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**ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ**

***ΤΟΜΕΑΣ ΜΗΧΑΝΙΚΗΣ, ΕΡΓΑΣΤΗΡΙΟ ΑΝΤΟΧΗΣ ΚΑΙ ΥΛΙΚΩΝ***

**Ηρώων Πολυτεχνείου 5, Κτίριο Θεοχάρη**
**Πολυτεχνειούπολη Ζωγράφου, 157 73 Ζωγράφου**

**Δρ Σταύρος Κ. Κουρκουλής, Καθηγητής Πειραματικής Μηχανικής**

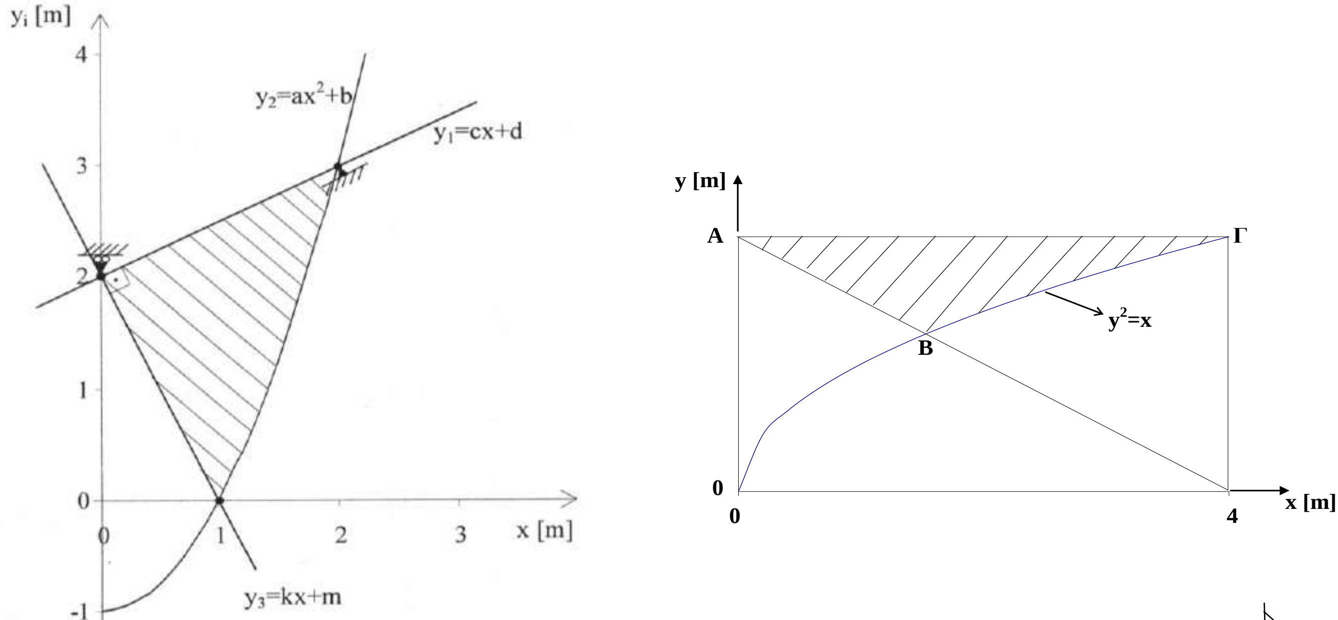
Τηλέφωνα: +210 772 1313, +210 772 1263 (γραφείο)
+210 772 4025, +210 772 4235, +210 772 1317, +210 7721310 (εργαστήρια)
Τηλεομοιότυπο (Fax): +210 7721302
Διεύθυνση ηλεκτρονικού ταχυδρομείου (e-mail): stakkour@central.ntua.gr

# ΜΗΧΑΝΙΚΗ Ι (ΣΤΑΤΙΚΗ)

## *10η σειρά ασκήσεων: Προσδιορισμός γεωμετρικού κέντρου επιπέδων επιφανειών*

### Άσκηση 1

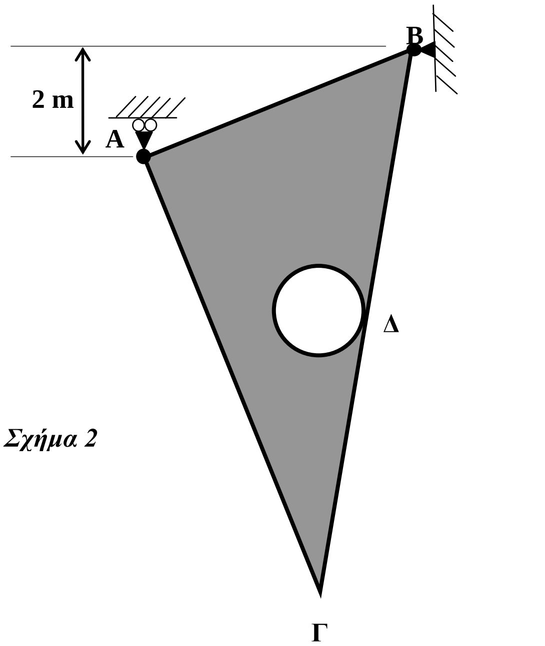
Να προσδιορισθούν τα γεωμετρικό κέντρα των γραμμοσκιασμένων επιφανειών του Σχ.1.



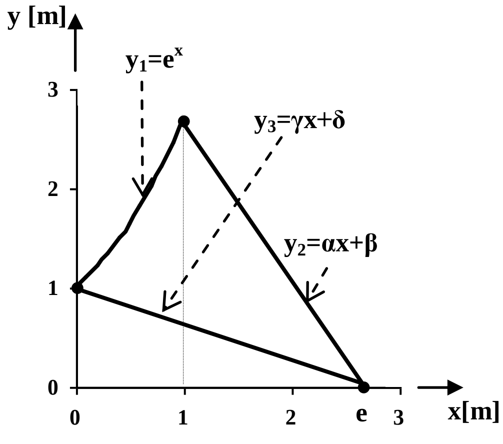

*Σχήμα 1*

### Άσκηση 2

Να προσδιορισθεί το γεωμετρικό κέντρο της σκιασμένης επιφάνειας του Σχ.2. Δίνεται ότι ΑΓ=2ΑΒ= 10m και ότι $\mathrm{B\hat{A}\Gamma} = 90^{\circ}$. Η κυκλική οπή ακτίνας 0.7 m, εφάπτεται στο μέσον Δ της ΒΓ.




*Σχήμα 2*

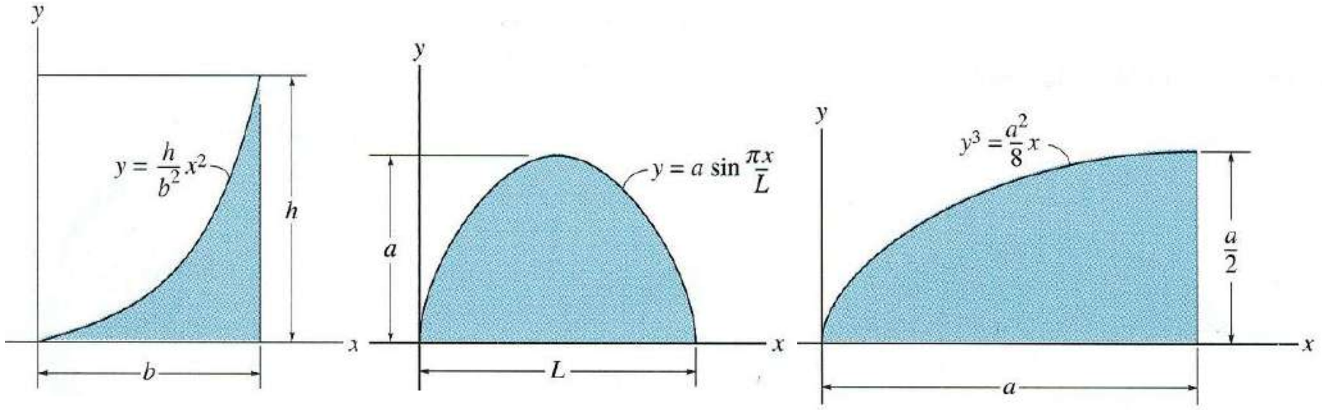



*Σχήμα 3*

### Άσκηση 3

Να ευρεθεί το γεωμετρικό κέντρο της επιφάνειας που περικλείεται μεταξύ των γραμμών $y_1$, $y_2$ και $y_3$ του παραπλεύρως Σχ.3.

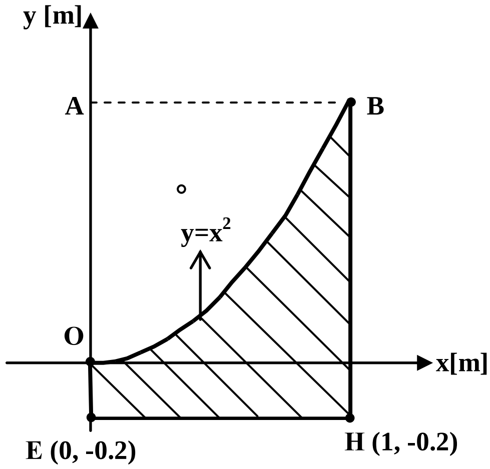
### Άσκηση 4
Να προσδιορισθούν τα γεωμετρικά κέντρα των κάτωθι επιφανειών (Σχ. 4):

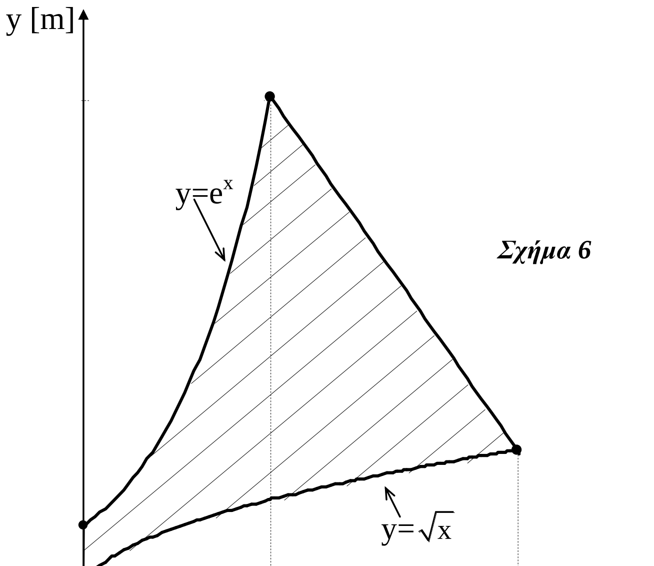



*Σχήμα 4*

### Άσκηση 5
Να προσδιορισθούν το γεωμετρικό κέντρο της γραμμοσκιασμένης επιφάνειας του Σχ.5.

*Σχήμα 5*

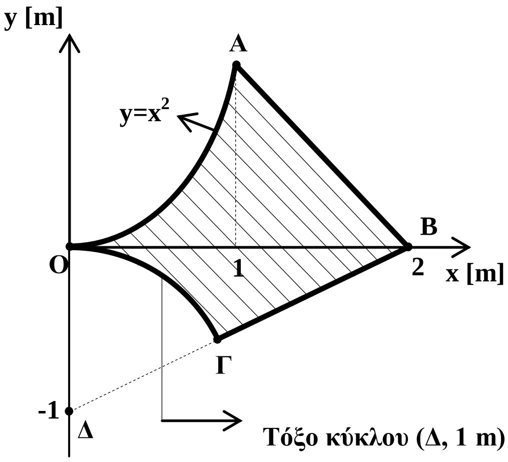



*Σχήμα 6*

### Άσκηση 6
Να προσδιορισθεί το γεωμετρικό κέντρο της επίπεδης επιφάνειας ΑΒΓ του Σχ.6.

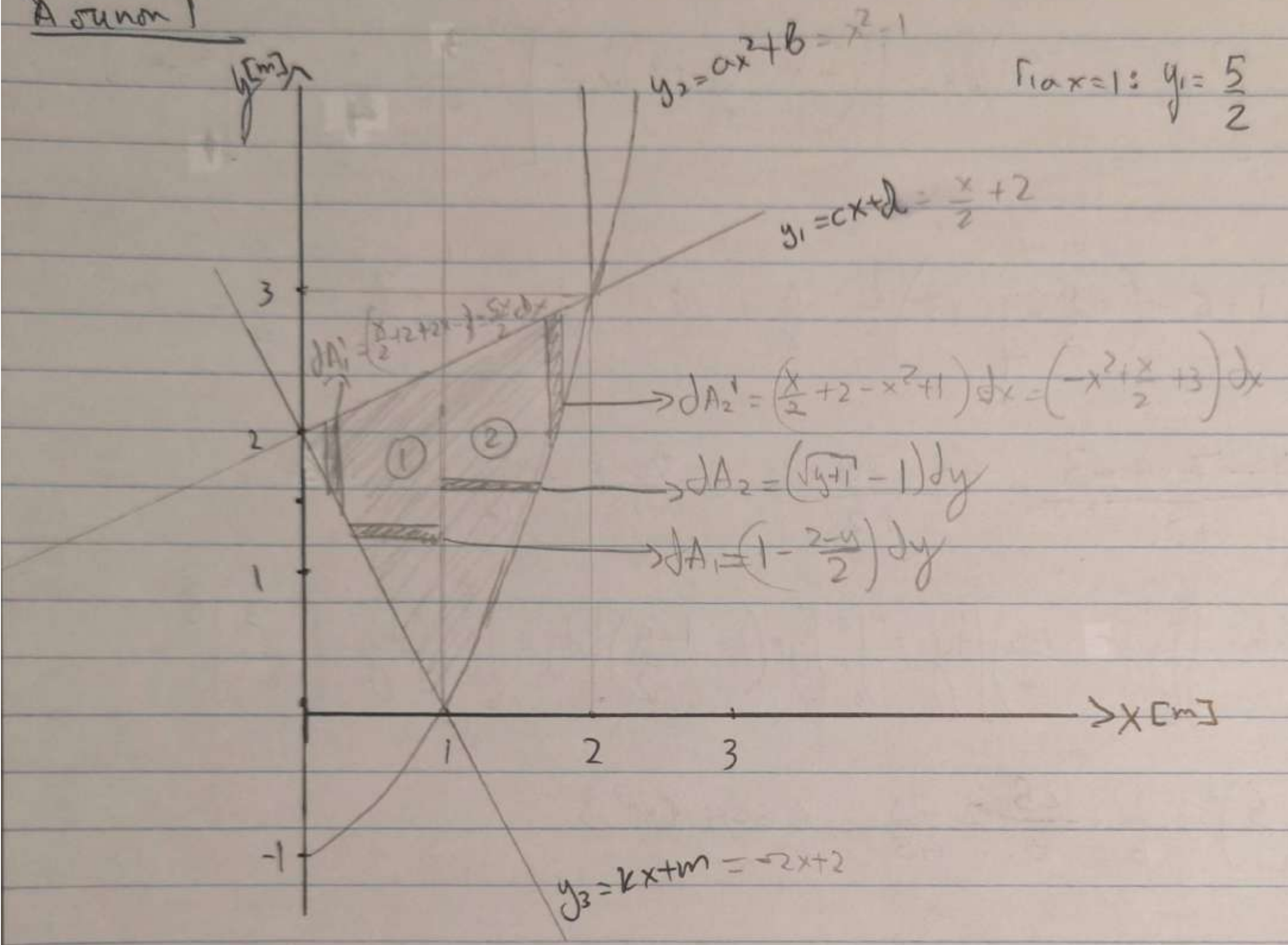
### Άσκηση 7
Να προσδιορισθεί το γεωμετρικό κέντρο της γραμμοσκιασμένης επιφάνειας (ΟΑΒΓΟ) του Σχ.7.




*Σχήμα 7*

10η Σειρά ασκήσεων: Προσδιορισμός γεωμετρικού κέντρου επιπέδων επιφανειών

Άσκηση 1




Έχω ότι για $y_3 = 2$, $x = 0$ άρα $2 = 0 + m \Leftrightarrow m = 2$
για $y_3 = 0$, $x = 1$ άρα $0 = k + 2 \Leftrightarrow k = -2$ $\Big\}$ $y_3 = -2x + 2$

Έχω ότι για $y_2 = -1$, $x = 0$ άρα $-1 = b$
$y_2 = 0$, $x = 1$ άρα $0 = a - 1 \Leftrightarrow a = 1$ $\Big\}$ $y_2 = x^2 - 1$

Έχω ότι για $y_1 = 2$, $x = 0$ άρα $2 = d$
$y_1 = 3$, $x = 2$ άρα $3 = 2c + 2 \Leftrightarrow c = \frac{1}{2}$ $\Big\}$ $y_1 = \frac{x}{2} + 2$

$$y_1 = y_2 \Leftrightarrow ax^2 + b = cx + d \Leftrightarrow x^2 - 1 = \frac{x}{2} + 2 \Leftrightarrow x^2 - \frac{x}{2} - 3 = 0 \Leftrightarrow 2x^2 - x - 6 = 0 \Rightarrow x = 2$$

Έχω $A_1 = \int_0^1 \frac{5}{2}x\,dx = \left[\frac{5}{4}x^2\right]_0^1 = \frac{5}{4} = 1{,}25$

και $A_2 = \int_1^2 \left(\frac{x}{2}+2-x^2+1\right)dx = \int_1^2\left(-x^2+\frac{x}{2}+3\right)dx = \left[-\frac{x^3}{3}+\frac{x^2}{4}+3x\right]_1^2 =$

$= -\frac{8}{3}+1+6-\left(-\frac{1}{3}+\frac{1}{4}+3\right) = -\frac{8}{3}+1+6+\frac{1}{3}-\frac{1}{4}-3 = -\frac{7}{3}+4-\frac{1}{4} =$

$= \frac{12\cdot 4-7\cdot 4-3}{12} = \frac{48-28-3}{12} = \frac{20-3}{12} = \frac{17}{12} = 1{,}417$

$Q_{x_1} = \int y\,dA_1 = \int_0^{\frac{5}{2}}\left[y\left(1-\frac{2-y}{2}\right)\right]dy = \int_0^{\frac{5}{2}}\left[y\cdot\left(1-1+\frac{y}{2}\right)\right]dy = \int_0^{\frac{5}{2}}\frac{y^2}{2}dy = \left[\frac{y^3}{6}\right]_0^{\frac{5}{2}} =$

$= \frac{1}{6}\cdot\left(\frac{5}{2}\right)^3 = \frac{1}{6}\cdot\frac{125}{8} = \frac{125}{48} = 2{,}604\ (m^3)$

$Q_{y_1} = \int x\,dA_1' = \int_0^1 x\cdot\frac{5x}{2}\,dx = \int_0^1\frac{5x^2}{4}\,dx = \left[\frac{5x^3}{12}\right]_0^1 = \frac{5}{12} = 0{,}417$

$Q_{x_2} = \int y\,dA_2 = \int_0^2\left[y\left(\sqrt{y+1}-1\right)\right]dy = \int_0^2\left(y\sqrt{y+1}-y\right)dy = \int_0^2 y\sqrt{y+1}\,dy - \int_0^2 y\,dy =$

$= \int_0^2 y\sqrt{y+1}\,dy - \left[\frac{y^2}{2}\right]_0^2 = \int_0^2 y\sqrt{y+1} - 2$ (*)

$I = \int_0^2 y\sqrt{y+1}\,dy$ Θέτω $u = y+1 \Rightarrow du = dy$, τα νέα όρια $u_1 = 1$, $u_2 = 3$

Άρα $I = \int_1^3 (u-1)\sqrt{u}\,du = \int_1^3\left(u\sqrt{u}-\sqrt{u}\right)du = \int_1^3\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du = \left[\frac{2u^{\frac{5}{2}}}{5}-\frac{2u^{\frac{3}{2}}}{3}\right]_1^3 =$

$$= \left[\frac{2}{3}u^2\sqrt{u} - \frac{2}{3}u\sqrt{u}\right]_1^3 = \frac{2}{5}\cdot 9\sqrt{3} - \frac{2}{\cancel{3}}\cdot\cancel{3}\sqrt{3} - \frac{2}{5} + \frac{2}{3} = \frac{18}{5}\sqrt{3} - 2\sqrt{3} + \frac{2}{3} - \frac{2}{5} =$$

$$= \frac{18\sqrt{3}}{5} - \frac{10\sqrt{3}}{5} + \frac{10}{15} - \frac{6}{15} = \frac{8\sqrt{3}}{5} + \frac{4}{15} = \frac{24\sqrt{3}+4}{15} = 3{,}038$$

Άρα (*) $\Rightarrow Q_{x_2} = 3{,}038 - 2 = 1{,}038\ m^3$

$$Q_{y_2} = \int x\,dA_2' = \int_1^2 x\left(-x^2 + \frac{x}{2} + 3\right)dx = \int_1^2\left(-x^3 + \frac{x^2}{2} + 3x\right)dx = \left[\frac{-x^4}{4} + \frac{x^3}{6} + \frac{3}{2}x^2\right]_1^2$$

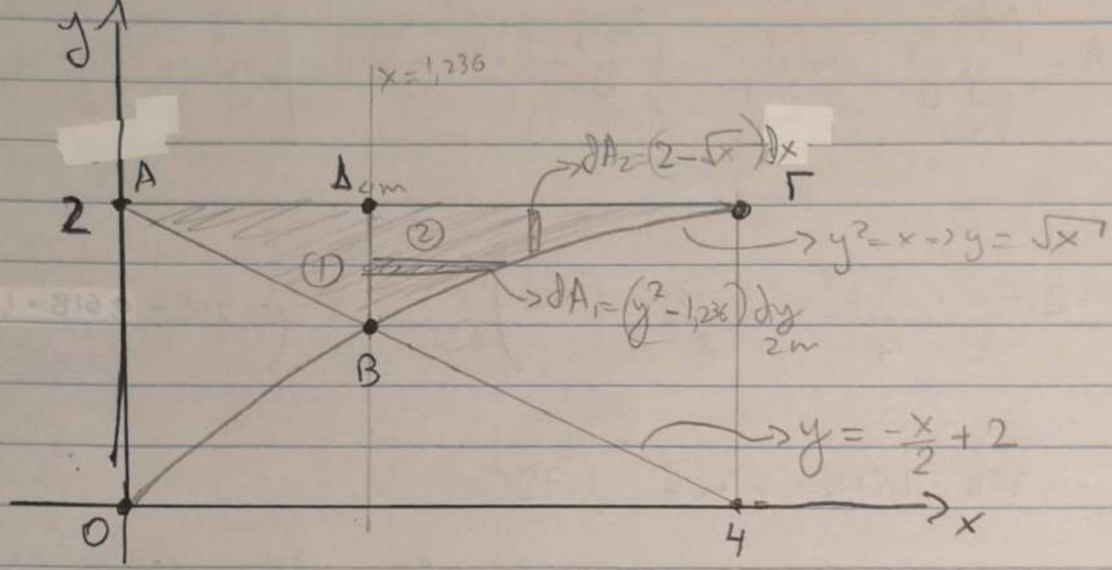
$$= -4 + \frac{4}{3} + 6 - \left(-\frac{1}{4} + \frac{1}{6} + \frac{3}{2}\right) = 2 + \frac{4}{3} + \frac{1}{4} - \frac{1}{6} - \frac{3}{2} = \frac{2\cdot 12 + 4\cdot 4 + 3\cdot 1 - 2\cdot 1 - 3\cdot 6}{12} =$$

$$= \frac{24+16+3-2-18}{12} = \frac{43-20}{12} = \frac{23}{12} = 1{,}917$$

Άρα $x_c = \frac{Q_{y_1} + Q_{y_2}}{A} = \frac{Q_{y_1} + Q_{y_2}}{A_1 + A_2} = \frac{0{,}417 + 1{,}917}{1{,}25 + 1{,}417} = \frac{2{,}334}{2{,}667} = 0{,}875\ m$

$$y_c = \frac{Q_{x_1} + Q_{x_2}}{A} = \frac{2{,}604 + 3{,}038}{2{,}667} = \frac{5{,}642}{2{,}667} = 2{,}115\ m$$

Άρα $C(0{,}875,\ 2{,}115)$




Έχω: $-\frac{x}{2}+2=\sqrt{x} \Leftrightarrow \frac{x}{2}+\sqrt{x}-2=0 \Leftrightarrow x+2\sqrt{x}-4=0$

$\Delta = 4+16=20 \quad x_{1,2}=\frac{-2\pm 2\sqrt{5}}{2} \Rightarrow x=\frac{-2+2\sqrt{5}}{2}=1,236 \Rightarrow y=1,382$

Άρα $B(1,236,\ 1,382) \Rightarrow \Delta(1,236,\ 2)$

$A_1=\frac{1,236\cdot(2-1,382)}{2}=0,382\ m^2$

$A_2=\int_{1,236}^{4}(2-\sqrt{x})dx=\left[2x-\frac{2x^{\frac{3}{2}}}{3}\right]_{1,236}^{4}=\left[2x-\frac{2x\sqrt{x}}{3}\right]_{1,236}^{4}=8-\frac{2\cdot 4\cdot 2}{3}-\left(2\cdot 1,236-\frac{2\cdot 1,236\sqrt{1,236}}{3}\right)$

$=8-\frac{16}{3}-\left(2,472-\frac{2,472\cdot 1,112}{3}\right)=2,667-2,472+0,916=1,111\ m^2$

Άρα $A=1,493\ m^2$

Έχω $A(0,2)$, $B(1,236,\ 1,382)$, $\Delta(1,236,\ 2)$

Άρα $C_1=(0,824,\ 1,794)$

Άρα $Q_{x_1}=A_1 y_{c_1}=0,382\cdot 1,794=0,685\ m^3$

$Q_{y_1}=A_1 x_{c_1}=0,382\cdot 0,824=0,315\ m^3$

$$Q_{x2} = \int y\,dA = \int_{1,382}^{2} y(y^2 - 1,236)\,dy = \int_{1,382}^{2} (y^3 - 1,236y)\,dy = \left[\frac{y^4}{4} - \frac{1,236}{2}y^2\right]_{1,382}^{2} =$$

$$= 4 - 1,236 \cdot 2 - \left(\frac{1,382^4}{4} - \frac{1,236}{2} \cdot 1,382^2\right) = 1,528 - (0,912 - 1,18) =$$

$$= 1,528 - (-0,268 = 1,528 + 0,268 = 1,796$$

$$Q_{y2} = \int x\,dA = \int_{1,236}^{4} [x(2 - \sqrt{x})]\,dx = \int_{1,236}^{4} (2x - x\sqrt{x})\,dx = \int_{1,236}^{4} (2x - x^{\frac{3}{2}})\,dx = \left[x^2 - \frac{2x^{\frac{5}{2}}}{5}\right]_{1,236}^{4} =$$

$$= \left[x^2 - \frac{2x^2\sqrt{x}}{5}\right]_{1,236}^{4} = 16 - \frac{2 \cdot 16 \cdot 2}{5} - \left(1,236^2 - \frac{2(1,236)^2\sqrt{1,236}}{5}\right) =$$

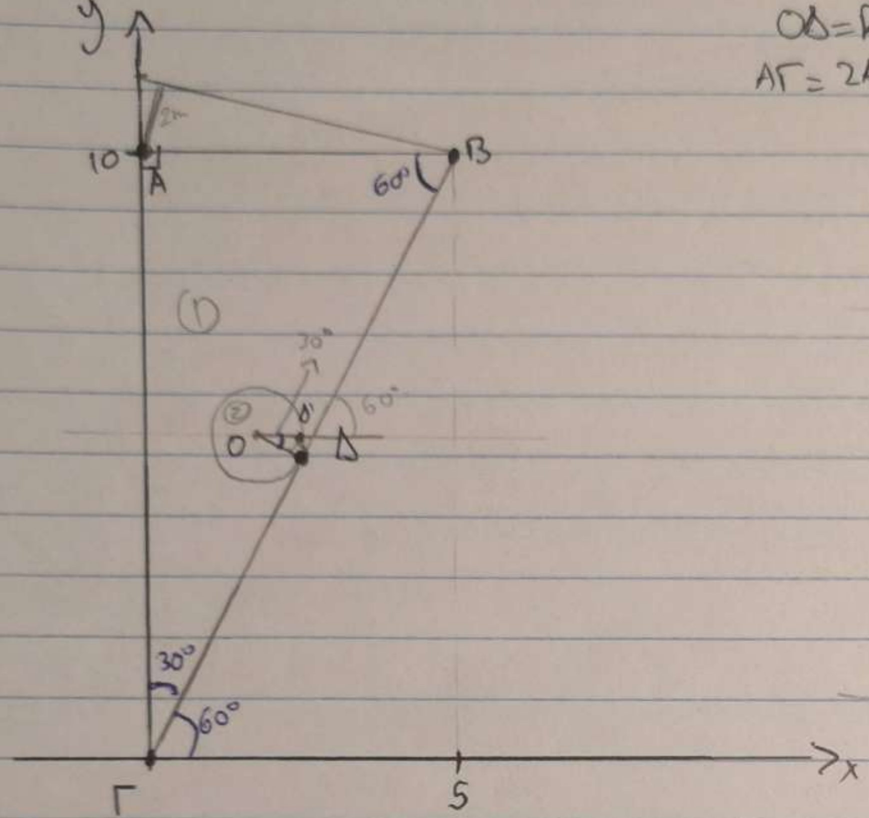
$$= 3,2 - (1,528 - 0,611 \cdot 1,112) = 3,2 - (1,528 - 0,679) = 3,2 - 0,849 = 2,351$$

Άρα $x_c = \dfrac{Q_{y_1} + Q_{y_2}}{A_1 + A_2} = \dfrac{0,315 + 2,351}{1,493} = \dfrac{2,666}{1,493} = 1,786$

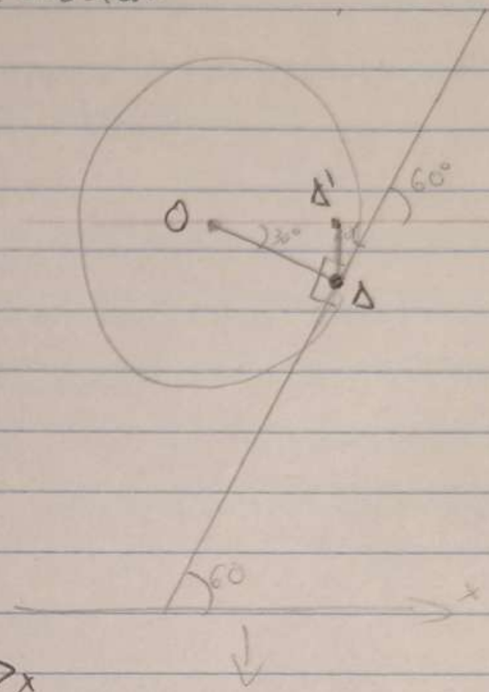
και $y_c = \dfrac{Q_{x_1} + Q_{x_2}}{A} = \dfrac{0,685 + 1,796}{1,493} = \dfrac{2,481}{1,493} = 1,662$

Άρα $C(1,786, 1,662)$

Άσκηση 2




OΔ = R = 0,7m
AΓ = 2AB = 10m




Αφού AΓ = 2AB έχω $A\hat{\Gamma}B = 30° \Rightarrow \Gamma\hat{B}A = 60°$

Έχω
A(0,10)
B(5,10)
Γ(0,0)
$\Big\}\ G\left(\frac{5}{3}, \frac{20}{3}\right) = (1{,}667,\ 6{,}667)$
Δ(2,5, 5)

$\sin(30°) = \frac{\Delta\Delta'}{O\Delta} \Leftrightarrow \Delta\Delta' = \frac{1}{2}R = \frac{1}{2}\cdot 0{,}7 = 0{,}35m$

$\cos(30°) = \frac{O\Delta'}{O\Delta} \Leftrightarrow O\Delta' = \frac{\sqrt{3}}{2}\cdot 0{,}7 = \sqrt{3}\cdot 0{,}35 = 0{,}61$

Άρα O(2,5−0,61, 5+0,35) = (1,89, 5,35)

Έχω O(1,89, 5,35), $A_2 = \pi R^2 = \pi(0{,}7)^2 = 1{,}54\ m^2$ και $A_1 = \frac{5\cdot 10}{2} = 25\ m^2$

Άρα $A = A_1 - A_2 = 23{,}46$

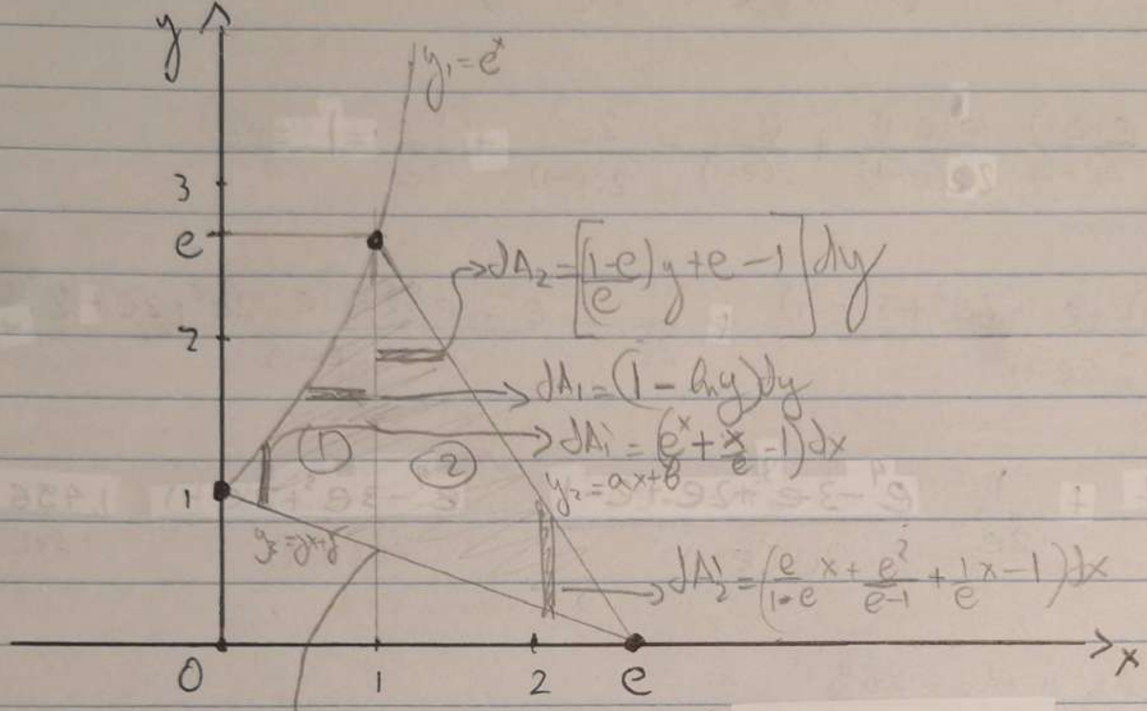
Άρα $Q_{x_2} = y_{c_2} A_2 = 5{,}35\cdot 1{,}54 = 8{,}24\ m^3$ | $Q_{y_1} = x_1 A_1 = 1{,}667\cdot 25 = 41{,}675\ m^3$

$Q_{y_2} = x_2 A_2 = 1{,}89\cdot 1{,}54 = 2{,}91\ m^3$ | $Q_{x_1} = y_1 A_1 = 6{,}667\cdot 25 = 166{,}675\ m^3$

Άρα $x_c = \frac{Q_{y_1} - Q_{y_2}}{A} = \frac{41{,}675 - 2{,}91}{23{,}46} = \frac{38{,}765}{23{,}46} = 1{,}65\ m$ και $y_c = \frac{Q_{x_1} - Q_{x_2}}{A} = \frac{158{,}435}{23{,}46} = 6{,}75\ m$

Άρα G(1,65, 6,75)

# Άσκηση 3




Για $x=1$: $y_3 = 1-\frac{1}{e}=\frac{e-1}{e}$

$$\left.\begin{matrix}\text{Για } y_2=0 \text{ έχω } x=e:\ 0=ae+b \\ \text{Για } y_2=e \text{ έχω } x=1:\ e=a+b\end{matrix}\right\} \iff \begin{cases} a=-\frac{b}{e} \\ a=e-b\end{cases} \iff \begin{cases} e-b=-\frac{b}{e} \\ a=e-b\end{cases} \iff \begin{cases} e^2-be=-b \\ a=e-b\end{cases} \iff \begin{cases} b(e-1)=e^2 \\ a=e-b\end{cases} \iff$$

$$\iff \begin{cases} b=\frac{e^2}{e-1} \\ a=e-\frac{e^2}{e-1}\end{cases} \iff \begin{cases} b=\frac{e^2}{e-1} \\ \frac{e(e-1)-e^2}{e-1}=a\end{cases} \iff \begin{cases} b=\frac{e^2}{e-1} \\ a=\frac{e^2-e-e^2}{e-1}\end{cases} \iff \begin{cases} b=\frac{e^2}{e-1} \\ a=\frac{-e}{e-1}\end{cases} \iff \begin{cases} b=\frac{e^2}{e-1} \\ a=\frac{e}{1-e}\end{cases}$$

Άρα $\boxed{y_2=\frac{e}{1-e}x+\frac{e^2}{e-1}} \Rightarrow y_2-\frac{e^2}{e-1}=\frac{e}{1-e}x \iff x=\left(y_2-\frac{e^2}{e-1}\right)\cdot\frac{1-e}{e}=\frac{1-e}{e}y+e$

$$\left.\begin{matrix}\text{Για } y_3=1 \text{ έχω } x=0:\ 1=\delta \\ \text{Για } y_3=0 \text{ έχω } x=e:\ 0=\gamma e+1 \iff \gamma=-\frac{1}{e}\end{matrix}\right\} \Rightarrow \boxed{y_3=-\frac{1}{e}x+1}$$

$$A_1=\int_0^1\left(e^x+\frac{1}{e}x-1\right)dx=\left[e^x+\frac{x^2}{2e}-x\right]_0^1=e+\frac{1}{2e}-1-1=\frac{2e^2+1-4e}{2e}=\frac{2e^2-4e+1}{2e}\simeq 0{,}9\,m^2$$

$$A_2=\int_1^e\left(\frac{e}{1-e}x+\frac{e^2}{e-1}+\frac{1}{e}x-1\right)dx=\left[\frac{ex^2}{2(1-e)}+\frac{e^2x}{e-1}+\frac{x^2}{2e}-x\right]_1^e =$$

$$= \frac{e^3}{2(1-e)} + \frac{e^3}{e-1} + \frac{e}{2} - e - \left(\frac{e}{2(1-e)} + \frac{e^2}{e-1} + \frac{1}{2e} - 1\right) =$$

$$= \frac{-e^3}{2(e-1)} + \frac{2e^3}{2(e-1)} + \frac{e(e-1)}{2(e-1)} - \frac{2e(e-1)}{2(e-1)} + \frac{e}{2(e-1)} - \frac{2e^2}{2(e-1)} - \frac{1}{2e} + \frac{2(e-1)}{2(e-1)} =$$

$$= \frac{e^3 - e(e-1) + e - 2e^2 + 2(e-1)}{2(e-1)} - \frac{1}{2e} = \frac{e^3 - e^2 + e + e - 2e^2 + 2e - 2}{2(e-1)} - \frac{1}{2e} =$$

$$= \frac{e^3 - 3e^2 + 4e - 2}{2(e-1)} - \frac{1}{2e} = \frac{e^4 - 3e^3 + 4e^2 - 2e - e + 1}{2e(e-1)} = \frac{e^4 - 3e^3 + 4e^2 - 3e + 1}{2e(e-1)} = \frac{16,74}{9,341} =$$

$= 1,79\ m^2 \quad \Rightarrow A = A_1 + A_2 = 2,69$

$$Q_{x_1} = \int y\,dA_1 = \int_{\frac{e-1}{e}}^{e} [y(1-\ln y)]\,dy = \int_{\frac{e-1}{e}}^{e} (y - y\ln y)\,dy = \int_{\frac{e-1}{e}}^{e} y\,dy - \int_{\frac{e-1}{e}}^{e} y\ln y\,dy =$$

$$= \left[\frac{y^2}{2}\right]_{\frac{e-1}{e}}^{e} - \int_{\frac{e-1}{e}}^{e} y\ln y\,dy$$

Εχω $I = \int_{\frac{e-1}{e}}^{e} y\ln y\,dy = \int_{\frac{e-1}{e}}^{e} \frac{1}{2}(y^2)'\ln y\,dy = \left[\frac{y^2}{2}\ln y\right]_{\frac{e-1}{e}}^{e} - \int_{\frac{e-1}{e}}^{e} \frac{y^2}{2}\cdot\frac{1}{y}\,dy = \left[\frac{y^2}{2}\ln y\right]_{\frac{e-1}{e}}^{e} - \left[\frac{y^2}{4}\right]_{\frac{e-1}{e}}^{e}$

Άρα $Q_{x_1} = \left[\frac{3y^2}{4}\right]_{\frac{e-1}{e}}^{e} - \left[\frac{y^2}{2}\ln y\right]_{\frac{e-1}{e}}^{e} = \left[\frac{3y^2}{4} - \frac{y^2}{2}\ln y\right]_{\frac{e-1}{e}}^{e} =$

$$= \frac{3e^2}{4} - \frac{e^2}{2}\ln e - \left(\frac{3(e-1)^2}{4e^2} - \frac{(e-1)^2}{2e^2}\ln\left(\frac{e-1}{e}\right)\right) = 1,85 - 0,29 - 0,2\cdot 0,46 = 1,56 - 0,09 = 1,47\ m^3$$

$$Q_{y_1} = \int x\, dA_1' = \int_0^1 \left[x\left(e^x + \frac{x}{e} - 1\right)\right] dx = \int_0^1 \left(xe^x + \frac{x^2}{e} - x\right) dx =$$

$$= \left[xe^x - e^x + \frac{x^3}{3e} - \frac{x^2}{2}\right]_0^1 = e - e + \frac{1}{3e} - \frac{1}{2} - (0 - 1 + 0 - 0) = \frac{1}{3e} - \frac{1}{2} + 1 = \frac{1}{3e} + \frac{1}{2} = 0{,}62$$

$$Q_{x_2} = \int y\, dA_2 = \int_0^e \left[y\left[\left(\frac{1-e}{e}\right)y + e - 1\right]\right] dy = \int_0^e \left[\left(\frac{1-e}{e}\right)y^2 + ey - y\right] dy =$$

$$= \left[\frac{y^3(1-e)}{3e} + \frac{e}{2}y^2 - \frac{1}{2}y^2\right]_0^e = \frac{e^{\cancel{3}2}(1-e)}{3\cancel{e}} + \frac{e^3}{2} - \frac{e^2}{2} = \frac{e^2 - e^3}{3} + \frac{e^3 - e^2}{2} =$$

$$= \frac{2e^2 - 2e^3}{6} + \frac{3e^3 - 3e^2}{6} = \frac{e^3 - e^2}{6} = 2{,}12$$

$$Q_{y_2} = \int x\, dA_2' = \int_1^e \left[x\left(\frac{ex}{1-e} + \frac{e^2}{e-1} + \frac{x}{e} - 1\right)\right] dx = \int_1^e \left(\frac{ex^2}{1-e} + \frac{e^2 x}{e-1} + \frac{x^2}{e} - x\right) dx =$$

$$= \left[\frac{ex^3}{3(1-e)} + \frac{e^2x^2}{2(e-1)} + \frac{x^3}{3e} - \frac{x^2}{2}\right]_1^e = \frac{e^4}{3(1-e)} + \frac{e^4}{2(e-1)} + \frac{e^3}{3e} - \frac{e^2}{2} - \left(\frac{e}{3(1-e)} + \frac{e^2}{2(e-1)} + \frac{1}{3e} - \frac{1}{2}\right)$$

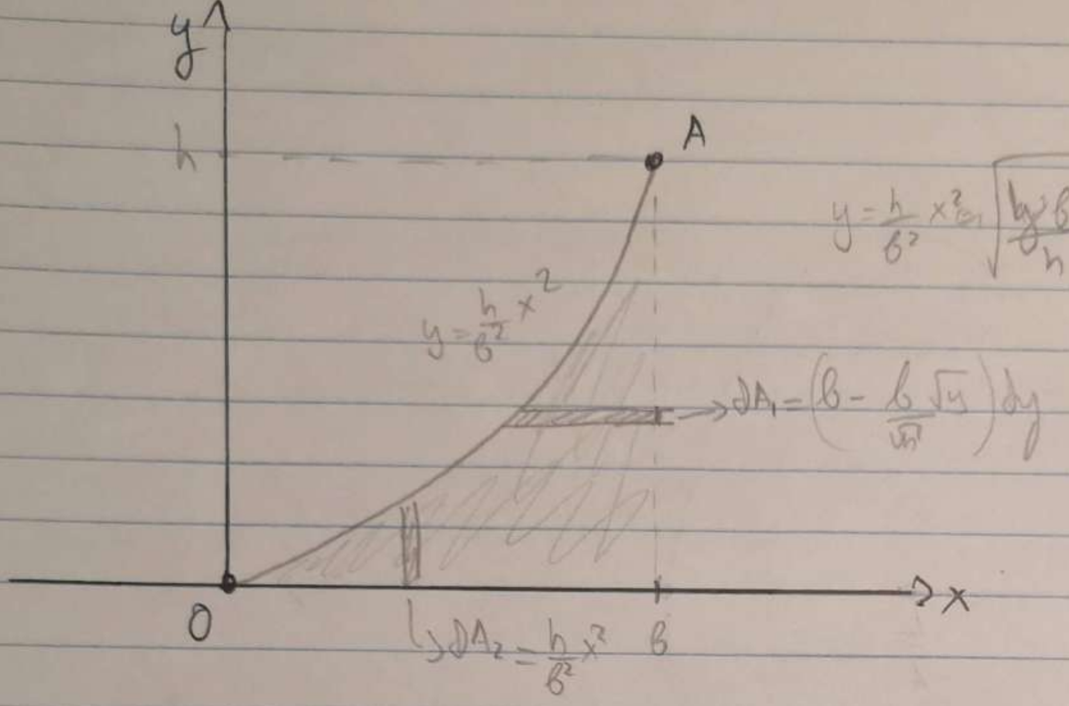
$$= -10{,}59 + 15{,}89 + 2{,}46 - 3{,}69 - (-0{,}53 + 2{,}15 + 0{,}12 - 0{,}5) = 4{,}07 - (1{,}24) = 2{,}83$$

Άρα $x_c = \frac{Q_{y_1} + Q_{y_2}}{A} = \frac{0{,}62 + 2{,}83}{2{,}69} = \frac{3{,}45}{2{,}69} = 1{,}28\ m$

και $y_c = \frac{Q_{x_1} + Q_{x_2}}{A} = \frac{2{,}12 + 1{,}47}{2{,}69} = \frac{3{,}59}{2{,}69} = 1{,}33$

Άρα $G(1{,}28, 1{,}33)$

Άσκηση 4




$y=\frac{h}{b^2}x^2 \Leftrightarrow \sqrt{\frac{y b^2}{h}}=x \Leftrightarrow x=b\sqrt{\frac{y}{h}}$

Έχω για $x=b \Rightarrow y=h$ : $h=\frac{h\cdot b^2}{b^2} \Leftrightarrow 1=\frac{1\cdot 1}{1}$,

$$A=\int_0^b \frac{h}{b^2}x^2\,dx=\left[\frac{hx^3}{3b^2}\right]_0^b=\frac{hb^3}{3b^2}=\frac{hb}{3}$$

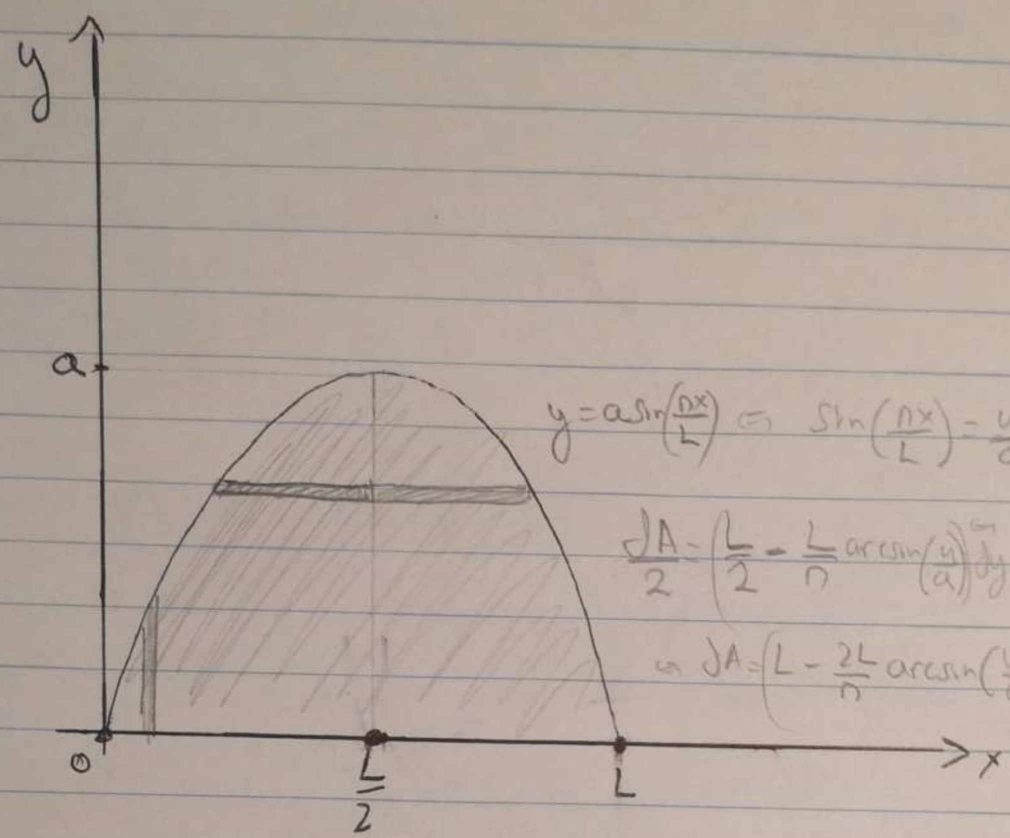
$$Q_x=\int y\,dA_1=\int_0^h\left[y\left(b-\frac{b}{\sqrt{h}}\sqrt{y}\right)\right]dy=\int_0^h\left(yb-\frac{by\sqrt{y}}{\sqrt{h}}\right)dy=\left[\frac{by^2}{2}-\frac{b}{\sqrt{h}}\frac{2}{5}y^{\frac{5}{2}}\right]_0^h=$$

$$=\left[\frac{b}{2}y^2-\frac{2b}{5\sqrt{h}}y^2\sqrt{y}\right]_0^h=\frac{bh^2}{2}-\frac{2bh^2\sqrt{h}}{5\sqrt{h}}=\frac{bh^2}{2}-\frac{2bh^2}{5}=\frac{5bh^2-4bh^2}{10}=\frac{bh^2}{10}$$

$$Q_y=\int x\,dA=\int_0^b\frac{h}{b^2}x^3\,dx=\left[\frac{hx^4}{4b^2}\right]_0^b=\frac{hb^4}{4b^2}=\frac{hb^2}{4}$$

Άρα $x_c=\frac{Q_y}{A}=\frac{\frac{hb^2}{4}}{\frac{hb}{3}}=\frac{3hb^2}{4hb}=\frac{3}{4}b$ και $y_c=\frac{Q_x}{A}=\frac{\frac{bh^2}{10}}{\frac{hb}{3}}=\frac{3bh^2}{10bh}=\frac{3}{10}h$

Άρα $G\left(\frac{3h}{10},\frac{3b}{4}\right)$




$$y = a\sin\left(\frac{\pi x}{L}\right) \Leftrightarrow \sin\left(\frac{\pi x}{L}\right) = \frac{y}{a} \Leftrightarrow \frac{\pi x}{L} = \arcsin\left(\frac{y}{a}\right) \Leftrightarrow x = \frac{L}{\pi}\arcsin\left(\frac{y}{a}\right)$$

$$\frac{dA}{2} = \left(\frac{L}{2} - \frac{L}{\pi}\arcsin\left(\frac{y}{a}\right)\right)dy$$

$$\Leftrightarrow dA = \left(L - \frac{2L}{\pi}\arcsin\left(\frac{y}{a}\right)\right)dy$$

$$A = \int_0^L a\sin\left(\frac{\pi x}{L}\right)dx = a\int_0^L \sin\left(\frac{\pi x}{L}\right)dx = a\left[-\frac{L}{\pi}\cos\left(\frac{\pi x}{L}\right)\right]_0^L = -\frac{aL}{\pi}\left[\cos\left(\frac{\pi x}{L}\right)\right]_0^L = \frac{-aL}{\pi}(-2) = \frac{2aL}{\pi}$$

$$Q_x = \int y\,dA = \int_0^a \left[y\left(L - \frac{2L}{\pi}\arcsin\left(\frac{y}{a}\right)\right)\right]dy$$

Θέτω $u = \frac{y}{a}$, $du = \frac{dy}{a} \Rightarrow u_1 = 0,\ u_2 = 1$

Άρα $Q_x = \int_0^1 \left[a^2u\left(L - \frac{2L}{\pi}\arcsin u\right)\right]du = \int_0^1 \left(a^2uL - \frac{2L}{\pi}a^2u\arcsin u\right)du =$

$$= \int_0^1 a^2Lu\,du - \frac{2L}{\pi}a^2\int_0^1 u\arcsin u\,du = \left[\frac{a^2L}{2}u^2\right]_0^1 - \frac{2La^2}{\pi}\int_0^1 u\arcsin u\,du =$$

$$= \frac{a^2L}{2} - \frac{2La^2}{\pi}\int_0^1 u\arcsin u\,du$$

Έχω $I=\int_0^1 u\arcsin u\,du$.

Έστω $f(x)=\arcsin x$, $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Ισχύει ότι $\sin(f(x))=x \Rightarrow (\sin f(x))'=(x)' \Leftrightarrow \cos(f(x))\cdot f'(x)=1 \Rightarrow$

$\Rightarrow f'(x)=\frac{1}{\cos f(x)}=\frac{1}{\cos(\arcsin x)}$

Ισχύει η ταυτότητα: $\cos a=\sqrt{1-\sin^2(a)}$

Άρα έχω $f'(x)=\frac{1}{\sqrt{1-(\cos(\arcsin x))^2}}=\frac{1}{\sqrt{1-x^2}}$

Άρα $I=\int_0^1 u\arcsin u\,du=\left[\frac{u^2}{2}\arcsin u\,du\right]_0^1-\int_0^1\frac{u^2}{2}\cdot\frac{1}{\sqrt{1-u^2}}du=\frac{\pi}{4}-\int_0^1\frac{u^2}{2}\cdot\frac{1}{\sqrt{1-u^2}}du$

Θέτω $u=\sin w \Rightarrow du=\cos w\,dw$ Τα νέα όρια: $w_1=0$, $w_2=\frac{\pi}{2}$

Άρα $I=\frac{\pi}{4}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin^2 w}{\sqrt{1-\sin^2 w}}\cdot\cos w\,dw=\frac{\pi}{4}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin^2 w}{\cancel{\cos w}}\cancel{\cos w}\,dw=\frac{\pi}{4}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 w\,dw$

$=\frac{\pi}{4}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1-\cos 2w}{2}dw=\frac{\pi}{4}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{2}-\frac{\cos 2w}{2}dw=\frac{\pi}{4}-\frac{1}{2}\left[\frac{w}{2}-\frac{\sin 2w}{4}\right]_0^{\frac{\pi}{2}}=$

$=\frac{\pi}{4}-\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}$

Άρα $Q_x=\frac{a^2L}{2}-\frac{2La^2}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{8}=\frac{a^2L}{2}-\frac{2a^2L}{8}=\frac{a^2L}{2}-\frac{a^2L}{4}=\frac{a^2L}{4}$

$$Q_y = \int x \, dA = \int_0^L x \cdot a\sin\left(\frac{\pi x}{L}\right) dx$$

Θέτω $u = \frac{\pi x}{L} \Rightarrow du = \frac{\pi}{L} dx \Leftrightarrow dx = \frac{L}{\pi} du$, Τα νέα όρια $u_1 = 0$, $u_2 = \pi$

Άρα $Q_y = \int_0^\pi \frac{L}{\pi} u \cdot a\sin(u) \cdot \frac{L}{\pi} du = \int_0^\pi \frac{aL^2}{\pi^2} u\sin u \, du =$

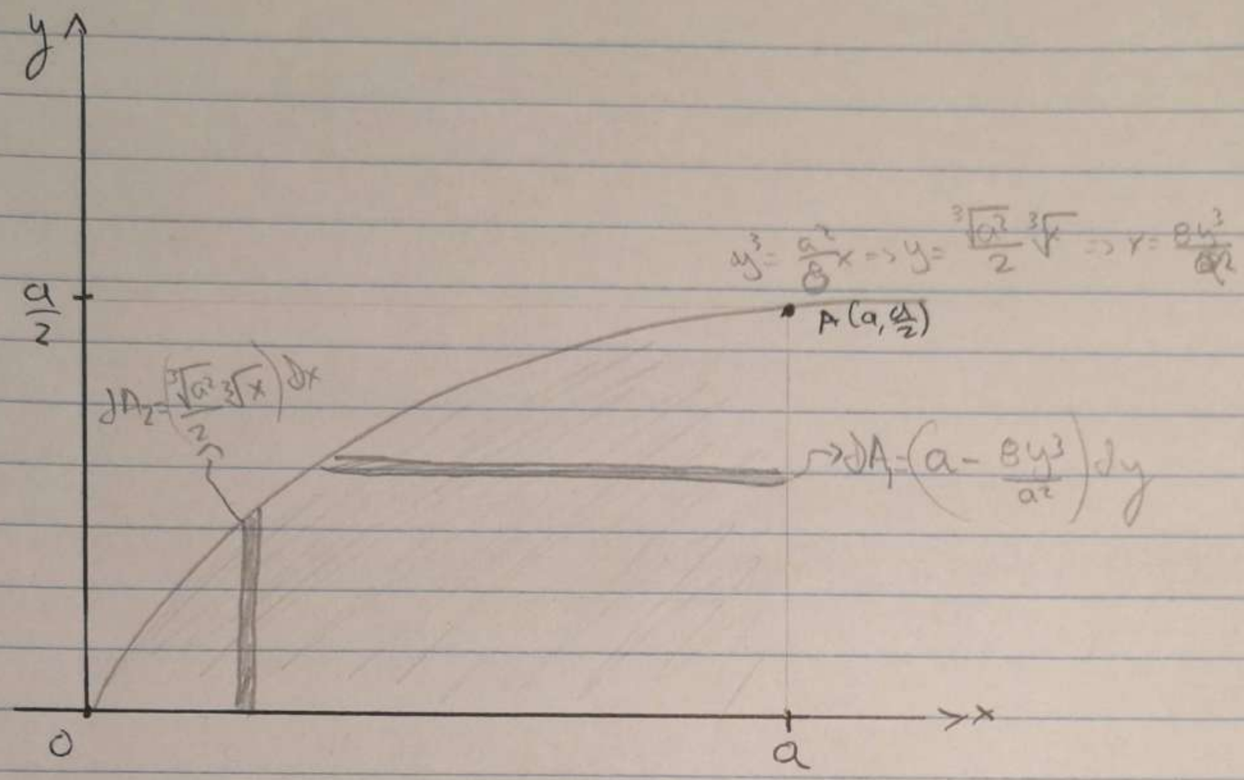
$$= \frac{aL^2}{\pi^2} \int_0^\pi u\sin u \, du = \frac{aL^2}{\pi^2} = \left[ \left[ -u\cos u \right]_0^\pi - \int_0^\pi -\cos u \, du \right] \frac{aL^2}{\pi^2} =$$

$$= \frac{aL^2}{\pi^2} \left( -\pi \cdot (-1) + [\sin u]_0^\pi \right) = \frac{aL^2}{\pi^2} \cdot \pi = \frac{aL^3}{\pi}$$

Άρα $x_c = \frac{Q_y}{A} = \frac{\frac{aL^{\cancel{3}}}{\cancel{\pi}}}{\frac{2aL}{\cancel{\pi}}} = \frac{L}{2}$

και $y_c = \frac{Q_x}{A} = \frac{\frac{a^2 L}{4}}{\frac{2aL}{\pi}} = \frac{a\pi}{8}$

$$C\left(\frac{L}{2}, \frac{\pi a}{8}\right)$$




$$A = \int_0^a \frac{\sqrt[3]{a^2}}{2} x^{\frac{1}{3}}\, dx = \left[\frac{\sqrt[3]{a^2}}{2}\frac{3x^{\frac{4}{3}}}{4}\right]_0^a = \left[\frac{3a^{\frac{2}{3}}}{8} x^{\frac{4}{3}}\right] = \frac{3}{8} a^{\frac{2}{3}} a^{\frac{4}{3}} = \frac{3a^2}{8}$$

$$Q_x = \int y\, dA_1 = \int_0^{\frac{a}{2}} \left[y\left(a - \frac{8y^3}{a^2}\right)\right] dy = \int_0^{\frac{a}{2}} \left(ay - \frac{8y^4}{a^2}\right) dy = \left[\frac{a}{2}y^2 - \frac{8}{5a^2}y^5\right]_0^{\frac{a}{2}} =$$

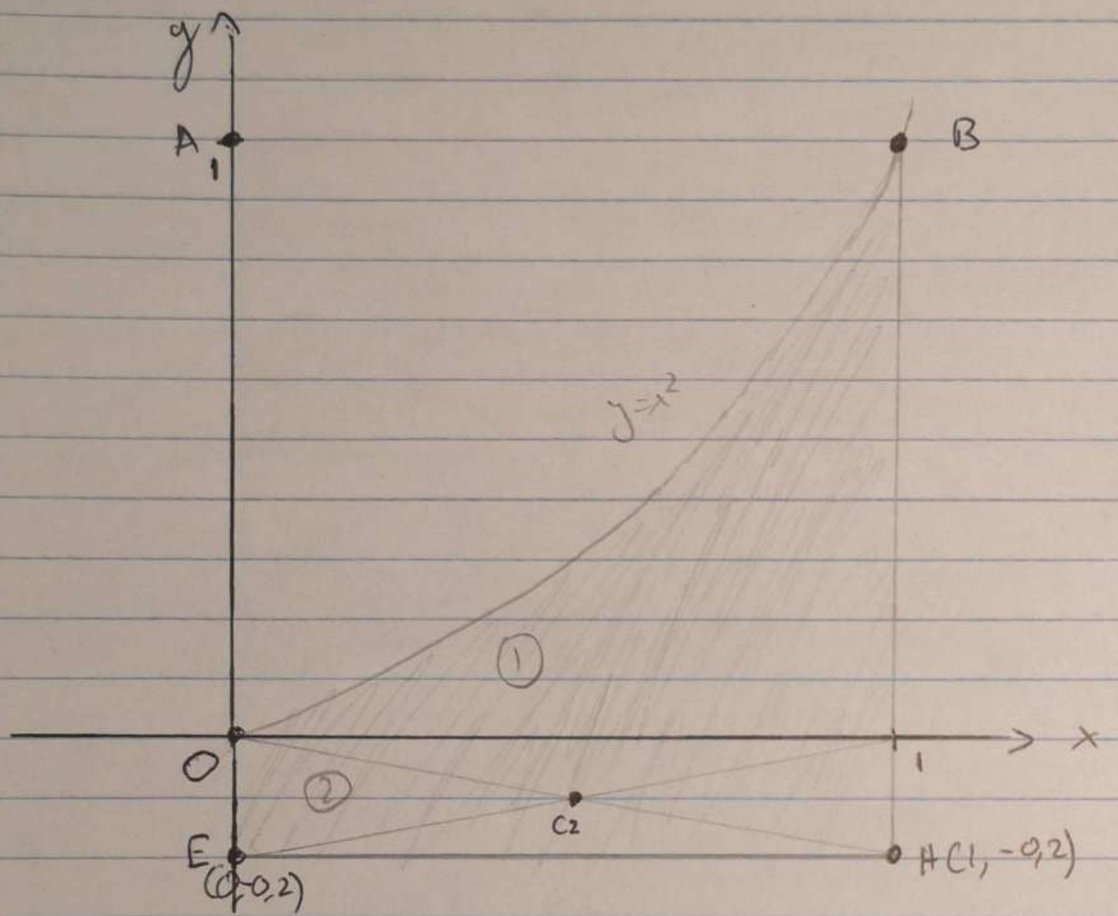
$$= \frac{a}{2}\cdot\frac{a^2}{4} - \frac{8}{5a^2}\frac{a^5}{32} = \frac{a^3}{8} - \frac{a^3}{20} = \frac{5a^3 - 2a^3}{40} = \frac{3a^3}{40}$$

$$Q_y = \int x\, dA_2 = \int_0^a x \cdot \frac{\sqrt[3]{a^2}}{2}\sqrt[3]{x}\, dx = \int_0^a \frac{x^{\frac{4}{3}} \cdot a^{\frac{2}{3}}}{2} dx = \left[\frac{3}{7} \cdot \frac{a^{\frac{2}{3}}}{2} \cdot x^{\frac{7}{3}}\right]_0^a = \frac{3}{14} \cdot a^{\frac{2}{3}} \cdot a^{\frac{7}{3}} =$$

$$= \frac{3a^3}{14}$$

Άρα $x_c = \frac{Q_y}{A} = \frac{\frac{3a^3}{14}}{\frac{3a^2}{8}} = \frac{8a}{14} = \frac{4a}{7}$ m και $y_c = \frac{Q_x}{A} = \frac{\frac{3a^3}{40}}{\frac{3a^2}{8}} = \frac{8a}{40} = \frac{a}{5}$ m Άρα $C\left(\frac{4a}{7}, \frac{a}{5}\right)$

Άσκηση 5




Για την εξίσωση $y = \frac{h}{b^2}x^2$ έχω, από άσκηση 4 ότι $Q_{x_1} = \frac{bh^2}{10}$ και $Q_{y_1} = \frac{hb^2}{4}$

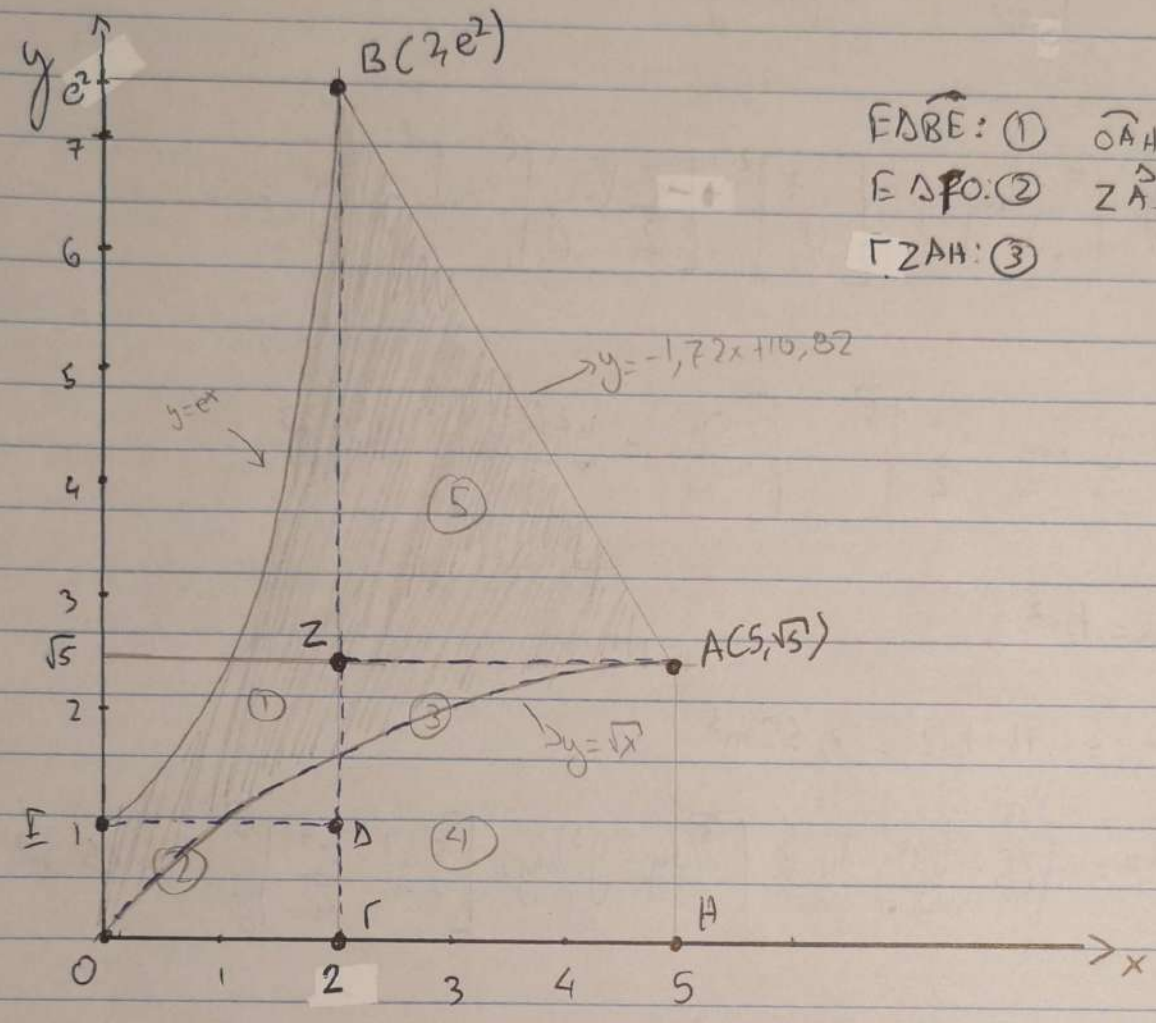
Άρα εδώ, για $h=b=1$ έχω $Q_{x_1} = \frac{1}{10} = 0{,}1\,m^3$ και $Q_y = \frac{1}{4} = 0{,}25\ m^3$

Επίσης έχω: $O(0,0)$ και $H(1,-0{,}2)$ άρα $C_2(0{,}5,-0{,}1) \Rightarrow x_2 = 0{,}5$, $y_2 = -0{,}1$

$A_1 = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} m^2$, $A_2 = 0{,}2 m^2 \Rightarrow A = 0{,}53 m^2$

$$\left.\begin{aligned} y_c &= \frac{Q_{x_1}+Q_{x_2}}{A_1+A_2} = \frac{Q_{x_1}+A_2 y_{c2}}{A} = \frac{0{,}1-0{,}2\cdot 0{,}1}{0{,}53} = \frac{0{,}08}{0{,}53} = 0{,}15\ m \\ x_c &= \frac{Q_y+Q_{y_2}}{A} = \frac{0{,}25+0{,}2\cdot 0{,}5}{0{,}53} = \frac{0{,}35}{0{,}53} = 0{,}66\ m \end{aligned}\right\} C\,(0{,}66,\ 0{,}15)$$

Άσκηση 6




ΕΔΒΕ: (1) ΟΔΗΟ: (4)
ΕΔΓΟ: (2) ΖΑΒ: (5)
ΓΖΑΗ: (3)

$$A=\int_0^2 (e^x-\sqrt{x})\,dx+\int_2^5(-1,72x+10,82-\sqrt{x})\,dx=$$

$$=\left[e^x-\frac{2}{3}x\sqrt{x}\right]_0^2+\left[-\frac{1,72x^2}{2}+10,82x-\frac{2}{3}x\sqrt{x}\right]_2^5=4,5+8,83=13,33m^2$$

$A_2=2m^2$, $A_3=3\cdot\sqrt{5}=6,71m^2$, $A_5=\dfrac{3\cdot(e^2-\sqrt{5})}{2}=1,5(e^2-\sqrt{5})=7,73\,m^2$

Έχω Δ(2,1) άρα $C_2(1,0,5)$

και Γ(2,0) και $A(5,\sqrt{5})$ άρα $C_3(3,5,\,1,12)$

και $Z(2,\sqrt{5})$, $A(5,\sqrt{5})$ και $B(2,e^2)$ άρα $C_5(3,\,3,95)$

$$Q_{x_1} = \int y\,dA = \int_1^{e^2} \left[ y\,(2-\ln y) \right] dy = \int_1^{e^2} (2y - y\ln y)\,dy =$$

$$= \int_1^{e^2} 2y\,dy - \int_1^{e^2} y\ln y\,dy = \left[ y^2 \right]_1^{e^2} - \left[ \frac{y^2}{2}\ln y \right]_1^{e^2} + \int_1^{e^2} \frac{y}{2}\,dy =$$

$$= \left[ y^2 - \frac{y^2}{2}\ln y + \frac{y^2}{4} \right]_1^{e^2} = \left[ \frac{5y^2}{4} - \frac{y^2 \ln y}{2} \right] = 12,4\,m^3$$

$$Q_{x_2} = A_2\, y_{c_2} = 1\,m^3$$

$$Q_{x_3} = A_3\, y_{c_3} = 6,71 \cdot 1,12 = 7,52\,m^3$$

$$Q_{x_4} = \int y\,dA = \int_0^{\sqrt{5}} (5-y^2)\,y\,dy = \int_0^{\sqrt{5}} (5y - y^3)\,dy = \left[ \frac{5}{2}y^2 - \frac{y^4}{4} \right]_0^{\sqrt{5}} = 6,25\,m^3$$

$$Q_{x_5} = A_5\, y_{c_5} = 7,73 \cdot 3,95 = 30,53\,m^3$$

Άρα $Q_x = Q_{x_1} + Q_{x_2} + Q_{x_3} - Q_{x_4} + Q_{x_5} = 45,2\,m^3$

$$Q_{y_1} = \int x\,dA = \int_0^2 \left[ x\,(e^x - 1) \right] dx = \int_0^2 (xe^x - x)\,dx = \int_0^2 xe^x dx - \int_0^2 x\,dx =$$

$$= \left[ xe^x \right]_0^2 - \int_0^2 e^x dx - \left[ \frac{x^2}{2} \right]_0^2 = \left[ xe^x - e^x - \frac{x^2}{2} \right]_0^2 = 6,39\,m^3$$

$$Q_{y_2} = A_2\, x_{c_2} = 2\,m^3$$

$$Q_{y_3} = A_3\, x_{c_3} = 6,71 \cdot 3,5 = 23,49$$

$$Q_{y_4} = \int x\,dA = \int_0^5 x\sqrt{x}\,dx = \int_0^5 x^{\frac{3}{2}}\,dx = \left[\frac{2x^{\frac{5}{2}}}{5}\right]_0^5 = \left[\frac{2x^2\sqrt{x}}{5}\right]_0^5 = \frac{2\cdot 5\cdot \cancel{5}\sqrt{5}}{\cancel{5}} = 10\sqrt{5} = 22,36\ m^3$$

$$Q_{y_5} = A_5 x_{c_5} = 7,73\cdot 3 = 23,19$$

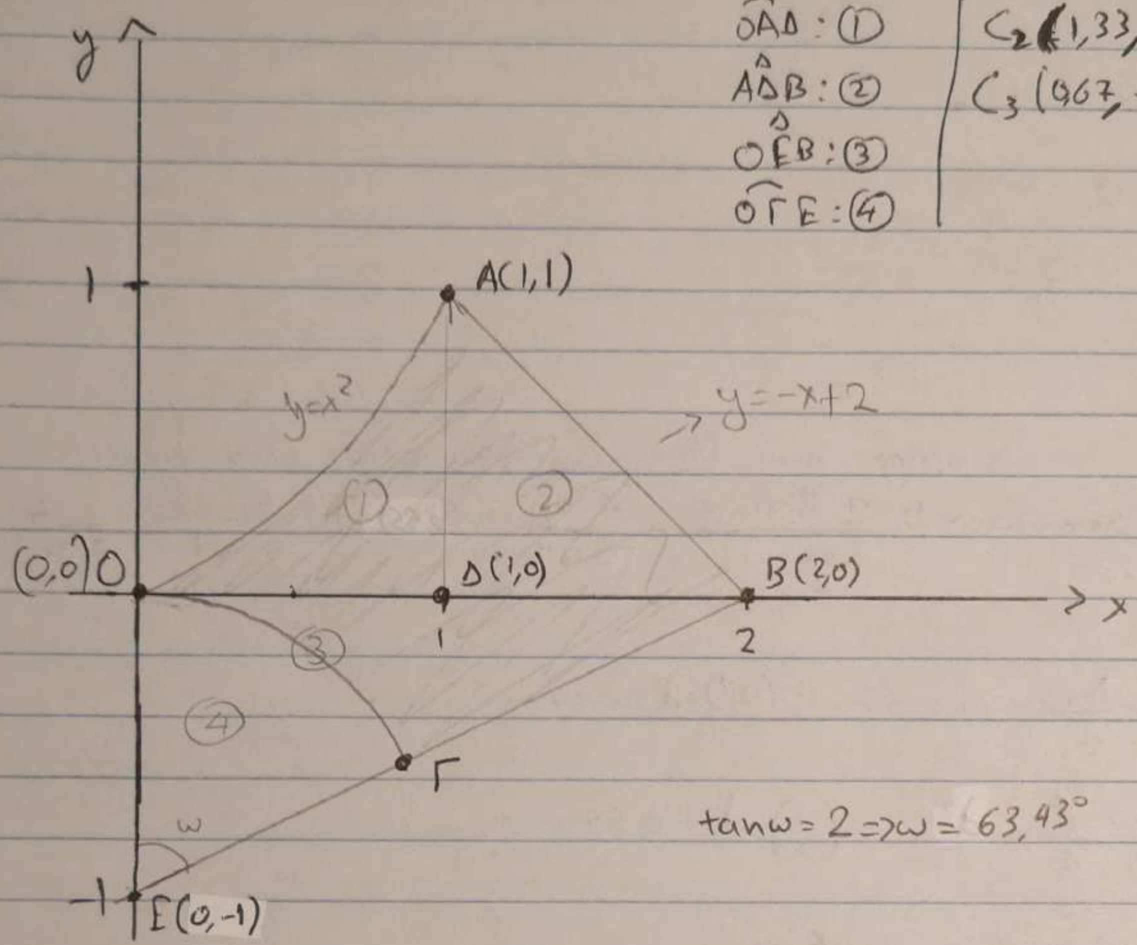
Άρα $Q_y = Q_{y_1} + Q_{y_2} + Q_{y_3} - Q_{y_4} + Q_{y_5} = 32,71$

Άρα $x_c = \dfrac{Q_y}{A} = \dfrac{32,71}{13,33} = 2,45$ m

και $y_c = \dfrac{Q_x}{A} = \dfrac{45,2}{13,33} = 3,39$ m

Άρα $C(2,28,\ 3,39)$

Άσκηση 7




$\widehat{OA\Delta}$ : (1)
$A\hat{\Delta}B$ : (2)
$O\hat{E}B$ : (3)
$\widehat{O\Gamma E}$ : (4)

$C_2(1{,}33,\ 0{,}33)$
$C_3(0{,}67,\ -0{,}33)$

$\tan\omega = 2 \Rightarrow \omega = 63{,}43^\circ$

Για την εξίσωση $y=\frac{h}{b^2}x^2$ έχω, από ασκ. 4, ότι $Q_x = \frac{bh^3}{10}$ και $Q_y = \frac{hb^2}{4}$

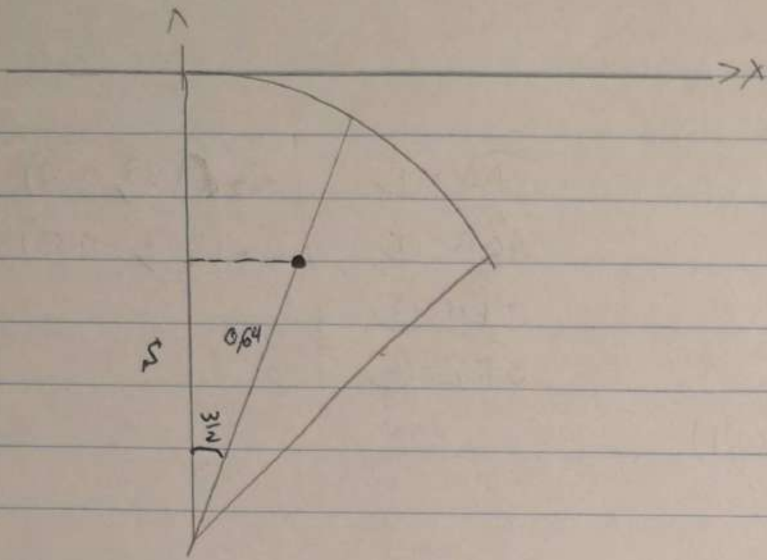
Άρα για $h=b=1$ έχω $Q_{x_1}=0{,}1$ και $Q_{y_1}=0{,}25$

$A_1 = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}\,m^2$, $A_2 = \frac{1\cdot 1}{2} = 0{,}5\,m^2$, $A_3 = \frac{1\cdot 2}{2} = 1\,m^2$

$A_4 = \pi R^2 \cdot \frac{63{,}43}{360} = \frac{\pi}{5{,}68} = 0{,}55\,m^2$

Άρα $A = A_1 + A_2 + A_3 - A_4 = 0{,}33 + 0{,}5 + 1 - 0{,}55 = 1{,}28\,m^2$

$Q_{x_2} = A_2 y_{c_2} = 0{,}5\cdot 0{,}33 = 0{,}17\,m^3$, $Q_{y_2} = A_2 x_{c_2} = 0{,}5\cdot 1{,}33 = 0{,}67\,m^3$
$Q_{x_3} = A_3 y_{c_3} = -0{,}33\,m^3$, $Q_{y_3} = A_3 x_{c_3} = 0{,}67\,m^3$




Ξέρω ότι το γ. κέντρο του (4) βρίσκεται πάνω στην διχοτόμο του και σε απόσταση $d = \dfrac{2R \cdot \sin\frac{\omega}{2}}{3\frac{\omega}{2}} = \dfrac{4 \cdot 1}{3 \cdot 1{,}11} \cdot \sin(31{,}715°) = \dfrac{4 \cdot 0{,}53}{3 \cdot 1{,}11} = \dfrac{2{,}12}{3{,}33} = 0{,}64$

Άρα $\sin\left(\frac{\omega}{2}\right) = \dfrac{x_{c4}}{d} \Leftrightarrow x_{c4} = \sin(31{,}715°) \cdot d = 0{,}34\ m$

και $\cos\left(\frac{\omega}{2}\right) = \dfrac{y'}{d} \Leftrightarrow y' = \cos\left(\frac{\omega}{2}\right) \cdot d = 0{,}54\ m$

Άρα $y_{c4} = -1 + 0{,}54 = -0{,}46\ m$

Άρα $Q_{x4} = A_4 y_{c4} = 0{,}55(-0{,}46) = -0{,}25\ m^3$

και $Q_{y4} = A_4 x_{c4} = 0{,}55 \cdot 0{,}34 = 0{,}19\ m^3$

Άρα $x_c = \dfrac{Q_y}{A} = \dfrac{Q_{y1} + Q_{y2} + Q_{y3} - Q_{y4}}{A} = \dfrac{0{,}25 + 0{,}67 + 0{,}67 - 0{,}19}{1{,}28} = \dfrac{1{,}4}{1{,}28} = 1{,}09\ m$

και $y_c = \dfrac{Q_x}{A} = \dfrac{Q_{x1} + Q_{x2} + Q_{x3} - Q_{x4}}{A} = \dfrac{0{,}1 + 0{,}17 - 0{,}33 + 0{,}25}{A} = \dfrac{0{,}19}{1{,}28} = 0{,}15\ m$