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Construction and sampling of Archimedean and nested Archimedean Lévy copulas

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ABSTRACT

The class of Archimedean Lévy copulas is considered with focus on the construction and sampling of the corresponding Lévy processes. Furthermore, the class of nested Archimedean Lévy copulas is introduced. This class allows one to model hierarchical dependences between Lévy processes. It also overcomes the symmetry of Archimedean Lévy copulas. Finally, a new sampling algorithm for multivariate Lévy processes with dependence structure specified by either Archimedean or nested Archimedean Lévy copulas is derived from a Marshall–Olkin-type algorithm. In contrast to the widely used conditional sampling method, this algorithm does not require (inverses of) conditional Lévy copulas to be known. It also does not suffer from an asymmetric bias introduced by the conditional sampling method in the Lévy framework.

1. Introduction

One approach how to construct hierarchical dependence structures in the classical distributional framework going back to Joe [25] is to plug Archimedean copulas into each other, so taking, e.g.,

$$C(\boldsymbol{u}) = C_0(C_1(u_{11}, \dots, u_{1d_1}), \dots, C_S(u_{S1}, \dots, u_{Sd_S})), \quad \boldsymbol{u} \in [0,1]^d, \tag{1}$$

where S denotes the number of groups, sectors or hierarchies to be modeled and $d = \sum_{s=1}^{S} d_s$; more hierarchical levels can be incorporated by further nesting of Archimedean copulas. Sufficient conditions for such constructions to lead to proper copulas, known as *nested Archimedean copulas*, have been given by Bandeen-Roche and Liang [3], Joe [27] and McNeil [32]. Nested Archimedean copulas such as (1) are attractive since they are easy to interpret: Pairs of variables (U_{sj}, U_{sk}) belonging to the same group s follow the copula C_s whereas pairs of variables (U_{sj}, U_{lk}) belonging to different groups $s \neq l$ follow C_0. This partially asymmetric construction generalizes the well-known and widely used class of (exchangeable) Archimedean copulas and typically provides more realistic dependence structures in applications; see, e.g., [24,22,40] or [37].

Modeling dependent Lévy processes can be reduced to the modeling of the dependence of their Lévy measures, i.e., their (joint) jumps. For that purpose, Tankov [44] and Cont and Tankov [12] introduce the notion of Lévy copulas which parallels the distributional theory of copulas in certain aspects. However, Lévy copulas are not copulas and we will therefore sometimes add the word *probabilistic* to the word *copulas* to emphasize this. Probabilistic copulas are distribution functions. They couple distribution functions of random variables to their joint distribution function. Lévy copulas refer to Radon measures instead of probability measures and couple integrals of univariate Lévy measures to multivariate ones. Since the

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Lévy measures determine the jumps of the Lévy processes, Lévy copulas capture the dependence between the jumps of the processes. Different parametric Lévy copulas have been presented in the literature and the subclass of Archimedean Lévy copulas has provided examples for Lévy copulas from the very beginning. Bäuerle and Blatter [4] explicitly construct Archimedean Lévy copulas in a way which parallels the distributional theory to a large degree (see also [7, pp. 78]).

Applications involving Lévy copulas stem, e.g., from the areas of insurance, finance and operational risk. In these areas, the advantage of Lévy copula models over copula models is that the former refer to multivariate stochastic processes (with heavy tails) instead of, e.g., loss distributions over a fixed time horizons. Lévy copula models thus enable to model losses, claims etc. in a direct and dynamic way instead of aggregating them. Recent applications in the literature are, e.g., [4] for reinsurance and investment policies and Böcker and Kluppelberg [9,8], or Biagini and Ulmer [6] for operational risk modeling; see also [11] for modeling ruin probabilities with Lévy copulas.

Still, most of the Lévy copulas applied in the literature are either bivariate or symmetric in their dependence structure, like, e.g., Archimedean Lévy copulas. Typical applications (e.g., in insurance or operational risk modeling), however, require very flexible high-dimensional models which are able to model different kinds of dependences between different branches or business lines of a company. Since (probabilistic) copulas and Lévy copulas are different functions, it is not immediately clear, which ideas of multivariate (probabilistic) copula constructions may be transferred to the Lévy copula world. Recently, Grothe and Nicklas [17] showed that multidimensional Lévy copulas may be constructed from bivariate dependence functions in the sense of a pair construction of (probabilistic) copulas going back to Joe [26].

The present paper aims to link the notion of Lévy copulas and the notion of nested Archimedean copulas to widen the classes of flexible high dimensional Lévy copulas. We construct nested Archimedean Lévy copulas, i.e., Lévy copulas which follow a hierarchical Archimedean construction such as the one given in (1). The nested Archimedean structure aims at overcoming symmetries and allows to incorporate hierarchies in models based on Lévy copulas; e.g., Avanzi et al. [2] recently presented low-dimensional nested structures in the context of multivariate loss processes for insurance and operational risk models. We then introduce a fast sampling algorithm for both Archimedean and nested Archimedean Lévy copulas which overcomes problems of "conditional approaches" used so far for sampling Lévy process with jump dependence structure specified by (Archimedean) Lévy copulas. This is of interest for financial applications; see, e.g., [41] or [42]. Furthermore, in contrast to the algorithm used in [2], the algorithm presented does not rely on a compound Poisson approximation of the Lévy process.

This paper is organized as follows. Section 2 recalls the notion of (nested) Archimedean copulas, Lévy processes and Lévy copulas. In Section 3, we first present some construction principles for Archimedean Lévy generators and then introduce nested Archimedean Lévy copulas. Section 4 introduces a fast Marshall–Olkin-type algorithm for sampling both Archimedean and nested Archimedean Lévy copulas. Section 5 addresses two simulation studies. Finally, Section 6 concludes.

2. Preliminaries

2.1. (Nested) Archimedean copulas

An *(Archimedean) generator* is a continuous, decreasing function $\psi : [0,\infty] \to [0,1]$ which satisfies $\psi(0) = 1$, $\psi(\infty) := \lim_{t\uparrow\infty} \psi(t) = 0$ and which is strictly decreasing on $[0, \inf\{t : \psi(t) = 0\}]$. The set of all such functions is denoted by Ψ. If $\psi(t) > 0$ for all $t \in [0,\infty)$, ψ is called *strict*. A d-dimensional copula C is an *Archimedean copula (AC)* if it admits the representation

$$C(\boldsymbol{u}) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)), \quad \boldsymbol{u} \in [0,1]^d, \tag{2}$$

for some $\psi \in \Psi$ with inverse $\psi^{-1} : [0,1] \to [0,\infty]$, where $\psi^{-1}(0) = \inf\{t : \psi(t) = 0\}$.

A necessary and sufficient condition for (2) to be a copula in a particular dimension $d \geq 2$ is that ψ is *d-monotone*, i.e., $(-1)^k \psi^{(k)}(t) \geq 0$ for all $k \in \{0,\dots,d-2\}$, $t \in (0,\infty)$, and $(-1)^{d-2}\psi^{(d-2)}(t)$ is decreasing and convex on $(0,\infty)$; see [33] for more details. In what follows, we mainly assume that ψ is *completely monotone*, i.e., $(-1)^k \psi^{(k)}(t) \geq 0$ for all $k \in \mathbb{N}_0$. This guarantees that (2) is a copula in all dimensions. The class of completely monotone generators is denoted by Ψ_∞. By Bernstein's Theorem, the class of completely monotone generators precisely coincides with the class of Laplace–Stieltjes transforms ($\mathcal{LS}$) of distribution functions F on the positive real line, that is, any completely monotone generator allows for the representation

$$\psi(t) = \mathcal{LS}[F](t) = \int_0^\infty \exp(-tx)\, dF(x); \tag{3}$$

see [14, p. 439]. This representation has an important consequence for sampling since it provides a stochastic representation for $\boldsymbol{U} \sim C$, given by

$$\boldsymbol{U} = \left(\psi\left(\frac{E_1}{V}\right), \dots, \psi\left(\frac{E_d}{V}\right)\right), \tag{4}$$

where $V \sim F = \mathcal{LS}^{-1}[\psi]$ and $E_j \sim \mathrm{Exp}(1)$, $j \in \{1,\dots,d\}$, are independent. The corresponding sampling algorithm is known as *Marshall–Olkin algorithm*; see [31].

Besides sharing many useful properties (such as an explicit functional form and different types of tail dependence), one major drawback of ACs is their symmetry, also known as exchangeability. Nested Archimedean copulas aim at relaxing this rather strong modeling assumption. A *nested Archimedean copula (NAC)* is simply an AC with some of its arguments replaced by other NACs. One popular example is given by (1). If, for each $s \in \{0, \dots, S\}$, the corresponding generator ψ_s is an element of Ψ_∞ and

$$(\psi_0^{-1} \circ \psi_j)' \text{ is completely monotone}, \tag{5}$$

then (1) is a copula; see [27,32,21]. Condition (5) typically implies that C_s, $s \in \{1, \dots, S\}$, is more concordant than C_0, see [19, pp. 59]; this is often the case in applications since groups are typically affected by the same macro-economic effects, political decisions, consumer trends etc.

2.2. Lévy processes and Lévy copulas

Detailed information about Lévy processes may be found in [38], [28] or [39]. Introductions to Lévy copulas are given in [12] or [29]. Here, we briefly summarize the main notions.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{B}([0,\infty)^d)$ the Borel σ-algebra on $[0,\infty)^d$. A d-dimensional *Lévy process* $\boldsymbol{L} = (\boldsymbol{L}_t)_{t\in[0,\infty)}$ is a stochastic process with stationary, independent increments, starting at zero. Lévy processes can be decomposed into a deterministic drift function, a Brownian motion part and a pure jump process with a possibly infinite number of small jumps; see, e.g., [28, Lévy Itô decomposition; Theorem 15.4]. The *Lévy triplet* $(\boldsymbol{\gamma}, \Sigma, \nu)$, consisting of the *drift* $\boldsymbol{\gamma} \in \mathbb{R}^d$, the *Gaussian covariance matrix* $\Sigma \in \mathbb{R}^{d\times d}$ and the *Lévy measure* ν, characterizes Lévy processes in a time-independent fashion. Here, the Lévy measure completely characterizes the jumps of size in $A \in \mathcal{B}([0,\infty)^d)$, where $\nu(A)$ is the expected number of jumps per unit time step.

A d-dimensional Lévy process with $\boldsymbol{\gamma} \geq \boldsymbol{0}$ (componentwise), $\Sigma = (0) \in \mathbb{R}^{d\times d}$ (null matrix) and Lévy measure ν concentrated on $[0,\infty)^d \setminus \{\boldsymbol{0}\}$ is called *subordinator*; see [43]. Such a Lévy process has no negative increments. General Lévy processes may be assembled by gluing together subordinators for each orthant of jump sizes of the process (assigned with the correct sign) and adding a Brownian Motion and a drift component; see [12, Section 5.6]. For the ease of notation, we consider the case with positive jumps in what follows.

We now introduce some basic notions about multivariate functions which we need for defining Lévy copulas for spectrally positive Lévy processes; see [29] for the general case.

Definition 2.1 (*Grounded, J-margin, Uniform Margins, H-volume, d-increasing*)**.** Let $H : [0,\infty]^d \to [0,\infty]$.

1. H is called *grounded* if $x_j = 0$ for at least one $j \in \{1, \dots, d\}$ implies that $H(\boldsymbol{x}) = 0$.
2. For any $\emptyset \neq J \subseteq \{1, \dots, d\}$, the *J-margin* of H is the function $H_J : [0,\infty]^{|J|} \to [0,\infty]$ defined by
$$H_J(\boldsymbol{x}_J) = H(\boldsymbol{x})\big|_{\boldsymbol{x}_{J^c}=\boldsymbol{\infty}},$$
where $\boldsymbol{x}_J$ denotes the vector of components of $\boldsymbol{x}$ with indices in J, $J^c = \{1, \dots, d\} \setminus J$ and $\boldsymbol{\infty}$ is a vector of ∞ of the necessary size.
3. H has *uniform margins* if $H_{\{j\}}(x) = x$ for any $j \in \{1, \dots, d\}$ and $x \in \mathbb{R}$; for simplicity, we write H_j for $H_{\{j\}}$.
4. For $\boldsymbol{a}, \boldsymbol{b} \in [0,\infty]^d$ with $\boldsymbol{a} \leq \boldsymbol{b}$, the *H-volume* is defined by
$$\Delta_{(\boldsymbol{a},\boldsymbol{b}]}H = \sum_{\boldsymbol{j}\in\{0,1\}^d} (-1)^{\sum_{k=1}^d j_k} H(a_1^{j_1} b_1^{1-j_1}, \dots, a_d^{j_d} b_d^{1-j_d}).$$

H is called *d-increasing* if $\Delta_{(\boldsymbol{a},\boldsymbol{b}]}H \geq 0$ for all $\boldsymbol{a}, \boldsymbol{b} \in (-\infty,\infty]^d$ with $\boldsymbol{a} \leq \boldsymbol{b}$.

We can now define Lévy copulas for spectrally positive Lévy processes (called *positive Lévy copulas*). Their definition resembles the definition of distributional copulas, the difference being that the latter are distribution functions and defined on the unit hypercube $[0,1]^d$ whereas Lévy copulas are defined on $[0,\infty]^d$ and are in general not distribution functions; the connection with distribution functions will become clear from Lemma 2.5.

Definition 2.2 (*Positive Lévy Copula*)**.** A function $\bar{C} : [0,\infty]^d \to [0,\infty]$ is a *positive Lévy copula* if it is grounded, has uniform margins and is d-increasing.

While distributional copulas couple marginal distribution functions to a joint distribution function, Lévy copulas couple marginal tail integrals of Lévy processes, which are defined as follows.

Definition 2.3 (*Marginal Lévy Process or Lévy Measure, (Marginal) Tail Integral*)**.** Let $\boldsymbol{L}$ be an $\mathbb{R}^d$-valued spectrally positive Lévy process with Lévy measure ν and let $\emptyset \neq J \subseteq \{1, \dots, d\}$.

1. The *J-margin* of $\boldsymbol{L}$ is the Lévy process $\boldsymbol{L}_J = (L_j)_{j\in J}$.
2. The *J-marginal Lévy measure* is defined by $\nu_J(A) = \nu(\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}_J \in A\})$ for all $A \in \mathcal{B}((\mathbb{R}\setminus\{0\})^{|J|})$.

3. The *tail integral* $\bar{\nu}$ of $\boldsymbol{L}$ (or of ν) is the function $\bar{\nu} : (0,\infty)^d \to \mathbb{R}$ defined by

$$\bar{\nu}(\boldsymbol{x}) = \begin{cases} \nu([x_1,\infty) \times \cdots \times [x_d,\infty)) & \text{if } (x_1,\ldots,x_d) \in [0,\infty)^d \setminus \{0\}, \\ 0 & \text{if } x_i = \infty \text{ for at least one } i, \\ \infty & \text{if } (x_1,\ldots,x_d) = 0. \end{cases}$$

It can be interpreted as the expected number of jumps per unit of time with jump sizes componentwise larger or equal to $\boldsymbol{x}$.

4. The *J-marginal tail integral* of $\boldsymbol{L}$ (or of ν), denoted by $\bar{\nu}_J$, is the tail integral of the Lévy process $\boldsymbol{L}_J$.

The tail integral $\bar{\nu}$ of a spectrally positive Lévy process uniquely determines its Lévy measure ν. The basic idea of Lévy copulas is to couple marginal tail integrals to multivariate tail integrals, as the following theorem shows.

Theorem 2.4 (*Sklar's Theorem for Lévy Copulas, Kallsen and Tankov [29]*)**.**

1. *Let $\boldsymbol{L}$ be an $\mathbb{R}^d$-valued (spectrally positive) Lévy process with Lévy measure ν. Then there exists a Lévy copula $\bar{C}$ such that, for all $\emptyset \neq J \subseteq \{1,\ldots,d\}$ and all $\boldsymbol{x}_J \in (\mathbb{R}\setminus\{0\})^{|J|}$, the J-marginal tail integral $\bar{\nu}_J$ of $\boldsymbol{L}$ satisfies*

$$\bar{\nu}_J(\boldsymbol{x}_J) = \bar{C}_J\big((\bar{\nu}_j(x_j))_{j\in J}\big), \tag{6}$$

where $\bar{C}_J$ denotes the J-margin of $\bar{C}$. The Lévy copula in (6) is uniquely defined on $\prod_{j=1}^d \overline{\operatorname{ran}\,\bar{\nu}_j}$, where ran *denotes the range.*

2. *Conversely, if $\bar{C}$ is a d-dimensional Lévy copula and $\bar{\nu}_j, j \in \{1,\ldots,d\}$, are tail integrals of real-valued Lévy processes, then there exists an $\mathbb{R}^d$-valued Lévy process $\boldsymbol{L}$ whose components have tail integrals $\bar{\nu}_j, j \in \{1,\ldots,d\}$, and whose marginal tail integrals satisfy (6) for all $\emptyset \neq J \subseteq \{1,\ldots,d\}$ and all $x_J \in (\mathbb{R}\setminus\{0\})^{|J|}$. The Lévy measure ν of $\boldsymbol{L}$ is uniquely determined by $\bar{C}$ and $\bar{\nu}_j, j \in \{1,\ldots,d\}$.*

Note that the theorem does not only hold for spectrally positive Lévy processes, but also in the general case. However, the definition of the tail integral is then notationally more involved. Moreover, similar to a Lévy process which may be constructed by gluing together subordinators, general Lévy copulas may be represented by gluing together positive Lévy copulas, each referring to one of the 2^d orthants; details are given in [45, Theorem 5.3].

In contrast to distributional copulas, Lévy copulas cannot be interpreted as distribution functions. However, conditioned on one component, a Lévy copula is the distribution function of the other components; see [46, Lemma 4.2]. This result is used extensively for simulation purposes; see Section 4.1.

Lemma 2.5 (*Probabilistic Interpretation of Positive Lévy Copulas*)**.** *Let $\bar{C}$ be a positive Lévy copula satisfying*

$$\lim_{\boldsymbol{x}_J \to \{\infty\}^{|J|}} \bar{C}(\boldsymbol{x}) = \bar{C}(\boldsymbol{x})|_{\boldsymbol{x}_J = \{\infty\}^{|J|}}. \tag{7}$$

Then, for $j \in \{1,\ldots,d\}$ and almost every fixed $w \in \mathbb{R}$,

$$H_w(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_d) = \frac{\partial}{\partial w}\bar{C}(x_1,\ldots,x_{j-1},w,x_{j+1},\ldots,x_d), \tag{8}$$

is a distribution function on the set of its continuity points.

3. Nested Archimedean Lévy copulas

3.1. Archimedean Lévy copulas

3.1.1. Definition and basic properties

There are different definitions of *Archimedean Lévy copulas (ALC)*. The one we adopt here parallels the probabilistic copula theory. However, in particular, Kallsen and Tankov [29] define ALCs differently. Their approach does not parallel the theory of probabilistic copulas and does not allow to generate positively associated Lévy processes, see [5]. Again, we focus on the subordinator framework here, ALCs on $(-\infty,\infty]^d$ can then be defined via Tankov [45, Theorem 5.3]; see also [12, Section 5.6] and [5].

Definition 3.1 (*PALC*)**.** An *Archimedean Lévy generator* is a continuous, decreasing function $\bar{\psi} : [0,\infty] \to [0,\infty]$ which satisfies $\bar{\psi}(0) = \infty$, $\bar{\psi}(\infty) = \lim_{t\uparrow\infty}\bar{\psi}(t) = 0$ and which is strictly decreasing on $[0, \inf\{t : \bar{\psi}(t) = 0\}]$. The set of all such functions is denoted by $\bar{\Psi}$. If $\bar{\psi}(t) > 0$ for all $t \in [0,\infty)$, $\bar{\psi}$ is called *strict*. A d-dimensional positive Lévy copula is a *positive Archimedean Lévy copula (PALC)* if it can be written as

$$\bar{C}(\boldsymbol{x}) = \bar{\psi}\left(\sum_{j=1}^d \bar{\psi}^{-1}(x_j)\right), \quad \boldsymbol{x} \in [0,\infty]^d, \tag{9}$$

for some $\bar{\psi} \in \bar{\Psi}$.

Similar to the distributional theory, we aim for a criterion when (9) is a proper positive Lévy copula. Bäuerle and Blatter [4] prove that this holds if and only if $\bar{\psi}$ is d-monotone. This implies that $\bar{C}$ is a Lévy copula in all dimensions if and only if $\bar{\psi}$ is completely monotone, i.e., $\bar{\psi}$ is continuous on $[0,\infty]$ and $(-1)^k\bar{\psi}^{(k)}(t) \geq 0$ for all $k \in \mathbb{N}_0$, $t \in (0,\infty)$. The set of all such functions is denoted by $\bar{\Psi}_\infty$. It is interesting that Bernstein's Theorem is also applicable in this "non-distributional" framework, so a completely monotone Archimedean Lévy generator can be written as

$$\bar{\psi}(t) = \mathcal{LS}[F](t) = \int_0^\infty \exp(-tx)\, dF(x), \tag{10}$$

where F is now a non-finite measure on $[0,\infty)$; see [14, p. 439]; it will become clear later why we also use the letter "F" in (10), although F is not a distribution function anymore. The following lemma provides some properties about F and $\bar{\psi} \in \bar{\Psi}_\infty$ which are used implicitly later in this work; see the Appendix for a proof.

Lemma 3.2 ($F(0) = 0$, $\bar{\psi}$ *Strict*)**.** *Let $\bar{\psi} \in \bar{\Psi}_\infty$ be completely monotone with $\bar{\psi} = \mathcal{LS}[F]$ for a non-finite measure F on $[0,\infty)$. Then $F(0) = 0$ and $\bar{\psi}$ is strict.*

Although $\bar{\psi} \in \bar{\Psi}_\infty$ allows for a representation as in (10), note that there is no direct analog to the stochastic representation (4) anymore, simply because F is not a distribution function. However, the simulation algorithm we introduce in Section 4.2 will also be based on (10).

We now derive an interesting connection between ACs and PALCs; for its proof, see the Appendix.

Theorem 3.3 (*Conditional PALCs Have ACs*)**.** *Let $\bar{C}$ be a d-dimensional PALC with d-monotone generator $\bar{\psi}$. For $j \in \{1,\dots,d\}$, let*

$$H_w(x_1,\dots,x_{j-1},x_{j+1},\dots,x_d) = \frac{\partial}{\partial x_j}\bar{C}(\boldsymbol{x})\Big|_{x_j=w}$$

be the $(d-1)$-dimensional distribution function as given by Lemma 2.5 with marginal distribution functions $F_w(x_k)$, $k \in \{1,\dots,j-1,j+1,\dots,d\}$; note that they are all equal due to symmetry. Then the copula C_w of H_w is Archimedean with $(d-1)$-monotone generator $\psi(t) = F_w(\bar{\psi}(t))$.

3.1.2. Examples and connections to Archimedean generators

Due to its simplicity, a positive Lévy copula often found in the literature is the positive Clayton Lévy copula.

Example 3.4 (*Positive Clayton Lévy Copula*)**.** A positive Clayton Lévy copula is a PALC with generator $\bar{\psi}(t) = t^{-1/\theta}$ for $\theta \in (0,\infty)$, see [4]. It is given by

$$\bar{C}(\boldsymbol{x}) = (x_1^{-\theta} + \cdots + x_d^{-\theta})^{-1/\theta}, \quad \boldsymbol{x} \in [0,\infty]^d.$$

The distribution H_w induced by a positive Clayton Lévy copula when conditioned on one component in the sense of Theorem 3.3 equals

$$H_w(x_1,\dots,x_{j-1},x_{j+1},\dots,x_d) = \left(w^{-\theta} + \sum_{\substack{k=1\\k\neq j}}^d x_k^{-\theta}\right)^{-\frac{\theta+1}{\theta}} w^{-(\theta+1)}$$

with margins $F_w(x) = (w^{-\theta} + x^{-\theta})^{-\frac{\theta+1}{\theta}} w^{-(\theta+1)}$. It is easily checked that the copula C_w of H_w is

$$\begin{aligned} C_w(u_1,\dots,u_{j-1},u_{j+1},\dots,u_d) &= H_w(F_w^{-1}(u_1),\dots,F_w^{-1}(u_{j-1}),F_w^{-1}(u_{j+1}),\dots,F_w^{-1}(u_d)) \\ &= \left(1-(d-1)+\sum_{\substack{k=1\\k\neq j}}^d u_k^{-\frac{\theta}{\theta+1}}\right)^{-\frac{\theta+1}{\theta}}, \end{aligned}$$

which is a Clayton copula with parameter $\theta/(\theta+1)$. The parameter θ of the Clayton Lévy copula determines the dependence of the jump sizes, where larger values of θ indicate a "stronger dependence". Here, the word "stronger" is understood in the sense of a larger parameter or in the sense of a "concordance ordering" for Lévy copulas. To see this, consider two PALCs with completely monotone, parametric generators $\bar{\psi}_1, \bar{\psi}_2$ such that $\theta_1 \leq \theta_2$ implies that $(\bar{\psi}_1^{-1} \circ \bar{\psi}_2)'$ is completely monotone (as is easily seen to hold in Clayton's case). Then $\bar{\psi}_1^{-1} \circ \bar{\psi}_2$ is concave and furthermore, $\bar{\psi}_1^{-1}(\bar{\psi}_2(0)) = 0$. The argument of Nelsen [35, p. 136] directly carries over to the positive Archimedean Lévy framework. We thus obtain that $\bar{\psi}_1^{-1} \circ \bar{\psi}_2$ is subadditive. This readily implies that $\bar{C}_1 \leq \bar{C}_2$ pointwise, see [35, p. 136] for the argument for ACs. Therefore, an increasing θ implies a pointwise larger positive Clayton Lévy copula. This, e.g., also implies a stronger jump tail dependence in the sense of Grothe [16] who defines the jump tail dependence coefficient for positive jumps by $\lambda_{PP} = \lim_{x\downarrow 0}\bar{C}(x,x)/x$. It measures the tendency to observe an extreme positive jump in one component of a Lévy process given a simultaneous extreme positive jump in another component and is determined by the Lévy copula of the process. For the positive Clayton Lévy copula one has $\lambda_{PP} = 2^{-1/\theta}$. Thus, the probability of observing joint extreme jumps grows with θ.

Examples for other Lévy generators leading to PALCs are:

- the Gumbel Lévy generator $\bar{\psi}(t) = \exp(t^{-1/\theta}) - 1$ for $\theta \in (0, \infty)$, see [4];
- the complementary Gumbel Lévy generator $\bar{\psi}(t) = (\log(1+t))^{-1/\theta}$ for $\theta \in (0, \infty)$, see [30]; and
- the Ali–Mikhail–Haq (AMH) Lévy generator $\bar{\psi}(t) = (1-\theta)/(e^t - 1)$ with $\theta \in [-1, 1)$, see [4]. Note that for $\theta \in [0, 1)$, $\bar{\psi}$ is completely monotone.

Besides resembling their Archimedean counterparts (as in Clayton's case, for example) or the resulting (Lévy) copulas sharing certain properties, there is no obvious reason why one should name certain Archimedean Lévy generators correspondingly. Nevertheless, it is an interesting question how to generate an Archimedean Lévy generator from an Archimedean generator as much more is known about the latter. This can help constructing new PALCs; see [20] for this approach in the context of ACs and how this can be used for sampling purposes. To this end, we now present two transformations which convert Archimedean generators to Lévy generators. The proof of the following result is a consequence (or slight extension) of Feller [14, p. 441] and Widder [47, p. 145]. Note that there are potentially many of such transformations leading to different Archimedean Lévy generators based on the same Archimedean generator. Also note that, as Archimedean generators, Archimedean Lévy generators are not unique: if $\bar{\psi}$ generates $\bar{C}$ then $c\bar{\psi}$ generates $\bar{C}$ as well, for any $c > 0$.

Proposition 3.5 (*A Transformation for Extendable Generators ψ*). *Let $h : [0, \infty) \to [0, \infty)$ be absolutely monotone with $h(0) = 0$ and $h(\infty) = \infty$. Let $g : \mathbb{R} \to \mathbb{R}$ with g' being completely monotone and $\lim_{t \downarrow 0} \psi(g(t)) = g(\infty) = \infty$. Furthermore, let $\psi \in \Psi_\infty$ such that its domain can be extended to an interval (t_0, ∞) with $t_0 = \inf\{t \in \mathbb{R} : \psi(t) < \infty\} < 0$. Then, $\bar{\psi}(t) = h\big(\psi(g(t))\big) \in \bar{\Psi}_\infty$.*

Example 3.6 (*Clayton, AMH, Frank*). Although Proposition 3.5 seems quite technical, it helps to understand how the Clayton and AMH Lévy generators arise. Furthermore, it allows us to construct a Lévy generator based on an Archimedean generator of the Frank copula.

1. It is easy to see that the Clayton Lévy generator $\bar{\psi}$ satisfies
$$\bar{\psi}(t) = \psi(t-1) = t^{-1/\theta},$$
where $\psi(t) = (1+t)^{-1/\theta}$ is the Clayton generator. Choosing h as the identity and $g(t) = t - 1$, we obtain by Proposition 3.5 that $\bar{\psi}$ is indeed a valid Lévy generator. Note that for the Clayton generator, $F = \mathcal{LS}^{-1}[\psi]$ has Gamma density $f(x) = x^{1/\theta - 1}\exp(-x)/\Gamma(1/\theta)$ (Γ denoting the Gamma function) whereas for the Clayton Lévy generator, $F = \mathcal{LS}^{-1}[\bar{\psi}]$ satisfies $F(z) = \int_0^z \bar{f}(x)\,dx$ with $\bar{f}(x) = x^{1/\theta-1}/\Gamma(1/\theta)$. We thus obtain
$$\bar{f}(x) = \exp(x) f(x), \quad x \in (0, \infty),$$
i.e., the "density" underlying $F = \mathcal{LS}^{-1}[\bar{\psi}]$ corresponding to the Clayton Lévy generator $\bar{\psi}$ is an exponentially tilted version of the density of F corresponding to the Clayton generator ψ.
2. For AMH, consider $\theta \in [0, 1)$, so that the AMH generator $\psi(t) = (1-\theta)/(e^t - \theta)$ is completely monotone. Setting $h(t) = \theta t$ and $g(t) = t + \log\theta$, we obtain the AMH Lévy generator
$$\bar{\psi}(t) = \theta\psi(t + \log\theta) = \frac{1-\theta}{e^t - 1}$$
from the AMH generator ψ. It is easy to check that the assumptions of Proposition 3.5 hold for the above h and g, so we can again construct $\bar{\psi}$ from ψ in this case. More importantly, we know a connection between the underlying F's. For AMH copulas, F is a discrete distribution with geometric probability mass function $p_k = (1-\theta)\theta^{k-1}$ at $k \in \mathbb{N}$; see [19, p. 62]. Since $\psi(t) = \sum_{k=1}^\infty p_k \exp(-tk)$, we obtain
$$\bar{\psi}(t) = h\big(\psi(g(t))\big) = \theta \sum_{k=1}^\infty p_k \exp(-g(t))^k = \sum_{k=1}^\infty \frac{p_k}{\theta^{k-1}} \exp(-tk) = \sum_{k=1}^\infty q_k \exp(-tk),$$
with the geometrically transformed sequence
$$q_k = \frac{p_k}{\theta^{k-1}} = 1 - \theta, \quad k \in \mathbb{N}.$$
Hence $F = \mathcal{LS}^{-1}[\bar{\psi}]$ is also discrete and puts constant mass $q_k = 1 - \theta$ at $k \in \mathbb{N}$. Note that we could have equally well chosen h as the identity to obtain another (constant) mass distribution on $\mathbb{N}$, but then we would not have obtained the AMH Lévy generator from Bäuerle and Blatter [4].

One can give a generalization of the AMH Lévy generator based on a generalization of the AMH generator (see [31] for the latter). Let $V = r + X$ where $X \sim \mathrm{NB}(r, 1-\theta)$ for $r > 0$, $\theta \in [0, 1)$, where the probability mass function of the negative binomial distribution $\mathrm{NB}(r, 1-\theta)$ is given by $p_k = \binom{k+r-1}{k}(1-\theta)^r\theta^k$, $k \in \mathbb{N}_0$. The Laplace–Stieltjes transform of V is given by $((1-\theta)/(e^t - \theta))^r$. With $h(t) = \theta^r t$ and $g(t) = t + \log\theta$, Proposition 3.5 implies that $\bar{\psi}(t) = ((1-\theta)/(e^t - 1))^r$ is a (two-parametric) Archimedean Lévy generator. One can easily verify that $\bar{\psi}(t) = \sum_{k=0}^\infty q_k \exp(-t(r+k))$ for $q_k = \theta^r p_k$, $k \in \mathbb{N}_0$, and hence $F = \mathcal{LS}^{-1}[\bar{\psi}]$ puts mass $q_k = \theta^r p_k$ at $r + k$, $k \in \mathbb{N}_0$. For $r = 1$, the AMH Lévy case follows.

3. A Lévy generator based on the completely monotone Frank generator $\psi(t) = -\log(1-(1-e^{-\theta})\exp(-t))/\theta, \theta \in (0,\infty), t \in (0,\infty)$, can be constructed as follows. Extend $\psi(t)$ to $t \in (\log(1-e^{-\theta}),\infty)$. Similar to the AMH case, we choose $g(t) = t + \log(1-e^{-\theta})$ and h as the identity. Applying Proposition 3.5 and again proceeding similarly as in the AMH case, we obtain the Lévy generator

$$\bar{\psi}(t) = \psi(t + \log(1-e^{-\theta})) = -\log(1-\exp(-t))/\theta.$$

Now consider F. For the Frank generator, F is a discrete logarithmic distribution which puts mass $p_k = (1-e^{-\theta})^k/(k\theta)$ at $k \in \mathbb{N}$; see [19, p. 62]. For the Frank Lévy generator, $F = \mathcal{LS}^{-1}[\bar{\psi}]$ is easily seen to be discrete as well, with mass

$$q_k = \frac{p_k}{(1-e^{-\theta})^k} = \frac{1}{k\theta}, \quad k \in \mathbb{N},$$

at the positive integers. We thus see that, as in the AMH case, multiplying the probability mass function $(p_k)_{k\in\mathbb{N}}$ corresponding to ψ with a specific geometric sequence leads to $(q_k)_{k\in\mathbb{N}}$ corresponding to $\bar{\psi}$.

Considering the way in which the densities or mass functions underlying $F = \mathcal{LS}^{-1}[\psi]$ are transformed in Example 3.6, one can directly infer new strategies for constructing Archimedean Lévy generators. Starting from any density or mass function concentrated on $(0,\infty)$, one can tilt (or "scale-up") these probabilistic quantities exponentially (as for Clayton's case) or geometrically (as for AMH's and Frank's case) to obtain $F = \mathcal{LS}^{-1}[\bar{\psi}]$. The advantage of such an approach (over choosing functions h, g in Proposition 3.5) for constructing new Archimedean Lévy generators is that one knows $F = \mathcal{LS}^{-1}[\bar{\psi}]$ which will be crucial for efficient sampling; see Section 4.

Note that not all Archimedean generators can be extended to some part of the negative real line, the Gumbel case being one example. A different transformation which encompasses the Gumbel (Lévy) generator can be given as follows.

Proposition 3.7 (*A Transformation for Non-extendable Generators ψ*). *Let $\bar{\psi}_0 \in \bar{\Psi}_\infty$ and $\psi \in \Psi_\infty$ such that $(-\log\psi)'$ is completely monotone. Then $\bar{\psi}(t) = \bar{\psi}_0(-\log\psi(t)) \in \bar{\Psi}_\infty$.*

The proof of Proposition 3.7 readily follows from Feller [14, p. 441] since $\bar{\psi}_0(-\log\psi(t))$ is a composition of the completely monotone function $\bar{\psi}_0$ with a function with completely monotone derivative $(-\log\psi)$ and thus $\bar{\psi}$ is completely monotone; trivially, $\bar{\psi}(0) = \infty$ and $\bar{\psi}(\infty) = 0$, so $\bar{\psi} \in \bar{\Psi}_\infty$. Note that Joe [27, p. 374] showed that $(-\log\psi)'$ being completely monotone is equivalent to $\psi^\alpha \in \Psi_\infty$ for all $\alpha \in (0,\infty)$; this provides another way of verifying this assumption of Proposition 3.7. All generators in [19, Table 2.1] share this property.

Example 3.8 (*Gumbel, Joe*).

1. Consider the generator $\psi(t) = \exp(-t^{1/\theta}), \theta \in [1,\infty)$, of a Gumbel copula ($\psi$ corresponds to a stable distribution of the form $S(1/\theta, 1, (\cos(\frac{\pi}{2\theta}))^\theta, 0; 1)$; see [18] for the parameterization). The Gumbel Lévy generator can be obtained from Proposition 3.7 by noting that $\bar{\psi}_0(t) = \exp(1/t) - 1$ is completely monotone (as a composition of the absolutely monotone function $t \mapsto e^t - 1$ with the completely monotone function $t \mapsto 1/t$) and that $-\log\psi(t) = t^{1/\theta}$ indeed has a completely monotone derivative. Therefore, $\bar{\psi}(t) = \bar{\psi}_0(-\log\psi(t)) = \exp(t^{-1/\theta}) - 1$ is an Archimedean Lévy generator; it is valid for all $\theta > 0$.
2. In the same spirit, Proposition 3.7 allows us to construct a (class of) Lévy generator(s) based on the (non-extendable) generator of a Joe copula $\psi(t) = 1-(1-\exp(-t))^{1/\theta}$; the corresponding $F = \mathcal{LS}^{-1}[\psi]$ is the distribution function of a Sibuya distribution with parameter $1/\theta$ which puts mass $p_k = \binom{1/\theta}{k}(-1)^{k-1}$ at k, $k \in \mathbb{N}$. Joe [27, p. 375] shows that $(-\log\psi)'$ is completely monotone. By Proposition 3.7, $\bar{\psi}_0(-\log\psi)$ is thus a valid Archimedean Lévy generator for any $\bar{\psi}_0 \in \bar{\Psi}_\infty$. Using the same $\bar{\psi}_0$ as in the Gumbel case, we obtain $\bar{\psi}(t) = \exp\big(\big(-\log(1-(1-\exp(-t))^{1/\theta})\big)^{-1}\big) - 1$.

Note that $\bar{\psi}$ in Proposition 3.7 equals $\bar{\psi}(t) = \int_0^\infty \psi(t)^x \, dF_0(x)$, where $F_0 = \mathcal{LS}^{-1}[\psi_0]$. Hence, $\bar{\psi}$ is a F_0-randomized power of ψ. The decomposition of the Lévy generator given by Proposition 3.7 also adapts to our simulation algorithm. We explicitly state the algorithm for the generators of Proposition 3.7 in Algorithm A.1 in the Appendix.

3.2. Nested Archimedean Lévy copulas

We are now able to introduce nested Archimedean Lévy copulas (NALCs). As before, we focus on the subordinator framework. A *positive nested Archimedean Lévy copula (PNALC)* is a PALC with arguments possibly replaced by other PNALCs. The question is, in which cases such a construction leads to a proper positive Lévy copula. We answer this question in the following theorem; its proof can be found in the Appendix. For ease of notation, we only consider the three-dimensional case, noting that the theorem easily extends to the general d-dimensional case with more hierarchical levels; see [32] for an inductive argument for distributional copulas and also Example 4.6.

Theorem 3.9 (*Three-dimensional PNALCs*). *Let $\bar{\psi}_s \in \bar{\Psi}_\infty$, $s \in \{0,1\}$. If $(\bar{\psi}_0^{-1} \circ \bar{\psi}_1)'$ is completely monotone, then*

$$\begin{aligned}\bar{C}(\boldsymbol{x}) &= \bar{C}_0(x_1, \bar{C}_1(x_2, x_3)) \\ &= \bar{\psi}_0\big(\bar{\psi}_0^{-1}(x_1) + \bar{\psi}_0^{-1}\big(\bar{\psi}_1(\bar{\psi}_1^{-1}(x_2) + \bar{\psi}_1^{-1}(x_3))\big)\big), \quad \boldsymbol{x} \in [0,\infty]^3, \end{aligned} \tag{11}$$

is a positive Lévy copula.

Note that the assumption that the combined generators, e.g., $(\bar{\psi}_0^{-1} \circ \bar{\psi}_1)'$, are completely monotone may be relaxed to m-monotone, where $m \in \{1, \dots, d\}$ depending on the particular position of the generators within the structures. This was first pointed out by Hofert [21, Remark 2.3] for distributional copulas and the most nested levels. It readily applies to the Lévy framework as well as will become clear later in this work; see also et al. [2] in the low-dimensional case. We restrict ourselves to the completely monotone case here for applying Bernstein's Theorem (see (10)) which we will use to construct an efficient sampling algorithm. For efficient sampling in the m-monotone case, one would need to generalize the formula for the distribution function F_R of the radial part R, given by McNeil and Nešlehová [33], to a non-finite measure in the Lévy framework. We have not investigated how feasible Step 3 in Algorithms 4.2 and 4.4 would then be. Similar to Bernstein's Theorem, however, the transform of Williamson [48] (required in the m-monotone case) indeed applies to more general functions than distribution functions.

Example 3.10 (*Positive Nested Clayton Lévy Copula*)**.** An example of a positive nested Clayton Lévy copula is

$$\bar{C}(\boldsymbol{x}; \theta_0, \theta_1) = \bar{C}_0(x_1, \bar{C}_1(x_2, x_3; \theta_1); \theta_0) = (x_1^{-\theta_0} + (x_2^{-\theta_1} + x_3^{-\theta_1})^{\frac{\theta_0}{\theta_1}})^{-\frac{1}{\theta_0}}, \quad \boldsymbol{x} \in [0,\infty]^3,$$

for $\theta_0 \le \theta_1$. Clearly, $(\bar{\psi}_0^{-1} \circ \bar{\psi}_1)'(t) = \frac{\theta_0}{\theta_1} t^{\theta_0/\theta_1 - 1}$ is completely monotone for all $\theta_0 \le \theta_1$.

Example 3.11 (*Positive Nested Gumbel Lévy Copula*)**.** An example of a positive nested Gumbel Lévy copula is

$$\begin{aligned}\bar{C}(\boldsymbol{x}; \theta_0, \theta_1) &= \bar{C}_0(x_1, \bar{C}_1(x_2, x_3; \theta_1); \theta_0) \\ &= \exp\left(\left(\log(x_1)^{-\theta_0} + \left(\log(x_2)^{-\theta_1} + \log(x_3)^{-\theta_1}\right)^{\frac{\theta_0}{\theta_1}}\right)^{-\frac{1}{\theta_0}}\right), \quad \boldsymbol{x} \in [0,\infty]^3, \end{aligned}$$

for $\theta_0 \le \theta_1$. Again, $(\bar{\psi}_0^{-1} \circ \bar{\psi}_1)'(t) = \frac{\theta_0}{\theta_1} t^{\theta_0/\theta_1 - 1}$ is completely monotone for all $\theta_0 \le \theta_1$.

Besides the specific examples, partially NACs of type (1) are often of interest. Their Lévy counterparts are given by

$$\bar{C}(\boldsymbol{x}) = \bar{C}_0(\bar{C}_1(x_{11}, \dots, x_{1d_1}), \dots, \bar{C}_S(x_{S1}, \dots, x_{Sd_S})), \quad \boldsymbol{x} \in [0,\infty]^d, \tag{12}$$

with S different *groups*, each of dimensions $d_s, s \in \{1, \dots, S\}$, where $d = \sum_{s=1}^S d_s$. Thus, if $(\bar{\psi}_0^{-1} \circ \bar{\psi}_s)'$ is completely monotone for all $s \in \{1, \dots, S\}$, then $\bar{C}$ is a valid PNALC. The appealing intuition behind such a hierarchical construction is repeated here in the context of multivariate Lévy processes: Marginal Lévy processes belonging to the same group $s \in \{1, \dots, S\}$ jump together with the dependence structure specified by the Lévy copula $\bar{C}_s$ while marginal Lévy processes belonging to different groups jump together with the dependence structure specified by the Lévy copula $\bar{C}_0$; this may easily be checked by calculating the corresponding bivariate margins.

Note that for PNALCs like (12) one can also derive a corresponding *density* expression

$$\bar{c}(\boldsymbol{x}) = \frac{\partial^d}{\partial x_{Sd_S} \cdots \partial x_{11}} \bar{C}(\boldsymbol{x}).$$

The calculations are tedious but have been addressed for NACs (see, e.g., [23]). They readily apply to PNALCs as well.

4. Simulating Archimedean and positive nested Archimedean Lévy copulas

Similar as before, we only consider the subordinator framework here since the general case may be inferred from a superposition of simulations of subordinators.

Sampling multidimensional Lévy processes with given Lévy copulas is generally achieved with a series representation of Lévy processes. For a one-dimensional subordinator L with tail integral $\bar{\nu}$, its ordered jump magnitudes $\Delta L_1 > \Delta L_2 > \cdots > 0$ have the same distribution as $\bar{\nu}^{-1}(\Gamma_1), \bar{\nu}^{-1}(\Gamma_2), \dots$, respectively, where $(\Gamma_k)_{k\in\mathbb{N}}$ denote the jump times of a standard Poisson process, that is, $\Gamma_k = \sum_{l=1}^k E_l$ for independent $E_l \sim \text{Exp}(1), l \in \{1, \dots, k\}$; see [12, Proposition 6.3]. The jump times are uniformly distributed over the time horizon, e.g., taken as $[0, 1]$. Therefore, for independent $(U_k)_{k\in\mathbb{N}} \sim \text{U}[0, 1]$ and $t \in [0, 1]$, the stochastic representation

$$L_t = \text{“Sum of all jumps in } [0, t]\text{”} = \sum_{k=1}^{\infty} \Delta L_k \mathbb{1}_{\{U_k \in [0,t]\}} = \sum_{k=1}^{\infty} \bar{\nu}^{-1}(\Gamma_k) \mathbb{1}_{\{U_k \in [0,t]\}} \tag{13}$$

holds; see [12, Proposition 6.3]. Note that $\bar{\nu}^{-1}$ is decreasing so that small values of Γ_k correspond to large jumps. For simulation purposes, small jumps are neglected, i.e., the series in (13) is truncated at K for some sufficiently large number K.

A similar result as (13) applying to multivariate Lévy processes can be found in [38]. For multivariate Lévy processes, representation (13) holds for any single dimension, where the process U_k determining the jump times is the “same” for all dimensions and a Lévy copula determines the dependence between the components of $\boldsymbol{\Gamma}_k = (\Gamma_{k1}, \dots, \Gamma_{kd})$ for each $k \in \mathbb{N}$.

For simulation, the task is then to sample the Γ_{kj}'s such that they are marginally, i.e., for fixed j, jump times of a standard Poisson process and that for fixed kth jump time, i.e., cross-dimensionally, they are "distributed" according to the desired Lévy copula. In Section 4.1, we discuss a standard sampling approach for this task, the idea being to first sample $(\Gamma_{kj})_{k\in\mathbb{N}}$ for one dimension j as a standard Poisson process and then to sample, conditionally on $(\Gamma_{kj})_{k\in\mathbb{N}}$, the values of Γ_{ki} for $i \neq j$ and all $k \in \mathbb{N}$ using Lemma 2.5 (conditionally on one component the Lévy copula provides the distribution of the other components). For distributional copulas, this approach is known as *conditional distribution method*; see [13]. For Lévy copulas, we will call it *conditional method*. The conditional method may be convenient for PALCs since, by Theorem 3.3, the distribution of Γ_{ki} given Γ_{kj}, $j \neq i$, has an AC. However, explicit forms of the conditional copulas and their inverses involved in this approach are typically not known (a rare exception being the case of Clayton Lévy copulas). Furthermore, for PNALCs this approach may be especially inconvenient, since the conditional copulas may not be of (nested) Archimedean type anymore. In Sections 4.2 and 4.3 we therefore introduce a Marshall–Olkin-type sampling algorithm for PALCs and PNALCs, respectively, which directly provides a sample of $\mathbf{\Gamma}_k$ without conditioning on one variable.

4.1. Sampling $(\mathbf{\Gamma}_k)_{k=1}^K$ conditionally on one component

In what follows, we (without loss of generality) condition on the first component. The idea is to simulate $(\Gamma_{k1})_{k\in\mathbb{N}}$ from an independent sequence of jump times of a standard Poisson process and then, conditionally on Γ_{k1}, to simulate the random vector $(\Gamma_{k2}, \dots, \Gamma_{kd})$. Note that the vector $\mathbf{\Gamma}_k$ is independent of Γ_{ij} for any $i \neq k$ and all $j \in \{1, \dots, d\}$ and that, according to Lemma 2.5, $(\Gamma_{k2}, \dots, \Gamma_{kd})$ is distributed in $\mathbb{R}^{d-1}$ with law $\frac{\partial}{\partial x_1}\bar{C}(\boldsymbol{x})|_{x_1=\Gamma_{k1}}$. Thus, a sampling algorithm can be given as follows.

Algorithm 4.1 (*Conditional Method for Sampling* $(\mathbf{\Gamma}_k)_{k=1}^K$).

1. Fix a (sufficiently large) number K of (large) jumps to be simulated.
2. Generate independent $Z_k \sim \mathrm{Exp}(1)$, $k \in \{1, \dots, K\}$, and set $\Gamma_{k1} = \Gamma_{k-1\,1} + Z_k$, $k \in \{1, \dots, K\}$, with $\Gamma_{01} = 0$; i.e., Γ_{k1}, $k \in \{1, \dots, K\}$, are the jump times of a standard Poisson process.
3. For $k \in \{1, \dots, K\}$, generate $(\Gamma_{k2}, \dots, \Gamma_{kd})$ from the distribution $\frac{\partial}{\partial x_1}\bar{C}(\boldsymbol{x})|_{x_1=\Gamma_{k1}}$.
4. Return $\mathbf{\Gamma}_k$, $k \in \{1, \dots, K\}$.

By Theorem 3.3, for PALCs the distribution of $(\Gamma_{k2}, \dots, \Gamma_{kd})$ given Γ_{k1} has an AC as dependence structure. This is a major advantage in contrast to the more general case of PNALCs for which the conditional distributions $\frac{\partial}{\partial x_j}\bar{C}(\boldsymbol{x})|_{x_j=\Gamma_{kj}}$ do not necessarily have an Archimedean (or even nested Archimedean) dependence structure underlying and may thus not be feasible to sample anymore. PNALCs will be addressed in Section 4.3.

Note that $\bar{\nu}_1^{-1}(K)$ is the smallest jump size simulated in the first dimension. Since the truncation holds only for the dimension on which we condition, it is well known that simulating by conditioning on one of the variables introduces an asymmetric bias, i.e., different dimensions are affected differently by the truncation. This bias tends to become larger, the smaller the dependence between the dimensions; see [15] or [30]. Intuitively, a large jump in the conditioned component (the first component here) may, by chance, lead to a very small jump (smaller than $\bar{\nu}_2^{-1}(K)$) in, e.g., the second component. On average, the second component should have K jumps of size larger than $\bar{\nu}_2^{-1}(K)$. However, the (accidentally) very small jump is missing now, since only an average number of K jumps are simulated (which, in the first component, are all larger than $\bar{\nu}_1^{-1}(K)$). In the next section, we therefore introduce a different sampling algorithm for PALCs which does not suffer from such an (with respect to the components) asymmetric bias. Also, this algorithm can be extended to sample PNALCs which we will address in Section 4.3.

4.2. Directly sampling $(\mathbf{\Gamma}_k)_{k=1}^K$ for positive Archimedean Lévy copulas

Similar to the so-called Marshall–Olkin algorithm for sampling ACs, see [31], we now introduce an algorithm for directly sampling from PALCs. In contrast to the conditional method presented in Section 4.1, the proposed algorithm directly provides samples from $\mathbf{\Gamma}_k$ which can be inserted into Eq. (13) instead of sampling $(\Gamma_{k2}, \dots, \Gamma_{kd})$ conditional on Γ_{k1}.

For an AC C with generator ψ the inverse Laplace–Stieltjes transform F in (3) is a distribution function. For sampling ACs, the idea is then to draw a variate $V \sim F$ and to use representation (4) to receive a d-dimensional random vector $\boldsymbol{U} \sim C$. This idea is not directly applicable to PALCs, because the inverse Laplace–Stieltjes transform F in (10) is not a distribution function anymore ($F(\infty) = \infty$). Furthermore, we deal with an infinite number of jumps, which, of course, cannot all be simulated. So we have to truncate (sufficiently small) jumps.

The basic idea of our algorithm is to use the inverse F^{-1} and a finite number of points of a standard Poisson process W_k, $k \in \{1, \dots, K\}$, to draw variates $V_k = F^{-1}(W_k)$. Given V_k, $k \in \{1, \dots, K\}$, we can then draw associated values of $\mathbf{\Gamma}_k$, $k \in \mathbb{N}$. Due to the underlying Archimedean structure, constructing the jumps in this way is possible because we can parameterize the tail integral of the process in such a way that simulating in the sense of infinitely divisible distributions is possible. For a proof of the following algorithm, see the Appendix.

Algorithm 4.2 (*Directly Sampling* $(\mathbf{\Gamma}_k)_{k=1}^K$ *for Sampling PALCs*).

1. Fix a (sufficiently large) number K of (large) jumps to be simulated.
2. Generate independent $Z_k \sim \mathrm{Exp}(1)$, $k \in \{1, \dots, K\}$ and set $W_k = W_{k-1} + Z_k$, $k \in \{1, \dots, K\}$, with $W_0 = 0$.
3. Set $V_k = F^{-1}(W_k)$, $k \in \{1, \dots, K\}$, where $F = \mathcal{LS}^{-1}[\bar{\psi}]$.

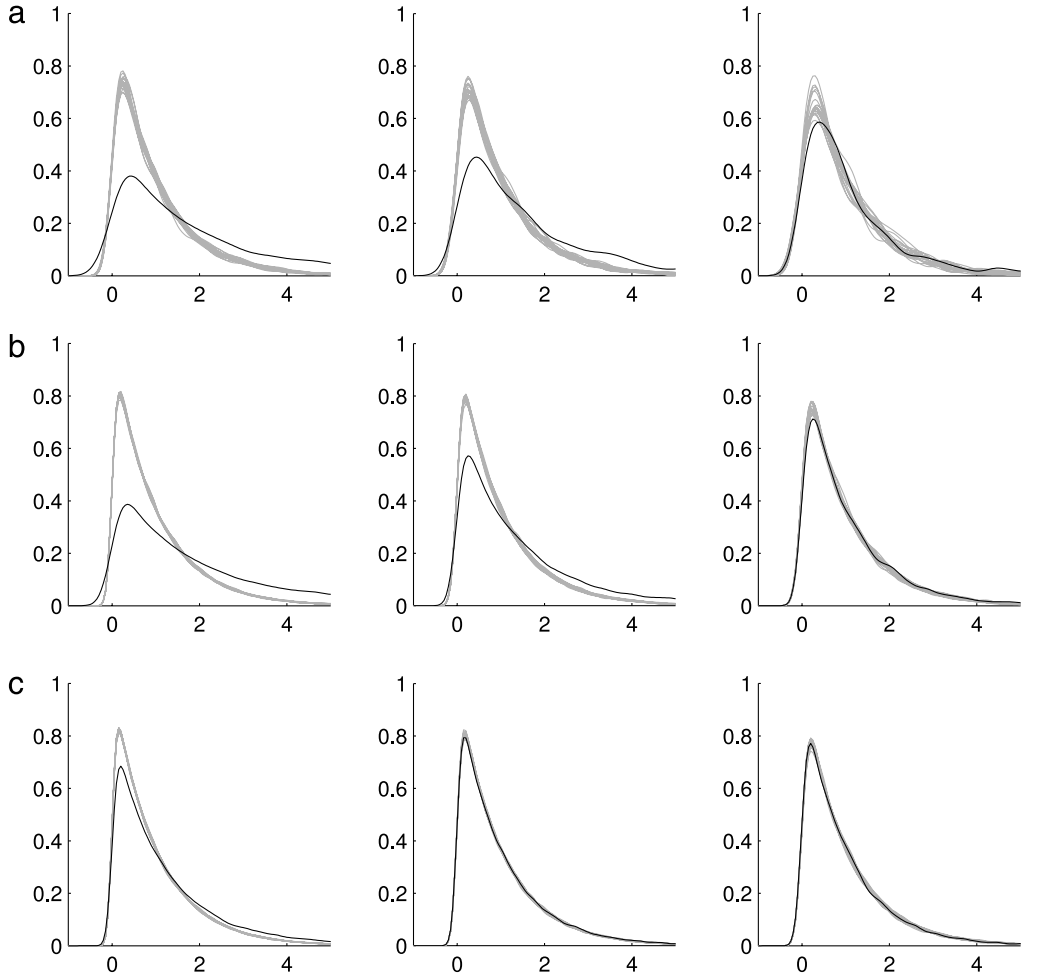

Fig. 1. The black lines show kernel density estimates of the differences $E_{kj} = \Gamma_{(k)j} - \Gamma_{(k-1)j}$, $k \in \{1 \dots K^*\}$, for a fixed dimension j from a three-dimensional positive Clayton Lévy copula with parameter θ for the following values of θ and K^*: (a): $\theta = 0.2$, $K^* \in \{3000, 1500, 500\}$ (from left to right); (b): $\theta = 0.4$, $K^* \in \{30\,000, 15\,000, 5000\}$ (from left to right); and (c): $\theta = 1.2$, $K^* \in \{60\,000, 30\,000, 10\,000\}$ (from left to right). The shaded lines stem from 20 kernel density estimates drawn from Exp(1) with the same sample size K^*. A total number of $K = 100\,000$ jumps are drawn in all cases. The higher K^*, the more of the simulated K joint jumps are used and thus the smaller the bias.

4. Generate independent $E_{kj} \sim \text{Exp}(1)$, $k \in \{1, \dots, K\}$, $j \in \{1, \dots, d\}$.
5. Return $\mathbf{\Gamma}_k = \left(\bar{\psi}\left(\frac{E_{k1}}{V_k}\right), \dots, \bar{\psi}\left(\frac{E_{kd}}{V_k}\right)\right)$, $k \in \{1, \dots, K\}$.

Note the remarkable resemblance of the stochastic representation of $\mathbf{\Gamma}_k$ in Algorithm 4.2 with (4). Further remarks are summarized below.

Remark 4.3.

1. Since larger values of W_k tend to produce larger entries of $\mathbf{\Gamma}_k$, i.e., smaller jump sizes, stopping the series of W_k at K (correctly) results in truncating (mostly) small jumps.
2. Fix a number $K^* \leq K$. The larger the truncation point K, i.e., the more W_k, $k = 1 \dots K$ are used in the simulation, the lower is the probability that W_k for $k > K$ would lead to jumps larger than $\bar{\nu}_j^{-1}(K)$.
3. Especially, when the dependence within the PALC is weak in the sense that simultaneous jumps of very different sizes are more likely to occur, the vector $\mathbf{\Gamma}_k$ may contain small values even for large values of W_k. In such cases K should be chosen larger than in cases with strong dependence, so that enough large jumps are simulated; see also Fig. 1.
4. Thus, it can be convenient not to use all K simulated jump sizes but only the largest $K^* \leq K$.
5. Concerning the choice of K, let for each dimension $j \in \{1, \dots, d\}$, $\Gamma_{(1)j} \leq \cdots \Gamma_{(K)j}$ denote the order statistics of Γ_{kj}, $k \in \{1, \dots, K\}$. Then, for the limit $K \to \infty$, any difference $\Gamma_{(k)j} - \Gamma_{(k-1)j}$ should be Exp(1) distributed. To decide whether K is large enough, choose a (sufficiently large) truncation point τ, where $\bar{\nu}_j^{-1}(\tau)$ corresponds to the smallest

jump size in dimension j up to which we want the simulation to be exact and test whether the sample $\Gamma_{(k)j} - \Gamma_{(k-1)j}$, $k \in \{1, \dots, K^*\}$ follows a unit exponential distribution; here $\Gamma_{(K^*)j}$ is the first $\Gamma_{(k)j}$ to be larger than τ.

6. Similar to the distributional Marshall–Olkin algorithm for sampling ACs, it is not always easy to find an efficient sampling algorithm for the inverse Laplace–Stieltjes transform F leading to the generator $\bar{\psi}$; see (10). In such a case, numerical inversion algorithms of Laplace transforms might be useful; see [18]. For a list of Laplace–Stieltjes transforms and their inverses, see [36]. See also Section 3.1.2.

4.3. Directly sampling $(\mathbf{\Gamma}_k)_{k=1}^K$ for positive nested Archimedean Lévy copulas

Our suggested procedure for sampling PNALCs is very similar to the procedure of sampling NACs. As shown by McNeil [32] and Hofert [21], for sampling NACs a Marshall–Olkin type algorithm can be used to sample all involved building blocks of the structure. Since combinations of Archimedean Lévy generators as they appear within the algorithm are proper (probabilistic) Archimedean generators, sampling PNALCs can be done as sampling NACs. The only exception is the "outermost" level ($\bar{C}_0$ in (12)), for which Algorithm 4.2 applies.

For simplicity of the argument, we first consider the three-dimensional PNALC (11) here. Sampling the corresponding NAC (replacing $\bar{\psi}_0$ by ψ_0 and $\bar{\psi}_1$ by ψ_1) can be done via

$$\left(\psi_0\left(\frac{E_1}{V_0}\right), \psi_1\left(\frac{E_2}{V_{01}}\right), \psi_1\left(\frac{E_3}{V_{01}}\right)\right), \tag{14}$$

where $V_0 \sim F_0 = \mathcal{LS}^{-1}[\psi_0]$ and $V_{01} \sim F_{01} = \mathcal{LS}^{-1}[\psi_{01}(\cdot; V_0)]$ are independent of $E_j \sim \mathrm{Exp}(1)$, $j \in \{1, 2, 3\}$. Here,

$$\psi_{01}(t; V_0) = \exp\big(-V_0 \psi_0^{-1}(\psi_1(t))\big), \quad t \in [0, \infty);$$

see [20,21] for more details.

In the Lévy copula framework, we use Algorithm 4.2 to address the "sampling step" for the outermost generator $\bar{\psi}_0$ and proceed in the same way as in the NAC framework for sampling

$$\psi_{01}(t; V_0) = \exp\big(-V_0 \bar{\psi}_0^{-1}(\bar{\psi}_1(t))\big), \quad t \in [0, \infty).$$

If $(\bar{\psi}_0^{-1} \circ \bar{\psi}_1)'$ is completely monotone, $\psi_{01}(\cdot; V_0)$ is indeed a classical Archimedean generator (being completely monotone and satisfying $\psi_{01}(0; V_0) = 1$; this is also the reason for the notation "F": on all but the outermost nesting level, we indeed have proper distribution functions). This allows us to proceed as in the distributional framework.

The first three steps for directly sampling $(\mathbf{\Gamma}_k)_{k=1}^K$ for PNALCs of type (11) are as in Algorithm 4.2, with V_k replaced by $V_{0,k} = F_0^{-1}(W_k)$, $k \in \{1, \dots, K\}$, where $F_0 = \mathcal{LS}^{-1}[\bar{\psi}_0]$. We can then proceed by sampling independent $U_{k1} \sim \mathrm{U}[0, 1]$, $k \in \{1, \dots, K\}$, and $(\tilde{U}_{k2}, \tilde{U}_{k3})$, $k \in \{1, \dots, K\}$, from a bivariate AC with (proper!) generator $\psi_{01}(t; V_{0,k}) = \exp\big(-V_{0,k} \bar{\psi}_0^{-1}(\bar{\psi}_1(t))\big)$. Applying the Marshall–Olkin algorithm to the latter sampling problem leads to the stochastic representation

$$(\tilde{U}_{k2}, \tilde{U}_{k3}) = \left(\psi_{01}\left(\frac{-\log U_{k2}}{V_{01,k}}; V_{0,k}\right), \psi_{01}\left(\frac{-\log U_{k3}}{V_{01,k}}; V_{0,k}\right)\right),$$

where $(U_{k2}, U_{k3}) \sim \mathrm{U}[0, 1]^2$ are independent of $V_{01,k} \sim F_{01} = \mathcal{LS}^{-1}[\psi_{01}(\cdot; V_{0,k})]$, $k \in \{1, \dots, K\}$. Putting the pieces together, we obtain the stochastic representation

$$\mathbf{\Gamma}_k = \left(\bar{\psi}_0\left(\frac{-\log U_{k1}}{V_{0,k}}\right), \bar{\psi}_0\left(\frac{-\log \tilde{U}_{k2}}{V_{0,k}}\right), \bar{\psi}_0\left(\frac{-\log \tilde{U}_{k3}}{V_{0,k}}\right)\right) \tag{15}$$

$$= \left(\bar{\psi}_0\left(\frac{-\log U_{k1}}{V_{0,k}}\right), \bar{\psi}_1\left(\frac{-\log U_{k2}}{V_{01,k}}\right), \bar{\psi}_1\left(\frac{-\log U_{k3}}{V_{01,k}}\right)\right). \tag{16}$$

In fact, we see that from (15) to (16) several terms cancel out and we obtain the simplified stochastic representation

$$\mathbf{\Gamma}_k = \left(\bar{\psi}_0\left(\frac{E_{k1}}{V_{0,k}}\right), \bar{\psi}_1\left(\frac{E_{k2}}{V_{01,k}}\right), \bar{\psi}_1\left(\frac{E_{k3}}{V_{01,k}}\right)\right)$$

for $E_{kj} \sim \mathrm{Exp}(1)$, $k \in \{1, \dots, K\}$, $j \in \{1, 2, 3\}$, independent of the other involved random variables. As for PALCs, note the remarkable resemblance of the stochastic representation of $\mathbf{\Gamma}_k$ with the stochastic representation (14).

To summarize, we obtain the following algorithm for sampling $(\Gamma_{k1}, \Gamma_{k2}, \Gamma_{k3})$ for (11).

Algorithm 4.4 (*Directly Sampling $(\mathbf{\Gamma}_k)_{k=1}^K$ for Sampling PNALCs of Type (11)*)**.**

1. Fix a (sufficiently large) number K of (large) jumps to be simulated.
2. Generate independent $Z_k \sim \mathrm{Exp}(1)$, $k \in \{1, \dots, K\}$ and set $W_k = W_{k-1} + Z_k$, $k \in \{1, \dots, K\}$, with $W_0 = 0$.
3. Set $V_{0,k} = F_0^{-1}(W_k)$, $k \in \{1, \dots, K\}$, where $F_0 = \mathcal{LS}^{-1}[\bar{\psi}_0]$.
4. Generate independent $V_{01,k} \sim F_{01} = \mathcal{LS}^{-1}[\psi_{01}(\cdot; V_{0,k})]$, $k \in \{1, \dots, K\}$; note again here that $\psi_{01}(\cdot; V_{0,k})$ is a proper Archimedean generator.

5. Generate independent $E_{kj} \sim \mathrm{Exp}(1)$, $k \in \{1, \dots, K\}$, $j \in \{1, 2, 3\}$.
6. Return $\boldsymbol{\Gamma}_k$, $k \in \{1, \dots, K\}$, where

$$\boldsymbol{\Gamma}_k = \left(\bar{\psi}_0\left(\frac{E_{k1}}{V_{0,k}}\right), \bar{\psi}_1\left(\frac{E_{k2}}{V_{01,k}}\right), \bar{\psi}_1\left(\frac{E_{k3}}{V_{01,k}}\right)\right). \tag{17}$$

With similar arguments as before, Algorithm 4.4 can be generalized to more complicated nesting structures, see Example 4.6; see also [21] in the case of NACs, the calculations there readily apply here as well.

Example 4.5 (*Sampling $(\boldsymbol{\Gamma}_k)_{k=1}^K$ for Positive Nested Clayton Lévy Copulas*)**.** Let $\bar{\psi}_0$ and $\bar{\psi}_1$ be Clayton Lévy generators with parameters θ_0 and θ_1, respectively, and, to ensure that $\bar{\psi}_0^{-1}(\bar{\psi}_1(t)) = t^{\theta_0/\theta_1}$ has completely monotone derivative, let $\theta_0 < \theta_1$. We have

$$\psi_{01}(t; V_0) = \exp\big(-V_0 \bar{\psi}_0^{-1}(\bar{\psi}_1(t))\big) = \exp(-V_0 t^{\theta_0/\theta_1}), \quad t \in [0, \infty).$$

This corresponds to an alpha-stable law of type $S(\alpha, 1, (\cos(\alpha\pi/2)V_0)^{1/\alpha}, V_0 \mathbb{1}_{\{\alpha=1\}}; 1)$, $\alpha = \theta_0/\theta_1$; see [20] or the R package `copula` for the exact parameterization.

Example 4.6 (*A More Sophisticated PNALC Example*)**.** In the spirit of Algorithm 4.4 and [21], a PNALC of the form

$$\bar{C}(\boldsymbol{x}) = \bar{C}_0\big(x_1, \bar{C}_1(x_2, x_3), \bar{C}_2(x_4, \bar{C}_3(x_5, x_6, x_7))\big)$$

allows for the stochastic representation of $\boldsymbol{\Gamma}_k$, $k \in \{1, \dots, K\}$, given by

$$\left(\bar{\psi}_0\left(\frac{E_{k1}}{V_{0,k}}\right), \bar{\psi}_1\left(\frac{E_{k2}}{V_{01,k}}\right), \bar{\psi}_1\left(\frac{E_{k3}}{V_{01,k}}\right), \bar{\psi}_2\left(\frac{E_{k4}}{V_{02,k}}\right), \bar{\psi}_3\left(\frac{E_{k5}}{V_{23,k}}\right), \bar{\psi}_3\left(\frac{E_{k6}}{V_{23,k}}\right), \bar{\psi}_3\left(\frac{E_{k7}}{V_{23,k}}\right)\right),$$

where $V_{0,k}$ and $V_{01,k}$ are as in Algorithm 4.4 and, similarly, $V_{02,k} \sim F_{02} = \mathcal{LS}^{-1}[\bar{\psi}_{02}(\cdot; V_{0,k})]$ with $\bar{\psi}_{02}(t; V_{0,k}) = \exp\big(-V_{0,k}\bar{\psi}_0^{-1}(\bar{\psi}_2(t))\big)$ and $V_{23,k} \sim F_{23} = \mathcal{LS}^{-1}[\bar{\psi}_{23}(\cdot; V_{02,k})]$ with $\bar{\psi}_{23}(t; V_{02,k}) = \exp\big(-V_{02,k}\bar{\psi}_2^{-1}(\bar{\psi}_3(t))\big)$.

5. Simulation study

First, we analyze the direct simulation algorithm, i.e., Algorithm 4.2, in an example. We then compare it to the conditional method, i.e., Algorithm 4.1.

5.1. Checking the distribution of Γ_{kj}, $k \in \{1, \dots, K^*\}$

To assess the quality of the simulated samples of the K^* largest jumps in component $j \in \{1, \dots, d\}$, which are used in representation (13), we focus on the differences $E_{kj} = \Gamma_{(k)j} - \Gamma_{(k-1)j}$, $k \in \{1 \dots K^*\}$, where we set $\Gamma_{(0)j} = 0$. For each $j \in \{1, \dots, d\}$, the samples E_{kj}, $k \in \{1, \dots, K^*\}$, should follow an Exp(1) distribution.

Fig. 1 shows examples of kernel density estimates for E_{kj}, $k \in \{1, \dots, K^*\}$, for a three-dimensional positive Clayton Lévy copula for different values of K^* together with 20 repetitions of kernel density estimates (gray lines) computed from samples of size K^* of a standard exponential random variable. We find that the smallest jump size above which the simulated jumps are adequate (measured by the largest K^* of the K simulated jumps in Algorithm 4.2) decreases with increasing dependence parameter of the positive Clayton Lévy copula. In short, a larger parameter θ allows to take a smaller K^*. This means that the bias introduced by simulating only a finite number K of jumps is the smaller the stronger the dependence between the jumps sizes of the components is (in the sense of Example 3.4).

5.2. Comparison with the conditional method

We now compare the effect of a finite K to the conditional method. To this end, we draw from a bivariate positive Clayton Lévy copula using Algorithms 4.1 and 4.2, both based on a series of jump times of a standard Poisson process. We let $\tau = 10\,000$, successively increase K and count, for each method, how many jumps larger than $\bar{\nu}_j^{-1}(\tau)$ in component j have been simulated. Note that this equals the number of Γ_{kj}, $k \in \{1, \dots, K\}$, less than τ. The expectation of this number coincides with τ if the Γ_{kj}'s are proper jump times of a standard Poisson process.

The results are shown in Fig. 2; the abscissas show K, the ordinates show the number of Γ_{kj} smaller than τ ($j = 1$: left column of plots; $j = 2$: right column of plots). In the conditional method we condition on the first component, so that Γ_{k1} is indeed a sequence of standard Poisson jump times. Therefore, the number of values Γ_{k1} smaller than τ equals K if $K \leq \tau$ and τ if $K > \tau$. This is shown in the top left panel of Fig. 2; this optimal benchmark case is displayed in the other panels with a dotted line. Each of the other three panels shows six solid lines, which, from top to bottom, refer to the parameters $\theta \in \{5, 3, 1.5, 0.8, 0.3, 0.2\}$ of the positive Clayton Lévy copula. As can be seen by comparing the top row (Algorithm 4.1) and the bottom row (Algorithm 4.2), it is obvious that the conditional method induces an asymmetric bias with respect to the components of the process. While the first component coincides with the best possible method (top left), the second component (top right) is biased for small values of K, especially if θ is small. In many applications such as, e.g., portfolio

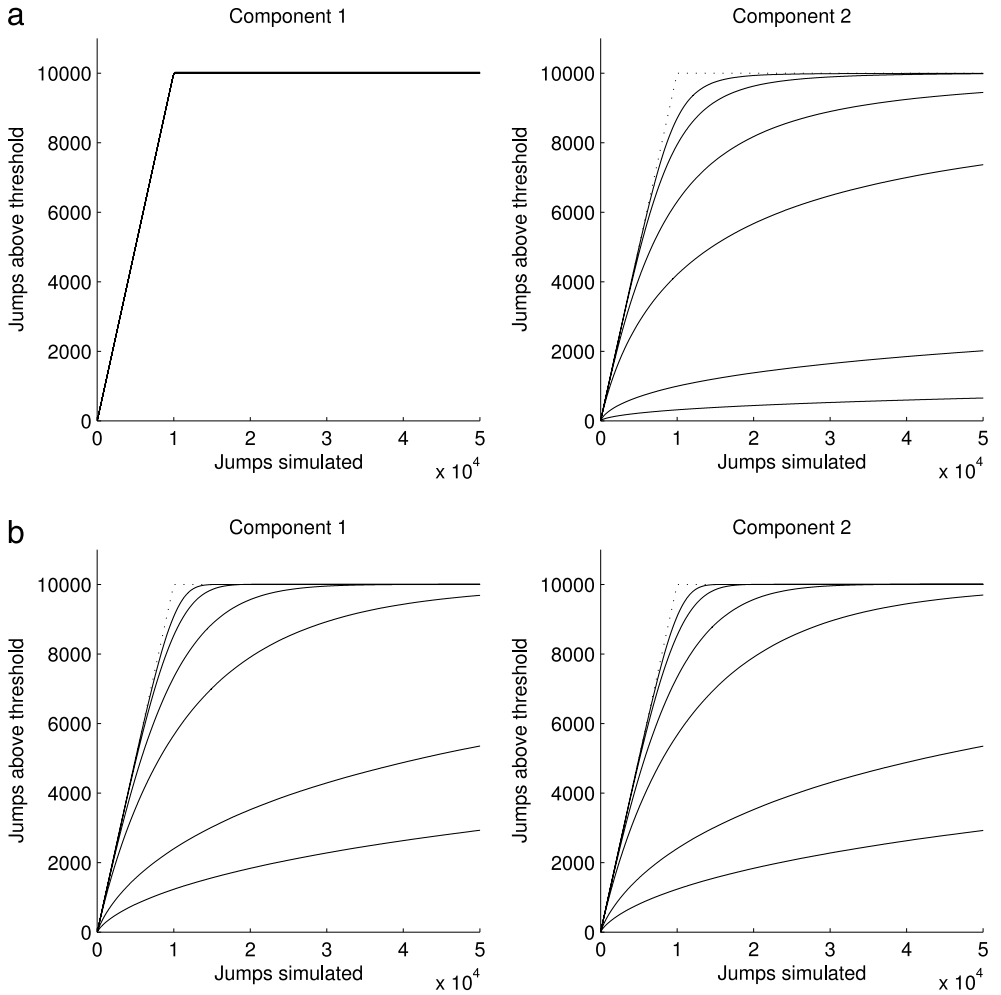

Fig. 2. Numbers of simulated values Γ_{kj} smaller than $\tau = 10\,000$ plotted against the number of simulated jumps K for a bivariate Clayton Lévy copula with parameter $\theta \in \{5, 3, 1.5, 0.8, 0.3, 0.2\}$ (corresponding to the curves from top to bottom). Left-hand graphs refer to the first component $j = 1$, right-hand graphs to the second component $j = 2$. Top row (a): Simulation with the conditional method of Algorithm 4.1 (conditioning on the first component); Bottom row (b): Simulation with the direct method of Algorithm 4.2.

optimization, the ratio of numbers of jumps between the different components is a key parameter and such asymmetric bias can highly affect the results. Our suggested direct simulation approach does not suffer from this asymmetric bias which can be inferred from the bottom row of the figure: The graphs for both components look identically. Furthermore, it can be seen that the lines tend to approach the threshold faster than for the conditional method, which shows that the convergence of Algorithm 4.2 is faster.

5.3. An example process

For illustration purposes we simulate a 4-dimensional nested Clayton Lévy copula

$$\bar{C}(\boldsymbol{x}) = \bar{C}_0(\bar{C}_1(x_1, x_2), \bar{C}_2(x_3, x_4)),$$

where $\bar{C}_0$, $\bar{C}_1$ and $\bar{C}_2$ are Clayton Lévy copulas with parameters $\theta_0 = 0.7$, $\theta_1 = 3$ and $\theta_2 = 2$. Thus, the dependence between jumps of the first and second (third and fourth) component is of Clayton-type with parameter $\theta_1 = 3$ ($\theta_2 = 2$). The dependence of jumps of any combination of a first- or second component with a third- or fourth component is of Clayton-type with parameter $\theta_0 = 0.7$. Note that for this example the generators $\psi_{01}(t; V_0)$ and $\psi_{02}(t; V_0)$ are given in Example 4.5.

To focus on the effect of the dependence only, we use the same pure jump Lévy process for the four margins. We use the positive jump parts of a variance gamma Lévy process (see, e.g., [1] for an application of this process in an option pricing context) and numerically compute the inverse of its tail integral. Its one-dimensional Lévy measure $\nu(x)$ depends on three parameters (θ, κ, σ) and is given by

$$\nu(x) = \frac{c}{x}\exp(-\lambda + x) \quad \text{with } c = \frac{1}{\kappa} \text{ and } \lambda = \frac{\sqrt{\vartheta^2 + 2\sigma^2/\kappa}}{\sigma^2} - \frac{\vartheta}{\sigma^2}.$$

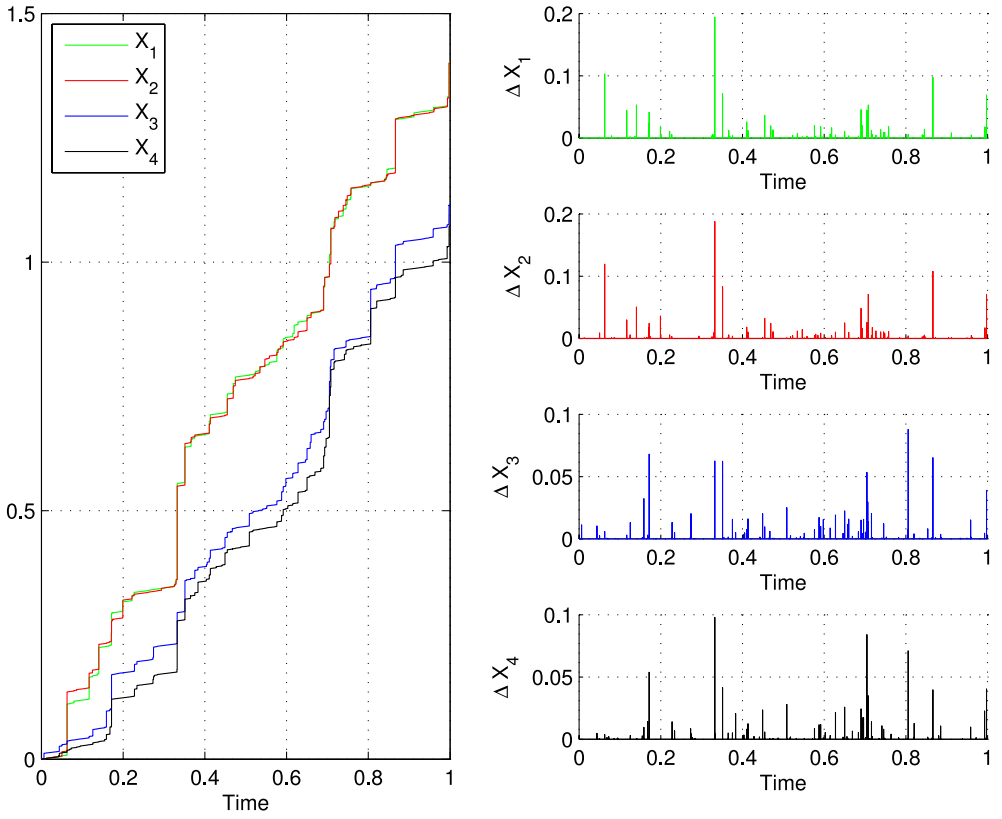

Fig. 3. A simulated 4-dimensional pure jump Lévy process with positive jumps on a unit time interval. The processes are shown on the left-hand side while the pure jumps of the processes are shown on the right hand side. The dependence structure is given by a 4-dimensional nested Clayton Lévy copula (see text). The dependence between the first two components (Group 1) is stronger than between the last two components (Group 2). The inter-group dependence is weakest, which can most conveniently be seen by comparing concurrent jump sizes on the right-hand side.

The parameters are chosen as $\vartheta = -0.2$, $\kappa = 0.05$ and $\sigma = 0.3$ which are typical values in the literature and give a good balance between the number of large and small jumps. The process is shown in Fig. 3. Note that the characteristics of a nested Archimedean Lévy copula can be seen especially in the left part of the figure: the dependence between pairs of components varies within different pairs in contrast to a pure Archimedean structure, where all pairs show the same dependence.

6. Conclusion

We addressed the construction of new positive Archimedean Lévy copulas (PALCs). We then widened the framework to the more general class of positive nested Archimedean Lévy copulas (PNALCs) which allows to model hierarchical dependences between the components of a multivariate Lévy process. Both constructions are of special interest in various high-dimensional applications such as, e.g., operational risk management or insurance. Models in these areas frequently rely on Lévy processes, but are still restricted to low dimensions due to the missing availability of flexible and easy-to-understand high-dimensional Lévy copulas. Furthermore, we considered sampling of Lévy processes with dependence structures given by PALCs or PNALCs. To this end, a general conditional sampling method is frequently applied, which is based on conditioning on the jumps sizes of one of the component processes. As we showed, for PALCs, the dependence between the other jump sizes is then given by an Archimedean copula (AC) which simplifies the application of such a conditional algorithm. However, for PNALCs this is not the case. Furthermore, the conditional approach is rarely practical due to the conditional copulas involved. Moreover, the conditional approach additionally introduces an asymmetric bias in all cases. We therefore introduced a new sampling algorithm similar to the Marshall–Olkin algorithm for sampling Archimedean copulas. This algorithm for sampling Lévy processes with dependence structure specified by a PALC or PNALC does not introduce an asymmetric bias.

Besides our findings in this paper, further research on the construction and sampling of PALCs and PNALCs is required. In particular, questions of the type "How can we construct (more) Archimedean Lévy generators which allow for both nesting and efficient sampling?" would be interesting, in order to provide even more flexible Lévy copulas which are easy to sample for practical applications.

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Appendix

Proof of Lemma 3.2. Assume $F(0) > 0$. Then, by Bernstein's Theorem,

$$\bar{\psi}(t) = \int_0^\infty \exp(-tx)\, dF(x) \geq \exp(-t \cdot 0)F(0) > 0,$$

which contradicts $\bar{\psi} \in \bar{\Psi}_\infty$. Now, by Bernstein's Theorem and since $F(0) = 0$, we have $\bar{\psi}(t) = \int_0^\infty \exp(-tx)\, dF(x) \geq \int_0^t \exp(-tx)\, dF(x) \geq \exp(-t^2) \int_0^t dF(x) = \exp(-t^2)F(t) \geq \exp(-t^2)/2$ for all $t \geq t_0$ for t_0 sufficiently large, thus $\bar{\psi}$ is strict. □

The proof of the following theorem relies on the proof of Mesfioui and Quessy [34, Theorem 1].

Proof of Theorem 3.3. Let $\bar{C}$ denote a d-dimensional PALC. From Lemma 2.5 we have that

$$\begin{aligned} H_w(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) &= \frac{\partial}{\partial w} \bar{C}(x_1, \ldots, x_{j-1}, w, x_{j+1}, \ldots, x_d) \\ &= \bar{\psi}'(\bar{\psi}^{-1}(x_1) + \cdots + \bar{\psi}^{-1}(x_{j-1}) + \bar{\psi}^{-1}(w) + \bar{\psi}^{-1}(x_{j+1}) \\ &\quad + \cdots + \bar{\psi}^{-1}(x_d))(\bar{\psi}^{-1})'(w) \end{aligned} \tag{18}$$

is a distribution function. It is symmetric in $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d$. The marginal distribution function(s) F_w are thus given by (let all except one x_k tend to ∞)

$$F_w(x) = \bar{\psi}'(\bar{\psi}^{-1}(x) + \bar{\psi}^{-1}(w))(\bar{\psi}^{-1})'(w), \tag{19}$$

where we used that $\bar{\psi}^{-1}(\infty) = 0$. From (18) and (19) we obtain that $H_w(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) = F_w(\bar{C}_{\bar{\psi}}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d))$ with $\bar{C}_{\bar{\psi}}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) = \bar{\psi}(\bar{\psi}^{-1}(x_1) + \cdots + \bar{\psi}^{-1}(x_{j-1}) + \bar{\psi}^{-1}(x_{j+1}) + \cdots + \bar{\psi}^{-1}(x_d))$. By Sklar's Theorem, the copula C_w of H_w is thus given by

$$\begin{aligned} C_w(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_d) &= H_w(F_w^{-1}(u_1), \ldots, F_w^{-1}(u_{j-1}), F_w^{-1}(u_{j+1}), \ldots, F_w^{-1}(u_d)) \\ &= F_w\big(\bar{C}_{\bar{\psi}}(F_w^{-1}(u_1), \ldots, F_w^{-1}(u_{j-1}), F_w^{-1}(u_{j+1}), \ldots, F_w^{-1}(u_d))\big). \end{aligned}$$

Letting $\psi_w(t) = F_w(\bar{\psi}(t)) = \bar{\psi}'(t + \bar{\psi}^{-1}(w))(\bar{\psi}^{-1})'(w)$, we see that C_w is a $(d-1)$-dimensional AC with generator ψ_w if ψ_w is a proper generator. Note that $\psi_w(0) = F_w(\infty) = 1$. Also, $\psi_w(\infty) = F_w(0) = \bar{\psi}'(\infty)(\bar{\psi}^{-1})'(w) = 0$. Furthermore, from

$$\psi_w^{(n)}(t) = \bar{\psi}^{(n+1)}(t + \bar{\psi}^{-1}(w))(\bar{\psi}^{-1})'(w), \quad n \in \{1, \ldots, d-3\}, \tag{20}$$

it follows that ψ_w is $(d-1)$-monotone because $\bar{\psi}$ is d-monotone. □

Proof of Theorem 3.9. By definition, (11) is grounded, has uniform margins and satisfies $\bar{C}(\boldsymbol{x}) \neq \infty$ for $\boldsymbol{x} \neq (\infty, \infty, \infty)$. We thus have left to show that $\bar{C}$ is 3-increasing. For this, let

$$G_0(z; x) = \exp(-x\bar{\psi}_0^{-1}(z)), \quad z \in [0, \infty),\ x \in [0, \infty),$$

and note that G_0 is a distribution function in z on $[0, \infty)$ for each $x \in [0, \infty)$. Furthermore, let

$$\psi_{01}(t; x) = \exp\big(-x\bar{\psi}_0^{-1}(\bar{\psi}_1(t))\big), \quad t \in [0, \infty),\ x \in (0, \infty).$$

Then $\psi_{01}(0; x) = 1$ and $\psi_{01}(\cdot; x)$ is completely monotone since a composition of a completely monotone function ($t \mapsto \exp(-xt)$) with a non-negative function with completely monotone derivative ($t \mapsto \bar{\psi}_0^{-1}(\bar{\psi}_1(t))$; by assumption) is again completely monotone; see [14, p. 441]. Therefore, $\psi_{01}(\cdot; x)$ is the generator of an AC in any dimension for each $x \in (0, \infty)$.

By Bernstein's Theorem, $\bar{\psi}_0 = \mathcal{LS}[F_0]$ for some non-finite measure F_0 on $[0, \infty)$. Hence, (11) can be written as

$$\begin{aligned} \bar{C}(\boldsymbol{x}) &= \int_0^\infty \exp(-x\bar{\psi}_0^{-1}(x_1)) \exp\big(-x\bar{\psi}_0^{-1}\big(\bar{\psi}_1(\bar{\psi}_1^{-1}(x_2) + \bar{\psi}_1^{-1}(x_3))\big)\big)\, dF_0(x) \\ &= \int_0^\infty G_0(x_1; x)\psi_{01}(\bar{\psi}_1^{-1}(x_2) + \bar{\psi}_1^{-1}(x_3); x)\, dF_0(x) \\ &= \int_0^\infty G_0(x_1; x)\psi_{01}\big(\psi_{01}^{-1}(G_0(x_2; x); x) + \psi_{01}^{-1}(G_0(x_3; x); x); x\big)\, dF_0(x) \\ &= \int_0^\infty G_0(x_1; x)C_{01}(G_0(x_2; x), G_0(x_3; x))\, dF_0(x), \end{aligned}$$

where C_{01} denotes the AC generated by ψ_{01}. By Sklar's Theorem, $C_{01}(G_0(x_2; x), G_0(x_3; x))$ is a bivariate distribution function for each $x \in (0, \infty)$. We thus see that C is an F_0-mixture of the product of the univariate distribution function $G_0(x_1; x)$ and the bivariate distribution function $C_{01}(G_0(x_2; x), G_0(x_3; x))$, hence 3-increasing; note that by Lemma 3.2 $F_0(0) = 0$, so F_0 puts no mass at 0 (hence the above argument for $x \in (0, \infty)$ suffices). □

Proof of Algorithm 4.2. The idea of the proof is to show that the tail integral $\bar{\nu}(\boldsymbol{x})$, i.e., the expected number of jumps (componentwise) larger than $\boldsymbol{x}$, of a Lévy process $\boldsymbol{L}_t$ with PALC may be represented in the form

$$\int_0^\infty \bar{H}(\boldsymbol{x}; z)\, dz, \tag{21}$$

where $\bar{H}(\boldsymbol{x}; z)$ is a conditional survival function given $z \in (0, \infty)$. It then follows from Bondesson [10] that the jump sizes of the process are given by $\boldsymbol{Z}(V_k)$, where $\boldsymbol{Z}(z)$ is a family of independent, non-negative random vectors with survival function $\bar{H}(\boldsymbol{x}; z)$ and V_k, $k \in \mathbb{N}$, are the jump points in increasing order of a standard Poisson process independent of $\boldsymbol{Z}(z)$. The jump times of the process are uniformly distributed over the considered time horizon (see the series representation (13)). Note that Bondesson [10] focuses on the one-dimensional case only, but the relevant Eqs. (2.1)–(2.4) in his paper equally hold in the multivariate case.

Now consider the PALC

$$\begin{aligned}\bar{C}(\boldsymbol{x}) &= \bar{\psi}(\bar{\psi}^{-1}(x_1) + \cdots + \bar{\psi}(x_d)) = \int_0^\infty \exp\big(-x(\bar{\psi}^{-1}(x_1) + \cdots + \bar{\psi}(x_d))\big)\, dF(x)\\ &= \int_0^\infty \prod_{j=1}^d G_0(x_j; x)\, dF(x) = \int_0^\infty \prod_{j=1}^d G_0(x_j; F^{-1}(z))\, dz,\end{aligned}$$

where $G_0(z; x) = \exp(-x\bar{\psi}_0^{-1}(z))$ is a distribution function in $z \in [0, \infty)$ for each $x \in [0, \infty)$. By Sklar's Theorem for Lévy copulas, we obtain the tail integral

$$\bar{\nu}(\boldsymbol{x}) = \bar{C}(\bar{\nu}_1(x_1), \ldots, \bar{\nu}_d(x_d)) = \int_0^\infty \prod_{j=1}^d G_0(\bar{\nu}_j(x_j); F^{-1}(z))\, dz,$$

where $\prod_{j=1}^d G_0(\bar{\nu}_j(x_j); F^{-1}(z))$ is a survival function in $x_j, j = 1 \ldots d$. To complete the proof, observe that $\Gamma_j = \bar{\nu}_j(x_j)$; i.e., one can either draw a jump size x_j from $G_0(\bar{\nu}_j(x_j); F^{-1}(z))$ and then calculate the Γ_j or equivalently draw the Γ_j from $G_0(\Gamma_j; F^{-1}(z))$ as is done in Algorithm 4.2. □

Algorithm A.1 (*Simulating from Generators Given by* Proposition 3.7)**.** Replace Step 3 in Algorithm 4.2 by

3. For $k \in \{1, \ldots, K\}$, set $\bar{V}_k = F_0^{-1}(W_k)$ (for $\bar{\psi}_0 = \mathcal{LS}[F_0]$) and simulate $V_k \sim F_k$ for $F_k = \mathcal{LS}^{-1}[\psi^{\bar{V}_k}]$.

Proof of Algorithm A.1. Let $\bar{\psi}(t) = \bar{\psi}_0(-\log \psi(t)) \in \bar{\Psi}_\infty$ be a Lévy generator with $\bar{\psi}_0 \in \bar{\Psi}_\infty$ and $\psi \in \Psi_\infty$ such that $(-\log \psi)'$ is completely monotone as assumed in Proposition 3.7. Note that $\bar{\psi}^{-1}(t) = \psi^{-1}\big(\exp(-\bar{\psi}_0^{-1}(t))\big) = \psi^{-1}(G_0(t; 1))$. It follows that

$$\begin{aligned}\bar{C}(\boldsymbol{x}) &= \bar{\psi}(\bar{\psi}^{-1}(x_1) + \cdots + \bar{\psi}^{-1}(x_d)) = \bar{\psi}_0\left(-\log \psi(\bar{\psi}^{-1}(x_1) + \cdots + \bar{\psi}^{-1}(x_d))\right)\\ &= \int_0^\infty \exp\big(x \log \psi(\bar{\psi}^{-1}(x_1) + \cdots + \bar{\psi}(x_d))\big)\, dF_0(x)\\ &= \int_0^\infty \psi(\psi^{-1}(G_0(x_1; 1)) + \cdots + \psi^{-1}(G_0(x_d; 1)))^x\, dF_0(x)\\ &= \int_0^\infty \psi(\psi^{-1}(G_0(x_1; 1)) + \cdots + \psi^{-1}(G_0(x_d; 1)))^{F^{-1}(z)}\, dz.\end{aligned}$$

We thus obtain that

$$\bar{\nu}(\boldsymbol{x}) = \bar{C}(\bar{\nu}_1(x_1), \ldots, \bar{\nu}_d(x_d)) = \int_0^\infty \bar{H}(\boldsymbol{x}; F^{-1}(z))\, dz$$

for

$$\begin{aligned}\bar{H}(\boldsymbol{x}; F^{-1}(z)) &= \psi\big(\psi^{-1}(G_0(\bar{\nu}_1(x_1); 1)) + \cdots + \psi^{-1}(G_0(\bar{\nu}_d(x_d); 1))\big)^{F^{-1}(z)}\\ &= \tilde{\psi}\big(\tilde{\psi}^{-1}\big(G_0(\bar{\nu}_1(x_1); F^{-1}(z))\big) + \cdots + \tilde{\psi}^{-1}\big(G_0(\bar{\nu}_d(x_d); F^{-1}(z))\big)\big),\end{aligned}$$

where $\tilde{\psi}(t) = \psi(t)^{F^{-1}(z)}$. Note that $\bar{H}(\boldsymbol{x}; F^{-1}(z))$ is thus a survival function in $\boldsymbol{x}$ with (survival) copula

$$\tilde{\psi}(\tilde{\psi}^{-1}(u_1) + \cdots + \tilde{\psi}^{-1}(u_d)) = \psi(\psi^{-1}(u_1^{1/F^{-1}(z)}) + \cdots + \psi^{-1}(u_d^{1/F^{-1}(z)}))^{F^{-1}(z)}$$

and marginal survival functions $G_0(\bar{\nu}_1(x_1); F^{-1}(z)), \ldots, G_0(\bar{\nu}_d(x_d); F^{-1}(z))$. □

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