   Given $f \in L_p$, find simple functions f_n tending to f in L_p. So $\psi(f_n) \to \psi(f)$ and $f_n g \to fg$ in L_1 (by Hölder for p, q). Hence $\psi(f) = \int_\Omega f g d\mu$, as wanted.

   Before going onto Case 2, for $A \in \mathcal{F}$, let $\mathcal{F}_A = \{B \in \mathcal{F} \mid B \subseteq A\}$ and $\mu_A = \mu \restriction_{\mathcal{F}_A}$ so that $(A, \mathcal{F}_A, \mu_A)$ is a measure space. Then $L_p(\mu_A) \subseteq L_p(\mu)$ (by extending $f \in L_p(\mu_A)$ by 0 outside A). Let $\psi_A = \psi \restriction_{L_p(\mu_A)}$.

Lecture 7

   Claim. If $A, B \in \mathcal{F}$ are disjoint, then

$$\|\psi_{A \cup B}\| = (\|\psi_A\|^q + \|\psi_B\|^q)^{\frac{1}{q}}$$

Proof.

$$\begin{aligned}
(\|\psi_A\|^q + \|\psi_B\|^q)^{\frac{1}{q}} &= \sup_{\substack{a,b\geq 0\\ a^p+b^p\leq 1}} a\,\|\psi_A\| + b\,\|\psi_B\| \\
&= \sup_{\substack{a,b\geq 0\\ a^p+b^p\leq 1\\ f\in B_{L_p(\mu_A)}\\ g\in B_{L_p(\mu_B)}}} a\,|\psi_A(f)| + b\,|\psi_B(g)| \\
&= \sup_{\substack{|a|^p+|b|^p\leq 1\\ f\in B_{L_p(\mu_A)}\\ g\in B_{L_p(\mu_B)}}} \underbrace{|a\psi_A(f) + b\psi_B(g)|}_{\psi_{A\cup B}(af+bg)} \\
&= \sup_{h\in L_p(\mu_{A\cup B})} |\psi_{A\cup B}(h)| \\
&= \|\psi_{A\cup B}\|
\end{aligned}$$

□

Case 2: μ is σ-finite
Find a measurable partition $\Omega = \bigcup_n A_n$ such that $\mu(A_n) < \infty$. By Case 1, find, for each n, $g_n \in L_q(A_n)$ such that $\psi_{A_n} = \phi_{g_n}$, ie

$$\psi(f) = \int_{A_n} fg_n d\mu \text{ for all } f \in L_q(\mu_{A_n})$$

If we define g on Ω by $g = g_n$ on A_n, then $g \in L_q$ and

$$\psi(f) = \phi_g(f) \text{ for all } f \in L_p(\mu_{A_n})$$

Hence $\psi = \phi_g$ on $\overline{\text{Span}} \bigcup_n L_p(\mu_{A_n}) = L_p(\mu)$.

Case 3: General n
First observe that, for $f \in L_p(\mu)$, $\{f \neq 0\}$ is σ-finite. Indeed,

$$\{f \neq 0\} = \bigcup_n \left\{\frac{1}{n} < |f|\right\}$$

and

$$\mu\left\{\frac{1}{n} < |f|\right\} \leq |n^p|\,\|f\|_p^p < \infty \text{ by Markov}$$

Choose $f_n \in B_{L_p}$ such that $\psi(f_n) \to \|\psi\|$. Then $A = \bigcup_n \{f_n \neq 0\}$ is σ-finite and $\|\psi_A\| = \|\psi\|$. By the claim,

$$\|\psi\| = (\|\psi_A\|^q + \|\psi_{A^c}\|^q)^{\frac{1}{q}}$$

So $\Psi_{A^c} = 0$. By Case 2, find $g \in L_q(\mu_A) \subseteq L_q(\mu)$ such that $\psi_A = \phi_g$, so that

$$\psi(f) = \psi_A f\restriction_A + \psi A^c(f \restriction A^c) = \int_A fgd\mu + 0 = \int_\Omega fgd\mu$$

2. **$p = 1, \mu$ is σ-finite**
ϕ is isometric
Let $g \in L_\infty$. We know $\|\phi_g\| \leq \|g\|_\infty$ (by Hölder) Fix $s < \|g\|_\infty$. Then $\mu\{s < |g|\} > 0$. Since μ is σ-finite, find $A \subseteq \{s < |g|\}$ such that $0 < \mu(A) < \infty$. Choose a

measurable function λ such that $|\lambda| = 1, \lambda g = |g|$. Then $\lambda 1_A \in L_1, \|\lambda 1_A\|_1 = \mu(A)$. Now,

$$\mu(A)\,\|\phi_g\| \geq |\phi_g(\lambda 1_A)| = \int_A |g|\, d\mu \geq s\mu(A)$$

So $\|\phi_g\| \geq s$. Taking the sup, $\|\phi_g\| \geq \|g\|_\infty$.

ϕ is onto
Fix $\psi \in L_q^*$. We seek $g \in L_\infty$ such that $\psi = \phi_g$.

Case 1: μ is finite
Define $\nu(A) = \psi(1_A)$ for all $A \in \mathcal{F}$. Follow the same steps as for $1 < p < \infty$.

Case 2: μ is σ-finite
This time, prove that

$$\|\psi_{A\cup B}\| = \max(\|\psi_A\|, \|\psi_B\|)$$

for all $A, B \in \mathcal{F}$ disjoint and proceed as before.

□

Corollary 2.6. For $1 < p < \infty$, $L_p(\mu)$ is reflexive.

Proof. Let $\psi \in L_p^{**}$. Then $g \mapsto \langle \phi_g, \psi\rangle : L_q \to \mathbb{K}$ is in L_q^*. By Theorem 2.5.i, find $f \in L_p$ such that

$$\langle \phi_g, \psi\rangle = \int_\Omega fg d\mu = \langle f, \phi_g\rangle = \left\langle \phi_g, \hat{f}\right\rangle$$

Since $L_p^* = \{\phi_g \mid g \in L_q\}$, this proves $\psi = \hat{f}$. □

2.2 Dual space of $C(K)$

Throughout, K will be a compact Hausdorff topological space. Define

$$
\begin{aligned}
C(K) &= \{f : K \to \mathbb{C} \mid f \text{ continuous}\} \\
C^{\mathbb{R}}(K) &= \{f : K \to \mathbb{R} \mid f \text{ continuous}\} \\
C^{+}(K) &= \{f : K \to \mathbb{R}^{+} \mid f \text{ continuous}\} \\
M(K) &= C(K)^* \\
M^{\mathbb{R}}(K) &= \{\phi \in M(K) \mid \forall f \in C^{\mathbb{R}}(K), \phi(f) \in \mathbb{R}\} \\
M^{+}(K) &= \{\phi : C(K) \to \mathbb{C} \mid \phi \text{ is } \mathbb{R}\text{-linear}, \forall f \in C^{+}(K), 0 \le \phi(f) \in \mathbb{R}\}
\end{aligned}
$$

$C(K), C^{\mathbb{R}}(K)$ are complex/real Banach spaces in the sup norm: $\|f\|_\infty = \sup_K |f|$. $M(K)$ is a complex Banach space in the operator norm. $M^{\mathbb{R}}(K)$ is a closed real-linear subspace of $M(k)$. Elements of $M^{+}(K)$ are called **positive linear functionals**.

Aim. Identify $M(K), M^{\mathbb{R}}(K)$.

Lecture 8

The next lemma tells us that it's enough to understand $M^{+}(K)$.

Lemma 2.7.

1. For all $\phi \in M(K)$, there are unique $\phi_1, \phi_2 \in M^{\mathbb{R}}(K)$ such that $\phi = \phi_1 + i\phi_2$.
2. $\phi \mapsto \phi \restriction_{C^{\mathbb{R}}(K)}: M^{\mathbb{R}}(K) \to C^{\mathbb{R}}(K)^*$ is an isometric isomorphism.
3. $M^{+}(K) \subseteq M(K)$ and $M^{+}(K) = \{\phi \in M(K) \mid \|\phi\| = \phi(1)\}$
4. For all $\phi \in M^{\mathbb{R}}(K)$, there are unique $\phi^{+}, \phi^{-} \in M^{+}(K)$ such that $\phi = \phi^{+} - \phi^{-}$ and $\|\phi\| = \|\phi^{+}\| + \|\phi^{-}\|$.

Proof.

1. Let $\phi \in M(K)$. Then $\overline{\phi}$ sending $f \mapsto \phi(\overline{f})$ is in $M(K)$ as well and $\phi \in M^{\mathbb{R}}(K) \iff \overline{\phi} = \phi$.

   Uniqueness
   Assume $\phi = \phi_1 + i\phi_2$ where $\phi_1, \phi_2 \in M^{\mathbb{R}}(K)$. Then $\overline{\phi} = \phi_1 - i\phi_2$, so

   $$\phi_1 = \frac{\phi + \overline{\phi}}{2}, \phi_2 = \frac{\phi - \overline{\phi}}{2i}$$

   Existence
   Check that the above works

2. Let $\phi \in M^{\mathbb{R}}(K)$. We show $\left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\| = \|\phi\|$. Clearly, $\left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\| \le \|\phi\|$. Let $f \in B_{C(K)}$. Choose $\lambda \in \mathbb{C}, |\lambda| = 1, \lambda\phi(f) = |\phi(f)|$, so that

   $$
   \begin{aligned}
   |\phi(f)| &= \lambda\phi(f) \\
   &= \phi(\lambda f) \\
   &= \phi(\mathrm{Re}(\lambda f)) + \cancelto{0}{\phi(\mathrm{Im}(\lambda f))} \\
   &\le \left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\| \|\mathrm{Re}(\lambda f)\|_\infty \\
   &\le \left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\|
   \end{aligned}
   $$

Hence $\|\phi\| \leq \left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\|$.
Finally, given $\psi \in C^{\mathbb{R}}(K)$, define $\phi(f) = \psi(\operatorname{Re} f) + i\psi(\operatorname{Im} f)$. Then $\phi \in M(K)$ and $\phi \restriction_{C^{\mathbb{R}}(K)} = \psi$.

3. $M^+(K) \subseteq M(K)$
Let $\phi \in M^+(K)$. For $f \in B_{C^{\mathbb{R}}(K)}$, we have $1 \pm f \geq 0$, so $\phi(1 \pm f \geq 0$. Hence $\phi(f) \in \mathbb{R}$ and $|\phi(f)| \leq \phi(1)$. So $\phi \restriction_{C^{\mathbb{R}}(K)} \in C^{\mathbb{R}}(K)^*$ and $\left\|\phi \restriction_{C^{\mathbb{R}}(K)}\right\| = \phi(1)$. By (ii), $\phi \in M(K), \|\phi\| = \phi(1)$.

   $M^+(K) = \{\phi \in M(K) \mid \|\phi\| = \phi(1)$
We have already checked one inclusion. Let $\phi \in M(K)$ with $\|\phi\| = \phi(1)$. WLOG $\|\phi\| = \phi(1) = 1$. Let $f \in B_{C^{\mathbb{R}}(K)}$ and write $\phi(f) = a + ib$ where $a, b \in \mathbb{R}$. We want $b = 0$. For $t \in \mathbb{R}$,

$$\begin{aligned} |\phi(f + it)|^2 &= a^2 + (b+t)^2 = a^2 + b^2 + t^2 + 2bt \\ &\leq \|f + it\|_\infty \leq 1 + t^2 \end{aligned}$$

   So $b = 0$.
Given $f \in C^+(K)$ with $0 \leq f \leq 1$, we have $-1 \leq 2f - 1 \leq 1$, so $|\phi(2f-1)| \leq \|2f-1\|_\infty \leq 1$, ie $-1 \leq 2\phi(f) - 1 \leq 1$. So $\phi(f) \geq 0$.

4. Let $\phi \in M^{\mathbb{R}}(K)$. Assume for a moment that $\phi = \psi_1 - \psi_2$ where $\psi_1, \psi_2 \in M^+(K)$. For $f, g \in C^+(K)$ with $0 \leq g \leq f$, we have $\psi_1(f) \geq \psi_1(g) = \phi(g) + \psi_2(g) \geq \phi(g)$. So

$$\psi_1(f) \geq \sup_{0 \leq g \leq f} \phi(g)$$

   For $f \in C^+(K)$, define

$$\phi^+(f) = \sup_{0 \leq g \leq f} \phi(g)$$

   Observe that $\phi^+ \geq 0, \phi^+(f) \leq \|\phi\| \|f\|_\infty, \phi^+(f) \geq \phi(f), \phi^+$ is linear.
Next, for $f \in C^{\mathbb{R}}(K)$, write $f = f_1 - f_2$ where $f_1, f_2 \in C^+(K)$ and define $\phi^+(f) = \phi^+(f_1) - \phi^+(f_2)$. This is well-defined and $\mathbb{R}$-linear. Then ϕ is $\mathbb{C}$-linear since $\phi^+(f) \geq 0$. For all $f \in C^+(K)$ and $\phi^+ \in M^+(K)$.
Define $\phi^- = \phi^+ - \phi$. For $f \in C^+(K)$, $\phi^+(f) \geq \phi(f)$, so $\phi^-(f) \geq 0$, namely $\phi^- \in M^+(K)$.
We now see that $\|\phi\| \leq \|\phi^+\| + \|\phi^-\|$. Given $f \in C^+(K), 0 \leq f \leq 1$, we have $-1 \leq 2f - 1 \leq 1$, so

$$2\phi(f) - \phi(1) = \phi(2f-1) \leq \|\phi\|$$

   Taking the sup over f, we thus check that

$$\left\|\phi^+\right\| + \left\|\phi^-\right\| = \phi^+(1) + \phi^-(1) = 2\phi^+(1) - \phi(1) \leq \|\phi\|$$

   Uniqueness
Assume $\phi = \psi_1 - \psi_2, \psi_1, \psi_2 \in M^+(K), \|\phi\| = \|\psi_1\| + \|\psi_2\|$. From the initial observation, $\psi_1 \geq \phi^+$, hence $\psi_2 = \psi_1 - \phi \geq \phi^+ - \phi = \phi^-$. Therefore $\psi_1 - \phi^+, \psi_2 - \phi^- \in M^+(K)$. By (iii),

$$\left\|\psi_1 - \phi^+\right\| + \left\|\psi_2 - \phi^-\right\| = \psi_1(1) - \phi^+(1) + \psi_2(1) - \phi^-(1) = \|\phi\| - \|\phi\| = 0$$

   Hence $\psi_1 = \phi^+, \psi_2 = \phi^-$.

□

Topological preliminaries

1. K being compact Hausdorff, it is **normal**: given disjoint closed sets E, F in K, there are disjoint open sets U, V such that $E \subseteq U, F \subseteq V$. Equivalently, given $E \subseteq U \subseteq K$, E, closed, U open, there exists V open such that $E \subseteq V \subseteq \overline{V} \subseteq U$.
2. Urysohn says: given disjoint closed sets E, F, there is a continuous function $f : K \to [0,1]$ such that $f = 0$ on E, $f = 1$ on F.
3. Write $f \prec U$ to mean that U is an open set, f is continuous and $\operatorname{supp} f \subseteq U$. Write $E \prec f$ to mean that E is closed, f is continuous and $f = 1$ on E.
4. Urysohn then becomes: Given $E \subseteq U$, there exists f such that $E \prec f \prec U$.

Lemma 2.8. Let E closed, $U_1, \dots, U_n$ open such that $E \subseteq \bigcup_n U_n$. Then

1. There exist open sets V_j such that $\overline{V_j} \subseteq U_j$ and $E \subseteq \bigcup_j V_j$.
2. There exist $f_j \prec U_j$ such that $0 \le \sum_j f_j \le 1$ and $\sum_j f_j = 1$ on E.

Proof.

1. Induction on n: $n = 0$
   Obvious.

   $n > 0$
   $E \setminus U_n \subseteq \bigcup_{j<n} U_j$ so, by induction, find open sets V_j such that $\overline{V_j} \subseteq U_j$ for all $j < n$ and $E \setminus U_n \subseteq \bigcup_{j<n} U_j$. So $\underbrace{E \setminus \bigcup_{j<n} V_j}_{\text{closed}} \subseteq \underbrace{U_n}_{\text{open}}$. By Urysohn, find an open V_n such that
   $$E \setminus \bigcup_{j<n} V_j \subseteq V_n \subseteq \overline{V_n} \subseteq U_n$$

2. Find the V_j as in (i) for $1 \le j \le n$ and by Urysohn find h_j such that $\overline{V_j} \prec h_j \prec U_j$. By Urysohn again, find h_0 such that $\left(\bigcup_j U_j\right)^c \prec h_0 \prec E^c$. Let $h = \sum_{j=0}^n h_j \ge 1$ and $f_j = \frac{h_j}{h}$ for $1 \le j \le n$. Then $0 \le \sum_{j=1}^n \le 1$, $f_j \prec U_j$ and $\sum_{j=1}^n f_j = 1$ on E.

□

Definition (Borel measures)**.** Let X be a Hausdorff space and $\mathcal{G}$ its family of open sets. The **Borel σ-algebra** is $\mathcal{B} := \sigma(\mathcal{G})$, the σ-algebra generated by open sets. Elements of $\mathcal{B}$ are called **Borel sets**. A **Borel measure** on X is a measure μ on $\mathcal{B}$. We say μ is **regular** if

1. $\mu(E) < \infty$ for all compact $E \subseteq X$
2. $\mu(A) = \inf_{\substack{U \text{ open} \\ A \subseteq U}} \mu(A)$ for all Borel set A
3. $\mu(U) = \sup_{\substack{E \text{ compact} \\ E \subseteq U}} \mu(E)$ for all open U

A complex Borel measure ν is **regular** if $|\nu|$ is regular.

If X is compact and μ is a Borel measure on X, then

$$\begin{aligned} \mu \text{ regular} &\iff \mu(X) < \infty \text{ and } \forall A \in \mathcal{B}, \mu(A) = \inf_{\substack{U \text{ open} \\ A \subseteq U}} \mu(U) \\ &\iff \mu(X) < \infty \text{ and } \forall A \in \mathcal{B}, \mu(A) = \sup_{\substack{E \text{ closed} \\ E \subseteq A}} \mu(E) \end{aligned}$$

Definition (Integration with respect to a complex measure)**.** Let Ω be a set, $\mathcal{F}$ a σ-algebra on Ω, ν a complex measure on $\mathcal{F}$. Write $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ the Jordan decomposition. Say a measurable function is **ν-integrable** if f is $|\nu|$-integrable, or equivalently if f is $\nu_1, \nu_2, \nu_3, \nu_4$-integrable. Define

$$\int_\Omega f d\nu = \int_\Omega f d\nu_1 - \int_\Omega f d\nu_2 + i \int_\Omega f d\nu_3 - i \int_\Omega f d\nu_4$$

Lecture 9

Proposition 2.9.

1. $\int_\Omega d\nu = \nu(A)$ for all $A \in \mathcal{F}$.
2. Linearity: If $f, g : \Omega \to \mathbb{C}$ are ν-integrable and $\lambda \in \mathbb{C}$, then
$$\int_\Omega f + g d\nu = \int_\Omega f d\nu + \int_\Omega g d\nu, \int_\Omega \lambda f d\nu = \lambda \int_\Omega f d\nu$$
3. Dominated Convergence: Let f_n, f, g be measurable functions $\Omega \to \mathbb{C}$ such that $f_n \to f$ ae (with respect to $|\nu|$), $g \in L_1$ and $\forall n, f_n \leq g$ ae. Then f is ν-integrable and $\int_\Omega f_n d\nu \to \int_\Omega f d\nu$
4. $\left|\int_\Omega f d\nu\right| \leq \int_\Omega |f| \, d\,|\nu|$ for all ν-integrable f. This is true for simple functions by properties 1 and 2. For general f, use Dominated Convergence.

Let ν be a complex Borel measure on K. Then for $f \in C(K)$ we have

$$\int_K |f| \, d\,|\nu| \leq \|f\|_\infty \, |\nu| \, (K) = \|f\|_\infty \, \|\nu\|_1$$

So f is ν-integrable. Define $\phi : C(K) \to \mathbb{C}$ by $\phi(f) = \int_\Omega f d\nu$. Then $\phi \in M(K)$ and $\|\phi\| \leq \|\nu\|_1$. If ν is a signed measure, then $\phi \in M^{\mathbb{R}}(K)$. If ν is a positive measure, then $\phi \in M^+(K)$.

Theorem 2.10 (Riesz Representation Theorem)**.** For every $\phi \in M^+(K)$, there exists a unique regular Borel measure μ on K that represents ϕ: $\phi(f) = \int_K f d\mu$ for all $f \in C(K)$. Moreover, $\|\phi\| = \mu(K) = \|\mu\|_1$.

Proof.
Uniqueness
Assume μ_1, μ_2 both represent ϕ. Let $E \subseteq U \subseteq K$ where E closed, U open. By Urysohn, find f such that $E \prec f \prec U$. Now,

$$\mu_1(E) \leq \int_K f d\mu_1 = \phi(f) = \int_K f d\mu_2 \leq \mu_2(U)$$

Taking the inf over U, we get $\mu_1(E) \leq \mu_2(E)$. By symmetry, $\mu_1(E) = \mu_2(E)$. By regularity, $\mu_1 = \mu_2$.

Existence
For U open, define $\mu^*(U) = \sup_{f \prec U} \phi(f)$. Note that

$$\mu^*(U) \geq 0, \mu \text{ monotone}, \mu^*(K) = \phi(1)$$

It follows that, for V open, $\mu^*(V) = \inf_{U \supseteq V} \mu^*(U)$. Hence extend the definition of μ^* to

$$\mu^*(A) = \inf_{U \supseteq A} \mu^*(U)$$

We will show that μ^* is an outer measure.

- $\mu(\varnothing) = 0$
- If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$.
- Do we have $\mu^*\left(\bigcup_n A_n\right) = \sum_n \mu^*(A_n)$?
  First assume that the $A_n = U_n$ are open. Let $U = \bigcup_n U_n$. Assume $f \prec U$ and let $E = \operatorname{supp} f$. $E \subseteq \bigcup_n U_n$, so by compactness find N such that $E \subseteq \bigcup_{n=1}^N U_n$. By Lemma 2.8, find $h_n \prec U_n$ with $\sum_{n=1}^N h_n \leq 1$ and $\sum_{n=1}^N h_n = 1$ on E. So $f = \sum_{n=1}^N f h_n$ and

$$\begin{aligned}\phi(f) &= \sum_{n=1}^N \phi(f h_n) \\ &\leq \sum_{n=1}^N \mu^*(U_j) \text{ as } f h_n \prec U_n \\ &\leq \sum_n \mu^*(U_n)\end{aligned}$$

  Taking the sup over f, we get $\mu^*(U) \leq \sum_n \mu^*(U_n)$. It follows that

$$\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$$

  We now let $\mathcal{M}$ be the set of μ^*-measurable sets. Then $\mathcal{M}$ is a σ-algebra and $\mu^* \restriction_{\mathcal{M}}$ is a measure on $\mathcal{M}$.
  To restrict it further to a Borel, we now show that $\mathcal{B} \subseteq \mathcal{M}$. It's enough to show that $\mathcal{G} \subseteq \mathcal{M}$.
  Let U open. We need

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U) \text{ for all } A$$

  First, let $A = V \in \mathcal{G}$. Fix $f \prec V \cap U$ and $g \prec V \setminus \operatorname{supp} f$. Then $f + g \prec V$, thus

$$\mu^*(V) \geq \phi(f+g) = \phi(f) + \phi(g)$$

  Taking the sup over g,

$$\mu^*(V) \geq \phi(f+g) = \phi(f) + \mu^*(V \setminus \operatorname{supp} f) \geq \phi(f) + \mu^*(V \setminus U)$$

  Taking the sup over f,

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$$

  Now let A be arbitrary. Fix V open such that $A \subseteq V$. then

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$$

  Taking the inf over V,

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$$

  Now, $\mu := \mu^* \restriction_{\mathcal{B}}$ is a Borel measure on K. We have

$$\mu(K) = \phi(1) = \|\phi\| < \infty$$

  and by definition μ is regular. It remains to show that $\phi(f) = \int_K f d\mu$ for all $f \in C(K)$. It is enough to check that for $f \in C^{\mathbb{R}}(K)$ and enough to check that

$\phi(f) \leq \int_K f d\mu$ (apply this to $-f$).
Fix $0 < a < b$ in $\mathbb{R}$ such that $\phi(1) \in [a, b]$. Let $\varepsilon > 0$. Choose $0 \leq y_0 < a \leq y_1 < \cdots < y_n = b$ such that

$$y_j < y_{j-1} + \varepsilon$$

Let $A_j = f^{-1}]y_{j-1}, y_j]$. Those sets form a measurable partition of K. Choose closed sets E_j and open sets U_j such that $E_j \subseteq A_j \subseteq U_j$ and $\mu(U_j \setminus E_j) < \frac{\varepsilon}{n}$ (by regularity) and $f(U_j) \subseteq]y_{j-1}, y_j]$. By Lemma 2.8, find $h_j \prec U_j$ for each j such that $\sum_j h_j = 1$. Now,

$$\begin{aligned}
\phi(f) &= \sum_j \phi(f_j) \\
&\leq \sum_j (y_j + \varepsilon)\phi(h_j) \\
&\leq \sum_j (y_{j-1} + 2\varepsilon)\left(\mu(E_j) + \frac{\varepsilon}{n}\right) \\
&= \sum_j y_{j-1}\mu(E_j) + \underbrace{\sum_j (b + \varepsilon) + 2\varepsilon\mu(K) + 2\varepsilon^2}_{o(1)} \\
&= \int_K \sum_j y_{j-1} 1_{E_j} d\mu + o(1) \qquad \leq \int_K f d\mu + o(1)
\end{aligned}$$

since $f \leq y_j + \varepsilon$ on U_j, $h_j \prec U_j$ and $\phi \in M^+(K)$. So $\phi(f) \leq \int_K f d\mu$.

$\square$

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Corollary 2.11. For every $\phi \in M(K)$, there exists a unique regular complex Borel measure ν on K that represents ϕ: $\phi(f) = \int_K f d\nu$ for all $f \in C(K)$. Moreover, $\|\phi\| = \|\nu\|_1$ and if $\phi \in M^{\mathbb{R}}(K)$ then ν is a signed measure.

Proof.
Existence
Apply Lemma 2.7 and Theorem 2.10 to obtain a regular complex Borel measure representing ϕ. We now want $\|\phi\| = \|\nu\|_1$.
We already know $\|\phi\| \leq \|\nu\|_1$. Take a measurable partition $K = \bigcup_{j=1}^n A_j$. Fix $\varepsilon > 0$ and closed sets E_j, open sets U_j such that $E_j \subseteq A_j \subseteq U_j$, $|\nu|(U_j \setminus E_j) < \frac{\varepsilon}{n}$ (ν is regular). We can also assume $U_i \subseteq \bigcap_{j \neq i} E_j^c$. Fix $\lambda_j \in \mathbb{C}$ such that $|\lambda_j| = 1, \lambda_j \nu(E_j) = |\nu(E_j)|$. By Lemma 2.8, find $h_j \prec U_j$ such that $\sum_{j=1}^n h_j = 1$. Then $E_j \prec h_j$, hence

$$\begin{aligned}
\left| \int_K \left(\sum_{j=1}^n \lambda_j 1_{E_j} - \sum_{j=1}^n \lambda_j h_j \right) d\nu \right| &\leq \sum_{j=1}^n \int_K \left| 1_{E_j} - h_j \right| d|\nu| \\
&\leq \sum_{j=1}^n |\nu| (U_j \setminus E_j) < \varepsilon
\end{aligned}$$

Now,

$$
\begin{aligned}
\sum_{j=1}^{n} |\nu(A_j)| &\leq \sum_{j=1}^{n} |\nu(E_j)| + \varepsilon \\
&= \sum_{j=1}^{n} \lambda_j \nu(E_j) + \varepsilon \\
&= \int_K \sum_{j=1}^{n} \lambda_j 1_{E_j} d\nu + \varepsilon \\
&\leq \left| \int_K \sum_{j=1}^{n} \lambda_j h_j d\nu \right| + 2\varepsilon \\
&\leq \left| \phi\left(\sum_{j=1}^{n} \lambda_j h_j \right) \right| + 2\varepsilon \\
&\leq \|\phi\| \left\| \sum_{j=1}^{n} \lambda_j h_j \right\|_\infty + 2\varepsilon \\
&\leq \|\phi\| + 2\varepsilon
\end{aligned}
$$

It follows that $\|\nu\|_1 \leq \|\phi\|$. □

Corollary 2.12. The space of regular real (resp. complex) Borel measures on K is a real (resp. complex) Banach space in $\|\cdot\|_1$ isomorphic to $M^{\mathbb{R}}(K)$ (resp. $M(K)$).

3 Weak topologies

Let X be a set and $\mathcal{F}$ a set of functions on X such that each $f \in \mathcal{F}$ is a function $X \to Y_f$ where Y_f is a topological space. The **weak topology $\sigma(X, \mathcal{F})$ on X generated by $\mathcal{F}$** is the smallest topology on X that makes each $f \in \mathcal{F}$ continuous.

Remarks.

1. $S = \{f^{-1}(U) \mid f \in \mathcal{F}, U \subseteq Y_f \text{ open}\}$ is a subbase of $\sigma(X, \mathcal{F})$. So

$$\begin{aligned} V \subseteq X \text{ open} &\iff \forall x \in V, \exists f_1, \ldots, f_n \in \mathcal{F}, U_i \subseteq Y_{f_i}, x \in \bigcap_i f_i^{-1}(U_i) \subseteq V \\ &\iff \forall x \in V, \exists f_1, \ldots, f_n \in \mathcal{F}, \\ &\qquad \text{open neighborhoods } U_i \text{ of } f_i(x), \bigcap_i U_i \subseteq V \end{aligned}$$

2. More generally, if S_f is a subbase in Y_f, then $\{f^{-1}(U) \mid f \in \mathcal{F}, U \in S_f\}$ is a subbase of $\sigma(X, \mathcal{F})$.
3. If Y_f is Hausdorff for all $f \in \mathcal{F}$ and $\mathcal{F}$ **separates points of** X ($\forall x \neq y, \exists f \in \mathcal{F}, f(x) \neq f(y)$), then $\sigma(X, \mathcal{F})$ is Hausdorff.
4. Let $Y \subseteq X, \mathcal{F}_Y = f \restriction_Y \mid f \in \mathcal{F}$. Then $\sigma(Y, \mathcal{F}_Y) = \sigma(X, \mathcal{F}) \restriction_Y$.
5. **Universal property**: Let Z be a topological space and $g : Z \to X$. then

$$g \text{ continuous} \iff \forall f \in \mathcal{F}, f \circ g : Z \to Y_f \text{ continuous}$$

Examples.

1. Let X be a topological space, $Y \subseteq X$ and $\iota : Y \to X$ the inclusion map. Then $\sigma(Y, \iota)$ is the subspace topology on Y.
2. Let Γ be a set, X_γ a topological space for each $\gamma \in \Gamma$, $X = \prod_{\gamma \in \Gamma} X_\gamma$. For each γ, we have $\pi_\gamma : X \to X_\gamma$ sending $x \mapsto x_\gamma$, the **evaluation map at** γ, or **projection onto** X_γ. The weak topology $\sigma(X, \{\pi_\gamma \mid \gamma \in \Gamma\})$ is called the **product topology** on X.

$$V \subseteq X \text{ open} \iff \substack{\forall x \in V, \exists s \subseteq \Gamma \text{ finite}, U_\gamma \text{ neighborhood of } x_\gamma, \\ \{y \mid \forall \gamma \in s, y_\gamma \in U_\gamma\} \subseteq V}$$

Proposition 3.1. Let X be a set. For each n, let (Y_n, d_n) be a metric space and $f_n : X \to Y_n$ be a separating family of functions. Then $\sigma(X, \{f_n \mid n \in \mathbb{N}\})$ is metrisable.

Proof. Call $\sigma := \sigma(X, \{f_n \mid n \in \mathbb{N}\})$. Define

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n} \min(d(f_n(x), f_n(y)), 1)$$

This is a metric on X. Given $0 < \varepsilon < 1$, if $d(x, y) < 2^{-n}\varepsilon$, then $d(f_n(x), f_n(y)) < \varepsilon$. So each f_n is continuous with respect to the topology τ induced by that metric. Hence $\sigma \subseteq \tau$.
Reciprocally, $y \mapsto d(x, y)$ is σ-continuous for each x by the Weierstrass M-test since

$$y \mapsto 2^{-n} \min(d(f_n(x), f_n(y)), 1)$$

is σ-continuous for each n. $\square$

Theorem 3.2 (Tychonoff)**.** The product of compact topological spaces is compact.

Proof. Assume each X_γ is compact. Let $\mathcal{E}$ be a family of closed subsets with the FIP (finite intersection property). We want $\bigcap_{E\in\mathcal{E}} E \neq \varnothing$.
By Zorn, find a maximal family $\mathcal{A}$ of sets in X such that $\mathcal{E} \subseteq \mathcal{A}$ and $\mathcal{A}$ has the FIP. We will show that $\bigcap_{A\in\mathcal{A}} \overline{A} \neq \varnothing$. Maximality of $\mathcal{A}$ means that

- $\mathcal{A}$ is closed under finite intersections.
- If B intersects every $A \in \mathcal{A}$, then $B \in \mathcal{A}$.

For each $\gamma \in \Gamma$, $\{\pi_\gamma(A) \mid A \in \mathcal{A}\}$ has the FIP, hence find by compactness of X_γ some $x_\gamma \in \bigcap_{A\in\mathcal{A}} \overline{\pi_\gamma(A)}$.
We show that all neighborhoods of x are in $\mathcal{A}$. Then $\forall A \in \mathcal{A}, x \in \overline{A}$.
It's enough to show it for neighborhoods of the form $U = \bigcap_{\gamma\in s} \pi_\gamma^{-1}(U_\gamma)$ for some $s \subseteq \Gamma$ finite where each U_γ is a neighborhood of x_γ. For such U, we see that $\pi_\gamma^{-1}(U_\gamma)$ intersects every $A \in \mathcal{A}$, so $\pi_\gamma^{-1}(U_\gamma)$ by the second remark. Hence $U \in \mathcal{A}$ by the first remark. $\square$

3.1 Weak topologies on vector spaces

Lecture 11

Let E be a real or complex vector space. Let F be a subspace of the space of all linear functionals on E that separates points of E, ie $\forall x \in E, x \neq 0 \implies \exists f \in F, f(x) \neq 0$. Consider the weak topology $\sigma(E, F)$

$$U \text{ open} \iff \forall x \in U, \exists f_1, \dots, f_n \in F, \varepsilon > 0, \{y \mid \forall i, |f_i(x-y)| < \varepsilon\} \subseteq U$$

For $f \in F$, $x \in E$, let $p_f(x) = |f(x)|$. Let $\mathcal{P} = \{p_f \mid f \in \mathcal{F}\}$. Then $(E, \mathcal{P})$ is a LCS whose topology is $\sigma(E, F)$. So $\sigma(E, F)$ is Hausdorff and vector addition and scalar multiplication are continuous.

Lemma 3.3. Let E be as above, $f, g_1, \dots, g_n$ linear functionals on E such that

$$\bigcap_i \ker g_i \subseteq \ker f$$

Then $f \in \operatorname{Span}\{g_1, \dots, g_n\}$.

Proof. Reinterpret the g_i as a single linear map $g : E \to \mathbb{K}^n$. Then

$$\ker g = \bigcap_i \ker g_i \subseteq f$$

Hence we have a factorisation $f = h \circ g$. Find $a_1, \dots, a_n$ such that $h(y) = \sum_i a_i y_i$ for all $y \in \mathbb{K}^n$. Then

$$f(x) = h(g(x)) = \sum_i a_i g_i(x)$$

for all x, so $f = \sum_i a_i g_i \in \operatorname{Span}\{g_1, \dots, g_n\}$. $\square$

Proposition 3.4. Let E, F be as above and f a linear functional on E. Then

$$f \text{ is } \sigma(E, F)\text{-continuous} \iff f \in F$$

Namely,

$$(E, \sigma(E, F))^* = F$$

Proof.
$\Longleftarrow$
True by definition.

$\Longrightarrow$
Find an open neighborhood U of 0 in E such that $\forall x \in U, |f(x)| < 1$. WLOG $U = \{x \mid \forall i, |g_i(x)| < \varepsilon\}$ for some $\varepsilon > 0$, $g_1, \dots, g_n \in F$.
If $x \in \bigcap_i \ker g_i$, then $\lambda x \in U$ for all λ, hence

$$|\lambda| \, |f(x)| = |f(\lambda x)| < 1$$

for all λ, so $f(x) = 0$. By Lemma 3.3, $f \in \mathrm{Span}\{g_1, \dots, g_n\} \subseteq F$. $\square$

Examples.

1. Let X be a normed space. The **weak topology** on X is the topology $\sigma(X, X^*)$ on X (X^* separates points of X by Hahn-Banach). We sometimes write (X, w) for $(X, \sigma(X, X^*))$. Open sets in $\sigma(X, X^*)$ are called **weak open** or **w-open**.

$$U \subseteq X \text{ is w-open}$$
$$\Longleftrightarrow$$
$$\forall x \in U, \exists \varepsilon > 0, f_1, \dots, f_n \in X^*, \{y \in X \mid \forall i, |f_i(y - x)| < \varepsilon\} \subseteq U$$

2. Let X be a normed space. The **weak star topology** or **w*-topology** on X^* is the topology $\sigma(X^*, X)$. Here we identify X with its image $\hat{X}$ in X^{**} under the canonical embedding. Open sets in $\sigma(X^*, X)$ are called **w*-open**.

$$U \subseteq X^* \text{ is w*-open}$$
$$\Longleftrightarrow$$
$$\forall f \in U, \exists \varepsilon > 0, x_1, \dots, x_n \in X, \{g \in X^* \mid \forall i, |g(x_i) - f(x)| < \varepsilon\} \subseteq U$$

Properties.

1. (X, w) and (X^*, w^*) are LCS, hence Hausdorff with continuous vector space operations.
2. $\sigma(X, X^*)$ is a subtopology of the norm topology, with equality iff X is finite dimensional.
3. $\sigma(X^*, X)$ is a subtopology of $\sigma(X^*, X^{**})$, with equality iff X is reflexive.
4. Let Y be a subspace of X. Then

$$\sigma(X, X^*) \restriction_Y = \sigma(Y, \{f \restriction_Y \mid f \in X^*\}) \overset{\text{Hahn-Banach}}{=} \sigma(Y, Y^*)$$

   Similarly,
$$\sigma(X^{**}, X^*) \restriction_X = \sigma(X, X^*) = \sigma(X, \{\hat{f} \restriction_X \mid f \in X^*\})$$

   So the canonical embedding is a homeomorphism $\sigma(X, X^*) \to \sigma(\hat{X}, X^*)$.

Proposition 3.5. Let X be a normed space.

1. A linear functional f on X is w-continuous iff $f \in X^*$. So $(X, w)^* = X^*$.
2. A linear functional Λ on X^* is w*-continuous iff $\Lambda \in \hat{X}$. So $(X^*, w^*)^* = X$.

It follows that $\sigma(X^*, X) = \sigma(X^*, X^{**})$ iff X is reflexive.

Definition. Let X be a normed space.

1. A set A in X is **weakly bounded** if $\{f(x) \mid x \in A\}$ is bounded for all $f \in X^*$, or equivalently if for all w-neighborhood U there exists λ such that $A \subseteq \lambda U$.
2. A set B in X^* is **w*-bounded** if $\{f(x) \mid f \in B\}$ is bounded for all $x \in X$, or equivalently if for all w*-neighborhood U there exists λ such that $B \subseteq \lambda U$.

Theorem (Principle of Uniform Boundedness, PUB)**.** Let X be a Banach space, Y a normed space $\mathcal{T} \subseteq \mathcal{B}(X,Y)$. If $\mathcal{T}$ is **pointwise bounded** ($\forall x \in X, \sup_{T\in\mathcal{T}} \|Tx\| < \infty$), then $\mathcal{T}$ is **uniformly bounded** ($\sup_{T\in\mathcal{T}} \|T\| < \infty$).

Proposition 3.6. Let X be a normed space.

1. If $A \subseteq X$ is weakly bounded, then A is norm-bounded.
2. If X is complete and $B \subseteq X^*$ is w*-bounded, then B is norm-bounded.

Proof.

1. A being weak bounded means that $\hat{A} = \{\hat{x} \mid x \in A\}$ is pointwise bounded. So we're done by PUB.
2. B being w*-bounded means that B is pointwise bounded. So we're done by PUB.

□

Notation. We write $x_n \xrightarrow{w} x$ if x_n converges to x in the weak topology. Note that

$$x_n \xrightarrow{w} x \iff \forall f \in X^*, \langle x_n, f\rangle \to \langle x, f\rangle$$

We write $f_n \xrightarrow{w*} f$ if f_n converges to f in the w*-topology. Note that

$$f_n \xrightarrow{w} f \iff \forall x \in X, \langle x, f_n\rangle \to \langle x, f\rangle$$

Theorem 3.7 (Consequence of PUB)**.** Let X be a Banach space, Y a normed space, T_n a sequence in $\mathcal{B}(X,Y)$. If T_n converges pointwise to some function $T : X \to Y$, then $T \in \mathcal{B}(X,Y)$, $\sup_n \|T_n\| < \infty$ and $\|T\| \le \liminf_n \|T_n\|$.

Proposition 3.8. Let X be a normed space.

1. If $x_n \xrightarrow{w} x$ in X, then $\sup_n \|x_n\| < \infty$ and $\|x\| \le \liminf \|x_n\|$.
2. If $f_n \xrightarrow{w*} f$ in X^*, then $\sup_n \|f_n\| < \infty$ and $\|f\| \le \liminf \|f_n\|$.

Proof.

1. $\widehat{x_n} \to \hat{x}$ pointwise in X^{**}. Result follows by PUB.
2. $f_n \to f$ pointwise in X^*. Result follows by PUB.

□

Lecture 12

The weak topology is weaker than the norm topology as we see by the fact that $e_n \xrightarrow{w} 0$ in ℓ_p ($1 \le p < \infty$) but $e_n \not\to 0$, where e_n is the vector with a single 1 in the n-th position.

3.2 Hahn-Banach Separation Theorems

Let $(X, \mathcal{P})$ be a locally convex space. Let C be a convex set such that $0 \in \operatorname{int} C$. Then define

$$\begin{aligned} \mu_C : X &\to \mathbb{R} \\ x &\mapsto \inf\{t > 0 \mid x \in tC\} \end{aligned}$$

This is well-defined since $\frac{1}{n}x \to 0$ and so $\frac{1}{n}x \in C$ for some n. μ_C is the **Minkowski functional** (aka **gauge functional**) of C.

Example. If X is a normed space and $C = B_X$, then $\mu_C = \|\cdot\|$.

Lemma 3.9. μ_c is positive homogeneous and subadditive. Moreover,

$$\{x \mid \mu_C(x) < 1\} \subseteq C \subseteq \{x \mid \mu_C(x) \leq 1\}$$

with the first equality holding iff C is open and the second equality holding iff C is closed.

Proof.
positive homogeneity
For $x \in X, s, t > 0$, we have $sx \in stC \iff x \in tC$. Hence $\mu_C(sx) = s\mu_C(x)$. It also holds for $s = 0$ since $\mu_C(0) = 0$.

subadditivity
First observe that $\mu_C(x) < t$ implies $x \in tC$. Indeed, there is some $s < t$ such that $x \in sC$. Then

$$\frac{x}{t} = \left(1 - \frac{s}{t}\right) \cdot 0 + \frac{s}{t} \cdot \frac{x}{s} \in C$$

by convexity. Now let $x, y \in X$. Fix $s > \mu_C(x), t > \mu_C(y)$. Then $x \in sC, y \in tC$, so

$$x + y \in sC + tC = (s + t)C$$

by convexity. So $\mu_C(x + y) < s + t$. Taking the infima over s and t, $\mu_C(x + y) \leq \mu_C(x) + \mu_C(y)$.

$\{x \mid \mu_C(x) < 1\} \subseteq C$ **with equality iff** C **open**
If $\mu_C(x) < 1$, then $x \in C$ by the observation. If C is open and $x \in C$, find n such that $\left(1 + \frac{1}{n}\right) x \in C$. Then

$$\mu_C(x) \leq \frac{1}{1 + \frac{1}{n}} < 1$$

$C \subseteq \{x \mid \mu_C(x) \leq 1\}$ **with equality iff** C **closed**
If $x \in C$, then $\mu_C(x) \leq 1$ by definition. If C is closed and $\mu_C(x) \leq 1$, then by homogeneity $\mu_C\left(\left(1 - \frac{1}{n}\right) x\right) < 1$ for all n, so $\left(1 - \frac{1}{n}\right) x \in C$, and $x \in C$ since C is closed. $\square$

Remark. If C is balanced, then μ_C is a seminorm. If further C is bounded, then μ_C is a norm.

Theorem 3.10 (Hahn-Banach Separation)**.** Let $(X, \mathcal{P})$ be a LCS and C be an open convex set with $0 \in C$. Let $x_0 \notin C$. Then there exists $f \in X^*$ such that $f(x_0) > f(x)$ for all $x \in C$.

TODO: Insert separation picture

Remark. From now on, we work with real scalars. The complex case follows from the fact that $\mathrm{Re} : X^* \to X^*_{\mathbb{R}}$ is a real-linear bijection.

Proof. Consider μ_C. By Lemma 3.9, $C = \{x \mid \mu_C(x) < 1\}$. So $\mu_C(x_0) \geq 1$. Let $Y = \mathrm{Span}(x_0)$ and $g : Y \to \mathbb{R}$ defined by $g(\lambda x_0) = \lambda$. g is linear and $g(x_0) = 1 \leq \mu_C(x_0)$. Hence $g \leq \mu_C$ on Y.
By Theorem 1.1, find $f : X \to \mathbb{R}$ linear such that $f \restriction_Y = g$ and $f \leq \mu_C$. For all $x \in C$, $f(x) \leq \mu_C(x) < 1 = f(x_0)$. further, f is continuous since $C \cap (-C)$ is a neighborhood of 0 on which $|f(x)| \leq 1$. □

Theorem 3.11. Let $(X, \mathcal{P})$ be a LCS. Let A, B be disjoint nonempty convex sets.

- If A is open, then there exists $f \in X^*$ such that $f(x) < \inf_B f$ for all $x \in A$.
- If A is compact and B is closed, then there exists $f \in X^*$ such that $\sup_A f < \inf_B f$.

Proof.

- Fix $a \in A, b \in B$. Let $C = (A - a) - (B - b)$ and $x_0 = b - a$. Then C is open, convex, $0 \in C$ and $x_0 \notin C$ (A, B are disjoint). By Theorem 3.10, find $f \in X^*$ such that $f(z) < f(x_0)$ for all $z \in C$. So for all $x \in A, y \in B$, $f(x - y + x_0) < f(x_0)$, namely $f(x) < f(y)$. In particular, $f \neq 0$. So find u such that $f(u) > 0$. Given $x \in A$, as A is open and $x + \frac{1}{n}u \to x$, find n such that $x + \frac{1}{n}u \in A$. Then

$$f(x) < f\left(x + \frac{1}{n}u\right) \leq \inf_B f$$

- **Claim.** There exists a convex open neighborhood U of 0 such that $A+U$ is disjoint from B.

  Proof. For $x \in A$, find U_x an open neighborhood of 0 such that $x + U_x$ is disjoint from B (since B is closed). By continuity of addition, find V_x an open neighborhood of 0 such that $V_x + V_x \subseteq U_x$. WLOG V_x is convex and symmetric. By compactness, find $x_1, \ldots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^n x_i + V_{x_i}$. We claim $U = \bigcap_{i=1}^n V_{x_i}$ works. Given $x \in A$, find i such that $x \in x_i + V_{x_i}$, so that

$$x + U \subseteq x_i + V_{x_i} + U \subseteq x_i + V_{x_i} + V_{x_i} \subseteq x_i + U_{x_i}$$

  is disjoint from B. Hence $A + U$ is disjoint from B. □

  Apply the first part with $A + U$ and B to get $f \in X^*$ such that $f(x + u) < f(y)$ for all $x \in A, y \in B, u \in U$. In particular, $f \neq 0$, so find z such that $f(z) > 0$. As $\frac{1}{n}z \to 0$, find n such that $\frac{1}{n}z \in U$. Then $f(x) + \frac{1}{n}f(z) < f(y)$ for all $x \in A, y \in B$. So

$$\sup_A f < \sup_A f + \frac{1}{n}f(z) \leq \inf_B f$$

□

Theorem 3.12 (Mazur)**.** Let C be a convex set in a normed space. Then $\overline{C}^{\|\cdot\|} = \overline{C}^{\mathrm{w}}$. In particular,

$$C \text{ norm-closed} \iff C \text{ w-closed}$$

Proof. WLOG C is nonempty. We already know $\overline{C}^{\|\cdot\|} \subseteq \overline{C}^{\mathrm{w}}$ as the weak topology is weaker than the norm-topology.
If $x \notin \overline{C}^{\|\cdot\|}$, then apply Theorem 3.11 to $A = \{x\}$ and $B = \overline{C}^{\|\cdot\|}$ to obtain $f \in X^*$ such that $f(x) < \inf_B f$. Then $\{z \mid f(z) < \inf_B f\}$ is a w-open neighborhood of x disjoint from B. So $x \notin \overline{C}^{\mathrm{w}}$. □

Corollary 3.13. If $x_n \xrightarrow{w} 0$ in a normed space, then for $\varepsilon > 0$ there is some x in the convex hull of the x_0 such that $\|x\| < \varepsilon$.

Proof.

$$0 \in \overline{\mathrm{conv}\{x_n \mid n \in \mathbb{N}\}}^{\mathrm{w}} = \overline{\mathrm{conv}\{x_n \mid n \in \mathbb{N}\}}^{\|\cdot\|}$$

□

Remark. It follows from this that there exist $p_1 < q_1 < p_2 < q_2 < \dots$ and convex combinations $z_n = \sum_{i=p_n}^{q_n} t_i x_i$ such that $z_n \to 0$.

Lecture 13

Theorem 3.14 (Banach-Alaoglu)**.** For any normed space X, (B_{X^*}, w^*) is compact.

Proof. For $x \in X$, let $K_x = \{\lambda \in \mathbb{K} \mid |\lambda| \le \|x\|\}$. Equip K with its product topology. Let $\pi_x : K \to K_x$ be the projection. Note

$$K = \{\lambda : X \to \mathbb{K} \mid \forall x \in X, |\lambda(x)| \le \|x\|\}$$

So $B_{X^*} \subseteq K$. By Tychonoff (Theorem 3.2), K is compact. So all we need to show is that B_{X^*} is closed in K.

$$\begin{aligned} B_{X^*} &= \{\lambda \in K \mid \forall a, b, x, y, \lambda(ax + by) = a\lambda(x) + b\lambda(y)\} \\ &= \bigcap_{a,b,x,y} \{\lambda \in \mathbb{K} \mid \pi_{ax+by}(\lambda) = a\pi_x(\lambda) + b\pi_y(\lambda)\} \\ &= \bigcap_{a,b,x,y} (\pi_{ax+by} - a\pi_x - b\pi_y)^{-1}\{0\} \end{aligned}$$

is closed in K since each π_x is continuous. □

Proposition 3.15. Let X be a normed space and K be a compact Hausdorff space.

1. X separable $\iff$ (B_{X^*}, w^*) metrisable
2. $C(K)$ separable $\iff$ K metrisable

Proof.

1. $\Rightarrow$ Fix a dense sequence x_n in X. Let $\mathcal{F} = \{\hat{x}_n \mid n \in \mathbb{N}\}$. Then $\mathcal{F}$ separates points of X^*, so $\sigma(B_{X^*}, \mathcal{F})$ is Hausdorff and contained in the w*-topology of B_{X^*}. So

$$\mathrm{id} : (B_{X^*}, w^*) \to (B_{X^*}, \sigma(B_{X^*}, \mathcal{F}))$$

is a continuous bijection from compact to Hausdorff, hence a homeomorphism. So $\sigma(B_{X^*}, \mathcal{F})$ is the w*-topology on B_{X^*}. This is metrisable by Proposition 3.1.

2. $\Rightarrow$ By the above, $(B_{C(K)^*}, w^*)$ is metrisable. For $k \in K$, define

$$\begin{aligned} \delta_k &: C(K) \to \mathbb{K} \\ F &\mapsto f(k) \end{aligned}$$

Then $\delta_k \in B_{C(K)^*}$. We thus have $\delta : K \to (B_{C(K)^*}, w^*)$.

δ continuous
By the universal property, it's enough to check that $\hat{f} \circ \delta$ is continuous for all $f \in C(K)$. But

$$(\hat{f} \circ \delta)(k) = \delta_k(f) = f(k)$$

So $\hat{f} \circ \delta = f$ is continuous.

δ injective
$C(K)$ separates points of K.

Now, $\delta : K \to (\delta(K), w^*)$ is a continuous bijection from compact to Hausdorff, hence a homeomorphism. Hence K is metrisable.

2. $\Leftarrow$ As K is compact metrisable, it is separable. Fix a sequence x_n dense in K. Let $f_n(x) = d(x, x_n)$. d is a metric inducing the topology of K. Let A be the subalgebra of $C(K)$ generated by 1 and the f_n. Then A is separable, closed under complex conjugation, separates points of K and $1 \in A$. By Stone-Weierstrass, $\overline{A} = C(K)$. So $C(K)$ is separable.

1. $\Leftarrow$ Let $K = (B_{X^*}, w^*)$. This is compact by Theorem 3.14. Since K is metrisable, $C(K)$ is separable. It's enough to show that X embeds isometrically into $C(K)$. Let

$$\begin{aligned} T &: X \to C(K) \\ x &\mapsto \hat{x} \restriction_{B_{X^*}} \end{aligned}$$

Then T is linear and $\|Tx\|_\infty = \|\hat{x}\| = \|x\|$.

□

Remarks.

1. If X is separable, then (B_{X^*}, w^*) is compact metrisable, hence w*-sequentially compact.
2. X separable $\implies$ X^* w*-separable ((B_{X^*}, w^*) compact metrisable, hence separable). Recall that, for any topological vector space Y,

$$Y \text{ separable} \iff \exists A \text{ countable}, \overline{\operatorname{Span}} A = Y$$

Hence Mazur tells us

$$X \text{ separable} \iff X \text{ w-separable}$$

So X w-separable $\implies$ X^* w*-separable. The converse is false, eg ℓ_∞.
3. The proof shows that $(B_{C(K)^*}, w^*)$ contains a homeomorphic copy of K.
4. The proof shows that every normed space X embeds isometrically into $C(K)$ for some compact Hausdorff space K, eg $K = (B_{X^*}, w^*)$.

Proposition 3.16. Let X be a normed space. Then

$$X^* \text{ separable} \iff (B_X, w) \text{ metrisable}$$

Proof.

$\Rightarrow$ By Proposition 3.15, $(B_{X^{**}}, w^*)$ is metrisable. Hence

$$(B_X, w) = (B_{X^{**}}, w^*) \restriction_{B_X}$$

is metrisable.

$\Leftarrow$ Let d metrise (B_X, w). For all n, find $F_n \subseteq X^*$ finite and $\varepsilon_n > 0$ such that

$$U_n = \{x \in B_X \mid \forall f \in F_n, |f(x)| < \varepsilon_n\} \subseteq \left\{x \mid d(x,0) < \frac{1}{n}\right\}$$

We claim $Z = \operatorname{Span}_n F_n$ is dense. Then we're done.
Let $g \in X^*, \varepsilon > 0$. Then $\{x \in B_X \mid |g(x)| < \varepsilon\}$ is a w-neighborhood of 0 in B_X, hence contains U_n for some n. Let $Y = \bigcap_{f \in F_n} \ker f$. For $x \in B_Y$, $x \in U_n$, so $|g(x)| < \varepsilon$. So $\|g \restriction_Y\| \le \varepsilon$. By Hahn-Banach, find $h \in X^*$ such that $h \restriction_Y = g \restriction_Y$ and $\|h\| \le \varepsilon$. Now

$$Y = \bigcap_{f \in F_n} \ker f \subseteq \ker(g - h)$$

hence $g - h \subseteq \operatorname{Span} F_n \subseteq Z$ by Lemma 3.3. So $d(g, Z) < \varepsilon$. Hence $g \in \overline{Z}$.

□

Theorem 3.17 (Goldstine)**.** For any normed space X,

$$\overline{B_X}^{w^*} = B_{X^{**}}$$

where B_X is thought of as a subspace of X^{**}.

Proof. $B_{X^{**}}$ is w*-closed (by Theorem 3.14) and $B_X \subseteq B_{X^{**}}$, so $\overline{B_X}^{w^*} \subseteq B_{X^{**}}$. Now, let $\phi \notin \overline{B_X}^{w^*}$. Apply Theorem 3.11.ii to $(X^{**}, w^*), A = \{\phi\}, B = \overline{B_X}^{w^*}$ and find $f \in X^*$? such that $\phi(f) > \sup_B \hat{f}$ (or $\operatorname{Re}\phi(f) > \sup_B \operatorname{Re}\hat{f}$ in the complex case).

$$\|\phi\| \, \|f\| \ge |\phi(f)| > \sup_B \left\|\hat{f}\right\| = \sup_B \|f\| \ge 1$$

So $\phi \notin B_{X^{**}}$. □

Lecture 14

Theorem 3.18. Let X be a Banach space. TFAE

1. X is reflexive.
2. (B_X, w) is compact.
3. X^* is reflexive.

Proof.

$1 \Rightarrow 2$ $(B_X, w) \cong (B_{X^{**}}, w^*)$ is compact by Banach-Alaoglu (Theorem 3.14).

2 ⇒ 1 $(B_X, w) = (B_{X^{**}}, w^*) \restriction_{B_X}$, so B_X is compact in the w*-topology of X^{**}. Hence it is w*-closed in X^{**}. By Goldstine (Theorem 3.17), $B_{X^{**}} = \overline{B_X}^{w^*} = B_X$.

1 ⇒ 3 $(B_{X^*}, w) = (B_{X^*}, w^*)$ is compact by Theorem 3.14. By 2 $\implies$ 1, X^* is reflexive.

3 ⇒ 1 By 1 ⇒ 3, X^{**} is reflexive. So by 1 ⇒ 2, $(B_{X^{**}}, w)$ is compact. Since X is complete, X is closed in X^{**}, hence w-closed in X^{**} (by Mazur). Hence $B_X = X \cap B_{X^{**}}$ is a w-closed subset of $B_{X^{**}}$ and thus is w-compact. By 2 ⇒ 1, X is reflexive.

□

Remark. If X is separable and reflexive, then (B_X, w) is compact metrisable. Hence B_X is sequentially compact.

Lemma 3.19. Let (K, d) be a nonempty compact metric space. There exists a continuous surjection $\phi : \{0,1\}^{\mathbb{N}} \to K$ where $\{0,1\}^{\mathbb{N}}$ is given the product topology.

Proof. Since K is totally bounded, if A is nonempty closed and $\varepsilon > 0$ there exist nonempty closed sets $B_1, \ldots, B_n$ such that $A = \bigcup_i B_i$ and $\operatorname{diam} B_i < \varepsilon$. Applying this repeatedly, find nonempty closed sets K_ε for all $\varepsilon \in \Sigma := \bigcup_{n=0}^{\infty} \{0,1\}^n$ such that

$$K_\varnothing = K, K_\varepsilon = K_{\varepsilon,0} \cup K_{\varepsilon,1}, \max_{\varepsilon \in \{0,1\}^n} \operatorname{diam} K_\varepsilon \to 0$$

Define

$$\begin{aligned} \phi : \{0,1\}^{\mathbb{N}} &\to K \\ \varepsilon &\mapsto \text{unique point in } \bigcap_n K_{\varepsilon_1, \ldots, \varepsilon_n} \end{aligned}$$

ϕ onto
Given $x \in K$, find ε such that $x \in K_{\varepsilon_1, \ldots, \varepsilon_n}$ for all n. Then $\phi(\varepsilon) = x$.

ϕ continuous
Let $\varepsilon, \delta \in \{0,1\}^{\mathbb{N}}, n \in \mathbb{N}$. If $\varepsilon_i = \delta_i$ for all $i \le n$, then

$$d(\phi(\varepsilon), \phi(\delta)) \le \operatorname{diam} K_{\varepsilon_1, \ldots, \varepsilon_n} \to 0$$

□

Remark. $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the middle third Cantor set Δ via

$$\varepsilon \mapsto \sum_{i=1}^{\infty} 2\varepsilon_i 3^{-i}$$

Theorem 3.20. Every separable Banach space X embeds isometrically into $C[0,1]$, namely $C[0,1]$ is isometrically universal for $\mathcal{SB}$.

Proof. From the proof of Proposition 3.15, X embeds isometrically into $C(K)$ where $K = (B_{X^*}, w^*)$. Since X is separable, K is metrisable. By Lemma 3.19, find $\phi : \Delta \to K$ a continuous surjection. Hence $C(K)$ embeds isometrically into $C(\Delta)$ via $f \mapsto f \circ \phi$. $C(\Delta)$ embeds isometrically into $C[0,1]$ via $f \mapsto \tilde{f}$ where $\tilde{f}$ linearly interpolates f between elements of the Cantor set. □

4 Convexity

Let X be a real or complex vector space and K a convex set. A point $x \in K$ is an **extreme point** of K if, whenever $x = ay + bz, a, b > 0, a + b = 1$, we have $x = y = z$. Denote $\operatorname{Ext} K$ the set of extreme points of K.

Examples. TODO: Pictures

- $\operatorname{Ext}(B_{\ell_1^2}) = \{\pm e_1, \pm e_2\}$
- $\operatorname{Ext}(B_{\ell_2^2}) = S_{\ell_2^2}$
- $\operatorname{Ext}(B_{c_0}) = \varnothing$. Indeed, if $x \in B_{c_0}$, we can find n such that $|x_n| < \frac{1}{2}$ and define $y = x + \frac{1}{2}e_n, z = x - \frac{1}{2}e_n$ so that $y, z \in B_{c_0}$ and $x + \frac{1}{2}y + \frac{1}{2}z, y \neq x, z \neq x$.

Theorem 4.1 (Krein-Milman)**.** Let K be a nonempty compact convex set in a LCS $(X, \mathcal{P})$. Then

$$K = \overline{\operatorname{conv}}(\operatorname{Ext} K)$$

In particular, $\operatorname{Ext} K$ is nonempty if K is nonempty.

Corollary 4.2. If X is a normed space, then $B_{X^*} = \overline{\operatorname{conv}}(\operatorname{Ext} B_{X^*})$ and $\operatorname{Ext} B_{X^*}$ is nonempty.

Remark. c_0 is not a dual space since $\operatorname{Ext} B_{c_0}$ is empty.

Definition. Let K be a nonempty compact convex set in a LCS $(X, \mathcal{P})$. A **face** of K is a nonempty compact convex set $E \subseteq K$ such that, for all $y, z \in K, a, b > 0, a + b = 1$, if $ay + bz \in E$ then $y, z \in E$.

Examples.

- K is a face of K.
- For $x \in K$, $\{x\}$ is a face of K iff $x \in \operatorname{Ext} K$.
- Let $f \in X^*, \alpha = \sup_K f$. Then $E = \{x \in K \mid f(x) = \alpha\}$ is a face of K.
- Let E be a face of K. If F is a face of E, then F is a face of K. In particular, $\operatorname{Ext} E \subseteq \operatorname{Ext} K$.

Proof of Theorem 4.1. First we show that any nonempty compact convex set K has an extreme point.

By Zorn, find a minimal face E of K.
If $|E| > 1$, then pick $x \neq y$ in E such that $f(x) > f(y)$. Then $F = \{z \in K \mid f(z) = \sup_E f\}$ is a face of E which does not contain y. Hence it is a strictly smaller face of K. Contradiction.
So F is a singleton and $\operatorname{Ext} E \neq \varnothing$.

Now WLOG K is nonempty and let $L = \overline{\operatorname{conv}}(\operatorname{Ext} K)$. Then L is a nonempty face of K. Assume $x_0 \neq K \setminus L$. By Theorem 3.10, find $f \in X^*$ such that $f(x_0) > \sup_L f$. Let $\alpha = \sup_K f$. Then $E = \{x \in K \mid f(x) = \alpha\}$ is a face of K. Find z an extreme point of E. Then $z \notin L$ is an extreme point of K. Contradiction. ☐

Lecture 15

Lemma 4.3. Let $(X, \mathcal{P})$ be a LCS. Let $K \subseteq X$ be compact and $x_0 \in K$. Then for a neighborhood V of x_0 in X, there exist $f_1, \dots, f_n \in X^*$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$x \in \{x \in X \mid \forall i, f_i(x) < \alpha_i\} \cap K \subseteq v$$

Proof. Let τ be the topology of X induced by $\mathcal{P}$. Let $\sigma = \sigma(X, X^*)$. Then $\mathrm{id} : (K, \tau) \to (K, \sigma)$ is a continuous bijection from compact to Hausdorff. Hence it is a homeomorphism and $\sigma = \tau$ on K. $\square$

Lemma 4.4. Let $(X, \mathcal{P})$ be a LCS, $K \subseteq X$ be compact convex. Let $x_0 \in \operatorname{Ext} K$. Then for a neighborhood V of x_0 in X there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$x_0 \in \{x \in X \mid f(x) < \alpha\}$$

Proof. Find f_i and α_i from Lemma 3. Let $K_i = \{x \in K \mid \alpha_i \leq f_i(x)\}$. This is compact and convex. $x_0 \notin V^c \subseteq \bigcup_i K_i$ and

$$\operatorname{conv} \bigcup_i K_i = \left\{ \sum_i t_i x_i \mid x_i \in K_i, t_i \geq 0, \sum_i t_i = 1 \right\}$$

Since x_0 is an extreme point of K, $x_0 \notin \operatorname{conv} \bigcup_i K_i$. Also,

$$K_1 \times \cdots \times K_n \times \left\{ t \geq 0 \mid \sum_i t_i = 1 \right\}$$

is compact and

$$(x_1, \ldots, x_n, t) \mapsto \sum_i t_i x_i$$

is continuous. So the image $B = \operatorname{conv} \bigcup_i K_i$ is compact. By Hahn-Banach Separation (Theorem 3.10), find $f \in X^*$ such that $f(x_0) < \inf_B f$. Choose $\alpha \in \mathbb{R}$ such that $f(x_0) < \alpha < \inf_B f$. Then

$$x \in \underbrace{\{x \in X \mid f(x) < \alpha\}}_{\text{disjoint from } B} \cap K \subseteq V$$

$\square$

Theorem 4.5 (Partial converse to Krein-Milman)**.** Let $(X, \mathcal{P})$ be a LCS, $K \subseteq X$ compact convex, $S \subseteq K$. If $K = \overline{\operatorname{conv}} S$, then $\operatorname{Ext} K \subseteq \overline{S}$.

Remark. The closure is necessary, eg let S be a dense subset of $B_{\ell_2^2}$. Then $\overline{\operatorname{conv}} S = B_{\ell_2^2}$ and $\operatorname{Ext} B_{\ell_2^2} = S_{\ell_2^2}$. Also $\operatorname{Ext} K$ need not be closed.

TODO: Insert picture

Proof. Assume $x_0 \in \operatorname{Ext} K \setminus \overline{S}$. Apply Lemma 4.4 with $V = \overline{S}^c$. So there exists $f \in X^*, \alpha \in \mathbb{R}$ such that

$$x_0 \in \{x \in X \mid f(x) < \alpha\} \cap K \subseteq V$$

Then $L = \{x \in K \mid \alpha \leq f(x)\}$ is compact convex with $S \subseteq L$. Hence $K = \overline{\operatorname{conv}} S \subseteq L$, contradicting $x_0 \notin L$. $\square$

Exercise. $\operatorname{Ext} B_{C(K)^*} = \{\lambda \delta_k \mid |\lambda| = 1, k \in K\}$ where K is compact Hausdorff. Use Theorem 4.5 for the inclusion.

Theorem 4.6 (Banach-Stone)**.** Let K, L be compact Hausdorff spaces. Then

$$C(K) \cong C(L) \iff K \cong L$$

Proof.

- $\Leftarrow$ If $\phi : K \to L$ is a homeomorphism, then $\phi^* : C(L) \to C(K)$ is an isometric isomorphism.
- $\Rightarrow$ Let $T : C(L) \to C(K)$ be an isometric isomorphism. Then so is $T^* : C(K)^* \to C(L)^*$. So $T^*(B_{C(K)^*}) = B_{C(L)^*}$ and $T^*(\operatorname{Ext} B_{C(K)^*}) = \operatorname{Ext} B_{C(L)^*}$. Thus, for each $k \in K$,
$$T^*(\delta_k) = \lambda(k)\delta_{\phi(k)}$$
for some scalar $\lambda(k)$ with $|\lambda(k)| = 1$ and some $\phi(k) \in L$. So we have functions
$$\begin{aligned}\lambda &: K \to \mathbb{K} \\ \phi &: K \to L\end{aligned}$$
For all k, $\lambda(k) = T^*(\delta_k)(1) = (T1)(k)$. So $\lambda = T1 \in C(K)$ is continuous.

  Recall $\delta : K \to (C(K), w^*)$ is continuous (in fact a homeomorphism from K to $\delta(K)$). Also, $T^* : C(K)^* \to C(L)^*$ is w*-to-w*-continuous. Hence $k \mapsto \overline{\lambda(k)}T^*(\delta_k) = \delta_{\phi(k)}$ is continuous and so is ϕ.

  ϕ injective
  Assume $\phi(k_1) = \phi(k_2)$. Then $\overline{\lambda(k_1)}T^*(\delta_{k_1}) = \overline{\lambda(k_2)}T^*(\delta_{k_2})$. Evaluate at 1 to get $\lambda(k_1) = \lambda(k_2)$. As T^* is injective, we get $\delta_{k_1} = \delta_{k_2}$ and hence $k_1 = k_2$.

  ϕ onto
  Given $\ell \in L$, find by surjectivity of T^* a scalar μ and $k \in K$ such that $|\mu| = 1, T^*(\mu\delta_k) = \delta_\ell$. So $\mu\lambda(k)\delta_{\phi(k)} = \delta_\ell$. Evaluate at 1 to get $\mu\lambda(k) = 1$. So $\phi(k) = \ell$.

$\square$

5 Banach Algebras

A real/complex **algebra** is a real/complex vector space A with multiplication $A \times A \to A$ such that

1. $a(bc) = (ab)c$
2. $a(b+c) = ab + ac$
3. $\lambda(ab) = (\lambda a)b = a(\lambda b)$

A is **unital** if there exists an element $1 \in A$ such that $1 \neq 0$ and $\forall a \in A, 1a = a1 = a$. This element is unique and is called the **unit** of A.

An **algebra norm** on A is a norm on A such that $\forall a, b, \|ab\| \leq \|a\| \|b\|$. A **normed algebra** is an algebra equipped with an algebra norm. A **Banach algebra** is a complete normed algebra. A **unital normed algebra** is a normed algebra which is unital and such that $\|1\| = 1$ ($1 \leq \|1\|$ always since $\|1\| = \|1 \cdot 1\| \leq \|1\| \cdot \|1\|$). A **unital Banach algebra** is a complete unital normed algebra.

If A is a normed algebra which is also a unital algebra (but not assuming $\|1\| = 1$), then $|a| = \sup_{\|b\| \leq 1} \|ab\|$ defines an equivalent norm that makes A into a unital normed algebra.

In the category of normed algebras, an isomorphism will mean a continuous homeomorphism with continuous inverse. But **a morphism need not be continuous**.

Note. From now on, the scalar field is $\mathbb{C}$.

Lecture 16

Examples.

1. $C(K)$ with K compact is a commutative unital normed algebra with pointwise multiplication and uniform norm.
2. Let K be compact Hausdorff. A **uniform normed algebra** on K is a closed subalgebra of $C(K)$ that separates points of K and contains the constant functions (if it's further closed under complex conjugation, Stone-Weierstrass says it's everything). Eg, the **disk algebra**

$$A(\Delta) = \{f \in C(\Delta) \mid f \text{ holomorphic on } \operatorname{int} \Delta\}$$

where

$$\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

More generally, let $K \subseteq \mathbb{C}$ be nonempty compact. We have the following uniform algebras on K:

$$\mathcal{P}(K) \subseteq \mathcal{R}(K) \subseteq \mathcal{O}(K) \subseteq A(K) \subseteq C(K)$$

where $\mathcal{P}(K), \mathcal{R}(K), \mathcal{O}(K)$ are the closures in $C(K)$ of polynomials, rational functions without poles in K, functions holomorphic on some open neighborhood of K respectively, and

$$A(K) = \{f \in C(K) \mid f \text{ holomorphic on } \operatorname{int} K\}$$

Later we will show that $\mathcal{R}(K) = \mathcal{O}(K)$ always, and

$$\mathcal{P}(K) = \mathcal{R}(K) \iff K^c \text{ connected}$$

In general, $A(K) \neq \mathcal{O}(K)$, and

$$A(K) = C(K) \iff \operatorname{int} K = \varnothing$$

3. $L_1(\mathbb{R})$ with L_1 norm and convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$$

is a commutative Banach without a unit (Riemann-Lebesgue lemma).

4. If X is a Banach algebra, then $\mathcal{B}(X)$ with composition and operator norm is a unital Banach algebra (it's not commutative if $\dim X > 1$). If X is a Hilbert space, then $\mathcal{B}(X)$ is a C*-algebra.

Definition (Elementary constructions)**.**

1. If A is a unital algebra, then a **unital subalgebra** is a subalgebra B of A that contains 1.

   If A is a normed algebra, then the closure of a subalgebra of A is a subalgebra.

2. **Unitisation**
   The unitisation of an algebra A is the vector space direct sum $A_+ = A \oplus \mathbb{C}$ with multiplication $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. Then A_+ is a unital algebra with unit $(0, 1)$. The ideal $\{(a, 0) \mid a \in A\}$ is isomorphic to A and will always be identified with A. We can write

   $$A_+ = \{a + \lambda 1 \mid a \in A, \lambda \in \mathbb{C}\}$$

   If A is a normed algebra, then A_+ becomes a unital normed algebra with

   $$\|a + \lambda 1\| = \|a\| + |\lambda|$$

   Then A is a closed ideal of A_+.

   If A is a Banach algebra, then A_+ is a unital Banach algebra.

3. The closure of an ideal of a normed algebra is an ideal. If J is a closed ideal of the normed algebra A, then A/J is a normed algebra in the quotient norm.

   If A is a unital normed algebra and J is a proper ($J \neq A$) closed ideal of A, then A/J is a unital normed algebra with unit $1 + J$ ($\|1 + J\| \le \|1\| = 1$).

4. Let $\tilde{A}$ be the Banach space completion of a normed algebra A. Then $\tilde{A}$ is a Banach algebra with the following multiplication. Given $a, b \in \tilde{A}$, choose a_n, b_n in A such that $a_n \to a, b_n \to b$. Then define $ab = \lim_n a_n b_n$. One can check this is well-defined and respects the algebra axioms.

5. Let A be a unital Banach algebra. Let $X = A$ thought of as a Banach space. FOr $a \in A$, define $L_a : X \to X, L_a(x) = ax$. Then $L_a \in \mathcal{B}(X)$ and $\|L_a\| = \|a\|$. The map $L : X \to \mathcal{B}(X)$ is an isometric unital homomorphism.

Lemma 5.1. Let A be a unital Banach algebra and $a \in A$. If $\|1 - a\| < 1$, then a is invertible and

$$\left\|a^{-1}\right\| \le \frac{1}{1 - \|1 - a\|}$$

Proof. For all n, $\|(1-a)^n\| \le \|1 - a\|^n$. So $\sum_n \|(1-a)^n\| < \infty$. Hence $b := \sum_n (1-a)^n$ converges absolutely. Then

$$(1 - a)b = b(1 - a) = b - 1$$

So $ab = ba = 1$. We see that

$$\|b\| \leq \sum_n \|(1-a)^n\| \leq \sum_n \|1-a\|^n = \frac{1}{1 - \|1-a\|}$$

$\square$

Notation. Let $G(A)$ denote the group of invertibles of a unital algebra A.

Corollary 5.2. Let A be a unital Banach algebra.

1. $G(A)$ is open in A.
2. $x \mapsto x^{-1}$ is continuous on $G(A)$.
3. If X_n in $G(A)$ converges to $x \notin G(A)$, then $\left\|x_n^{-1}\right\| \to \infty$.
4. If $x \in \partial G(A)$, then there exist z_n in A such that $\|z_n\| = 1$ and $z_n x, x z_n \to 0$. It follows that x has no left or right inverse in A, nor even in any unital Banach algebra B containing A as a (not necessarily unital) subalgebra.

Proof.

1. Let $x \in G(A)$. If $y \in A$ and $\|y - x\| < \left\|x^{-1}\right\|^{-1}$, then

$$\left\|1 - x^{-1}y\right\| = \left\|x^{-1}(x-y)\right\| \leq \left\|x^{-1}\right\| \|x - y\| < 1$$

   Hence $x^{-1}y \in G(A)$ by Lemma 5.1. So $y \in G(A)$.

2. Let's fix $x \in G(A)$. For $y \in G(A)$;

$$y^{-1} - x^{-1} = y^{-1}(x-y)x^{-1}$$

   So

$$\left\|y^{-1} - x^{-1}\right\| \leq \left\|y^{-1}\right\| \|x-y\| \left\|x^{-1}\right\|$$

   If $\|x - y\| < \frac{1}{2\|x^{-1}\|}$, then

$$\left\|y^{-1} - x^{-1}\right\| \leq 2\left\|x^{-1}\right\|^2 \|x-y\| \to 0$$

   as $y \to x$.

3. From the proof of 1, if $\|x - x_n\| < \left\|x_n^{-1}\right\|^{-1}$, then $x \in G(A)$. Contradiction.

4. Given $x \in \partial G(A)$, find x_n in $G(A)$ such that $x_n \to x$. By 3, $\left\|x_n^{-1}\right\| \to \infty$. Set $z_n = \frac{x_n^{-1}}{\left\|x_n^{-1}\right\|}$. Then

$$z_n x = z_n x_n + z_n(x - x_n) = \left\|x_n^{-1}\right\|^{-1} + z_n(x - x_n) \to 0$$

   as $\|z_n(x - x_n)\| \leq \|z_n\| \|x - x_n\| \to 0$. Similarly, $x z_n \to 0$.

   Assume B is a unital Banach algebra and A is a subalgebra of B. If $y \in B$ and $yx = 1$, then $yxz_n = z_n$. So

$$1 = \|z_n\| = \|yxz_n\| \leq \|y\| \|xz_n\| \to 0$$

   Similarly, we can't have $y \in B$ and $xy = 1$.

$\square$

Definition. Let A be an algebra and $x \in A$. The **spectrum** σ_A of x in A is

- $\sigma_A(x) = \{\lambda \in \mathbb{C} \mid \lambda 1 - x \notin G(A)\}$ if A is unital.
- $\sigma_A(x) = \sigma_{A_+}(x)$ if A is non-unital.

Examples.

1. If $A = M_n(\mathbb{C})$, then $\sigma_A(x)$ is the set of eigenvalues of x.
2. If $A = C(K)$ where K is compact Hausdorff, then $\sigma_A(f) = f(K)$.
3. If $A = \mathcal{B}(X)$ where X is a Banach space, then

$$\sigma_A(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ not an isomorphism}\}$$

Theorem 5.3. Let A be a Banach algebra, $x \in A$. Then $\sigma_A(x)$ is a nonempty compact subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \le \|x\|\}$.

Proof. WLOG A is a unital Banach algebra.

If $|\lambda| > \|x\|$, then $\left\|\frac{x}{\lambda}\right\| < 1$. So by Lemma 5.1, $1 - \frac{x}{\lambda} \in G(A)$ and $\lambda 1 - x = \lambda\left(1 - \frac{x}{\lambda}\right) \in G(A)$. Hence $\sigma_A(x) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \le \|x\|\}$. Also, $\sigma_A(x)$ is the preimage of the closed set $G(A)^c$ (Corollary 5.2.i) under the continuous function $\lambda \mapsto \lambda 1 - x$. Hence it is closed. It follows that $\sigma_A(x)$ is compact.

Consider

$$\begin{aligned} f : \sigma_A(x)^c &\to A \\ \lambda &\mapsto (\lambda 1 - x)^{-1} \end{aligned}$$

By Corollary 5.2.ii, f is continuous. For $\lambda \neq \mu$,

$$f(\lambda) - f(\mu) = f(\lambda)((\mu 1 - x) - (\lambda 1 - x))f(\mu) = (\mu - \lambda)f(\lambda)f(\mu)$$

So

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -f(\lambda)f(\mu) \underset{\lambda \to \mu}{\to} -f(\mu)^2$$

Thus f is holomorphic.

If $|\lambda| > \|x\|$, then $\lambda 1 - x \in G(A)$ and

$$\left\|(\lambda 1 - x)^{-1}\right\| = \frac{1}{|\lambda|}\left\|\left(1 - \frac{x}{\lambda}\right)^{-1}\right\| \le \frac{1}{|\lambda|}\frac{1}{1 - \left\|\frac{x}{\lambda}\right\|} = \frac{1}{|\lambda| - \|x\|} \to 0$$

as $|\lambda| \to \infty$.
If $\sigma_A(x) = \varnothing$, then f is an entire function. By vector-valued Liouville (Theorem 1.8), f is constant. But then $f(x) = x^{-1} \neq 0$ contradicts $f(\lambda) \to 0$ as $\lambda \to 0$. $\square$

Corollary 5.4 (Gelfand-Mazur)**.** A complex unital normed division algebra A ($G(A) = \{0\}^c$) is isometrically isomorphic to $\mathbb{C}$.

Proof. Let's define

$$\begin{aligned} \theta : \mathbb{C} &\to A \\ \lambda &\mapsto \lambda 1 \end{aligned}$$

Then θ is an isometric homomorphism. Is it onto?

Fix $a \in A$ and consider B the completion of A. B is a unital Banach algebra. Hence Theorem 5.3 tells us that $\sigma_B(x)$ is nonempty, namely there exists $\lambda \in \mathbb{C}$ such that $\lambda 1 - x$ is not invertible in B. But then $\lambda 1 - x$ is not invertible in A, so $\lambda 1 - x = 0$ and $\theta(\lambda) = x$. ☐

Definition. Let A be a Banach algebra and $x \in A$. The **spectral radius** $r_A(x)$ of x in A is

$$r_A(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|$$

From Theorem 5.3, $r_A(x)$ is well-defined and $r_A(x) \leq \|x\|$.

Note. Let x, y be commuting elements of a unital algebra A. Then

$$xy \in G(A) \iff x, y \in G(A)$$

Indeed, if $z(xy) = (xy)z = 1$, then

$$x(yz) = 1, (yz)x = zxyyzx = zyxyzx = zyx = zxy = 1$$

So $x \in G(A)$. Similarly, $y \in G(A)$.

Lemma 5.5 (Spectral Mapping Theorem for polynomials)**.** Let A be a unital Banach algebra and $x \in A$. Then for a complex polynomial $p = \sum_{k=0}^{n} a_k z^k$ we have

$$\sigma_A(p(x)) = \{p(\lambda) \mid \lambda \in \sigma_A(x)\} = p(\sigma_A(x))$$

Proof. WLOG $n \geq 1$ and $a_n \neq 0$ ($\sigma_A(\lambda 1) = \{\lambda\}$). Fix $\mu \in \mathbb{C}$. Write

$$\mu - p(z) = c \prod_{k=1}^{n} (\lambda_k - z)$$

for some $c, \lambda_1, \ldots, \lambda_n \in \mathbb{C}, c \neq 0$. Note that

$$p^{-1}(\mu) = \{\lambda_1, \ldots, \lambda_n\}$$

Now,

$$\begin{aligned} \mu \notin \sigma_A(p(x)) &\iff \mu 1 - p(x) = c \prod_{k=1}^{n} (\lambda_k 1 - x) \text{ invertible} \\ &\iff \forall k, \lambda_k 1 - x \text{ invertible (the factors commute)} \\ &\iff \forall \lambda \in \sigma_A(x), p(\lambda) \neq \mu \end{aligned}$$

The result follows. ☐

Theorem 5.6 (Beurling-Gelfand Spectral Radius Formula)**.** Let A be a Banach algebra, $x \in A$. Then

$$r_A(x) = \lim_n \|x^n\|^{\frac{1}{n}} = \inf_n \|x^n\|^{\frac{1}{n}}$$

Proof. WLOG A is unital.

By Lemma 5.5, if $\lambda \in \sigma_A(x)$ and $n \in \mathbb{N}$, then $\lambda^n \in \sigma_A(x^n)$. By Theorem 5.3, $|\lambda^n| \leq \|x^n\|$. So $|\lambda| \leq \|x^n\|^{1/n}$. It follows that

$$r_A(x) \leq \inf_n \|x^n\|^{1/n}$$

Consider

$$\begin{aligned} f : \sigma_A(x)^c &\to \mathbb{C} \\ \lambda &\mapsto (\lambda 1 - x)^{-1} \end{aligned}$$

By the proof of Theorem 5.3, f is holomorphic. Note

$$\sigma_A(x)^c \supseteq \{\lambda \mid r_A(x) < |\lambda|\} \supseteq \{\lambda \mid \|x\| < |\lambda|\}$$

If $|\lambda| > \|x\|$, then

$$f(\lambda) = \frac{1}{\lambda}\left(1 - \frac{x}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty} \frac{x^n}{\lambda^n}$$

by the proof of Theorem 5.1.

Fix $\varphi \in A^*$. Then $\varphi \circ f$ is holomorphic on $\sigma_A(x)^c$ and if $|\lambda| > \|x\|$ then

$$\varphi(f(\lambda)) = \sum_{n=0}^{\infty} \varphi\left(\frac{x^n}{\lambda^n}\right)$$

So, for $|\lambda| > r_A(x)$ and $\varphi \in A^*$, we have $\varphi(\frac{x^n}{\lambda^n}) \to 0$. Hence $\frac{x^n}{\lambda^n} \xrightarrow{w} 0$. By Proposition 3.6, there exists $M > 0$ such that $\left\|\frac{x^n}{\lambda^n}\right\| \le M$ for all n. So $\|x^n\| \le M^{1/n}|\lambda|$. Hence $\limsup \|x^n\|^{1/n} \le |\lambda|$.

Thus we proved

$$r_A(x) \le \inf_n \|x^n\|^{1/n} \le \liminf_n \|x^n\|^{1/n} \le \limsup \|x^n\|^{1/n} \le r_A(x)$$

The result follows. □

Theorem 5.7. Let A be a unital Banach algebra and B be a closed unital subalgebra of A. Let $x \in B$. Then

$$\sigma_A(x) \subseteq \sigma_B(x), \quad \partial\sigma_B(x) \subseteq \partial\sigma_A(x)$$

It follows that $\sigma_B(x)$ is the union of $\sigma_A(x)$ and some of the bounded components of $\sigma_A(x)^c$.

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Proof. Let $\lambda \notin \sigma_B(x)$. Then $\lambda 1 - x$ is invertible in B, hence is invertible in A and $\lambda \notin \sigma_A(x)$.

Assume $\lambda \in \partial\sigma_B(x) = \sigma_B(x) \setminus \operatorname{int}\sigma_B(x)$. On one hand, $\lambda \notin \operatorname{int}\sigma_B(x) \supseteq \operatorname{int}\sigma_A(x)$. On the other hand, $\lambda 1 - x \in \partial G(B) = \overline{G(B)} \setminus G(B)$ (pick a sequence $\lambda_n \to \lambda$ with $\lambda_n \notin \sigma_B(x)$, then $\lambda_n 1 - x \to \lambda 1 - x$ and $\lambda_n 1 - x \in G(B)$), so Corollary 5.2 tells us that $\lambda 1 - x$ is not invertible in any Banach superalgebra, in particular in A, meaning that $\lambda \in G(A)$. Hence $\lambda \in \sigma_A(x) \setminus \operatorname{int}\sigma_A(x) = \partial\sigma_A(x)$. □