Part III – Combinatorics (Incomplete)

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Michaelmas 2023

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0 Introduction

For a finite set A, we write its cardinality $|A|$.

For a graph $G = (V, E)$ and $A, B \subseteq V$, we denote $\Gamma(A) = \{b | \exists a \in A, a \sim b\}$ the set of neighbors of A and $e(A, B)$ the number of edges between A and B.

1 Basic Results

1.1 Chains, Antichains and Scattered Sets of Vectors

Lecture 1

During WW2, Littlewood and Offord were interested in roots of polynomials with random coefficients. They came up with the following neat theorem.

Theorem (Littlewood-Offord, 1943)**.** If $z_1, \dots, z_n \in \mathbb{C}$ with $|z_i| \geq 1$, then, for any disk D of radius r,

$$\#\{\varepsilon \in \{-1,1\}^n | \sum_i \varepsilon_i z_i \in D\} \leq c \log n \frac{2^n}{\sqrt{n}}$$

for some constant c depending only on r.

Upon seeing this theorem, Erdős immediately knew he could drastically improve the bound if the z_i were real.

Theorem (Erdős, 1945)**.** If $x_1, \dots, x_n \in \mathbb{R}$, $|x_i| \geq 1$, then, for any interval I of length 2,

$$\#\{\varepsilon \in \{-1,1\}^n | \sum_i \varepsilon_i z_i \in I\} \leq \binom{n}{\frac{n}{2}}$$

This is best possible, as we see by taking $x_1 = \cdots = x_n = 1$.

Let G be a bipartite graph with parts U and W. A **complete matching** from U to W is an injective function $f : U \to W$ such that $\forall u \in U, u \sim f(u)$.
If G has a complete matching, then certainly $|A| \leq |\Gamma(A)|$. Surprisingly, this is enough.

Theorem 1.1 (Kőnig-Egerváry-Hall Theorem, Hall's Marriage Theorem)**.**

$$G \text{ has a complete matching} \iff \forall A \subseteq U, |A| \leq |\Gamma(A)|$$

Proof. Exercise □

Let $\mathcal{F} = (F_1, \dots, F_m)$ where the F_i are finite sets. We say $a_1, \dots, a_m$ is a **set of distinct representatives**, aka **SDR** if they are distinct and $\forall i, a_i \in F_i$. Certainly, if $\mathcal{F}$ has SDR, then $|I| \leq \left|\bigcup_{i \in I} F_i\right|$ for all $I \subseteq [m]$.

Theorem 1.2.

$$\mathcal{F} \text{ is a SDR} \iff \forall I \subseteq [m], |I| \leq \left|\bigcup_{i \in I} F_i\right|$$

Proof. Define a bipartite graph G with parts $[m]$ and $\bigcup_i F_i$ by $i \sim a \iff a \in F_i$. For all $I \subseteq [m]$, $|I| \leq \left|\bigcup_{i \in I} F_i\right| = |\Gamma(I)|$, so Theorem 1.1 applies. □

Theorem 1.3. If G is a bipartite graph with parts U, W such that $\deg(u) \geq \deg(w)$ for all $u \in U, w \in W$, then there is a complete matching from U to W.

Proof. Find d such that $\deg(u) \geq d \geq \deg(w)$ for all $u \in U, w \in W$. For all $A \subseteq U$, we have

$$d\,|A| \leq e(A, \Gamma(A)) \leq d\,|\Gamma(A)|$$

Hence $|A| \leq |\Gamma(A)|$. We're done by Theorem 1.1. □

For $A \subseteq U, B \subseteq W$, define $w(A) = \frac{|A|}{|U|}, w(B) = \frac{|B|}{|W|}$.

Say a bipartite graph G with parts U, W is (k, ℓ)-**biregular** if $\deg(u) = k, \deg(w) = \ell$ for all $u \in U, w \in W$.

Lemma 1.4. If G is biregular with parts U, W and $A \subseteq U$, then $w(A) \leq w(\Gamma(A))$.

Proof. First, $k\,|U| = e(G) = \ell\,|W|$. Second,

$$k\,|A| = e(A, \Gamma(A)) \leq \ell\,|\Gamma(A)|$$

Dividing the inequality by the equality gives the result. $\square$

Lecture 2

Corollary 1.5. Let G be a (k, ℓ)-biregular graph with parts U, W. If $k \geq \ell$ (or equivalently $|U| \leq |W|$), then there is a complete matching from U to W.

Corollary 1.6. If $\left|s - \frac{n}{2}\right| \leq \left|r - \frac{n}{2}\right|$, then there exists an injection $f : X^{(r)} \hookrightarrow X^{(s)}$ such that either

- $r \leq s$ and $A \subseteq f(A)$ for all $A \in X^{(r)}$
- $s \leq r$ and $f(A) \subseteq A$ for all $A \in X^{(r)}$

Theorem 1.7 (Sperner, 1928)**.** Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an antichain. Then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

Proof. A chain and an antichain can intersect in at most one element. If we manage to partition $\mathcal{P}(X)$ into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains, we win.

But we can repeatedly use Corollary 1.6 to construct matchings $X^{(0)}$ to $X^{(1)}$, $X^{(1)}$ to $X^{(2)}$, ..., $X^{(\lceil \frac{n}{2} \rceil - 1)}$ to $X^{(\lceil \frac{n}{2} \rceil)}$ and $X^{(n)}$ to $X^{(n-1)}$, $X^{(n-1)}$ to $X^{(n-2)}$, ..., $X^{(\lfloor \frac{n}{2} \rfloor + 1)}$ to $X^{(\lfloor \frac{n}{2} \rfloor)}$, then "stack" the matchings together to make chains (if an element of the middle layer). Each chain goes through $X^{(\lfloor fracn2 \rfloor)}$, so we made $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. $\square$

We can now understand the observation of Erdős (1945) about Littlewood-Offord (1943).

Corollary 1.8. Let $x_1, \dots, x_n \in \mathbb{R}$ be such that $|x_i| \geq 1$. Then at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ of the sums $\sum_i \varepsilon_i x_i$, $\varepsilon_i = \pm 1$ fall into the interior of an interval I of length 2.

Proof. WLOG $\forall i, x_i \geq 1$. Set $F_\varepsilon = \{i | \varepsilon_i = 1\}$. $\{F\varepsilon | \sum_i \varepsilon_i x_i \in I\}$ is an antichain (if $F_\varepsilon \subsetneq F_{\varepsilon'}$, then $\sum_i \varepsilon'_i x_i \geq \sum_i \varepsilon_i x_i + 2$, so both sums can't lie in I). $\square$

Definition. A partial order P is **graded** if it has a partition P_i such that

- if $x < y$, $x \in P_i$, $y \in P_j$, then $i < j$ (in particular each P_i is an antichain)
- if $x < y$, $x \in P_i$, $y \in P_j$, $i + 2 \leq j$, then there exists z such that $x < z < y$.

For $a \in P$, we call the unique i for which $a \in P_i$ the **grade** or **rank** of a.
A graded order is **regular** if for every i there exists p_i such that every $x \in P_i$ is less than exactly p_k elemenets of P_{i+1}.
For $A \subseteq P$, define $A_i = A \cap P_i$ and $w(A) = \sum_i \frac{|A_i|}{|P_i|}$.

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Theorem 1.9. Let A be an antichain in a connected regular graded order P. Then $w(A) \leq 1$.

Proof. The regularity condition means that for each i the bipartite graph G_i with parts P_{i-1}, P_i and $x \sim y \iff x < y$ is (p_{i-1}, q)-biregular. In particular, $w(A_i) \leq w(\Gamma_{G_i}(A_i))$. Now, write r the maximal rank of an element of A and define

$$B := A \setminus A_r \cup \Gamma_{G_r}(A_r)$$

The fact that A is an antichain means that B is an antichain as well and $\Gamma_{G_r}(A_r)$ is disjoint from A_{r-1}. Hence

$$\begin{aligned} w(A) &= w(A_r) + w(A_{r-1}) + \sum_{i<r-1} w(A_i) \\ &\leq w(\Gamma_{G_r}(A_r)) + w(A_{r-1}) + \sum_{i<r-1} w(A_i) \\ &= w(B_{r-1}) + \sum_{i<r-1} w(B_i) \\ &= w(B) \end{aligned}$$

We therefore have decreased the maximal rank without decreasing the weight. We can repeat the process until the antichain is contained in some P_i, in which case its weight is clearly at most 1. $\square$

Lecture 3

Consider maximal chains in our regular graded order. Say there are M of them. Each $x \in P_h$ lies in the same number of chains $m(x)$, namely $\frac{m}{|P_h|}$.

Second proof. No two elements of A lie in the same maximal chain. Hence

$$M \geq \sum_{x \in A} m(x) = \sum_{x \in A} \frac{M}{|P_{\mathrm{rank}(x)}|} = Mw(A)$$

$\square$

The following is a corollary of the above, but we provide a proof using Katona's circle method.

Theorem 1.10 (Lubell-Yamamoto-Meshalkin Inequality). If $\mathcal{A} \subseteq 2^{[n]}$ is an antichain, then $\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1$

Proof. We say that $A \in 2^{[n]}$ is **contained** in a permutation π if $A = \{\pi_1, \dots, \pi_{|A|}\}$. Every permutation contains at most one element of $\mathcal{A}$ and every $A \in \mathcal{A}$ is contained in $|A|!(n - |A|)!$ permutations. $\square$

We say a chain $C_i \subseteq C_{i+1} \subseteq \dots \subseteq$ is **symmetric** if $|C_j| = j$ for all j.

Example. $\{1\} \subseteq \{1, 4\} \subseteq \{1, 3, 4\} \subseteq \{1, 3, 4, 6\} \subseteq \{1, 3, 4, 5, 6\}$ and $\{2, 4, 5\}$ are symmetric chains in $2^{[6]}$. $\{2, 5, 6\} \subseteq \{2, 4, 5, 6\}$ is a symmetric in $2^{[7]}$ but not in any other $2^{[n]}$.

Theorem 1.11 (Partition into Symmetric Chains). Every finite powerset can be partitioned into symmetric chains.

Proof. Induction on n:

- $\{\{\}\}$ is a PSC for $n = 0$.
- Assume we have a PSC for n. For every chain $\mathcal{C} = \{C_i, \ldots, C_{n-i}\}$ in our PSC for n, add the following two chains to our PSC for $n+1$:

$$\begin{aligned} \mathcal{C}' &= \{C_i, \ldots, C_{n-i}, C_{n-i} \cup \{n\}\} \\ \mathcal{C}'' &= \{C_i \cup \{n\}, \ldots, C_{n-i-1} \cup \{n\}\} \end{aligned}$$

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□

The number of symmetric chains of length $n + 1 - 2i$ in a PSC is

$$\binom{n}{i} - \binom{n}{i-1}$$

Theorem 1.12. Let $x_1, \ldots, x_n$ be vectors of norm at least 1 a normed space. For $A \subseteq [n]$, set $x_A = \sum_{i \in A} x_i$. Let $\mathcal{A} \subseteq 2^{[n]}$ such that

$$\forall A, B \in \mathcal{A}, \|x_A - x_B\| < 1$$

Then $\mathcal{A} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. Call $\mathcal{B} \subseteq 2^{[n]}$ **sparse** or **scattered** if $\forall A, B, \mathcal{B}, A \neq B, \|x_A - x_B\| \geq 1$. $\mathcal{A}$ intersects every sparse family in at most one set, so we would be done if there existed a partition of $2^{[n]}$ into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ sparse chains. This is the next theorem. □

Theorem 1.13 (Kleitman)**.** $2^{[n]}$ has a partition into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ sparse chains.

Proof. Induction on n:

- $\{\{\}\}$ is a sparse partition for $n = 0$.
- Assume we have a sparse partition for n. Let f be a support functional at x_n ($\forall x, f(x) \leq \|x\|$, with equality if $x = x_n$). For every sparse family $\mathcal{D} = \{D_1, \ldots, D_k\}$ in our sparse partition for n, find i maximising $f(x_{D_i})$ and add the following two sparse families to our sparse partition for $n+1$:

$$\begin{aligned} \mathcal{D}' &= \mathcal{D} \cup \{D_i \cup \{n\}\} \\ \mathcal{D}'' &= \{D_j \cup \{n\} | j \neq i\} \end{aligned}$$

  $\mathcal{D}''$ is clearly sparse. $\mathcal{D}'$ is also sparse because for all $D \in \mathcal{D}$

$$\begin{aligned} \left\|x_{D_i \cup \{n\}} - x_D\right\| &= \|x_{D_i} + x_n - x_D\| \\ &\geq f(x_{D_i} + x_n - x_D) \\ &= f(x_{D_i}) - f(x_D) + \|x_n\| \\ &\geq 1 \end{aligned}$$

  The number of sparse partitions of length $n + 1 - 2i$ is again $\binom{n}{i} - \binom{n}{i-1}$.

□

Lecture 4