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ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΤΟΜΕΑΣ ΜΗΧΑΝΙΚΗΣ, ΕΡΓΑΣΤΗΡΙΟ ΑΝΤΟΧΗΣ ΚΑΙ ΥΛΙΚΩΝ

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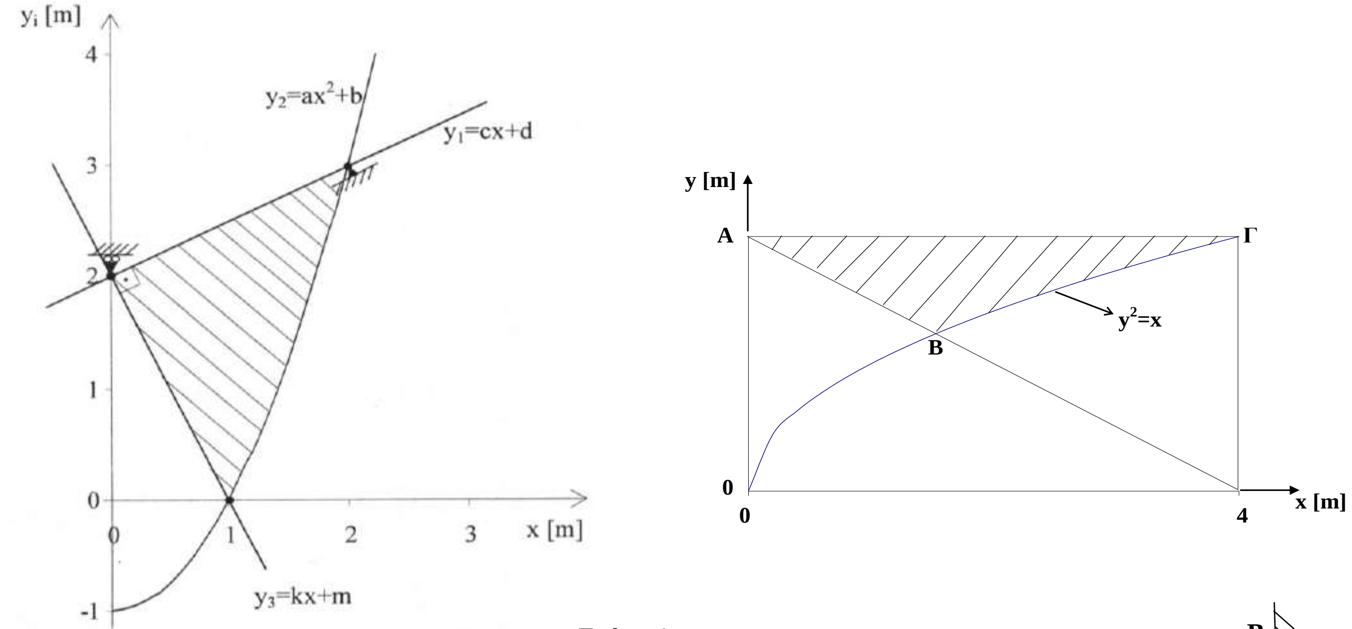
ΜΗΧΑΝΙΚΗ Ι (ΣΤΑΤΙΚΗ)

<u>10η σειρά ασκήσεων: Προσδιορισμός γεωμετρικού κέντρου επιπέδων επιφανειών</u>

Άσκηση 1

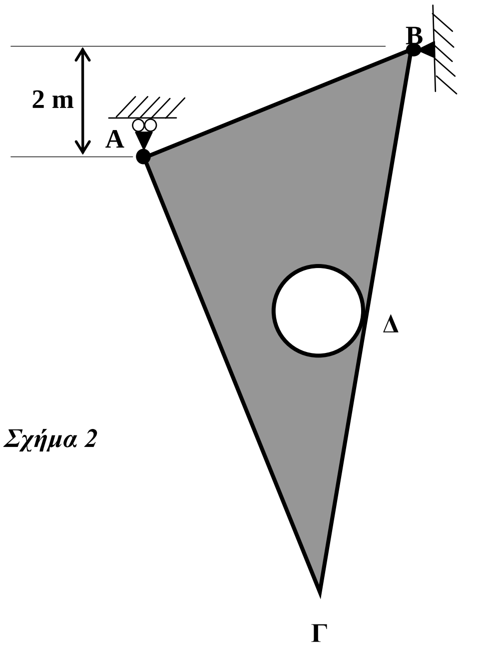
Να προσδιορισθούν τα γεωμετρικό κέντρα των γραμμοσκιασμένων επιφανειών του Σχ.1.



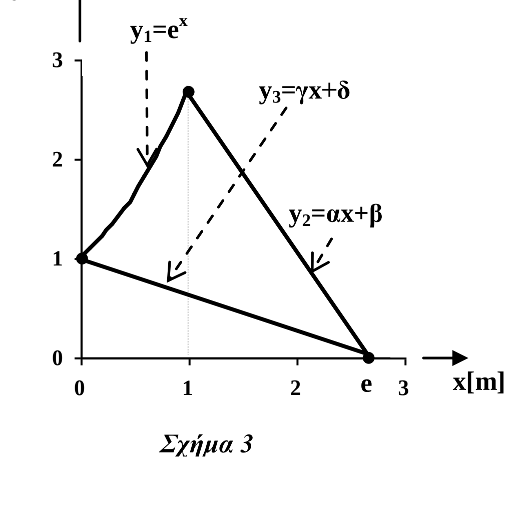
Σχήμα 1

Άσκηση 2

Να προσδιορισθεί το γεωμετρικό κέντρο της σκιασμένης επιφάνειας του Σχ.2. Δίνεται ότι ΑΓ=2ΑΒ= 10m και ότι $\widehat{ΒΑΓ}=90^o$. Η κυκλική οπή ακτίνας 0.7 m, εφάπτεται στο μέσον Δ της ΒΓ.



Σχήμα 2



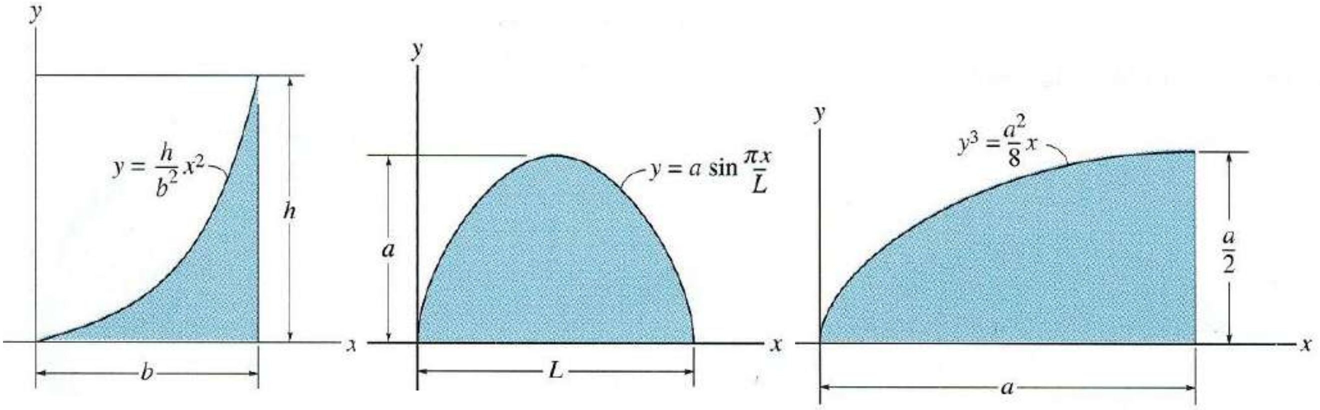
Σχήμα 3

Άσκηση 3

Να ευρεθεί το γεωμετρικό κέντρο της επιφάνειας που περικλείεται μεταξύ των γραμμών y_1, y_2 και y_3 του παραπλεύρως Σχ.3.

Άσκηση 4

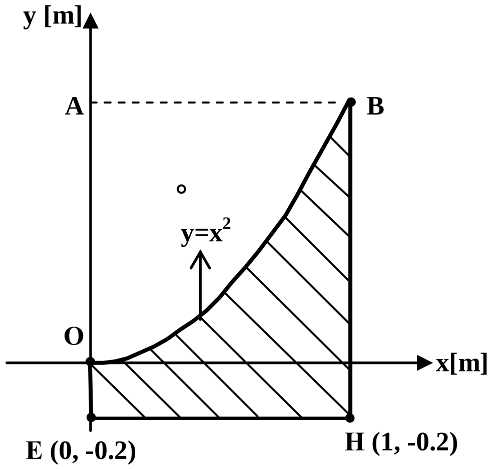
Να προσδιορισθούν τα γεωμετρικά κέντρα των κάτωθι επιφανειών (Σχ. 4):



Σχήμα 4

Άσκηση 5

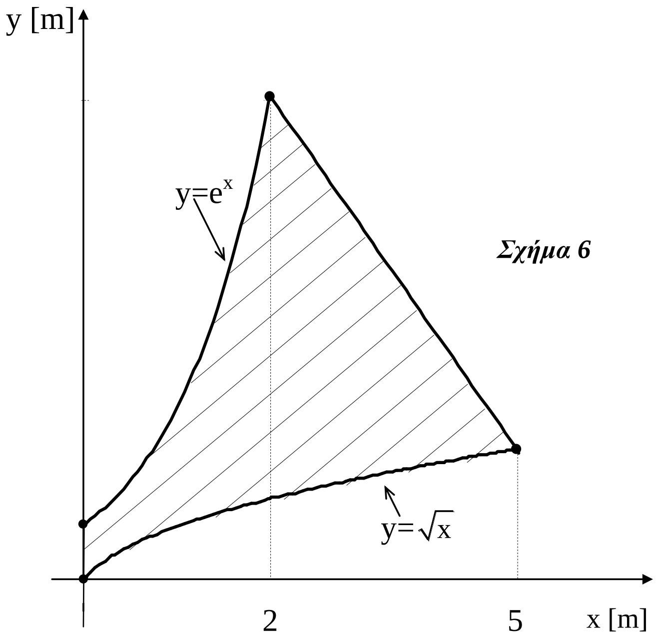
Να προσδιορισθούν το γεωμετρικό κέντρο της γραμμοσκιασμένης επιφάνειας του Σχ.5.



Σχήμα 5

Άσκηση 6

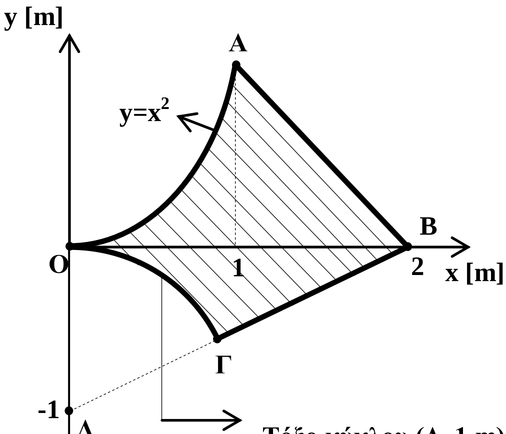
Να προσδιορισθεί το γεωμετρικό κέντρο της επίπεδης επιφάνειας ΑΒΓ του Σχ.6.



Σχήμα 6

Άσκηση 7

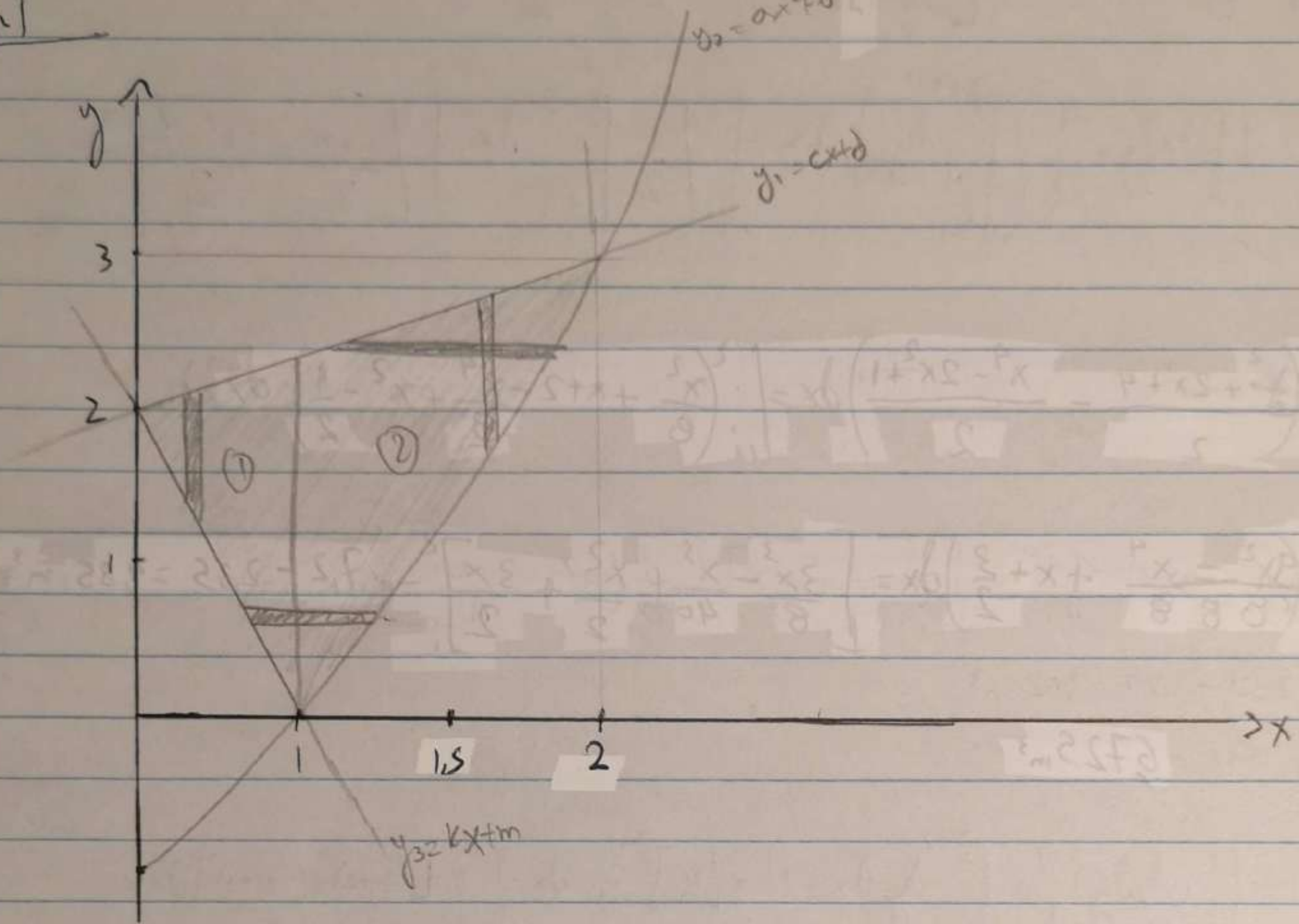
Να προσδιορισθεί το γεωμετρικό κέντρο της γραμμοσκιασμένης επιφάνειας (ΟΑΒΓΟ) του Σχ.7.



Σχήμα 7

10η Σειρά ασκήσεων: Προσδιορισμός γεωμετρικού κέντρου επιπέδων επιφανειών

Άσκηση 1



Για $y_3=2, x=0$ άρα $m=2$
για $y_3=0, x=1$ άρα $k=-2$
$\Big\}\ y_3=-2x+2$

Για $y_2=-1, x=0$ άρα $b=-1$
για $y_2=0, x=1$ άρα $a=1$
$\Big\}\ y_2=x^2-1$

Για $y_1=2, x=0$ άρα $d=2$
Για $y_1=3, x=2$ άρα $c=\frac{1}{2}$
$\Big\}\ y_1=\frac{x}{2}+2$

$y_1=y_2 \Leftrightarrow x^2-1=\frac{x}{2}+2 \Rightarrow x=2$

$$Q_{x_1}=\iint y\,dA=\iint_{A_1} y\,dx\,dy=\int_0^1\int_{-2x+2}^{\frac{x}{2}+2} y\,dy\,dx=\int_0^1\left[\frac{y^2}{2}\right]_{-2x+2}^{\frac{x}{2}+2}dx=\int_0^1\left(\frac{\left(\frac{x}{2}+2\right)^2}{2}-\frac{(-2x+2)^2}{2}\right)dx$$

$$=\int_0^1 \frac{\frac{x^2}{4}+2x+4}{2}-\frac{4x^2-8x+4}{2}\,dx=\int_0^1\left[\frac{x^2}{8}+x+2-(2x^2-4x+2)\right]dx=\int_0^1\left(\frac{x^2}{8}+x+2-2x^2+4x-2\right)dx$$

$$= \int_0^1 \left(-\frac{15x^2}{8} + 5x \right) dx = \left[\frac{-15x^3}{24} + \frac{5}{2}x^2 \right]_0^1 = \frac{-15}{24} + \frac{5}{2} = -0{,}625 + 2{,}5 = 1{,}875 \ m^3$$

$$Q_{x_2} = \iint_{A_2} y\,dx\,dy = \int_1^2 \left(\int_{x^2-1}^{\frac{x}{2}+2} y\,dy \right) dx = \int_1^2 \left[\frac{y^2}{2} \right]_{x^2-1}^{\frac{x}{2}+2} dx = \int_1^2 \left[\frac{(\frac{x}{2}+2)^2}{2} - \frac{(x^2-1)^2}{2} \right] dx =$$

$$= \frac{1}{8} \int_1^2 (x+4)^2 dx - \frac{1}{2} \int_1^2 (x^2-1)^2 dx = \frac{1}{8} \int_1^2 (x^2+8x+16) dx - \frac{1}{2} \int_1^2 (x^4-2x^2+1) dx =$$

$$= \frac{1}{8} \left[\frac{x^3}{3} + 4x^2 + 16x \right]_1^2 - \frac{1}{2} \left[\frac{x^5}{5} - \frac{2}{3}x^3 + x \right]_1^2 = \frac{1}{8}(50{,}67 - 20{,}33) - \frac{1}{2}(3{,}07 - 0{,}53) = \frac{1}{8} \cdot 30{,}34 - \frac{1}{2} \cdot 2{,}54 =$$

$$= 3{,}7925 - 1{,}27 = 2{,}52 \ m^3$$

Άρα $Q_x = 4{,}395 \ m^3$

$$Q_{y_1} = \iint_{A_1} x\,dx\,dy = \int_0^1 x \int_{-2x+2}^{\frac{x}{2}+2} dy\,dx = \int_0^1 x \left[y \right]_{-2x+2}^{\frac{x}{2}+2} dx = \int_0^1 x \left[\left(\frac{x}{2}+2\right) - (-2x+2) \right] dx =$$

$$= \int_0^1 x \left(\frac{x}{2} + 2 + 2x - 2 \right) dx = \int_0^1 x \left(2x + \frac{x}{2} \right) dx = \int_0^1 x \cdot \frac{5x}{2} dx = \left[\frac{5x^3}{6} \right]_0^1 = \frac{5}{6} \ m^3 = 0{,}83 \ m^3$$

$$Q_{y_2} = \iint_{A_2} x\,dx\,dy = \int_1^2 x \int_{x^2-1}^{\frac{x}{2}+2} dy\,dx = \int_1^2 x \left[y \right]_{x^2-1}^{\frac{x}{2}+2} dx = \int_1^2 x \left[\left(\frac{x}{2}+2\right) - (x^2-1) \right] dx =$$

$$= \int_1^2 x \left(\frac{x}{2} + 2 - x^2 + 1 \right) dx = \int_1^2 x \left(-x^2 + \frac{x}{2} + 3 \right) dx = \int_1^2 \left(-x^3 + \frac{x^2}{2} + 3x \right) dx = \left[\frac{-x^4}{4} + \frac{x^3}{6} + \frac{3x^2}{2} \right]_1^2 =$$

$$= 3{,}33 - 1{,}42 = 1{,}91 \ m^3$$

Άρα $Q_y = 2{,}74 \ m^3$

Έτσι $A_1 = \int_0^1 \left(\frac{\cancel{x}}{2} + 2 + 2x - 2\right) dx = \int_0^1 \frac{5x}{2} dx = \left[\frac{5x^2}{4}\right]_0^1 = \frac{5}{4} = 1,25 m^2$

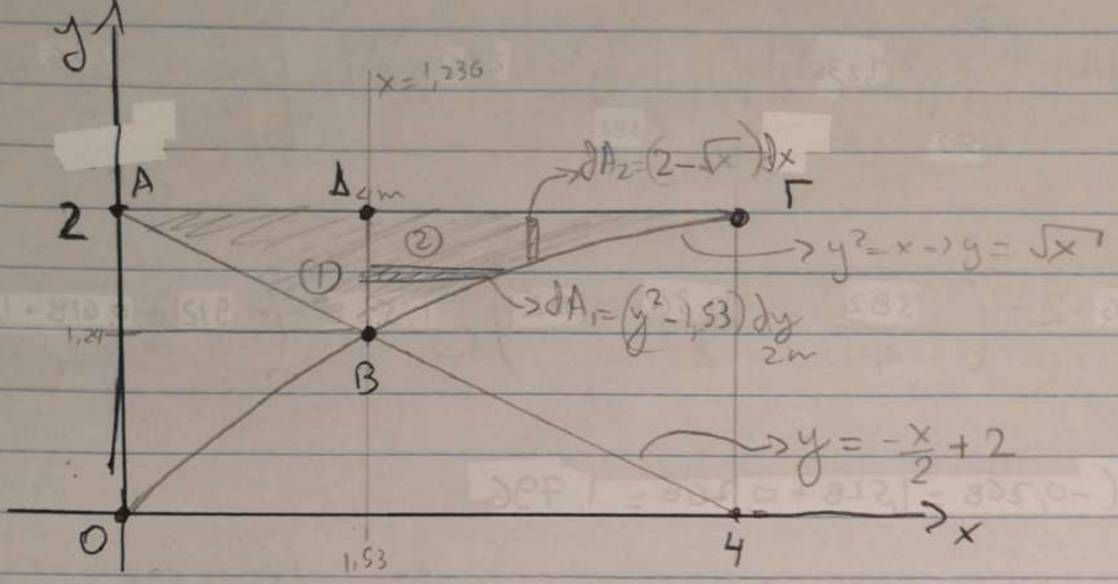
και $A_2 = \int_1^2 \left(\frac{x}{2} + 2 - x^2 + 1\right) dx = \int_1^2 \left(-x^2 + \frac{x}{2} + 3\right) dx = \left[\frac{-x^3}{3} + \frac{x^2}{4} + 3x\right]_1^2 = 4,33 - 2,92 = 1,41 m^2$

Άρα $A = 2,66 m^2$

Άρα $x_c = \frac{Q_y}{A} = \frac{2,74}{2,66} = 1,03 m$

και $y_c = \frac{Q_x}{A} = \frac{4,395}{2,66} = 1,65 m$

$C(1,03, 2,53)$



Έχω: $-\frac{x}{2}+2=\sqrt{x} \Leftrightarrow \frac{x}{2}+\sqrt{x}-2=0 \Leftrightarrow x+2\sqrt{x}-4=0$

$\Delta = 4+16=20 \quad \sqrt{x}_{1,2} = \frac{-2\pm 2\sqrt{5}}{2} \Rightarrow \sqrt{x} = \frac{-2+2\sqrt{5}}{2} = 1,236 \Rightarrow x = 1,53 \Rightarrow y = 1,24$

Άρα $B(1,53\ ,\ 1,24) \Rightarrow \Delta(1,53\ ,\ 2)$

$A_1 = \frac{1,53 \cdot (2-1,24)}{2} = 0,58 \ m^2$

$A_2 = \int_{1,53}^{4} (2-\sqrt{x})dx = \left[2x - \frac{2x^{\frac{3}{2}}}{3}\right]_{1,53}^{4} = \left[2x - \frac{2x\sqrt{x}}{3}\right]_{1,53}^{4} = 8 - \frac{2\cdot 4\cdot 2}{3} - \left(2\cdot 1,53 - \frac{2\cdot 1,53\sqrt{1,53}}{3}\right)$

$= 8 - \frac{16}{3} - \left(3,06 - \frac{3,79}{3}\right) = 2,67 - 3,06 + 1,26 = 0,87 \ m^2$

Άρα $A = 1,45 \ m^2$

Έχω $A(0,2)\ ,\ B(1,53\ ,\ 1,24)\ ,\ \Delta(1,53\ ,\ 2)$

Άρα $C_1 = (1,02\ ,\ 1,75)$

Άρα $Q_{x_1} = A_1 y_{c_1} = 0,58 \cdot 1,75 = 1,02 \ m^3$

$Q_{y_1} = A_1 x_{c_1} = 0,58 \cdot 1,02 = 0,59 \ m^3$

$$Q_{x_2} = \int y\,dA = \int_{1,24}^{2} y(y^2 - 1,53)\,dy = \int_{1,24}^{2} (y^3 - 1,53\,y)\,dy = \left[\frac{y^4}{4} - \frac{1,53}{2}y^2\right]_{1,24}^{2} =$$

$$= 4 - 1,53 \cdot 2 - \left(\frac{1,24^4}{4} - \frac{1,53}{2} \cdot 1,24^2\right) = 0,94 - (0,59 - 1,18) =$$

$$= 0,94 + 0,59 = 1,53\ m^3$$

$$Q_{y_2} = \int x\,dA = \int_{1,53}^{4} [x(2 - \sqrt{x})]\,dx = \int_{1,53}^{4} (2x - x\sqrt{x})\,dx = \int_{1,53}^{4} (2x - x^{\frac{3}{2}})\,dx = \left[x^2 - \frac{2x^{\frac{5}{2}}}{5}\right]_{1,53}^{4} =$$

$$= \left[x^2 - \frac{2x^2\sqrt{x}}{5}\right]_{1,53}^{4} = 16 - \frac{2 \cdot 16 \cdot 2}{5} - \left(1,53^2 - \frac{2(1,53)^2\sqrt{1,53}}{5}\right) =$$

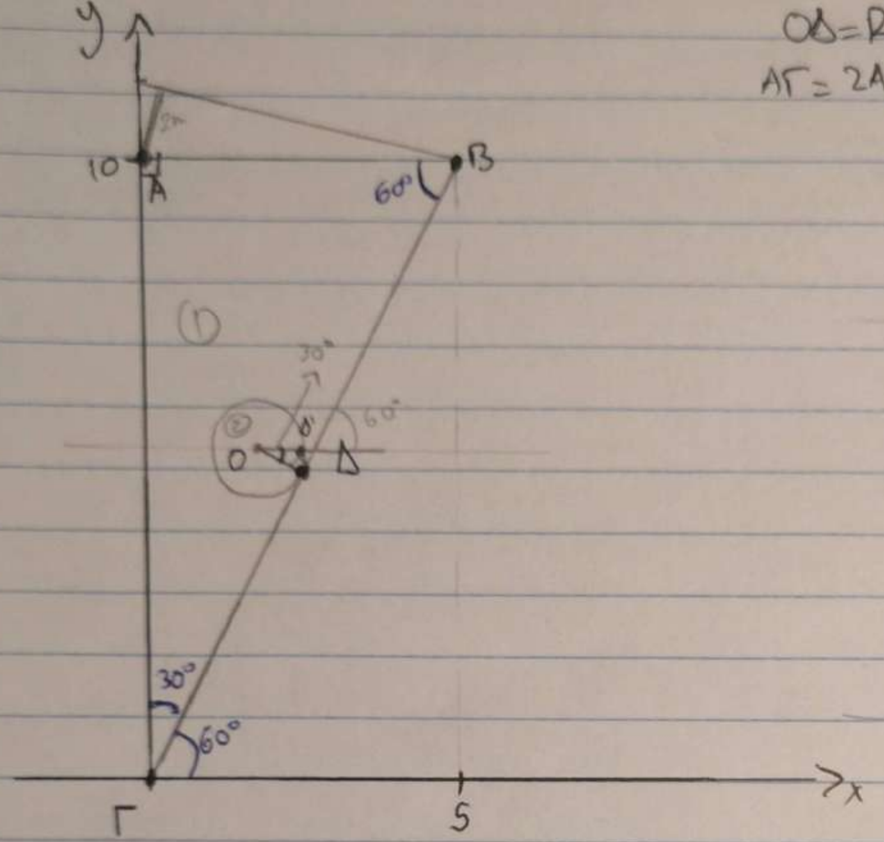
$$= 3,2 - (2,34 - 1,15) = 3,2 - (1,19) = 2,01\ m^3$$

Άρα $x_c = \dfrac{Q_{y_1} + Q_{y_2}}{A_1 + A_2} = \dfrac{0,59 + 2,01}{1,45} = \dfrac{2,6}{1,45} = 1,793\ m$

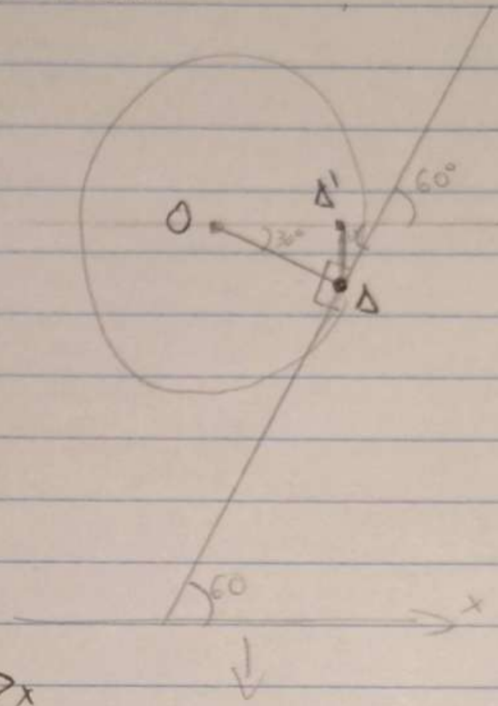
και $y_c = \dfrac{Q_{x_1} + Q_{x_2}}{A} = \dfrac{1,02 + 1,53}{1,45} = \dfrac{2,55}{1,45} = 1,759$

Άρα $C(1,793\,,\ 1,759)$

Άσκηση 2



OΔ = R = 0,7m
AΓ = 2AB = 10m



Αφού $A\Gamma = 2AB$ έχω $A\hat{\Gamma}B = 30° \Rightarrow \Gamma\hat{B}A = 60°$

Έχω
$A(0,10)$
$B(5,10)$
$\Gamma(0,0)$
$\Delta(2{,}5, 5)$

$G\left(\frac{5}{3}, \frac{20}{3}\right) = (1{,}667,\ 6{,}667$

$\sin(30°) = \frac{\Delta\Delta'}{O\Delta} \Leftrightarrow \Delta\Delta' = \frac{1}{2}R = \frac{1}{2}\cdot 0{,}7 = 0{,}35m$

$\cos(30°) = \frac{O\Delta'}{O\Delta} \Leftrightarrow O\Delta' = \frac{\sqrt{3}}{2}\cdot 0{,}7 = \sqrt{3}\cdot 0{,}35 = 0{,}61$

Άρα $O(2{,}5-0{,}61,\ 5+0{,}35) = (1{,}89,\ 5{,}35)$

Έχω $O(1{,}89,\ 5{,}35)$, $A_2 = \pi R^2 = \pi(0{,}7)^2 = 1{,}54\ m^2$ και $A_1 = \frac{5\cdot 10}{2} = 25\ m^2$

Άρα $A = A_1 - A_2 = 23{,}46$

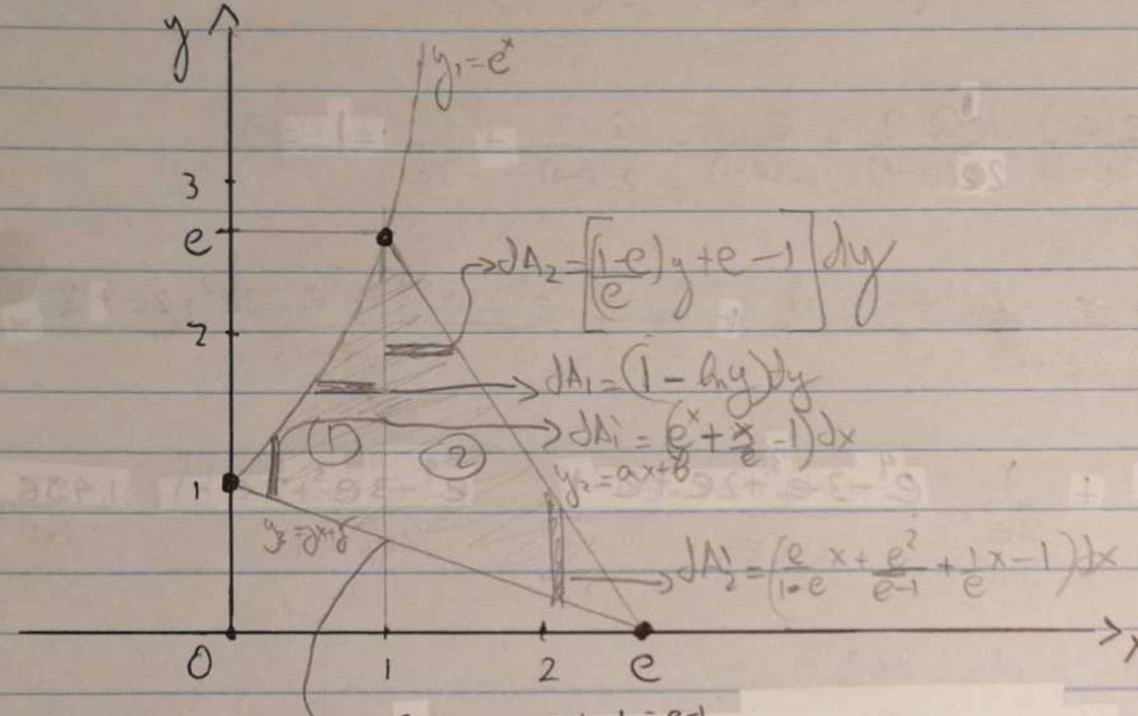
Άρα $Q_{x_2} = y_{c_2} A_2 = 5{,}35 \cdot 1{,}54 = 8{,}24\ m^3$ | $Q_{y_1} = x_1 A_1 = 1{,}667 \cdot 25 = 41{,}675\ m^3$

$Q_{y_2} = x_{c_2} A_2 = 1{,}89 \cdot 1{,}54 = 2{,}91\ m^3$ | $Q_{x_1} = y_1 A_1 = 6{,}667 \cdot 25 = 166{,}675\ m^3$

Άρα $x_c = \frac{Q_{y_1} - Q_{y_2}}{A} = \frac{41{,}675 - 2{,}91}{23{,}46} = \frac{38{,}765}{23{,}46} = 1{,}65\ m$ και $y_c = \frac{Q_{x_1} - Q_{x_2}}{A} = \frac{158{,}435}{23{,}46} = 6{,}75\ m$

Άρα $G(1{,}65,\ 6{,}75)$

Άσκηση 3



Για $x=1$: $y_3=1-\frac{1}{e}=\frac{e-1}{e}$

Για $y_2=0$ έχω $x=e$: $0=ae+b$
Για $y_2=e$ έχω $x=1$: $e=a+b$

$\Leftrightarrow \begin{cases} a=-\frac{b}{e} \\ a=e-b \end{cases} \Leftrightarrow \begin{cases} e-b=-\frac{b}{e} \\ a=e-b \end{cases} \Leftrightarrow \begin{cases} e^2-be=-b \\ a=e-b \end{cases} \Leftrightarrow \begin{cases} b(e-1)=e^2 \\ a=e-b \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} b=\frac{e^2}{e-1} \\ a=e-\frac{e^2}{e-1} \end{cases} \Leftrightarrow \begin{cases} b=\frac{e^2}{e-1} \\ \frac{e(e-1)-e^2}{e-1}=a \end{cases} \Leftrightarrow \begin{cases} b=\frac{e^2}{e-1} \\ a=\frac{e^2-e-e^2}{e-1} \end{cases} \Leftrightarrow \begin{cases} b=\frac{e^2}{e-1} \\ a=\frac{-e}{e-1} \end{cases} \Leftrightarrow \begin{cases} b=\frac{e^2}{e-1} \\ a=\frac{e}{1-e} \end{cases}$

Άρα $\boxed{y_2=\frac{e}{1-e}x+\frac{e^2}{e-1}} \Leftrightarrow y_2-\frac{e^2}{e-1}=\frac{e}{1-e}x \Leftrightarrow x=\left(y_2-\frac{e^2}{e-1}\right)\cdot\frac{1-e}{e}=\frac{(1-e)}{e}y+e$

Για $y_3=1$ έχω $x=0$: $1=\delta$
Για $y_3=0$ έχω $x=e$: $0=\gamma e+1 \Leftrightarrow \gamma=-\frac{1}{e}$

$\Rightarrow \boxed{y_3=-\frac{1}{e}x+1}$

$$A_1=\int_0^1\left(e^x+\frac{1}{e}x-1\right)dx=\left[e^x+\frac{x^2}{2e}-x\right]_0^1=e+\frac{1}{2e}-1-1=\frac{2e^2+1-4e}{2e}=\frac{2e^2-4e+1}{2e}=0{,}9m^2$$

$$A_2=\int_1^e\left(\frac{e}{1-e}x+\frac{e^2}{e-1}+\frac{1}{e}x-1\right)dx=\left[\frac{ex^2}{2(1-e)}+\frac{e^2x}{e-1}+\frac{x^2}{2e}-x\right]_1^e =$$

$$= \frac{e^3}{2(1-e)} + \frac{e^3}{e-1} + \frac{e}{2} - e - \left(\frac{e}{2(1-e)} + \frac{e^2}{e-1} + \frac{1}{2e} - 1 \right) =$$

$$= \frac{-e^3}{2(e-1)} + \frac{2e^3}{2(e-1)} + \frac{e(e-1)}{2(e-1)} - \frac{2e(e-1)}{2(e-1)} + \frac{e}{2(e-1)} - \frac{2e^2}{2(e-1)} - \frac{1}{2e} + \frac{2(e-1)}{2(e-1)} =$$

$$= \frac{e^3 - e(e-1) + e - 2e^2 + 2(e-1)}{2(e-1)} - \frac{1}{2e} = \frac{e^3 - e^2 + e + e - 2e^2 + 2e - 2}{2(e-1)} - \frac{1}{2e} =$$

$$= \frac{e^3 - 3e^2 + 4e - 2}{2(e-1)} - \frac{1}{2e} = \frac{e^4 - 3e^3 + 4e^2 - 2e - e + 1}{2e(e-1)} = \frac{e^4 - 3e^3 + 4e^2 - 3e + 1}{2e(e-1)} = \frac{16,74}{9,341} =$$

$= 1,79 \ m^2 \quad \Rightarrow A = A_1 + A_2 = 2,69$

$$Q_{x_1} = \int y \, dA_1 = \int_{\frac{e-1}{e}}^{e} [y(1 - \ln y)] dy = \int_{\frac{e-1}{e}}^{e} (y - y \ln y) dy = \int_{\frac{e-1}{e}}^{e} y \, dy - \int_{\frac{e-1}{e}}^{e} y \ln y \, dy =$$

$$= \left[\frac{y^2}{2} \right]_{\frac{e-1}{e}}^{e} - \int_{\frac{e-1}{e}}^{e} y \ln y \, dy$$

Έχω $I = \int_{\frac{e-1}{e}}^{e} y \ln y \, dy = \int_{\frac{e-1}{e}}^{e} \frac{1}{2}(y^2)' \ln y \, dy = \left[\frac{y^2}{2} \ln y \right]_{\frac{e-1}{e}}^{e} - \int_{\frac{e-1}{e}}^{e} \frac{y^2}{2} \cdot \frac{1}{y} dy = \left[\frac{y^2}{2} \ln y \right]_{\frac{e-1}{e}}^{e} - \left[\frac{y^2}{4} \right]_{\frac{e-1}{e}}^{e}$

Άρα $Q_{x_1} = \left[\frac{3y^2}{4} \right]_{\frac{e-1}{e}}^{e} - \left[\frac{y^2}{2} \ln y \right]_{\frac{e-1}{e}}^{e} = \left[\frac{3y^2}{4} - \frac{y^2}{2} \ln y \right]_{\frac{e-1}{e}}^{e} =$

$$= \frac{3e^2}{4} - \frac{e^2}{2} \ln e - \left(\frac{3(e-1)^2}{4e^2} - \frac{(e-1)^2}{2e^2} \ln\left(\frac{e-1}{e}\right) \right) = 1,85 - 0,29 - 0,2 \cdot 0,46 = 1,56 - 0,09 = 1,47 \ m^3$$

$$Q_{y_1} = \int x\, dA_1' = \int_0^1 \left[x\left(e^x + \frac{x}{e} - 1\right)\right] dx = \int_0^1 \left(xe^x + \frac{x^2}{e} - x\right) dx =$$

$$= \left[xe^x - e^x + \frac{x^3}{3e} - \frac{x^2}{2}\right]_0^1 = e - e + \frac{1}{3e} - \frac{1}{2} - (0 - 1 + 0 - 0) = \frac{1}{3e} - \frac{1}{2} + 1 = \frac{1}{3e} + \frac{1}{2} = 0{,}62$$

$$Q_{x_2} = \int y\, dA_2 = \int_0^e \left[y\left[\left(\frac{1-e}{e}\right)y + e - 1\right]\right] dy = \int_0^e \left[\left(\frac{1-e}{e}\right)y^2 + ey - y\right] dy =$$

$$= \left[\frac{y^3(1-e)}{3e} + \frac{e}{2}y^2 - \frac{1}{2}y^2\right]_0^e = \frac{e^{\cancel{3}2}(1-e)}{3\cancel{e}} + \frac{e^3}{2} - \frac{e^2}{2} = \frac{e^2 - e^3}{3} + \frac{e^3 - e^2}{2} =$$

$$= \frac{2e^2 - 2e^3}{6} + \frac{3e^3 - 3e^2}{6} = \frac{e^3 - e^2}{6} = 2{,}12$$

$$Q_{y_2} = \int x\, dA_2' = \int_1^e \left[x\left(\frac{ex}{1-e} + \frac{e^2}{e-1} + \frac{x}{e} - 1\right)\right] dx = \int_1^e \left(\frac{ex^2}{1-e} + \frac{e^2 x}{e-1} + \frac{x^2}{e} - x\right) dx =$$

$$= \left[\frac{ex^3}{3(1-e)} + \frac{e^2x^2}{2(e-1)} + \frac{x^3}{3e} - \frac{x^2}{2}\right]_1^e = \frac{e^4}{3(1-e)} + \frac{e^4}{2(e-1)} + \frac{e^3}{3e} - \frac{e^2}{2} - \left(\frac{e}{3(1-e)} + \frac{e^2}{2(e-1)} + \frac{1}{3e} - \frac{1}{2}\right)$$

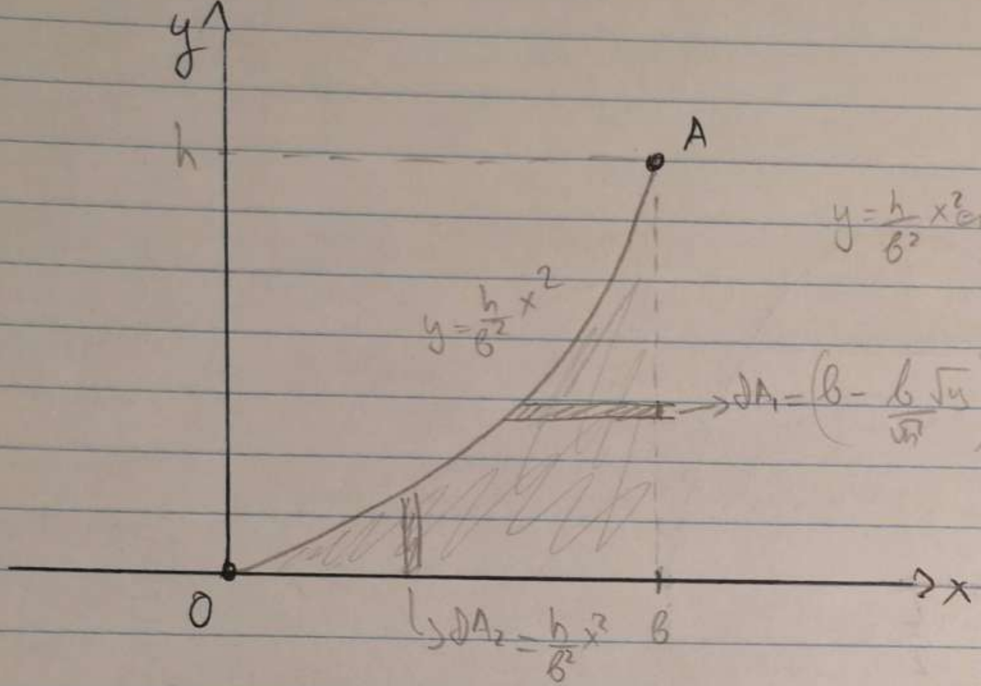
$$= -10{,}59 + 15{,}89 + 2{,}46 - 3{,}69 - (-0{,}53 + 2{,}15 + 0{,}12 - 0{,}5) = 4{,}07 - (1{,}24) = 2{,}83$$

Άρα $x_c = \dfrac{Q_{y_1} + Q_{y_2}}{A} = \dfrac{0{,}62 + 2{,}83}{2{,}69} = \dfrac{3{,}45}{2{,}69} = 1{,}28\ m$

και $y_c = \dfrac{Q_{x_1} + Q_{x_2}}{A} = \dfrac{2{,}12 + 1{,}47}{2{,}69} = \dfrac{3{,}59}{2{,}69} = 1{,}33$

Άρα $G(1{,}28,\ 1{,}33)$

Άσκηση 4



$y = \frac{h}{b^2}x^2 \Leftrightarrow \sqrt{\frac{y b^2}{h}} = x \Leftrightarrow x = b\sqrt{\frac{y}{h}}$

Έχω για $x = b \Rightarrow y = h$: $h = \frac{h \cdot b^2}{b^2} \Leftrightarrow 1 = \frac{1\cdot 1}{1}$,

$$A = \int_0^b \frac{h}{b^2}x^2\,dx = \left[\frac{h x^3}{3b^2}\right]_0^b = \frac{hb^3}{3b^2} = \frac{hb}{3}$$

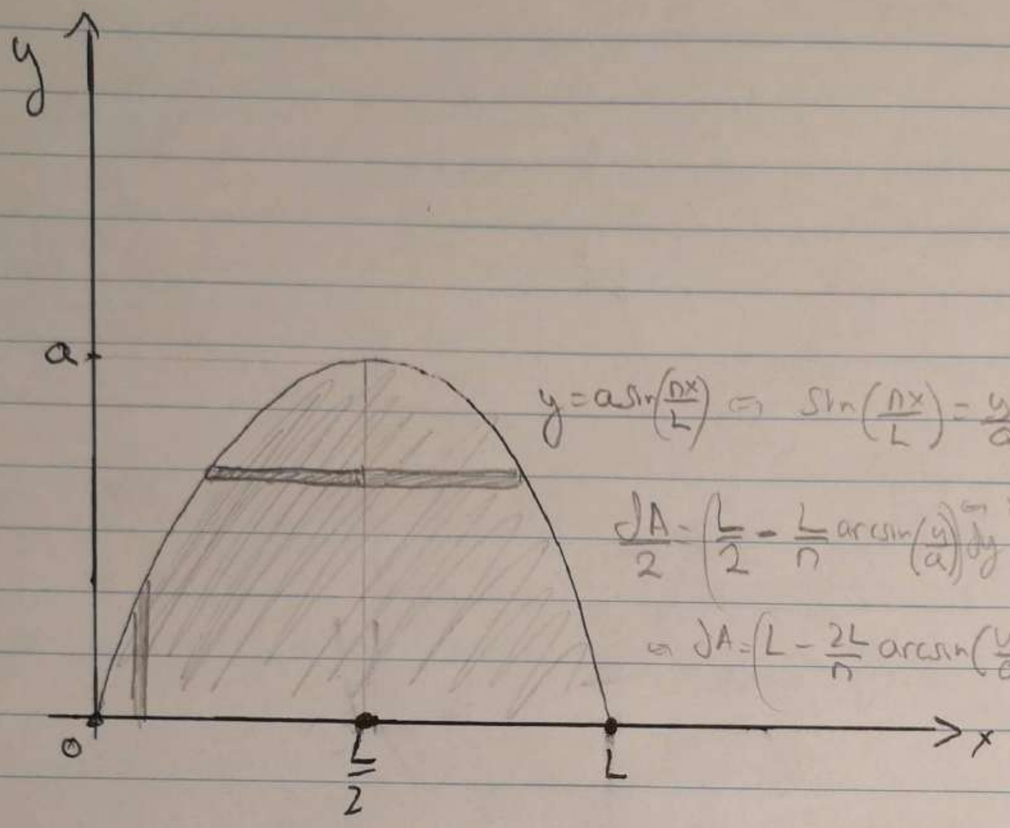
$$Q_x = \int y\,dA_1 = \int_0^h \left[y\left(b - \frac{b}{\sqrt{h}}\sqrt{y}\right)\right]dy = \int_0^h \left(yb - \frac{b y\sqrt{y}}{\sqrt{h}}\right)dy = \left[\frac{b y^2}{2} - \frac{b}{\sqrt{h}}\,\frac{2}{5}\,y^{\frac{5}{2}}\right]_0^h =$$

$$= \left[\frac{b}{2}y^2 - \frac{2b}{5\sqrt{h}}y^2\sqrt{y}\right]_0^h = \frac{bh^2}{2} - \frac{2b\,h^2\sqrt{h}}{5\sqrt{h}} = \frac{bh^2}{2} - \frac{2bh^2}{5} = \frac{5bh^2 - 4bh^2}{10} = \frac{bh^2}{10}$$

$$Q_y = \int x\,dA = \int_0^b \frac{h}{b^2}x^3\,dx = \left[\frac{h x^4}{4b^2}\right]_0^b = \frac{hb^4}{4b^2} = \frac{hb^2}{4}$$

Άρα $x_c = \frac{Q_y}{A} = \frac{\frac{hb^2}{4}}{\frac{hb}{3}} = \frac{3hb^2}{4hb} = \frac{3}{4}b$ και $y_c = \frac{Q_x}{A} = \frac{\frac{bh^2}{10}}{\frac{hb}{3}} = \frac{3bh^2}{10bh} = \frac{3}{10}h$

Άρα $G\left(\frac{3h}{10}, \frac{3b}{4}\right)$



$$y = a\sin\left(\frac{\pi x}{L}\right) \Leftrightarrow \sin\left(\frac{\pi x}{L}\right) = \frac{y}{a} \Leftrightarrow \frac{\pi x}{L} = \arcsin\left(\frac{y}{a}\right) \Leftrightarrow x = \frac{L}{\pi}\arcsin\left(\frac{y}{a}\right)$$

$$\frac{dA}{2} = \left(\frac{L}{2} - \frac{L}{\pi}\arcsin\left(\frac{y}{a}\right)\right)dy$$

$$\Leftrightarrow dA = \left(L - \frac{2L}{\pi}\arcsin\left(\frac{y}{a}\right)\right)dy$$

$$A = \int_0^L a\sin\left(\frac{\pi x}{L}\right)dx = a\int_0^L \sin\left(\frac{\pi x}{L}\right)dx = a\left[-\frac{L}{\pi}\cos\left(\frac{\pi x}{L}\right)\right]_0^L = -\frac{aL}{\pi}\left[\cos\left(\frac{\pi x}{L}\right)\right]_0^L = -\frac{aL}{\pi}(-2) = \frac{2aL}{\pi}$$

$$Q_x = \int y\,dA = \int_0^a \left[y\left(L - \frac{2L}{\pi}\arcsin\left(\frac{y}{a}\right)\right)\right]dy$$

Θέτω $u = \frac{y}{a}$, $du = \frac{dy}{a} \Rightarrow u_1 = 0,\ u_2 = 1$

Άρα $Q_x = \int_0^1 \left[a^2u\left(L - \frac{2L}{\pi}\arcsin u\right)\right]du = \int_0^1 \left(a^2uL - \frac{2L}{\pi}\cdot a^2u\arcsin u\right)du =$

$$= \int_0^1 a^2Lu\,du - \frac{2L}{\pi}a^2\int_0^1 u\arcsin u\,du = \left[\frac{a^2L}{2}u^2\right]_0^1 - \frac{2La^2}{\pi}\int_0^1 u\arcsin u\,du =$$

$$= \frac{a^2L}{2} - \frac{2La^2}{\pi}\int_0^1 u\arcsin u\,du$$

Έχω $I = \int_0^1 u \arcsin u \, du$.

Έστω $f(x) = \arcsin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Ισχύει ότι $\sin(f(x)) = x \Rightarrow (\sin(f(x)))' = (x)' \Leftrightarrow \cos(f(x)) \cdot f'(x) = 1 \Rightarrow$

$\Rightarrow f'(x) = \frac{1}{\cos f(x)} = \frac{1}{\cos(\arcsin x)}$

Ισχύει η ταυτότητα: $\cos a = \sqrt{1 - \sin^2(a)}$

Άρα έχω $f'(x) = \frac{1}{\sqrt{1 - (\cos(\arcsin x))^2}} = \frac{1}{\sqrt{1 - x^2}}$

Άρα $I = \int_0^1 u \arcsin u \, du = \left[\frac{u^2}{2} \arcsin u \, du\right]_0^1 - \int_0^1 \frac{u^2}{2} \cdot \frac{1}{\sqrt{1-u^2}} du = \frac{\pi}{4} - \int_0^1 \frac{u^2}{2} \cdot \frac{1}{\sqrt{1-u^2}} du$

Θέτω $u = \sin w \Rightarrow du = \cos w \, dw$ Τα νέα όρια: $w_1 = 0$, $w_2 = \frac{\pi}{2}$

Άρα $I = \frac{\pi}{4} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^2 w}{\sqrt{1 - \sin^2 w}} \cdot \cos w \, dw = \frac{\pi}{4} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^2 w}{\cancel{\cos w}} \cancel{\cos w} \, dw = \frac{\pi}{4} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 w \, dw$

$= \frac{\pi}{4} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2w}{2} dw = \frac{\pi}{4} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{\cos 2w}{2} dw = \frac{\pi}{4} - \frac{1}{2} \left[\frac{w}{2} - \frac{\sin 2w}{4}\right]_0^{\frac{\pi}{2}} =$

$= \frac{\pi}{4} - \frac{1}{2} \left(\frac{\pi}{4}\right) = \frac{\pi}{8}$

Άρα $Q_x = \frac{a^2 L}{2} - \frac{2 L a^2}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{8} = \frac{a^2 L}{2} - \frac{2 a^2 L}{8} = \frac{a^2 L}{2} - \frac{a^2 L}{4} = \frac{a^2 L}{4}$

$$Q_y = \int x \, dA = \int_0^L x \cdot a\sin\left(\frac{\pi x}{L}\right) dx$$

Θέτω $u = \frac{\pi x}{L} \Rightarrow du = \frac{\pi}{L} dx$ άρα $dx = \frac{L}{\pi} du$, Τα νέα όρια $u_1 = 0$, $u_2 = \pi$

Άρα $Q_y = \int_0^\pi \frac{L}{\pi} u \cdot a\sin(u) \cdot \frac{L}{\pi} du = \int_0^\pi \frac{aL^2}{\pi^2} u\sin u \, du =$

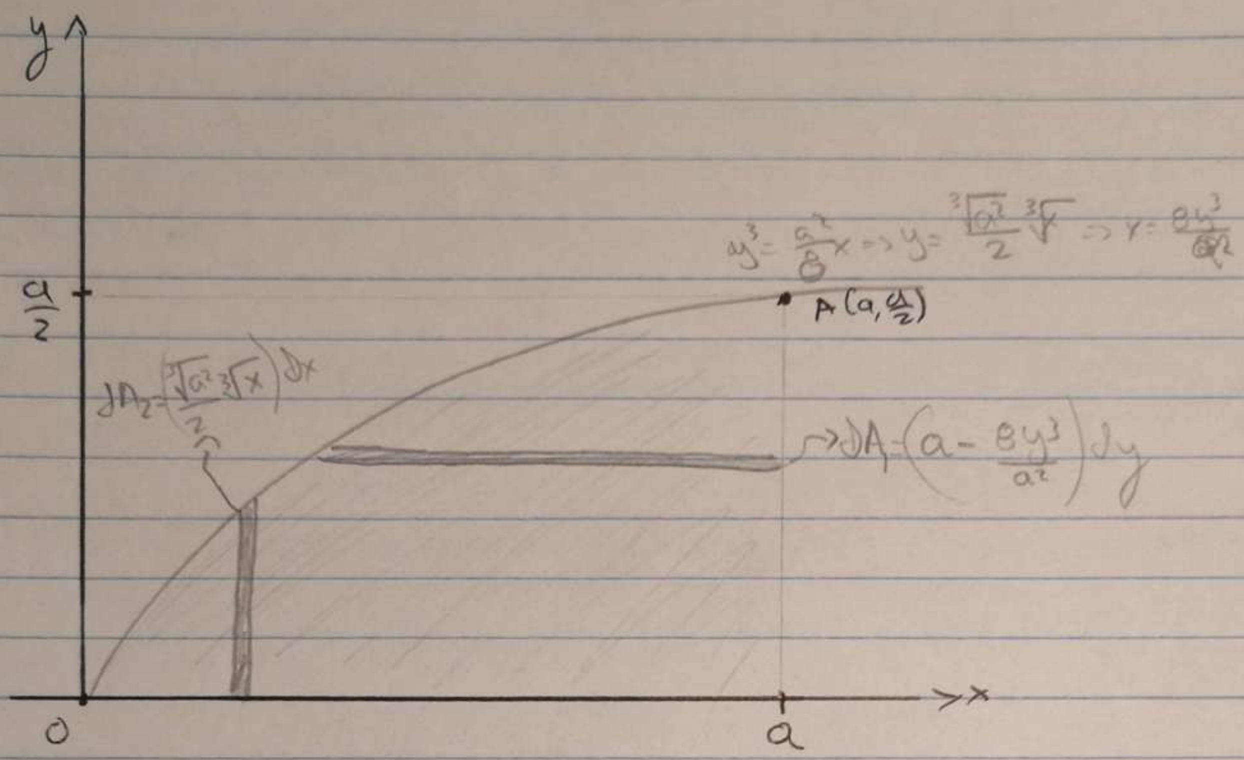
$$= \frac{aL^2}{\pi^2} \int_0^\pi u\sin u \, du = \frac{aL^2}{\pi^2} = \left[\left[-u\cos u\right]_0^\pi - \int_0^\pi -\cos u \, du\right] \frac{aL^2}{\pi^2} =$$

$$= \frac{aL^2}{\pi^2}\left(-\pi \cdot (-1) + [\sin u]_0^\pi\right) = \frac{aL^2}{\pi^2} \cdot \pi = \frac{aL^3}{\pi}$$

Άρα $x_c = \frac{Q_y}{A} = \frac{\frac{aL^{\cancel{3}}}{\cancel{\pi}}}{\frac{2aL}{\cancel{\pi}}} = \frac{L}{2}$

και $y_c = \frac{Q_x}{A} = \frac{\frac{\cancel{a}^2 L}{4}}{\frac{2aL}{\pi}} = \frac{a\pi}{8}$

$$C\left(\frac{L}{2}, \frac{\pi a}{8}\right)$$



$$A = \int_0^a \frac{\sqrt[3]{a^2}}{2} x^{\frac{1}{3}}\, dx = \left[\frac{\sqrt[3]{a^2}}{2}\frac{3}{4}x^{\frac{4}{3}}\right]_0^a = \left[\frac{3a^{\frac{2}{3}}}{8}x^{\frac{4}{3}}\right] = \frac{3}{8}a^{\frac{2}{3}}a^{\frac{4}{3}} = \frac{3a^2}{8}$$

$$Q_x = \int y\, dA_1 = \int_0^{\frac{a}{2}} \left[y\left(a - \frac{8y^3}{a^2}\right)\right]dy = \int_0^{\frac{a}{2}}\left(ay - \frac{8y^4}{a^2}\right)dy = \left[\frac{a}{2}y^2 - \frac{8}{5a^2}y^5\right]_0^{\frac{a}{2}} =$$

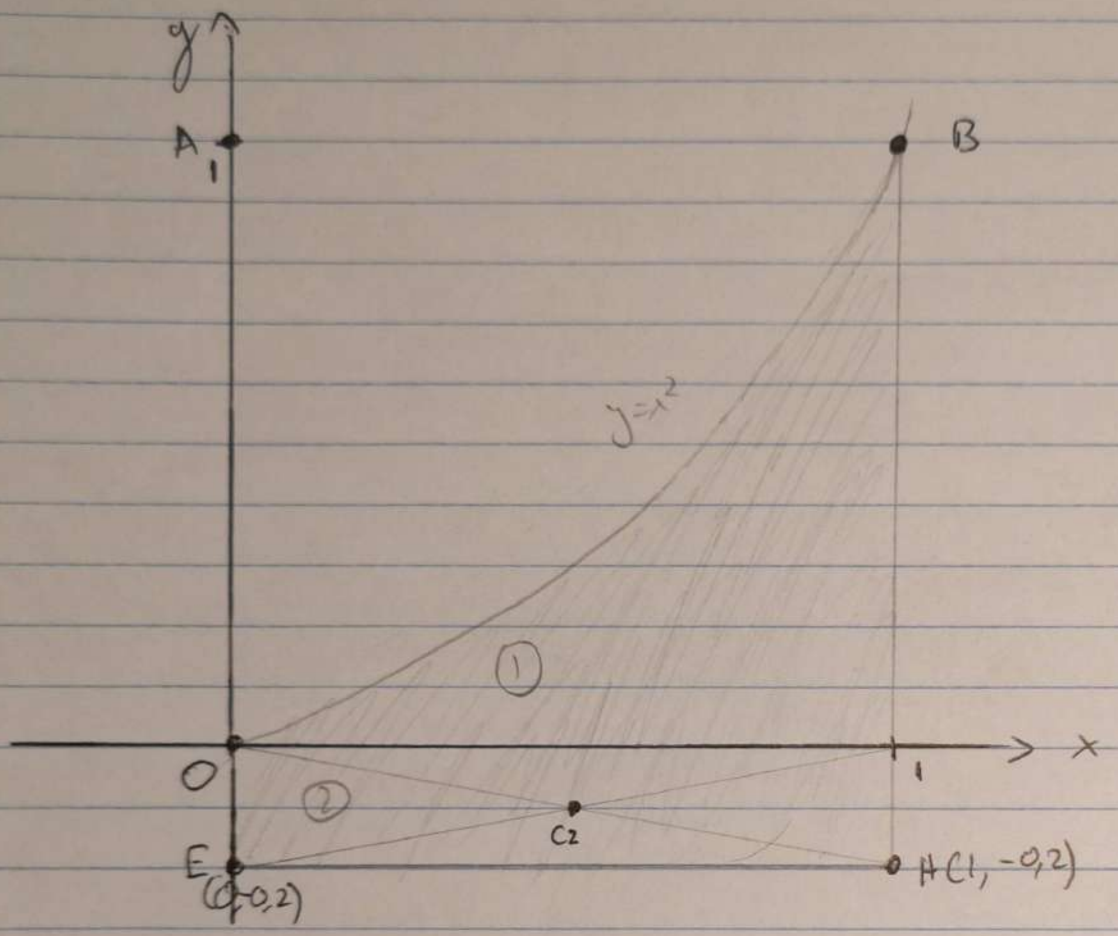
$$= \frac{a}{2}\cdot\frac{a^2}{4} - \frac{8}{5a^2}\frac{a^5}{32} = \frac{a^3}{8} - \frac{a^3}{20} = \frac{5a^3 - 2a^3}{40} = \frac{3a^3}{40}$$

$$Q_y = \int x\, dA_2 = \int_0^a x\cdot\frac{\sqrt[3]{a^2}\sqrt[3]{x}}{2}\, dx = \int_0^a \frac{x^{\frac{4}{3}}\cdot a^{\frac{2}{3}}}{2}\, dx = \left[\frac{3}{7}\cdot\frac{a^{\frac{2}{3}}}{2}\cdot x^{\frac{7}{3}}\right]_0^a = \frac{3}{14}\cdot a^{\frac{2}{3}}\cdot a^{\frac{7}{3}} =$$

$$= \frac{3a^3}{14}$$

Άρα $x_c = \frac{Q_y}{A} = \frac{\frac{3a^3}{14}}{\frac{3a^2}{8}} = \frac{8a}{14} = \frac{4a}{7}$ m και $y_c = \frac{Q_x}{A} = \frac{\frac{3a^3}{40}}{\frac{3a^2}{8}} = \frac{8a}{40} = \frac{a}{5}$ m Άρα $C\left(\frac{4a}{7}, \frac{a}{5}\right)$

Άσκηση 5



Για την εξίσωση $y=\frac{h}{b^2}x^2$ έχω, από άσκηση 4 ότι $Q_{x_1}=\frac{bh^2}{10}$ και $Q_y=\frac{hb^2}{4}$

Άρα εδώ, για $h=b=1$ έχω $Q_{x_1}=\frac{1}{10}=0{,}1\ m^3$ και $Q_y=\frac{1}{4}=0{,}25\ m^3$

Επίσης έχω: $O(0,0)$ και $H(1,-0{,}2)$ άρα $C_2(0{,}5,-0{,}1) \Rightarrow x_2=0{,}5$, $y_2=-0{,}1$

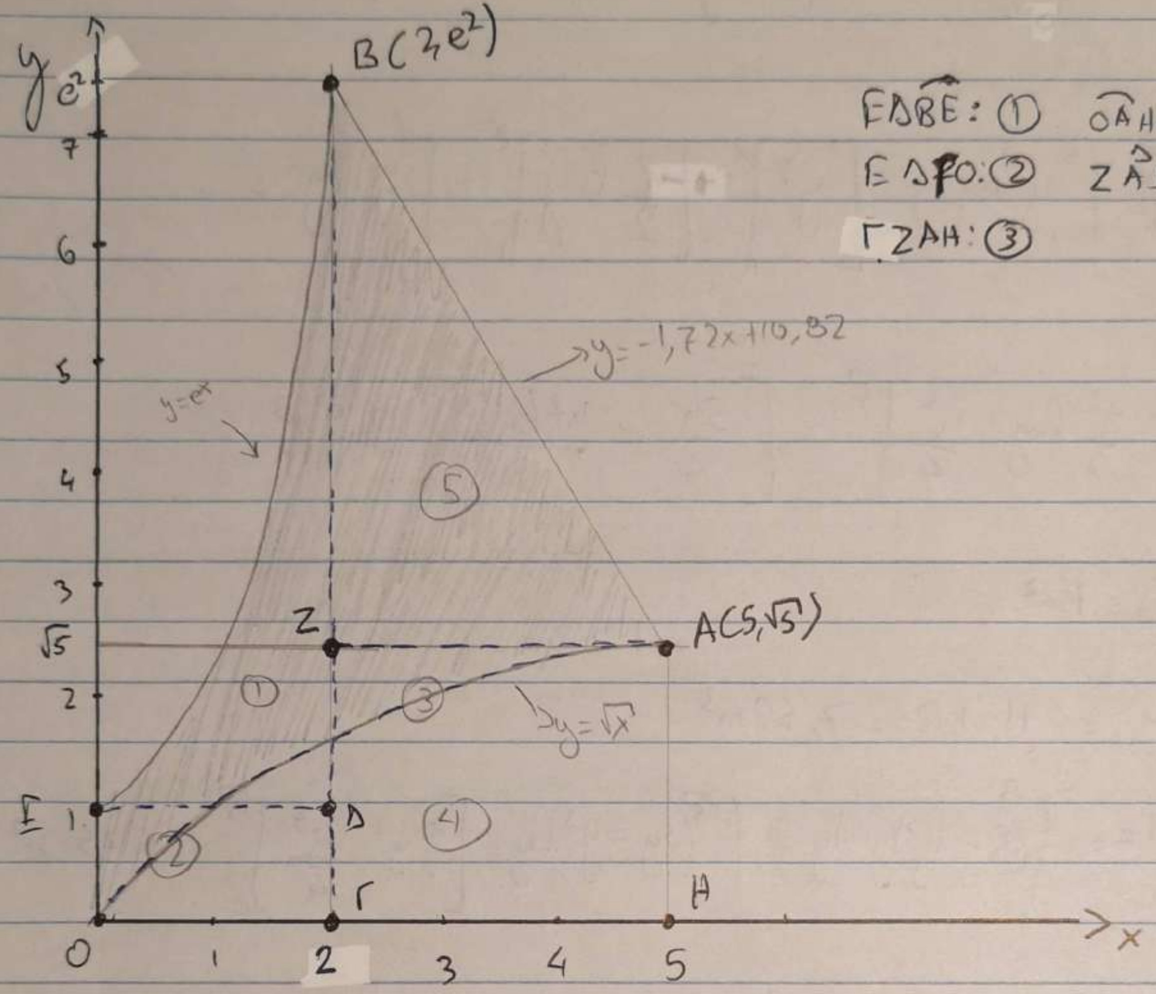
$$A_1=\int_0^1 x^2dx=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}m^2,\ A_2=0{,}2m^2 \Rightarrow A=0{,}53m^2$$

$$y_c=\frac{Q_{x_1}+Q_{x_2}}{A_1+A_2}=\frac{Q_{x_1}+A_2y_{c2}}{A}=\frac{0{,}1-0{,}2\cdot0{,}1}{0{,}53}=\frac{0{,}08}{0{,}53}=0{,}15\ m$$

$$x_c=\frac{Q_y+Q_{y_2}}{A}=\frac{0{,}25+0{,}2\cdot0{,}5}{0{,}53}=\frac{0{,}35}{0{,}53}=0{,}66\ m$$

$C(0{,}66,\ 0{,}15)$

Άσκηση 6



ΕΔΒΕ: ① ΟΑΗΟ: ④
ΕΔΓΟ: ② ΖΑΒ: ⑤
ΓΖΑΗ: ③

$$A = \int_0^2 (e^x - \sqrt{x})\,dx + \int_2^5 (-1{,}72x + 10{,}82 - \sqrt{x})\,dx =$$

$$= \left[e^x - \frac{2}{3} x\sqrt{x} \right]_0^2 + \left[\frac{-1{,}72x^2}{2} + 10{,}82x - \frac{2}{3} x\sqrt{x} \right]_2^5 = 4{,}5 + 8{,}83 = 13{,}33\,m^2$$

$A_2 = 2m^2$, $A_3 = 3 \cdot \sqrt{5} = 6{,}71 m^2$, $A_5 = \frac{3 \cdot (e^2 - \sqrt{5})}{2} = 1{,}5\,(e^2 - \sqrt{5}) = 7{,}73\, m^2$

Έχω $Δ(2,1)$ άρα $C_2(1, 0{,}5)$

και $Γ(2,0)$ και $A(5,\sqrt{5})$ άρα $C_3(3{,}5,\ 1{,}12)$

και $Z(2,\sqrt{5})$, $A(5,\sqrt{5})$ και $B(2,e^2)$ άρα $C_5(3,\ 3{,}95)$

$$Q_{x_1} = \int y\,dA = \int_1^{e^2} \left[y\,(2-\ln y)\right] dy = \int_1^{e^2} (2y - y\ln y)\,dy =$$

$$= \int_1^{e^2} 2y\,dy - \int_1^{e^2} y\ln y\,dy = \left[y^2\right]_1^{e^2} - \left[\frac{y^2}{2}\ln y\right]_1^{e^2} + \int_1^{e^2} \frac{y}{2}\,dy =$$

$$= \left[y^2 - \frac{y^2}{2}\ln y + \frac{y^2}{4}\right]_1^{e^2} = \left[\frac{5y^2}{4} - \frac{y^2\ln y}{2}\right] = 12,4\,m^3$$

$$Q_{x_2} = A_2\,y_{c_2} = 1\,m^3$$

$$Q_{x_3} = A_3\,y_{c_3} = 6,71 \cdot 1,12 = 7,52\,m^3$$

$$Q_{x_4} = \int y\,dA = \int_0^{\sqrt{5}} (5-y^2)\,y\,dy = \int_0^{\sqrt{5}} (5y - y^3)\,dy = \left[\frac{5}{2}y^2 - \frac{y^4}{4}\right]_0^{\sqrt{5}} = 6,25\,m^3$$

$$Q_{x_5} = A_5\,y_{c_5} = 7,73 \cdot 3,95 = 30,53\,m^3$$

Άρα $Q_x = Q_{x_1} + Q_{x_2} + Q_{x_3} - Q_{x_4} + Q_{x_5} = 45,2\,m^3$

$$Q_{y_1} = \int x\,dA = \int_0^{2} \left[x\,(e^x - 1)\right] dx = \int_0^2 (xe^x - x)\,dx = \int_0^2 xe^x\,dx - \int_0^2 x\,dx =$$

$$= \left[xe^x\right]_0^2 - \int_0^2 e^x\,dx - \left[\frac{x^2}{2}\right]_0^2 = \left[xe^x - e^x - \frac{x^2}{2}\right]_0^2 = 6,39\,m^3$$

$$Q_{y_2} = A_2\,x_{c_2} = 2\,m^3$$

$$Q_{y_3} = A_3\,x_{c_3} = 6,71 \cdot 3,5 = 23,49$$

$$Q_{y_4} = \int x\,dA = \int_0^5 x\sqrt{x}\,dx = \int_0^5 x^{\frac{3}{2}}\,dx = \left[\frac{2x^{\frac{5}{2}}}{5}\right]_0^5 = \left[\frac{2x^2\sqrt{x}}{5}\right]_0^5 = \frac{2\cdot 5\cdot \cancel{5}\sqrt{5}}{\cancel{5}} = 10\sqrt{5} = 22,36\ m^3$$

$$Q_{y_5} = A_5\, x_{c5} = 7,73\cdot 3 = 23,19$$

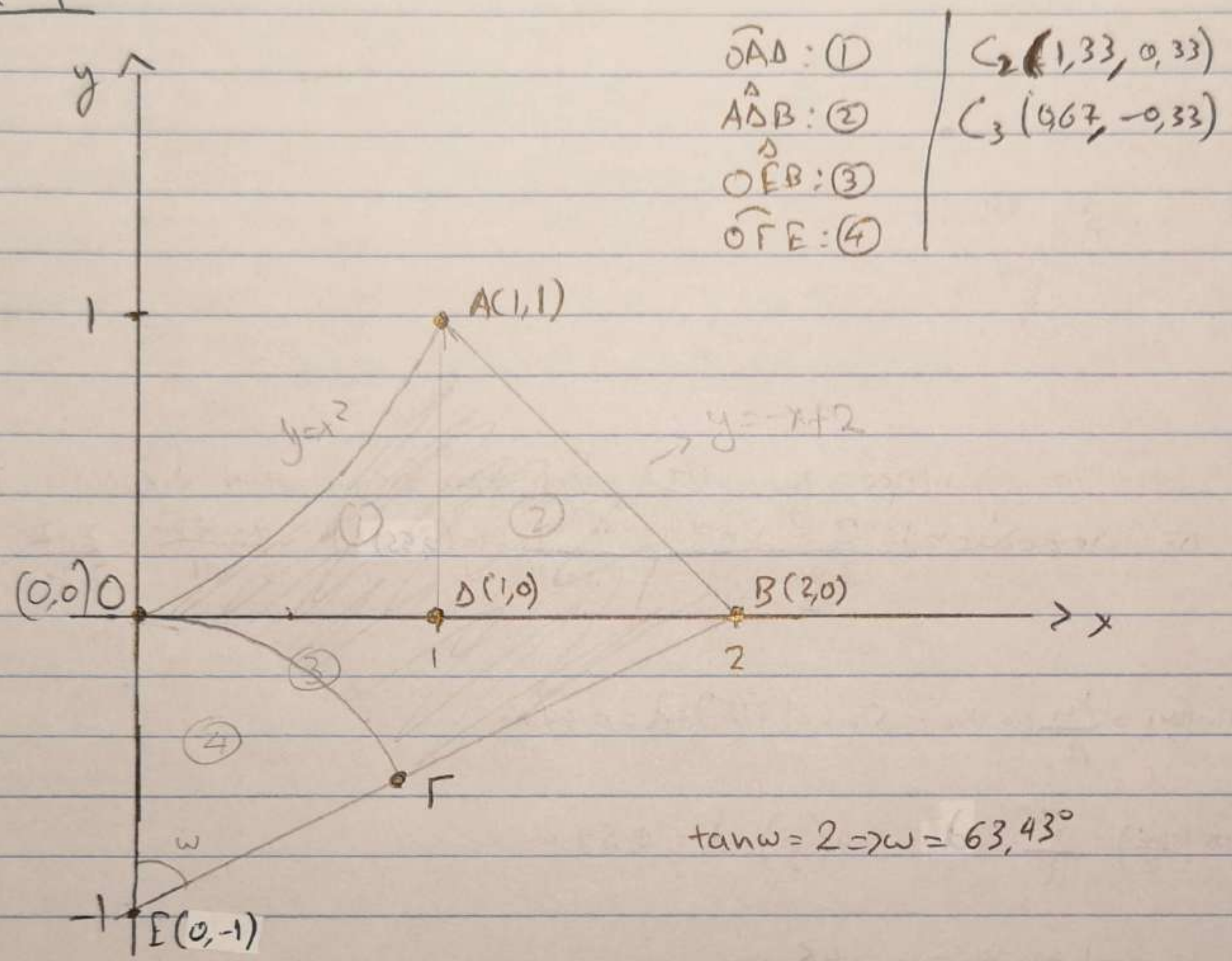
Άρα $Q_y = Q_{y_1} + Q_{y_2} + Q_{y_3} - Q_{y_4} + Q_{y_5} = 32,71$

Άρα $x_c = \dfrac{Q_y}{A} = \dfrac{32,71}{13,33} = 2,45$ m

και $y_c = \dfrac{Q_x}{A} = \dfrac{45,2}{13,33} = 3,39$ m

Άρα $C(2,28,\ 3,39)$

Άσκηση 7



Για την εξίσωση $y=\frac{h}{b^2}x^2$ έχω, από ασκ.4, ότι $Q_{x_1}=\frac{bh^2}{10}$ και $Q_{y_1}=\frac{hb^2}{4}$

Άρα για $h=b=1$ έχω $Q_{x_1}=0,1$ και $Q_{y_1}=0,25$

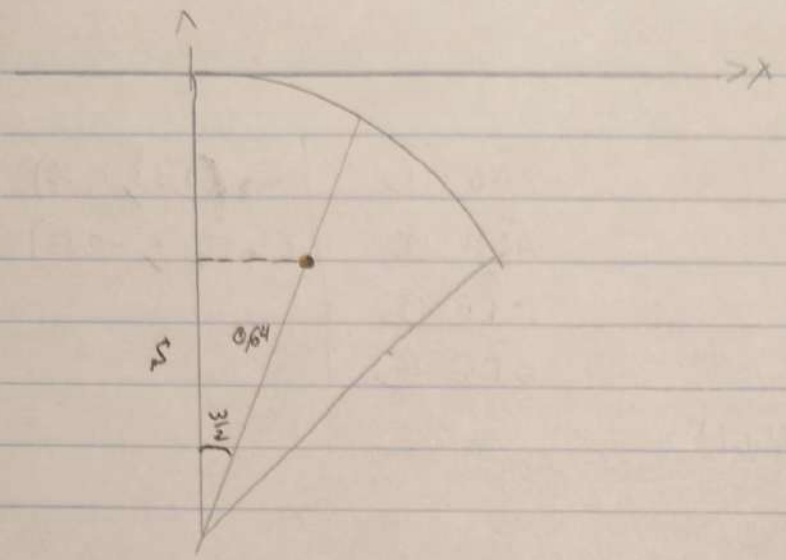
$A_1=\int_0^1 x^2dx=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}m^2$, $A_2=\frac{1\cdot1}{2}=0,5m^2$, $A_3=\frac{1\cdot 2}{2}=1m^2$

$A_4=\pi R^2\cdot\frac{63,43}{360}=\frac{\pi}{5,68}=0,55m^2$

Άρα $A=A_1+A_2+A_3-A_4=0,33+0,5+1-0,55=1,28m^2$

$Q_{x_2}=A_2y_{c_2}=0,5\cdot0,33=0,17m^3$, $Q_{y_2}=A_2x_{c_2}=0,5\cdot1,33=0,67m^3$

$Q_{x_3}=A_3y_{c_3}=-0,33\ m^3$, $Q_{y_3}=A_3x_{c_3}=0,67m^3$



Ξέρω ότι το γ. κέντρο του (4) βρίσκεται πάνω στην διχοτόμο του
και σε απόσταση $d = \frac{2 R \cdot \sin\frac{\omega}{2}}{3\frac{\omega}{2}} = \frac{4 \cdot 1 \cdot \sin(31,715^\circ)}{3 \cdot 1,11} = \frac{4 \cdot 0,53}{3 \cdot 1,11} = \frac{2,12}{3,33} = 0,64$

Άρα $\sin\left(\frac{\omega}{2}\right) = \frac{x_{c4}}{d} \Leftrightarrow x_{c4} = \sin(31,715^\circ) \cdot d = 0,34\ m$

και $\cos\left(\frac{\omega}{2}\right) = \frac{y'}{d} \Leftrightarrow y' = \cos\left(\frac{\omega}{2}\right) \cdot d = 0,54\ m$

Άρα $y_{c4} = -1 + 0,54 = -0,46\ m$

Άρα $Q_{x4} = A_4 y_{c4} = 0,55(-0,46) = -0,25\ m^3$
και $Q_{y4} = A_4 x_{c4} = 0,55 \cdot 0,34 = 0,19\ m^3$

Άρα $x_c = \frac{Q_y}{A} = \frac{Q_{y1} + Q_{y2} + Q_{y3} - Q_{y4}}{A} = \frac{0,25 + 0,67 + 0,67 - 0,19}{1,28} = \frac{1,4}{1,28} = 1,09\ m$

και $y_c = \frac{Q_x}{A} = \frac{Q_{x1} + Q_{x2} + Q_{x3} - Q_{x4}}{A} = \frac{0,1 + 0,17 - 0,33 + 0,25}{A} = \frac{0,19}{1,28} = 0,15\ m$