Αν βρείτε κάποιο λάθος PM me να το διορθώσω: Georgepan

ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΤΟΜΕΑΣ ΜΗΧΑΝΙΚΗΣ, ΕΡΓΑΣΤΗΡΙΟ ΑΝΤΟΧΗΣ ΚΑΙ ΥΛΙΚΩΝ

Ηρώων Πολυτεχνείου 5, Κτίριο Θεοχάρη
Πολυτεχνειούπολη Ζωγράφου, 157 73 Ζωγράφου

Δρ Σταύρος Κ. Κουρκουλής, Καθηγητής Πειραματικής Μηχανικής

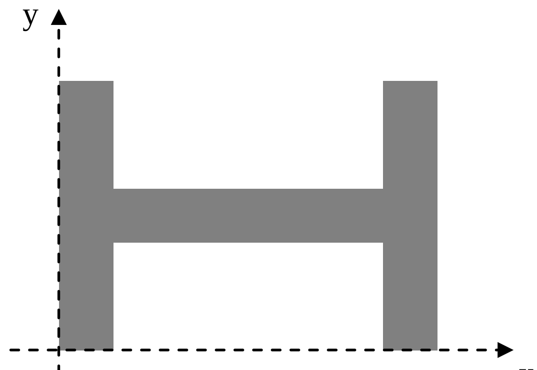
Τηλέφωνα: +210 772 1313, +210 772 1263 (γραφείο)
+210 772 4025, +210 772 4235, +210 772 1317, +210 7721310 (εργαστήρια)
Τηλεομοιότυπο (Fax): +210 7721302
Διεύθυνση ηλεκτρονικού ταχυδρομείου (e-mail): stakkour@central.ntua.gr

ΜΗΧΑΝΙΚΗ Ι (ΣΤΑΤΙΚΗ)

114η σειρά ασκήσεων: Επιφανειακές ροπές 2ης τάξης

Άσκηση 1

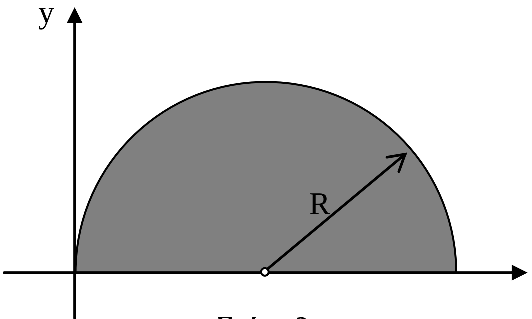
Η επιφάνεια του Σχ.1 αποτελείται από τρία ίδια ορθογώνια παραλληλόγραμμα διαστάσεων 5x 25 cm^2. Υπολογίστε τις επιφανειακές ροπές 2ης τάξης I_{ij}, i,j=x,y.



Σχήμα 1

Άσκηση 2

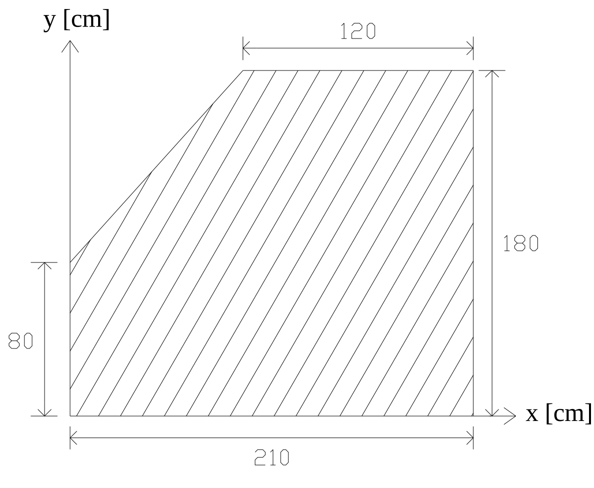
Για τον ημικυκλικό δίσκο του Σχ.2, ακτίνας R= 15 cm να υπολογισθούν οι επιφανειακές ροπές 2ης τάξης I_{xx} και I_{yy}.



Σχήμα 2

Άσκηση 3

Να υπολογισθούν οι επιφανειακές ροπές 2ης τάξης I_{ij}, i,j=x,y (Σχ.3).



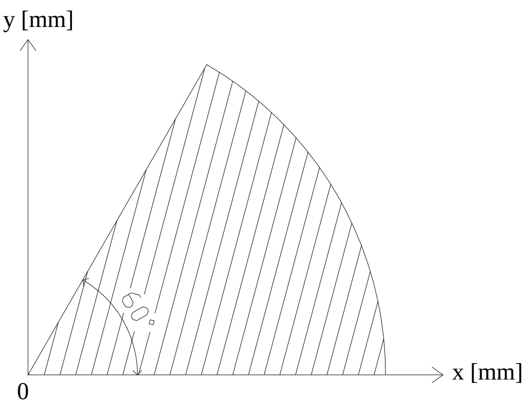
Σχήμα 3

Άσκηση 4

Η γραμμοσκιασμένη επιφάνεια του Σχ.4 είναι κυκλικός τομέας ακτίνας 200 mm. Υπολογίστε:

α. Τις επιφανειακές ροπές 2ης τάξης I_{xx} και I_{yy}.

β. Τις επιφανειακές ροπές 2ης τάξης $I_{x_cx_c}$ και $I_{y_cy_c}$ όπου x_c και y_c άξονες διερχόμενοι από το γεωμετρικό κέντρο C του σχήματος παράλληλοι με τους άξονες x και y του Σχ.4.



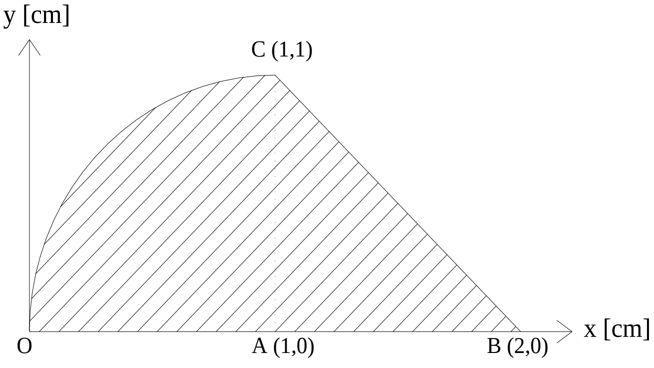
Σχήμα 4

Άσκηση 5

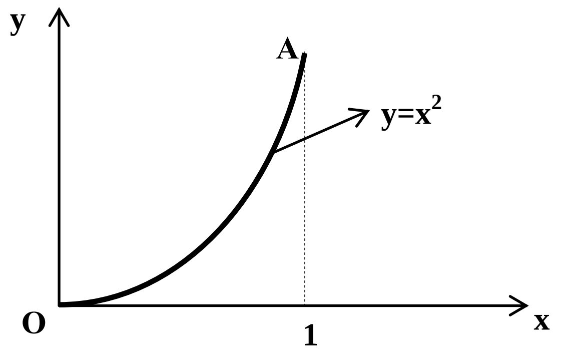
Για τη γραμμοσκιασμένη επιφάνεια του Σχ.5 (η καμπύλη OC είναι τεταρτοκύκλιο) υπολογίστε:

α. Τις επιφανειακές ροπές 2^{ης} τάξης I_{xx} και I_{yy}.

β. Τις επιφανειακές ροπές 2^{ης} τάξης I_{xGxG} και I_{yGyG} όπου x_c και y_c άξονες διερχόμενοι από το γεωμετρικό κέντρο G της επιφάνειας παράλληλοι με τους άξονες x και y αντίστοιχα.



Σχήμα 5



Σχήμα 6

Άσκηση 6

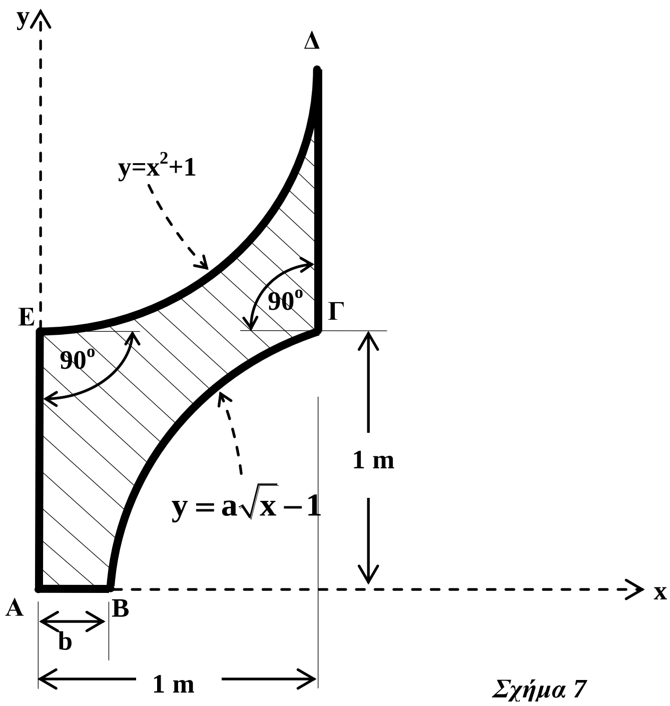
Για τη επιφάνεια (OA1O) του Σχ.6 να υπολογισθούν οι επιφανειακές ροπές 2^{ης} τάξης I_{ij}, i,j=x,y.

Άσκηση 7

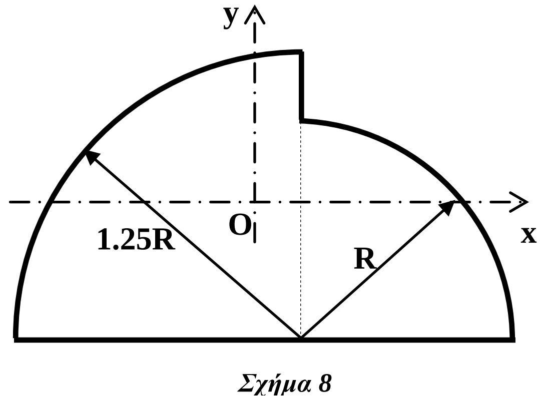
Για τη γραμμοσκιασμένη επιφάνεια του Σχ.7 να υπολογισθούν:

α. Οι επιφανειακές ροπές 2^{ης} τάξης I_{xx} και I_{yy}.

β. Οι επιφανειακές ροπές 2^{ης} τάξης I_{xGxG} και I_{yGyG} όπου x_c, y_c άξονες διερχόμενοι από το γεωμετρικό κέντρο G της επιφάνειας παράλληλοι με τους άξονες x και y, αντίστοιχα.



Σχήμα 7



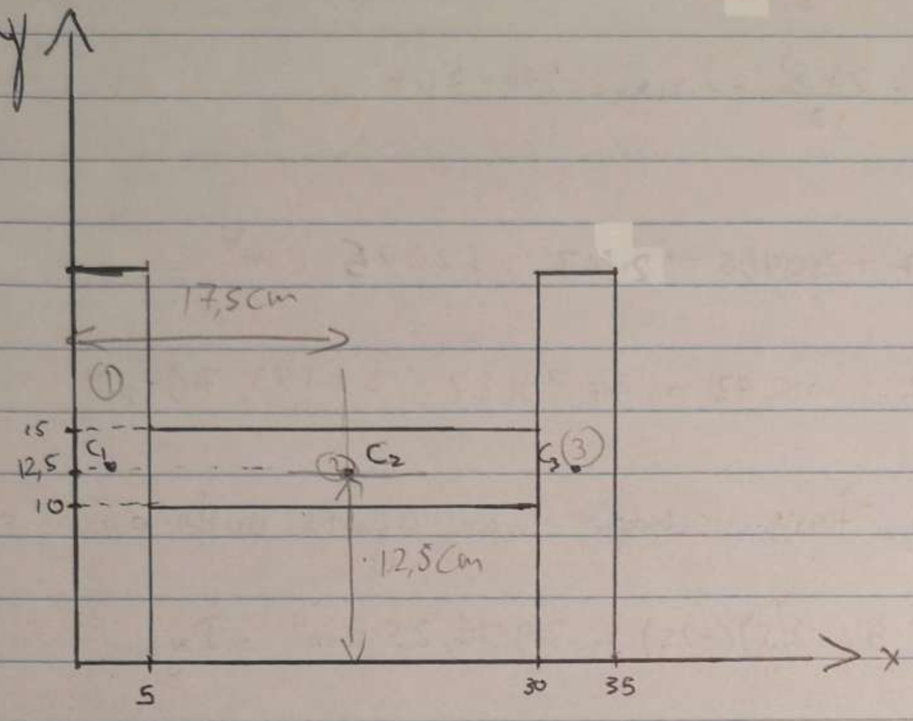
Σχήμα 8

Άσκηση 8

Για τη επιφάνεια του Σχ.8 να υπολογισθεί ο τανυστής των επιφανειακών ροπών 2^{ης} τάξης I_{ij}, i,j= x,y όπου O το γεωμετρικό κέντρο της επιφάνειας.

11η Σειρά ασκήσεων: Επιφανειακές ροπές 2ης τάξης

Άσκηση 1



Ξέρω ότι για παραλληλόγραμμο με βάση στον άξονα x ισχύει:

$I_{xx} = \frac{bh^3}{3}$,

και για παραλληλόγραμμο επίσης ισχύει για άξονες που διέρχονται από το κέντρο του:

$I_{x_cx_c} = \frac{bh^3}{12}$, $I_{y_cy_c} = \frac{hb^3}{12}$

Ψάχνω $I_{ij} = \begin{bmatrix} I_{xx}, & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix}$

$I_{xx} = I_{xx_1} + I_{xx_2} + I_{xx_3}$, $I_{yy} = I_{yy_1} + I_{yy_2} + I_{yy_3}$

Έχω $I_{xx_1} = \frac{5 \cdot 25^3}{3} = 26041,67 \text{ cm}^4$, $I_{yy_1} = \frac{25 \cdot 5^3}{3} = 1041,67 \text{ cm}^4$

Έχω $I_{x_{c_2}x_{c_2}} = \frac{25 \cdot 5^3}{12} = 260,42 \text{ cm}^4$ και $I_{y_{c_2}y_{c_2}} = \frac{5 \cdot 25^3}{12} = 6510,42 \text{cm}^3$

Άρα από Θ. Steiner έχω $I_{xx_2} = I_{x_cx_c} + 25 \cdot 5 \cdot 12,5^2 = 260,42 + 281,25 = 541,67 \text{cm}^4$

και $I_{yy_2} = I_{y_cy_{c_2}} + 25 \cdot 5 (17,5)^2 = 6510,42 + 38281,25 = 44791,67 \text{ cm}^4$

Έχω $I_{xx_3} = \frac{bh^3}{3} = \frac{5\cdot 25^3}{3} = I_{xx_1} = 26041,67\ cm^4$

και $I_{y_{c_3}y_{c_3}} = \frac{hb^3}{12} = \frac{25\cdot 5^3}{12} = I_{x_2x_2} = 240,42\,cm^4$

Άρα $I_{xx} = 26041,67 + 26041,67 + 541,67 = 52625\ cm^4$

και $I_{yy} = 1041,67 + 6510,42 + 44.791,67 = 52343,76\,cm^4$

Έχω $I_{x_{c_1}y_{c_1}} = I_{x_{c_2}y_{c_2}} = I_{x_{c_3}y_{c_3}} = 0$ αφού είναι άξονες συμμετρίας των ορθ/μων.

Άρα $I_{xy_1} = I_{x_{c_1}y_{c_1}} + \overbrace{25\cdot 5}^{A}\,\overbrace{(-12,5)}^{d_1}\,\overbrace{(-2,5)}^{d_2} = 3906,25\ cm^4 = I_{yx_1}$

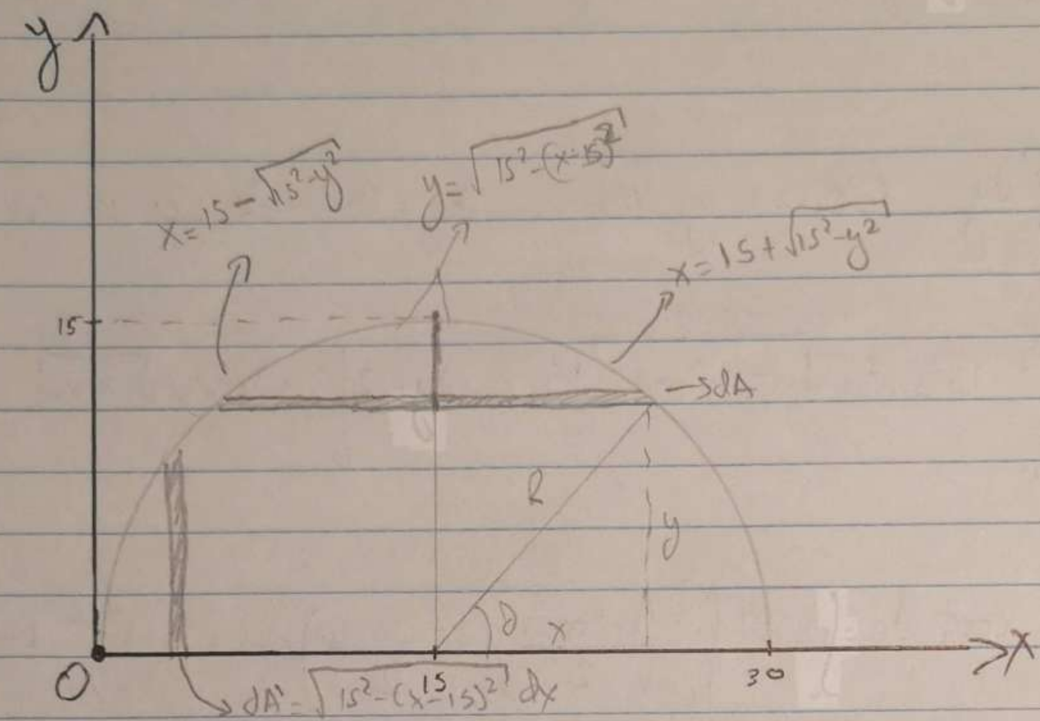
και $I_{xy_2} = I_{x_{c_2}y_{c_2}} + 25\cdot 5\cdot(-12,5)(-17,5) = 27343,75\,cm^4 = I_{yx_2}$

και $I_{xy_3} = I_{x_{c_3}y_{c_3}} + 25\cdot 5\,(-32,5)(-12,5) = 50781,25\ cm^4 = I_{yx_3}$

Άρα $I_{xy} = I_{yx} = I_{xy_1} + I_{xy_2} + I_{xy_3} = 82031,25\,cm^4$

Άρα $I_{ij} = \begin{bmatrix} 52625 & 82031,25 \\ 82031,25 & 52343,76 \end{bmatrix}\ cm^4,\quad i,j = x,y$

Άσκηση 2



1ος Τρόπος

Η εξίσωση του κύκλου είναι η: $y^2+(x-15)^2=15^2$

Λύνοντας ως προς y προκύπτει ότι η συνάρτηση του ημικυκλίου είναι $y=\sqrt{15^2-(x-15)^2}$

Λύνοντας ως προς x προκύπτει: $(x-15)^2=(15^2-y^2) \Leftrightarrow x-15=\pm\sqrt{15^2-y^2} \Leftrightarrow$

$\Leftrightarrow x=15\pm\sqrt{15^2-y^2}$

Οπότε έχω $\frac{dA}{2}=\left[15-\left(15-\sqrt{15^2-y^2}\right)\right]dy \Leftrightarrow \frac{dA}{2}=\sqrt{15^2-y^2}\,dy \Leftrightarrow dA=2\sqrt{15^2-y^2}\,dy$

ή $\frac{dA}{2}=\left(15+\sqrt{15^2-y^2}-15\right)dy=\sqrt{15^2-y^2}\,dy \Leftrightarrow dA=2\sqrt{15^2-y^2}\,dy$

$$\text{Άρα } I_{xx} = \iint_A y^2 dA = \int_0^{15} y^2 \cdot 2\sqrt{15^2 - y^2}\, dy =$$

$$= 2\int_0^{15} y^2 \sqrt{15^2 - y^2}\, dy \quad \text{Θέτω } y = 15\sin u \Rightarrow dy = 15\cos u\, du,\ u_1 = 0,\ u_2 = \frac{\pi}{2}$$

$$\text{Άρα } I_{xx} = 2\int_0^{\frac{\pi}{2}} 15^2 \sin^2 u \sqrt{15^2 - 15^2\sin^2 u} \cdot 15\cos u\, du = 2\int_0^{\frac{\pi}{2}} 15^4 \sin^2 u \sqrt{1-\sin^2 u} \cdot \cos u\, du =$$

$$= 2\cdot 15^4 \int_0^{\frac{\pi}{2}} \sin^2 u \cdot \cos^2 u\, du = 2\cdot 15^4 \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2u}{2}\right)\left(\frac{1+\cos 2u}{2}\right) du = 2\cdot 15^4 \int_0^{\frac{\pi}{2}} \frac{1-\cos^2 2u}{4}\, du$$

$$= \frac{15^4}{2}\int_0^{\frac{\pi}{2}} \left(1-\cos^2(2u)\right) du \quad \text{Θέτω } w = 2u \Rightarrow dw = 2du \Rightarrow du = \frac{dw}{2},\ w_1 = 0,\ w_2 = \pi$$

$$\text{Άρα } I_{xx} = \frac{15^4}{2}\int_0^{\pi} \frac{1-\cos^2 w}{2}\, dw = \frac{15^4}{4}\int_0^{\pi} \sin^2 w\, dw = \frac{15^4}{4}\int_0^{\pi} \frac{1-\cos 2w}{2}\, dw =$$

$$= \frac{15^4}{8}\int_0^{\pi} (1-\cos 2w)\, dw = \frac{15^4}{8}\left[w - \frac{\sin 2w}{2}\right]_0^{\pi} = \frac{15^4}{8}\pi = 19880{,}39\ m^4$$

$$I_{yy} = \iint_A x^2 dA = \int_0^{30} x^2 \sqrt{15^2 - (x-15)^2}\, dx \quad \text{Θέτω } u = x - 15 \Rightarrow du = dx,\ u_1 = -15,\ u_2 = 15$$

$$\text{Άρα } I_{yy} = \int_{-15}^{15} (u+15)^2 \sqrt{15^2 - u^2}\, du = \int_{-15}^{15} (u^2 + 30u + 15^2)\sqrt{15^2 - u^2}\, du =$$

$$= \int_{-15}^{15} u^2 \sqrt{15^2 - u^2}\, du + 30\int_{-15}^{15} u\sqrt{15^2 - u^2}\, du + 15\int_{-15}^{15} \sqrt{15^2 - u^2}\, du = I_1 + 30 I_2 + 15 I_3$$

Έχω $I_1 = \int_{-15}^{15} u^2 \sqrt{15^2-u^2}\,du$

Όπως η συνάρτηση $f(u) = u^2\sqrt{15^2-u^2}$ είναι άρτια και τα άκρα του I_1 συμμετρικά (ως προς το 0) άρα έχω:

$$I_1 = 2\int_0^{15} u^2\sqrt{15^2-u^2}\,du = I_{xx} = 19880{,}39\ m^4$$

$$I_2 = \int_{-15}^{15} u\sqrt{15^2-u^2}\,du$$

Όπως η συνάρτηση $g(u) = u\sqrt{15^2-u^2}$ είναι περιττή και τα άκρα του I_2 συμμετρικά (ως προς το 0), άρα ισχύει: ότι $I_2 = 0$

$$I_3 = \int_{-15}^{15} \sqrt{15^2-u^2}\,du$$

Η $h(u) = \sqrt{15^2-u^2}$ είναι άρτια και τα όρια του I_3 συμμετρικά (ως προς το 0) άρα ισχύει

$I_3 = 2\int_0^{15} \sqrt{15^2-u^2}\,du$ Θέτω $u = 15\sin w \Rightarrow du = 15\cos w\,dw$, $w_1 = 0$, $w_2 = \frac{\pi}{2}$

Άρα $I_3 = 2\int_0^{\frac{\pi}{2}} \sqrt{15^2-15^2\sin^2 w}\cdot 15\cos w\,dw = 2\int_0^{\frac{\pi}{2}} 15^2\sqrt{1-\sin^2 w}\cos w\,dw =$

$= 2\int_0^{\frac{\pi}{2}} 15^2\cos^2 w\,dw = 2\cdot 15^2\int_0^{\frac{\pi}{2}} \frac{1+\cos 2w}{2}\,dw = 15^2\int_0^{\frac{\pi}{2}} (\cos 2w+1)\,dw = 15^2\cdot\left[\frac{\sin 2w}{2}+w\right]_0^{\frac{\pi}{2}} =$

$= 15^2\cdot\frac{\pi}{2} = \frac{15^2\pi}{2} = 353{,}43\ m^4$

Άρα $I_{yy} = I_1 + 30I_2 + 15I_3 = 19880{,}39 + 5301{,}44 = 25181{,}83\ m^4$

2ος Τρόπος

Έστω $x = 15 + R\cos\theta$
$y = R\sin\theta$

Άρα $\frac{\partial x}{\partial R} = \cos\theta \quad \frac{\partial x}{\partial \theta} = -R\sin\theta$
$\frac{\partial y}{\partial R} = \sin\theta \quad \frac{\partial y}{\partial \theta} = R\cos\theta$

Άρα: $\begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix} = R$

$$I_{xx} = \iint_A y^2\,dx\,dy = \iint R^2\sin^2\theta\, R\,dR\,d\theta = \int_0^{\pi} \sin^2\theta \left(\int_0^{15} R^3\,dR\right)d\theta =$$

$$= \int_0^{\pi} \sin^2\theta \left[\frac{R^4}{4}\right]_0^{15} d\theta = \frac{15^4}{4}\int_0^{\pi} \sin^2\theta\,d\theta = \frac{15^4}{4}\int_0^{\pi} \frac{1-\cos 2\theta}{2}\,d\theta =$$

$$= \frac{15^4}{8}\int_0^{\pi} (1-\cos 2\theta)\,d\theta = \frac{15^4}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{15^4}{8}\pi = 19880{,}39\ m^4$$

$$I_{yy} = \iint_A x^2\,dx\,dy = \iint (R\cos\theta + 15)^2 R\,dR\,d\theta = \iint (R^2\cos^2\theta + 30R\cos\theta + 15^2) R\,dR\,d\theta =$$

$$= \iint (R^3\cos^2\theta + 30R^2\cos\theta + 15R^2)\,dR\,d\theta = \iint R^3\cos^2\theta\,dR\,d\theta + \iint 30R^2\cos\theta\,dR\,d\theta + \iint 15R^2\,dR\,d\theta =$$

$$= \int_0^{\pi}\cos^2\theta\left(\int_0^{15} R^3\,dR\right)d\theta + 30\int_0^{\pi}\cos\theta\left(\int_0^{15} R^2\,dR\right)d\theta + 15\int_0^{\pi}\int_0^{15} R^2\,dR\,d\theta$$

$$= \int_0^{\pi} \cos^2\theta \left[\frac{\rho^4}{4}\right]_0^{15} d\theta + 30 \int_0^{\pi} \cos\theta \left[\frac{\rho^3}{3}\right]_0^{15} d\theta + 15 \int_0^{\pi} \left[\frac{\rho^2}{2}\right]_0^{15} d\theta =$$

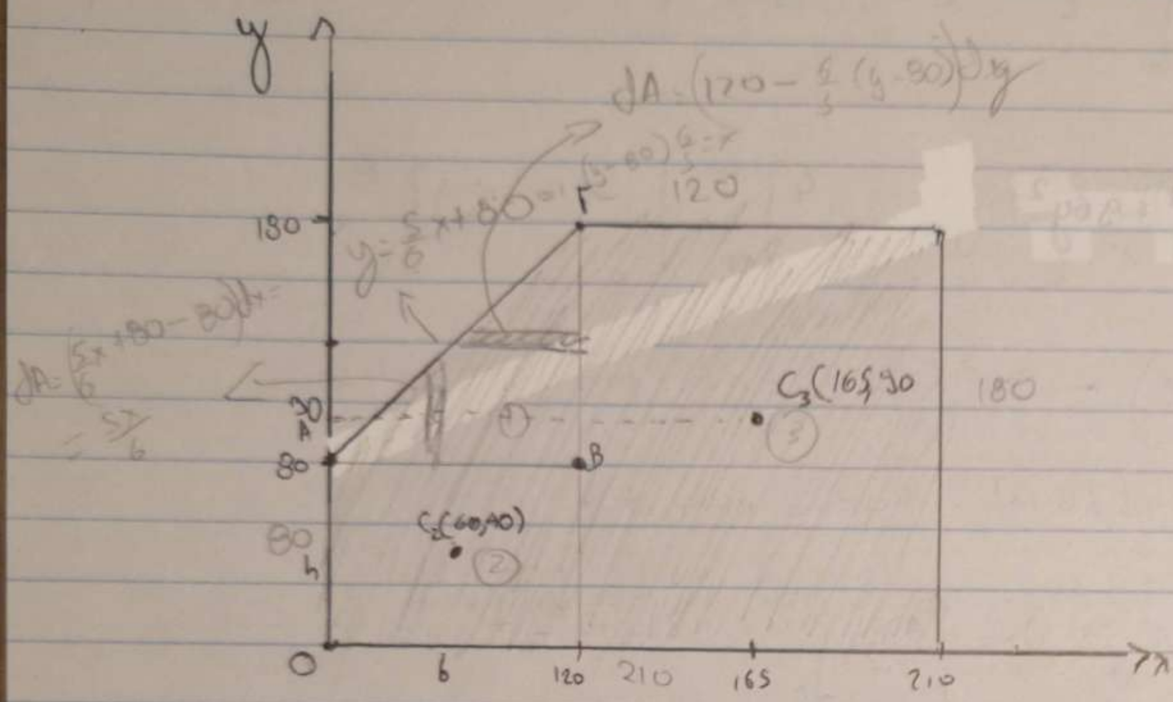
$$= \int_0^{\pi} \frac{15^4}{4} \cos^2\theta \, d\theta + 30 \int_0^{\pi} \frac{15^3}{3} \cos\theta \, d\theta + 15 \int_0^{\pi} \frac{15^2}{2} \, d\theta =$$

$$= \frac{15^4}{4} \int_0^{\pi} \frac{\cos 2\theta + 1}{2} \, d\theta + 15^3 \cdot 10 \int_0^{\pi} \cos\theta \, d\theta + \frac{15^3}{2} \int_0^{\pi} d\theta =$$

$$= \frac{15^4}{4} \left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\pi} + 15^3 \cdot 10 \left[\sin\theta\right]_0^{\pi} + \frac{15^3}{2} \left[\theta\right]_0^{\pi} =$$

$$= \frac{15^4}{4} \pi + 0 + \frac{15^3 \pi}{2} = 19880{,}39 + 5301{,}44 = 25181{,}83 \, m^4$$

Άσκηση 3



$I_{xx} = I_{xx_1} + I_{xx_2} + I_{xx_3}$

$I_{yy} = I_{yy_1} + I_{yy_2} + I_{yy_3}$

Έχω $I_{xx_1} = \frac{bh^3}{3} = \frac{120 \cdot 80^3}{3} = 40 \cdot 80^3 = 2048 \cdot 10^4 \, cm^4$

και $I_{yy_1} = \frac{hb^3}{3} = \frac{80 \cdot 120^3}{3} = 4608 \cdot 10^4 \, cm^4$

$I_{xx_3} = \frac{90 \cdot 180^3}{3} = 30 \cdot 180^3 = 17496 \cdot 10^4 \, cm^4$

Έχω $I_{y_{c3}y_3} = \frac{hb^3}{12} = \frac{180 \cdot 90^3}{12} = 1093{,}5 \cdot 10^4$

Άρα $I_{yy_3} = I_{y_{c3}y_{c3}} + Ad^2 = 1093{,}5 \cdot 10^4 + 90 \cdot 180 \cdot 165^2 = 1093{,}5 \cdot 10^4 + 44104{,}5 \cdot 10^4 =$

$= 45198 \cdot 10^4 \, cm^4$

Έχω $I_{xx_2} = \int y^2 dA = \int_{80}^{180} y^2 \left[120 - \frac{6}{5}(y-80)\right] dy = \int_{80}^{180} y^2 \left(120 - \frac{6}{5}y + 96\right) dy =$

$$= \int_{80}^{180} \left(216y^2 - \frac{6}{5}y^3\right) dy = \left[72y^3 - \frac{3}{10}y^4\right]_{80}^{180} = \left[y^3\left(72 - \frac{3y}{10}\right)\right]_{80}^{180} =$$

$$= 180^3\left(72 - \frac{3\cdot 18\cancel{0}}{1\cancel{0}}\right) - 80^3\left(72 - \frac{3\cdot 8\cancel{0}}{1\cancel{0}}\right) = 180^3\cdot 18 - 80^3\cdot 48 =$$

$$= 104976\cdot 10^3 - 24576\cdot 10^3 = 80400\cdot 10^3 = 8040\cdot 10^4 \text{ cm}^4$$

$$I_{yy_2} = \int x^2 dA = \int_0^{120} x^2 \frac{5x}{6} dx = \left[\frac{5x^4}{24}\right]_0^{120} = 4320\cdot 10^4 \text{ cm}^4$$

Άρα $I_{xx} = (2048 + 17496 + 8040)10^4 = 27584\cdot 10^4 \text{ cm}^4$

και $I_{yy} = (4608 + 45198 + 4320)10^4 = 54126\cdot 10^4 \text{ cm}^4$

Για τα ② και ③ έχω ότι οι x_c, y_c τους είναι άξονες συμμετρίας.
Άρα $I_{x_{c2}y_{c2}} = I_{y_{c2}x_{c2}} = I_{x_{c3}y_{c3}} = I_{y_{c3}x_{c3}} = 0$

Άρα $I_{xy_2} = I_{x_{c2}y_{c2}} + A_2 d_1 d_2 = 0 + 80\cdot 120\cdot(-40)(-60) = 2304\cdot 10^4 \text{ cm}^4 = I_{yx_2}$

και $I_{xy_3} = I_{x_{c3}y_{c3}} + A_3 d_1 d_2 = 0 + 90\cdot 180(-30)(-165) = 24057\cdot 10^4 = I_{yx_3}$

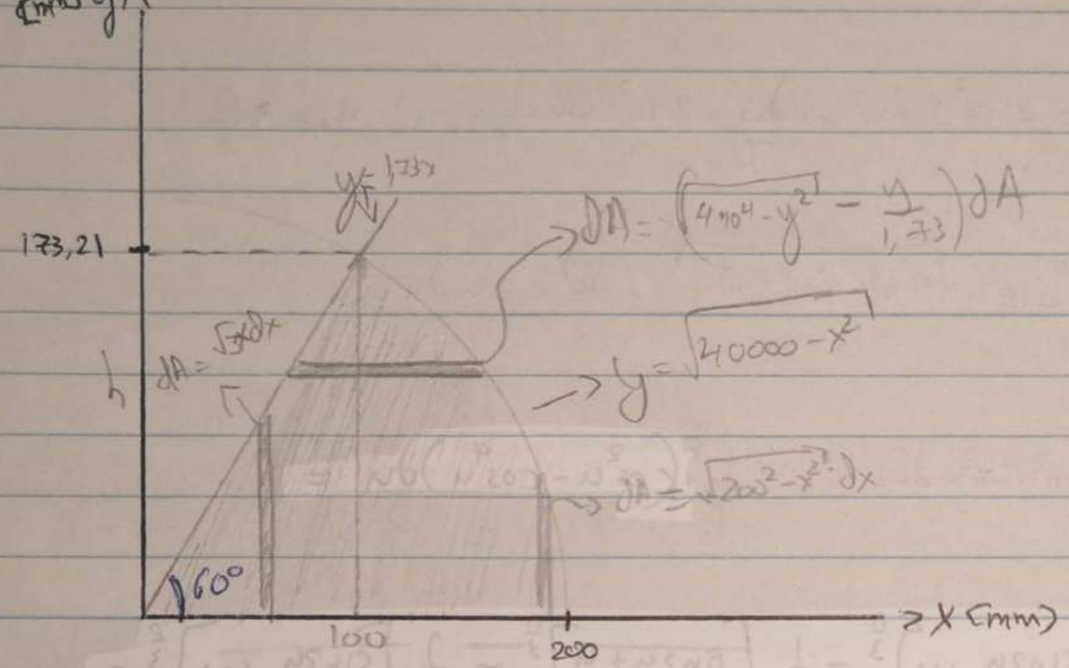
Για I_{xy_2} έχω $I_{xy_2} = \int xy\, dA = \iint xy\, dx dy = \int_0^{120} x \int_0^{\frac{5x}{6}+80} y\, dy dx = \int_0^{120} x \left[\frac{y^2}{2}\right]_0^{\frac{5x}{6}+80} dx$

$$= \int_0^{120} x\left(\frac{\left(\frac{5x}{6}+80\right)^2}{2}\right) dx = \int_0^{120} x\left(\frac{\frac{25}{36}x^2 + \frac{400x}{3} + 6400}{2}\right) dx = \int_0^{120} x\left(\frac{25x^2}{72} + \frac{200x}{3} + 3200\right) dx$$

$$= \int_0^{120}\left(\frac{25x^3}{72} + \frac{200x^2}{3} + 3200x\right) dx = \left[\frac{25}{288}x^4 + \frac{200}{9}x^3 + 1600x^2\right]_0^{120} = 1800\cdot 10^4 + 3840\cdot 10^4 + 2304\cdot 10^4 = 7944\cdot 10^4$$

Άρα $I_{ij} = \begin{bmatrix} 27584\cdot 10^4 & 7944\cdot 10^4 \\ 7944\cdot 10^4 & 54126\cdot 10^4 \end{bmatrix} \text{ cm}^4$

Ασκηση 4 (1ος Τροπος)



$\sin 60 = \frac{h}{R} \Leftrightarrow h = \sin 60 \cdot R = \frac{\sqrt{3}}{2} \cdot 200 = 100\sqrt{3} = 173,21$

$\tan(60) = \frac{\sin 60}{\cos 60} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} = 1,73$

Αρα $y = 1,73x$

Ο κύκλος έχει εξίσωση $x^2 + y^2 = 40000 \Leftrightarrow y^2 = 40000 - x^2 \Rightarrow y = \sqrt{40000 - x^2}$

$$\text{Αρα } I_{xx} = \int y^2 dA = \int_0^{100\sqrt{3}} y^2\left(\sqrt{4\cdot 10^4 - y^2} - 0,58y\right) dy = \int_0^{100\sqrt{3}} \left(y^2\sqrt{4\cdot 10^4 - y^2} - 0,58y^3\right) dy =$$

$$= \int_0^{100\sqrt{3}} y^2\sqrt{4\cdot 10^4 - y^2}\,dy - \int_0^{100\sqrt{3}} 0,58y^3 = \int_0^{100\sqrt{3}} y^2\sqrt{4\cdot 10^4 - y^2}\,dy - \left[0,145\, y^4\right]_0^{100\sqrt{3}} = \int_0^{100\sqrt{3}} y^2\sqrt{4\cdot 10^4 - y^2}\,dy - 126\cdot 10^6$$

$$= I - 126\cdot 10^6$$

Έχω $I = \int_0^{100\sqrt{3}} y^2 \sqrt{4\cdot 10^4 - y^2}\, dy$

Θέτω $y = 2\cdot 10^2 \sin u \Rightarrow dy = 2\cdot 10^2 \cos u\, du,\ u_1 = 0,\ u_2 = \frac{\pi}{3}$

Άρα $I = \int_0^{\frac{\pi}{3}} 4\cdot 10^4 \sin^2 u \sqrt{4\cdot 10^4 - 4\cdot 10^4 \sin^2 u} \cdot 2\cdot 10^2 \cos u\, du =$

$= \int_0^{\frac{\pi}{3}} 16\cdot 10^8 \sin^2 u \cos^2 u\, du = 16\cdot 10^8 \int_0^{\frac{\pi}{3}} \left(\frac{1-\cos 2u}{2}\right)\left(\frac{1+\cos 2u}{2}\right) du$

$= 4\cdot 10^8 \int_0^{\frac{\pi}{3}} (1-\cos^2(2u))\, du$ Θέτω $w = 2u \Rightarrow dw = 2du,\ w_1 = 0,\ w_2 = \frac{2\pi}{3}$

Άρα $I = 4\cdot 10^8 \int_0^{\frac{2\pi}{3}} \frac{(1-\cos^2 w)}{2}\, dw = 2\cdot 10^8 \int_0^{\frac{2\pi}{3}} \sin^2 w\, dw = 2\cdot 10^8 \int_0^{\frac{2\pi}{3}} \frac{1-\cos 2w}{2}\, dw = 10^8 \left[w - \frac{\sin 2w}{2}\right]_0^{\frac{2\pi}{3}}$

$= 10^8 \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{4}\right) = 252\cdot 10^6\ m^4$

Άρα $I_{xx} = 128\cdot 10^6$

$I_{yy} = \iint_A x^2\, dA = \int_0^{100} x^3\sqrt{3}\, dx + \int_{100}^{200} x^2 \sqrt{200^2 - x^2}\, dx = \left[\frac{\sqrt{3}x^4}{4}\right]_0^{100} + \int_{100}^{200} x^2\sqrt{200^2 - x^2}\, dx =$

$= 43\cdot 10^6 + \int_{100}^{200} x^2 \sqrt{200^2 - x^2}\, dx = 43\cdot 10^6 + J$

$$J=\int_{100}^{200} x^2\sqrt{200^2-x^2}\,dx \quad \text{Θέτω } u=200\sin u \Rightarrow du=200\cos u\,du,\ u_1=\frac{\pi}{6},\ u_2=\frac{\pi}{2}$$

$$\text{Άρα } J=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 200^2\sin^2 u\sqrt{200^2-200^2\sin^2 u}\cdot 200\cos u\,du=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 200^4\sin^2 u\cos^2 u\,du=$$

$$=200^4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sin^2 u\cos^2 u\,du=200^4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{(1-\cos 2u)(1+\cos 2u)}{2\cdot 2}\,du=\frac{200^4}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1-\cos^2 2u)\,du$$

$$\text{Θέτω } 2u=w \Rightarrow dw=2du \Rightarrow du=\frac{dw}{2},\ w_1=\frac{\pi}{3},\ w_2=\pi$$

$$\text{Άρα } J=\frac{200^4}{4}\int_{\frac{\pi}{3}}^{\pi}\frac{1-\cos^2 w}{2}\,dw=\frac{200^4}{8}\int_{\frac{\pi}{3}}^{\pi}\sin^2 w\,dw=\frac{200^4}{8}\int_{\frac{\pi}{3}}^{\pi}\frac{1-\cos 2w}{2}\,dw=$$

$$=\frac{200^4}{16}\int_{\frac{\pi}{3}}^{\pi}(1-\cos 2w)\,dw=\frac{200^4}{16}\left[w-\frac{\sin 2w}{2}\right]_{\frac{\pi}{3}}^{\pi}=\frac{200^4}{16}\left(\pi-\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)=\frac{200^4}{16}\left(\frac{2\pi}{3}+\frac{\sqrt{3}}{4}\right)$$

$$=\frac{200^4}{16}\cdot 2,53=0,16\cdot 200^4=256\cdot 10^6\ m^4$$

Άρα $I_{yy}=299\cdot 10^6\,mm^4$ $(\approx 296\cdot 10^6)$

β) α' Τρόπος

$$\text{Έχω } Q_x=\iint_A y\,dA=\int_0^{100\sqrt{3}} y\left(\sqrt{200^2-y^2}-\frac{y}{1,73}\right)dy=\int_0^{100\sqrt{3}}\left(y\sqrt{200^2-y^2}-\frac{y^2}{1,73}\right)dy=$$

$$=\int_0^{100\sqrt{3}} y\sqrt{200^2-y^2}\,dy-\int_0^{100\sqrt{3}}0,58y^2\,dy=\int_0^{100\sqrt{3}} y\sqrt{200^2-y^2}\,dy-\left[0,19y^3\right]_0^{100\sqrt{3}}=\int_0^{100\sqrt{3}} y\sqrt{200^2-y^2}\,dy-987269$$

$$\text{Θέτω } u=200^2-y^2 \Rightarrow du=-2y\,dy \Rightarrow y\,dy=-\frac{du}{2},\ u_1=200^2,\ u_2=10^4$$

$$\text{Άρα } Q_x=\int_{200^2}^{10^4}-\frac{1}{2}\sqrt{u}\,du-987269=\left[\frac{u\sqrt{u}}{3}\right]_{10^4}^{4\cdot 10^4}=\frac{4\cdot 10^4\cdot 2\cdot 10^2}{3}-\frac{10^4\cdot 10^2}{3}-98,73\cdot 10^4=$$

$$=\frac{7}{3}\cdot 10^6-98,73\cdot 10^4=(233,33-98,73)\cdot 10^4=134,6\cdot 10^4\,mm^3$$

$$Q_y = \iint x\,dA = \int_0^{100} x\cdot\sqrt{3}\,x\,dx + \int_{100}^{200} x\sqrt{200^2-x^2}\,dx = \left[\frac{\sqrt{3}x^3}{3}\right]_0^{100} + \int_{100}^{200} x\sqrt{200^2-x^2}\,dx =$$

$$= \frac{10^6\sqrt{3}}{3} + \int_{100}^{200} x\sqrt{200^2-x^2}\,dx$$ Θέτω $u = 200^2 - x^2 \Rightarrow du = -2x\,dx \Leftrightarrow x\,dx = -\frac{du}{2}$

$u_1 = 3\cdot 10^4,\ u_2 = 0$

Άρα $$Q_y = \frac{\sqrt{3}}{3}10^6 + \int_0^{3\cdot 10^4} \frac{\sqrt{u}}{2}\,du = \frac{\sqrt{3}}{3}\cdot 10^6 + \left[\frac{u\sqrt{u}}{3}\right]_0^{3\cdot 10^4} = \frac{\sqrt{3}}{3}\cdot 10^6 + \frac{3\cdot 10^4\cdot 10^2\sqrt{3}}{3} =$$

$$= \frac{\sqrt{3}\cdot 10^6 + 3\sqrt{3}\cdot 10^6}{3} = \frac{4\sqrt{3}\cdot 10^6}{3} = 2{,}31\cdot 10^6\,mm^3 = 231\cdot 10^4\,mm^3$$

Έχω $$A = \int_0^{100} \sqrt{3}\,x\,dx + \int_{100}^{200} \sqrt{200^2-x^2}\,dx = \left[\frac{\sqrt{3}}{2}x^2\right]_0 + I = \frac{\sqrt{3}}{2}\cdot 10^4 + I = 0{,}87\cdot 10^4 + I$$

$$I = \int_{100}^{200} \sqrt{200^2-x^2}\,dx$$ Θέτω $x = 200\sin u \Rightarrow dx = 200\cos u\,du$, $u_1 = \frac{\pi}{6}$, $u_2 = \frac{\pi}{2}$

Άρα $$I = \int_{\pi/6}^{\pi/2} \sqrt{200^2 - 200^2\sin^2 u}\cdot 200\cos u\,du = 200^2\int_{\pi/6}^{\pi/2} \cos^2 u\,du = 200^2\int_{\pi/6}^{\pi/2} \frac{1+\cos 2u}{2}\,du =$$

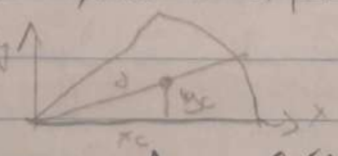
$$= \frac{200^2}{2}\int_{\pi/6}^{\pi/2} (1+\cos 2u)\,du = \frac{200^2}{2}\left[u + \frac{\sin 2u}{2}\right]_{\pi/6}^{\pi/2} = \frac{200^2}{2}\left(\frac{\pi}{2} - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = 200^2\cdot 0{,}31 = 124\cdot 10^4\,mm^2$$

Άρα $A = 2{,}11\cdot 10^4\,mm^2$

Άρα $x_c = \frac{Q_y}{A} = \frac{231\cdot 10^4}{2{,}11\cdot 10^4} = 109{,}48\,m$ και $y_c = \frac{Q_x}{A} = \frac{134{,}6\cdot 10^4}{2{,}11\cdot 10^4} = 63{,}79\ m$

β' Τρόπος: Βάσει γνωστού τύπου έχω ότι το κέντρο κυκλικού τόξου βρίσκεται πάνω στην διχοτόμο του σε απόσταση $d = \frac{2R}{3\omega}\sin\omega$, όπου ω η μισή γωνία του τόξου.

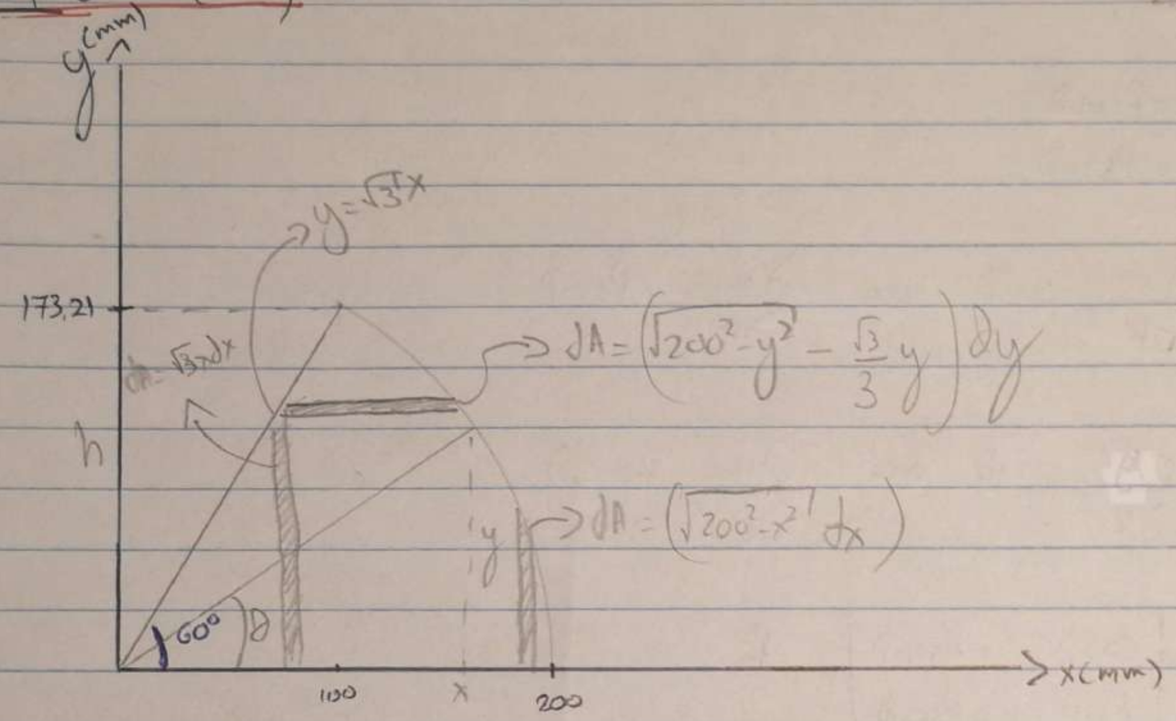
Άρα $d = \frac{2\cdot 200}{3\cdot\frac{\pi}{6}}\cdot\sin\frac{\pi}{6} = \frac{400}{\frac{\pi}{2}}\cdot\frac{1}{2} = \frac{400}{\pi} = 127{,}32$



Άρα $y_c = d\sin\omega = \frac{127{,}32}{2} = 63{,}66m$ και $x_c = \frac{127{,}32\sqrt{3}}{2} = 110{,}12\,m$ Άρα $C(110{,}32,\ 63{,}66)$

Ο υπολογισμός των $I_{x_cx_c}$ και $I_{y_cy_c}$ είναι κοινός και για τους 2 τρόπους οπότε γίνεται μια φορά στο τέλος.

Άσκηση 4 (2^{ος} Τρόπος)



$\tan(60°)=\frac{h}{100} \Leftrightarrow h=100\sqrt{3}=173{,}21$

Η εξίσωση του κύκλου: $x^2+y^2=200^2$

Η συνάρτηση του άνω ημικυκλίου: $y=\sqrt{200^2-x^2}$

Λύνοντας ως προς x για το δεξί ημικύκλιο έχω την εξίσωση: $x=\sqrt{200^2-y^2}$

1^{ος} Τρόπος (Υπάρχει λυμένο παρακάτω)

Για I_{xx} λύνω $I_{xx}=\iint_A y^2\,dA=\int_0^{100\sqrt{3}}\left[y^2\cdot\left(\sqrt{200^2-y^2}-\frac{\sqrt{3}}{3}y\right)\right]dy$

και για $I_{yy}=\iint_A x^2\,dA=\int_0^{100}x^2\sqrt{3}\,x\,dx+\int_{100}^{200}x^2\sqrt{200^2-x^2}\,dx$

2ος Τρόπος

Έστω $x = R\cos\theta$

$y = R\sin\theta$

Άρα $\dfrac{\partial x}{\partial R} = \cos\theta$ και $\dfrac{\partial x}{\partial \theta} = -R\sin\theta$

και $\dfrac{\partial y}{\partial R} = \sin\theta$ και $\dfrac{\partial y}{\partial \theta} = R\cos\theta$

Άρα $\begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix} = R$

Άρα $I_{xx} = \iint_A y^2 \, dx\, dy = \iint R^2 \sin^2\theta \cdot R\, dR\, d\theta = \int_0^{\frac{\pi}{3}} \sin^2\theta \left(\int_0^{200} R^3\, dR \right) d\theta =$

$= \int_0^{\frac{\pi}{3}} \sin^2\theta \left[\dfrac{R^4}{4} \right]_0^{200} d\theta = \int_0^{\frac{\pi}{3}} \dfrac{200^4}{4} \sin^2\theta \, d\theta = \dfrac{200^4}{4} \int_0^{\frac{\pi}{3}} \dfrac{1-\cos 2\theta}{2} d\theta =$

$= \dfrac{200^4}{8} \int_0^{\frac{\pi}{3}} (1-\cos 2\theta)\, d\theta = \dfrac{200^4}{8} \left[\theta - \dfrac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{3}} = \dfrac{200^4}{8} \left(\dfrac{\pi}{3} - \dfrac{\sin\frac{2\pi}{3}}{2} \right) =$

$= \dfrac{200^4}{8} \left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4} \right) = \dfrac{200^4}{8} \cdot 0{,}61 = 0{,}08 \cdot 200^4 = 128 \cdot 10^6 \text{ mm}^4$

$I_{yy} = \iint_A x^2\, dx\, dy = \iint R^2 \cos^2\theta \cdot R\, dR\, d\theta = \int_0^{\frac{\pi}{3}} \cos^2\theta \left(\int_0^{200} R^3\, dR \right) d\theta = \int_0^{\frac{\pi}{3}} \cos^2\theta \left[\dfrac{R^4}{4} \right]_0^{200} d\theta =$

$= \int_0^{\frac{\pi}{3}} \cos^2\theta \cdot \dfrac{200^4}{4} d\theta = \dfrac{200^4}{4} \int_0^{\frac{\pi}{3}} \dfrac{\cos 2\theta + 1}{2} = \dfrac{200^4}{8} \left[\dfrac{\sin 2\theta}{2} + \theta \right]_0^{\frac{\pi}{3}} = \dfrac{200^4}{8} \left(\dfrac{\sin\frac{2\pi}{3}}{2} + \dfrac{\pi}{3} \right) = \dfrac{200^4}{8} \left(\dfrac{\sqrt{3}}{4} + \dfrac{\pi}{3} \right)$

$= \dfrac{200^4}{8} \cdot 1{,}48 = 0{,}185 \cdot 200^4 = 296 \cdot 10^6 \text{ mm}^4$

β) Για την εύρεση του γ. κέντρου:

$$Q_x = \iint_A y\,dA = \iint_A R\sin\theta \cdot R\,dR\,d\theta = \int_0^{\frac{\pi}{3}} \sin\theta \left(\int_0^{200} R^2\,dR\right) d\theta = \int_0^{\frac{\pi}{3}} \sin\theta \left[\frac{R^3}{3}\right]_0^{200} d\theta =$$

$$= \int_0^{\frac{\pi}{3}} \frac{200^3}{3} \sin\theta\,d\theta = \frac{200^3}{3} \int_0^{\frac{\pi}{3}} \sin\theta\,d\theta = \frac{200^3}{3}\left[-\cos\theta\right]_0^{\frac{\pi}{3}} = \frac{200^3}{3}\left(-\frac{1}{2}+1\right) = \frac{200^3}{6} =$$

$$= 133 \cdot 10^4\ mm^3$$

$$Q_y = \iint_A x\,dA = \iint_A R\cos\theta\,R\,dR\,d\theta = \int_0^{\frac{\pi}{3}} \cos\theta \left(\int_0^{200} R^2\,dR\right) d\theta = \int_0^{\frac{\pi}{3}} \cos\theta \left[\frac{R^3}{3}\right]_0^{200} d\theta =$$

$$= \int_0^{\frac{\pi}{3}} \frac{200^3}{3}\cos\theta\,d\theta = \frac{200^3}{3}\left[\sin\theta\right]_0^{\frac{\pi}{3}} = \frac{200^3}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{6} 200^3 = 231 \cdot 10^4\ mm^3$$

Επίσης $A = \iint dA = \iint R\,dR\,d\theta = \int_0^{\frac{\pi}{3}} \left(\int_0^{200} R\,dR\right) d\theta = \int_0^{\frac{\pi}{3}} \left[\frac{R^2}{2}\right]_0^{200} d\theta = \int_0^{\frac{\pi}{3}} \frac{200^2}{2}\,d\theta =$

$$= \left[\frac{200^2}{2}\theta\right]_0^{\frac{\pi}{3}} = \frac{200^2 \pi}{6} = 2{,}1 \cdot 10^4\ mm^2$$

Άρα $x_c = \frac{Q_y}{A} = \frac{231 \cdot 10^4}{2{,}1 \cdot 10^4} = 110\,m$ και $y_c = \frac{Q_x}{A} = \frac{133 \cdot 10^4}{2{,}1 \cdot 10^4} = 63{,}33\ m$

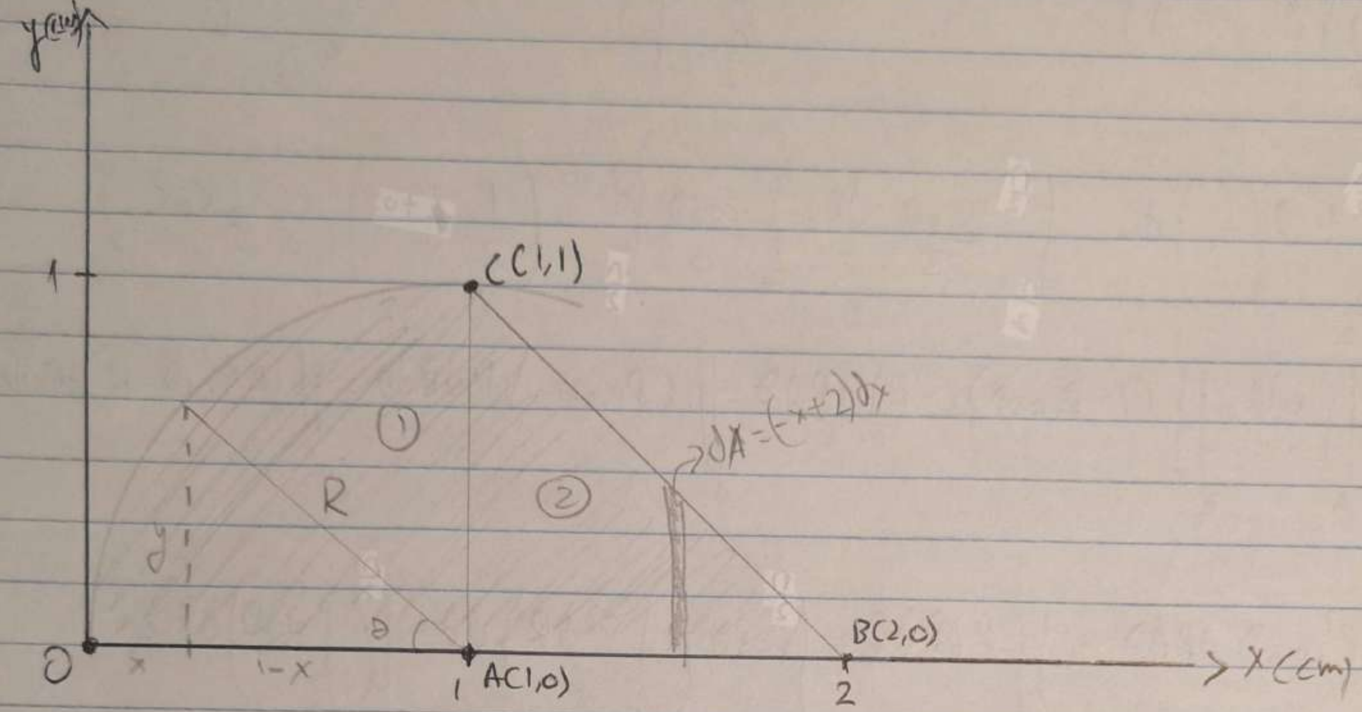
Άρα $C(110,\ 63{,}33)$

Υπολογίζω $I_{x_cx_c}$ και $I_{y_cy_c}$

$$I_{x_cx_c} = I_{xx} - A d_1^2 = 128 \cdot 10^6 - 2{,}1 \cdot 10^4 \cdot y_c^2 = 128 \cdot 10^6 - 2{,}1 \cdot 10^4 \cdot (63{,}33)^2 = 43{,}78 \cdot 10^6 \, mm^4$$

$$I_{y_cy_c} = I_{yy} - A d_2^2 = 296 \cdot 10^6 - 2{,}1 \cdot 10^4 \cdot x_c^2 = 296 \cdot 10^6 - 2{,}1 \cdot 10^4 \cdot 110^2 = 41{,}9 \cdot 10^6 \, mm^4$$

Άσκηση 5



Έχω $1-x=R\cos\theta \Leftrightarrow x=1-R\cos\theta$
$y=R\sin\theta$

Άρα $\frac{\partial x}{\partial R}=-\cos\theta$, $\frac{\partial x}{\partial \theta}=R\sin\theta$
$\frac{\partial y}{\partial R}=\sin\theta$, $\frac{\partial y}{\partial \theta}=R\cos\theta$ $\Bigg\} \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}-\cos\theta & R\sin\theta\\ \sin\theta & R\cos\theta\end{vmatrix}=-R$

Έχω $A_1=\iint_A \partial A=\iint_A -R\,\partial R\,\partial\theta=\int_{\frac{\pi}{2}}^{0}\left(\int_0^1 -R\,\partial R\right)\partial\theta=\int_{\frac{\pi}{2}}^{0}\left[-\frac{R^2}{2}\right]_0^1\partial\theta=\int_{\frac{\pi}{2}}^{0}-\frac{1}{2}\,\partial\theta=-\frac{1}{2}\left[\theta\right]_{\frac{\pi}{2}}^{0}=$

$=-\frac{1}{2}\left(-\frac{\pi}{2}\right)=\frac{\pi}{4}=0,79\ cm^2$

$A_2=\frac{1}{2}=0,5\ cm^2$

Άρα $A=A_1+A_2=1,29\ cm^2$

Έχω $G\left(1-\frac{4R}{3\pi},\ \frac{4R}{3\pi}\right)=(0,58,\ 0,42)$

Εναλλακτικά:

$$Q_{x_1} = \iint_A y\,dA = \iint R\sin\theta(-R)\,dR\,d\theta = \int_{\frac{\pi}{2}}^{0} \sin\theta\left(\int_0^1 -R^2\,dR\right)d\theta = \int_{\frac{\pi}{2}}^{0} \sin\theta\left[-\frac{R^3}{3}\right]_0^1 d\theta$$

$$= \int_{\frac{\pi}{2}}^{0} \sin\theta\left(-\frac{1}{3}\right)d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{3}\,d\theta = \left[\frac{-\cos\theta}{3}\right]_0^{\frac{\pi}{2}} = \left(\frac{1}{3}\right) = \frac{1}{3}\,cm^3$$

$$Q_{y_1} = \iint_A x\,dA = \iint_A (1-R\cos\theta)(-R)\,dR\,d\theta = \iint_A (R\cos\theta-1)R\,dR\,d\theta = \iint_A (R^2\cos\theta - R)\,dR\,d\theta =$$

$$= \iint_A R^2\cos\theta\,dR\,d\theta - \iint_A R\,dR\,d\theta = \int_{\frac{\pi}{2}}^{0} \cos\theta\left(\int_0^1 R^2\,dR\right)d\theta - \int_{\frac{\pi}{2}}^{0}\left(\int_0^1 R\,dR\right)d\theta =$$

$$= \int_{\frac{\pi}{2}}^{0} \cos\theta\left[\frac{R^3}{3}\right]_0^1 d\theta - \int_{\frac{\pi}{2}}^{0}\left[\frac{R^2}{2}\right]_0^1 d\theta = \int_{\frac{\pi}{2}}^{0} \frac{\cos\theta}{3}\,d\theta - \int_{\frac{\pi}{2}}^{0} \frac{1}{2}\,d\theta =$$

$$= \left[\frac{\sin\theta}{3}\right]_{\frac{\pi}{2}}^{0} + \left[\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} = -\frac{1}{3} + \frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{3} = 0,45$$

• Για το γινόμενο του τριγώνου έχω $G_2\left(\frac{4}{3}, \frac{1}{3}\right)$

Άρα $Q_{x_2} = A_2 y_{c_2} = 0,5 \cdot \frac{1}{3} = 0,17\,m^3$

$Q_{y_2} = A_2 x_{c_2} = 0,5 \cdot \frac{4}{3} = \frac{2}{3} = 0,67\,m^3$

Άρα $Q_x = Q_{x_1} + Q_{x_2} = 0,5\,m^3$ και $Q_y = Q_{y_1} + Q_{y_2} = 1,12\,m^3$

Άρα $x_c = \frac{Q_y}{A} = \frac{1,12}{1,29} = 0,87\,m$

και $y_c = \frac{Q_x}{A} = \frac{0,5}{1,29} = 0,39\,m$

$$\left.\begin{matrix} x_c = 0,87\,m \\ y_c = 0,39\,m \end{matrix}\right\} C(0,87,\ 0,39)$$

a) $E_{x\omega}$ $\quad I_{xx_1} = \iint_A y^2 dA = \iint_A R^2 \sin^2\theta(-R)\, dR\, d\theta = \int_{\frac{\pi}{2}}^{0} \sin^2\theta \left(\int_0^1 -R^3\, dR\right) d\theta =$

$= \int_{\frac{\pi}{2}}^{0} \sin^2\theta \left[\frac{-R^4}{4}\right]_0^1 d\theta = \int_{\frac{\pi}{2}}^{0} -\frac{\sin^2\theta}{4}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\, d\theta}{4} = \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{8}\, d\theta =$

$= \frac{1}{8}\int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)\, d\theta = \frac{1}{8}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}} = \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16} = 0{,}2\ cm^4$

$I_{yy_1} = \iint_A x^2 dA = \iint_A (1 - R\cos\theta)^2 (-R)\, dR\, d\theta = \iint_A (1 - 2R\cos\theta + R^2\cos^2\theta)(-R)\, dR\, d\theta =$

$= \iint_A (-R + 2R^2\cos\theta - R^3\cos^2\theta)\, dR\, d\theta = \iint_A -R\, dR\, d\theta + 2\iint_A R^2 \cos\theta\, dR\, d\theta - \iint_A R^3 \cos^2\theta\, dR\, d\theta =$

$= \int_{\frac{\pi}{2}}^{0}\left(\int_0^1 -R\, dR\right) d\theta + 2\int_{\frac{\pi}{2}}^{0} \cos\theta \left(\int_0^1 R^2\, dR\right) d\theta - \int_{\frac{\pi}{2}}^{0} \cos^2\theta \left(\int_0^1 R^3\, dR\right) d\theta =$

$= \int_{\frac{\pi}{2}}^{0} \left[-\frac{R^2}{2}\right]_0^1 d\theta + 2\int_{\frac{\pi}{2}}^{0} \cos\theta \left[\frac{R^3}{3}\right]_0^1 d\theta - \int_{\frac{\pi}{2}}^{0} \cos^2\theta \left[\frac{R^4}{4}\right]_0^1 d\theta =$

$= \int_{\frac{\pi}{2}}^{0} -\frac{1}{2}\, d\theta + 2\int_{\frac{\pi}{2}}^{0} \frac{\cos\theta}{3}\, d\theta - \int_{\frac{\pi}{2}}^{0} \frac{\cos^2\theta\, d\theta}{4} = \left[\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} + \frac{2}{3}\left[\sin\theta\right]_{\frac{\pi}{2}}^{0} + \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{8}\, d\theta =$

$= \frac{\pi}{4} - \frac{2}{3} + \frac{1}{8}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} - \frac{2}{3} + \frac{1}{8}\cdot\frac{\pi}{2} = \frac{\pi}{4} + \frac{\pi}{16} - \frac{2}{3} = 0{,}32\ cm^4$

$$I_{xx_2} = \frac{bh^3}{12} = \frac{1 \cdot 1^3}{12} = \frac{1}{12} = 0{,}08 \text{ cm}^4$$

$$I_{yy_2} = \iint_A x^2 dA = \int_1^2 x^2 (2-x) dx = \int_1^2 (2x^2 - x^3) dx = \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 = \frac{2^4}{3} - \frac{2^4}{4} - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{2^4 - 2}{3} + \frac{1 - 2^4}{4} = 4{,}67 - 3{,}75 = 0{,}92 \text{ cm}^4$$

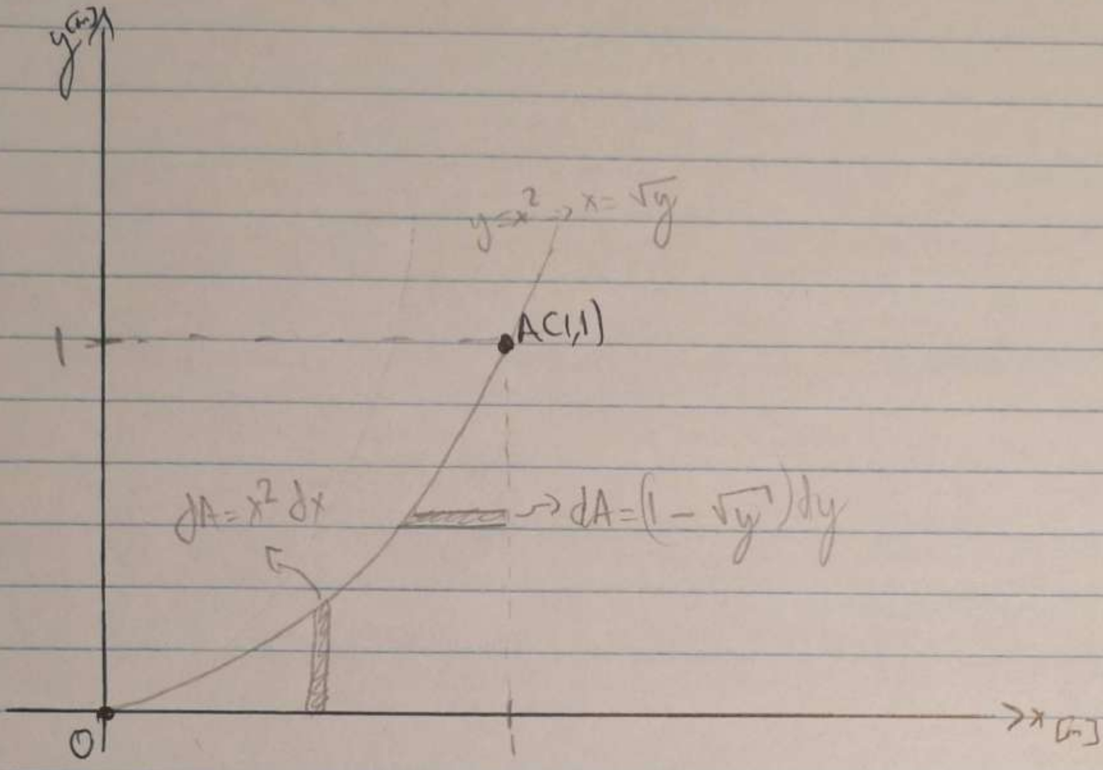
Άρα $I_{xx} = I_{xx_1} + I_{xx_2} = 0{,}2 + 0{,}08 = 0{,}28 \text{ cm}^4$

$I_{yy} = I_{yy_1} + I_{yy_2} = 0{,}32 + 0{,}92 = 1{,}24 \text{ cm}^4$

Άρα $I_{x_c x_c} = I_{xx} - A \cdot y_c^2 = 0{,}28 - 1{,}29 \cdot (0{,}39)^2 = 0{,}28 - 0{,}2 = 0{,}08 \text{ cm}^4$

και $I_{y_c y_c} = I_{yy} - A x_c^2 = 1{,}24 - 1{,}29 \cdot (0{,}87)^2 = 1{,}27 - 0{,}98 = 0{,}29 \text{ cm}^4$

Άσκηση 6



$$I_{xx}=\int y^2\,dA=\int_0^1 y^2(1-y^{\frac{1}{2}})\,dy=\int_0^1\left(y^2-y^{\frac{5}{2}}\right)dy=\left[\frac{y^3}{3}-\frac{2y^{\frac{7}{2}}}{7}\right]_0^1=\left[\frac{y^3}{3}-\frac{2}{7}y^3\sqrt{y}\right]_0^1=$$

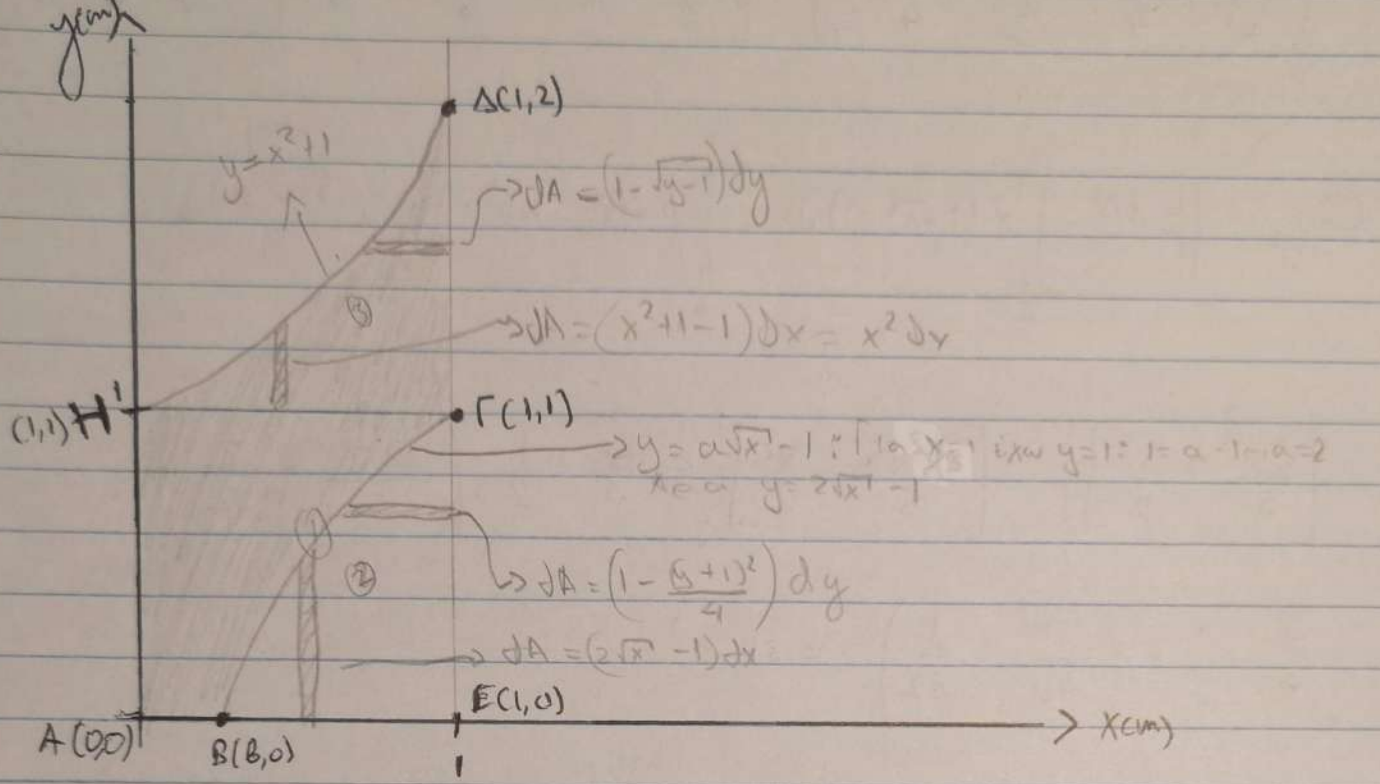
$$=\frac{1}{3}-\frac{2}{7}=\ldots=0{,}05\ m^4$$

$$I_{yy}=\int x^2\,dA=\int_0^1 x^2\cdot x^2\,dx=\int_0^1 x^4\,dx=\left[\frac{x^5}{5}\right]_0^1=\frac{1}{5}=0{,}2\ m^4$$

$$I_{xy}=I_{yx}=\int xy\,dA=\iint xy\,dx\,dy=\int_0^1 x\int_0^{x^2} y\,dy\,dx=\int_0^1 x\left[\frac{y^2}{2}\right]_0^{x^2}dx=\int_0^1\frac{x^5}{2}\,dx=\left[\frac{x^6}{12}\right]_0^1=\frac{1}{12}=0{,}08\,m^4$$

Άρα $I_{i,j}=\begin{bmatrix}0{,}05 & 0{,}08\\ 0{,}08 & 0{,}2\end{bmatrix}$ $i,j=x,y$ $[m^4]$

Άσκηση 7



Έχω: ΑΕΓΗ: ① Για $x=B$ και $y=0$ έχω: $0=2\sqrt{B}-1 \Rightarrow \sqrt{B}=\frac{1}{2} \Rightarrow B=\frac{1}{4}$

$\widehat{B\Gamma E}$: ②

$\widehat{\Gamma\Delta H}$: ③

$A_1 = 1\cdot 1 = 1\,m^2$

Έχω $G_1=(0{,}5,\ 0{,}5) \Rightarrow Q_{x_1} = y_{G_1}\cdot A_1 = 0{,}5\,m^3$ και $Q_{y_1} = x_{G_1}\cdot A_1 = 0{,}5$

$$A_2 = \int_{\frac{1}{4}}^{1} (2\sqrt{x}-1)\,dx = \int_{\frac{1}{4}}^{1} (2x^{\frac{1}{2}}-1)\,dx = \left[\frac{4x^{\frac{3}{2}}}{3} - x\right]_{\frac{1}{4}}^{1} = \left[\frac{4x\sqrt{x}}{3} - x\right]_{\frac{1}{4}}^{1} = \frac{4}{3} - 1 - \frac{1}{6} + \frac{1}{4} = 0{,}42\,m^2$$

$$A_3 = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} = 0{,}33\,m^2$$

Άρα $A = A_1 - A_2 + A_3 = 1 - 0{,}42 + 0{,}33 = 0{,}91\,m^2$

$$Q_{x_2} = \iint_A y\,dA = \int_0^1 y\left(1 - \frac{(y+1)^2}{4}\right)dy = \int_0^1 \left(y - \frac{y(y^2+2y+1)}{4}\right)dy = \int_0^1 \left(y - \frac{y^3+2y^2+y}{4}\right)dy =$$

$$= \int_0^1 \left(y - \frac{y^3}{4} - \frac{y^2}{2} - \frac{y}{4} \right) dy = \left[\frac{y^2}{2} - \frac{y^4}{16} - \frac{y^3}{6} - \frac{y^2}{8} \right]_0^1 = \frac{1}{2} - \frac{1}{16} - \frac{1}{6} - \frac{1}{8} = 0{,}15\,m^3$$

$$Q_{y_2} = \iint_A x\, dA = \int_{\frac{1}{4}}^1 x\,(2\sqrt{x} - 1)\, dx = \int_{\frac{1}{4}}^1 (2x\sqrt{x} - x)\, dx = \int_{\frac{1}{4}}^1 \left(2x^{\frac{3}{2}} - x\right) dx =$$

$$= \left[\frac{4x^{\frac{5}{2}}}{5} - \frac{x^2}{2} \right]_{\frac{1}{4}}^1 = \left[\frac{4x^2\sqrt{x}}{5} - \frac{x^2}{2} \right]_{\frac{1}{4}}^1 = \frac{4}{5} - \frac{1}{2} - \left(\frac{4\frac{1}{16}\cdot\frac{1}{2}}{5} - \frac{\frac{1}{16}}{2} \right) =$$

$$= \frac{4}{5} - \frac{1}{2} - \left(\frac{1}{40} - \frac{1}{32} \right) = 0{,}31\ m^3$$

$$Q_{x_3} = \iint_A y\, dA = \int_1^2 y\,(1 - \sqrt{y-1})\, dy = \int_1^2 (y - y\sqrt{y-1})\, dy = \left[\frac{y^2}{2} \right]_1^2 - \int_1^2 y\sqrt{y-1}\, dy$$

$$= 2 - \frac{1}{2} - \int_1^2 y\sqrt{y-1}\, dy = \frac{3}{2} - \int_1^2 y\sqrt{y-1}\, dy$$ Θέτω $u = y-1 \Rightarrow du = dy$, $u_1 = 0$, $u_2 = 1$

Άρα $$Q_{x_3} = \frac{3}{2} - \int_0^1 (u+1)\sqrt{u}\, du = \frac{3}{2} - \int_0^1 (u\sqrt{u} + \sqrt{u})\, du = \frac{3}{2} - \int_0^1 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}} \right) du =$$

$$= \frac{3}{2} - \left[\frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} \right]_0^1 = \frac{3}{2} - \frac{2}{5} + \frac{2}{3} = 0{,}43\ m^4$$

$$Q_{y_3} = \iint_A x\, dA = \int_0^1 x^3\, dA = \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4} = 0{,}25$$

Άρα $Q_x = Q_{x_1} - Q_{x_2} + Q_{x_3} = 0{,}5 - 0{,}15 + 0{,}43 = 0{,}78\ m^3$

$Q_y = Q_{y_1} - Q_{y_2} + Q_{y_3} = 0{,}5 - 0{,}31 + 0{,}25 = 0{,}44\ m^3$

Άρα $x_G = \frac{Q_y}{A} = \frac{0,44}{0,91} = 0,48\,m$

$y_G = \frac{Q_x}{A} = \frac{0,78}{0,91} = 0,86\,m$

$\Bigg\} G(0,48,\ 0,86)$

$$I_{xx_2} = \iint_{A_2} y^2 dA = \int_0^1 y^2\left(1 - \frac{(y+1)^2}{4}\right) dy = \int_0^1 \left[y^2 - \frac{y^2}{4}(y^2+2y+1)\right] dy =$$

$$= \int_0^1 \left(y^2 - \frac{y^4}{4} - \frac{y^3}{2} - \frac{y^2}{4}\right) dy = \left[\frac{y^3}{3} - \frac{y^5}{20} - \frac{y^4}{8} - \frac{y^3}{12}\right]_0^1 =$$

$$= \frac{1}{3} - \frac{1}{20} - \frac{1}{8} - \frac{1}{12} = 0,08\ m^4$$

$$I_{yy_2} = \iint x^2 dA = \int_0^1 x^2(2\sqrt{x} - 1) dx = \int_0^1 \left(2x^{\frac{5}{2}} - x^2\right) dx = \left[\frac{4x^{\frac{7}{2}}}{7} - \frac{x^3}{3}\right]_0^1 = \frac{4}{7} - \frac{1}{3} = 0,24\ m^4$$

$$I_{xx_3} = \iint_{A_3} y^2 dA = \int_1^2 y^2\left(1 - \sqrt{y-1}\right) dy = \int_1^2 \left(y^2 - y^2\sqrt{y-1}\right) dy = \left[\frac{y^3}{3}\right]_1^2 - \int_1^2 y^2\sqrt{y-1}\, dy =$$

$$= \frac{8}{3} - \frac{1}{3} - \int_1^2 y^2 \sqrt{y-1}\, dy \quad \text{Θέτω } u = y-1 \Rightarrow du = dy,\ u_1 = 0,\ u_2 = 1$$

$$\text{Άρα } I_{xx_3} = \frac{7}{3} - \int_0^1 (u+1)^2\sqrt{u}\, du = \frac{7}{3} - \int_0^1 \left(u^2+2u+1\right)\sqrt{u}\, du =$$

$$= \frac{7}{3} - \int_0^1 u^{\frac{5}{2}} du - 2\int_0^1 u^{\frac{3}{2}} du - \int_0^1 u^{\frac{1}{2}} du = \frac{7}{3} - \left[\frac{2u^{\frac{7}{2}}}{7}\right]_0^1 - 2\left[\frac{2u^{\frac{5}{2}}}{5}\right]_0^1 - \left[\frac{2u^{\frac{3}{2}}}{3}\right]_0^1 =$$

$$= \frac{7}{3} - \frac{2}{7} - \frac{4}{5} - \frac{2}{3} = 0,58\ m^4$$

$$I_{yy_3} = \iint_{A_3} x^2\, dA = \int_0^1 x^2 \cdot x^2\, dA = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5} = 0,2$$

Έχω $I_{xx_1} = I_{yy_1} = \frac{1}{3}\ m^4$ $\left(I_{xx} = \frac{bh^3}{3} \text{ και } I_{yy} = \frac{hb^3}{3} \right)$

Άρα $I_{xx} = I_{xx_1} - I_{xx_2} + I_{xx_3} = \frac{1}{3} - 0,08 + 0,58 = 0,83\ m^4$

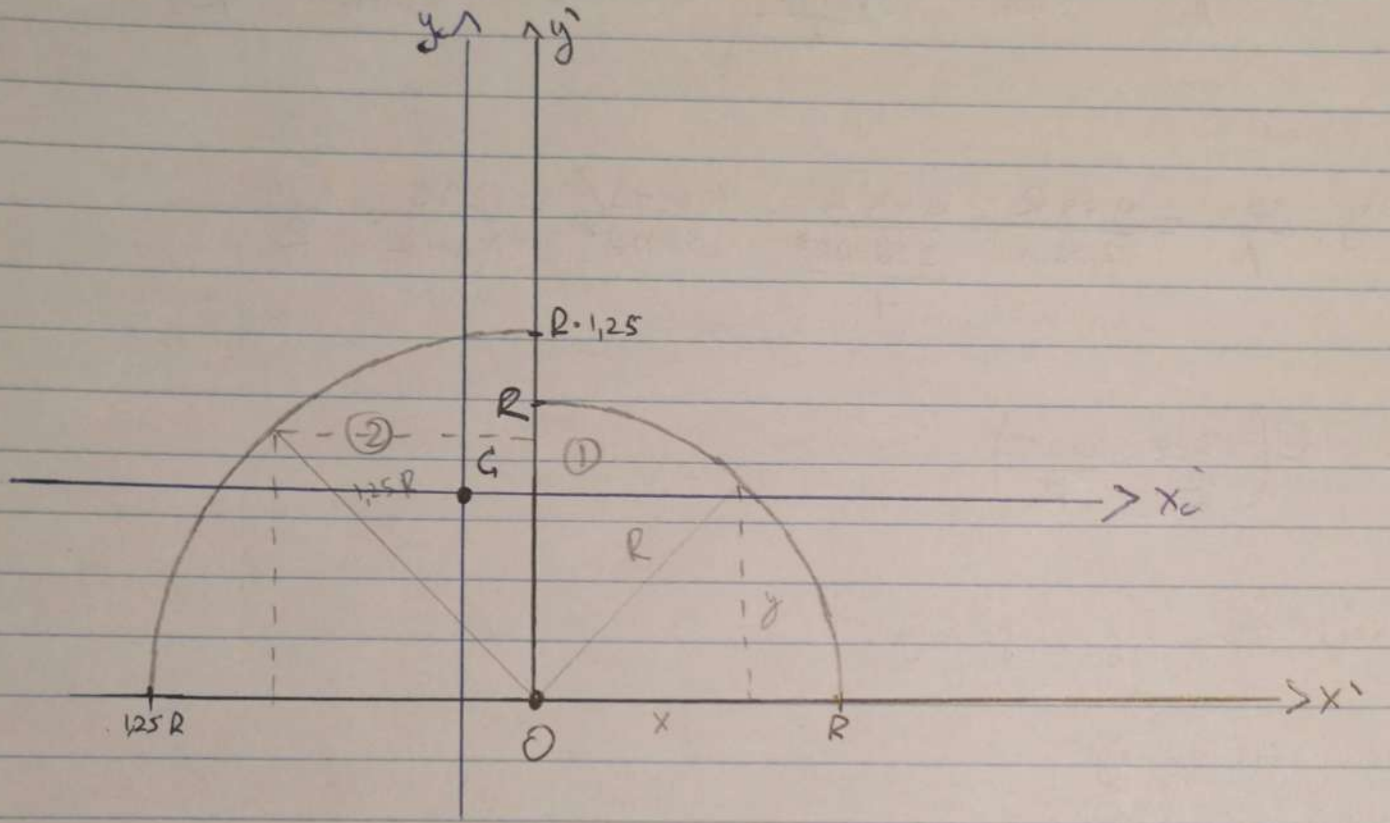
και $I_{yy} = I_{yy_1} - I_{yy_2} + I_{yy_3} = \frac{1}{3} - 0,24 + 0,2 = 0,29\ m^4$

β) Έχω $I_{xx} = 0,32\ m^4$ και $G(0,48,\ 0,86)$
$I_{yy} = 0,29\ m^4$

Άρα $I_{x_G x_G} = I_{xx} - A \cdot y_G^2 = 0,83 - 0,91 \cdot (0,86)^2 = 0,16\ m^4$

και $I_{y_G y_G} = I_{yy} - A \cdot x_G^2 = 0,29 - 0,91 \cdot (0,48)^2 = 0,08\ m^4$

Άσκηση 8



Για το τεταρτοκύκλιο (1) έχω ότι $x_1 = y_{c1} = \frac{4R}{3\pi} \Rightarrow C_1\left(\frac{4R}{3\pi}, \frac{4R}{3\pi}\right)$

Για το τεταρτοκύκλιο (2) έχω ότι $-y_{c2} = x_{c2} = \frac{-4\cdot 1{,}25R}{3\pi} = \frac{-4(\frac{5}{4})R}{3\pi} = \frac{-5R}{3\pi} \Rightarrow C_2 = \left(\frac{-5R}{3\pi}, \frac{5R}{3\pi}\right)$

Επίσης έχω $A_1 = \frac{\pi R^2}{4}$ και $A_2 = \frac{\pi(1{,}25R)^2}{4} = \frac{\pi\left(\frac{5}{4}\right)^2 R^2}{4} = \left(\frac{5}{4}\right)^2 A_1 = \frac{25}{16}A_1 = 1{,}56A_1$

Άρα $A = A_1 + A_2 = 2{,}56A_1$

Άρα $Q_x = \frac{A_1\cdot y_{c1} + A_2\cdot y_{c2}}{A} = \frac{A_1\cdot\frac{4R}{3\pi} + 1{,}56A_1\cdot\left(\frac{+5R}{3\pi}\right)}{2{,}56A_1} = \frac{\frac{4RA_1}{3\pi} + \frac{7{,}8RA_1}{3\pi}}{2{,}56A_1} = \frac{11{,}8A_1\cdot R}{3\pi\cdot 2{,}56A_1} =$

$= 0{,}49R$

$Q_y = \frac{A_1\cdot x_{c1} + A_2\cdot x_{c2}}{A} = \frac{A_1\cdot\frac{4R}{3\pi} + A_2\left(-\frac{5R}{3\pi}\right)}{A} = \frac{A_1\frac{4R}{3\pi} - 1{,}56\cdot A_1\cdot\frac{5R}{3\pi}}{2{,}56A_1} = \frac{\frac{4A_1R}{3\pi} - \frac{7{,}8A_1R}{3\pi}}{2{,}56A_1} =$

$= \frac{-3{,}8A_1R}{3\pi\cdot 2{,}56A_1} = \frac{-3{,}8R}{3\pi\cdot 2{,}56} = -0{,}16R$

Άρα $x_c = \frac{Q_y}{A} = \frac{-0,16R}{2,56A_1} = \frac{-0,16R}{\frac{2,56\cdot \pi R^2}{4}} = \frac{-0,16\cdot 4\cdot \cancel{R}}{\pi\cdot 2,56\cdot R^{\cancel{2}}} = \frac{-0,64}{8,04R} = \frac{-0,08}{R}$

και $y_c = \frac{Q_x}{A} = \frac{0,49R}{2,56\cdot A_1} = \frac{0,49R}{\frac{2,56\cdot \pi R^2}{4}} = \frac{4\cdot 0,49\cancel{R}}{2,56\cdot \pi R^{\cancel{2}}} = \frac{1,96}{8,04R} = \frac{0,24}{R}$

Άρα $C\left(\frac{-0,08}{R}, \frac{0,24}{R}\right)$

Ζητείται το I_{ij}, $i,j = x_c, y_c$.

Βρίσκω I_{ij}, $i,j = x'y'$

Για (1) έχω: $\left.\begin{array}{l} x = R\cos\theta \\ y = R\sin\theta \end{array}\right\} \frac{\partial(x,y)}{\partial(r,\theta)} = R$

Άρα $I_{x_1x_1} = \iint_A y^2\, dx\, dy = \iint_A R^2\sin^2\theta \cdot R\, dR\, d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta \left(\int_0^R R^3\, dR\right) d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cdot R^4}{4}\, d\theta =$

$= \frac{R^4}{4}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \frac{R^4}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{R^4}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{R^4\cdot\pi}{16}$

$I_{y_1y_1} = \iint_A x^2\, dx\, dy = \iint_A R^2\cos^2\theta\, R\, dR\, d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta \left(\int_0^R R^3\, dR\right) d\theta = \int_0^{\frac{\pi}{2}} \frac{R^4}{4}\cos^2\theta\, d\theta =$

$= \frac{R^4}{4}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{R^4}{8}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{R^4\cdot\pi}{16}$

Για το (2) I_{xx} : $x = 1,25 R\cos\theta$

$y = 1,25 R\sin\theta$

$$\frac{\partial x}{\partial R} = 1,25\cos\theta \quad \frac{\partial x}{\partial \theta} = -1,25 R\sin\theta$$

$$\frac{\partial y}{\partial R} = 1,25\sin\theta \quad \frac{\partial y}{\partial \theta} = 1,25 R\cos\theta$$

$$\frac{\partial(x,y)}{\partial(R,\theta)} = \begin{vmatrix} 1,25\cos\theta & -1,25R\sin\theta \\ 1,25\sin\theta & 1,25R\cos\theta \end{vmatrix} =$$

$$= R(1,25\cos\theta)^2 + R(1,25\sin\theta)^2 = 1,25^2 R(\sin^2\theta + \cos^2\theta) = 1,25^2 R = 1,56R$$

Άρα $I_{xx_2} = \iint_A y^2\,dx\,dy = \iint_A 1,56R^2\sin^2\theta \cdot 1,56R\,\partial R\,\partial\theta = 1,56^2 \int_{\frac{\pi}{2}}^{\pi} \sin^2\theta \int_0^R R^3\,\partial R\,\partial\theta =$

$$= 1,56^2 \cdot \frac{R^4}{4} \int_{\frac{\pi}{2}}^{\pi} \frac{1-\cos 2\theta}{2} = \frac{1,56^2}{8} \cdot R^4 \left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{\pi} = \frac{1,56^2 \cdot R^4}{8}\left(\pi - \frac{\pi}{2}\right) =$$

$$= \frac{1,56^2 \cdot R^4}{16}\pi$$

$$I_{yy_2} = \iint_A x^2\,dx\,dy = \iint_A 1,56R^2\cos^2\theta \cdot 1,56R\,\partial R\,\partial\theta = 1,56^2 \iint_A \cos^2\theta\, R^3\,\partial R\,\partial\theta =$$

$$= 1,56^2 \int_{\frac{\pi}{2}}^{\pi} \cos^2\theta \int_0^R R^3\,\partial R\,\partial\theta = \frac{1,56^2 \cdot R^4}{4} \int_{\frac{\pi}{2}}^{\pi} \frac{\cos 2\theta + 1}{2}\,\partial\theta = \frac{1,56^2 R^4}{8}\left[\frac{\sin 2\theta}{2} + \theta\right]_{\frac{\pi}{2}}^{\pi} =$$

$$= \frac{1,56^2 R^4}{8} \cdot \frac{\pi}{2} = \frac{1,56^2 R^4 \pi}{16}$$

Άρα $I_{xx} = I_{xx_1} + I_{xx_2} = \frac{\pi R^4}{16} + \frac{1,56\cdot\pi R^4}{16} = \frac{2,56\pi R^4}{16}$

$I_{y'y'} = I_{yy_1} + I_{y'y'_2} = \frac{\pi R^4}{16} + \frac{1,56\cdot\pi R^4}{16} = \frac{2,56\pi R^4}{16}$

Επίσης $I_{x'y'_1} = \iint_{A_1} xy\,dA = \iint_{A_1} R^2\cos\theta\sin\theta\cdot R\,dR\,d\theta = \int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\int_0^R R^3\,dR\,d\theta =$

$= \frac{R^4}{8}\int_0^{\frac{\pi}{2}}\sin 2\theta\,d\theta = \frac{R^4}{8}\left[\frac{-\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{R^4}{16}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{R^4}{16}(1+1) = \frac{R^4}{8} = I_{yx_1}$

και $I_{x'y'_2} = \iint_{A_2} xy\,dA = \iint_{A_2} 1,56R^2\cos\theta\sin\theta\cdot 1,56R\,dR\,d\theta = \iint_{A_2} 1,56^2\sin\theta\cos\theta R^3\,dR\,d\theta =$

$= 1,56^2\int_{\frac{\pi}{2}}^{\pi}\frac{\sin 2\theta}{2}\left(\int_0^R R^3\,dR\right)d\theta = \frac{1,56^2R^4}{8}\int_{\frac{\pi}{2}}^{\pi}\sin 2\theta\,d\theta = \frac{1,56^2R^4}{8}\left[\frac{-\cos 2\theta}{2}\right]_{\frac{\pi}{2}}^{\pi} =$

$= \frac{1,56^2R^4}{16}\left[-\cos 2\theta\right]_{\frac{\pi}{2}}^{\pi} = \frac{1,56^2R^4}{16}(-1-1) = -\frac{1,56^2R^4}{8} = I_{yx_2}$

Άρα $I_{x'y'} = I_{x'y'_1} + I_{x'y'_2} = \frac{R^4}{8} - \frac{1,56^2R^4}{8} = \frac{-1,43R^4}{8} = -0,18R^4$

Άρα $I_{ij} = \begin{bmatrix} \frac{2,56\pi R^4}{16} & -0,18R^4 \\ -0,18R^4 & \frac{2,56\pi R^4}{16} \end{bmatrix}, ij = x'y'$

Έχω $I_{x_cx_c} = I_{xx} - Ay_c^2 = \frac{2,56\cdot\pi R^4}{16} - \frac{2,56\cdot\pi \cancel{R^2}}{4}\cdot\frac{(0,24)^2}{\cancel{R^2}} = -\frac{2,56\pi}{4}\cdot(0,24)^2 + \frac{2,56\pi R^4}{16} = 0,5R^4 - 0,12$

$I_{y_cy_c} = I_{yy} - Ax_c^2 = \frac{2,56\cdot\pi R^4}{16} - \frac{2,56\cdot\pi \cancel{R^2}}{4}\cdot\frac{(0,08)^2}{\cancel{R^2}} = 0,5R^4 - 0,01$

$I_{x_cy_c} = I_{x'y'} - A(-x_c)(-y_c) = -0,18R^4 - \frac{2,56\cdot\pi \cancel{R^2}}{4}\cdot\frac{0,08}{\cancel{R}}\cdot\frac{(-0,24)}{\cancel{R}} = -0,18R^4 + 0,04$

Άρα $I_{ij} = \begin{bmatrix} 0,5R^4 - 0,12 & -0,18R^4 + 0,04 \\ -0,18R^4 + 0,04 & 0,5R^4 - 0,01 \end{bmatrix}$