Neap

HSC Trial Examination 2020

Mathematics Extension 2

General Instructions	• Reading time – 5 minutes • Working time – 3 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • In Questions 11–16, show relevant mathematical reasoning and/or calculations
Total marks: 100	**Section I – 10 marks** (pages 2–5) • Attempt Questions 1–10 • Allow about 15 minutes for this section **Section II – 90 marks** (pages 6–12) • Attempt Questions 11–16 • Allow about 2 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. Consider the following statement for $n \in Z$:

$$\text{If } n^2 + 4n + 1 \text{ is even, then } n \text{ is odd.}$$

Which of the following statements is the contrapositive of this statement for $n \in Z$?

(A) If n is even, then $n^2 + 4n + 1$ is odd.

(B) If $n^2 + 4n + 1$ is odd, then n is even.

(C) If n is odd, then $n^2 + 4n + 1$ is even.

(D) If $n^2 + 4n + 1$ is even, then n is even.

2. Which of the following expressions is equal to $\int x^2 e^{-x} dx$?

(A) $-x^2 e^{-x} + \int 2xe^{-x} dx$

(B) $-2xe^{-x} - \int 2xe^{-x} dx$

(C) $-x^2 e^{-x} - \int 2xe^{-x} dx$

(D) $-2xe^{-x} + \int 2xe^{-x} dx$

3. Which of the following expressions is the partial fraction form of the algebraic fraction $\dfrac{x-4}{(x-3)^2(x^2+2)}$?

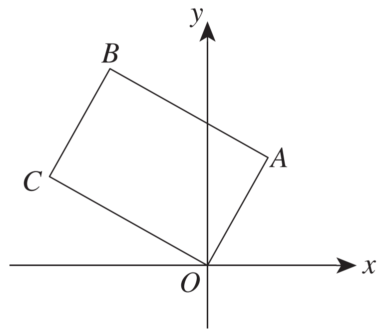
(A) $\dfrac{A}{(x-3)^2} + \dfrac{B}{x^2+2}$

(B) $\dfrac{A}{(x-3)^2} + \dfrac{Bx+C}{x^2+2}$

(C) $\dfrac{A}{x-3} + \dfrac{B}{(x-3)^2} + \dfrac{C}{x^2+2}$

(D) $\dfrac{A}{x-3} + \dfrac{B}{(x-3)^2} + \dfrac{Cx+D}{x^2+2}$

4. The Argand diagram shows the rectangle $OABC$ where $OC = 2OA$. Vertex A corresponds to the complex number w.



Which of the following complex numbers corresponds to vertex C?

(A) $-2iw$

(B) $2iw$

(C) $-2w$

(D) $2w$

5. If x, y and z are any real numbers and $x > y$, which of the following statements must be true?

(A) $x^2 > y^2$

(B) $\frac{1}{x} < \frac{1}{y}$

(C) $x + z > y + z$

(D) $xz > yz$

6. A particle moves in simple harmonic motion along the x-axis about the origin. Initially, the particle is at its extreme positive position. The amplitude of the motion is 12 metres and the particle returns to its initial position every 3 seconds.

What is the equation for the position of the particle at time t seconds?

(A) $x = 12\cos\frac{2\pi t}{3}$

(B) $x = 24\cos\frac{2\pi t}{3}$

(C) $x = 12\cos 3t$

(D) $x = 24\cos 3t$

7. What is the Cartesian equation of a sphere with centre $\underset{\sim}{c} = -3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ that passes through $\underset{\sim}{a} = 3\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$?

(A) $(x-3)^2 + (y+1)^2 + (z-2)^2 = 7$

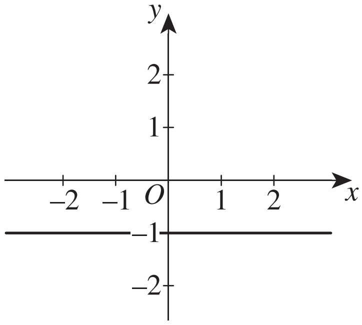
(B) $(x+3)^2 + (y-1)^2 + (z+2)^2 = 7$

(C) $(x-3)^2 + (y+1)^2 + (z-2)^2 = 49$

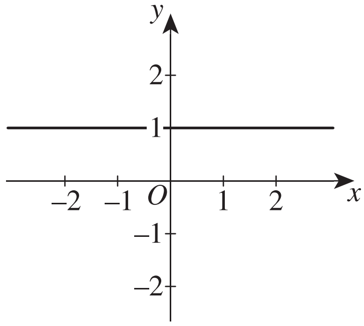
(D) $(x+3)^2 + (y-1)^2 + (z+2)^2 = 49$

8. Which of the following diagrams shows the subset of the complex plane satisfied by the relation $i\bar{z} - iz = 2$?

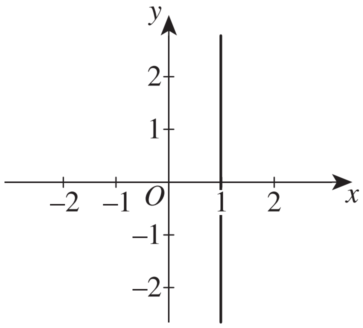
(A)



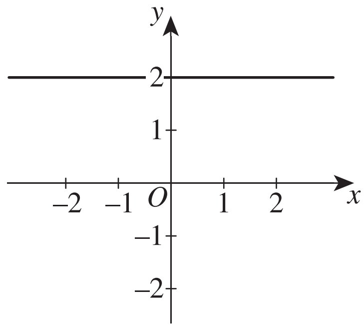
(B)



(C)



(D)



9. A particle is moving along a straight line. At time t, its velocity is v and its displacement from a fixed origin is x.

If $\dfrac{dv}{dx} = \dfrac{1}{2v}$, which of the following best describes the particle's acceleration and velocity?

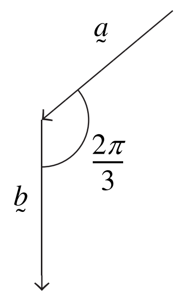
(A) constant acceleration and constant velocity

(B) constant acceleration and decreasing velocity

(C) constant acceleration and increasing velocity

(D) increasing acceleration and increasing velocity

10. In the diagram, the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are such that $|\underset{\sim}{a}| = |\underset{\sim}{b}|$.



Given that $|\underset{\sim}{a}| = a$, which of the following expressions is equal to $\underset{\sim}{a} \cdot \underset{\sim}{b}$?

(A) $-\frac{\sqrt{3}}{2}a^2$

(B) $-\frac{1}{2}a^2$

(C) $\frac{1}{2}a^2$

(D) $\frac{\sqrt{3}}{2}a^2$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find the value of b given that $-i$ is a root of the equation $z^2 + bz + (1 - i) = 0$. **2**

(b) Consider the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = m\underset{\sim}{i} + n\underset{\sim}{j}$.

(i) Find the values of m and n such that $(\underset{\sim}{a} + \underset{\sim}{c})$ is parallel to $\underset{\sim}{b}$. **2**

(ii) Find the values of m and n such that $\underset{\sim}{c}$ is a unit vector perpendicular to $\underset{\sim}{a}$. **2**

(c) Using the substitution $u = 1 - \sin 2x$, evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 - \sin 2x}\,(1 - 2\cos^2 x)\,dx$. **4**

(d) Let $z = \sqrt{3} + i$.

(i) Express z in modulus-argument form. **2**

(ii) Find the smallest positive integer n such that $z^n - \bar{z}^n = 0$. **3**

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Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider two complex numbers, u and v, such that $\text{Im}(u) = 2$ and $\text{Re}(v) = -1$. **2**

Given that $u + v = -uv$, find the values of u and v.

(b) Solve the equation $\left|e^{2i\theta} + e^{-2i\theta}\right| = 1$ where $-\pi < \theta \leq \pi$. **3**

(c) Given that $y = \dfrac{1}{1+x}$, prove by mathematical induction that $\dfrac{d^n y}{dx^n} = \dfrac{(-1)^n n!}{(1+x)^{n+1}}$ for all positive integers n. **3**

(d) A subset of the complex plane is described by the relation $\text{Arg}(z - 2i) = \dfrac{\pi}{6}$.

(i) Show that the Cartesian equation of this relation is $y = \dfrac{1}{\sqrt{3}}x + 2, x > 0$. **2**

(ii) Draw a sketch of this relation. **1**

(iii) Given that z is a complex number that satisfies the relation $\text{Arg}(z - 2i) = \dfrac{\pi}{6}$, find the least possible exact value of $|z - 3 + i|$. **4**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Prove that $\log_2 5$ is an irrational number. **3**

(b) Relative to a fixed origin, O, a particle is moving in a straight line with simple harmonic motion of period $\dfrac{2\pi}{n}$ seconds and amplitude a metres. Initially, the particle is $\dfrac{a}{2}$ metres from O and is moving away from O.

(i) Find an expression for the particle's displacement, x, at time t. Give your answer in the form $x = a\sin(nt + \alpha)$. **1**

(ii) Find the time when the particle will first reach an extreme position. **1**

(iii) The particle has speed V m s^{-1} when it is $\dfrac{a}{3}$ metres from an extreme position. **2**

Find, in terms of V, the particle's maximum speed.

(c) Consider two lines, l_1 and l_2, with vector equations $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ respectively.

(i) Find $\underset{\sim}{r}_1$, the vector equation of l_1, in the direction of $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and passing through the point $(-1, 2, -3)$. **1**

The line l_2 has the vector equation $\underset{\sim}{r}_2 = (-t + 1)\underset{\sim}{i} + (2t - 2)\underset{\sim}{j} + (3t + 6)\underset{\sim}{k}$ where $t \in R$.

(ii) Find a vector parallel to l_2. **1**

(iii) Find the point of intersection of l_1 and l_2. **3**

(iv) Find the acute angle between l_1 and l_2. Give your answer in degrees correct to one decimal place. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) From an origin O, a parachutist of mass m kg falls vertically downwards. The forces acting on the parachutist are his weight and a force due to air resistance of magnitude mkv^2 newtons, where k is a constant and v m s^{-1} is the parachutist's velocity. Let x be the parachutist's displacement in metres below O.

(i) Show that the parachutist's equation of motion is $\ddot{x} = g - kv^2$ where g is the acceleration of gravity. **1**

(ii) Use integration to show that $v^2 = \frac{g}{k}(1 - e^{-2kx})$. **2**

(iii) Given that the parachutist's terminal velocity is $6g$ m s^{-1}, find the value of k. **1**

When the parachutist's velocity is $5g$ m s^{-1}, he opens his parachute. The motion is now modelled by assuming that the magnitude of the force due to air resistance instantaneously changes to $\frac{mgv}{10}$ newtons. The time from the parachute opening is t seconds.

(iv) Use integration to show that $v = 10 + 5(g - 2)e^{-\frac{gt}{10}}$. **3**

(v) It takes T seconds for the parachutist's velocity to decrease to $2g$ m s^{-1}. **1**

Show that $T = \frac{10}{g}\ln\left[\frac{5(g-2)}{2(g-5)}\right]$ seconds.

(b) Consider the equation $z^5 = (z + 1)^5$ where $z \in \mathbb{C}$.

(i) Explain why this equation does NOT have five roots. **1**

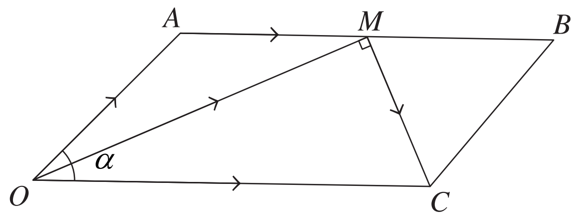
(ii) Solve $z^5 = (z + 1)^5$, giving your answer in the form $a + bi\cot\theta$. **4**

(iii) Describe the geometrical relationship between the roots of the equation $iz^5 = (iz + 1)^5$ and the roots of the equation $z^5 = (z + 1)^5$. **2**

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows parallelogram $OABC$ where $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{b}$ and $\left|\overrightarrow{OC}\right| = 2\left|\overrightarrow{OA}\right|$. The angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is α.

M is a point on AB such that $\overrightarrow{AM} = k\overrightarrow{AB}$, $0 \le k \le 1$ and $\overrightarrow{OM} \cdot \overrightarrow{MC} = 0$.



(i) Use a vector method to show that $|\underset{\sim}{a}|^2(1-2k)(2\cos\alpha - (1-2k)) = 0$. **4**

(ii) Find the set of values for α such that there are two possible positions for M. **2**

(b) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{2}{3 + 5\cos x}\,dx$.

(i) Using the substitution $t = \tan\dfrac{x}{2}$, show that $I = \displaystyle\int_0^1 \frac{2}{4 - t^2}\,dt$. **2**

(ii) Hence find the value of I. Give your answer in the form $\ln\sqrt{k}$ where k is a positive integer. **2**

Question 15 continues on page 11

Question 15 (continued)

(c) A particle is projected from a point O above horizontal ground. At time t seconds, the particle's position vector is $\underset{\sim}{r} = gt\cos\theta\underset{\sim}{i} + \left(\frac{g}{4} + gt\sin\theta - \frac{g}{2}t^2\right)\underset{\sim}{j}$ where θ is the angle of projection and g is the acceleration due to gravity.

(i) The particle's time of flight is T seconds. **3**

Show that $T = \frac{1}{\sqrt{2}}(\sqrt{1-\cos 2\theta} + \sqrt{2-\cos 2\theta})$.

(ii) The particle's range is R metres. **1**

Show that $R = \frac{g}{2}(\sqrt{1-\cos^2 2\theta} + \sqrt{2+\cos 2\theta - \cos^2 2\theta})$.

(iii) The particle's maximum range occurs when $\cos 2\theta = \frac{1}{5}$. (Do NOT prove this.) **1**

Find the extra distance attained by projecting the particle at this angle rather than at an angle of 45°. Give your answer in the form $\frac{g}{2}(\sqrt{a} - \sqrt{b} - c)$ where a, b and c are positive integers.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int_0^1 x^n \tan^{-1} x\,dx$ where $n = 0, 1, 2, \ldots$.

(i) Show that $(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx$ for $n \geq 0$. **2**

(ii) Hence, or otherwise, find the value of I_0. **1**

(iii) Show that $(n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \frac{1}{n+2}$. **2**

(iv) Hence find the value of I_4. **2**

(b) Consider two positive real numbers a_1 and a_2.

(i) Prove that $\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$. **1**

Let $a_1, a_2, \ldots, a_n$ be n positive real numbers.

If $a_1 a_2 \ldots a_n = 1$ then $a_1 + a_2 + \ldots + a_n \geq n$. (Do NOT prove this.)

(ii) Prove that $\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$. **3**

(iii) Hence prove that $2^n - 1 > n\sqrt{2^{n-1}}$ for integers $n \geq 1$. **4**

End of paper

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\text{and } \alpha\beta\gamma = -\frac{d}{a}$$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2}ab\sin C$

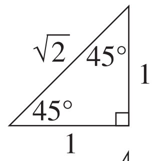
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

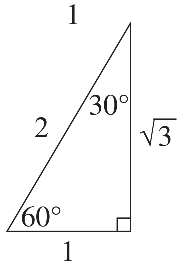
$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$





Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}[\cos(A - B) + \cos(A + B)]$

$\sin A\sin B = \dfrac{1}{2}[\cos(A - B) - \cos(A + B)]$

$\sin A\cos B = \dfrac{1}{2}[\sin(A + B) + \sin(A - B)]$

$\cos A\sin B = \dfrac{1}{2}[\sin(A + B) - \sin(A - B)]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

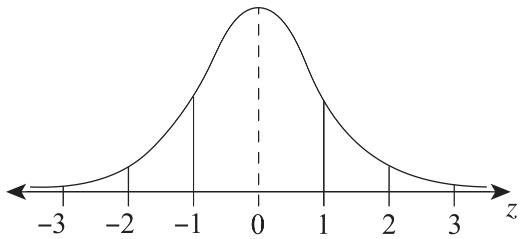
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E[(X - \mu)^2] = E(X^2) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, P(B) \neq 0$

Continuous random variables

$P(X \leq r) = \displaystyle\int_a^r f(x)dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n - r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow P(X = x)$

$= \dbinom{n}{x}p^x(1 - p)^{n - x}, x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

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Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1 - [f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1 - [f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1 + [f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x) \sin f(x) dx = -\cos f(x) + c$$

$$\int f'(x) \cos f(x) dx = \sin f(x) + c$$

$$\int f'(x) \sec^2 f(x) dx = \tan f(x) + c$$

$$\int f'(x) e^{f(x)} dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$$\int f'(x) a^{f(x)} dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx} dx = uv - \int v\frac{du}{dx} dx$$

$$\int_a^b f(x) dx$$

$$\approx \frac{b-a}{2n}\{f(a) + f(b) + 2[f(x_1) + \ldots + f(x_{n-1})]\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$^nP_r = \dfrac{n!}{(n-r)!}$

$\dbinom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

$(x+a)^n = x^n + \dbinom{n}{1}x^{n-1}a + \ldots + \dbinom{n}{r}x^{n-r}a^r + \ldots + a^n$

Vectors

$|\underset{\sim}{u}| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta = x_1x_2 + y_1y_2$,

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

Complex Numbers

$z = a + ib = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

$= r^n e^{in\theta}$

Mechanics

$\dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$x = a\cos(nt + \alpha) + c$

$x = a\sin(nt + \alpha) + c$

$\ddot{x} = -n^2(x - c)$

Neap

HSC Trial Examination 2020

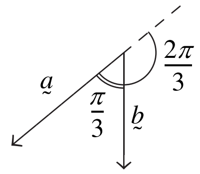
Mathematics Extension 2

Solutions and marking guidelines

Section I

Sample answer	Syllabus content, outcomes and targeted performance bands
Question 1 **A** Let P be $n^2 + 4n + 1$ is even and let Q be n is odd. The contrapositive of $P \Rightarrow Q$ is $(\sim Q) \Rightarrow (\sim P)$. So the contrapositive is "If n is even, then $n^2 + 4n + 1$ is odd".	MEX-P1 The Nature of Proof MEX12–8 Bands E2–E3
Question 2 **A** Integration by parts takes the form $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Let $u = x^2$ and $\frac{dv}{dx} = e^{-x}$. So $\frac{du}{dx} = 2x$ and $v = -e^{-x}$. Substituting into $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ gives: $\int x^2e^{-x}dx = -x^2e^{-x} - \int(-e^{-x})(2x)dx$ So $\int x^2e^{-x}dx = -x^2e^{-x} + \int 2xe^{-x}dx$.	MEX-C1 Further Integration MEX12–5 Bands E2–E3
Question 3 **D** In general, if the factor in the denominator is $(ax+b)^n$, then the corresponding term(s) in the partial fraction decomposition is $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$. Here, $(x-3)^2$ corresponds to $\frac{A}{x-3} + \frac{B}{(x-3)^2}$. If the factor in the denominator is $ax^2 + bx + c$ (with no linear factors), then the corresponding term in the partial fraction decomposition is $\frac{Cx+D}{ax^2+bx+c}$. So the partial fraction form is $\frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+2}$.	MEX-C1 Further Integration MEX12–5 Bands E2–E3
Question 4 **B** Geometrically, to form OC from OA, we rotate OA anticlockwise through an angle of $\frac{\pi}{2}$ and then double the length of OA. Anticlockwise rotation through $\frac{\pi}{2}$ is equivalent to multiplying by i, and doubling the length is achieved by multiplying by 2. So the complex number that corresponds to vertex C is $2iw$.	MEX-N1 Introduction to Complex Numbers MEX12–4 Bands E2–E3

Sample answer	Syllabus content, outcomes and targeted performance bands
Question 5 **C** To disprove a statement of the form $P \Rightarrow Q$, we require a counterexample for which P is true and Q is not true. Here P is $x > y$. For **A**, Q is $x^2 > y^2$. For example, with $x = -2$ and $y = -3$, $-2 > -3$ is true but $4 > 9$ is not true, and so Q is not true. **A** is incorrect. For **B**, Q is $\frac{1}{x} < \frac{1}{y}$. For example, with $x = 3$ and $y = -2$, $3 > -2$ is true but $\frac{1}{3} < -\frac{1}{2}$ is not true and so Q is not true. **B** is incorrect. For **C**, Q is $x + z > y + z$. An inequality is unchanged when the same amount is added to both sides, so Q is true and hence **C** is correct. For **D**, Q is $xz > yz$. For example, with $x = 3$, $y = 2$ and $z = -2$, $3 > 2$ is true $-6 > -4$ is not true and so Q is not true. **D** is incorrect.	MEX-P1 The Nature of Proof MEX12–2 Bands E2–E3
Question 6 **A** The period is 3 seconds. $3 = \frac{2\pi}{n} \Rightarrow n = \frac{2\pi}{3}$ When $t = 0$, $x = 12$. So the equation of motion is $x = 12\cos\frac{2\pi t}{3}$.	MEX-M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4
Question 7 **D** The radius of the sphere is given by $r^2 = \lvert\underset{\sim}{a} - \underset{\sim}{c}\rvert^2$. $\underset{\sim}{a} - \underset{\sim}{c} = (3\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}) - (-3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$ $= 6\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ $r^2 = 6^2 + 2^2 + 3^2$ $= 49$ The equation of the sphere is $\lvert\underset{\sim}{v} - \underset{\sim}{c}\rvert^2 = r^2$ where $\underset{\sim}{v} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$. In Cartesian form, the equation of the sphere is $(x+3)^2 + (y-1)^2 + (z+2)^2 = 49$.	MEX-V1 Further Work with Vectors MEX12–3 Bands E3–E4
Question 8 **B** If $z = x + yi$, then $\bar{z} = x - yi$. Substituting these into $i\bar{z} - iz = 2$ gives: $i(x - yi) - i(x + yi) = 2$ $-2i^2 y = 2$ $2y = 2$ $y = 1$	MEX-N1 Introduction to Complex Numbers MEX12–4 Bands E3–E4

Sample answer	Syllabus content, outcomes and targeted performance bands
Question 9 **C** $\frac{dv}{dx}=\frac{1}{2v}$ and so $v\frac{dv}{dx}=a=v\left(\frac{1}{2v}\right)=\frac{1}{2}$. Therefore, the acceleration is constant. Since the acceleration is also positive, the velocity is increasing.	MEX-M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4
Question 10 **C** The angle between the two vectors is defined as the angle between their positive directions – the angle formed when the two vectors are placed ‘tail-to-tail’.  Hence the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\pi-\frac{2\pi}{3}=\frac{\pi}{3}$. $\underset{\sim}{a}\cdot\underset{\sim}{b}=\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta$ $=a^2\cos\frac{\pi}{3}$ since $\lvert\underset{\sim}{a}\rvert=\lvert\underset{\sim}{b}\rvert$ and $\lvert\underset{\sim}{a}\rvert=a$ $=\frac{1}{2}a^2$ since $\cos\frac{\pi}{3}=\frac{1}{2}$	MEX-V1 Further Work with Vectors MEX12–3 Bands E3–E4

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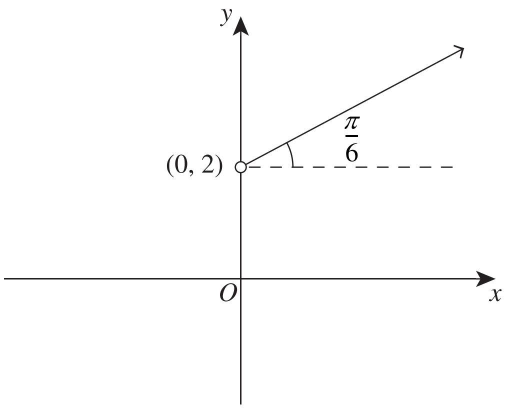
Section II

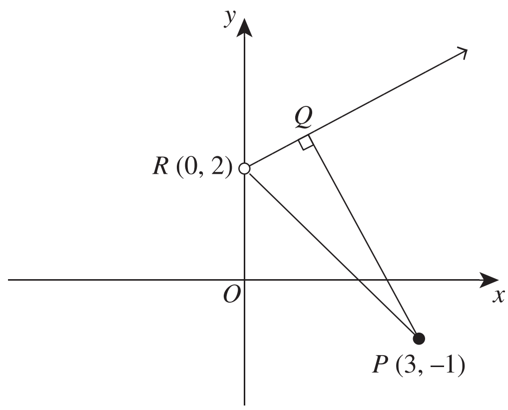
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 11	
(a) Let the other root be β. $-i\beta = 1 - i$ (product of roots) $\beta = \frac{1-i}{-i} \times \frac{i}{i}$ $= 1 + i$ $-i + 1 + i = -b$ (sum of roots) $b = -1$	MEX-N1 Introduction to Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution . . . . .2 • Uses product of roots to attempt to find the other root 1
(b) (i) $\underset{\sim}{a} + \underset{\sim}{c} = (2+m)\underset{\sim}{i} + (2+n)\underset{\sim}{j} + \underset{\sim}{k}$ As $\underset{\sim}{a} + \underset{\sim}{c}$ is parallel to $\underset{\sim}{b}$, $\underset{\sim}{a} + \underset{\sim}{c} = \lambda\underset{\sim}{b}$. $\lambda\underset{\sim}{b} = 2\lambda\underset{\sim}{j} + 2\lambda\underset{\sim}{k}$ $(2+m)\underset{\sim}{i} + (2+n)\underset{\sim}{j} + \underset{\sim}{k} = 2\lambda\underset{\sim}{j} + 2\lambda\underset{\sim}{k}$ Equating components of $\underset{\sim}{a} + \underset{\sim}{c}$ and $\lambda\underset{\sim}{b}$ and solving for λ gives $\lambda = \frac{1}{2}$. So $2 + m = 0$ and $2 + n = 1$. Hence $m = -2$ and $n = -1$.	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . .2 • Obtains correct expressions for $\underset{\sim}{a} + \underset{\sim}{c}$ and $\lambda\underset{\sim}{b}$ 1
(ii) If $\underset{\sim}{c}$ is a unit vector, then $\lvert\underset{\sim}{c}\rvert = 1$, and so $\sqrt{m^2 + n^2} = 1$. Squaring both sides gives $m^2 + n^2 = 1$. (1) Applying the condition $\underset{\sim}{c} \cdot \underset{\sim}{a} = 0$ gives: $(m\underset{\sim}{i} + n\underset{\sim}{j}) \cdot (2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) = 0$ $2m + 2n = 0$ $m = -n$ (2) Substituting (2) into (1) and solving for n gives: $2n^2 = 1 \Rightarrow n = \pm\frac{1}{\sqrt{2}}$ Substituting into (2) gives: $m = -\frac{1}{\sqrt{2}}, n = \frac{1}{\sqrt{2}}$ or $m = \frac{1}{\sqrt{2}}, n = -\frac{1}{\sqrt{2}}$	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . .2 • Obtains two valid equations in m and n. . .1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) $u = 1 - \sin 2x$ $du = -2\cos 2x\,dx$ $= -2(2\cos^2 x - 1)dx$ $= 2(1 - 2\cos^2 x)dx$ When $x = \frac{\pi}{4},\ u = 0$ and when $x = \frac{\pi}{2},\ u = 1$. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 - \sin 2x}(1 - 2\cos^2 x)dx = \int_0^1 u^{\frac{1}{2}}\left(\frac{1}{2}du\right)$ $= \frac{1}{2} \times \frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^1$ $= \frac{1}{3}(1 - 0)$ $= \frac{1}{3}$	MEX-C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution . . . . . 4 • Obtains appropriate integral OR equivalent merit . . . . . 3 • Correctly rewrites integrand and limits OR equivalent merit . . . . . 2 • Makes suggested substitution and rewrites integrand 1
(d) (i) $\lvert z\rvert = \sqrt{(\sqrt{3})^2 + 1^2}$ $= 2$ $\text{Arg}\, z = \tan^{-1}\frac{1}{\sqrt{3}}$ $= \frac{\pi}{6}$ So $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$.	MEX-N1 Introduction to Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution . . . . . 2 • Gives exact argument in radians OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $\bar{z} = \sqrt{3} - i$ $= 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$	MEX-N1 Introduction to Complex Numbers MEX12–4 Bands E2–E4 • Gives the correct solution 3
Applying de Moivre's theorem gives $z^n = 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right)$ and	• Obtains $z^n - \bar{z}^n = 2^{n+1} i\sin\frac{n\pi}{6}$ OR equivalent merit..................2
$\bar{z}^n = 2^n\left(\cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)\right)$. Using $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ to form $z^n - \bar{z}^n$ gives: $z^n - \bar{z}^n = 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right) -$ $2^n\left(\cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)\right)$ $= 2^n\cos\frac{n\pi}{6} - 2^n\cos\frac{n\pi}{6} + 2^n i\sin\frac{n\pi}{6} + 2^n i\sin\frac{n\pi}{6}$ $= 2^{n+1} i\sin\frac{n\pi}{6}$ $z^n - \bar{z}^n = 0 \Rightarrow \sin\frac{n\pi}{6} = 0$ $\sin\frac{n\pi}{6} = \sin k\pi$ $k = 1 \Rightarrow n = 6$ Hence the smallest possible integer is 6.	• Applies de Moivre's theorem on z and $\bar{z}$ to form expressions for z^n and $\bar{z}^n$ 1
Question 12	
(a) Let $u = u_1 + 2i$ and $v = -1 + v_2 i$. $u + v = (u_1 - 1) + (2 + v_2)i$ (1) $-uv = (u_1 + 2v_2) + (2 - u_1 v_2)i$ (2)	MEX-N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution 2
Equating the real components of (1) and (2) gives $u_1 - 1 = u_1 + 2v_2 \Rightarrow v_2 = -\frac{1}{2}$. Equating the imaginary components of (1) and (2) with $v_2 = -\frac{1}{2}$ gives $-\frac{1}{2} = \frac{1}{2}u_1 \Rightarrow u_1 = -1$. So the two complex numbers are $u = -1 + 2i$ and $v = -1 - \frac{1}{2}i$.	• Uses $u + v$ and $-uv$ to form two valid equations......................1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) $\left\lvert e^{2i\theta}+e^{-2i\theta}\right\rvert = 1$ Use of Euler's formula and $\cos(-2\theta)=\cos 2\theta$ and $\sin(-2\theta)=-\sin 2\theta$ gives: $\lvert\cos 2\theta + i\sin 2\theta + \cos(-2\theta) + i\sin(-2\theta)\rvert = 1$ $\lvert\cos 2\theta + \cos 2\theta + i\sin 2\theta - i\sin 2\theta\rvert = 1$ $2\lvert\cos 2\theta\rvert = 1$ So $\cos 2\theta = \pm\frac{1}{2}$ and $-2\pi < 2\theta \le 2\pi$. $\cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = \pm\frac{\pi}{3}, \pm\left(2\pi - \frac{\pi}{3}\right)$ and $\cos 2\theta = -\frac{1}{2} \Rightarrow 2\theta = \pm\frac{2\pi}{3}, \pm\left(2\pi - \frac{2\pi}{3}\right)$ $\theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$	MEX-N2 Using Complex Numbers MEX12–7, 12–8 Bands E2–E4 • Finds all EIGHT correct values for θ. . . . 3 • Obtains $\cos 2\theta = \pm\frac{1}{2}$ . . . . 2 • Uses Euler's formula OR equivalent merit 1
(c) Consider $n = 1$. $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$ and $\frac{(-1)^1 1!}{(1+x)^{1+1}} = -\frac{1}{1+x^2} = \frac{dy}{dx}$ True when $n = 1$. Suppose true for $n = k$. So $\frac{d^k y}{dx^k} = \frac{(-1)^k k!}{(1+x)^{k+1}}$. Required to show it is true for $n = k+1$. That is, $\frac{d^{k+1}y}{dx^{k+1}} = \frac{(-1)^{k+1}(k+1)!}{(1+x)^{k+2}}$. $\text{LHS} = \frac{d^{k+1}y}{dx^{k+1}}$ $= \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$ $= \frac{d}{dx}\left(\frac{(-1)^k k!}{(1+x)^{k+1}}\right)$ $= (-1)^k k!(-(k+1)(1+x)^{-(k+2)})$ $= \frac{(-1)^{k+1}(k+1)!}{(1+x)^{k+2}}$ $= \text{RHS}$ If true for $n = k$, then true for $n = k+1$. Hence, by mathematical induction, true for $n \ge 1$.	MEX-P2 Further Proof by Mathematical Induction MEX12–2 Bands E2–E3 • Gives the correct proof . . . . 3 • Establishes an inductive step OR equivalent merit . . . . 2 • Establishes the $n = 1$ case OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(d) (i) Substituting $z = x + yi$ into $\text{Arg}(z - 2i) = \frac{\pi}{6}$ gives: $\text{Arg}(x + (y-2)i) = \frac{\pi}{6}$ $\frac{y-2}{x} = \tan\frac{\pi}{6}$ $\frac{y-2}{x} = \frac{1}{\sqrt{3}}$ $y - 2 = \frac{1}{\sqrt{3}}x$ $y = \frac{1}{\sqrt{3}}x + 2,\ x > 0$	MEX-N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution . . . . .2 • Substitutes $z = x + yi$ and attempts to express y in terms of x OR equivalent merit 1
(ii) 	MEX-N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Correctly sketches the relation 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) 	MEX-N2 Using Complex Numbers MEX12–4 Bands E3–E4 • Gives the correct solution . . . . 4 • Obtains the exact value of $\sin\frac{5\pi}{12}$ . . . . . . 3 • Obtains RP and $\angle QRP$ OR equivalent merit . . . . 2 • Recognises that the minimum distance from P to Q occurs when $\angle RQP = \frac{\pi}{2}$ . . . 1

The complex number z lies on the ray $\text{Arg}(z - 2i) = \frac{\pi}{6}$.

$|z - 3 + i|$ is the distance from the point $P(3, -1)$ to the ray.

The minimum distance from point P to a point Q on the ray occurs when $\angle RQP = \frac{\pi}{2}$. This is the condition for the least possible value of $|z - 3 + i|$.

$$RP = \sqrt{3^2 + 3^2} = 3\sqrt{2}$$

$$\angle QRP = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$$

$$\frac{QP}{3\sqrt{2}} = \sin\frac{5\pi}{12}$$

The compound angle formula $\sin(A + B) = \sin A\cos B + \cos A\sin B$ can be used to find the exact value of $\sin\frac{5\pi}{12}$.

$$\begin{aligned}\sin\frac{5\pi}{12} &= \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\\ &= \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3} + 1}{2\sqrt{2}}\end{aligned}$$

$$\begin{aligned}QP &= \frac{3\sqrt{2}(\sqrt{3} + 1)}{2\sqrt{2}}\\ &= \frac{3(\sqrt{3} + 1)}{2}\end{aligned}$$

So the least possible exact value of $|z - 3 + i|$ is $\frac{3(\sqrt{3} + 1)}{2}$.

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 13	
(a) Assume that there exists $p, q \in N$ such that $\log_2 5 = \frac{p}{q}$ where $q \geq 1$ and the highest common factor of p and q is 1. $\log_2 5 = \frac{p}{q} \Rightarrow 5 = 2^{\frac{p}{q}}$ Taking the qth power of both sides gives $5^q = 2^p$. **EITHER:** Therefore, 5 is a factor of 5^q but not a factor of 2^p. **OR** 2 is a factor of 2^p but not a factor of 5^q. **OR** 5^q is odd and 2^p is even. Then: No p and q such that $p, q \in N$ satisfies the equation $5^q = 2^p$ and this equation must be a contradiction. So $\log_2 5$ is an irrational number.	MEX-P1 The Nature of Proof MEX12–2 Bands E2–E3 • Gives the correct proof .... 3 • Determines $\log_2 5 = \frac{p}{q} \Rightarrow 5 = 2^{\frac{p}{q}} \Rightarrow 5^q = 2^p$ .... 2 • Attempts proof by contradiction 1
(b) (i) $x = a\sin(nt + \alpha)$ When $t = 0$, $x = \frac{a}{2}$ and so $\frac{a}{2} = a\sin\alpha \Rightarrow \alpha = \frac{\pi}{6}$. So $x = a\sin\left(nt + \frac{\pi}{6}\right)$.	MEX-M1 Applications of Calculus to Mechanics MEX12–6, MEX12–7 Bands E2–E4 • Gives the correct solution............. 1
(ii) Find the value of t when $x = a$. $a = a\sin\left(nt + \frac{\pi}{6}\right) \Rightarrow \sin\left(nt + \frac{\pi}{6}\right) = 1$ $nt + \frac{\pi}{6} = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{3n}$	MEX-M1 Applications of Calculus to Mechanics MEX12–6, MEX12–7 Bands E2–E4 • Gives the correct solution............. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) $v^2 = n^2(a^2 - x^2)$ and when $x = \frac{2a}{3}$, $\lvert v\rvert = V$. *Note: $v^2 = n^2(a^2 - x^2)$ can be derived from the differential equation $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x - c)$, where $c = 0$.* $V^2 = \frac{5n^2a^2}{9}$ $a^2 = \frac{9V^2}{5n^2}$ $a = \frac{3V}{\sqrt{5}n}$ Using $v_{\max} = na$ gives: $v_{\max} = n\left(\frac{3V}{\sqrt{5}n}\right)$ $= \frac{3V}{\sqrt{5}}$ (m s^{-1})	MEX-M1 Applications of Calculus to Mechanics MEX12–6, MEX12–7 Bands E2–E4 • Gives the correct solution . . . . 2 • Obtains a valid expression for V^2 in terms of a and n 1
(c) (i) The position vector of the point $(-1, 2, -3)$ is $\begin{pmatrix}-1\\2\\-3\end{pmatrix}$. So the vector equation of l_1 is $\underset{\sim}{r}_1 = \begin{pmatrix}-1\\2\\-3\end{pmatrix} + s\begin{pmatrix}1\\-2\\2\end{pmatrix}$.	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution 1
(ii) The vector equation of l_2 can be written as $\underset{\sim}{r}_2 = \begin{pmatrix}1\\-2\\6\end{pmatrix} + t\begin{pmatrix}-1\\2\\3\end{pmatrix}$. So $\begin{pmatrix}-1\\2\\3\end{pmatrix}$ is a vector parallel to l_2.	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution 1

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Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) At the point of intersection, $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$. Equating the three components: $s - 1 = -t + 1 \Rightarrow s + t = 2 \quad (1)$ $-2s + 2 = 2t - 2 \Rightarrow -2s - 2t = -4 \quad (2)$ $2s - 3 = 3t + 6 \Rightarrow 2s - 3t = 9 \quad (3)$ (2) + (3) gives: $-5t = 5 \Rightarrow t = -1$ Substituting $t = -1$ into (3) and solving gives: $2s + 3 = 9 \Rightarrow s = 3$ Substituting $s = 3$ and $t = -1$ into (1) gives: $3 - 1 = 2$ Hence the lines intersect. Thus from $\underset{\sim}{r}_2$: $\underset{\sim}{r}_2 = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} - 1\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$ So the point of intersection is $(2, -4, 3)$. Note that substituting $s = 3$ into $\underset{\sim}{r}_1$ gives: $\underset{\sim}{r}_1 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$ This gives the same point.	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . .3 • Attempts to solve two of the equations AND checks that the lines intersect. . . . . .2 • Equates three components AND forms three valid equations 1
(iv) The angle between l_1 and l_2 is the angle between the two direction vectors of the lines. l_1 is parallel to $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and l_2 is parallel to $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. Let θ be the required angle. $\cos\theta = \dfrac{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\right\|\left\|\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}\right\|}$ $= \dfrac{-1 - 4 + 6}{\sqrt{1^2 + (-2)^2 + 2^2} \times \sqrt{(-1)^2 + 2^2 + 3^2}}$ $= \dfrac{1}{3\sqrt{14}}$ So $\theta = \cos^{-1}\left(\dfrac{1}{3\sqrt{14}}\right)$. Converting to degrees, $\theta = 84.9°$ (correct to one decimal place).	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . .3 • Obtains the correct value for $\cos\theta$ . . . . . .2 • Attempts to find the angle between the two direction vectors 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 14	
(a) (i) The forces acting on the parachutist are the force due to weight (acting in the same direction as the motion) and the force due to air resistance (acting in the opposite direction to the motion). From Newton's second law, the equation of motion is $m\ddot{x} = mg - mkv^2$. Hence $\ddot{x} = g - kv^2$.	MEX-M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution 1
(ii) $\ddot{x} = v\dfrac{dv}{dx} = g - kv^2$ Separating variables gives: $\int dx = \int \dfrac{v}{g - kv^2} dv$ Using integration by substitution (or recognition) gives: $x + c_1 = -\dfrac{1}{2k}\ln\left\|g - kv^2\right\|$ $Ae^{-2kx} = g - kv^2$ where $A = e^{-2kc_1}$ When $x = 0$, $v = 0$ and so $A = g$. $ge^{-2kx} = g - kv^2$ $kv^2 = g(1 - e^{-2kx})$ $v^2 = \dfrac{g}{k}(1 - e^{-2kx})$	MEX-C1 Further Integration MEX12–5, 12–7 Bands E2–E4 • Gives the correct solution . . . . 2 • Obtains $x + c_1 = -\dfrac{1}{2k}\ln\left\|g - kv^2\right\|$ OR equivalent merit 1
(iii) Terminal velocity is achieved when $\ddot{x} = 0$. $g - kv^2 = 0 \Rightarrow k = \dfrac{g}{v^2}$ Substituting $v = 6g$ gives: $k = \dfrac{g}{36g^2}$ $= \dfrac{1}{36g}$ **OR** $v^2 = \dfrac{g}{k}(1 - e^{-2kx})$ As $x \to \infty$, $1 - e^{-2kx} \to 1$. $v^2 = \dfrac{g}{k} \Rightarrow k = \dfrac{g}{v^2}$ Substituting $v = 6g$ gives: $k = \dfrac{g}{36g^2}$ $= \dfrac{1}{36g}$	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iv) From Newton's second law, the equation of motion is $m\ddot{x} = mg - \frac{1}{10}mgv$. $\ddot{x} = \frac{dv}{dt} = g\left(1 - \frac{v}{10}\right)$ Separating variables gives: $\int g\,dt = \int \frac{1}{1 - \frac{v}{10}}\,dv$ $gt + c_2 = -10\ln\left\|1 - \frac{v}{10}\right\|$ $Be^{-\frac{gt}{10}} = 1 - \frac{v}{10}$ where $B = e^{-\frac{c_2}{10}}$ When $t = 0$, $v = 5g$ and so $B = 1 - \frac{g}{2}$. $\frac{v}{10} = 1 - \left(1 - \frac{g}{2}\right)e^{-\frac{gt}{10}}$ $v = 10 + 5(g-2)e^{-\frac{gt}{10}}$	MEX-C1 Further Integration MEX12–5, 12–6, 12–7 Bands E2–E4 • Gives the correct solution . . . . .3 • Obtains $gt + c_2 = -10\ln\left\|1 - \frac{v}{10}\right\|$ OR equivalent merit. . . . . .2 • Obtains $\frac{dv}{dt} = g\left(1 - \frac{v}{10}\right)$ OR equivalent merit. 1
(v) Substituting $v = 2g$ and $t = T$ into $v = 10 + 5(g-2)e^{-\frac{gt}{10}}$ and solving for T gives: $2(g-5) = 5(g-2)e^{-\frac{gT}{10}}$ $e^{-\frac{gT}{10}} = \frac{2(g-5)}{5(g-2)}$ $e^{\frac{gT}{10}} = \frac{5(g-2)}{2(g-5)}$ $\frac{gT}{10} = \ln\left[\frac{5(g-2)}{2(g-5)}\right]$ $T = \frac{10}{g}\ln\left[\frac{5(g-2)}{2(g-5)}\right]$	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution 1

		Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b)	(i)	The z^5 terms cancel, leaving a quartic equation that will have only four roots. Hence the equation does not have five roots. **OR** As $e^{2k\pi i} = 1$, the equation can be written as $e^{2k\pi i}z^5 = (z+1)^5$. Taking the fifth root of each side of the equation gives: $e^{\frac{2k\pi i}{5}}z = z + 1, k = 0, 1, 2, 3, 4$ However, $k = 0$ must be excluded because this gives $z = z + 1$. So $k = 1, 2, 3, 4$ corresponds to four roots. Hence the equation does not have five roots.	MEX-N2 Using Complex Numbers MEX12–4, 12–7, 12–8 Bands E2–E4 • Gives a correct explanation 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Using $e^{2k\pi i} = 1$ gives $e^{2k\pi i} z^5 = (z+1)^5$. Hence $e^{\frac{2k\pi i}{5}} z = z + 1,\ k = 1, 2, 3, 4$. $k = 0$ is excluded because this gives $z = z + 1$. *Note: The above work may be seen in part (b) (i).* Solving for z where $k = 1, 2, 3, 4$ gives: $z\left(e^{\frac{2k\pi i}{5}} - 1\right) = 1$ $z = \dfrac{1}{e^{\frac{2k\pi i}{5}} - 1}$ Multiplying the numerator and denominator of the RHS by $e^{-\frac{k\pi i}{5}}$ gives: $z = \dfrac{e^{-\frac{k\pi i}{5}}}{\left(e^{\frac{2k\pi i}{5}} e^{-\frac{k\pi i}{5}}\right) - e^{-\frac{k\pi i}{5}}}$ $= \dfrac{e^{-\frac{k\pi i}{5}}}{e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}}}$ Using $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ gives $e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}} = 2i\sin\frac{k\pi}{5}$. $z = \dfrac{\cos\left(-\frac{k\pi}{5}\right) + i\sin\left(-\frac{k\pi}{5}\right)}{2i\sin\frac{k\pi}{5}}$ $= \dfrac{\cos\frac{k\pi}{5} - i\sin\frac{k\pi}{5}}{2i\sin\frac{k\pi}{5}}$ $= \frac{1}{2i}\cot\frac{k\pi}{5} - \frac{1}{2}$ $= -\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5},\ k = 1, 2, 3, 4$	MEX-N2 Using Complex Numbers MEX12–4, 12–7, 12–8 Bands E2–E4 • Gives the correct solutions in the correct form . . . . .4 • Determines $e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}} = 2i\sin\frac{k\pi}{5}$ and finds $z = \dfrac{\cos\left(-\frac{k\pi}{5}\right) + i\sin\left(-\frac{k\pi}{5}\right)}{2i\sin\frac{k\pi}{5}}$ OR equivalent merit. . . . . .3 • Obtains $z = \dfrac{e^{-\frac{k\pi i}{5}}}{e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}}}$ OR equivalent merit. . . . . .2 • Obtains $z = \dfrac{1}{e^{\frac{2k\pi i}{5}} - 1}$ OR equivalent merit. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) The roots of $z^5 = (z+1)^5$ are $z = -\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5},\ k = 1, 2, 3, 4.$ The roots of $iz^5 = (iz+1)^5$ are $iz = -\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5},\ k = 1, 2, 3, 4.$ $z = -i\left(-\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5}\right),\ k = 1, 2, 3, 4$ Multiplication by $-i$ is equivalent to a clockwise rotation through $\frac{\pi}{2}$. Hence the roots of $iz^5 = (iz+1)^5$ can be obtained by rotating the points $z = -\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5},\ k = 1, 2, 3, 4$ through $\frac{\pi}{2}$ clockwise about the origin.	MEX-N2 Using Complex Numbers MEX12–4, 12–7, 12–8 Bands E2–E4 • Gives a correct geometric description . . . 2 • Obtains $z = -i\left(-\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5}\right),\ k = 1, 2, 3, 4$ 1
Question 15	
(a) (i) Using $\overrightarrow{OM} \cdot \overrightarrow{MC} = 0$ gives: $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$ $= \underset{\sim}{a} + k\underset{\sim}{b}$ $\overrightarrow{MC} = \overrightarrow{MA} + \overrightarrow{AO} + \overrightarrow{OC}$ $= -k\underset{\sim}{b} - \underset{\sim}{a} + \underset{\sim}{b}$ $= (1-k)\underset{\sim}{b} - \underset{\sim}{a}$ $\overrightarrow{OM} \cdot \overrightarrow{MC} = (\underset{\sim}{a} + k\underset{\sim}{b}) \cdot ((1-k)\underset{\sim}{b} - \underset{\sim}{a})$ $= (1-k)(\underset{\sim}{a} \cdot \underset{\sim}{b}) - (\underset{\sim}{a} \cdot \underset{\sim}{a}) + k(1-k)(\underset{\sim}{b} \cdot \underset{\sim}{b}) - k(\underset{\sim}{a} \cdot \underset{\sim}{b})$ $= (1-k)(\underset{\sim}{a} \cdot \underset{\sim}{b}) - \lvert\underset{\sim}{a}\rvert^2 + k(1-k)\lvert\underset{\sim}{b}\rvert^2 - k(\underset{\sim}{a} \cdot \underset{\sim}{b})$ $= (1-2k)(\underset{\sim}{a} \cdot \underset{\sim}{b}) - \lvert\underset{\sim}{a}\rvert^2 + k(1-k)\lvert\underset{\sim}{b}\rvert^2$ $= 2(1-2k)\cos\alpha\lvert\underset{\sim}{a}\rvert^2 - \lvert\underset{\sim}{a}\rvert^2 + 4k(1-k)\lvert\underset{\sim}{a}\rvert^2$ (as $\lvert\underset{\sim}{b}\rvert = 2\lvert\underset{\sim}{a}\rvert$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2\lvert\underset{\sim}{a}\rvert^2\cos\alpha$) $= 2(1-2k)\cos\alpha\lvert\underset{\sim}{a}\rvert^2 - (1-4k+4k^2)\lvert\underset{\sim}{a}\rvert^2$ $= 2(1-2k)\cos\alpha\lvert\underset{\sim}{a}\rvert^2 - (1-2k)^2\lvert\underset{\sim}{a}\rvert^2$ $= (1-2k)\lvert\underset{\sim}{a}\rvert^2(2\cos\alpha - (1-2k))$ Given $\overrightarrow{OM} \cdot \overrightarrow{MC} = 0$, so $(1-2k)\lvert\underset{\sim}{a}\rvert^2(2\cos\alpha - (1-2k)) = 0.$	MEX-V1 Further Work with Vectors MEX12–3, 12–7 Bands E2–E4 • Gives the correct solution. . . . . 4 • Uses $\lvert\underset{\sim}{b}\rvert = 2\lvert\underset{\sim}{a}\rvert$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2\lvert\underset{\sim}{a}\rvert^2\cos\alpha$ AND attempts to simplify . . . . 3 • Obtains $(1-2k)(\underset{\sim}{a} \cdot \underset{\sim}{b}) - \lvert\underset{\sim}{a}\rvert^2 + k(1-k)\lvert\underset{\sim}{b}\rvert^2$ OR equivalent merit . . . . 2 • Obtains ONE of $\overrightarrow{OM}$ OR $\overrightarrow{MC}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and k 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $(1-2k)\lvert\underset{\sim}{a}\rvert^2(2\cos\alpha-(1-2k))=0$ $1-2k=0$ or $2\cos\alpha-(1-2k)=0$ $(\lvert\underset{\sim}{a}\rvert^2\neq 0)$ $k=\frac{1}{2}$ or $k=\frac{1}{2}-\cos\alpha$ As $0\le k\le 1$, $0\le\frac{1}{2}-\cos\alpha\le 1$. $-\frac{1}{2}\le\cos\alpha\le\frac{1}{2}$ So $\frac{\pi}{3}\le\alpha\le\frac{2\pi}{3}$, $\alpha\neq\frac{\pi}{2}$. *Note: If* $\alpha=\frac{\pi}{2}$, *there is only one possible position for M.*	MEX-V1 Further Work with Vectors MEX12–3, 12–7 Bands E2–E4 • Gives the correct solution . . . . .2 • Determines $k=\frac{1}{2}$ OR $k=\frac{1}{2}-\cos\alpha$ 1
(b) (i) Let $t=\tan\frac{x}{2}$ and so $\frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}$. Using the identity $\sec^2\frac{x}{2}=1+\tan^2\frac{x}{2}$ and knowing $t=\tan\frac{x}{2}$ gives: $\frac{dt}{dx}=\frac{1}{2}(1+t^2)$ $dx=\frac{2dt}{1+t^2}$ From t-formulae, $\cos x=\frac{1-t^2}{1+t^2}$. When $x=0$, $t=0$ and $x=\frac{\pi}{2}$, $t=1$. Substituting into $I=\int_0^{\frac{\pi}{2}}\frac{2}{3+5\cos x}dx$ gives: $I=\int_0^1\frac{2}{3+5\left(\frac{1-t^2}{1+t^2}\right)}\times\frac{2dt}{1+t^2}$ $=\int_0^1\frac{4}{8-2t^2}dt$ $=\int_0^1\frac{2}{4-t^2}dt$	MEX-C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution . . . . .2 • Makes suggested substitution and rewrites integrand and limits OR equivalent merit. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $\dfrac{2}{4-t^2} = \dfrac{2}{(2+t)(2-t)} = \dfrac{A}{2+t} + \dfrac{B}{2-t}$	MEX-C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution 2
$2 = A(2-t) + B(2+t)$ Using the cover up method: Substituting $t = 2$ and solving for B and substituting $t = -2$ and solving for A. $2 = 4B \Rightarrow B = \frac{1}{2}$; $2 = 4A \Rightarrow A = \frac{1}{2}$ Using the equating coefficients method: Forming $2 = 2A + 2B$ and $0 = -A + B$ and solving for A and B. $A = \frac{1}{2}$ and $B = \frac{1}{2}$ Then: $I = \frac{1}{2}\int_0^1 \frac{1}{2+t} + \frac{1}{2-t}dt$ $= \frac{1}{2}\Big[\ln\lvert 2+t\rvert - \ln\lvert 2-t\rvert\Big]_0^1$ $= \frac{1}{2}(\ln 3 - 0 - (\ln 2 - \ln 2))$ $= \ln\sqrt{3}$	• Obtains $A = B = \frac{1}{2}$ 1

 TENME2_SS_20.FM

		Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c)	(i)	We are required to find T such that $\frac{g}{4} + gT\sin\theta - \frac{g}{2}T^2 = 0$. $T = \dfrac{-g\sin\theta \pm \sqrt{g^2\sin^2\theta - 4\left(-\frac{g}{2}\right)\left(\frac{g}{4}\right)}}{-g}$ $= \sin\theta \pm \frac{1}{\sqrt{2}}\sqrt{2\sin^2\theta + 1}$ $\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow 2\sin^2\theta = 1 - \cos 2\theta$ $\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}\sqrt{1 - \cos 2\theta}$ $(\sin\theta > 0)$ Substituting for $2\sin^2\theta$ and $\sin\theta$ into $T = \sin\theta \pm \frac{1}{\sqrt{2}}\sqrt{2\sin^2\theta + 1}$ gives: $T = \frac{1}{\sqrt{2}}\sqrt{1 - \cos 2\theta} \pm \frac{1}{\sqrt{2}}\sqrt{2 - \cos 2\theta}$ $\sqrt{2 - \cos 2\theta} > \sqrt{1 - \cos 2\theta}$. We require $T > 0$ and so $T = \frac{1}{\sqrt{2}}\sqrt{1 - \cos 2\theta} + \frac{1}{\sqrt{2}}\sqrt{2 - \cos 2\theta}$. Hence $T = \frac{1}{\sqrt{2}}(\sqrt{1 - \cos 2\theta} + \sqrt{2 - \cos 2\theta})$.	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution . . . . .3 • Substitutes for $2\sin^2\theta$ and $\sin\theta$ into $T = \sin\theta \pm \frac{1}{\sqrt{2}}\sqrt{2\sin^2\theta + 1}$ OR equivalent merit . . . . .2 • Attempts to solve $\frac{g}{4} + gT\sin\theta - \frac{g}{2}T^2 = 0$ for T 1
	(ii)	$R = (g\cos\theta)T$ where $T = \frac{1}{\sqrt{2}}(\sqrt{1 - \cos 2\theta} + \sqrt{2 - \cos 2\theta})$. $\cos 2\theta = 2\cos^2\theta - 1$ $\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}\sqrt{1 + \cos 2\theta}$ $(\cos\theta > 0)$ Substituting for T and $\cos\theta$ into $R = (g\cos\theta)T$ gives: $R = \frac{g}{\sqrt{2}}\sqrt{1 + \cos 2\theta}\left(\frac{1}{\sqrt{2}}(\sqrt{1 - \cos 2\theta} + \sqrt{2 - \cos 2\theta})\right)$ So $R = \frac{g}{2}(\sqrt{1 - \cos^2 2\theta} + \sqrt{2 + \cos 2\theta - \cos^2 2\theta})$.	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) When $\theta = 45°$, $R = \frac{g}{2}(1 + \sqrt{2})$. When $\cos 2\theta = \frac{1}{5}$, $R = \frac{g}{2}\left(\sqrt{\frac{24}{25}} + \sqrt{\frac{54}{25}}\right)$. Let d represent the extra distance attained. $\sqrt{24} = 2\sqrt{6}$ and $\sqrt{54} = 3\sqrt{6}$. $d = \frac{g}{2}\left(\frac{2\sqrt{6}}{5} + \frac{3\sqrt{6}}{5}\right) - \frac{g}{2}(1 + \sqrt{2})$ $= \frac{g}{2}(\sqrt{6} - \sqrt{2} - 1)$ So the extra distance attained is $\frac{g}{2}(\sqrt{6} - \sqrt{2} - 1)$ metres.	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution 1
Question 16	
(a) (i) Integration by parts takes the form $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Use $u = \tan^{-1}x$ and $\frac{dv}{dx} = x^n$. So $\frac{du}{dx} = \frac{1}{1 + x^2}$ and $v = \frac{x^{n+1}}{n+1}$. Substituting into $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ gives: $I_n = \int_0^1 x^n \tan^{-1}x\,dx$ $= \left[(\tan^{-1}x)\left(\frac{x^{n+1}}{n+1}\right)\right]_0^1 - \int_0^1 \left(\frac{x^{n+1}}{n+1}\right)\left(\frac{1}{1+x^2}\right)dx$ $= \left(\frac{\pi}{4}\right)\left(\frac{1}{n+1}\right) - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}dx$ Multiply both sides by $(n + 1)$. So $(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx$.	MEX-C1 Further Integration MEX12–5, 12–8 Bands E2–E4 • Gives the correct solution . . . . 2 • Uses integration by parts with correct substitutions 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Using part (a) (i): Setting $n = 0$ gives: $I_0 = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx$ $= \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$ So $I_0 = \frac{\pi}{4} - \frac{1}{2}\ln 2$. **OR** Integration by parts takes the form $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Put $u = \tan^{-1}x$ and $\frac{dv}{dx} = 1$. So $\frac{du}{dx} = \frac{1}{1+x^2}$ and $v = x$. Substituting into $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ gives: $I_0 = \int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$ $I_0 = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx$ $= \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$ $= \frac{\pi}{4} - \frac{1}{2}\ln 2$	MEX-C1 Further Integration MEX12–5 Bands E2–E3 • Gives the correct solution 1
(iii) $(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx$ Replacing n with $n + 2$ gives: $(n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}dx$ Attempting to form $(n+3)I_{n+2} + (n+1)I_n$: $= \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}dx + \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx$ $= \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}(1+x^2)}{1+x^2}dx$ $= \frac{\pi}{2} - \int_0^1 x^{n+1}dx$ $= \frac{\pi}{2} - \frac{1}{n+2}$	MEX-C1 Further Integration MEX12–5, 12–8 Bands E2–E4 • Gives the correct solution . . . . .2 • Replaces n with $n + 2$ AND attempts to form a valid expression for $(n+3)I_{n+2} + (n+1)I_n$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iv) $(n+3)I_{n+2}+(n+1)I_n=\frac{\pi}{2}-\frac{1}{n+2}$ Substituting $n=0$ gives: $3I_2+I_0=\frac{\pi}{2}-\frac{1}{2}$ (1) Substituting $n=2$ gives: $5I_4+3I_2=\frac{\pi}{2}-\frac{1}{4}$ (2) $(2)-(1)$ gives $5I_4-I_0=\frac{1}{4}$. $I_4=\frac{1}{5}\left(I_0+\frac{1}{4}\right)$ Substituting $I_0=\frac{\pi}{4}-\frac{1}{2}\ln 2$ into $I_4=\frac{1}{5}\left(I_0+\frac{1}{4}\right)$ gives: $I_4=\frac{1}{5}\left(\frac{\pi}{4}-\frac{1}{2}\ln 2+\frac{1}{4}\right)$ $=\frac{1}{20}(1+\pi-2\ln 2)$	MEX-C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution . . . . 2 • Obtains the two equations AND attempts to solve for I_4 1
(b) (i) Since squares cannot be negative, $(\sqrt{a_1}-\sqrt{a_2})^2\geq 0$. Expanding the LHS gives: $a_1-2\sqrt{a_1a_2}+a_2\geq 0$ $a_1+a_2\geq 2\sqrt{a_1a_2}$ So $\frac{a_1+a_2}{2}\geq\sqrt{a_1a_2}$. **OR** $\frac{a_1+a_2}{2}-\sqrt{a_1a_2}=\frac{1}{2}(a_1+a_2-2\sqrt{a_1a_2})$ $=\frac{1}{2}((\sqrt{a_1})^2+(\sqrt{a_2})^2-2\sqrt{a_1a_2})$ $=\frac{1}{2}(\sqrt{a_1}-\sqrt{a_2})^2$ ≥ 0 So $\frac{a_1+a_2}{2}\geq\sqrt{a_1a_2}$.	MEX-P1 The Nature of Proof MEX12–2 Bands E2–E3 • Gives the correct proof 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Let $x = (a_1 a_2 \dots a_n)^{\frac{1}{n}} > 0$. Taking the nth power of both sides of the equality gives: $x^n = a_1 a_2 \dots a_n \Rightarrow \dfrac{a_1 a_2 \dots a_n}{x^n} = 1$ $\left(\dfrac{a_1}{x}\right)\left(\dfrac{a_2}{x}\right)\dots\left(\dfrac{a_n}{x}\right) = 1$ Note that $\dfrac{a_1}{x} > 0, \dfrac{a_2}{x} > 0, \dots, \dfrac{a_n}{x} > 0$. If $a_1 a_2 \dots a_n = 1$, then $a_1 + a_2 + \dots + a_n \ge n$. $\dfrac{a_1}{x} + \dfrac{a_2}{x} + \dots + \dfrac{a_n}{x} \ge n$ $\dfrac{a_1 + a_2 + \dots + a_n}{n} \ge x$ $\dfrac{a_1 + a_2 + \dots + a_n}{n} \ge (a_1 a_2 \dots a_n)^{\frac{1}{n}}$	MEX-P1 The Nature of Proof MEX12–2, 12–7 Bands E3–E4 • Gives the correct proof . . . . .3 • Obtains $\left(\dfrac{a_1}{x}\right)\left(\dfrac{a_2}{x}\right)\dots\left(\dfrac{a_n}{x}\right) = 1$ AND attempts to use the given result . . . . .2 • Obtains $\dfrac{a_1 a_2 \dots a_n}{x^n} = 1$ OR equivalent merit. 1
(iii) $2^n - 1 = 2^{n-1} + 2^{n-2} + \dots + 2 + 1$ $2^{n-1}, 2^{n-2}, \dots, 2, 1$ are unequal positive numbers. Hence using the strong inequality $\dfrac{a_1 + a_2 + \dots + a_n}{n} > (a_1 a_2 \dots a_n)^{\frac{1}{n}}$ gives: $\dfrac{2^n - 1}{n} > (2^{n-1} \times 2^{n-2} \times \dots 2 \times 1)^{\frac{1}{n}}$ $= (2^{(n-1)+(n-2)+\dots+2+1})^{\frac{1}{n}}$ $= \left(2^{\frac{n(n-1)}{2}}\right)^{\frac{1}{n}}$ $= 2^{\frac{n-1}{2}}$ $\dfrac{2^n - 1}{n} > 2^{\frac{n-1}{2}} \Rightarrow 2^n - 1 > n\left(2^{\frac{n-1}{2}}\right)$ So $2^n - 1 > n\sqrt{2^{n-1}}$ for integers $n \ge 1$.	MEX-P1 The Nature of Proof MEX12–2, 12–7 Bands E3–E4 • Gives the correct proof . . . . .4 • Obtains $(2^{n-1} \times 2^{n-2} \times \dots 2 \times 1)^{\frac{1}{n}} = 2^{\frac{n-1}{2}}$ . . . . .3 • Establishes $\dfrac{2^n - 1}{n} > (2^{n-1} \times 2^{n-2} \times \dots 2 \times 1)^{\frac{1}{n}}$ . . . . .2 • Obtains $2^n - 1 = 2^{n-1} + 2^{n-2} + \dots + 2 + 1$ OR equivalent merit. 1