Question 1

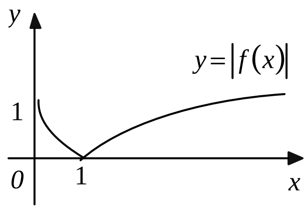
a. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
i • shows correct shape and intercepts on coordinate axes	1
ii • shows correct shape and intercepts on coordinate axes	1
iii • shows y intercept and vertical asymptote $x=1$	1
• shows correct shape of curve with horizontal asymptote $y=0$ as $x\to+\infty$	1
iv • shows correct shape and intercepts on coordinate axes	1
• shows horizontal asymptote $y=\frac{\pi}{2}$ as $x\to+\infty$	1

Answer

i



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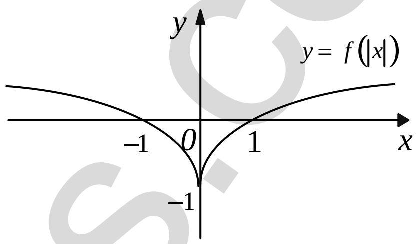

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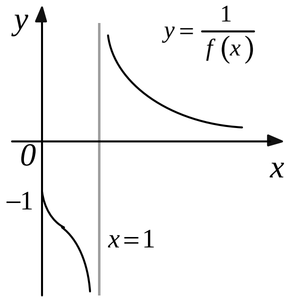

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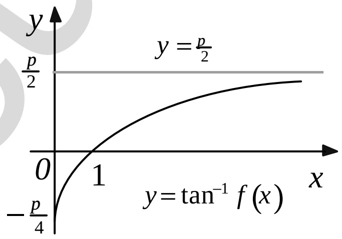


b. Outcomes assessed : E6, E8, HE4

Marking Guidelines

Criteria	Marks
i • writes two expressions for gradient OP (using coordinates of O and P; using calculus)	1
• solves resulting equation to obtain result	1
ii • uses intersection of line through O with curve to deduce $0<k<4e^{-2}$	1
iii • obtains indefinite integral using integration by parts	1
• expresses area as difference between $2e^2$ and definite integral between x values 1 and e^2	1
• evaluates definite integral in exact form	1
• gives exact area in simplest form	1
iv • finds equation of inverse function	1
v • uses reflection in $y=x$ to deduce equation of tangent	1

Answer

i. $gradient\ OP = \dfrac{y_1}{x_1} = \dfrac{(\ln x_1)^2}{x_1}$

$y = (\ln x)^2 \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{2\ln x}{x}$

$\therefore gradient\ OP = \dfrac{2\ln x_1}{x_1}$

Hence at P, $\dfrac{(\ln x_1)^2}{x_1} = \dfrac{2\ln x_1}{x_1}$

$\ln x_1(\ln x_1 - 2) = 0$

$x_1 \neq 1 \quad \Rightarrow \quad \ln x_1 = 2$

$\therefore\ x_1 = e^2$ and $y_1 = 2^2$

$\therefore (e^2, 4)$ are the coordinates of P.

ii. $f(x) = kx$ has two distinct real roots if the line $y = kx$ cuts the curve in two points, that is for $0 < k < gradient\ OP$. Hence $0 < k < 4e^{-2}$.

iii.

$$\begin{aligned}\int 1.(\ln x)^2\ dx &= x(\ln x)^2 - \int x \,.\, 2\ln x \frac{1}{x}\, dx\\ &= x(\ln x)^2 - 2\int 1 \,.\, \ln x\ dx\\ &= x(\ln x)^2 - 2\left\{x\ln x - \int x \,.\, \frac{1}{x} dx\right\}\\ &= x(\ln x)^2 - 2\left\{x\ln x - \int 1\ dx\right\}\\ &= x(\ln x)^2 - 2x\ln x + 2x + c\end{aligned}$$

Required area A is given by

$$\begin{aligned}A &= \frac{1}{2} \,.\, e^2 \,.\, 4 - \int_1^{e^2} (\ln x)^2\, dx\\ &= 2e^2 - \Big[x(\ln x)^2 - 2x\ln x + 2x\Big]_1^{e^2}\\ &= 2e^2 - \left\{(4e^2 - 0) - 2(2e^2 - 0) + 2(e^2 - 1)\right\}\\ &= 2\end{aligned}$$

Area is 2 sq. units

iv. $y = (\ln x)^2, \quad x \geq 1$

$\sqrt{y} = \ln x$

$e^{\sqrt{y}} = x$

Interchanging x and y,

$f^{-1}(x) = e^{\sqrt{x}}$

v. The required tangent is the reflection of OP in the line $y = x$. It passes through $(4, e^2)$ and has equation $y = \frac{1}{4}e^2\, x$.

Question 2

a. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• finds the primitive function	1
• evaluates, giving exact answer	1

Answer

$$\int_0^4 \frac{1}{\sqrt{x^2+9}}\, dx = \left[\ln\left(x + \sqrt{x^2+9}\right)\right]_0^4 = \ln 9 - \ln 3 = \ln 3$$

b. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• finds the primitive function • evaluates, giving answer to required accuracy	1 1

Answer

$$\int_0^1 e^x \cos\left(e^x\right)\ dx = \left[\sin\left(e^x\right)\right]_0^1 = \sin e - \sin 1 \approx -0 \cdot 4307 \quad (\textit{to 4 significant figures})$$

c. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• expresses integrand in partial fraction form • finds primitive of one fraction as log function • finds primitive of other fraction as inverse tan function • evaluates by substitution	1 1 1 1

Answer

$$\frac{x(x-16)}{(4x+1)(x^2+4)} \equiv \frac{a}{(4x+1)} + \frac{bx+c}{(x^2+4)}$$

$$x(x-16) \equiv a(x^2+4) + (bx+c)(4x+1)$$

sub. $x = -\frac{1}{4}$: $\quad \frac{65}{16} = \frac{65}{16}a \quad \therefore a = 1$

equating constant terms : $4a + c = 0 \quad \therefore c = -4$

equating coeff of x^2 : $\quad a + 4b = 1 \quad \therefore b = 0$

$$\int_0^2 \frac{x(x-16)}{(4x+1)(x^2+4)}\, dx$$

$$= \int_0^2 \frac{1}{4x+1} - \frac{4}{x^2+4}\, dx$$

$$= \left[\tfrac{1}{4}\ln(4x+1) - 2\tan^{-1}\tfrac{x}{2}\right]_0^2$$

$$= \tfrac{1}{4}(\ln 9 - \ln 1) - 2\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= \tfrac{1}{2}(\ln 3 - p)$$

d. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• writes dx in terms of du and converts x limits to u limits • writes integrand in terms of u • finds primitive • evaluates	1 1 1 1

Answer

$$u = \tan\tfrac{x}{2}$$

$$du = \tfrac{1}{2}\sec^2\tfrac{x}{2}\, dx$$

$$2\,du = \left(1+u^2\right)dx$$

$$dx = \frac{2}{1+u^2}\, du$$

$x = 0 \Rightarrow u = 0$

$x = \frac{p}{2} \Rightarrow u = 1$

$$3\cos x - 4\sin x + 5$$

$$= \frac{3\left(1-u^2\right) + 4(2u) + 5\left(1+u^2\right)}{1+u^2}$$

$$3\cos x-4\sin x+5=\frac{2\left(u^2+4u+4\right)}{1+u^2}=\frac{2\left(u+2\right)^2}{1+u^2}$$

$$\begin{aligned}\int_0^{\frac{p}{2}}\frac{1}{3\cos x-4\sin x+5}\,dx&=\int_0^1\frac{1+u^2}{2\left(u+2\right)^2}\cdot\frac{2}{1+u^2}\,du\\&=\int_0^1\left(u+2\right)^{-2}\,du\\&=-\left[\left(u+2\right)^{-1}\right]_0^1\\&=-\left(\tfrac{1}{3}-\tfrac{1}{2}\right)\\&=\tfrac{1}{6}\end{aligned}$$

e. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• performs substitution in integral between x limits $-a$ and 0	1
• uses property of odd function to write integrand in terms of $f\left(u\right)$	1
• uses property of definite integral to replace variable of integration by x and deduce result.	1

Answer

$u=-x$

$du=-dx$

$x=-a\ \Rightarrow\ u=a$

$x=a\ \Rightarrow\ u=-a$

Function f is odd, hence

$f\left(x\right)=f\left(-u\right)=-f\left(u\right)$

$$\begin{aligned}\int_{-a}^{a}f\left(x\right)\,dx&=\int_{a}^{-a}-f\left(u\right)\cdot -du\\&=-\int_{-a}^{a}f\left(u\right)\,du\\&=-\int_{-a}^{a}f\left(x\right)\,dx\end{aligned}$$

$$\therefore\ 2\int_{-a}^{a}f\left(x\right)\,dx=0$$

$$\therefore\ \int_{-a}^{a}f\left(x\right)\,dx=0$$

Question 3

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
• realizes denominators	1
• equates real parts to find b	1
• equates imaginary parts to find a	1

Answer

$$\frac{a}{i}+\frac{b}{1+i}=-ai+\frac{b\left(1-i\right)}{2}$$

$$\therefore\quad 1=\tfrac{1}{2}b+\left(-a-\tfrac{1}{2}b\right)i$$

Equating real and imaginary parts, $b=2,\ a=-1.$

b. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes z in modulus/argument form	1
• uses de Moivre's theorem to write z^9 in modulus/argument form then deduces result	1
ii • writes expression in terms of z and $\bar{z}$	1
• evaluates expression	1

Answer

i.

$$z=\sqrt{2}\left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}\,i\right)$$

$$z=\sqrt{2}\left(\cos\tfrac{\pi}{4}+i\sin\tfrac{\pi}{4}\right)$$

$$z^9=\left(\sqrt{2}\right)^9\left(\cos\tfrac{9\pi}{4}+i\sin\tfrac{9\pi}{4}\right)$$

$$=16\sqrt{2}\left\{\cos\left(2\pi+\tfrac{\pi}{4}\right)+i\sin\left(2\pi+\tfrac{\pi}{4}\right)\right\}$$

$$=16\left\{\sqrt{2}\left(\cos\tfrac{\pi}{4}+i\sin\tfrac{\pi}{4}\right)\right\}$$

$$=16z$$

ii.

$$(1+i)^9+(1-i)^9=z^9+\bar{z}^{\,9}$$

$$=z^9+\overline{z^9}$$

$$=16\left(z+\bar{z}\right)$$

$$=16\left(2\operatorname{Re} z\right)$$

$$=32$$

c. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • relates differences in complex numbers to vectors representing sides of $ABCD$	1
• applies an appropriate test to deduce $ABCD$ is a parallelogram	1
ii • uses properties of a square to deduce that $\overrightarrow{BC}$ is a rotation of $\overrightarrow{AB}$ by $\frac{\pi}{2}$ anticlockwise	1
• writes this relation in terms of differences in complex numbers then rearranges	1

Answer

i. $a+g=b+d$

$a-b=d-g$

$\therefore\ \overrightarrow{BA}=\overrightarrow{CD}$

$\therefore\ ABCD$ is a parallelogram

(*one pair of opp. sides both equal and parallel*)

ii. If $ABCD$ is a square with vertices in anticlockwise order, $AB=BC$ and $\angle ABC=\frac{\pi}{2}$.

Hence $\overrightarrow{BC}$ is rotation of $\overrightarrow{AB}$ by $\frac{\pi}{2}$ anticlockwise.

$\therefore\ g-b=i\,(b-a)$

$\therefore\ g+ia=b+ib$

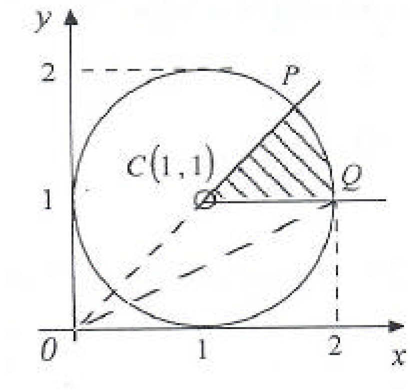
d. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • shades a region lying inside the circle of radius 1 centred at $(1,1)$	1
• shades the appropriate sector of this circle, excluding the centre of the circle.	1
ii • states the possible values of the modulus of z	1
• states the possible values of the argument of z	1

Answer

i.



ii. $OC < |z| \le OP$

$\therefore \sqrt{2} < |z| \le 1 + \sqrt{2}$

$\arg z$ takes its max and min values at P and Q respectively

$\therefore \tan^{-1}\tfrac{1}{2} \le \arg z \le \tfrac{\pi}{4}$

Question 4

a. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	Marks
i • finds the gradient of the tangent in terms of θ by differentiation	1
• uses the gradient to find the equation of the tangent	1
ii • solves simultaneously equations of tangent and asymptote to find coordinates of Q	1
• solves simultaneously equations of tangent and asymptote to find coordinates of R	1
iii • shows coordinates of midpoint of QR are same as coordinates of P	1
iv • finds expression for OQ (or its square)	1
• finds expression for OR (or its square)	1
• simplifies product of OQ and OR to show required result	1

Answer

i. $x = a\sec\theta \qquad y = b\tan\theta$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta \qquad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

Tangent at P has gradient $\dfrac{b\sec\theta}{a\tan\theta}$

and equation

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}\left(x - a\sec\theta\right)$$

$$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$$

$$ab\left(\sec^2\theta - \tan^2\theta\right) = bx\sec\theta - ay\tan\theta$$

$$bx\sec\theta - ay\tan\theta = ab$$

ii. At Q on the tangent, $ay = bx$

$$\therefore \quad bx\left(\sec\theta - \tan\theta\right) = ab$$

$$bx\left(\sec^2\theta - \tan^2\theta\right) = ab\left(\sec\theta + \tan\theta\right)$$

$$x = a\left(\sec\theta + \tan\theta\right)$$

$$y = b\left(\sec\theta + \tan\theta\right)$$

At R on the tangent, $ay = -bx$

$$\therefore \quad bx\left(\sec\theta + \tan\theta\right) = ab$$

$$bx\left(\sec^2\theta - \tan^2\theta\right) = ab\left(\sec\theta - \tan\theta\right)$$

$$x = a\left(\sec\theta - \tan\theta\right)$$

$$y = -b\left(\sec\theta - \tan\theta\right)$$

iii. At midpoint of QR, $x = \tfrac{1}{2}\left\{a\left(\sec\theta + \tan\theta\right) + a\left(\sec\theta - \tan\theta\right)\right\} = a\sec\theta$

$$y = \tfrac{1}{2}\left\{b\left(\sec\theta + \tan\theta\right) - b\left(\sec\theta - \tan\theta\right)\right\} = b\tan\theta$$

Hence P is the midpoint of QR.

iv. $b^2 = a^2\left(e^2 - 1\right) \Rightarrow a^2 + b^2 = a^2 e^2$. Hence

$$OQ^2 = \left(a^2 + b^2\right)\left(\sec q + \tan q\right)^2 \quad \text{and} \quad OR^2 = \left(a^2 + b^2\right)\left(\sec q - \tan q\right)^2$$

$$OQ = ae\left(\sec q + \tan q\right) \qquad OR = ae\left(\sec q - \tan q\right)$$

$$\therefore OQ \times OR = \left(ae\right)^2\left(\sec^2 q - \tan^2 q\right) = \left(ae\right)^2 = OS^2$$

b. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	Marks
i • finds the gradient of the chord PQ	1
• uses the gradient to find the equation of the chord PQ	1
ii • uses the formula for distance from the origin to a line to obtain required result	1
iii • writes expressions for x and y coordinates of M in terms of p and q	1
• uses the relationship between p and q to obtain Cartesian equation of locus of M	1
• states the domain	1
• states the range	1

Answer

i. Chord PQ has gradient

$$\frac{\frac{1}{p} - \frac{1}{q}}{p - q} = \frac{q - p}{pq\left(p - q\right)} = \frac{-1}{pq}$$

and equation $\quad y - \dfrac{1}{p} = \dfrac{-1}{pq}\left(x - p\right)$

$$pqy - q = -x + p$$

$$x + pqy - \left(p + q\right) = 0$$

ii. $$\left|\frac{-\left(p + q\right)}{\sqrt{1^2 + \left(pq\right)^2}}\right| = \sqrt{2}$$

$$\therefore \left(p + q\right)^2 = 2\left(1 + p^2 q^2\right)$$

iii. At M,

$$x = \tfrac{1}{2}\left(p + q\right) \quad \text{and} \quad y = \tfrac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right) = \frac{\frac{1}{2}\left(p + q\right)}{pq}$$

$$\therefore \frac{x^2}{y^2} = p^2 q^2 = \tfrac{1}{2}\left(p + q\right)^2 - 1 = 2x^2 - 1$$

$$\therefore y^2 = \frac{x^2}{2x^2 - 1}$$

This relation has domain $\left\{x : \left|x\right| > \frac{1}{\sqrt{2}}\right\}$.

Rearrangement gives $x^2 + y^2 = 2x^2 y^2$, which is symmetric in x and y.

Hence the relation has range $\left\{y : \left|y\right| > \frac{1}{\sqrt{2}}\right\}$.

Question 5

a. Outcomes assessed : PE2, PE3

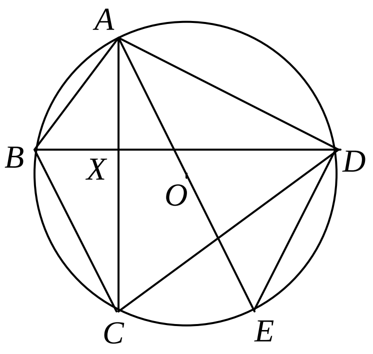
Marking Guidelines

Criteria	Marks
ii • uses circle property to explain why $\angle AXB = \angle ADE$	1
• uses circle property to explain why $\angle ABX = \angle AED$	1
• deduces required similarity and notes that $\angle BAC = \angle EAD$	1
• uses circle property to deduce $BC = ED$	1
iii • applies Pythagoras' theorem in triangle ADE	1
• applies Pythagoras' theorem in triangle AXD	1
• applies Pythagoras' theorem in triangle BXC	1
	1

• uses facts that $BC = AD$ and AE is a diameter to deduce required result	

Answer

i.



ii. In ΔABX , ΔAED

$\angle AXB = 90°$ (*given*)

and $\angle ADE = 90°$ ($\angle$ *in a semicircle is* $90°$)

$\therefore \angle AXB = \angle ADE$

Also $\angle ABD = \angle AED$ ($\angle$'*s subtended at the circumference by the same arc AD are equal*)

$\therefore \angle ABX = \angle AED$ (*B, X, D collinear*)

$\angle BAX = \angle EAD$ (*remaining* $\angle$'*s equal since* $\angle$ *sum of each* Δ *is* $180°$)

$\therefore \Delta ABX \; ||| \; \Delta AED$ (*equiangular*)

Also $\angle BAC = \angle EAD$ (*A, X, C collinear)*

$\therefore BC = ED$ (*chords subtending equal* $\angle$'*s at the circumference are equal*)

iii. $AD^2 + ED^2 = AE^2$ (*Pythagoras' theorem in* ΔADE)

$\therefore AD^2 + BC^2 = AE^2$ ($BC = ED$ *proved above*)

But $AD^2 = AX^2 + DX^2$ (*Pythagoras' theorem in* ΔAXD)

$BC^2 = BX^2 + CX^2$ (*Pythagoras' theorem in* ΔBXC)

and $AE^2 = d^2$ (*AE is a diameter*)

$\therefore AX^2 + BX^2 + CX^2 + DX^2 = d^2$

b. Outcomes assessed : H5, PE3

Marking Guidelines

Criteria	Marks
i • uses the sine rule in each of the designated triangles	1
• uses the fact that supplementary angles have the same value of sine	1
• deduces the relationship between $\sin q$, $\sin 2q$ and x by using $AB = AC$	1
• uses the double angle identity to obtain the required result	1
ii • shows that $\cos q$ lies between $\frac{1}{2}$ and 1	1
• obtains two simultaneous inequalities for x	1
• solves to obtain required result	1

Answer

i. $\dfrac{\sin q}{x} = \dfrac{\sin \angle ADB}{AB}$ (*sine rule in* ΔADB)

$\dfrac{\sin 2q}{1-x} = \dfrac{\sin \angle ADC}{AC}$ (*sine rule in* ΔADC)

But $\sin \angle ADC = \sin\left(180° - \angle ADB\right) = \sin \angle ADB$

and $AB = AC$.

$\therefore \dfrac{\sin q}{x} = \dfrac{\sin 2q}{1-x}$, $\sin q \neq 0$

$\dfrac{1-x}{x} = \dfrac{2\sin q \cos q}{\sin q}$

$\therefore \cos q = \dfrac{1-x}{2x}$

ii. $0° < 3q < 180°$

$0° < q < 60°$

$1 > \cos q > \frac{1}{2}$

$1 > \dfrac{1-x}{2x} > \frac{1}{2}$, $x > 0$

$\therefore 2x > 1 - x > x$

$3x > 1 > 2x$

$3x > 1$ *and* $2x < 1$

$x > \frac{1}{3}$ *and* $x < \frac{1}{2}$

$\therefore \frac{1}{3} < x < \frac{1}{2}$

Question 6

a. Outcomes assessed : H5, HE2, E9

Marking Guidelines

Criteria	Marks
i • defines the sequence of statements and shows the first two are true	1
• uses the given recurrence relation to write S_{k+1} in terms of S_k , S_{k-1}	1
• writes S_k , S_{k-1} in terms of powers of 4 and 2, conditional on truth of statements for $n \le k$	1
• rearranges resulting expression for S_{k+1} into form $4^{k+1} + 2^{k+1}$	1
• deduces the required result invoking the process of Mathematical Induction	1
ii • states $T_1 = 6$	1
• writes expression for T_n in terms of S_n , S_{n-1} for $n \ge 2$	1
• substitutes for S_n , S_{n-1} and simplifies resulting expression	1

Answer

i. Let $U(n)$, $n = 1, 2, 3, \ldots$ be the sequence of statements $S_n = 4^n + 2^n$, $n = 1, 2, 3, \ldots$

Consider $U(n)$, $n \le 2$: $S_1 = 6 = 4^1 + 2^1$ and $S_2 = 20 = 4^2 + 2^2$. $\therefore U(n)$ is true for $n \le 2$.

If $U(n)$ is true for $n \le k$: $S_n = 4^n + 2^n$, $n = 1, 2, 3, \ldots, k$ **

Consider $U(k+1)$ where $k \ge 2$:
$$\begin{aligned} S_{k+1} &= 6S_k - 8S_{k-1} \\ &= 6\left(4^k + 2^k\right) - 8\left(4^{k-1} + 2^{k-1}\right) \quad \textit{if } U(n) \textit{ true for } n \le k\textit{, using **} \\ &= 6\left(4^k + 2^k\right) - 2\times 4^k - 4\times 2^k \\ &= 4\times 4^k + 2\times 2^k \\ &= 4^{k+1} + 2^{k+1} \end{aligned}$$

Hence for $k \ge 2$, if $U(n)$ is true for $n \le k$ then $U(k+1)$ is true. But $U(n)$ is true for $n \le 2$, hence $U(3)$ is true and then $U(4)$ is true and so on. Hence, by Mathematical Induction, $U(n)$ is true for all positive integers *n*. $\therefore S_n = 4^n + 2^n$, $n = 1, 2, 3, \ldots$.

ii. $T_1 = S_1 = 6$ and for $n \ge 2$,
$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= \left(4^n + 2^n\right) - \left(4^{n-1} + 2^{n-1}\right) \\ &= 3\times 4^{n-1} + 2^{n-1} \end{aligned}$$

Hence $T_1 = 6$ and $T_n = 3\times 4^{n-1} + 2^{n-1}$, $n = 2, 3, 4, \ldots$

b. Outcomes assessed : E1, E7

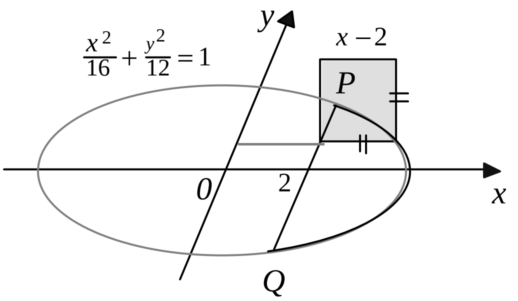
Marking Guidelines

Criteria	Marks
i • states the area of cross section and volume of a typical slice	1
• writes the volume of the solid as a limiting sum of slice volumes	1
• writes this limiting sum as an integral, explaining the values of the *y* limits.	1
ii • expands the integrand, writing x^2 and x in terms of y	1
• evaluates definite integral involving powers of y	1
• evaluates definite integral involving square root function	1
	1

• gives exact value of volume by combining these results	

Answer

i.



Area of cross section is $(x-2)^2$

Hence volume of slice is $dV=(x-2)^2 dy$

Also when $x=2,\ y=\pm 3$

$\therefore$ Volume of solid is given by

$$V=\lim_{d\ y\to 0}\sum_{y=-3}^{y=3}(x-2)^2 dy$$

$$=\int_{-3}^{3}(x-2)^2 dy$$

ii. $V=2\int_0^3 (x-2)^2\ dy=2\left\{\int_0^3 (x^2+4)\ dy-\int_0^3 4x\ dy\right\}$

$$\int_0^3 (x^2+4)\ dy=\int_0^3 \left(20-\tfrac{4}{3}y^2\right)\ dy$$

$$=\left[20y-\tfrac{4}{9}y^3\right]_0^3$$

$$=60-12$$

$$=48$$

$$\int_0^3 4x\ dy=\frac{8}{\sqrt{3}}\int_0^3\sqrt{12-y^2}\ dy$$

Using the substitution $y=\sqrt{12}\sin q$, $-\frac{p}{2}<q<\frac{p}{2}$

$y=2\sqrt{3}\sin q$ $\qquad$ $y=0\Rightarrow q=0$

$dy=2\sqrt{3}\cos q\,dq$ $\qquad$ $y=3\Rightarrow q=\frac{p}{3}$

$$\int_0^3 4x\ dy=8\sqrt{3}\int_0^{\frac{p}{3}} 4\cos^2 q\ \ dq$$

$$=8\sqrt{3}\int_0^{\frac{p}{3}}(2+2\cos 2q)\ dq$$

$$=8\sqrt{3}\ \left[2q+\sin 2q\right]_0^{\frac{p}{3}}$$

$$=8\sqrt{3}\ \left\{\tfrac{2p}{3}+\tfrac{\sqrt{3}}{2}\right\}$$

Hence volume is $\quad 2\left\{48-8\sqrt{3}\left(\frac{2p}{3}+\frac{\sqrt{3}}{2}\right)\right\}=72-\frac{32p\sqrt{3}}{3}$ cu. units.

Question 7

a. Outcomes assessed : E5

Marking Guidelines

Criteria	Marks
i • quotes Newton's second law to obtain expression for a	1
ii • expresses t as an integral in terms of v	1
• finds the primitive function (by substitution or otherwise)	1
• finds the constant of integration in terms of V to obtain expression for t in terms of v	1
iii • expresses x as an integral in terms of v	1
• finds the primitive function (by substitution or otherwise)	1
• finds an expression for x in terms of v and V	1
iv • finds the distance travelled in terms of V	1
• finds the time taken in terms of V	1

Answer

i. By Newton's second law, $ma = -\frac{1}{10} m \sqrt{v}\left(1+\sqrt{v}\right)$ $\quad \therefore a = -\frac{1}{10}\sqrt{v}\left(1+\sqrt{v}\right)$

ii. $\dfrac{dv}{dt} = -\frac{1}{10}\sqrt{v}\left(1+\sqrt{v}\right)$

$$\therefore \frac{dt}{dv} = \frac{-10}{\sqrt{v}\left(1+\sqrt{v}\right)}$$

$$t = -10\int \frac{1}{\sqrt{v}\left(1+\sqrt{v}\right)}\, dv$$

$$\therefore t = -20\int \frac{\frac{1}{2}v^{-\frac{1}{2}}}{1+\sqrt{v}}\, dv$$

$$= -20 \ln\left\{\left(1+\sqrt{v}\right)A\right\} \quad A\,const.$$

$$\left.\begin{array}{l} t=0 \\ v=V \end{array}\right\} \Rightarrow A = \frac{1}{1+\sqrt{V}}$$

$$\therefore t = 20\ln\left(\frac{1+\sqrt{V}}{1+\sqrt{v}}\right)$$

iii. $v\dfrac{dv}{dx} = -\frac{1}{10}\sqrt{v}\left(1+\sqrt{v}\right)$

$$\therefore \frac{dv}{dx} = -\frac{1}{10}\,\frac{1+\sqrt{v}}{\sqrt{v}}$$

$$\frac{dx}{dv} = -10\,\frac{\sqrt{v}}{1+\sqrt{v}}$$

$$x = -10\int \frac{\sqrt{v}}{1+\sqrt{v}}\, dv$$

$$\left.\begin{array}{l} v=u^2 \\ dv = 2u\,du \end{array}\right\} \Rightarrow x = -20\int \frac{u^2}{1+u}\, du$$

But $\dfrac{u^2}{1+u} = \dfrac{u^2-1}{1+u} + \dfrac{1}{1+u}$

$$\therefore x = -20\int \left\{u - 1 + \frac{1}{1+u}\right\} du$$

$$= -20\left\{\tfrac{1}{2}u^2 - u + \ln\left(1+u\right)\right\} + c, \quad c\,const.$$

$$\therefore x = -10\left\{v - 2\sqrt{v} + 2\ln\left(1+\sqrt{v}\right)\right\} + c$$

$$\left.\begin{array}{l} x=0 \\ v=V \end{array}\right\} \Rightarrow 0 = -10\left\{V - 2\sqrt{V} + 2\ln\left(1+\sqrt{V}\right)\right\} + c$$

$$x = -10\left\{v - V - 2\left(\sqrt{v}-\sqrt{V}\right) + 2\ln\frac{1+\sqrt{v}}{1+\sqrt{V}}\right\}$$

$$\therefore x = 10\left\{\left(V - v\right) - 2\left(\sqrt{V}-\sqrt{v}\right) + 2\ln\frac{1+\sqrt{V}}{1+\sqrt{v}}\right\}$$

iv.

$$v = 0 \Rightarrow \left\{\begin{array}{l} x = 10\left\{V - 2\sqrt{V} + 2\ln\left(1+\sqrt{V}\right)\right\} \\ t = 20\ln\left(1+\sqrt{V}\right) \end{array}\right.$$

Distance travelled is $10\left\{V - 2\sqrt{V} + 2\ln\left(1+\sqrt{V}\right)\right\}$ m.

Time taken is $20\ln\left(1+\sqrt{V}\right)$ s.

b. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
i • explains why a, b, g satisfy the given equation	1
ii • uses the product of the roots and the sum of products taken 2 at a time to write 2 equations	1
• finds the value of b	1
• finds the value of k	1
iii • writes a cubic equation in $x^{\frac{1}{2}}$ with required roots	1
• rearranges this equation to form a monic cubic equation in x	1

Answer

i. At P, Q, R $\quad xy = 2$ and $y = x(k - x)$ $\qquad \therefore x^2(k - x) = 2$.

Rearranging this equation, x coordinates a, b, g of P, Q, R are the roots of $x^3 - kx^2 + 2 = 0$.

ii. If a, b, g are consecutive terms in an AP, since $a < b < g$, let $a = b - d$, $g = b + d$ where $d > 0$.

Then $\sum ab = 0 \Rightarrow b(b - d) + b(b + d) + (b - d)(b + d) = 0 \qquad \therefore 3b^2 - d^2 = 0 \quad (1)$

$abg = -2 \Rightarrow b(b - d)(b + d) = -2 \qquad \therefore b^3 - bd^2 = -2 \quad (2)$

Substituting for d^2 in (2) gives $-2b^3 = -2$. $\quad \therefore b = 1$

Then $k = a + b + g = 3b = 3$

iii. Consider the equation $\left(x^{\frac{1}{2}}\right)^3 - k\left(x^{\frac{1}{2}}\right)^2 + 2 = 0$. Clearly a^2, b^2, g^2 satisfy this equation.

Rearrangement gives $\qquad x^{\frac{3}{2}} = kx - 2$

Squaring both sides $\qquad x^3 = k^2x^2 - 4kx + 4$

Hence required equation is $\quad x^3 - k^2x^2 + 4kx - 4 = 0$

Question 8

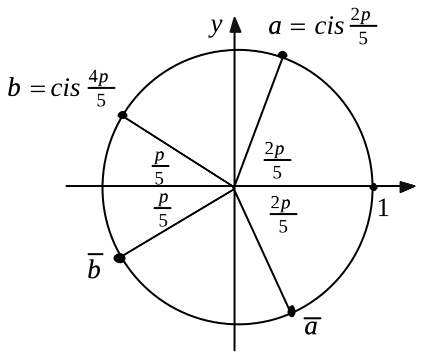
a. Outcomes assessed : E2, E3

Marking Guidelines

Criteria	Marks
i • recognizes that the roots are the complex 5th roots of unity, one of which is 1	1
• writes down the four non-real roots	1
ii • factors $z^5 - 1$ over the complex field	1
• takes products of factors involving complex conjugate roots	1
iii • compares the given factorization with $(z-1)(z^4 + z^3 + z^2 + z + 1)$	1
• substitutes $z = 1$ in resulting identity (with factor $(z-1)$ removed)	1
iv • substitutes $x = \cos\frac{2p}{5}$ in LHS of cubic equation and uses double angle formula for cosine	1
• rearranges and uses result from (iii) to show $x = \cos\frac{2p}{5}$ satisfies the equation	1

Answer

i. The five complex 5th roots of 1 are equally spaced by $\frac{2p}{5}$ around the unit circle in the Argand diagram.



Hence $z^5 - 1 = 0$ has roots

$$1,\quad \cos\tfrac{2p}{5} + i\sin\tfrac{2p}{5},\qquad \cos\tfrac{4p}{5} + i\sin\tfrac{4p}{5}$$
$$\cos\left(-\tfrac{2p}{5}\right) + i\sin\left(-\tfrac{2p}{5}\right),\quad \cos\left(-\tfrac{4p}{5}\right) + i\sin\left(-\tfrac{4p}{5}\right)$$

ii. $(z-a)(z-\overline{a}) = z^2 - (a+\overline{a})z + a\overline{a} = z^2 - (2\operatorname{Re} a)z + |a|^2$

Hence $z^5 - 1 = (z-1)(z-a)(z-\overline{a})(z-b)(z-\overline{b})$

$$= (z-1)\left(z^2 - 2\cos\tfrac{2p}{5}.\,z + 1\right)\left(z^2 - 2\cos\tfrac{4p}{5}.\,z + 1\right)$$

iii. $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$

$\therefore \left(z^2 - 2\cos\tfrac{2p}{5}.\,z + 1\right)\left(z^2 - 2\cos\tfrac{4p}{5}.\,z + 1\right) \equiv z^4 + z^3 + z^2 + z + 1$

Substituting $z = 1$: $\left(2 - 2\cos\tfrac{2p}{5}\right)\left(2 - 2\cos\tfrac{4p}{5}\right) = 5$

$$\therefore 4\left(1 - \cos\tfrac{2p}{5}\right)\left(1 - \cos\tfrac{4p}{5}\right) = 5$$

iv. If $x = \cos\tfrac{2p}{5}$, $\quad 1 - \cos\tfrac{4p}{5} = 2\sin^2\tfrac{2p}{5}$

$$= 2(1 - x^2)$$

Then, using (iii), $\quad 4(1-x).\,2(1-x^2) = 5$

$$8(x^3 - x^2 - x + 1) = 5$$
$$8x^3 - 8x^2 - 8x + 3 = 0$$

Hence $\cos\tfrac{2p}{5}$ is a root of the given cubic equation.

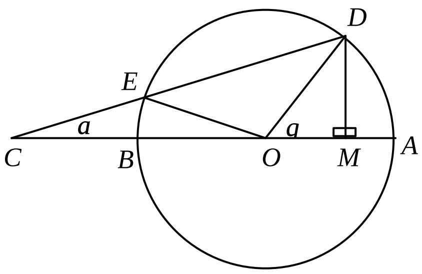
b. Outcomes assessed : H5, PE3

Marking Guidelines

Criteria	Marks
i • writes DM and OM in terms of trig. ratios of q	1
• uses ΔCMD to write required expression for $\tan a$ in terms of q	1
ii • compares sides of ΔCOE to deduce that $\angle BOE > a$	1
• uses exterior angle theorem and equal angles in isosceles triangle to find $\angle ODC$	1
• uses exterior angle theorem again to obtain required expression for q in terms of a, e	1
iii • deduces that $\tan a < \tan\frac{q}{3}$	1
• deduces required inequality for q	1

Answer

i.



In ΔOMD, $DM = \sin q$ and $OM = \cos q$

In ΔCMD, $\tan a = \dfrac{DM}{CM} = \dfrac{\sin q}{2+\cos q}$

ii. In ΔCOE, $CE + EO > CO$ $\quad \therefore CE + 1 > 2$

$\therefore CE > 1$ and hence $CE > OE$.

$\therefore \angle COE > \angle OCE$ (*larger $\angle$ opp. longer side*)

$\therefore \angle BOE = a + e$ for some $e > 0$.

Then $\angle DEO = 2a + e$ (*Exterior $\angle$ is sum of interior opp. $\angle$'s in ΔCOE*)

$\therefore \angle EDO = 2a + e$ (*$\angle$'s opp. equal sides are equal in ΔEOD*)

$\therefore q = 3a + e$ (*Exterior $\angle$ is sum of interior opp. $\angle$'s in ΔCOD*)

iii. $e > 0 \Rightarrow q > 3a$ $\quad$ Then $3a < q \Rightarrow a < \dfrac{q}{3}$

But $f(x) = \tan x$ is an increasing function. $\quad \therefore \tan a < \tan \frac{q}{3}$

Hence for the diagram above, $\dfrac{\sin q}{2+\cos q} = \tan a < \tan \frac{q}{3}$.

However, such a diagram can be drawn for any angle q such that $0 < q < \frac{p}{2}$.

Hence $\dfrac{\sin q}{2+\cos q} < \tan \frac{q}{3}$ for $0 < q < \frac{p}{2}$.