Extremes (2013) 16:303–324
DOI 10.1007/s10687-012-0162-1

Jump tail dependence in Lévy copula models

Oliver Grothe

Received: 28 June 2011 / Revised: 2 May 2012 /
Accepted: 25 September 2012 / Published online: 19 October 2012

Abstract This paper investigates the dependence of extreme jumps in multivariate Lévy processes. We introduce a measure called *jump tail dependence*, defined as the probability of observing a large jump in one component of a process given a concurrent large jump in another component. We show that this measure is determined by the Lévy copula alone and that it is independent of marginal Lévy processes. We derive a consistent nonparametric estimator for jump tail dependence and establish its asymptotic distribution. Regarding the economic relevance of the measure, a simulation study illustrates that jump tail dependence has a substantial impact on financial portfolio distributions and optimal portfolio weights.

Keywords Multivariate Lévy processes · Dependence of jumps ·
Nonparametric estimation · Strong consistency ·
High frequency financial data · Portfolios

AMS 2000 Subject Classifications 60G51 · 62G32 · 91G70

1 Introduction

This paper proposes a measure of the dependence of extreme changes in multivariate Lévy processes. In contrast to the existing literature, the proposed measure is constant with respect to different return frequencies and therefore allows for a clear interpretation. The measure is based on the theory of Lévy copulas and is independent of the marginal distributions of the processes.

O. Grothe (✉)
Department of Economic and Social Statistics, University of Cologne,
Albertus-Magnus-Platz, 50923 Köln, Germany
e-mail: grothe@statistik.uni-koeln.de

In the literature, the tendency of returns of time series to show common extreme realizations is measured by upper and lower tail dependence coefficients (see, e.g., Embrechts et al. 2003 or Poon et al. 2004). These coefficients asymptotically correspond to the probability of observing an extreme realization of a random variable given an extreme realization of another random variable. A significant limitation of this concept is that in the case of multivariate Lévy processes, the coefficients are different for each time interval of returns considered, which complicates their interpretation.

This limitation is illustrated in Fig. 1. The first subgraph depicts scatter plots of realizations of a two-dimensional variance gamma Lévy process $(X_t, Y_t)_{t\geq 0}$ with $(X_0, Y_0) = (0, 0)$ for different times t from 1 to 28 days. In financial applications, this process is often used to model the log-return of a price process over a time interval from 0 to t (see, e.g., Madan et al. 1998). The second subgraph shows scatterplots of the corresponding empirical copulas of the random variables (X_t, Y_t) for fixed t. As can be seen in the figure, the copulas change over time. Because the tail dependence coefficients are functions of these copulas (see, e.g., Joe 1997 or Nelsen 2006), their values differ as well. In this sense, tail dependence characterizes the distribution

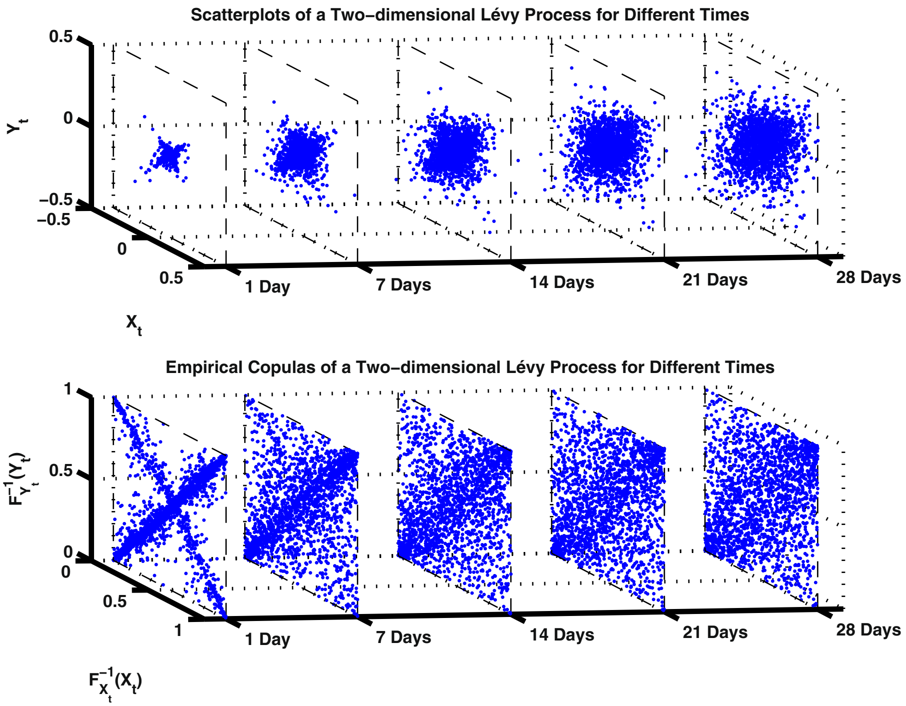

Fig. 1 Scatter plots of 2,000 realizations of a variance gamma Lévy process $(X_t, Y_t)_{t\geq 0}$ with $(X_0, Y_0) = (0, 0)$ for different points in time and corresponding empirical copulas. The copulas change with the time considered and with them the tail dependence coefficients of $(X_t, Y_t)_{t\geq 0}$. For the exact specification of the process see scenario B in Section 3

of a Lévy process on a given time horizon but is not able to characterize the entire Lévy process itself.

To mitigate the illustrated limitation of tail dependence, we propose a new measure, which we call jump tail dependence. Jump tail dependence measures the tendency to observe an extreme jump in one component given a simultaneous extreme jump in another component. The main goal of the measure is to focus directly on the jumps of the Lévy processes rather than to rely on distributional properties. We show that jump tail dependence is a function of the Lévy copula that characterizes a given Lévy process. Moreover, jump tail dependence is independent of the marginal Lévy processes, i.e., it characterizes only the dependence without being affected by the parameters of the marginal univariate Lévy processes. Furthermore, jump tail dependence is independent of the point in time t respectively the return interval considered and therefore characterizes the risk of extreme concurrent jumps for the entire process. To our knowledge, no similar measure of extreme dependence in Lévy processes has yet been proposed in the literature.

To illustrate the economic relevance of jump tail dependence, a simulation study is performed which examines its influence on asset allocation. We also discuss the relationship of jump tail dependence to tail dependence. For a set of assets in which prices are driven by exponential Lévy processes, we compare portfolio distributions for different values of jump tail dependence coefficients. The results show that the influence of jump tail dependence on portfolio distributions and the resulting optimal portfolio weights is substantial, especially for shorter investment horizons.

Regarding the estimation of jump tail dependence, for parametric Lévy copulas, jump tail dependence can be expressed by the parameters of the Lévy copula. Thus, if estimates of these parameters exist, jump tail dependence may be computed from these estimates (see the recent papers of Esmaeili and Klüppelberg 2010, 2011, 2012 for the estimation of Lévy copulas). If the Lévy copula is not parametric or is unknown, nonparametric estimators of jump tail dependence are required. We propose a consistent nonparametric estimator based on ranks of high-frequency observations of the process and derive its asymptotic distribution. The estimator is related to nonparametric estimators of tail dependence (see, e.g., Schmidt and Stadtmüller 2006) but is applied to the jumps of the process instead of to its distribution. The estimator counts the relative number of concurrent extreme jumps for two components of the process. Because the jumps may not be directly observable, high-frequency samples are used as proxies for the jumps (as in the literature on nonparametric inference for Lévy processes, e.g., Basawa and Brockwell 1982, Figueroa-López and Houdré 2006). The applicability of the estimator in finite sample sizes is shown in a simulation study. We also examine the applicability of the bootstrap to estimate the variance of the estimates.

The remainder of the paper is organized as follows. Section 2 introduces the jump tail dependence coefficients. Background information on Lévy processes and proofs are summarized in Appendix A and B. Section 3 contains illustrations of jump tail dependence in the context of financial portfolio optimization. In Section 4, we introduce the jump tail dependence estimator. Proofs and selected theorems can be found in Appendix C. Section 5 analyzes the finite sample behavior of the estimator in a simulation study. Section 6 provides a conclusion.

2 Jump tail dependence coefficients

The concept of jump tail dependence is motivated by the concept of tail dependence (see Sibuya 1960). Tail dependence describes the tendency of two jointly distributed random variables X and Y to show concurrent extreme values. Because the extreme values may be at the upper or at the lower tail of the distribution, upper and lower tail dependence coefficients λ_U and λ_L are considered. If the limits exist, λ_U and λ_L are defined by

$$\lambda_U = \lim_{u \to 1^-} P\left(X > G_X^{-1}(u) | Y > G_Y^{-1}(u)\right), \tag{1}$$

$$\lambda_L = \lim_{u \to 0^+} P\left(X < G_X^{-1}(u) | Y < G_Y^{-1}(u)\right), \tag{2}$$

where $G_X(x)$ and $G_Y(y)$ denote the marginal distribution functions of X and Y, respectively. Tail dependence is associated with the probability of observing an extreme value of X (with a probability value of $u \to 0$ or $u \to 1$), given that we observe an extreme value of Y with the same probability level.

Denote by C the copula of X and Y, i.e., $G(x, y) = C(G_X(x), G_Y(y))$, where $G(x, y)$ is the joint distribution function of X and Y. Both tail dependence coefficients are determined by C and are independent of the respective marginal distributions $G_X(x)$ and $G_Y(y)$:

$$\lambda_L = \lim_{u \to 0^+} \frac{C(u, u)}{u}, \qquad \lambda_U = \lim_{u \to 0^+} \frac{\overline{C}(u, u)}{u}, \tag{3}$$

where $\overline{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ is the survival copula of C (see Joe 1997 or Nelsen 2006). In financial analysis, tail dependence is a widely used measure of extreme events (see, e.g., Embrechts et al. 2003, Malevergne and Sornette 2004 or Poon et al. 2004).

To introduce the measure of jump tail dependence, the notion of tail integrals is needed. As is well known, a d-dimensional Lévy process $(X_t)_{t \geq 0}$ is entirely characterized by the triplet (b^*, c, ν), where b^* is a constant vector, c a covariance matrix and $\nu(A)$ is the Lévy measure, which is the expected number of jumps per time unit with jump sizes in A. The dependence of the jumps of a Levý process is determined by its Lévy copula F (see Appendix A for a more detailed introduction to multivariate Lévy processes). The tail integral $U : (\mathbb{R} \setminus \{0\})^d \to \mathbb{R}$ is defined by

$$U(x_1, \ldots, x_d) := \prod_{i=1}^{d} \operatorname{sgn}(x_i) \nu \left(\prod_{j=1}^{d} \mathcal{I}(x_j) \right),$$

with $\mathcal{I}(x) := [x, \infty)$, for $x \geq 0$ and $\mathcal{I}(x) := (-\infty, x)$ for $x < 0$ ($x \in \mathbb{R}$). For $i = 1 \ldots d$ we define marginal tail integrals U_i to be the tail integrals of the Lévy process consisting of the i-th dimension of X_t only. Thus, for positive x_i, $U_i(x_i)$ is the expected number of jumps per time unit with jump sizes larger than x_i.

We now introduce the concept of jump tail dependence, which focuses on joint extreme jumps of the process. Four types of scenarios are of interest: concurrent large positive jumps (*PP*), concurrent large negative jumps (*NN*), negative-positive (*NP*) and positive-negative (*PN*) jumps.

Definition 1 (Jump Tail Dependence Coefficients) Let $(X_t, Y_t)_{t\geq 0}$ be a two-dimensional real valued Lévy process with marginal tail integrals U_1 and U_2. Let t^* be the time of the largest jump of the component X_t in $t \in [0, 1]$. If the limit exists, define the jump tail dependence coefficient of large positive jumps as

$$\begin{aligned}\lambda_{PP} &= \lim_{u\to 0^+} P\left(\Delta Y_{t^*} > U_2^{-1}(u) | \Delta X_{t^*} > U_1^{-1}(u)\right)\\ &= \lim_{u\to 0^+} P\left(0 < U_2(\Delta Y_{t^*}) < u | 0 < U_1(\Delta X_{t^*}) < u\right), \qquad (4)\end{aligned}$$

where $\Delta X_t := X_t - X_{t^-}$ and $\Delta Y_t := Y_t - Y_{t^-}$ denote the jump sizes of the first and second components at time t, respectively. Analogously, we define λ_{NP}, λ_{PN}, and λ_{NN} :

$$\begin{aligned}\lambda_{NP} &= \lim_{u\to 0^+} P\left(0 > U_2(\Delta Y_{t^*}) > -u | 0 < U_1(\Delta X_{t^*}) < \quad u\right),\\ \lambda_{PN} &= \lim_{u\to 0^+} P\left(0 < U_2(\Delta Y_{t^*}) < \quad u | 0 > U_1(\Delta X_{t^*}) > -u\right),\\ \lambda_{NN} &= \lim_{u\to 0^+} P\left(0 > U_2(\Delta Y_{t^*}) > -u | 0 > U_1(\Delta X_{t^*}) > -u\right).\end{aligned}$$

Note that the tail integrals have singularities with changing signs at zero and that they are decreasing. Positive large jumps of the Lévy processes correspond to small positive values of the marginal tail integrals U_i and large negative jumps to absolutely small negative values. The interpretation of the jump tail dependence coefficients is analogous to the interpretation of tail dependence coefficients. For instance, λ_{PP} asymptotically provides the probability of observing a large positive jump in one component, when observing a large positive jump in another component, where both jump sizes refer to positive tail integral values smaller than or equal to u. The following theorem states that the jump tail dependence coefficients are entirely determined by the Lévy copula F of the process.

Theorem 1 (Jump Tail Dependence and Lévy Copula) *Let $(X_t, Y_t)_{t\geq 0}$ be a real-valued two-dimensional Lévy process with Lévy copula F. Then, if they exist, the four jump tail dependence coefficients are determined by the Lévy copula F by*

$$\begin{aligned}\lambda_{PP} = \lim_{u\to 0^+} \frac{F(u,u)}{u}, \quad \lambda_{PN} = \lim_{u\to 0^+} \frac{F(u,-u)}{-u},\\ \lambda_{NN} = \lim_{u\to 0^+} \frac{F(-u,-u)}{u}, \quad \lambda_{NP} = \lim_{u\to 0^+} \frac{F(-u,u)}{-u}.\end{aligned}$$

See Appendix B for a proof of this theorem. As an example of Theorem 1, we evaluate the jump tail dependence coefficients of the Clayton Lévy copula

$$F(u, v) = \left(|u|^{-\theta} + |v|^{-\theta}\right)^{-1/\theta} \left(\eta 1_{\{uv \geq 0\}} - (1-\eta) 1_{\{uv<0\}}\right) \tag{5}$$

which was introduced by Kallsen and Tankov (2006). Here, $\theta > 0$ determines the dependence of the absolute jump sizes, where larger values of θ correspond to stronger dependence, and $0 < \eta < 1$ determines the dependence of their signs. We get:

$$\lambda_{PP} = \lim_{u \to 0^+} \frac{F(u, u)}{u} = \lim_{u \to 0^+} \left(|u|^{-\theta} + |u|^{-\theta}\right)^{-1/\theta} \eta = 2^{-1/\theta} \cdot \eta = \lambda_{NN},$$

$$\lambda_{NP} = \lambda_{PN} = 2^{-1/\theta} \cdot (1-\eta).$$

For instance, the coefficient λ_{PP}, which is equal to $\lambda_{PP} = 2^{-1/\theta} \cdot \eta$, is the value of the lower tail dependence coefficient of the probabilistic Clayton copula ($\lambda_L = 2^{-1/\theta}$, see, e.g., Nelsen 2006) multiplied by η, the dependence parameter of the signs.

3 Illustration of jump tail dependence

In this section, we illustrate the effect of jump tail dependence on the distribution of a Lévy process and demonstrate the relationship of jump tail dependence to tail dependence. We use a financial example consisting of two portfolio scenarios (A and B). The portfolios are two-dimensional with the price processes modeled by exponential Lévy processes coupled by a Clayton Lévy copula. Parameters are chosen such that the variance-covariance structures of the portfolios are the same in both scenarios, indicating that the two portfolios have the same marginal Lévy processes and the same covariance matrix of returns. Because of these specifications, scenarios A and B are not distinguishable in terms of their drift or covariance matrices, which are the key inputs of the Markowitz (1952) mean-variance portfolio optimization theory. However, the two scenarios are chosen with different levels of jump tail dependence: scenario A has a relatively low jump tail dependence ($\lambda_{PP} = 0.389$), and scenario B has a relatively high jump tail dependence ($\lambda_{PP} = 0.7$).

The asset prices in portfolios A and B are modeled by exponential pure jump variance gamma Lévy processes, $100 \exp\left(r_1 t + X_t^{(1)}\right)$ and $100 \exp\left(r_2 t + X_t^{(2)}\right)$, as introduced by Madan and Seneta (1990). Here $X_t^{(1)}$ and $X_t^{(2)}$ are variance gamma Lévy processes and r_1 and r_2 additional drift coefficients. Because the returns of such processes are heavy tailed, they capture certain stylized facts of asset returns and have thus been widely used for modeling financial processes and asset valuation (see, e.g., Madan et al. 1998 or Almendral and Oosterlee 2007 or Barndorff-Nielsen and Shephard 2004). The one-dimensional Lévy measure, $\nu(x)$, of a variance gamma Lévy process depends on three parameters (θ, κ, σ) and is given by:

$$\nu(x) = \frac{c}{|x|}\exp(-\lambda_-|x|)1_{x<0} + \frac{c}{|x|}\exp(-\lambda_+|x|)1_{x\geq 0} \tag{6}$$

$$\text{with } c = \frac{1}{\kappa} \text{ and } \lambda_\pm = \frac{\sqrt{\vartheta^2 + 2\sigma^2/\kappa}}{\sigma^2} \mp \frac{\vartheta}{\sigma^2}.$$

The parameters of the marginal distributions of $X_t^{(1)}$ and $X_t^{(2)}$ are chosen as $\vartheta^{(1)} = \vartheta^{(2)} = -0.2$, $\kappa^{(1)} = \kappa^{(2)} = 0.04$, $\sigma^{(1)} = 0.3$ and $\sigma^{(2)} = 0.25$, for the first and second components, respectively. Thus, asset 1 is more volatile than asset 2 because it has larger average absolute jump sizes. This parameter selection corresponds to an annual volatility of approximately 25 and 21 %, respectively, if one time unit is interpreted as a year. The martingale property for both exponential Lévy processes is ensured by

$$r_i = c_i \log\left(1 - \frac{1}{\lambda_+^i} + \frac{1}{\lambda_-^i} - \frac{1}{\lambda_+^i\lambda_-^i}\right) \quad \text{for } i = 1, 2. \tag{7}$$

The dependence of $X_t^{(1)}$ and $X_t^{(2)}$ is modeled by a Clayton Lévy copula (see Eq. 5) with different parameters for scenarios A and B. The parameters are chosen such that the processes of the two scenarios have the same (empirical) covariance structure while scenario A has lower jump tail dependence than scenario B. The parameters are summarized in Table 1.

Figure 2 depicts scatter plots of the values of the two assets in each scenario after 1 week (blue) and after 1 year (green), i.e., $t = \frac{1}{50}$ and $t = 1$ for a yearly time unit with 250 trading days, 5 days a week and 8 hours per day. The left graph shows the scenario with lower jump tail dependence (scenario A), and the right graph shows the scenario with higher jump tail dependence (scenario B). In scenario B, on the 1-week horizon, the joint distribution of the two assets shows tail dependence. This is expected because concurrent large jumps in both assets lead to large, concurrent 1-week returns in both assets. Thus, when single jumps strongly affect the distribution of the returns, jump tail dependence induces tail dependence. With an increasing time horizon, however, the impact of large jumps on the asset values decreases

Table 1 Parameters θ and η of the Clayton Lévy copula used in scenarios A and B

	θ	η	$\lambda_{PP} = \lambda_{NN}$	$\lambda_{NP} = \lambda_{PN}$	$\hat{\rho}$	$\widehat{\text{stdev}}(\hat{\rho})$
Scenario A	0.900	0.841	0.389	0.074	0.501	0.010
Scenario B	10	0.750	0.700	0.233	0.502	0.011

The resulting jump tail dependence coefficients are calculated analytically whereas the estimates $\hat{\rho}$ of the correlation coefficients and their variances are based on the sample covariance matrix. The estimated variance of the estimator is the sample variance of $\hat{\rho}_i$ for $i = 1, \ldots, 100$ different simulations of each scenario with 10,000 trajectories

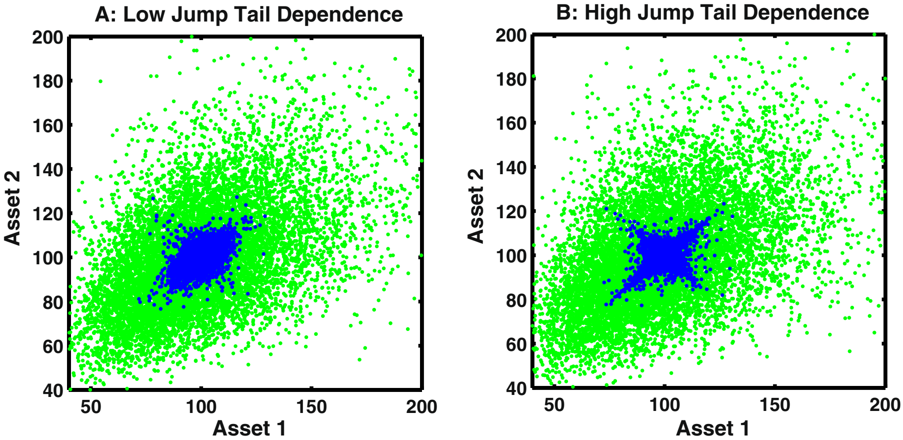

Fig. 2 Scatter plot of portfolio values of the two scenarios for a 1-week (*blue*) and a 1-year (*green*) horizon for 10,000 trajectories. *Left-hand side:* scenario A, low jump tail dependence; *right-hand side:* scenario B, higher jump tail dependence. In both scenarios, assets 1 and 2 have the same marginal Lévy processes and asset 2 has a lower variance than asset 1. On the 1-week horizon, jump tail dependence leads to tail dependence within the joint distribution of the assets. On the 1-year horizon, the variation of the processes is mainly driven by smaller jumps, and the influence of the (less numerous) large jumps on the tails of the distributions is smaller

because these values are driven to a larger extent by smaller (but more numerous) jumps. Therefore, on the 1-year horizon, the tail dependence pattern in the joint distribution vanishes, and the distribution becomes similar to the distribution of scenario A with lower jump tail dependence.

Because jump tail dependence has a larger impact on the joint distribution of asset prices on short time intervals, its role for risk management strategies and asset allocation for these horizons is particularly important. Figure 3 shows the optimal portfolio weights of asset 1 for the discussed portfolios on 1-week and 1-year horizons. The weights are optimal when they minimize the expected shortfall for different levels from 1 to 10 %. Note that both portfolios A and B fulfill the martingale property due to condition 7 and have the same covariance structure. Therefore, the weights of a classical (Markowitz 1952) minimum variance portfolio based on a multivariate normal distribution with the same covariance matrix serve as benchmarks for both portfolios (dashed line in Fig. 3). For the 1-year horizon, the weights in Portfolios A and B do not significantly deviate from the Markowitz case. For the 1-week horizon, however, the weight in asset 1 is lower in scenario B, i.e., the scenario with higher jump tail dependence, while the weights of scenario A are still similar to the Markowitz weights. Thus, for a decreasing time horizon, when the portfolio distributions become more influenced by single large jumps, the weights diverge further from the Markowitz weights when jump tail dependence increases. In cases such as these, considering jump tail dependence in the asset allocation processes is especially important.

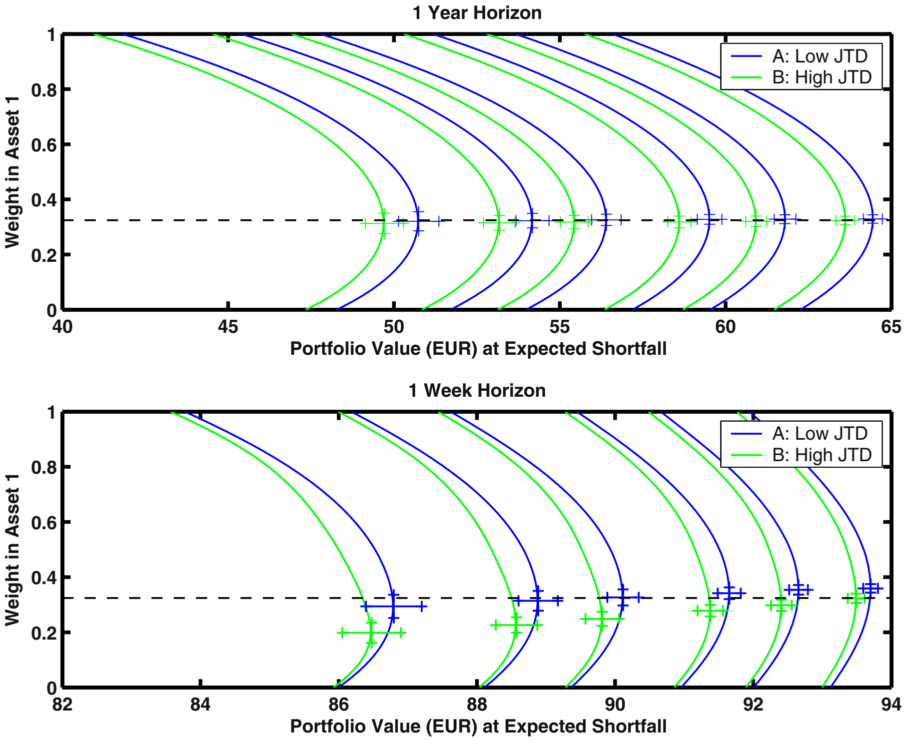

Fig. 3 Portfolio weight in asset 1 (riskier asset) against expected shortfall of a portfolio consisting of asset 1 and asset 2 for different security levels (1, 2, 3, 5, 7, 10 %,) on a 1-week and a 1-year horizon. The *blue lines* correspond to scenario A, and the *green lines* correspond to scenario B. The *error bars* denote the standard deviations of the optima based on 100 simulations with 10,000 trajectories per simulation. On shorter time horizons, jump tail dependence (scenario B) leads to lower weights in the riskier asset. The *dashed line* indicates the weight of asset 1 in a minimal variance portfolio assuming bivariate normal distributions with the same covariance structure

4 Estimation of jump tail dependence coefficients

This section introduces a nonparametric estimator of the jump tail dependence coefficients. The proposed estimator is motivated by the similarity of jump tail dependence coefficients to tail dependence coefficients. Both measures are limits of conditional probabilities: large, concurrent jumps in the case of jump tail dependence and simultaneous extreme values of random variables in the case of tail dependence. Whereas tail dependence is usually estimated by the proportion of realizations of two random variables in the tails of their distributions (see, e.g., Schmidt and Stadtmüller 2006), the proposed estimator for λ_{PP} relies on the proportion of pairs of large jumps in all large jumps of one component. In particular, the conditional probability from Eq. 4 is estimated by the proportion of concurrent jumps that belong to the $100 \cdot \frac{k}{n}$ percent largest observed jumps in all n observed jumps of the respective component divided by $\frac{k}{n}$. Because the jumps cannot be observed directly, returns of the process on small time increments of length Δt are used. Note that for $\Delta t \to 0$, the part of the process which is purely driven by diffusion vanishes, and the probability of observing more than one jump larger than an arbitrary $\varepsilon > 0$ approaches zero. Therefore, observations on small time increments of length Δt roughly correspond to observations of the large jumps of the process (see, e.g., the discussions in Basawa and Brockwell 1982, Nishiyama 2008, Figueroa-López and Houdré 2006 and the limit results of Rüschendorf and Woerner 2002).

Theorem 2 (Estimator for Jump Tail Dependence and Asymptotics) *Let $(X_t, Y_t)_{t\geq 0}$ be a $\mathbb{R}^2$-valued Lévy process with Lévy copula F. We assume the existence of the limits in Theorem 1 as well as the existence of the partial derivatives F_X and F_Y around zero. For $j = 1 \ldots n$ denote by $\left(X_{\Delta t}^{(j)}, Y_{\Delta t}^{(j)}\right)$ n independent increments of length Δt of a single sample path, e.g.,*

$$\left(X_{\Delta t}^{(j)}, Y_{\Delta t}^{(j)}\right) := (X_{j\Delta t}, Y_{j\Delta t}) - (X_{(j-1)\Delta t}, Y_{(j-1)\Delta t}).$$

Furthermore, we denote by $R_{n,X}^{(j)}$ and $R_{n,Y}^{(j)}$ the ranks of $X_{\Delta t}^{(j)}$ and $Y_{\Delta t}^{(j)}$ within each of the dimensions of the observations, i.e.,

$$R_{n,X}^{(j)} = \#\left\{X_{\Delta t}^{(i)} | X_{\Delta t}^{(i)} \leq X_{\Delta t}^{(j)}, i = 1, \ldots, n\right\},$$
$$R_{n,Y}^{(j)} = \#\left\{Y_{\Delta t}^{(i)} | Y_{\Delta t}^{(i)} \leq Y_{\Delta t}^{(j)}, i = 1, \ldots, n\right\}.$$

Then, for $k \in \mathbb{N}$ and $\frac{k}{n}$ small an estimator for λ_{PP} is

$$\begin{aligned}\widehat{\lambda}_{PP,n} &:= \frac{1}{k/n} \sum_{j=1}^{n} \frac{1_{\left\{R_{n,X}^{(j)} > n-k\right\}} 1_{\left\{R_{n,Y}^{(j)} > n-k\right\}}}{n} \\ &= \frac{1}{k} \sum_{j=1}^{n} 1_{\left\{R_{n,X}^{(j)} > n-k\right\}} 1_{\left\{R_{n,Y}^{(j)} > n-k\right\}}, \qquad (8)\end{aligned}$$

where $1_{\{.\}}$ denotes the indicator function. Moreover, for $\Delta t \to 0$ and $n \to \infty$ such that $n\Delta t \to \infty$ and $k = k(n) \to \infty$ with $k/n \to 0$, the estimator is strongly consistent and asymptotically normally distributed with

$$\sqrt{k}\left(\widehat{\lambda}_{PP,n} - \lambda_{PP}\right) \xrightarrow{d} Z \sim N(0, \sigma_{PP}^2),$$

where Z is a centered Gaussian random variable with variance

$$\begin{aligned}\sigma_{PP}^2 = \lambda_{PP} &+ \left(F_X(0^+, 0^+)\right)^2 + \left(F_Y(0^+, 0^+)\right)^2 \\ &+ 2\lambda_{PP}\left(\left(F_X(0^+, 0^+) - 1\right)\left(F_Y(0^+, 0^+) - 1\right) - 1\right). \qquad (9)\end{aligned}$$

Here, $F_X(0^+, 0^+)$ *and* $F_Y(0^+, 0^+)$ *denote the partial derivatives of the Lévy copula, evaluated at the right-hand limit of* 0.

A proof can be found in Appendix C. Estimators for λ_{NN}, λ_{NP} and λ_{PN} are derived analogously by focusing on the respective quadrants of the distributions. For λ_{NP}, e.g., use $1_{\left\{R_{n,X}^{(j)}<k\right\}} 1_{\left\{R_{n,Y}^{(j)}>n-k\right\}}$ in the estimator Eq. 8 and absolute values of derivatives of F at $(0^-, 0^+)$ in Eq. 9.

Note that the double limit $\Delta t \to 0$ and $n \to \infty$ with $n\Delta t \to \infty$ implies that the interval Δt gets finer and at the same time the overall time of the sample path considered, i.e., $n\Delta t$, goes to infinity. This means, that we observe the Lévy process on an infinitely fine sampling grid for an infinitely long time. The estimator then uses only the largest k observations, where k is a function of n, and $k \to \infty$ ensures that the number of observations used is infinite in the limit. At the same time $k/n \to 0$ implies that only the far end of the tail of the jump sizes is taken into account.

In practical applications, a small but fixed value of Δt and a fixed n will be used. In this case, the estimators resemble the conventional estimators for tail dependence, and the threshold parameter k may be chosen in the same way as in the context of estimating tail dependence. Schmidt and Stadtmüller (2006) draw the estimation results for successive increasing values of k and search for a characteristic plateau to choose k. We identify such a plateau for the jump tail dependence estimator in the simulation study in Section 5.

The asymptotic variance σ_{PP} depends on the Lévy copula F, which is unknown in general. Therefore, in most cases, the variance has to be estimated, e.g., by means of a nonparametric bootstrap. For fixed sampling rate Δt and fixed n, where Eq. 8 resembles an estimator of tail dependence, the bootstrap is justified by its justification for the tail dependence estimators (see, e.g., Schmid and Schmidt 2007 for a proof).

5 Simulation study: estimation of jump tail dependence

In this section, we analyze the applicability of the proposed estimator of jump tail dependence in finite sample sizes and for finite sampling intervals. First, we focus on a pure jump Lévy process, i.e., a Lévy process without a diffusion component. Because this process is driven purely by jumps, we expect the estimator to have the best possible performance. Second, we examine a Lévy process that includes diffusion components. Although the estimators are asymptotically not affected by the diffusion, we find that for too large sampling intervals Δt, the estimators become biased, and this bias increases with Δt.

For every estimate of jump tail dependence, e.g., for $\widehat{\lambda}_{PP}$, we use a standard nonparametric bootstrap to estimate the variance of the estimator. Let $\left(X_{\Delta t}^{(j)}, Y_{\Delta t}^{(j)}\right)$ with $j = 1 \ldots n$ be the n vectors of increments of the process given in Theorem 2. A total of M bootstrap samples $\left(X_{\Delta t}^{(j)^*}, Y_{\Delta t}^{(j)^*}\right)$, $j = 1 \ldots n$ of size n are generated by drawing (with replacement) from $\left(X_{\Delta t}^{(j)}, Y_{\Delta t}^{(j)}\right)$, $j = 1 \ldots n$. Let $\widehat{\lambda}_{PP}^{(m)}$ denote the estimator of jump tail dependence (Eq. 8) based on the m-th bootstrap sample

$m = 1, \ldots, M$. The bootstrap estimator $\widehat{\text{var}}^M_{\text{Boot}}(\hat{\lambda}_{PP})$ for the variance of the estimator $\hat{\lambda}_{PP}$ is then given by:

$$\widehat{\text{var}}^M_{\text{Boot}}(\hat{\lambda}_{PP}) = \text{var}_m\big(\hat{\lambda}^{(m)}_{PP}\big),$$

where $\text{var}_m(\hat{\lambda}^{(m)}_{PP})$ is the sample variance of the M bootstrap estimates $\hat{\lambda}^{(m)}_{PP}$, $m = 1 \ldots M$.

For the first part (i.e., the Lévy process without diffusion), we use the two-dimensional Lévy process introduced in scenario B in Section 3 and apply the estimator on data sampled at $\Delta t = 1/2000$. Assuming a yearly time unit as before, this corresponds to one price observation every hour (assuming 250 trading days, 8 hours per day). The theoretical values of the jump tail dependence coefficients are $\lambda_{PP} = \lambda_{NN} = 0.7$ and $\lambda_{PN} = \lambda_{NP} = 0.233$. Figure 4 depicts the estimation results for the unidirectional coefficients λ_{PP} and λ_{NN}, and Fig. 5 for the contradirectional coefficients λ_{PN} and λ_{NP}. Both graphs show the mean estimates of 1,000 simulations (solid lines). In each case, the threshold parameter k of the tail dependence estimator varies from $k = 10$ to $k = 180$. The sample standard deviation of the 1,000 estimates is indicated by confidence bands around the means (dashed lines). The mean values of the bootstrap estimates of the standard deviation ($B = 100$) are marked by confidence bands with dotted lines whereas the solid confidence bands denote the asymptotic standard deviation of the estimator derived in Theorem 2. In all cases, the mean of the bootstrap estimates and the asymptotic standard deviations align with the sample standard deviations. In the case of lower jump tail dependence,

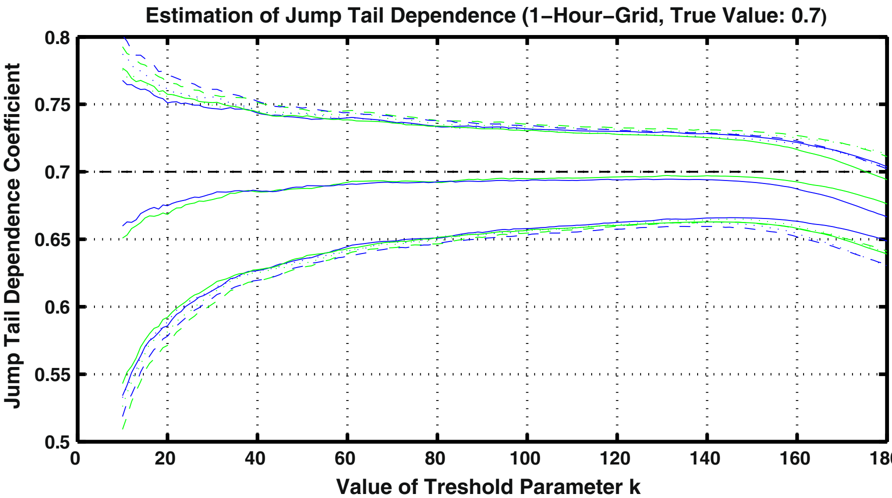

Fig. 4 Estimation of the jump tail dependence coefficients λ_{PP} (*green*) and λ_{NN} (*blue*) of the processes of scenario B for different values of the threshold k on a 1-hour grid. Mean estimates are shown of 1,000 simulations (*solid lines*), ± 1 sample standard deviation (*dashed lines*), $\pm$ mean of bootstrap estimates of the standard deviations (*dotted lines*) and $\pm$ asymptotic standard deviation (*solid line*). The true values of the respective jump tail dependence coefficients are $\lambda_{PP} = \lambda_{NN} = 0.7$ (*dashed-dotted horizontal line*)

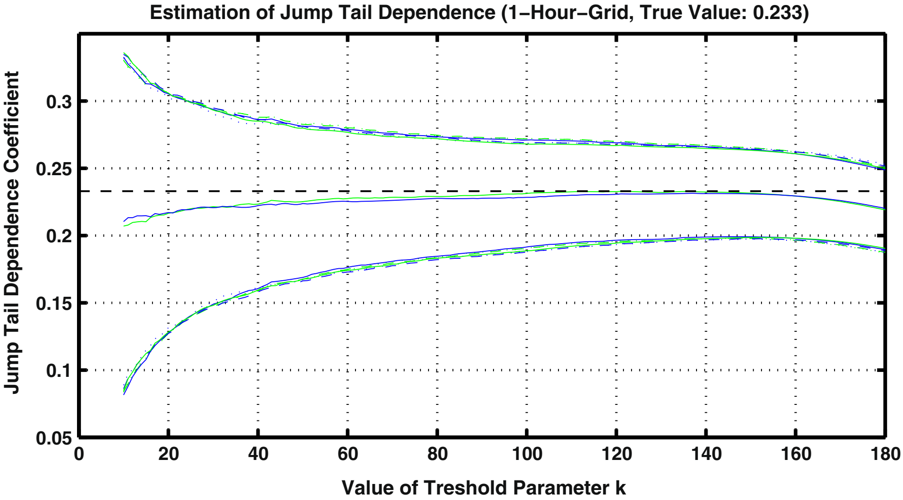

Fig. 5 Estimation of the jump tail dependence coefficients λ_{PN} (*green*) and λ_{NP} (*blue*) of the processes of scenario B for different values of the threshold k on a 1-hour grid. Mean estimates are shown of 1,000 simulations (*solid lines*), ± 1 sample standard deviation (*dashed lines*), $\pm$ mean of bootstrap estimates of these standard deviations (*dotted lines*) and $\pm$ asymptotic standard deviation (*solid line*). The true values of the respective jump tail dependence coefficients are $\lambda_{NP} = \lambda_{PN} = 0.233$ (*dashed-dotted horizontal line*)

the estimates rely on fewer jumps around the threshold, resulting in smoother results. Therefore, the estimates for $\lambda_{PN} = \lambda_{NP} = 0.233$ seem to be superior to those for $\lambda_{PP} = \lambda_{NN} = 0.7$. As expected, the results depend on the choice of the threshold parameter k. However, the plateau visible in the graphs suggests that a parameter between $k = 60$ and $k = 120$ is suitable in all cases (see the discussion in Schmidt and Stadtmüller 2006).

We now add diffusion components, i.e., Brownian motion, to the pure jump process. The covariance matrix of the added Brownian motion (symbol c of Eq. 10) is chosen to be diagonal with standard deviations of 0.05 (first simulation) and 0.2 (second simulation). In our design, this corresponds to annual volatility of 5 and 20 %, which is induced by diffusion, while the jumps of the processes induce an additional volatility of 21 to 25 %.

We apply the estimator for different time intervals ranging from $\Delta t = 2.5$ min to $\Delta t = 60$ min again assuming 250 days of data with 8 trading hours each. The results are shown in Fig. 6. For convenience, we show only the mean estimates and not the variance and bootstrap estimates. Again, bootstrap and sample variances match very well and may be obtained from the author on request. For the lower level of diffusion (5 % volatility), the estimator works sufficiently well on all considered sampling frequencies up to a threshold parameter of $k = 30$. For higher thresholds, the estimates are increasingly affected. As expected, the estimator performs better for smaller sampling intervals Δt. For the larger level diffusion case (20 % volatility), the estimator performs well for a sampling interval lower than 5 min but becomes biased for larger sampling intervals.

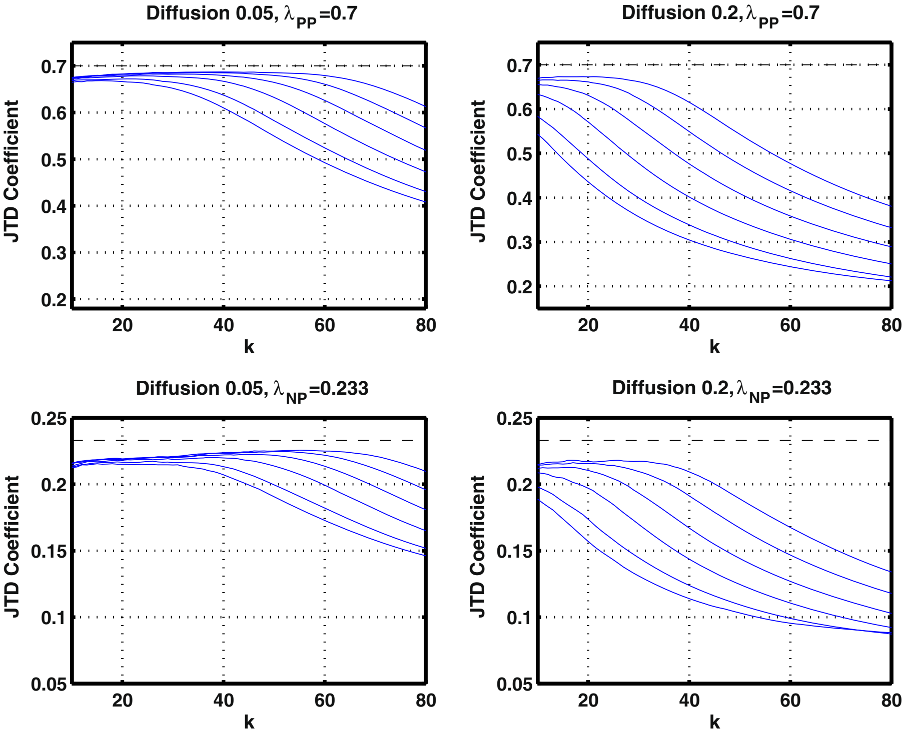

Fig. 6 Estimation of the jump tail dependence (JTD) coefficients for the Lévy process with diffusion on 5 different sampling intervals Δt and different threshold parameter k. Shown are mean estimates of 1,000 repetitions. In each of the subplots the time intervals Δt are (from *right* to *left*) 2.5, 5, 10, 20, 40 and 60 min

The bias of the estimator is related to the fact that returns of the processes on small time increments serve as proxies for jumps. The considered variance gamma process consists of relatively small jumps such that the variation of the diffusion on small time intervals is of similar order to the jump sizes of the jump process, and jumps cannot be distinguished from changes due to diffusion. If, however, the jump process has larger expected jump sizes, the estimator is expected to perform better. Thus, the influence of the diffusion on the estimators' performance decreases with lower sampling time Δt, with increasing expected jump sizes of the Lévy process and with lower threshold parameter k.

6 Conclusion

This paper introduces the concept of jump tail dependence in multivariate Lévy processes. Jump tail dependence corresponds to the probability of observing a large jump in one component of the process given a concurrent large jump in another component. In the theoretical part of the paper, we show that jump tail dependence is a function of the Lévy copula and that it is independent of the marginal jump distributions of the process. For econometric inference, we present a strongly consistent nonparametric estimator and derive its asymptotic distribution. We also demonstrate the applicability of the estimator in finite samples.

The economic relevance of our measure is analyzed in a simulation study, which shows a substantial effect of jump tail dependence on portfolio distributions and on optimal portfolio weights in a financial context. In particular, the role of jump tail dependence in risk management applications is especially large for shorter investment horizons, in which single jumps have a large impact on asset values.

To our knowledge, jump tail dependence provides the first measure that allows the capture of the dependence of extreme jumps in Lévy processes in a clearly interpretable way and that is, as opposed to tail dependence, consistent for varying sampling horizons.

Appendix A: Multivariate Lévy processes and Lévy copulas

This section presents a brief overview of Lévy processes. A more detailed description can be found in Rosinski (2001), Kallenberg (2002) or Cont and Tankov (2004). Lévy processes $(X_t)_{t\geq 0}$ are stochastic processes with stationary, independent increments. According to the Lévy Itô decomposition (see Kallenberg 2002, Theorem 15.4), these processes can be decomposed into a deterministic drift function $D_t = bt$, Brownian motion B_t, and a pure jump process with a possibly infinite number of small jumps. At time t, the characteristic function of the marginal distribution of an $\mathbb{R}^d$-valued Lévy process X is given by the Lévy-Khinchin formula (see Kallenberg 2002, Corollary 15.8):

$$\Phi_{X_t}(\xi) = \exp\left[t\left(i\xi' b^* - \frac{1}{2}\xi' c\xi + \int (e^{i\xi x} - 1 - i\xi' h(x))\nu(dx)\right)\right], \qquad (10)$$

where $h(x) := x1_{\{|x|\leq 1\}}$ is a truncation function ensuring the convergence of the integral and b^* is a drift-like part depending on the choice of $h(x)$. The parameter c is the covariance matrix of the Brownian motion at time $t = 1$ (B_1), and $\nu(A)$ is the Lévy measure, the expected number of jumps per time unit with jump sizes in A. Together, (b^*, c, ν) is called the characteristic triplet of the process.

The tail integral $U : (\mathbb{R} \setminus \{0\})^d \to \mathbb{R}$ is defined by

$$U(x_1, \ldots, x_d) := \prod_{i=1}^{d} \operatorname{sgn}(x_i)\nu\left(\prod_{j=1}^{d} \mathcal{I}(x_j)\right),$$

with $\mathcal{I}(x) := [x, \infty)$, for $x \geq 0$ and $\mathcal{I}(x) := (-\infty, x)$ for $x < 0$ ($x \in \mathbb{R}$). Further, we define marginal tail integrals. Let $I \subset \{1, ...d\}$ be a non-empty subset. Then, the I-marginal tail integral U^I of X is the tail integral of the process $X^I := (X^i)_{i\in I}$. For convenience, one-dimensional margins are denoted by $U_i := U^{\{i\}}$. For positive x_i, $U_i(x_i)$ is the expected number of jumps per time unit (e.g., $t \in [0, 1]$) with jump sizes larger than x_i. The Lévy measure ν of X is determined by its tail integral U and all of its marginal tail integrals U^I.

Further, we define the I-margin F^I of a d-increasing function $F : (\mathbb{R} \setminus \{0\})^d \to \mathbb{R}$ by

$$F^I((u_i)_{i \in I}) = \lim_{a \to \infty} \sum_{(u_i)_{i \in I^c} \in \{-a, \infty\}^{|I^c|}} F(u) \prod_{i \in I^c} \operatorname{sgn}(u_i),$$

where $I^c := \{1, \ldots, d\} \setminus I$. A function $F : S \to \mathbb{R}$ is called d-increasing on $S \subset \mathbb{R}^d$ if the F-volume $V_F((a, b]) \geq 0$ for all segments $\overline{[a, b]} \in S$, $a < b$ (componentwise, i.e., $a_k \leq b_k$ for $k = 1 \ldots d$) with

$$V_F((a, b]) := \sum_{u \in \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}} (-1)^{N(u)} F(u),$$

where $N(u) := \#\{k : u_k = a_k\}$. For example, for $d = 2$ we get $V_F((a, b]) = F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1) + F(a_2, b_2)$.

The dependence of multivariate Lévy processes is determined by the Lévy copula. The concept of Lévy copulas was introduced by Kallsen and Tankov (2006). Intuitively, one could imagine the tail integral as a type of survival function of the jump sizes. In the same way that a probabilistic copula couples marginal distribution functions to the joint distribution function, the Lévy copula couples the marginal tail integrals to the joint tail integral. However, because the tail integral is not a probabilistic distribution function, the definition of a Lévy copula differs from the definition of a probabilistic copula. In particular, the latter does not serve to couple the marginal tail integrals to the joint tail integral.

Define $\mathbb{R}^* := (-\infty, \infty]$. According to Kallsen and Tankov (2006), a Lévy copula F is a function $F : \mathbb{R}^{*d} \to \mathbb{R}^*$ with:

1. $F(u_1, \ldots, u_d) \neq \infty$ for $(u_1, \ldots, u_d) \neq (\infty, \ldots, \infty)$,
2. $F(u_1, \ldots, u_d) = 0$ if at least one $u_i = 0$ for $i \in \{1, \ldots, d\}$,
3. F is d-increasing, and
4. the $\{i\}$-margins $F^{\{i\}}(u) = u$ for any $i \in \{1, \ldots, d\}$ and $u \in \mathbb{R}^*$.

For every $\mathbb{R}^{*d}$-valued Lévy process, there exists a Lévy copula F such that the tail integrals of X satisfy

$$U^I\left((x_i)_{i \in I}\right) = F^I\left((U_i(x_i))_{i \in I}\right)$$

for any non-empty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R}^* \setminus \{0\})^I$ (see Kallsen and Tankov 2006, Theorem 3.6).

The Lévy copula is unique on $\prod_{i=1}^d \overline{\operatorname{Ran} U_i}$. Determining all marginal tail integrals, F together with U_i, $i = 1, \ldots, d$ determines the Lévy measure of the process. Let F be a d-dimensional Lévy copula and U_i, $i = 1, \ldots, d$ be the tail integrals of real-valued Lévy processes. There exists an $\mathbb{R}^{*d}$-valued Lévy process X whose components have the tail integrals $U_1, \ldots, U_d$ and whose marginal tail integrals satisfy the above equation for any non-empty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R}^* \setminus \{0\})^I$. Thus, analogous to probabilistic copulas, Lévy copulas separate the dependence of jumps from their marginal distributions. Note that in contrast to probabilistic copulas, Lévy copulas cannot be interpreted as distribution functions. Because the Lévy measure and the tail integral may be infinite at zero, they cannot be normalized to the [0, 1] interval.

Appendix B: Proof of Theorem 1

For convenience, we cite several theorems, which we use for the proof of Theorem 1, before we start the proof of Theorem 1. There exists a series representation of a univariate Lévy process X_t, which is derived in Rosinski (2001):

Theorem 3 (Series Representation of Lévy Processes) *Let $\{O_i\}$ be a sequence of i.i.d. random variables, uniformly distributed on* [0, 1], *and let $\{\Gamma_i\}$ be a Poisson point process on $\mathbb{R}$ with unit rate. Furthermore, let $\{O_i\}$ and $\{\Gamma_i\}$ be independent. For large $\tau > 0$ such that $U^{(-1)}(\tau) < 1$, let*

$$\nu_\tau := 1_{\{x \in (-\infty, U^{(-1)}(-\tau)] \cup [U^{(-1)}(\tau), \infty)\}} \nu(dx)$$

be the truncated Lévy measure, and define

$$A(\tau) = \int_{|x| \leq 1} x \nu_\tau(dx).$$

Then, for $\tau \to \infty$ and $t \in [0, 1]$, the process

$$X_{\tau,t} = \sum_{-\tau \leq \Gamma_i \leq \tau} U^{(-1)}(\Gamma_i) 1_{O_i \leq t} - t A(\tau) \tag{11}$$

converges in law to a Lévy process X, with characteristic function

$$\Phi_{X_t}(\xi) = \exp\left[t \left(\int \left(e^{i \xi x} - 1 - i \xi h(x) \right) \nu(dx) \right) \right],$$

where $h(x) := x 1_{\{|x| \leq 1\}}$. If the tail integral is not invertible, the inverse refers to the generalized inverse tail integral:

$$U^{(-1)}(u) := \begin{cases} sup\{x > 0 : U(x) \geq u\} \vee 0, & u \geq 0, \\ sup\{x < 0 : U(x) \geq u\}, & u < 0. \end{cases} \tag{12}$$

The sequence $\{\Gamma_i\}$ may be constructed separately for positive and negative values by the jump times of a standard Poisson process, i.e., from two sequences of jump times $\left\{P_j^{1,2}\right\}_{j \geq 1}$ of standard Poisson processes by $\Gamma_i = (-1) P_j^1$ for $i = 2j$ and $\Gamma_i = P_j^2$ for $i = 2j - 1$.

Remark 1 A direct consequence of the theorem is that the jump size of the largest positive jump of a univariate Lévy process within a time unit is distributed as $U^{(-1)}(Z)$, where $U^{(-1)}$ is the inverse of the tail integral and Z is a standard exponentially distributed random variable.

For two-dimensional Lévy processes, the distribution of the jump sizes of concurrent jumps conditioned on one of the components of the process is determined by the Lévy copula (see, e.g., Cont and Tankov 2004):

Theorem 4 (Conditional Distribution and Lévy Copula) *Let F be a Lévy copula satisfying $\lim_{(u_i)_{i\in I}\to\infty} F(u_1,\ldots,u_d) = F(u_1,\ldots,u_d)|_{(u_i)_{i\in I}=\infty}$ for every non-empty $I \subset \{1,\ldots,d\}$. Then, for every fixed $\xi \in \mathbb{R}^* \setminus N$, where N is a set of zero Lebesgue measures on $\mathbb{R}^*$,*

$$F_\xi(u_2,\ldots,u_d) = sgn(\xi)\frac{\partial}{\partial \xi} V_F\left((\xi \wedge 0, \xi \vee 0] \times (-\infty, x_2], \ldots, (-\infty, x_d]\right) \quad (13)$$

is the conditional distribution function of $(u_2,\ldots,u_d)$ given that $u_1 = \xi$ in every point $(u_2,\ldots,u_d)$. F_ξ is continuous.

Remark 2 Let $\Delta X_{t^*} := X_{t^*} - X_{t^*-}$ and $\Delta Y_{t^*} := Y_{t^*} - Y_{t^*-}$ denote jump sizes of the first and second components of a two-dimensional Lévy process $(X_t, Y_t)_{t\geq 0}$ at time t^* with marginal tail integrals U_1 and U_2. A direct consequence of Theorem 4 is that the conditional distribution function of $U_2(\Delta Y_{t^*})$ on the positive domain, given $U_1(\Delta X_{t^*})$, is

$$P\left(0 < U_2(\Delta Y_{t^*}) \leq u | U_1(\Delta X_{t^*}) = x\right) = \frac{\partial}{\partial x} F(x,u).$$

Proof of Theorem 1 We provide a proof only for the expression of λ_{PP} because the proof for the other expressions is analogous. Let $f_{U_1}(x) = \exp(-x)$ be the probability density of $U_1(\Delta X_{t^*})$, i.e., the tail integral value of the largest jump in component 1, which is given by Theorem 3.

Integration yields

$$\begin{aligned}
&P\left(0 < U_2(\Delta Y_{t^*}) < u | 0 < U_1(\Delta X_{t^*}) < u\right)\\
&\quad = \frac{\int_0^u P\left(0 < U_2(\Delta Y_{t^*}) \leq u | U_1(\Delta X_{t^*}) = x\right) f_{U_1}(x)dx}{\int_0^u f_{U_1}(x)dx}\\
&\quad = \frac{\int_0^u \frac{\partial}{\partial x} F(x,u)\exp(-x)dx}{1-\exp(-u)}\\
&\quad = \frac{F(u,u)}{u}\frac{u}{\exp(u)-1} + \frac{\int_0^u F(x,u)\exp(-x)dx}{1-\exp(-u)},
\end{aligned}$$

where $F(u, v)$ is the Lévy copula of the process $(X_t, Y_t)_{t\geq 0}$. The conditional probability in the second line results from Theorem 4 (see Remark 2). Because $F(x, u)$ is bounded by 0 and u, we find

$$\left| \frac{\int_0^u F(x,u)\exp(-x)dx}{1-\exp(-u)} \right| \leq |u| \,,$$

and the expression for λ_{PP} as given by Theorem 1 follows from taking the limit

$$\begin{aligned}\lambda_{PP} &= \lim_{u\to 0^+} \left(\frac{F(u,u)}{u} \frac{u}{\exp(u)-1} + O(u) \right) \\ &= \lim_{u\to 0^+} \frac{F(u,u)}{u}.\end{aligned}$$

$\square$

Appendix C: Proof of Theorem 2

The proof requires a relationship between the Lèvy copula F of a d-dimensional Lévy process $(X_t, Y_t)_{t\geq 0}$ and the probabilistic copula C_t of the d-dimensional distribution of $X_t - X_0$, which is summarized in the following theorem and proved in Kallsen and Tankov (2006):

Theorem 5 (Lévy Copula and Copula) *Let X be an $\mathbb{R}^d$-valued Lévy process with marginal tail integrals $U_1, \ldots, U_d$, and denote its Lévy copula by F. The probabilistic copula of $\left(-\alpha_1 X_t^1, \ldots, -\alpha_d X_t^d\right)$ (or, equivalently, a survival copula of $\left(\alpha_1 X_t^1, \ldots, \alpha_d X_t^d\right))$ for $t \in (0,\infty)$, $\alpha_i \in \{-1, 1\}$, is denoted by $C_t^{(\alpha_1,\ldots,\alpha_d)}$: $[0,1]^d \to [0,1]$. Then,*

$$F(u_1,\ldots,u_d) = \lim_{t\to 0} \frac{1}{t} C_t^{(\operatorname{sgn} u_1,\ldots,\operatorname{sgn} u_d)} \left(t|u_1|,\ldots,t|u_d|\right) \prod_{i=1}^d \operatorname{sgn} u_i \tag{14}$$

for any $(u_1,\ldots,u_d) \in \prod_{i=1}^d \overline{RanU_i}$.

Proof of Theorem 2 First, we show that $\widehat{\lambda}_{PP,n}$ is a strongly consistent estimator for λ_{PP}, and second, we show the asymptotic normality.

(i) Strong consistency. From Theorem 5, together with the definitions of jump tail dependence, follows:

$$\begin{aligned}\lambda_{PP} = \lim_{u\to 0}\left[\frac{F(u,u)}{u}\right] &= \lim_{u\to 0}\left[\frac{1}{u}\lim_{t\to 0}\frac{1}{t}\overline{C_t}(tu,tu)\right] \\ &= \lim_{t\to 0}\frac{1}{t}\lim_{u\to 0}\frac{\overline{C_t}(tu,tu)}{u} \\ &= \lim_{t\to 0}\frac{1}{t}\Lambda_U^{C_t}(t,t) = \lim_{t\to 0}\Lambda_U^{C_t}(1,1),\end{aligned} \tag{15}$$

where $\overline{C_t} = C_t^{(1,1)}$ and $C_t = C_t^{(-1,-1)}$ are the survival copula and the copula of the marginal distribution of $(X_t, Y_t) - (X_0, Y_0)$ at time t, respectively. The function

$$\Lambda_U^{C_t}(x, y) := \lim_{u \to 0} \frac{\overline{C_t}(xu, yu)}{u}$$

is the upper tail copula of a distribution with copula C_t (see, e.g., Schmidt and Stadtmüller 2006 for the definition and discussion of tail copulas). If F is such that λ_{PP} exists, the limits are interchangeable because both inner limits exist and the convergence for $u \to 0$ in the last line is locally uniform (see Schmidt and Stadtmüller 2006). In the final step, we use the homogeneity of the tail copula.

Let $\left(X_{\Delta t}^{(j)}, Y_{\Delta t}^{(j)}\right)$ and the ranks $R_{n,X}^{(j)}$ and $R_{n,Y}^{(j)}$ for $j = 1 \ldots n$ be as in Theorem 2. Let $C_{\Delta t}$ be the probabilistic copula of $\left(X_{\Delta t}^{(j)}, Y_{\Delta t}^{(j)}\right)$. Since Lévy processes have independent increments, $C_{\Delta t}$ is the same for all j of $\left(X_{\Delta t}^{(j)}, Y_{\Delta t}^{(j)}\right)$, and it holds $C_t = C_{\Delta t}$ if we set $t = t - 0 = \Delta t$. Based on the empirical copula representation of Genest et al. (1995), Schmidt and Stadtmüller (2006) show that for small values of k/n

$$\widehat{\Lambda}_{U,n}^{C_t}(x, y) = \frac{1}{k} \sum_{j=1}^{n} 1_{\left\{R_{n,X}^{(j)} > n - kx\right\}} 1_{\left\{R_{n,Y}^{(j)} > n - ky\right\}} \tag{16}$$

is a consistent estimator for the upper tail copula $\Lambda_U^{C_t}$ of (X_t, Y_t). Because for fixed t, this estimator equals the estimator of jump tail dependence from Theorem 1, strong consistency of $\widehat{\lambda}_{PP,n}$ follows directly from strong consistency of the estimator Eq. 16 together with Eq. 15.

(ii) Asymptotics. We now prove the second part of the theorem, the asymptotics. Based on the weak convergence of empirical copula processes (see Fermanian 2004), for $\widehat{\Lambda}_{U,n}^{C_t}(1, 1)$, Schmidt and Stadtmüller (2006) derive the asymptotics (for $k = k(n) \to \infty$ with $k/n \to 0$ as $n \to \infty$)

$$\sqrt{k}\left(\widehat{\Lambda}_{U,n}^{C_t}(1, 1) - \Lambda_U^{C_t}(1, 1)\right) \xrightarrow{d} Z \sim \mathbb{N}\left(0, \sigma_U^2\right),$$

where

$$\begin{aligned}
\sigma_U^2 &= \Lambda_U^{C_t}(1, 1) \\
&\quad + \left(\frac{\partial}{\partial x}\Lambda_U^{C_t}(1, 1)\right)^2 + \left(\frac{\partial}{\partial y}\Lambda_U^{C_t}(1, 1)\right)^2 \\
&\quad + 2\Lambda_U^{C_t}(1, 1)\left(\left(\frac{\partial}{\partial x}\Lambda_U^{C_t}(1, 1) - 1\right)\left(\frac{\partial}{\partial y}\Lambda_U^{C_t}(1, 1) - 1\right) - 1\right).
\end{aligned} \tag{17}$$

We focus on the case $\Delta t \to 0$. For fixed $t = \Delta t$, the estimator above matches the estimator of λ_{PP} and thus has the same distribution. Note that because $\Lambda_U^{C_t}(1, 1)$

is homogenous of degree one, the respective derivative is of degree zero, i.e., $\frac{\partial}{\partial x}\Lambda_U^{C_t}(1,1) = \frac{\partial}{\partial x}\Lambda_U^{C_t}(\epsilon,\epsilon)$, for $\epsilon > 0$. For $t \to 0$, we then get

$$\begin{aligned}
\lim_{t\to 0}\frac{\partial}{\partial x}\Lambda_U^{C_t}(\epsilon,\epsilon) &= \lim_{t\to 0}\lim_{h\to 0}\frac{\Lambda_U^{C_t}(\epsilon+h,\epsilon)-\Lambda_U^{C_t}(\epsilon,\epsilon)}{h}\\
&= \lim_{t\to 0}\lim_{h\to 0}\lim_{m\to 0}\frac{\overline{C}_t(m(\epsilon+h),m\epsilon)-\overline{C}_t(m\epsilon,m\epsilon)}{mh}\\
&= \lim_{h\to 0}\frac{1}{h}\lim_{t\to 0}\lim_{m\to 0}\left(\frac{\overline{C}_t(m(\epsilon+h),m\epsilon)}{m}-\frac{\overline{C}_t(m\epsilon,m\epsilon)}{m}\right)\\
&= \lim_{h\to 0}\frac{F(\epsilon+h,\epsilon)-F(\epsilon,\epsilon)}{h}\\
&= \frac{\partial}{\partial x}F(\epsilon,\epsilon) = F_X(\epsilon,\epsilon),
\end{aligned} \tag{18}$$

where we used the definition of the upper tail copula and Theorem 5. Again, the limits are interchangeable because all of the limits exist and the limit in $t \to 0$ is locally uniform. Inserting the above results into Eq. 17 yields the asymptotics of Theorem 2. □

References

Almendral, A., Oosterlee, C.W.: On american options under the variance gamma process. Appl. Math. Finance **14**(2), 131–152 (2007)

Barndorff-Nielsen, O.E., Shephard, N.: Financial Volatility in Continuous Time. Cambridge University Press, Cambridge (2013, forthcoming)

Basawa, I.V., Brockwell, P.J.: Non-parametric estimation for non-decreasing levy processes. J. R. Stat. Soc. Ser. B (Method.) **44**(2), 262–269 (1982)

Cont, R., Tankov, P.: Financial Modelling with Jump Processes. Chapman and Hall, Boca Raton, FL (2004)

Embrechts, P., Lindskog, F., McNeil, A.: Handbook of heavy tailed distibutions in finance, chap. Modelling Dependence with Copulas and Applications to Risk Management, pp. 329–384. Elsevier (2003)

Esmaeili, H., Klüppelberg, C.: Parameter estimation of a bivariate compound poisson process. Insur. Math. Econ. **2**(47), 224–233 (2010)

Esmaeili, H., Klüppelberg, C.: Parametric estimation of a bivariate stable lévy process. J. Multivar. Anal. **102**(5), 918–930 (2011)

Esmaeili, H., Klüppelberg, C.: Two-step estimation of a multivariate Lévy process. (2012, preprint)

Fermanian, J.D., Radulovic, D., Wegkamp, M.: Weak convergence of empirical copula processes. Bernoulli **10**(5), 847–860 (2004)

Figueroa-López, J.E., Houdré, C.: Risk bounds for the non-parametric estimation of lévy processes. IMS Lect. Notes **51**, 96–116 (2006)

Genest, C., Ghoudi, K., Rivest, L.P.: A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. Biometrika **82**(3), 543–552 (1995)

Joe, H.: Multivariate Models and Dependence Concepts. Chapman & Hall, London (1997)

Kallenberg, O.: Foundation of Modern Probability. Springer (2002)

Kallsen, J., Tankov, P.: Characterization of dependence of multidimensional Lévy processes using Lévy copulas. J. Multivar. Anal. **97**, 1551–1572 (2006)

Madan, D.B., Carr, P., Chang, E.: The variance gamma process and option pricing. Eur. Financ. Rev. **2**, 79–105 (1998)

Madan, D.B., Seneta, E.: The variance gamma (v.g.) model for share market returns. J. Bus. **63**(4), 511–524 (1990)

Malevergne, Y., Sornette, D.: How to account for extreme co-movements between individual stocks and the market. J. Risk **6**(3), 71–116 (2004)

Markowitz, H.: Portfolio selection. J. Financ. **7**(1), 77–91 (1952)

Nelsen, R.: An Introduction to Copulas, 2nd edn. Springer, New York (2006)

Nishiyama, Y.: Nonparametric estimation and testing time-homogeneity for processes with independent increments. Stoch. Process. their Appl. **118**(6), 1043–1055 (2008)

Poon, S.H., Rockinger, M., Tawn, J.: Extreme value dependence in financial markets: diagnostics, models, and financial implications. Rev. Financ. Stud. **17**, 581–610 (2004)

Rosinski, J.: Lévy Processes – Theory and Applications, chap. Series Representation of Lévy Processes from the Perspective of Point Processes. Birkhäuser, Boston (2001)

Rüschendorf, L., Woerner, J.H.C.: Expansion of transition distributions of Lévy processes in small time. Bernoulli **8**, 81–96 (2002)

Schmid, F., Schmidt, R.: Multivariate conditional versions of spearman's rho and related measures of tail dependence. J. Multivar. Anal. **98**(6), 1123–1140 (2007)

Schmidt, R., Stadtmüller, U.: Nonparametric estimation of tail dependence. Scand. J. Statist. **33**, 307–355 (2006)

Sibuya, M.: Bivariate extreme statistics. Ann. Inst. Statist. Math. **11**, 195–210 (1960)