Part III – Ramsey Theory on Graphs (Incomplete)

Based on lectures by Dr Julian Sahasrabudhe
Notes taken by Yaël Dillies

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0 Introduction

Lecture 1

Notation. We write

- $[n] = \{1, \ldots, n\}$
- K_n for the complete graph on n vertices.
- For X a set, $r \in \mathbb{N}$, $X^{(r)} = \{S \subseteq X | |S| = r\}$
- χ for a k-coloring of the edges of K_n

$$\begin{aligned} &\chi : E(K_n) \to [k] \\ &\chi : E(K_n) \to \{\text{red}, \text{blue}\} && (\text{if } k = 2) \end{aligned}$$

Ramsey theory is usually concerned with the following question:

Can we find some order in enough disorder?

In this course, we will specialise this question to graphs. We are thus interested in the following:

What can we say about the structure of an arbitrary 2-coloring of the edges of K_n?

Definition 0.1. Define the **Ramsey number** $R(\ell, k)$ to be the least n for which every 2-edge coloring contains either a blue K_ℓ or a red K_k, and the **diagonal Ramsey number** $R(k) = R(k, k)$ to be the least n for which every 2-edge coloring contains a monochromatic K_k.

It is unclear that such a n even exists! We shall prove it in due course.

$R(\ell, k) = R(k, \ell)$. By convention, we will usually assume $\ell \leq k$.

Example 0.2. $R(3) = 6$ because

- The following coloring shows that $R(3) > 5$. TODO: add picture
- If we have 6 vertices, we can pick a vertex v. By pigeonhole, three of the neighbors of v are connected to v via the same color, say red. Now either two of those neighbors are connected with a red edge, in which case they form a red triangle with v, or they are connected with blue edges to each other, in which case they form a blue triangle. As a way to remember this proof, we encourage you to watch the following music video: Everybody's looking for Ramsey

1 Old bounds on $R(\ell, k)$

Theorem 1.1 (Erdős-Szekeres, 1935)**.**

$$R(\ell, k) \leq \binom{k + \ell - 2}{\ell - 1}$$

In particular, $R(\ell, k)$ is well-defined.

Lemma 1.2. For all $k, \ell \geq 3$,

$$R(\ell, k) \leq \underbrace{R(\ell - 1, k)}_{a} + \underbrace{R(\ell, k - 1)}_{b}$$

Proof. Let $n = a + b$. Pick a vertex v. By pigeonhole, either

- v has at least a red neighbors. Either these neighbors contain a red $K_{\ell-1}$ (in which case we chuck v in), or contain a blue K_k (in which case we already won).
- v has at least b blue neighbors. Either these neighbors contain a blue K_{k-1} (in which case we chuck v in), or contain a red K_ℓ (in which case we already won).

□

Proof of Erdős-Szekeres. Use that $R(\ell, 2) = \ell$ and induct on k and ℓ. □

Corollary 1.3.

$$R(k) \leq \binom{2k}{k} \leq C\frac{4^k}{\sqrt{k}}$$

for some constant C.

1.1 Lower bounds

Can we find edge colorings on many vertices without a monochromatic K_k? Certainly, we can at least do so on $(k-1)^2$ vertices.

TODO: Insert figure

This polynomial lower bound is eons away from our exponential upper bound. For quite some time (in the 1930s), people thought that the lower bound was closer to the truth than the upper bound. Surprisingly, it is possible to show an exponential lower bound without actually exhibiting such a coloring!

Theorem 1.4 (Erdős, 1948)**.**

$$R(k) \geq \frac{k-1}{e\sqrt{2}} 2^{\frac{k}{2}}$$

Fact.

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

Proof. Let $n = \left\lceil \frac{k-1}{e\sqrt{2}} 2^{\frac{k}{2}} \right\rceil$ and χ be a random red/blue edge coloring of K_n (each edge is independently colored red or blue with probability $\frac{1}{2}$). We see that

$$\begin{aligned}
\mathbb{P}(\chi \text{ contains a monochromatic } K_k) &= \mathbb{P}\left(\bigcup_{S\in[n]^{(k)}} \{S \text{ monochromatic}\}\right) \\
&\le \binom{n}{k}\mathbb{P}([k] \text{ monochromatic}) \\
&= \binom{n}{k}2^{-\binom{k}{2}+1} \\
&\le 2\left(\frac{en}{k}\right)^k 2^{-\frac{k(k-1)}{2}} \\
&= 2\left(\frac{en}{k}2^{-\frac{k-1}{2}}\right)^k \\
&\le 2\left(1-\frac{1}{k}\right)^k \\
&< 1
\end{aligned}$$

Hence

$$2^{\frac{k}{2}} \le R(k) \le 4^k$$

□

This proof is remarkable by the fact that it proves that the probability of some object existing is high, without actually constructing such an object. In fact it is still an important open problem to explicitly construct a K_k-free edge-coloring of K_n with n exponential in k. In other words, *even though K_k-free edge-colorings are abundant, we don't know how to write down a single one.*

Remark. The use of "constructive" here is quite different to that in other areas of mathematics. We do not mean that the proof requires the Law of Excluded Middle or the Axiom of Choice, nor that we do not provide an algorithm to find a graph without monochromatic K_k.
Since there are only finitely many red/blue edge-colorings of K_n for a fixed n, there trivially is an algorithm to find such a coloring: enumerate them all and try them one by one. Less obviously, there is a procedure to systematically remove any use of the axiom of choice from the proofs of most of the results in this course. Excluded Middle is also redundant since the case splits we consider can be decided in finite time (again, everything is finite).
A more careful definition of "constructive" here is about complexity of the description of the object: Erdős' lower bound does not provide any better *deterministic* algorithm than "Try all edge-colorings", and this has complexity $\Omega\left(2^{\binom{n}{2}}\right)$ (without even accounting for the time it takes to check whether a coloring contains a monochromatic K_k). In contrast, we would expect a constructive lower bound to yield an edge-coloring in a polynomial number of operations in n.

Question 1. What's the base of the exponent here? Is there even such a base?

Lecture 2

We know

$$R(3,k) \le \binom{k+1}{2} \le (k+1)^2$$

Definition 1.5. An **independent set** in a graph is a set of vertices that does not contain an edge. The **independence number** $\alpha(G)$ is the maximum size of an independent set of G.

Definition 1.6 (Binomial Random Graph)**.** For $n \in \mathbb{N}, 0 \leq p \leq 1$, we define $G(n,p)$ the probability space of graphs where each edge is independently present with probability p.

Theorem 1.7 (Erdős)**.**

$$R(3,k) \geq c\left(\frac{k}{\log k}\right)^{\frac{3}{2}}$$

for some constant $c > 0$.

Idea. We will look at a binomial random graph and choose the parameters so that there are very few red K_k and the number of blue K_3 is at most some fixed proportion of n. Then we will remove one vertex from each red K_k and one vertex from each blue K_3. The resulting graph will have neither and most likely will still contain a fixed proportion of the vertices we started with.

Proof. Change the language. Discuss the blue graph. We are now looking for the maximum number of edges of a graph with no triangles and no independent set of size k.
Take $n = \left(\frac{k}{\log k}\right)^{\frac{3}{2}}, p = n^{-\frac{2}{3}} = \frac{\log k}{k}$. Now sample $G \sim G(n,p)$ and define $\tilde{G}$ to be G with one vertex removed from each triangle and independent set of size k. By construction, $K_3 \nsubseteq \tilde{G}$ and $\alpha(\tilde{G}) < k$. We will show $\mathbb{E}|\tilde{G}| \geq \frac{n}{2}$ using

$$|\tilde{G}| \geq n - \#\text{triangles in } G - \#\text{independent sets of size } k \text{ in } G$$

First,

$$\mathbb{E}\#\text{triangles in } G = \sum_{T \in [n]^{(3)}} \mathbb{P}(T \text{ triangle in } G) = \binom{n}{3} p^3 \leq \frac{(np)^3}{6} = \frac{n}{6}$$

Second,

$$\begin{aligned}
\mathbb{E}\#\text{independent sets of size } k \text{ in } G &= \binom{n}{k}(1-p)^{\binom{k}{2}} \\
&\leq \left(\frac{en}{k}\right)^k e^{-p\binom{k}{2}} \\
&\sim \left(\frac{en}{k} e^{-\frac{pk}{2}}\right)^k \\
&= \left(\frac{ek^{\frac{3}{2}}}{k \log^{\frac{3}{2}} k} e^{-\frac{\log k}{2}}\right)^k \\
&= \left(\frac{e}{\log^{\frac{3}{2}} k}\right)^k \longrightarrow 0
\end{aligned}$$

Hence, for large enough k,

$$\mathbb{E}|\tilde{G}| \geq n - \frac{n}{6} - 1 \geq \frac{n}{2} = \frac{1}{2}\left(\frac{k}{\log k}\right)^{\frac{3}{2}}$$

By adjusting $c > 0$, we have proved the theorem. □

Remark. The values of n and p come from the constraints

$$n^3p^3 \ll n, \quad \frac{\log n}{k} \ll p$$

We are being wasteful here. Why throw an entire vertex away when we could get away with removing a single edge? Because we might accidentally create an independent set of size k. But we can be smarter...

Idea. Take a maximal collection of edge-disjoint triangles in $G \sim G(n,p)$ and remove all edges from these triangles.

Theorem 1.8 (Erdős)**.**

$$R(3,k) \geq c\left(\frac{k}{\log k}\right)^2$$

for some constant $c > 0$.

Lemma 1.9. Let $\mathcal{F} = \{A_1, \dots, A_m\}$ be a family of events in a probability space. Let $\mathcal{E}_t$ be the event that t *independent* events from $\mathcal{F}$ occur. then

$$\mathbb{P}(\mathcal{E}_t) \leq \frac{1}{t!}\left(\sum_{i=1}^{m} \mathbb{P}(A_i)\right)^t$$

Proof. Note that

$$1_{\mathcal{E}_t} \leq \frac{1}{t!} \sum_{\substack{i \in [m]^t \\ A_{i_1}, \dots, A_{i_t} \text{ independent}}} 1_{A_{i_1}} \dots 1_{A_{i_t}}$$

So

$$\begin{aligned}\mathbb{P}(\mathcal{E}_t) &\leq \frac{1}{t!} \sum_{\substack{i \in [m]^t \\ A_{i_1}, \dots, A_{i_t} \text{ independent}}} \mathbb{P}(A_{i_1}) \dots \mathbb{P}(A_{i_t}) \\ &\leq \frac{1}{t!} \sum_{i \in [m]^t} \mathbb{P}(A_{i_1}) \dots \mathbb{P}(A_{i_t}) \\ &= \frac{1}{t!}\left(\sum_{i=1}^{m} \mathbb{P}(A_i)\right)^t\end{aligned}$$

□

Lecture 3

Idea. Instead of killing vertices, we will kill edges. We only need to kill edges from a maximal set of edge-disjoint triangles. For this killing to create an independent set I of size k, we must have killed all edges in I. But with high probability all sets of size k contain a fixed fraction of the expectation $p\binom{k}{2}$ of the number of edges, and having so many edge-disjoint triangles each with two vertices among k fixed vertices is unlikely.

Lemma 1.10. Let $n, k \in \mathbb{N}, p \in [0,1]$ be such that $pk \geq 16 \log n$. Then with high probability every subset of size k of $G \sim G(n,p)$ contains at least $\frac{pk^2}{8}$ edges.

Proof of Erdős' bound. We fix $n = \left(\frac{c_1 k}{\log k}\right)^2, p = c_2 n^{-\frac{1}{2}} = \frac{c_2}{c_1}\frac{\log k}{k}$. Let $G \sim G(n,p), \mathcal{T}$ a maximal collection of edge-disjoint triangles in G, $\tilde{G}$ be G with all edges of $\mathcal{T}$ removed. Note, $\tilde{G}$ contains no triangle. We show

$$\mathbb{P}(\alpha(\tilde{G}) \geq k) < 1$$

Let Q be the event that every set of k vertices of G contains $\geq \frac{pk^2}{8}$ edges. Setting $\frac{c_2}{c_1} = 48$, we get

$$pk = \frac{c_2}{c_1}\frac{\log k}{k} k = 48 \log k > 16 \log n$$

so that $\mathbb{P}(Q) = 1 - o(1)$ by the lemma. Now note that

$$\mathbb{P}(\alpha(\tilde{G}) \geq k) \leq \mathbb{P}(\alpha(\tilde{G}) \geq k, Q) + \cancel{\mathbb{P}(Q^c)} \to 0$$

So we focus on $\mathbb{P}(\alpha(\tilde{G}) \geq k, Q)$. Observe that if $\tilde{G}$ contains an independent set I of size k and Q holds, then I contains $\geq \frac{pk^2}{8}$ edges of G by assumption. But I is an independent set in $\tilde{G}$, so all $\frac{pk^2}{8}$ edges must have belonged to some triangle $T \in \mathcal{T}$ and been removed. Therefore

$$\begin{aligned}\mathbb{P}(\alpha(\tilde{G}) \geq k, Q) &\leq \mathbb{P}\left(\exists S \in [n]^{(k)}, \mathcal{T} \text{ meets } S \text{ in } \geq \frac{pk^2}{8} \text{ edges}\right) \\ &\leq \binom{n}{k}\mathbb{P}\left(\underbrace{\substack{\text{at least } t \text{ triangles of } \mathcal{T} \\ \text{meet } [k] \text{ in at least two vertices}}}_{B}\right)\end{aligned}$$

where $t = \frac{pk^2}{24}$. Let $\{T_i\}$ be the collection of triangles in K_n that meet $[k]$ in at least two vertices. Let $A_i = \{T_i \subseteq G\}$. Note that if $T_{i_1}, \ldots, T_{i_k}$ are edge-disjoint, then $A_{i_1}, \ldots, A_{i_k}$ are independent. So

$$\begin{aligned}\mathbb{P}(B) &\leq \mathbb{P}(\mathcal{E}_t) \\ &\leq \frac{1}{t!}\left(\sum_{\substack{T_i \subseteq K_n \text{ intersects } [k] \\ \text{in at least two vertices}}} \mathbb{P}(T_i \subseteq G)\right)^t \\ &\leq \frac{1}{t!}(k^2 n p^3)^t \\ &\leq \left(\frac{ek^2np^3}{t}\right)^t \\ &= (24enp^2)^t = (24ec_2^2)^t = e^{-t}\end{aligned}$$

by choosing $c_2 = \frac{1}{\sqrt{24e}}$. To finish, observe that

$$t = \frac{pk^2}{24} = 2k \log k \geq k \log n$$

Hence

$$\binom{n}{k}\mathbb{P}(B) \leq \binom{n}{k} e^{-t} \leq \left(\frac{en}{k}e^{-\log n}\right)^k = \left(\frac{e}{k}\right)^k \to 0$$

□

1.2 Large deviation inequalities

Let Z be a gaussian random variable.

$$\mathbb{P}(Z - \mathbb{E}Z \geq t) \leq e^{\frac{-t}{2 \operatorname{Var} Z}}$$

Let $X_1, \dots, X_n$ be iid Bernoulli random variables. We denote this $X_i \sim \operatorname{Ber}(p)$. Write $S_n = X_1 + \cdots + X_n$. Note $\mathbb{E}S_n = np, \operatorname{Var}(S_n) = np(1-p)$.

Idea. Often, the tail of S_n looks like a gaussian tail.

Theorem 1.11 (Chernoff inequality)**.** Let $X_1, \dots, X_n \sim \operatorname{Ber}(p)$. Then

$$\mathbb{P}(|S_n - pn| \geq t) \leq 2 \exp \left(\underbrace{-\frac{t^2}{2pn}}_{\text{meat}} + \underbrace{\frac{t^3}{(pn)^2}}_{\text{error term}} \right)$$

Lecture 4

Proof of the $\frac{pk^2}{8}$ lemma. Using Chernoff on $e(G[[k]])$, namely with $p := p, n := \binom{k}{2}, t := \frac{pk^2}{4}$, we get

$$\begin{aligned}
\mathbb{P}(G \text{ fails the statement}) &= \mathbb{P}\left(\exists S \in [n]^{(k)}, e(G[s]) < \frac{pk^2}{8} \right) \\
&\leq \binom{n}{k} \mathbb{P}\left(e(G[[k]]) < \frac{pk^2}{8} \right) \\
&\leq \binom{n}{k} \mathbb{P}\left(\frac{pk^2}{4} < \left| e(G[[k]]) - p\binom{k}{2} \right| \right) \\
&\leq 2 \left(\frac{en}{k} \right)^k \exp\left(-\frac{pk^2}{16} + \frac{1}{8} \right) \\
&\ll \left(\frac{en}{k} \right)^k \exp(-k \log n) \\
&= \left(\frac{e}{k} \right)^k
\end{aligned}$$

which tends to 0 as k tends to infinity. □

1.3 The Local Lemma

The probabilistic method is like finding the hay in the hay stack. What if we want to find the needle?

Definition 1.12. Let $\mathcal{F} = \{A_1, \dots, A_n\}$ be a family of events in a probability space. A **dependency graph** Γ is a graph with vertices $\mathcal{F}$ such that the event A_i is independent of $\sigma(A_j \mid j \not\sim i)$ for all $i \in [n]$.

Remarks.

- A dependency graph is not unique.
- The complete graph is always a dependency graph.
- The empty graph is a dependency graph iff the A_i are globally independent.

Theorem 1.13 (The Local Lemma, symmetric version)**.** Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a family of events in a probability space, let Γ be a dependency graph for $\mathcal{F}$ with maximum degree Δ. If $\mathbb{P}(A_i) \leq \frac{1}{e(\Delta+1)}$ for all i, then

$$\mathbb{P}\left(\bigcap_i A_i^c\right) > 0$$

Theorem 1.14 (Spencer)**.**

$$R(k) \geq (1 - o(1))\frac{\sqrt{2}k}{e}2^{\frac{k}{2}}$$

Proof. Let $n = (1-\varepsilon)\frac{\sqrt{2}k}{e}2^{\frac{k}{2}}$ for some $\varepsilon > 0$. Let χ be a random edge-coloring of K_n uniformly over all colorings. Define, for $S \in [n]^{(k)}$, the event

$$A_S = \{S \text{ is monochromatic in } \mathbb{Z}\}$$

Note we want $\mathbb{P}\left(\bigcap_{S\in[n]^{(k)}} A_S^c\right) > 0$. Define the dependency graph Γ by

$$S \sim T \iff 1 < |S \cap T| < k$$

The maximum degree of Γ Is

$$\Delta = \sum_{t=2}^{k-1}\binom{k}{t}\binom{n-k}{k-t} = \binom{n}{k} - k\binom{n-k}{k-1} - \binom{n-k}{k} - 1$$

To apply the Local Lemma, we just check

$$\mathbb{P}(A_S) = 2^{-\binom{k}{2}+1} \leq \frac{1}{e(\Delta+1)}$$

$\square$

Theorem 1.15 (Lopsided Local Lemma)**.** Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a family of events on a probability space, Γ a dependency graph for $\mathcal{F}$, $0 \leq x_1, \ldots, x_n < 1$ satisfying

$$\mathbb{P}(A_i) \leq x_i \prod_{j\sim i}(1 - x_j)$$

Then

$$\mathbb{P}\left(\bigcap_i A_i^c\right) \geq \prod_i (1 - x_i) > 0$$

Theorem 1.16 (Erdős, 1961)**.**

$$R(3, k) \geq c\left(\frac{k}{\log k}\right)^2$$

for some $c > 0$.

Proof. Let $n = \varepsilon^4 \left(\frac{k}{\log k}\right)^2, p = \frac{\varepsilon}{\sqrt{n}} = \frac{\log k}{\varepsilon k}, G \sim G(n,p)$. For all $T \in [n]^{(3)}$ and $I \in [n]^{(k)}$, define $A_T = \{T \subseteq G\}$ and $B_I = \{I \subseteq G^c\}$. We want

$$\mathbb{P}\left(\bigcap_{T\in[n]^{(3)}} A_T^c \cap \bigcap_{I\in[n]^{(k)}} B_I^c\right) > 0$$

Define the dependency graph Γ by

$$\begin{aligned} T \sim T' &\iff |T \cap T'| = 2 \\ T \sim I &\iff 2 \le |I \cap T| \\ I \sim I' &\iff 2 \le |I \cap I'| < k \end{aligned}$$

We see that

$$\#\{T' \in [n]^{(3)} \mid T' \sim T\} \le 3n, \#\{I \in [n]^{(k)} \mid I \sim T\} \le 3n^{k-2}$$
$$\#\{T \in [n]^{(3)} \mid T \sim I\} \le k^2 n, \#\{I' \in [n]^{(k)} \mid I' \sim I\} \le k^2 n^{k-2}$$

We therefore check that

1. $\mathbb{P}(A_T) \le x_T \prod_{T'\sim T}(1 - x_{T'}) \prod_{I\sim T}(1 - x_I)$. Indeed,

$$\begin{aligned} \text{LHS} &= p^3 \\ \text{RHS} &\ge 3p^3(1 - 3p^3)^{3n}(1 - n^{-k})^{3n^{k-2}} \end{aligned}$$

and, using $1 - x \ge e^{-2x}$ for small enough x,

$$(1 - 3p^3)^{3n}(1 - n^{-k})^{3n^{k-2}} \ge \exp(-18p^3 n - 6n^{-2}) = \exp(-18\varepsilon^2 p - 6n^{-2}) \to 1$$

Lecture 5

2. $\mathbb{P}(B_I) \le x_I \prod_{T\sim I}(1 - x_T) \prod_{I'\sim I}(1 - x_{I'})$. Indeed,

$$\begin{aligned} \text{LHS} &= (1-p)^{\binom{k}{2}} \\ \text{RHS} &= n^{-k}(1 - n^{-k})^{k^2 n^{k-2}}(1 - 3p^3)^{k^2 n} \end{aligned}$$

and, using $1 - x \ge e^{-2x}$ for small enough x,

$$\begin{aligned} \log \text{RHS} &\ge -k \log n - 2k^2 n^{-2} - 6k^2 p^3 n \\ &\ge -2k \log k - 6k^2 p \varepsilon^2 + o(1) \\ &\ge -\frac{k \log k}{4\varepsilon} + o(1) \\ &\ge -p\binom{k}{2} \\ &\ge \log \text{LHS} \end{aligned}$$

$\square$

Proof of the Local Lemma. Applying $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B \mid A)$ repeatedly, write

$$\begin{aligned} \mathbb{P}\left(\bigcap_{i=1}^n A_i^c\right) &= \prod_{i=1}^n \mathbb{P}(A_i^c \mid A_1^c \cap \ldots \cap A_{i-1}^c) \\ &= \prod_{i=1}^n (1 - \mathbb{P}(A_i \mid A_1^c \cap \ldots \cap A_{i-1}^c)) \end{aligned}$$

It is enough to show that $\mathbb{P}(A_i \mid A_1^c \cap \ldots \cap A_{i-1}^c) \leq x_i$. We prove

$$\mathbb{P}(A_i \mid \bigcap_{j \in S} A_j^c) \leq x_i$$

for all $S \subseteq [n]$ by induction:

- $S = \varnothing$. Done by assumption.
- Write
$$I = \bigcap_{\substack{j \in S \\ j \not\sim i}} A_j^c, D = \bigcap_{\substack{j \in S \\ j \sim i}} A_j^c$$

  So

$$\mathbb{P}(A_i \mid I \cap D) = \frac{\mathbb{P}(A_i \cap I \cap D)}{\mathbb{P}(I \cap D)} \leq \frac{\mathbb{P}(A_i \cap I)}{\mathbb{P}(I \cap D)} = \frac{\mathbb{P}(A_i)\mathbb{P}(I)}{\mathbb{P}(I \cap D)} = \frac{\mathbb{P}(A_i)}{\mathbb{P}(D \mid I)}$$

  Now, write $D = A_{i_1}^c \cap \ldots \cap A_{i_m}^c$ and

$$\mathbb{P}(D \mid I) = \prod_{j=1}^{m} (1 - \mathbb{P}(A_{i_j} \mid I \cap A_{i_1}^c \cap \ldots \cap A_{i_{j-1}}^c)) \geq \prod_{j=1}^{m} (1 - x_{i_j}) \geq \prod_{j \sim i} (1 - x_j)$$

□

Theorem 1.17 (Lovasz Local Lemma, symmetric version)**.** For $\mathcal{F} = \{A_1, \ldots, A_n\}$, Γ a dependency graph with maximum degree Δ, if $\mathbb{P}(A_i) \leq \frac{1}{e(\Delta+1)}$ for all i, then

$$\mathbb{P}\left(\bigcap_i A_i^c\right) \geq \left(1 - \frac{1}{e(\Delta+1)}\right)^n > 0$$

Proof. We use the Lopsided Local Lemma with $x_i = \frac{1}{e(\Delta+1)}$. Note

$$x_i \prod_{j \sim i} (1 - x_j) \geq \frac{1}{\Delta + 1}\left(1 - \frac{1}{\Delta+1}\right)^{\Delta} \geq \frac{1}{e(\Delta+1)} \geq \mathbb{P}(A_i)$$

So Lopsided Local Lemma applies. □

We now know

$$\frac{ck^2}{(\log k)^2} \leq R(3,k) \leq (k+1)^2$$

Theorem 1.18 (State of the art on $R(3,k)$)**.**

$$\underbrace{\left(\frac{1}{4} + o(1)\right)\frac{k^2}{\log k}}_{\substack{\text{Fiz Pontiveros} \\ \text{Griffiths} \\ \text{Morris}} + \substack{\text{Bohman} \\ \text{Keevash}}} \leq R(3,k) \leq \underbrace{(1 + o(1))\frac{k^2}{\log k}}_{\substack{\text{Ajtai} \\ \text{Komlós} \\ \text{Szemerédi}} + \text{Shearer}}$$

1.4 Upper bounds on $R(3,k)$

Lecture 6

Theorem 1.19 (Ajtai, Komlós, Szemerédi)**.**

$$R(3,k) \le c\frac{k^2}{\log k}$$

for some $c > 0$. In fact, we will see $c = 1 + o(1)$.

Theorem 1.20 (Ajtai, Komlós, Szemerédi)**.** Let G be a triangle-free graph on n vertices with maximum degree Δ. Then

$$\alpha(G) \ge c\frac{n}{\Delta}\log\Delta$$

for some absolute constant $c > 0$.

Remark. For general graphs, we know $\alpha(G) \ge \frac{n}{\chi(G)} \ge \frac{n}{\Delta+1}$ by the naïve greedy algorithm and this is basically best possible. The extra $\log d$ factor will come from tracking how sparse our graph is becoming as we remove vertices from it.

We apply a random greedy algorithm to prove the following theorem due to Shearer.

Define

$$f(x) = \frac{x\log x - x + 1}{(x-1)^2}$$

extending continuously to $[0,1]$ by $f(0) = 1, f(1) = \frac{1}{2}$. We remark that

- f is continuous and differentiable.
- f is antitone and convex.
- $0 < f(x) < 1$
- $(x+1)f(x) = 1 + (x - x^2)f(x)$

Theorem 1.21 (Shearer)**.** Let G be a triangle-free graph on n vertices with average degree $d > 0$. Then

$$\alpha(G) \ge nf(d) \ge (1 - o(1))\frac{n}{\Delta}\log\Delta$$

Proof. Induction on n.

Let x a vertex to be chosen later and $G' := G - x - N(x)$. Writing d' the average degree of G' and applying induction to G', we get

$$\begin{aligned}
\alpha(G) &\ge 1 + \alpha(G') \\
&\ge 1 + (n - \deg x - 1)f(d') \\
&\ge 1 + (n - \deg x - 1)(f(d) + (d' - d)f'(d)) \\
&= nf(d) + 1 - (\deg x + 1)f(d) + (d\deg x + d + n(d' - d) - d'\deg x - d')f'(d) \\
&= nf(d) + 1 - (\deg x + 1)f(d) + \left(d\deg x + d - 2\sum_{y\sim x}\deg y\right)f'(d)
\end{aligned}$$

where we used that $N(x)$ is an independent set to get that

$$\begin{aligned}
n(d-d') &= n\left(\frac{2e(G)}{n} - \frac{2e(G')}{n-\deg x - 1}\right) \\
&= 2(e(G)-e(G')) - (\deg x + 1)d' \\
&= 2\sum_{y\sim x} \deg y - (\deg x + 1)d'
\end{aligned}$$

So we want to choose x such that

$$(\deg x + 1) f(d) \leq 1 + (d \deg x + d - 2\sum_{y\sim x} \deg y) f'(d)$$

We average over x:

$$\mathop{\mathbb{E}}_x \text{LHS} = (d+1) f(d)$$

$$\mathop{\mathbb{E}}_x \text{RHS} = 1 + (d^2 + d - 2\mathop{\mathbb{E}}_x \sum_{y\sim x} \deg y) f'(d)$$

Notice that

$$\mathop{\mathbb{E}}_x \sum_{y\sim x} \deg y = \frac{1}{n}\sum_x \sum_{y\sim x} \deg y = \frac{1}{n}\sum_x \deg x^2 \geq \left(\frac{1}{n}\sum_x \deg x\right)^2$$

Since $f'(d) \leq 0$, we get

$$\mathop{\mathbb{E}}_x \text{RHS} \geq 1 + (d^2 + d - 2d^2) f'(d) = 1 + (d - d^2) f'(d) = (d+1) f(d) = \mathop{\mathbb{E}}_x \text{LHS}$$

So such x exists. □

Proof of the AKS bound using Shearer. Let G be a graph on n vertices with neither a triangle nor an independent set of size k. We have

$$d \leq \Delta \leq \alpha(G) < k$$

where the middle inequality holds by triangle freeness. Hence Shearer says

$$k > \alpha(G) \geq \frac{n}{d}\log d \geq \frac{n}{k}\log k$$

And $n \leq \frac{k^2}{\log k}$ as wanted. □

Lecture 7

Second proof of AKS. Let I be an independent set in G sampled uniformly among all independent sets of G. We will show

$$\mathbb{E}\,|I| \geq c\frac{n}{d}\log n$$

Let v be a vertex. We define the random variable

$$X_v = d1_{v\in I} + |N(v)\cap I|$$

For any independent set I,
$$\sum_v X_v \le 2d\,|I|$$
So
$$\sum_v \mathop{\mathbb{E}}_I X_v \le 2d \mathop{\mathbb{E}}_I |I|$$
So we want to show that $\mathbb{E}_I X_v \ge c \log d$ for all v.
Let $G' = G - v - N(v)$, find J an independent set in G' minimising $\mathbb{E}_I[X_v \mid I \setminus (N(v) \cup \{v\}) = J]$ and let $F = \{w \in N(v) \mid N(w) \cap J = \varnothing\}$ and $t = |F|$. Note carefully that $I \cap (F \cup \{v\})$ is uniform over all independent sets in $G[F \cup v]$, and that the independent sets of $G[F \cup v]$ are exactly $\{v\}$ and the 2^t subsets of F. Hence
$$\mathop{\mathbb{E}}_I X_v \ge \mathop{\mathbb{E}}_{I \setminus (N(v) \cup \{v\}) = J} X_v = \mathop{\mathbb{E}}_{I \subseteq F \cup \{v\}} X_v = \frac{1}{2^t+1} d + \frac{2^t}{2^t+1} \frac{t}{2} \ge c \log d$$
for some $c > 0$ by optimising over t. □

Theorem 1.22 (Ajtai-Komlós-Szemerédi)**.** Let $\ell \in \mathbb{N}$. Then for sufficiently large $k \in \mathbb{N}$ we have
$$R(\ell, k) \le \left(\frac{4}{\log k}\right)^{\ell-2} k^{\ell-1}$$
We have seen that the power of k is correct for $\ell = 3$. As of recently, we also know that it is correct for $\ell = 4$.

2 3-uniform hypergraph Ramsey numbers

We define $K_n^{(r)}$ to be the **complete r-uniform hypergraph** on n vertices. The **r-uniform hypergraph Ramsey number** $R^{(r)}(\ell, k)$ is the minimal n such that every edge coloring of $K_n^{(r)}$ contains either a blue $K_\ell^{(r)}$ or a red $K_k^{(r)}$. As before, the **hypergraph diagonal Ramsey number** is $R^{(r)}(k) = R^{(r)}(k, k)$.

Theorem 2.1 (Erdős-Rado)**.**

$$R^{(3)}(\ell, k) \leq 2^{\binom{R(\ell-1,k-1)}{2}}$$

Eg $R^{(3)}(k) \leq 2^{16^k}$.

Proof. Let $t = R(\ell - 1, k - 1)$ and $n = 2^{\binom{t}{2}}$. Let χ be a red/blue edge coloring of $K_n^{(3)}$. Let $v_1, v_2 \in [n]$. Define

$$A_{1,2} = \{w \in [n] \mid \chi(v_1, v_2, w) = c_{1,2}\}$$

where $c_{1,2}$ is the **majority color**, chosen so that

$$|A_{1,2}| \geq \frac{n}{2} - 1$$

Let $v_3 \in A_{1,2}$. Define

$$A_{1,3} = \{w \in A_{1,2} \mid \chi(v_1, v_3, w) = c_{1,3}\}$$

where $c_{1,3}$ is the majority color. Now define

$$A_{2,3} = \{w \in A_{1,3} \mid \chi(v_2, v_3, w) = c_{2,3}\}$$

where $c_{2,3}$ is the majority color, and so on...
After t steps, our world has size $|A_{t-1,t}| \geq n2^{-\binom{t}{2}} \geq 1$. We thus have $\{v_1, \ldots, v_t\}$ such that $\chi(\{v_i, v_j, v_k\}) = c_{i,j}$ if $k > i, j$. c is a coloring of $\{v_1, \ldots, v_t\}^{(2)}$. By definition of Ramsey, we're done. □

2.1 Off-diagonal

Lecture 8

Erdős-Rado gives

$$R^{(3)}(4,k) \le 2^{ck^4}$$

Theorem 2.2 (Conlon, Fox, Sudakov, 2010)**.**

$$R^{(3)}(4,k) \le k^{ck^2}$$

Erdős-Rado makes us shrink our world by a factor of 2 at every query. Can we ask fewer questions? This suggests the following game.

Definition 2.3 (The Ramsey game)**.** At each question, we expose a new vertex v_i and get to ask our adversary to expose the color of a collection of edges $v_j v_i$ where $j < i$. Our goal is to force a blue K_3 or a red K_k with as few queries as possible.

Lemma 2.4. In the Ramsey game, we can force a blue K_3 or a red K_k in at most $2k^3$ queries whose answer is "red" and k^2 queries whose answer is "blue".

Proof. As we expose vertices, we sort them into **levels** $1, 2, 3, \dots$ The first vertex at level i is called the **root** of level i and denoted r_i.

We start by putting v_1 into level 1 and setting $r_1 := v_1$. When we expose vertex v_i, we ask for the color of $v_i r_1, \dots, v_i r_r$ until we get replied "blue".

- If we get a blue response to $v_i r_j$ for some j, stick v_i in level j and expose all edges to previous vertices of level j.
- If all $v_i r_j$ get replied "red", make v_i a new root.

TODO: Insert picture

Assuming we have not encountered a blue K_3 or red K_k, every level contains at most k vertices and there are at most k levels. We have exposed at most k^2 red edges and k blue edges in each level, and k^3 red edges between levels, so in total at most $2k^3$ red edges and k^2 blue edges. □

The idea now is that our adversary wants to reply "red" most of the time, so we're willing to shrink our world much more when they reply "blue".

Proof that $R^{(3)}(4,k) \le k^{ck^2}$. The proof follows the proof of Erdős-Rado but we now only refine our world based on the pairs that we query in the Ramsey game. We also have a different rule about when to refine our world to be blue vs red.

Start with $A_0 = [n]$ where $n = k^{ck^2}$ and define

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$$

where e_j is the edge coming from the Ramsey game and

$$A_j^B = \{x \in A_j \mid \chi(e_{j+1} \cup \{x\}) = \text{blue}\}$$
$$A_j^R = \{x \in A_j \mid \chi(e_{j+1} \cup \{x\}) = \text{red}\}$$
$$A_{j+1} = \begin{cases} A_j^B & \text{if } \left|A_j^B\right| \ge \frac{1}{k}\left|A_j\right| \\ A_j^R & \text{if } \left|A_j^R\right| \ge \left(1 - \frac{1}{k}\right)\left|A_j\right| \end{cases}$$

At the end of time,

$$|A_m| \geq n\left(\frac{1}{k}\right)^{k^2}\left(1-\frac{1}{k}\right)^{2k^3} \geq k^{(c-1)k^2}e^{-4k^2} \geq 1$$

if we pick c large enough. $\square$

What about lower bounds? Let's try the probabilistic method.

Color triples blue with probability p.

$$\mathbb{E}\left[\#\text{red } K_k^{(3)}\right] = \binom{n}{k}(1-p)^{\binom{k}{3}} \leq \left(\frac{en}{k}\right)^k e^{-p\binom{k}{3}} = \left(\frac{en}{k}e^{-\frac{pk^2}{6}}\right)^k$$

This is nontrivial if $p \gg \frac{1}{k^2}$. Then

$$\mathbb{E}\left[\#\text{blue } K_4^{(3)}\right] = \binom{n}{4}p^4 \geq \left(\frac{n}{4}\right)^4 \gg \left(\frac{n}{k^2}\right)^4$$

So the (naïve) probabilistic approach looks useless for anything better than polynomial in k.

Lecture 9

Theorem 2.5.

$$R^{(3)}(4,k) \geq 2^{\frac{k-1}{2}}$$

Proof. Let $n = 2^{\frac{k-1}{2}} - 1$ and T a random tournament on $[n]$. For $x, y \in [n]$ distinct, define

$$\chi(\{x,y,z\}) = \begin{cases} \text{blue} & \text{if } x,y,z \text{ oriented} \\ \text{red} & \text{if } x,y,z \text{ acyclic} \end{cases}$$

TODO: Insert oriented and acyclic pictures

Observation. A tournament on 4 points has at least one transitive triple.

This immediately implies there is no blue $K_4^{(3)}$.

Observation. If K is a tournament where every triple is transitive, then K itself is transitive.

Hence

$$\mathbb{E}\#\text{red } K_k^{(3)} = \mathbb{E}\#\text{transitive } K_k = \binom{n}{k}k!2^{-\binom{k}{2}} \leq n^k 2^{-\binom{k}{2}} = \left(n2^{-\frac{k-1}{2}}\right)^k < 1$$

Hence there exists a tournament with no red $K_k^{(3)}$. $\square$

Theorem 2.6 (Conlon, Fox, Sudakov, 2010)**.**

$$R^{(3)}(4,k) \geq k^{\frac{k}{5}}$$

for sufficiently large k.

Proof (Stepping up). Set $n = k^{\frac{k}{5}}, r = R(3, \frac{k}{4}) - 1$. Let θ be an edge coloring of K_r with no blue K_3 or red $K_{\frac{k}{4}}$. Now let σ be a random edge coloring in r colors. For $x < y < z$, define

$$\chi(\{x, y, z\}) = \begin{cases} \theta(\{\sigma(xy), \sigma(xz)\}) & \text{if } \sigma(xy) \neq \sigma(xz) \\ \text{red} & \text{if } \sigma(xy) = \sigma(xz) \end{cases}$$

Idea. The fact that θ has no blue triangle will imply that χ has no blue $K_4^{(3)}$. The fact that θ has no red $K_{\frac{k}{4}}$ will imply that χ has no red $K_k^{(3)}$ with high probability.

Assume that $x < y_1 < y_2 < y_3$ form a blue $K_4^{(3)}$. Note that $\sigma(xy_1)$, $\sigma(xy_2)$, $\sigma(xy_3)$ are distinct. So $\sigma(xy_1)\sigma(xy_2), \sigma(xy_2)\sigma(xy_3), \sigma(xy_3)\sigma(xy_1)$ are the edges of a triangle in K_r which is blue in θ. Contradiction.

TODO: Insert figure

Now assume that $K = \{x_1, \dots, x_k\}$ is a red $K_k^{(3)}$ in χ. For each $i \in [k]$,

$$|\{\sigma(x_i x_j) \mid j > i\}| < \frac{k}{4}$$

as otherwise one can find $i < j_1 < \cdots < j_{\frac{k}{4}}$ such that $\sigma(x_i x_{j_1}), \dots, \sigma(x_i x_{j_{\frac{k}{4}}})$ are all distinct, meaning that they are vertices of a red $K_{\frac{k}{4}}$ in θ.

TODO: Insert figure

Call such a set $K \in [n]^{[k]}$ **sad**. We consider

$$\begin{aligned}
\mathbb{E}\# \text{ sad sets} &\le n^k \prod_{i=1}^{k} \binom{r}{\frac{k}{4}} \left(\frac{k}{4r}\right)^{k-i} \\
&= n^k \binom{r}{\frac{k}{4}}^k \left(\frac{k}{4r}\right)^{\sum_{i=1}^k k-i} \\
&\le n^k \left(\frac{4er}{k}\right)^{\frac{k^2}{4}} \left(\frac{k}{4r}\right)^{\frac{k^2}{4}} \\
&= \left(n \left(\frac{e}{4}\frac{k}{r}\right)^{\frac{k}{4}}\right)^k \\
&\le \left(nk^{-\frac{k}{4}+o(k)}\right)^k \text{ since } r > k^{2-o(1)} \\
&< \left(nk^{-\frac{k}{5}}\right)^k \\
&= 1 \text{ since } n = k^{\frac{k}{5}}
\end{aligned}$$

Hence find some σ such that no set is sad. We're done. □

2.2 Diagonal

Lecture 10

The state of the art for $R^{(3)}(k)$ is

$$2^{ck^2} \le R^{(3)}(k) \le 2^{2^{2k+1}}$$

for some $c > 0$.

Conjecture 1 (Erdős, Hajnal, Rado, 1965)**.**

$$R^{(3)}(k) \geq 2^{2^{ck}}$$

for some $c > 0$.

Theorem 2.7 (Erdős, Hajnal, 1980)**.**

$$\underbrace{R_4^{(3)}(k)}_{4 \text{ colors}} \geq 2^{2^{ck}}$$

for some $c > 0$.

This is an example of "stepping up".

Lemma 2.8. For all k,

$$R_4^{(3)}(k) \geq 2^{R(k-1)-1}$$

Proof. Let $r = R(k-1)-1$ and let θ be a red-blue edge coloring with no monochromatic K_{k-1}. We now color triples on the ground set $\{0,1\}^r$. For $x, y \in \{0,1\}^r, x \neq y$, define

$$f(x,y) = \max\{i \mid x_i \neq y_i\}$$

so that

$$x < y \iff x_{f(x,y)} < y_{f(x,y)}$$

This is the reverse lexicographic order. Note that $f(x,y) \neq f(y,z)$ if $x < y < z$. We now define our coloring ξ of $\{x,y,z\}, x < y < z$ to be one of four colors depending on which of the following hold:

$$f(x,y) < f(y,z), \quad \theta(f(x,y), f(y,z)) = \text{red}$$

Let $x_1 < \cdots < x_k$ be the vertices of a monochromatic $K_k^{(3)}$, WLOG of the color where both conditions hold. We claim that $f(x_1,x_2) < f(x_2,x_3) < \ldots f(x_{k-1},x_k)$ are the vertices of a monochromatic, in fact red, K_{k-1} in θ. Indeed, since the x_i are increasing,

$$f(x_{i+1}, x_{j+1}) = \max_{i < \ell \leq j} f(x_\ell, x_{\ell+1}) = f(x_j, x_{j+1})$$

So

$$\theta(f(x_i, x_{i+1}), f(x_j, x_{j+1})) = \theta(f(x_i, x_{i+1}), f(x_{i+1}, x_{j+1})) = \text{red}$$

$\square$

Remark. We know a version of this stepping up (due to Erdős-Hajnal) for 2-colorings when the uniformity is ≥ 4.

Theorem 2.9 (Conlon, Fox, Sudakov, 2010)**.**

$$R^4(2k+1) \geq 2^{R^{(3)}(k-1)-1}$$

and for $s \geq 4$ we have

$$R^{(s+1)}(k+1) \geq 2^{R^{(s)}(k)-1}$$

The state of the art for $R^{(3)}(k)$ is

$$\underbrace{2^{\cdot^{\cdot^{\cdot^{2^{c_0 k}}}}}}_{s-2} \leq R^{(s)}(k) \leq \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{c_1 k}}}}}}_{s-1}$$

3 The Szemerédi Regularity Lemma

Informal Statement
For all $\varepsilon > 0$, there exist numbers $\ell(\varepsilon), L(\varepsilon)$ such that every graph can be partitioned into $V_1, \dots, V_k$, where $\ell(\varepsilon) \le k \le L(\varepsilon)$, $||V_i| - |V_j|| \le 1$ such that, for all but $\varepsilon\binom{k}{2}$ pairs (i, j), the graph between V_i and V_j is "random-like" up to some coarseness ε.

Definition 3.1. In a graph G, let X, Y be disjoint sets of vertices. We say that (X, Y) is **ε-uniform** (aka **ε-regular**) if for all $X' \subseteq X, Y' \subseteq Y$ such that $|X'| \ge \varepsilon |X|, |Y'| \ge \varepsilon |Y|$ we have

$$|d(X', Y') - d(X, Y)| < \varepsilon$$

where

$$d(X', Y') = \frac{e(X, Y)}{|X| |Y|}$$

is the **density** of edges.

Remark. We need some lower bound on $|X'|, |Y'|$ in the definition, else uniformity will basically never hold.

Lecture 11

Proposition 3.2. For $\varepsilon > 0$, let (X, Y) be an ε-regular pair in a graph G. Let $p = d(X, Y)$. Then

$$\#\{x \in X \mid |N(x) \cap Y| < (p - \varepsilon) |Y|\} < \varepsilon |X|$$
$$\#\{x \in X \mid |N(x) \cap Y| > (p + \varepsilon) |Y|\} < \varepsilon |X|$$

Proof. Let $X' = \{x \in X \mid |N(x) \cap Y| < (p - \varepsilon) |Y|\}$. By definition of X',

$$d(X', Y) = \frac{e(X', Y)}{|X| |Y|} < \frac{(p - \varepsilon) |Y| |X'|}{|X'| |Y|} = p - \varepsilon$$

So $|X'| < \varepsilon |X|$ by definition of ε-uniformity. □

Lemma 3.3 (Embedding lemma for triangles)**.** Let $\varepsilon > 0$ and $p \ge 2\varepsilon$. Let G be a graph on $V = V_1 \cup V_2 \cup V_3$ where V_1, V_2, V_3 are disjoint of size $m \ge 1$, (V_i, V_j) are ε-uniform for $i \ne j$ and $d(V_i, V_j) \ge p \ge 2\varepsilon$. Then there are at least

$$(1 - 2\varepsilon)(p - \varepsilon)^3 m^3$$

triangles in G.

Proof. We look at triangles with one vertex in each V_i. The number of $x \in V_1$ such that

$$|N(x) \cap V_2| \ge (p - \varepsilon)m, \quad |N(x) \cap V_3| \ge (p - \varepsilon)m$$

is at least $(1 - 2\varepsilon)m$. For each such x, the number of V_1, V_2, V_3 triangles containing x is at least

$$e(N(x) \cap V_2, N(x) \cap V_3) \ge (p - \varepsilon) |N(x) \cap V_2| |N(x) \cap V_3| \ge (p - \varepsilon)^3 m^2$$

since $|N(x) \cap V_2|, |N(x) \cap V_3| \ge (p - \varepsilon)m \ge \varepsilon m$. So we get $(1 - 2\varepsilon)(p - \varepsilon)^3 m^3$ triangles in total.

TODO: Insert figure □

Definition 3.4. We say a partition $V = V_1 \cup \ldots \cup V_k$ is an **equipartition** if $||V_i| - |V_j|| \leq 1$ for all i, j. Such a partition is **ε-uniform** if (V_i, V_j) is ε-uniform for all but $\varepsilon\binom{k}{2}$ pairs $\{i, j\}$.

Theorem 3.5 (Szemerédi Regularity Lemma)**.** For all $\varepsilon > 0, \ell \in \mathbb{N}$, there exists $L \in \mathbb{N}$ such that every graph has an ε-uniform equipartition in $k \in [\ell, L]$ parts.

Remarks.

- It is really important that L does not depend on $|G|$.
- This does not directly say anything about graphs with $e(G) = o(n^2)$
- The proof of the regularity lemma gives

$$L \leq \underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{\varepsilon^{-5}}$$

- Gowers showed that

$$L \geq \underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{\varepsilon^{-1/16}}$$

  is needed.

Lemma 3.6 (Triangle removal lemma)**.** For all $\varepsilon > 0$, there exists $\delta > 0$ such that every graph with at most δn^3 triangles contains εn^2 edges that together kill all triangles.

Proof. Let $\varepsilon' = \varepsilon/4, \ell = 10\varepsilon^{-1}, L = L(\varepsilon, \ell), \delta = 2^{-16}\varepsilon^4 L^{-3}$. By Szemerédi Regularity Lemma, find a ε'-uniform equipartition

$$V = V_1 \cup \ldots \cup V_k$$

where $\ell \leq k \leq L$. We say (i, j) is **bad** if either of the following holds:

1. $d(V_i, V_j) \leq \varepsilon/2$
2. (V_i, V_j) is not ε'-uniform
3. $i = j$

Now define

$$T = \{xy \in E(G) \mid x \in V_i, y \in V_j, (i, j) \text{ bad}\}$$

Then

$$\begin{aligned}
|T| \leq &\#\text{edges between pairs with } d(V_i, V_j) \leq \varepsilon/2 \\
&+ \#\text{edges between } \varepsilon'\text{-non-uniform pairs} \\
&+ \sum_i \#\text{edges inside } V_i \\
\leq &\frac{\varepsilon}{2}\binom{k}{2}\left(\frac{n}{k}\right)^2 + \frac{\varepsilon}{4}\left(\frac{n}{k}\right)^2\binom{k}{2} + k\left(\frac{n}{k}\right)^2 \\
\leq &\frac{\varepsilon}{4}n^2 + \frac{\varepsilon}{8}n^2 + \frac{n^2}{k} \\
\leq &\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{10}\right)\varepsilon n^2 \\
\leq &\varepsilon n^2
\end{aligned}$$

Now, $G - T$ is triangle-free. Indeed if x, y, z is a triangle then there must be a, b, c distinct such that $(V_a, V_b), (V_b, V_c), (V_c, V_a)$ are $\varepsilon/4$-uniform and

$$d(V_a, V_b), d(V_b, V_c), d(V_c, V_a) \geq \frac{\varepsilon}{2}$$

So the triangle embedding lemma tells us that there are at least

$$\left(1 - 2\frac{\varepsilon}{4}\right)\left(\frac{\varepsilon}{4}\right)^2\left(\frac{n}{k}\right)^3 \geq \left(1 - \frac{\varepsilon}{2}\right)2^{-6}L^{-3}\varepsilon^3 n^3 > \delta n^3$$

triangles. Contradiction. □

Lecture 12

Theorem 3.7 (Roth's theorem)**.** For $\varepsilon > 0$, there exists N such that for all $n \geq N$ any $A \subseteq [n]$ of density at least ε contains a non-trivial arithmetic progression of length three.

Proof. Assume $\varepsilon > 0$. Let δ be the δ from the triangle removal lemma applied to 3ε, and set $N = 3\delta^{-1}$. Define a tripartite graph on $[3n], [3n], [3n]$ by declaring that $(x, x + a, x + 2a)$ are the edges of a triangle for each $x \in [3n]$ and $a \in A$. We call such a triangle **explicit**. There are $3n\,|A| \geq 3\varepsilon n^2$ explicit triangles in our graph and they are edge-disjoint. Therefore triangle removal tells us that there are at least $\delta n^3 = 3N^{-1}n^3 > 3n^2 \geq 3n\,|A|$ triangles in our graph. Hence there must be some triangle that's not explicit. But a non-explicit triangle $(x, x + a, x + a + b) = (x, x + a, x + 2c)$ exactly corresponds to a non-trivial arithmetic progression. □

Theorem 3.8 (Turán)**.** Let G be a K_{r+1}-free graph. Then

$$e(G) \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2}$$

and this bound is sharp for the Turán graph.

But the Turán graph is a bit pathological in the sense that it has huge independence number $\alpha = \frac{n}{r}$. What if we require $\alpha(G) = o(n)$?

For $r = 2$, the neighborhood of any vertex is an independent set, so $\deg v = o(n)$ and $e(G) = o(n^2)$. For $r = 3$, we get an interesting problem.

Theorem 3.9 (Szemerédi)**.** For $\varepsilon > 0$, there exists $\delta > 0$ such that any K_4-free graph G on n vertices with $\alpha(G) < \delta n$ has

$$e(G) \leq \frac{n^2}{8} + \varepsilon n^2$$

In other words, if G is K_4-free and $\alpha(G) = o(n)$, then $e(G) \leq \frac{n^2}{8} + o(n^2)$.

Remarks. The $\frac{1}{8}$ is sharp!

Definition 3.10. For $0 < \alpha < 1, \varepsilon > 0$ and a partition

$$V(G) = V_1 \cup \ldots \cup V_k$$

the **reduced graph** $R_{\alpha,\varepsilon}$ is the graph whose vertices are $[k]$ and

$$i \sim j \iff (V_i, V_j)\varepsilon-\text{uniform and } d(V_i, V_j) \geq \alpha$$

Proof of Theorem 3.9. Let $\varepsilon > 0$. We may assume $\varepsilon < 1/4$. Let $L = L(10\varepsilon^{-1}, \varepsilon/4), \delta = \varepsilon^2 L^{-1}/10$. Assume G is K_4-free and $\alpha(G) < \delta n$. We want

$$e(G) \leq \frac{n^2}{8} + \varepsilon n^2$$

Regularity with $\varepsilon/4$ and $\ell = 10/\varepsilon$ gives a partition $V = V_1 \cup \ldots \cup V_k$ where $\ell \leq k \leq L$. Let $R = R_{\varepsilon,\varepsilon/4}(V_1, \ldots, V_k)$ be the corresponding reduced graph and $m = n/k$.

Claim. R is triangle-free.

Proof. Assume V_a, V_b, V_c form a triangle in R. Since the pairs are $\varepsilon/4$-uniform and have density at least ε, the triangle embedding lemma tells us that there are at least

$$(1 - 2\varepsilon)\varepsilon^3 m^3$$

triangles on V_a, V_b, V_c, where $m = \frac{n}{k}$. Find therefore $x \in V_a, y \in V_b$ such that $x \sim y$ and

$$\#\{z \in V_c \mid x, y, z \text{ is a triangle}\} \geq (1 - 2\varepsilon)\varepsilon^3 m \geq \frac{1}{2}\varepsilon^3 \frac{n}{L} > \delta n$$

Hence Z can't be an independent set and there are some $u, v \in Z$ such that $u \sim v$. But then x, y, u, v is a K_4. Contradiction. □

Lecture 13

Claim. All $\varepsilon/4$-uniform pairs (V_i, V_j) have $d(V_i, V_j) \leq 1/2 + \varepsilon$.

Proof. Assume (V_i, V_j) contradicts the claim. Then let

$$U = \{v \in V_i \mid |N(v) \cap V_j| \geq \frac{1}{2} + \frac{\varepsilon}{2}\}$$

so that $|U| \geq (1 - \varepsilon/4)m > \delta n$ by uniformity. By assumption, there therefore exist $x, y \in U$ such that $x \sim y$. Now note that

$$|N(x) \cap N(y)| \geq \varepsilon |V_j| = \varepsilon m \geq \delta n$$

So there exist $u, v \in N(x) \cap N(y)$ such that $u \sim v$. But now x, y, u, v is a K_4. Contradiction. □

The first claim gives us a bound on the number of edges in the reduced graph, the second claim gives us a bound on the density of those edges, and all other edges have negligible density. So we are morally done. Formally, by splitting the edges of G according to which kind of pair of the partition it belongs to,

$$\begin{aligned}
e(G) &\leq \underbrace{e(R_{\varepsilon,\varepsilon/4})\left(\frac{1}{2} + \varepsilon\right) m}_{\text{in the reduced graph}} + \underbrace{\frac{\varepsilon}{4}\binom{k}{2} m^2}_{\text{not } \varepsilon/4\text{-uniform}} + \underbrace{\varepsilon \binom{k}{2} m^2}_{\text{low density}} + \underbrace{km^2}_{\text{within a part}} \\
&\leq \frac{1}{2} e(R_{\varepsilon,\varepsilon/4}) m^2 + C\varepsilon n^2 \\
&\leq \frac{n^2}{8} + C'\varepsilon n^2 \text{ by Turán}
\end{aligned}$$

□

Theorem 3.11 (Chvátal-Rödl-Szemerédi-Trotter, 1983).

$$r(H) \leq C_d \, |H|$$

for some constant C_d where H has max degree d.

Remark. The naïve Ramsey bound gives

$$r(H) \leq R(|H|) \leq 4^{|H|}$$

Lemma 3.12 (Embedding lemma). Let H be a r-partite graph with max degree d and r-partition

$$V(H) = W_1 \cup \ldots \cup W_r$$

where $|W_i| \leq s$. Now let $m \in \mathbb{N}, \varepsilon, \lambda \in]0,1[$ be such that

$$\varepsilon(d+1) < (\lambda - \varepsilon)^d, \quad s \leq \varepsilon m$$

where $|V_i| = m$, (V_i, V_j) is ε-uniform and $d(V_i, V_j) \geq \lambda$. Then $H \subseteq G$.

Proof. We define an algorithm to embed H into G. In each step of the algorithm, we will choose a new vertex in $V(H)$ that has not been embedded yet and find some vertex in $V(G)$ to map it to.

At step t, for each vertex $u \in V(H)$, define

$$E_t = \text{ already embedded vertices}, \quad \mathcal{C}_t(u) = V_\ell \cap \bigcap_{\substack{w \in E_t \\ w \sim_H u}} N_G(w) \text{ where } u \in V_\ell$$

At each step of the algorithm, ensure the following:

1. If $x, y \in E_t$ and $x \sim_H y$, then $x \sim_G y$.
2. For all $u \notin E_t$,
$$\mathcal{C}_t(u) \geq m(\lambda - \varepsilon)^{|N(u) \cap E_t|}$$

At step $t+1$, let $v \notin E_t$. We want to find a vertex in $\mathcal{C}_t(v)$ to map v to. This ensure condition 1. Now, for $u \notin E_t \cup \{v\}$,

$$\mathcal{C}_{t+1}(u) = \begin{cases} \mathcal{C}_t(u) \cap N_G(v) & \text{if } u \sim_H v \\ \mathcal{C}_t(u) & \text{if } u \not\sim_H v \end{cases}$$

So we just need to exclude vertices from $\mathcal{C}_t(u)$ that have

1. $|\mathcal{C}_t(u) \cap N_G(v)| < (\lambda - \varepsilon) \, |\mathcal{C}_t(u)|$
2. are already identified with vertices in E_t.

By uniformity, there are at most $d\varepsilon m$ vertices of the first type. Since $|W_\ell| \leq s \leq \varepsilon m$, there are at most εm vertices of the second type. Thus there are at least

$$|\mathcal{C}_t(u)| - \varepsilon(d+1)m \geq (\lambda - \varepsilon)^d m - \varepsilon(d+1)m > 0$$

good choices for v. Pick one. Done. □

Lecture 14

Lemma 3.13. Let H be a n-vertex graph with max degree d. Then there exists a partition

$$V(H) = V_1 \cup \ldots \cup V_{10d^2}$$

so that all edges are between parts and

$$|V_i| \leq \frac{100n}{d}$$

Proof. H is $d+1$-colorable, say with parts $W_1, \ldots, W_{d+1}$. Split each W_i into at most $\lceil d/100 \rceil$ parts of size at most $100n/d$. This gives at most

$$(d+1)\left\lceil \frac{d}{100} \right\rceil \leq \frac{d^2}{25} \leq 10d^2$$

parts for big enough d. □

Proof of the theorem. Let $r = 10d^2, t = R(10d^2)$. Let $\varepsilon < \frac{1}{t+1}$ be such that

$$(d+1)\varepsilon < \left(\frac{1}{2} - \varepsilon\right)^d$$

Let $\ell > t+1$ and $L(\ell, \varepsilon)$ be the Regularity Lemma constant. Writing $|H| = n$. We will show that

$$r(H) \leq \frac{Ln}{\varepsilon}$$

Let $N > \max(100/d, 1)Ln/\varepsilon$ and χ be a red/blue edge coloring of K_N. Let G be the red graph. Apply the Regularity Lemma with parameters ε, ℓ to get a partition $V_1, \ldots, V_k$ with $\ell \leq k \leq L$.

Step 1
There are at least $(1-\varepsilon)\binom{k}{2} > (1 - \frac{1}{t+1})\binom{k}{2}$ ε-uniform pairs, so we can find a K_t in the graph of ε-uniform pairs. WLOG $V_1, \ldots, V_t$ is that K_t.

Step 2
Color each pair (V_i, V_j) with the majority color between V_i and V_j in G. Apply Ramsey to find a monochromatic K_{10d^2}. WLOG $V_1, \ldots, V_{10d^2}$ are all majority red. Partition H into parts $W_1, \ldots, W_{10d^2}$ of size at most $100n/d$. Since

$$|W_i| \leq \frac{100n}{d} \leq \frac{\varepsilon N}{L} \leq \frac{\varepsilon N}{k}$$

we can apply the Embedding Lemma with $\lambda = 1/2, m = N/L, s = \varepsilon N/L$ to finish. □

4 Dependent Random Choice

Definition 4.1. Let G be a graph. We say $R \subseteq V(G)$ is (s,k)**-rich** if, for all $x_1, \ldots, x_k \in R$,

$$k \leq |N(x_1) \cap \ldots \cap N(x_k)|$$

TODO: Insert picture

Theorem 4.2. Let G be a graph on n vertices with m edges. Let $t, s, r, k > 0$ satisfy

$$\frac{(2m)^t}{n^{2t-1}} - \binom{n}{s}\left(\frac{k}{n}\right)^t \geq r$$

Then G contains a (s,k)-rich set of size at least r.

Remark. t is a free parameter in the statement, so we get to optimise over t in applications.

Idea. Choose $v_1, \dots, v_t \in V(G)$ uniformly at random and consider $R = N(v_1) \cap \dots \cap N(v_t)$. neighborhoods of high degree often are in R, so R is likely to be rich.

Proof. Let $v_1, \dots, v_t \in V(G)$ be chosen uniformly at random. First consider

$$\begin{aligned}
\mathop{\mathbb{E}}_{v_1,\dots,v_t} |N(v_1) \cap \dots \cap N(v_t)| &= \mathop{\mathbb{E}}_{v_1,\dots,v_t} \sum_y 1_{y \sim v_1,\dots,v_t} \\
&= \sum_y \mathbb{P}(y \sim v_1, \dots, v_t) \\
&= \sum_y \mathbb{P}(y \sim v_1) \dots P(y \sim v_t) \\
&= \sum_y \left(\frac{d(y)}{n} \right)^t \\
&\geq n \left(\frac{2m}{n^2} \right)^t \\
&= \frac{(2m)^t}{n^{2t-1}}
\end{aligned}$$

Let

$$Y = \{(y_1, \dots, y_s) \in (N(v_1) \cap \dots \cap N(v_t))^s \mid |N(y_1) \cap \dots \cap N(y_s)| < k\}$$

We compute

$$\begin{aligned}
\mathop{\mathbb{E}}_{v_1,\dots,v_t} |Y| &= \mathop{\mathbb{E}}_{v_1,\dots,v_t} \sum_{|N(y_1)\cap\dots\cap N(y_s)|<k} 1_{y_1,\dots,y_s \in N(v_1)\cap\dots\cap N(v_t)} \\
&= \sum_{|N(y_1)\cap\dots\cap N(y_s)|<k} \mathbb{P}(y_1, \dots, y_s \in N(v_1) \cap \dots \cap N(v_t)) \\
&\leq \binom{n}{s} \left(\frac{k}{n} \right)^t
\end{aligned}$$

Define R to be $N(v_1) \cap \dots \cap N(v_t)$ with a vertex removed from each tuple $(y_1, \dots, y_s)$ such that $|N(y_1) \cap \dots N(y_s)| < k$. Then

$$\mathbb{E}\, |R| \geq \mathbb{E}[|N(v_1) \cap \dots \cap N(v_t)| - Y] \geq \frac{(2m)^t}{n^{2t-1}} - \binom{n}{s} \left(\frac{k}{n} \right)^t \geq r$$

So $|R| \geq r$ for some $y_1, \dots, y_s$, and we have found a (s, k)-rich set of size at least r. $\square$

In Part II Graph Theory, we saw

$$\mathrm{ex}(n, K_{t,t}) \leq C_t n^{2-1/t}$$

Theorem 4.3 (Füredi, 1991)**.** Let H be a bipartite graph with bipartition $A \cup B$ where $\deg y \leq s$ for all $y \in B$. Then

$$\mathrm{ex}(n, H) \leq C_H n^{2-1/s}$$

for some constant C_H.

Lemma 4.4 (Rich Set Embedding Lemma)**.** Let H be a bipartite graph on parts A and B such that $\deg y \le s$ for all $y \in B$. Let G be a graph and $R \subseteq V(G)$ be a $(s, |B|)$-rich set with $|R| \ge |A|$. Then $H \subseteq G$.

Proof. Let $A = \{x_1, \ldots, x_a\}, B = \{y_1, \ldots, y_b\}$. Find $\tilde{x}_1, \ldots, \tilde{x}_a \in R$ distinct. Start by embedding $x_i \mapsto \tilde{x}_i$. We choose $\tilde{y}_i$ such that $y_i \mapsto \tilde{y}_i$ inductively as follows. Assume we have already determined $\tilde{y}_1, \ldots, \tilde{y}_i$. Consider $N(y_{i+1}) = \{x_{i_1}, \ldots, x_{i_\ell}\}$ and note that

$$|N(\tilde{x}_{i_1} \cap \ldots \cap N(\tilde{x}_{i_\ell}))| \ge |H|$$

by richness. Thus simply choose any

$$\tilde{y}_{i+1} \in N(\tilde{x}_{i_1}) \cap \ldots \cap N(\tilde{x}_{i_\ell}) \setminus \{\tilde{y}_1, \ldots, \tilde{y}_i\}$$

□

Proof of Füredi. We want to apply our Rich Set Embedding Lemma to find a $(s, |H|)$-rich set $R \subseteq V(G)$ such that $|R| \ge |H|$. We know $m \ge C_H n^{2-1/s}$ for some constant C_H to be chosen and we want to find t such that

$$\frac{(2m)^t}{n^{2t-1}} - \binom{n}{s}\left(\frac{|H|}{N}\right)^t \ge |H|$$

Take $C_H = 2|H|, t = s$ so that

$$\text{LHS} \ge \frac{(2C_H)^t n^{2t-t/s}}{n^{2t-1}} - \left(\frac{en}{s}\right)^s \left(\frac{|H|}{n}\right)^t = \left(4^s - \left(\frac{e}{s}\right)^s\right)|H|^s \ge |H|$$

□

Let Q_d be the graph on $\{0,1\}^d$ such that $x \sim y$ iff they differ in one coordinate.

Theorem 4.5. For all large enough d,

$$r(Q_d) \le 2^{3d}$$

Proof. We want to apply DRC + Rich Set Embedding Lemma. Let $N = 2^{3d}$ and χ be a red-blue coloring of K_N. Let G be the graph of the majority color, so that

$$e(G) \ge \frac{1}{2}\binom{N}{2} =: m$$

By the Rich Set Embedding Lemma, we are done if we find a $(d, 2^d)$-rich set $R \subseteq V(G)$ with $|R| \ge 2^d$. By DRC, there exists such a rich set if we can find t such that

$$\frac{(2m)^t}{N^{2t-1}} - \binom{N}{d}\left(\frac{2^d}{N}\right)^t \ge 2^d$$

Pick $t = \frac{3}{2}d$. Then

$$\begin{aligned}
\text{LHS} &\ge \frac{1}{N^{2t-1}}\left(\frac{N(N-1)}{2}\right)^t - \left(\frac{eN}{D}\right)^d \left(\frac{2^d}{N}\right)^t \\
&= \left(1 - \frac{1}{N}\right)^t \frac{N}{2^t} - \left(\frac{e}{d}\right)^d N^{d - \frac{2}{3}t} \\
&= (1 - o(1))2^{\frac{3}{2}d} - \left(\frac{e}{d}\right)^d \\
&\gg 2^d
\end{aligned}$$

$\square$

Remark. The same argument shows

$$r(Q_d) \leq 2^{(\varphi+1+o(1))d}$$

where $\varphi + 1 = \frac{3+\sqrt{5}}{2} = 2.618\dots$.

How does $r(Q_d)$ grow?

Conjecture 2 (Erdős)**.**

$$r(Q_d) \leq C2^d$$

Theorem 4.6 (Conlon-Fox-Sudakov)**.**

$$r(Q_d) \leq 2^{2d+o(d)}$$

Theorem 4.7 (Tikharimov)**.**

$$r(Q_d) \leq 2^{(2-c)d}$$

for some $c > 0$ and all large enough d.

4.1 Ramsey-Turán

Definition 4.8. For $n, k \in \mathbb{N}$ and a graph H, we define

$$\mathrm{RT}(n, H, t) = \max\{e(G) \mid |G| = n, G \not\supseteq H, \alpha(G) \leq k\}$$

We showed that

$$\mathrm{RT}(n, K_4, o(n)) \leq \frac{n^2}{8} + o(n^2)$$

Theorem 4.9 (Sudakov)**.** Let $\delta(n) = \exp(-\omega(n)\sqrt{\log n})$ where $\omega(n) \to \infty$. Then

$$\mathrm{RT}(n, K_4, \delta(n)n) = o(n^2)$$

Proof. Let G be a n-vertex graph with $G \not\supseteq K_4$ and $\alpha(G) \leq \delta(n)n$. Let's first find a $(2, \delta(n)n)$-rich set $R \subseteq V(G)$ such that $|R| > \delta(n)n$. By DRC, we are looking for t such that

$$\frac{(2\varepsilon n^2)^t}{n^{2t-1}} - \binom{n}{2}\left(\frac{\delta(n)n}{n}\right)^t \geq \delta(n)n$$

Choose $t = 2\sqrt{\log n}/\omega(n)$ so that $\delta(n)^t = n^{-2}$. Since $\omega(n)^2 \gg 2\log\varepsilon^{-1}$, we have

$$\begin{aligned}
\mathrm{LHS} &\geq \varepsilon^t n - n^2\delta(n)^t \\
&= \exp\left(-\frac{3\log\varepsilon^{-1}}{\omega(n)}\sqrt{\log n}\right) n - 1 \\
&\geq \exp\left(-\omega(n)\sqrt{\log n}\right) n \quad \text{for large enough } n \\
&= \delta(n)n
\end{aligned}$$

$\square$

5 Exponential improvement on Ramsey numbers

Lecture 16