CSSA

2021

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

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Centre Number

Student Number

Mathematics Extension 2

Morning Session
Thursday, 29 July 2021

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 100

Section I Pages 2 - 5

10 marks

- Attempt Questions 1 - 10
- Allow 15 minutes for this section

Section II Pages 6 - 14

90 marks

- Attempt Questions 11 - 16
- Allow about 2 hours and 45 minutes for this section

6300-1

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1–10.

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1 Consider the statement "If an animal is a bird then it can fly or swim."

What is the contrapositive statement?

(A) If an animal cannot fly or cannot swim then it is a bird

(B) If an animal cannot fly or cannot swim then it is not a bird

(C) If an animal cannot fly and cannot swim then it is a bird

(D) If an animal cannot fly and cannot swim then it is not a bird

2 What value of z satisfies $z^2 = 7 + 24i$?

(A) $4 + 3i$

(B) $-4 + 3i$

(C) $-3 + 4i$

(D) $3 + 4i$

3 What is the angle between vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, to the nearest degree?

(A) $77°$

(B) $83°$

(C) $84°$

(D) $96°$

4 $A(1,2,2)$, $B(3,-12,4)$, $C(1,2,0)$ and $D(3,-12,0)$ are four positional vectors. What is the vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{CD}$?

(A) $2\underset{\sim}{i}-14\underset{\sim}{j}+2\underset{\sim}{k}$

(B) $2\underset{\sim}{i}-14\underset{\sim}{j}+4\underset{\sim}{k}$

(C) $2\underset{\sim}{i}-14\underset{\sim}{j}$

(D) $-2\underset{\sim}{i}+14\underset{\sim}{j}$

5 For all non-zero integers x and y, if $x>y$ then $\frac{1}{x}<\frac{1}{y}$. What is a counter example to the statement above?

(A) $x=2, y=-1$

(B) $x=0, y=0$

(C) $x=4, y=3$

(D) $x=-2, y=1$

6 A particle is moving in simple harmonic motion with displacement x metres. Its acceleration, $\ddot{x}$, is given by $\ddot{x}=-4x+3$. What are the centre and period of motion?

(A) centre of motion $=3$, period $=\frac{\pi}{2}$

(B) centre of motion $=-3$, period $=\pi$

(C) centre of motion $=\frac{3}{4}$, period $=\pi$

(D) centre of motion $=\frac{3}{4}$, period $=\frac{\pi}{2}$

7 It is given that $z = 2 + i$ is a root of $z^3 + az^2 - bz + 5 = 0$, where a and b are real numbers.

What is the value of a?

(A) -5

(B) -3

(C) 3

(D) 5

8 Which integral has the smallest value?

(A) $\int_0^{\frac{\pi}{4}} \sin^2 x dx$

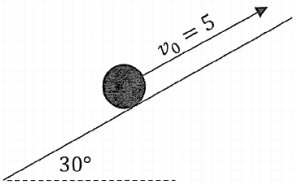
(B) $\int_0^{\frac{\pi}{4}} \cos^2 x dx$

(C) $\int_0^{\frac{\pi}{4}} \sin x \cos x dx$

(D) $\int_0^{\frac{\pi}{4}} \sin x \tan x dx$

9 A ball is rolled up a frictionless 30° ramp, with an initial velocity of 5 m/s.

Assuming $g = 10$ m/s^2, what is the net acceleration on the ball?



(A) $5\sqrt{3}$ m/s^2 directed down the ramp

(B) $5\sqrt{3}$ m/s^2 directed up the ramp

(C) 5 m/s^2 directed down the ramp

(D) 5 m/s^2 directed up the ramp

10 What value of a will minimise the integral $\int_0^1 (x^2 - a)^2 dx$?

(A) $a = \frac{1}{2}$

(B) $a = \frac{1}{\sqrt{2}}$

(C) $a = \frac{4}{45}$

(D) $a = \frac{1}{3}$

End of Section I

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Given $w = 2 + 5i$ and $z = 4 - 3i$, evaluate

(i) $|w + \overline{z}|$. **2**

(ii) $(w + \overline{z})(\overline{w} + z)$. **2**

(b) Find the square roots of $15 - 8i$. **3**

(c) Use the substitution $t = \tan\frac{\theta}{2}$, evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \sin\theta + \cos\theta}$. **4**

(d) The acceleration, a, of a particle moving in a straight line is given by $a = 6\left(1 - \frac{1}{2}x^2\right)$, where x is its displacement in metres. The particle is initially at the origin and travelling with velocity of 2 m/s.

(i) Show that the velocity of the particle is described by $v^2 = 4 + 12x - 2x^3$. **2**

(ii) Show that the particle returns to the origin. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Use integration by parts to find $\int x3^x dx$. **3**

(b) By writing $\frac{8-2x}{(1+x)(4+x^2)}$ in the form $\frac{a}{1+x}+\frac{bx+c}{4+x^2}$, **4**

evaluate $\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}dx$.

(c) On the same Argand diagram, draw a neat sketch of $|z-4-4i|=2$ and $\arg(z)=\frac{\pi}{4}$. **4**

Hence write down all the values of z which satisfy simultaneously

$|z-4-4i|=2$ and $\arg(z)=\frac{\pi}{4}$.

(d) Find the scalar projection of the vector $\underset{\sim}{u}=\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$ onto the vector $4\underset{\sim}{i}-4\underset{\sim}{j}+7\underset{\sim}{k}$. **2**

(e) Given $\underset{\sim}{a}=\begin{pmatrix}-1\\3\\4\end{pmatrix}$ and $\underset{\sim}{b}=\begin{pmatrix}-2\\1\\-4\end{pmatrix}$, and $\underset{\sim}{a}-\underset{\sim}{b}+2\underset{\sim}{c}=0$, find $\underset{\sim}{c}$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Suppose a and b are positive integers, where a is even and b is odd. Show that the sum of a and b is odd. **1**

(ii) Let P be the proposition **1**

"For all positive integers m and n, if $m+n$ is even then m and n are both even or m and n are both odd."

Using (i) above, prove that the proposition P is true by proving that the contrapositive statement is true.

(b) Let $I_1 = \int_0^{\pi} \frac{x\sin x dx}{1+\cos^2 x}$ and $I_2 = \int_0^{\pi} \frac{(\pi - x)\sin x dx}{1+\cos^2 x}$

(i) Using the substitution, $u = \pi - x$, show that $I_1 = I_2$. **2**

(ii) Hence, or otherwise, evaluate I_1. **2**

(c) (i) If a and b are positive integers with $a > b$, prove that $6(a+b)^2 - 2(a-b)^2$ is divisible by 4. **2**

(ii) It is given that x and y are positive integers with $x > y$, $M = 6x^2 - 2y^2$ and $D = x - y$. **2**

The result in (i) proves only one of the following statements to be true.

P: M is a multiple of 4 if D is an odd integer only, or

Q: M is a multiple of 4 if D is an even integer only, or

R: M is a multiple of 4 if D is an odd integer or an even integer.

Which of these statements is true? Justify your answer.

Question 13 continues on page 9

Question 13 (continued)

(d) A particle is travelling in a straight line. Its displacement, x cm, from O at a given time, t seconds after the start of motion, is given by $x = 3 + \sin^2 t$.

(i) Prove that the particle is undergoing simple harmonic motion. **2**

(ii) Find the period of the motion. **1**

(iii) Find the total distance travelled by the particle in the first π seconds. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Prove that $\log_3 7$ is irrational. **2**

(b) The scalar product of $\underset{\sim}{i} - 2\lambda\underset{\sim}{j} - \underset{\sim}{k}$, and the sum of $\underset{\sim}{i} - \lambda\underset{\sim}{k}$ and $\lambda\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, is 6. **2**

Find λ.

(c) Prove by mathematical induction that $(2n)! < (n!)^2 \, 4^{n-1}$ for $n \geq 5$. **3**

(d) Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = 3\underset{\sim}{a} + 2\underset{\sim}{b}$.

(i) Prove that if $\overrightarrow{OD} = \frac{1}{5}\overrightarrow{OC}$ then D lies on AB. **3**

(ii) Is the point D closer to point A or point B? Justify your answer. **1**

(e) (i) Given $z = \cos\theta + i\sin\theta$ prove that $z^n - \frac{1}{z^n} = 2i\sin n\theta$. **1**

(ii) Hence by considering the expansion $\left(z - \frac{1}{z}\right)^5$, show that **3**

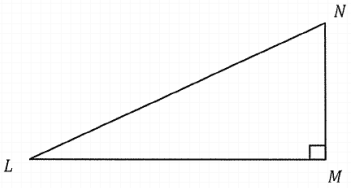
$$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta.$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove that for non-zero vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ that $(\underset{\sim}{a}+\underset{\sim}{b}).(\underset{\sim}{a}+\underset{\sim}{b}) = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$ if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular. **2**

(ii)



In ΔLMN, let $\overrightarrow{LM} = \underset{\sim}{a}$ and $\overrightarrow{MN} = \underset{\sim}{b}$.

By finding an expression for the side LN in terms of the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, or otherwise, prove that $\left|\overrightarrow{LN}\right|^2 = \left|\overrightarrow{LM}\right|^2 + \left|\overrightarrow{MN}\right|^2$. **2**

(b) Find $\int x^2\sqrt{1-x^2}\,dx$. **3**

(c) It is given that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ for $k \in \mathbb{N}$.

(i) Prove $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$. **1**

(ii) If $x_1, x_2, \ldots, x_n$ are positive integers, not necessarily consecutive, such that $1 < x_1 < x_2 < \ldots < x_n$, **2**

prove that $\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \ldots + \frac{1}{x_{n-1}^2} < 1$.

Question 15 continues on page 12

Question 15 (continued)

(d) A particle of unit mass is moving vertically downward in a medium which exerts a resistance force proportional to the square of the speed, v, of the particle. It is released from rest at O and its terminal velocity is U.

(i) Show that the distance it has fallen below O is given by **2**

$$x = \frac{1}{2k} \ln \left| \frac{g}{g - kv^2} \right|.$$

(ii) Prove that the time taken, T, for the particle to fall from O to when its velocity is half of its terminal velocity, U, is given by **3**

$$T = \frac{U}{2g} \ln 3.$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) State all the roots of $z^7 - 1 = 0$ in exponential form. **2**

(ii) Using $z^7 - 1 = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$, or otherwise, **3**

prove that $\frac{2\pi}{7}$, $\frac{4\pi}{7}$ and $\frac{6\pi}{7}$ are solutions to

$$2\cos 3\theta + 2\cos 2\theta + 2\cos\theta + 1 = 0.$$

(iii) Hence or otherwise, prove that **2**

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}.$$

(b) (i) Given that $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2} dx$ where n is a positive integer, **3**

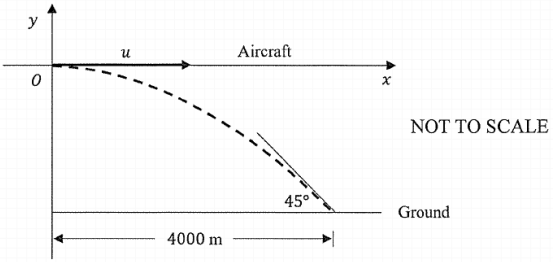
prove that $I_{2n+1} = \frac{e}{2} - nI_{2n-1}$.

(ii) Hence, or otherwise, prove that $2\int_0^1 x^{2n-1}(1 + x^2)e^{x^2} dx \leq e$ for $n \geq 1$. **2**

Question 16 continues on page 14

Question 16 (continued)

(c) An aircraft flying horizontally at u m/s delivers an emergency medical supply package that hits the ground 4000 m away, measured horizontally. The package experiences an air resistance of $0.1v$ where v is the velocity at time t and g is the acceleration due to gravity. The package hits the ground an angle of $45°$ to the horizontal.



Assume that t seconds after release, the position vector is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} 10u(1-e^{-0.1t}) \\ 100g(1-e^{-0.1t}) - 10gt \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

(i) Show that the velocity vector $\underset{\sim}{v}(t)$ of the particle is given by **1**

$$\underset{\sim}{v}(t) = \begin{pmatrix} ue^{-0.1t} \\ -10g(1-e^{-0.1t}) \end{pmatrix}.$$

(ii) Find the time when the package hits the ground and the speed on impact, where **2** $g = 10 \text{ m/s}^2$.

End of examination

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EXAMINERS

Piv Huot (Convenor)	Mount St Joseph Catholic College, Milperra
Eric Choy	Educational Consultant
Khiet Hoang	John Edmondson High School, Horningsea Park
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Praneetha Singh	The Hills Grammar School, Kenthurst
Vito Zurlo	St Scholastica's College, Glebe

CSSA

2021
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MARKING GUIDELINES

Mathematics Extension 2

Section I
10 marks
Multiple Choice Answer Key

Question	Answer
1	D
2	A
3	C
4	C
5	A
6	C
7	B
8	A
9	C
10	D

6300-2

Question 1 (1 mark)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E2-E3

Solution	Mark
The parts of the statement $P \Rightarrow Q$ are: P: is a bird and Q: fly or swim The negations are: $\neg P$: is not a bird and $\neg Q$: cannot fly and cannot swim The contrapositive is $\neg Q \Rightarrow \neg P$, so: "If an animal cannot fly and cannot swim then it is not a bird" Hence (D)	1

Question 2 (1 mark)
Outcomes Assessed: N1.1/MEX12-4
Targeted Performance Bands: E2-E3

Solution	Mark
$(4+3i)^2 = 16 + 24i + 9i^2$ $= 7 + 24i$ Hence (A)	1

Question 3 (1 mark)
Outcomes Assessed: V1.2/MEX12-3
Targeted Performance Bands: E2-E3

Solution	Mark
$\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$ $\underset{\sim}{u} \cdot \underset{\sim}{v} = 2 - 3 + 2 = 1$ $\left\lvert\underset{\sim}{u}\right\rvert = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$ and $\left\lvert\underset{\sim}{v}\right\rvert = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$ Angle θ between $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is given by: $\theta = \cos^{-1}\left(\frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\left\lvert\underset{\sim}{u}\right\rvert \left\lvert\underset{\sim}{v}\right\rvert}\right)$ $= \cos^{-1}\left(\frac{1}{\sqrt{14} \times \sqrt{6}}\right) = 83.73604728 = 84^0$ Hence (C)	1

6300-2

Question 4 (1 mark)
Outcomes Assessed: V1.1/MEX12-3
Targeted Performance Bands: E2-E3

Solution	Mark
$A(1,2,2)$, $B(3,-12,4)$, $C(1,2,0)$ and $D(3,-12,0)$ $\overrightarrow{AB} = 2\underset{\sim}{i} - 14\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\overrightarrow{CD} = 2\underset{\sim}{i} - 14\underset{\sim}{j}$ When $\overrightarrow{AB}$ is projected onto $\overrightarrow{CD} = \frac{\overrightarrow{AB}\times\overrightarrow{CD}}{\lvert\overrightarrow{CD}\rvert^2} \times \overrightarrow{CD}$ Therefore the vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{CD}$ is $2\underset{\sim}{i} - 14\underset{\sim}{j}$ Hence (C)	1

Question 5 (1 mark)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E2-E3

Solution	Mark
A counter example is one which shows that the statement is false (or not true). For option A , $x = 2,\ y = -1 \Rightarrow\ 2 > -1$ then $\frac{1}{2} < \frac{1}{-1}$ but this is false. Hence (A)	1

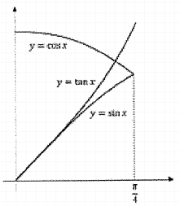
Question 6 (1 mark)
Outcomes Assessed: M1.1/MEX12-6
Targeted Performance Bands: E3-E4

Solution	Mark
$\ddot{x} = -4x + 3$ $= -2^2(x - \frac{3}{4})$ $\therefore\ n = 2,$ centre of motion $= \frac{3}{4}$ period $= \frac{2\pi}{2}$ $= \pi$ Hence (C)	1

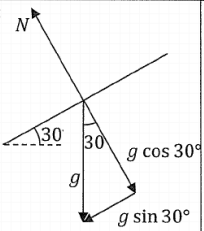
Question 7 (1 mark)
Outcomes Assessed: N2.1/MEX12-4
Targeted Performance Bands: E3-E4

Solution	Mark
Since $z_1 = 2+i$ is a root then $z_2 = 2-i$ is also a root since all the coefficients are real. Let the roots be z_1, z_2, α Using product of roots, $z_1 z_2 \alpha = -5$ $(2+i)(2-i)\alpha = -5$ $5\alpha = -5$ $\alpha = -1$ Using the sum of roots, $z_1 + z_2 + \alpha = -a$ $2+i+2-i+-1 = -a$ $a = -3$ Hence (B)	1

Question 8 (1 mark)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Solution	Mark
For $\left[0, \frac{\pi}{4}\right]$ $\sin x < \cos x$ and $\sin x < \tan x$ $\therefore$ $\sin^2 x < \sin x \cos x < \cos^2 x$ and $\sin^2 x < \sin x \tan x$ $\therefore$ the smallest integral is $\int_0^{\frac{\pi}{4}} \sin^2 x dx$ Hence (A) 	1

Question 9(1 mark)
Outcomes Assessed: M1.2/MEX12-6
Targeted Performance Bands: E3-E4

Solution	Mark
The acceleration due to gravity can be split into two components: • the component perpendicular to the ramp, $g\cos 30°$, is balanced by the normal reactive force • the component parallel to the ramp, $g\sin 30°$, points down the ramp • The acceleration is given by $a = 10\sin 30° = 5$ m/s^2 down the ramp  Hence (C)	1

Question 10 (1 mark)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Solution	Mark
$\int_0^1 (x^2 - a)^2\,dx$ $= \int_0^1 (x^4 - 2ax^2 + a^2)\,dx$ $= \left[\frac{x^5}{5} - \frac{2a}{3}x^3 + a^2x\right]_0^1$ $= \left(\frac{1}{5} - \frac{2a}{3} + a^2\right) - (0)$ $= a^2 - \frac{2a}{3} + \frac{1}{5}$ Minimum value at the axis of symmetry when $a = -\frac{-\frac{2}{3}}{2(1)} = \frac{1}{3}$ Hence (D)	1

Section II
90 marks

Question 11 (15 marks)

11 (a) (i) (2 marks)
Outcomes Assessed: N1.1/MEX12-4
Targeted Performance: E2-E3

Criteria	Marks
• Provides correct solution	2
• Obtains the correct expression of the sum, or equivalent merit	1

Sample Answer:

$$\begin{aligned} w+\bar{z} &= (2+5i)+\overline{(4-3i)} \\ &= 2+5i+4+3i \\ &= 6+8i \end{aligned}$$

$$|w+\bar{z}| = \sqrt{6^2+8^2} = 10$$

11 (a) (ii) (2 marks)
Outcomes Assessed: N1.1/MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Arrives at $(w+\bar{z})\overline{(w+\bar{z})}$ OR • Attempts to apply result from part(i), or equivalent merit	1

Sample Answer:

$$\begin{aligned} (w+\bar{z})(\bar{w}+z) &= (w+\bar{z})\overline{(w+\bar{z})} \\ &= |(w+\bar{z})|^2 = 100 \end{aligned}$$

11 (b) (3 marks)
Outcomes Assessed: N1.1/MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	3
• Obtains a value for either a or b	2
• Obtains the 2 simultaneous equations OR • Attempts to find a value for either a or b, or equivalent merit	1

Sample Answer:
Let a square root of $15-8i$ be a + bi

$15-8i=(1+bi)^2$
$a^2-b^2=15$
$ab=-4$
$a=\pm 4, b=\mp 1$ (by inspection)
The square roots of $15-8i$ are $\pm(4-i)$

Alternatively:

$$15-18i=(a+bi)^2 \quad a,b\in R$$
$$a^2-b^2=15$$
$$ab=-4$$
$$b=\frac{-4}{a}$$
$$a^2-\frac{16}{a^2}=15$$
$$a^4-15a^2-16=0$$
$$(a^2+1)(a^2-16)=0$$
$$a^2=-1, NS \qquad a=\pm 4, b=\mp 1$$
$$\therefore \sqrt{15-18i}=\pm(4-i)$$

11 (c) (4 marks)
Outcomes assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	4
• Obtains correct integrand, or equivalent merit	3
• Correctly substitutes and attempts to simplify (ignoring limits), or equivalent merit	2
• Uses given substitution, or equivalent merit	1

Sample Answer:

$t = \tan\frac{\theta}{2} \Rightarrow \theta = 2\tan^{-1}t$

$d\theta = \frac{2}{1+t^2}dt$

$\theta = 0, t = 0 \text{ and } \theta = \frac{\pi}{2}, t = 1$

$$\int_0^1 \frac{1}{(1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2})} \times \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2dt}{1+t^2+2t+1-t^2}$$

$$= \int_0^1 \frac{dt}{1+t}$$

$$= [\ln|1+t|]_0^1$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

11 (d)(i) (2 marks)
Outcomes Assessed: M1.2/MEX12-6
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Obtains integral for v^2 in terms of x, or equivalent merit	1

Sample Answer:

$$a = 6\left(1+\frac{1}{2}x^2\right)$$

i.e. $$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6-3x^2$$

$$\frac{1}{2}v^2 = \int (6-3x^2)\,dx$$

$$= 6x - x^3 + C$$

At $x = 0, v = 2$

$$\therefore \frac{1}{2}(4) = 6(0) - 0^3 + C$$

$$\therefore C = 2$$

$$\frac{1}{2}v^2 = 6x - x^3 + 2$$

$$v^2 = 12x - 2x^3 + 4$$

11 (d)(ii) (2 marks)
Outcomes Assessed: M1.2/MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Arrives at $v = \pm 2$ m/s	1

Sample Answer:
$v^2 = 4 + 12x - 2x^3 = 0$ at approximately $x = -2.3, -0.3$ and 2.6 (using trial and error), so the particle oscillates between -0.3 and 2.6 given initial conditions, with the particle repeatedly passing through the origin.

Alternatively:
The particle is at rest when $x = 0$

$$\therefore v^2 = 4 + 12(0) - 0$$

$$= 4$$

$$v = \pm 2 \text{ m/s}$$

Since the particle was initially at the origin with a velocity of 2 m/s, a velocity of -2 m/s indicates it passes through the origin in the reverse direction to that of its original motion

Question 12

12 (a) (2 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	3
• Correctly applies integration by parts, or equivalent merit	2
• Attempts to use integration by parts, or equivalent merit	1

Sample Answer:

$$\int x3^x\,dx$$

$u = x$	$\frac{dv}{dx} = 3^x$
$\frac{du}{dx} = 1$	$v = \frac{3^x}{\ln 3}$

$$= \frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\int 3^x\,dx$$

$$= \frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\left(\frac{3^x}{\ln 3}\right) + c$$

$$= \frac{x3^x}{\ln 3} - \frac{3^x}{\ln^2 3} + c$$

12 (b) (4 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E2-E4

Criteria	Marks
• Provides correct solution	4
• Integrates correctly in terms of ln, or equivalent merit	3
• Obtains 3 values correctly	2
• Obtains at least 2 values correctly, or equivalent merit	1

Sample Answer:

$$\frac{8-2x}{(1+x)(4+x^2)} = \frac{a}{1+x} + \frac{bx+c}{4+x^2}$$

$$8-2x = a(4+x^2) + (bx+c)(1+x)$$

let $x=-1$

$$10 = 5a$$

$$a = 2$$

let $x=0$

$$c = 0$$

Equating the coefficient of x^2 :

$$0 = 2+b$$

$$b = -2$$

$$\int_0^4 \left(\frac{2}{1+x} + \frac{-2x}{4+x^2}\right) dx$$

$$= \left[2\ln|1+x| - \ln|4+x^2|\right]_0^4$$

$$= 2\ln 5 - \ln 20 - 2\ln 1 + \ln 4$$

$$= 2(\ln 5 - \ln 1) - (\ln 20 - \ln 4)$$

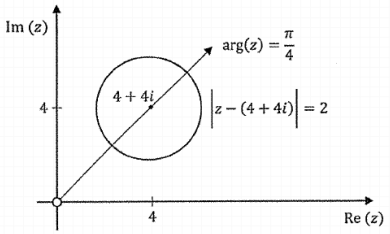
$$= \ln 5$$

12 (c) (5 marks)
Outcomes Assessed: N2.2/MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	4
• Correct sketch for $\lvert z-(4+4i)\rvert = 2$ and $\arg z = \frac{\pi}{4}$ and correctly finds one intersection point	3
• Correct sketch for $\lvert z-(4+4i)\rvert = 2$ and $\arg z = \frac{\pi}{4}$	2
• Correct sketch for $\lvert z-(4+4i)\rvert = 2$ or $\arg z = \frac{\pi}{4}$	1

Sample Answer:
$|z-(4+4i)| = 2 \Rightarrow$ circle centre at $(4,4)$ and radius 2 units $\Rightarrow (x-4)^2+(y-4)^2 = 4$

$\arg z = \frac{\pi}{4} \Rightarrow$ The angle that the vector from 0 to z makes with the positive direction of the x axis is always $\frac{\pi}{4} \Rightarrow$ The line $y = x$ (NOT including the origin as arg (0) is undefined)



$|z-(4+4i)|=2 \quad \Rightarrow \quad$ circle centre at $(4,4)$ and radius 2 units

$\Rightarrow (x-4)^2+(y-4)^2=4 \qquad (1)$

$\arg(z)=\frac{\pi}{4} \quad \Rightarrow \quad y=x \quad$ (for $x>0$) $\quad (2)$

Solving (1) and (2) simultaneously

$$(x-4)^2+(x-4)^2=4$$

$$x=4\pm\sqrt{2}$$

$x=4+\sqrt{2}$, $y=4+\sqrt{2}$ and $x=4-\sqrt{2}$, $y=4-\sqrt{2}$

Hence the values of z which satisfy simultaneously $|z-(4+4i)|=2$ and $\arg(z)=\frac{\pi}{4}$ are

$z=(4-\sqrt{2})+(4-\sqrt{2})i$ and $z=(4+\sqrt{2})+(4+\sqrt{2})i$

12(d) (2 marks)
Outcomes Assessed: V1.1/MEX12-3
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Correctly finds $\underset{\sim}{u}\cdot\underset{\sim}{v}$, or equivalent merit	1

Sample Answer:

Let $\underset{\sim}{u}=\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$ and $\underset{\sim}{v}=4\underset{\sim}{i}-4\underset{\sim}{j}+7\underset{\sim}{k}$.

Scalar projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}=\dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|}$

$$=\frac{\left(\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}\right)\cdot\left(4\underset{\sim}{i}-4\underset{\sim}{j}+7\underset{\sim}{k}\right)}{\left|4\underset{\sim}{i}-4\underset{\sim}{j}+7\underset{\sim}{k}\right|}$$

$$=\frac{4+8+7}{\sqrt{4^4+(-4)^2+7^2}}$$

$$=\frac{19}{9}$$

12(e) (2 marks)
Outcomes Assessed: V1.1/MEX12-3
Targeted Performance Bands: E2-E4

Criteria	Marks
• Provides correct solution	2
• Attempts to use $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ in a calculation	1

Sample Answer:

$$\begin{pmatrix} -1+2 \\ 3-1 \\ 4+4 \end{pmatrix} + 2\underset{\sim}{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} + 2\underset{\sim}{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2\underset{\sim}{c} = \begin{pmatrix} -1 \\ -2 \\ -8 \end{pmatrix}$$

$$\underset{\sim}{c} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ -4 \end{pmatrix}$$

Question 13 (15 marks)
13 (a) (i) (1 mark)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	1

Sample Answer:
By the definition of even and odd, $\exists r, s$ such that $a = 2r$ and $b = 2s + 1$ for $r, s \in \mathbb{N}$.

$$\therefore a + b = 2r + (2s + 1)$$
$$= 2(r + s) + 1$$

Let $k = r + s$ then $k \in \mathbb{N}$ because r and s are integers and the sum of integers are integers.

$$\therefore a + b = 2k + 1$$

$\Rightarrow a + b$ is odd.

13 (a) (ii) (1 mark)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	1

Sample Answer:
Proof by contrapositive.
Suppose m and n are positive integers such that one of m and n is even and the other is odd.
From part (i), the sum of any even integer and any odd integer is an odd.
$\Rightarrow m+n$ is odd
$\therefore P$ is true.

13 (b) (i) (2 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Attempts to use given substitution, or equivalent merit	1

Sample Answer:

$$I_1 = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$u = \pi - x \Rightarrow du = -dx$
$x = 0, u = \pi$ and $x = \pi, u = 0$

$$I_1 = \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}(-du)$$

$$= \int_0^{\pi} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}\,du$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$$

$$= I_2$$

since u is a dummy variable and $\sin(\pi - A) = \sin A$

$\Rightarrow \sin(\pi - u) = \sin x$

13 (b) (ii) (2 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Arrives at a correct integrated expression for $I_1 + I_2$	1

Sample Answer:

$$I_1 + I_2 = \int_0^{\pi} \frac{x\sin x + (\pi - x)\sin x}{1+\cos^2 x}\,dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= -\pi \int_0^{\pi} \frac{-\sin x}{1+\cos^2 x}\,dx$$

$$= -\pi \left[\tan^{-1}(\cos x)\right]_0^{\pi}$$

$$= -\pi\left(\tan^{-1}(-1) - \tan^{-1} 1\right)$$

$$= -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right)$$

$$= \frac{\pi^2}{2}$$

As $I_1 = I_2$

$$I_1 = \frac{1}{2} \times \frac{\pi^2}{2}$$

$$= \frac{\pi^2}{4}$$

13 (c) (i) (2 marks)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Correctly simplify the expression	1

Sample Answer:

$6(a+b)^2-2(a-b)^2$

$=6(a^2+2ab+b^2)-2(a^2-2ab+b^2)$

$=6a^2+12ab+6b^2-2a^2+4ab-2b^2$

$=4(a^2+b^2+4ab)$

$=4p$ for integral p since a, b integral

$\therefore$ if a and b are integers, then $6(a+b)^2-2(a-b)^2$ is divisible by 4

13 (c) (ii) (2 marks)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Attempts to use part (i), or equivalent merit	1

Sample Answer:

Let $x=a+b, y=a-b$

$D=x-y$

$=a+b-a+b$ from (i)

$=2b$

$\therefore$ D is even since b is integral

$\therefore$ part (i) proves that M is a multiple of 4 when D is even, so statement Q is correct.

13 (d)(i) (2 marks)
Outcomes Assessed: M1.1/MEX12-6
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Provides correct derivative in terms of t	1

Sample Answer:

$$x = 3 + \sin^2 t$$

$$v = 2\sin t \cos t$$

$$\frac{dv}{dt} = \cos t\left(2\cos t\right) + 2\sin t\left(-\sin t\right)$$

$$= 2\left[\cos^2 t - \sin^2 t\right]$$

$$= 2\left[1 - \sin^2 t - \sin^2 t\right]$$

$$= 2\left[1 - 2\sin^2 t\right]$$

$$= 2\left[1 - 2\left(x - 3\right)\right]$$

$$= 2\left[1 - 2x + 6\right]$$

$$= 14 - 4x$$

$$= -4\left(x - \frac{7}{2}\right)$$

Hence the particle is undergoing SHM.

13 (d)(ii) (1 mark)
Outcomes Assessed: M1.1/MEX12-6
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	1

Sample Answer:

$$n = 2, \ \therefore \text{ Period} = \frac{2\pi}{n}$$

$$= \frac{2\pi}{2}$$

$$= \pi$$

13 (d)(iii) (2 marks)
Outcomes Assessed: M1.1/MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Correctly finds at least two values of t when $v = 0$, or equivalent merit	1

Sample Answer:

$v = 0$

$v = 2\sin t \cos t$

$2\sin t \cos t = 0$

$\therefore \sin t = 0 \qquad \cos t = 0$

$t = 0, \pi, 2\pi, 3\pi... \qquad t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.....$

Using $x = 3 + \sin^2 t$

$t = 0 \rightarrow x = 3$

$t = \frac{\pi}{2} \rightarrow x = 4$

$t = \pi \rightarrow x = 3$

$\therefore$ total distance $= 2$ cm

Question 14 (15 marks)

14 (a) (2 marks)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E2-E4

Criteria	Marks
• Provides correct solution	2
• Assumes $\log_3 7$ is rational and attempts to eliminate the logarithm, or equivalent merit	1

Sample Answer:
Assume $\log_3 7$ is a rational number

Assume $\log_3 7 = \frac{a}{b}$ where a and b are integers with $HCF = 1$ and $b \neq 0$ $(*)$

$$\therefore\ 3^{\frac{a}{b}} = 7$$

$$\therefore\ 3^a = 7^b$$

Since 7 is not divisible by 3 then $3^a = 7^b$ only when $a = b = 0$ which contradicts $(*)$.

Hence $\log_3 7$ is an irrational number.

14 (b) (2 marks)
Outcomes Assessed: V1.2/MEX12-3
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Obtains a correct expression for the scalar product OR • Obtains the correct response from the incorrect order of operations	1

Sample Answer:

$$\begin{pmatrix} 1 \\ -2\lambda \\ -1 \end{pmatrix} . \begin{pmatrix} 1+\lambda \\ 0+2 \\ -\lambda-1 \end{pmatrix} = 6$$

$$1 + \lambda - 4\lambda + \lambda + 1 = 6$$

$$-2\lambda = 4$$

$$\lambda = -2$$

14 (c) (3 marks)
Outcomes Assessed: P2/MEX12-2
Targeted Performance Bands: E2-E4

Criteria	Marks
• Provides correct solution	3
• Proves base case, states the induction hypothesis and uses it in simplifying $P(k+1)$	2
• Proves base case OR • States the induction hypothesis and uses it in simplifying $P(k+1)$	1

Sample Answer:
Let $P(n)$ represent the proposition

$P(5)$ is true since LHS $= (2\times5)! = 3\ 628\ 800$, RHS $= (5!)^2 \times 4^{5-1} = 3\ 686\ 000$, $\therefore$ LHS < RHS

If $P(k)$ is true for some $k \geq 5$ then $(2k)! < (k!)^2 4^{k-1}$

RTP : $P(k+1)$ $(2(k+1))! < ((k+1)!)^2 4^k$

$$\begin{aligned}
\text{LHS} &= (2(k+1))! \\
&= (2k+2)(2k+1)(2k)! \\
&< (2k+2)(2k+1)\times(k!)^2 4^{k-1} \quad \text{from } P(k) \\
&= 2(k+1)(2k+1)\times(k!)^2 4^{k-1} \\
&< 2(k+1)2(k+1)\times(k!)^2 4^{k-1} \quad \text{since } 2k+1 < 2k+2 = 2(k+1) \text{ for } k \geq 5 \\
&= 4(k+1)^2(k!)^2 4^{k-1} \\
&= ((k+1)!)^2 4^k
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

Hence $P(n)$ is true for $n \geq 5$ by induction.

14 (d) (i) (3 marks)
Outcomes Assessed: V1.3/MEX12-3
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Obtains an expression for $\overrightarrow{OD}$ and the equation of the line through A and B	2
• Obtains an expression for $\overrightarrow{OD}$ or the equation of the line through A and B	1

Sample Answer:

Let $\underset{\sim}{d} = \overrightarrow{OD}$

$$\underset{\sim}{d} = \frac{1}{5}\left(3\underset{\sim}{a} + 2\underset{\sim}{b}\right)$$

$$= \frac{3}{5}\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}$$

Let the line AB be $\underset{\sim}{r}_{AB} = \underset{\sim}{a} + \lambda\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$

$$= (1-\lambda)\underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Let $\lambda = \frac{2}{5}$

$$\therefore \underset{\sim}{r}_{AB} = \left(1 - \frac{2}{5}\right)\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}$$

$$= \frac{3}{5}\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}$$

$$= \underset{\sim}{d}$$

$\therefore$ D lies on AB

14 (d) (ii) (1 mark)
Outcomes Assessed: V1.3/MEX12-3
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	1

Sample Answer:

D is $\frac{2}{5}$ of the distance from A to B, so D is closer to A since $\frac{2}{5} < \frac{1}{2}$

14 (e) (i) (1 mark)
Outcomes Assessed: N2.1/MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	1

Sample Answer:

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

$z^n - z^{-n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= 2i\sin(n\theta)$

Alternatively:

$z^{-n} = (\overline{z^n})$ since $|z^n| = |z|^n = 1$

$\therefore\ z^n - z^{-n} = z^n - (\overline{z^n})$

$= 2\,Im(z^n)$

$= 2i\,sin(n\theta)$

14 (e) (ii) (3 marks)
Outcomes Assessed: N2.1/MEX12-4
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Rearranges the equation into conjugate pairs and uses the result from (i), or equivalent merit	2
• Expands $\left(z-\frac{1}{z}\right)^5$ correctly	1

Sample Answer:

$\left(z-\frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$

$(2i\sin\theta)^5 = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$

$\therefore\ \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$

Question 15 (15 marks)

15 (a) (i) (2 marks)
Outcomes Assessed: V1.2/MEX12-3
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Obtains the correct expansion	1

Sample Answer:

$$(\underset{\sim}{a}+\underset{\sim}{b}).(\underset{\sim}{a}+\underset{\sim}{b}) = \underset{\sim}{a}.\underset{\sim}{a}+2\underset{\sim}{a}.\underset{\sim}{b}+\underset{\sim}{b}.\underset{\sim}{b}$$
$$= |\underset{\sim}{a}|^2+2\underset{\sim}{a}.\underset{\sim}{b}+|\underset{\sim}{b}|^2$$
$$= |\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2 \text{ since } \underset{\sim}{a}.\underset{\sim}{b}=0$$

15 (a) (ii) (2 marks)
Outcomes Assessed: V1.2/MEX12-3
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Correctly finds $\overrightarrow{LN}$	1

Sample Answer:

$$\left|\overrightarrow{LN}\right|^2 = \left|\overrightarrow{LM}+\overrightarrow{MN}\right|^2$$
$$= |\underset{\sim}{a}+\underset{\sim}{b}|^2$$
$$= (\underset{\sim}{a}+\underset{\sim}{b}).(\underset{\sim}{a}+\underset{\sim}{b})$$
$$= |\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2 \text{ from } (i)$$
$$= \left|\overrightarrow{LM}\right|^2+\left|\overrightarrow{MN}\right|^2$$

15 (b) (3 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Finds $-\frac{1}{4}-\frac{\pi}{2}-\left[\frac{1}{2}\sin 2x\frac{2}{2}\right]_0^{\frac{\pi}{2}}$ OR • Finds $\frac{\pi}{4}-\frac{1}{4}\left[x-\frac{1}{2}\sin 2x\frac{2}{2}\right]_0^{\frac{\pi}{2}}$	2
• Rewrites the integrand as $x\sin 2x$ OR • Determines $u=x\sin x$ and $\frac{dv}{dx}=\cos x$ or $u=x\cos x$ and $\frac{dv}{dx}=\sin x$	1

Sample Answer:

$$\int x^2\sqrt{1-x^2}\,dx = \int \sin^2\theta\sqrt{1-\sin^2\theta}\times\cos\theta\,d\theta$$

$x=\sin\theta$
$dx=\cos\theta\,d\theta$

$$= \int \sin^2\theta\cos^2\theta\,d\theta = \frac{1}{4}\int \sin^2 2\theta\,d\theta$$
$$= \frac{1}{4}\int \frac{1}{2}(1-\cos 4\theta)\,d\theta = \frac{\theta}{8}-\frac{\sin 4\theta}{32}+c$$
$$= \frac{\sin^{-1}x}{8}-\frac{2\sin 2\theta\cos 2\theta}{32}+c$$
$$= \frac{\sin^{-1}x}{8}-\frac{2\sin\theta\cos\theta\,(1-2\sin^2\theta)}{16}+c$$
$$= \frac{\sin^{-1}x}{8}-\frac{x\sqrt{1-x^2}(1-2x^2)}{8}+c$$
$$= \frac{\sin^{-1}x+(2x^3-x)\sqrt{1-x^2}}{8}+c$$

15 (c) (i) (1 mark)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	1

Sample Answer:

$$\text{LHS} = \frac{1}{(k+1)^2}$$

$$= \frac{1}{(k+1)(k+1)}$$

$$< \frac{1}{k(k+1)}$$

$$= \frac{1}{k} - \frac{1}{k+1}$$

$$\therefore \frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$$

15 (c) (ii) (2 marks)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Attempts to apply result from part(i), or equivalent merit	1

Sample Answer:

Since $1 < x_1 < x_2 < \ldots\ldots\ldots\ldots\ldots < x_n$,

$$\therefore \frac{1}{{x_1}^2}+\frac{1}{{x_2}^2}+\frac{1}{{x_3}^2}+\ldots+\frac{1}{{x_{n-1}}^2} < \frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{{x_n}^2}$$

Now from a(i)

$$\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} = \frac{1}{k}-\frac{1}{k+1}$$

$$\therefore \quad \frac{1}{2^2} < \frac{1}{1}-\frac{1}{2}$$

$$\frac{1}{3^2} < \frac{1}{2}-\frac{1}{3}$$

$$\ldots.$$

$$\frac{1}{{x_n}^2} < \frac{1}{x_{n-1}}-\frac{1}{x_n}$$

$$\therefore \frac{1}{{x_1}^2}+\frac{1}{{x_2}^2}+\frac{1}{{x_3}^2}+\ldots+\frac{1}{{x_{n-1}}^2} < \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{x_{n-1}}-\frac{1}{x_n}\right)$$

$$< 1-\frac{1}{x_n}$$

$$< 1$$

Alternatively:

$$\frac{1}{x_1^2} < \frac{1}{x_1 - 1} - \frac{1}{x_1} \text{ from (i)}$$

$$\leq \frac{1}{1} - \frac{1}{x_1} \quad \text{since } 1 < x_1$$

$$\therefore \frac{1}{x_1^2} < 1 - \frac{1}{x_1}$$

Similarly:

$$\frac{1}{x_2^2} < \frac{1}{x_1} - \frac{1}{x_2}, \ldots, \frac{1}{x_{n-1}^2} < \frac{1}{x_{n-2}} - \frac{1}{x_{n-1}}$$

$$\therefore \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \ldots + \frac{1}{x_{n-1}^2} < 1 - \frac{1}{x_1} + \frac{1}{x_1} - \frac{1}{x_2} + \frac{1}{x_2} - \frac{1}{x_3} + \ldots + \frac{1}{x_{n-2}} - \frac{1}{x_{n-1}}$$

$$= 1 - \frac{1}{x_{n-1}}$$

$$< 1 \qquad \text{since } x_{n-1} > 1$$

15 (d) (i) (2 marks)
Outcomes Assessed: M1.3/MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Correctly separates the variables, or equivalent merit	1

Sample Answer:
Choose point 0 as origin and $\downarrow$ as positive direction.

Equation of motion: $\dot{v} = g - kv^2$.
Initial conditions: $t = 0,\ x = 0,\ v = 0$.
Terminal velocity U hence $g = kU^2 \Rightarrow k = g / U^2$.
Relation between v and x:

$$\dot{v} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$-2k\,dx = \frac{-2kv\,dv}{g - kv^2}$$

$$-2kx + C = \ln\left|g - kv^2\right|$$

$$x = 0,\ v = 0 \Rightarrow C = \ln g,$$

$$x = \frac{1}{2k}\ln\left|\frac{g}{g - kv^2}\right|.$$

Alternatively:

$$
\begin{aligned}
v\frac{dv}{dx} &= g - kv^2 \\
\frac{dv}{dx} &= \frac{g - kv^2}{v} \\
\frac{dx}{dv} &= \frac{v}{g - kv^2} \\
x &= \int_0^v \frac{v}{g - kv^2}\, dv \\
&= -\frac{1}{2k}\int_0^v -\frac{2kv}{g - kv^2}\, dv \\
&= \frac{1}{2k}\Big[\ln\left|g - kv^2\right|\Big]_v^0 \\
&= \frac{1}{2k}\left(\ln|g| - \ln\left|g - kv^2\right|\right) \\
&= \frac{1}{2k}\ln\left|\frac{g}{g - kv^2}\right|
\end{aligned}
$$

15 (d) (ii) (3 marks)
Outcomes Assessed: M1.3/MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Finds $t=\frac{1}{2\sqrt{kg}}\ln\left\lvert\frac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v}\right\rvert$	2
• Correctly separates the variables OR • Finds $\frac{dt}{dv}=\frac{1}{g-kv^2}$	1

Sample Answer:
Relation between v and t :

$$\frac{dv}{dt}=g-kv^2$$

$$\sqrt{k}dt=\frac{\sqrt{k}dv}{g-\left(\sqrt{k}v\right)^2}$$

$$\sqrt{k}dt=\frac{\sqrt{k}dv}{g-kv^2}$$

$$\sqrt{k}dt=\left\{\frac{1}{\sqrt{g}-\sqrt{k}v}+\frac{1}{\sqrt{g}+\sqrt{k}v}\right\}\frac{\sqrt{k}dv}{2\sqrt{g}}$$

$$2\sqrt{kg}t+C=\ln\left\lvert\frac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v}\right\rvert,$$

$t=0, v=0 \Rightarrow C=0,$

$$t=\frac{1}{2\sqrt{kg}}\ln\left\lvert\frac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v}\right\rvert \qquad (1)$$

If $v=\frac{U}{2}$ and $k=\frac{g}{U^2}$, then from (1) $t=\frac{U}{2g}\ln 3$.

$$\frac{1}{g-kv^2}=\frac{1}{\left(\sqrt{g}-\sqrt{k}v\right)\left(\sqrt{g}+\sqrt{k}v\right)}$$

$$\frac{1}{\left(\sqrt{g}-\sqrt{k}v\right)\left(\sqrt{g}+\sqrt{k}v\right)}=\frac{A}{\left(\sqrt{g}-\sqrt{k}v\right)}+\frac{B}{\left(\sqrt{g}+\sqrt{k}v\right)}$$

$$v=A\left(\sqrt{g}+\sqrt{k}v\right)+B\left(\sqrt{g}-\sqrt{k}v\right)$$

$$v=\frac{\sqrt{g}}{\sqrt{k}},\quad 1=2A\sqrt{g} \Rightarrow A=\frac{1}{2\sqrt{g}}$$

$$v=-\frac{\sqrt{g}}{\sqrt{k}},\quad 1=2B\sqrt{g} \Rightarrow B=\frac{1}{2\sqrt{g}}$$

$$\frac{v}{g-kv^2}=\frac{1}{2\sqrt{g}\left(\sqrt{g}-\sqrt{k}v\right)}+\frac{1}{2\sqrt{g}\left(\sqrt{g}+\sqrt{k}v\right)}$$

Alternatively:

$$\frac{dv}{dt} = g - kv^2$$

$$\frac{dt}{dv} = \frac{1}{g - kv^2}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$t = \frac{1}{2\sqrt{g}} \int_0^{\frac{U}{2}} \left(\frac{1}{\sqrt{g} - \sqrt{k}v} + \frac{1}{\sqrt{g} + \sqrt{k}v} \right) dv$$

$$= \frac{1}{2\sqrt{kg}} \left[-\ln\left|\sqrt{g} - \sqrt{k}v\right| + \ln\left|\sqrt{g} + \sqrt{k}v\right| \right]$$

$$= \frac{1}{2\sqrt{kg}} \left[\ln\left|\frac{\sqrt{g} + \sqrt{k}v}{\sqrt{g} - \sqrt{k}v}\right| \right]_v^{\frac{U}{2}}$$

$$= \frac{1}{2\sqrt{kg}} \left(\ln\left|\frac{\sqrt{g} + \sqrt{k} \times \frac{U}{2}}{\sqrt{g} - \sqrt{k} \times \frac{U}{2}}\right| - \ln\left|\frac{\sqrt{g}}{\sqrt{g}}\right| \right)$$

$$= \frac{1}{2\sqrt{kg}} \ln\left|\frac{2\sqrt{g} + U\sqrt{k}}{2\sqrt{g} - U\sqrt{k}}\right|$$

$$\frac{1}{g - kv^2} = \frac{1}{\left(\sqrt{g} - \sqrt{k}v\right)\left(\sqrt{g} + \sqrt{k}v\right)}$$

$$\frac{1}{\left(\sqrt{g} - \sqrt{k}v\right)\left(\sqrt{g} + \sqrt{k}v\right)} = \frac{A}{\left(\sqrt{g} - \sqrt{k}v\right)} + \frac{B}{\left(\sqrt{g} + \sqrt{k}v\right)}$$

$$v = A\left(\sqrt{g} + \sqrt{k}v\right) + B\left(\sqrt{g} - \sqrt{k}v\right)$$

$$v = \frac{\sqrt{g}}{\sqrt{k}}, \quad 1 = 2A\sqrt{g} \quad \Rightarrow A = \frac{1}{2\sqrt{g}}$$

$$v = -\frac{\sqrt{g}}{\sqrt{k}}, \quad 1 = 2B\sqrt{g} \quad \Rightarrow B = \frac{1}{2\sqrt{g}}$$

$$\frac{v}{g - kv^2} = \frac{1}{2\sqrt{g}\left(\sqrt{g} - \sqrt{k}v\right)} + \frac{1}{2\sqrt{g}\left(\sqrt{g} + \sqrt{k}v\right)}$$

At terminal velocity:

$$0 = g - kU^2 \Rightarrow \sqrt{k} = \frac{\sqrt{g}}{U}$$

$$\therefore t = \frac{1}{2\sqrt{g}\left(\frac{\sqrt{g}}{U}\right)} \ln\left|\frac{2\sqrt{g} + U\frac{\sqrt{g}}{U}}{2\sqrt{g} - U\frac{\sqrt{g}}{U}}\right|$$

$$= \frac{U}{2g} \ln\frac{3\sqrt{g}}{\sqrt{g}}$$

$$= \frac{U}{2g} \ln 3.$$

Question 16 (15 marks)

16 (a) (i) (2 marks)
Outcomes Assessed: N2.2/MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Demonstrates significant progress towards solution	1

Sample Answer:

$z^7 - 1 = 0 \Rightarrow z^7 = 1$

$z_1 = 1 = \cos 0 + i \sin 0 = e^{0i}$

$z_2 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} = e^{\frac{2\pi}{7}i}$

$z_3 = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} = e^{\frac{4\pi}{7}i}$

$z_4 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7} = e^{\frac{6\pi}{7}i}$

$z_5 = \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7} = \cos\left(-\frac{6\pi}{7}\right) + i\sin\left(-\frac{6\pi}{7}\right) = \overline{z_4} = e^{-\frac{6\pi}{7}i}$

$z_6 = \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7} = \cos\left(-\frac{4\pi}{7}\right) + i\sin\left(-\frac{4\pi}{7}\right) = \overline{z_3} = e^{-\frac{4\pi}{7}i}$

$z_7 = \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7} = \cos\left(-\frac{2\pi}{7}\right) + i\sin\left(-\frac{2\pi}{7}\right) = \overline{z_2} = e^{-\frac{2\pi}{7}i}$

16 (a) (ii) (3 marks)
Outcomes Assessed: N2.2/MEX12-4
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Shows some of the roots by substitution, or equivalent merit	2
• Shows one of the roots by substitution, or equivalent merit	1

Sample Answer:

$z^7 - 1 = 0$

$(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$

$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0 \quad (1)$

$\dfrac{z^6 + z^5 + z^4 + z^3 + z^2 + z + 1}{z^3} = \dfrac{0}{z^3}$

$z^3 + z^2 + z + 1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \dfrac{1}{z^3} = 0$

$\left(z^3 + \dfrac{1}{z^3}\right) + \left(z^2 + \dfrac{1}{z^2}\right) + \left(z + \dfrac{1}{z}\right) + 1 = 0$

since $z^3 + \dfrac{1}{z^3} = 2\cos 3\theta$, $z^2 + \dfrac{1}{z^2} = 2\cos 2\theta$ and $z + \dfrac{1}{z} = 2\cos\theta$

$\Rightarrow \; 2\cos 3\theta + 2\cos 2\theta + 2\cos\theta + 1 = 0 \quad (2)$

The solutions of (1) are the non-real seventh roots of unity, so $\theta = \pm\dfrac{2\pi}{7}, \pm\dfrac{4\pi}{7}, \pm\dfrac{6\pi}{7}$

$\therefore$ Since (1) and (2) are equivalent then $\dfrac{2\pi}{7}$, $\dfrac{4\pi}{7}$ and $\dfrac{6\pi}{7}$ are also solutions to (2).

16 (a) (iii) (2 marks)
Outcomes Assessed: N2.2/MEX12-4
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Substitute $\theta = \frac{2\pi}{7}$ in (ii) OR • Use roots and coefficients to show sum of the roots is $-\frac{1}{2}$ only	1

Sample Answer:

Let $\theta = \frac{2\pi}{7}$ in (ii):

$$\therefore\ 2\cos 3\left(\frac{2\pi}{7}\right) + 2\cos 2\left(\frac{2\pi}{7}\right) + 2\cos\left(\frac{2\pi}{7}\right) + 1 = 0$$

$$2\left(\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right) = -1$$

$$\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$$

16 (b)(i) (3 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Obtains $I_{2n+1} = \frac{1}{2}\left[x^{2n}e^{x^2}\frac{2}{2}\right]_0^1 - n\int_0^1 x^{2n-1}e^{x^2}\,dx$	2
• Obtains correct expressions for u and $\frac{dv}{dx}$	1

Sample Answer:

$$I_{2n+1} = \int_0^1 x^{2n+1}e^{x^2}\,dx$$
$$= \frac{1}{2}\left[x^{2n}e^{x^2}\frac{2}{2}\right]_0^1 - n\int_0^1 x^{2n-1}e^{x^2}\,dx$$
$$= \frac{1}{2}(e-0) - nI_{2n-1}$$
$$= \frac{e}{2} - nI_{2n-1}$$

$u = \frac{1}{2}x^{2n}$	$\frac{dv}{dx} = 2xe^{x^2}$
$\frac{du}{dx} = nx^{2n-1}$	$v = e^{x^2}$

16 (b)(ii) (2 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Criteria	Marks
Provides the correct solution	2
Finds the integral in terms of I_{2n-1} and I_{2n+1}, or equivalent merit	1

Sample Answer:

$$\text{LHS} = 2\int_0^1 x^{2n-1}(1+x^2)e^{x^2}\,dx$$
$$= 2\left[\int_0^1 x^{2n-1}e^{x^2}\,dx + \int_0^1 x^{2n+1}e^{x^2}\,dx\right]$$
$$= 2\left[I_{2n-1} + I_{2n+1}\frac{2}{2}\right]$$
$$\leq 2\left[I_{2n+1}\frac{2}{2} + nI_{2n-1}\frac{2}{2}\right] \qquad I_{2n-1} > 0 \text{ since } x^{2n-1} \geq 0, e^{x^2} > 0 \text{ for } [0,1]$$
$$\leq 2\left[\frac{e}{2} - nI_{2n-1} + nI_{2n-1}\frac{2}{2}\right] \quad \text{from (i)}$$
$$\leq e$$

16 (c) (i) (1 mark)
Outcomes Assessed: M1.4/MEX12-6
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	1

Sample Answer:

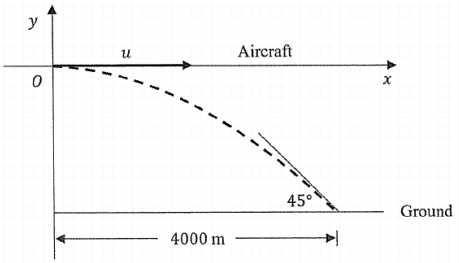
$$x = 10u(1-e^{-0.1t}) \Rightarrow \dot{x} = 0.1\times 10ue^{-0.1t}$$
$$= ue^{-0.1t}$$
$$y = 100g(1-e^{-0.1t}) - 10gt \Rightarrow \dot{y} = 10ge^{-0.1t} - 10g$$
$$= -10g\left(1-e^{-0.1t}\right)$$

$$\therefore \underset{\sim}{v}(t) = \begin{pmatrix} ue^{-0.1t} \\ -10g(1-e^{-0.1t}) \end{pmatrix}$$

16 (c) (ii) (2 marks)
Outcomes Assessed: M1.4/MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Finds the correct time on impact or the velocity on impact	1

Sample Answer:



$x = 10u(1 - e^{-0.1t})$

When $x = 4000$, $4000 = 10u(1 - e^{-0.1t})$

$$u = \frac{400}{1 - e^{-0.1t}} \qquad (1)$$

$$\tan\theta = \frac{\dot{y}}{\dot{x}}$$

$$\Rightarrow \tan(-45°) = \frac{-100(1 - e^{-0.1t})}{ue^{-0.1t}}$$

$$-1 = \frac{-100(1 - e^{-0.1t})}{ue^{-0.1t}} \qquad (2)$$

Sub (1) into (2),

$$-1 = \frac{-100(1-e^{-0.1t})}{\left(\frac{400}{1-e^{-0.1t}}\right)e^{-0.1t}}$$

$$4 = \frac{(1-e^{-0.1t})^2}{e^{-0.1t}}$$

Let $m = e^{-0.1t} \Rightarrow 4m = (1-m)^2$

$$m^2 - 6m + 1 = 0$$

$$m = \frac{6 \pm \sqrt{36-4}}{2}$$

$$= \frac{6 \pm 4\sqrt{2}}{2}$$

$$= 3 \pm 2\sqrt{2}$$

$$e^{-0.1t} = 3 \pm 2\sqrt{2}$$

when $e^{-0.1t} = 3 + 2\sqrt{2}$, $\quad t = -\dfrac{\ln(3+2\sqrt{2})}{0.1} = -17.6$ sec

when $e^{-0.1t} = 3 - 2\sqrt{2}$, $\quad t = -\dfrac{\ln(3-2\sqrt{2})}{0.1} = 17.6$ sec (1 dec)

$$t = 17.6,\ u = \frac{400}{1-e^{-0.1(17.6)}}$$

$$= 483$$

$$\dot{x} = ue^{-0.1t} \Rightarrow \dot{x} = 483e^{-0.1(17.6)}$$

$$= 83.1$$

$$\dot{y} = -100(1-e^{-0.1t}) \Rightarrow \dot{y} = -100(1-e^{-0.1(17.6)})$$

$$= -82.8$$

$$|\underset{\sim}{v}| = \sqrt{|\dot{x}|^2 + |\dot{y}|^2} \Rightarrow |\underset{\sim}{v}| = \sqrt{83.1^2 + (-82.8)^2}$$

$$= 117.2 \text{ m/s (1 dp).}$$

Alternatively:

On impact $x = 4000$ (given) and $\dot{x} = -\dot{y}$ since the angle of impact is 45°.

$$\therefore 10u(1-e^{-0.1t}) = 4000$$

$$1-e^{-0.1t} = \frac{400}{u}$$

$$e^{-0.1t} = \frac{400}{1-u} \qquad (1)$$

Also $\quad ue^{-0.1t} = 100(1-e^{-0.1t}) \qquad (2)$

Sub (1) in (2):

$$u\left(\frac{u-400}{u}\right) = 100\left(1-\frac{u-400}{u}\right)$$

$$u-400 = \frac{100(u-u+400)}{u}$$

$$u-400 = \frac{40000}{u}$$

$$u^2-400u = 40000$$

$$u^2-400u-40000 = 0$$

$$u = \frac{400 \pm \sqrt{400^2-4(1)(-40000)}}{2}$$

$$= \frac{400 \pm 400\sqrt{2}}{2}$$

$$= 200\left(\sqrt{2}+1\right) \text{ since } u > 0$$

Sub in (1):

$$e^{-0.1t} = \frac{200\left(\sqrt{2}+1\right)-400}{2\left(\sqrt{2}+1\right)}$$

$$e^{0.1t} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$t = 10\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\approx 17.6 \text{ sec (1 dp)}$$

On impact $v = \sqrt{2}\dot{x}$ since angle impact is 45°

$\therefore\ v = \sqrt{2}ue^{-0.1t}$

$$= \frac{\sqrt{2}u(u-400)}{u}$$

$$= \sqrt{2}\left(200\left(\sqrt{2}+1\right)-400\right)$$

$$= \sqrt{2}\left(200\left(\sqrt{2}-1\right)\right)$$

$$= 200(2-\sqrt{2})$$

$$\approx 117.2 \text{ m/s (1 dp).}$$

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