Neap

Trial Examination 2023

HSC Year 12 Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 3 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • For questions in Section II, show relevant mathematical reasoning and/or calculations
Total Marks: 100	**Section I – 10 marks (pages 2–6)** • Attempt Questions 1–10 • Allow about 15 minutes for this section **Section II – 90 marks (pages 7–13)** • Attempt Questions 11–16 • Allow about 2 hours and 45 minutes for this section

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2023 HSC Year 12 Mathematics Extension 2 examination.

SECTION I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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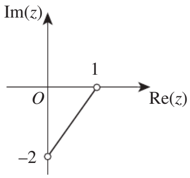
1 The line $L = \begin{pmatrix} 0 \\ -1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$ forms an angle with the positive y-axis.

What is the size of this angle?

A. 35.8°
B. 36.8°
C. 37.2°
D. 37.8°

2 Which of the following diagrams best represents the solution to the equation $\arg\left(\dfrac{z-1}{z+2i}\right) = \pi$?

A.


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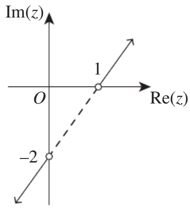

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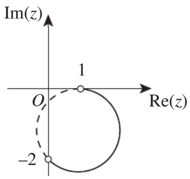

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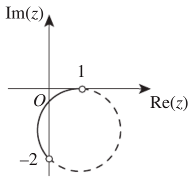


3 Consider the statement.

'If I do not complete my homework, then my teacher will give me a detention.'

Which of the following is the converse of the contrapositive of the statement?

A. 'If my teacher did not give me a detention, then I did complete my homework.'

B. 'My teacher will give me a detention, if I do not complete my homework.'

C. 'My teacher will give me a detention, if I do complete my homework.'

D. 'If I do complete my homework, then my teacher will not give me a detention.'

4 If $(a+bi)(2-i)=3+i$, what are the values of a and b?

A. $a=-1,\ b=-1$

B. $a=1,\ b=-1$

C. $a=-1,\ b=1$

D. $a=1,\ b=1$

5 Consider the complex number $z=\cos\theta+i\sin\theta$ where $0<\theta<90°$.
Which of the following is the modulus of $z+1$?

A. $2\cos\left(\frac{\theta}{2}\right)$

B. $2\cos\left(\frac{\theta}{2}\right)+1$

C. $2\cos(\theta)$

D. $2\cos(\theta)+1$

6 Consider a line that passes through the point (5, 2, 1) and is parallel to the *x*–*y* plane and *x*–*z* plane. Which of the following is the vector equation of the line?

A. $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

B. $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

C. $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

D. $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

7 A particle is moving in a straight line such that its velocity, in m s^{-1}, is given by $v^2 = 20 - 16x - 4x^2$, where *x* is the displacement of the particle from a fixed point, *O*.

Which of the following statements about the motion of the particle is true?

A. The particle moves in a simple harmonic motion, oscillating about the centre $x = -2$ with a period of π and an amplitude of 3.

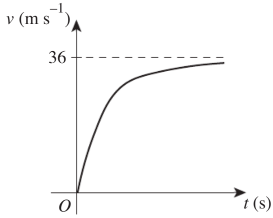
B. The particle moves in a simple harmonic motion, oscillating about the centre $x = -2$ with a period of $\frac{\pi}{2}$ and an amplitude of 3.

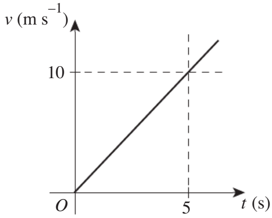
C. The particle moves in a simple harmonic motion, oscillating about the centre $x = 2$ with a period of π and an amplitude of 3.

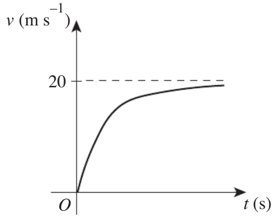
D. The particle does not move in simple harmonic motion and has a turning point at $x = -2$.

8 A mass of 1 kg is dropped from a height in a resistive medium under a constant gravitational acceleration of 10 m s^{-2}. The resistive force is directly proportional to the speed v.

If the constant of proportionality is 0.5, which of the following best represents the velocity–time graph of the mass?

A.


B.


C.


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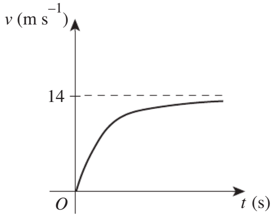


9 Which of the following statements about inequality proofs is true?

A. If $a > b$ and $c > d$, then $a + c > b + d$.

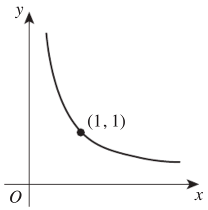
B. If $a > b$ and $c > d$, then $a - c > b - d$.

C. If $a > b$ and $c > d$, then $ac > bd$.

D. If $a > b$ and $c > d$, then $\frac{a}{c} > \frac{b}{d}$.

10 Which of the following shows the graph of $x = \sqrt{t-2},\ y = \dfrac{1}{2-t}$?

A.



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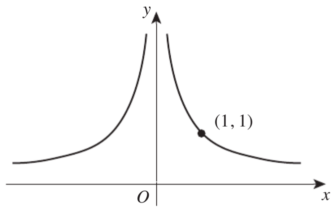
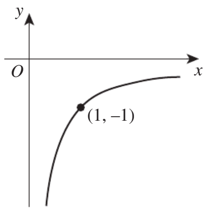


C.



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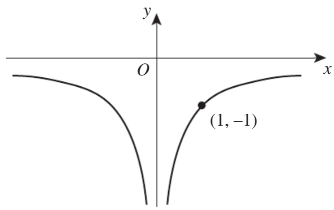


SECTION II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

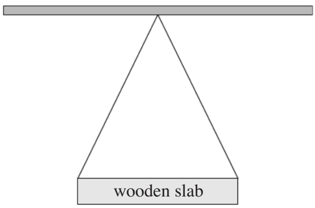
Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) If $n \in \mathbb{Z}^+$, prove that $\sqrt{3n+1}$ is always irrational. **2**

(b) Find the quadratic equation with the roots $\sqrt{3}cis\left(\frac{\pi}{3}\right), \sqrt{3}cis\left(-\frac{\pi}{3}\right)$. **2**

(c) Find $\int \sin^{-1} 3x dx$. **2**

(d) A swing is to be designed so that two inelastic ropes of negligible weight and equal length will hang from the same point from an iron bar to support a horizontal wooden slab of negligible weight, as shown in the diagram. **2**



The swing can support a maximum mass of 50 kg. Let acceleration due to gravity be $g = 10$ m s^{-2}.

If each rope can withstand a maximum tension of 326 N, calculate the maximum angle between the two ropes, correct to the nearest degree.

Question 11 continues on page 8

Question 11 (continued)

(e) Consider two lines $L_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ and $L_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(i) Show that lines L_1 and L_2 will never intersect. **1**

Points P and Q are positioned on lines L_1 and L_2 respectively.

(ii) Find the vector equation of $\overrightarrow{PQ}$ in terms of λ and μ. **1**

(iii) Hence, or otherwise, find the shortest distance between lines L_1 and L_2. **3**

(f) Let z be a complex number such that $2|z-1| = |z-4|$. **2**
Find $|z|$.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

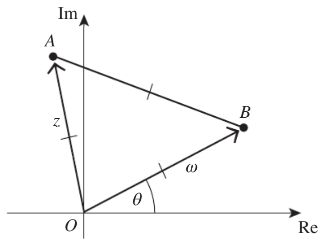
(a) A missile is launched vertically upwards from the surface of the Earth at a velocity of 8 km s^{-1}. The acceleration of the rocket is given by $\ddot{x} = -\dfrac{96\,000}{x^2}$ km s^{-2}, where x is the distance of the missile from the centre of the Earth in kilometres. Let the radius of the Earth be 6400 km.

(i) Find the distance that the missile travelled when its speed is 6.5 km s^{-1}, correct to the nearest kilometre. **3**

(ii) What is the terminal velocity of the missile? **1**

(b) Use mathematical induction to prove that $2n > 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{3^n} > \dfrac{1}{3n}$, $\forall n \in \mathbb{Z}^+$. **4**

(c) Let $z = \overrightarrow{OA}$ and $\omega = \overrightarrow{OB}$ represent the two sides of an equilateral triangle OAB, as shown.



(i) Find the modulus and argument of $\dfrac{z}{\omega}$. **2**

(ii) Hence, find $z^3 + \omega^3$. **2**

(d) Find $\displaystyle\int \frac{x^3 + 4x^2 - 2x - 33}{x^2 - 9}\,dx$. **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Let $a, b \in \mathbb{R}^+$, $a > b$. **3**

Prove that $(a+b+1)^2 > 3b(b+1)$.

(b) Using the substitution $t = \tan\frac{x}{2}$, find $\int \frac{\cos x\,dx}{4+3\cos x}$. **3**

(c) By considering conjugate pairs, or otherwise, solve $9z^4 - 18z^3 + 5z^2 - 18z + 9 = 0$. **4**

(d) A sphere has the parametric equations $x = 3\sin\theta + 2$, $y = 3\cos^2\theta + 1$ and $z = 3\cos\theta\sin\theta + 5$.

(i) Show that the Cartesian equation of the sphere is $(x-2)^2 + (y-1)^2 + (z-5)^2 = 9$. **2**

The vector equation of line L is $(2\lambda - 2)\underset{\sim}{i} + (2\lambda - 3)\underset{\sim}{j} + (3+\lambda)\underset{\sim}{k}$.

(ii) Show that line L passes through the centre of the sphere. **1**

(iii) Find the coordinates of the intersection points of line L and the surface of the sphere. **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Find the primitive function of $f(x) = \dfrac{x+3}{\sqrt{9-8x-x^2}}$. **3**

(b) If $\text{Im}(z) \neq 0$ and $z + \dfrac{1}{z}$ is purely real, show that $|z| = 1$. **3**

(c) (i) Prove that for $x \geq 2$, $x \geq 2\sqrt{2(x-2)}$. **1**

(ii) Hence, prove that for $a > 0$, $a^4 + 4a^2 + 4 \geq 8a^2$. **2**

(d) A helicopter leaves its base at point $(-25, 124, 28)$ at 8 am. Its velocity vector is $\begin{pmatrix} 18 \\ 12 \\ 4 \end{pmatrix}$ km h^{-1}.

At 9 am, a practice missile is fired from an airbase at point $(-8, -238, 3)$. The velocity vector of the missile is $\begin{pmatrix} 20 \\ 280 \\ 25 \end{pmatrix}$ km h^{-1}.

(i) Show that the missile will NOT collide with the helicopter. **2**

(ii) At what time would the missile need to be fired so that it collides with the helicopter? **4**

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Let the complex number z be the root of the equation $|z| = |z+2|$. Let p be all points on the Argand diagram that represent z.

(i) Find the Cartesian equation of p. **1**

(ii) If $|z| = 2$, find all values of z in the form $re^{i\theta}$, $-\pi \le \theta \le \pi$. **2**

(iii) Let υ be z in the second quadrant, and ω be z in the third quadrant. **2**
Find the value of the real number k, such that $\dfrac{\upsilon\omega^k}{ki}$ is purely imaginary.

(b) Consider $I_n = \int x^n \cos x dx$.

(i) Find the recurrence relation for I_n. **3**

(ii) Hence, evaluate $\int_0^{\pi} x^4 \cos x dx$. **2**

(c) A sequence is defined by the recursive formula $T_n = \dfrac{T_{n-1} \times (2n+1)}{2n-3}$, where $T_1 = 3$ and $n > 1$.

(i) Use mathematical induction to prove that $T_n = 4n^2 - 1$. **3**

(ii) Using the formula $\sum_1^k n^2 = \dfrac{k(k+1)(2k+1)}{6}$, prove that the sum of n terms from part (i) is $\dfrac{1}{3}n\left(4n^2 + 6n - 1\right)$. **2**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) A rock is projected from ground level with an initial velocity of $\begin{pmatrix} 5 \\ 13 \end{pmatrix}$ m s^{-1}, where acceleration due to gravity is 10 m s^{-2}.

(i) If the only resistance to the motion of the rock is gravity, find the parametric equations of the position of the rock at time t seconds. **1**

The rock is projected a second time with the same initial velocity as the first projection into a medium that resists the motion, where the resistive force is proportional to the velocity of the rock. The constant of proportionality is $k = 0.5$.

(ii) Find the parametric equations for the velocity of the rock at time t seconds. **2**

(iii) Hence, find the maximum height to which the rock can be projected above ground level, correct to two decimal places. **3**

(b) The random movement of a particle under a specific set of conditions is given by $v = (k + v_0)a^{bt} - k$, where v_0 is the initial speed of the particle and k, a, b are all constants. **3**

Show that the displacement of the particle from its initial position is

$$x = \frac{1}{b \ln a}\left(v - v_0 - k \ln\left|\frac{v + k}{v_0 + k}\right|\right).$$

(c) (i) Show that $\left(e^{i\theta} + e^{-i\theta}\right)^4 = e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}$. **1**

(ii) Hence, prove that $\left(e^{i\theta} + e^{-i\theta}\right)^n = 2\sum_{r=0}^{n}\binom{n}{r}\cos\left((n-2r)\theta\right)$. **3**

(iii) Hence, evaluate $\int \left(e^{i\theta} + e^{-i\theta}\right)^6 d\theta$. **2**

End of paper

MATHEMATICS ADVANCED
MATHEMATICS EXTENSION 1
MATHEMATICS EXTENSION 2
REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, r \neq 1$$

$$S = \frac{a}{1-r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

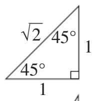
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2}ab \sin C$



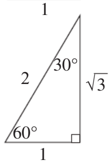
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$\cos A = \dfrac{1-t^2}{1+t^2}$

$\tan A = \dfrac{2t}{1-t^2}$

$\cos A \cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A \sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A \cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A \sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

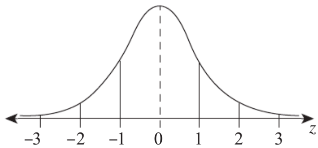
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le r) = \displaystyle\int_a^r f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$

$X \sim \text{Bin}(n, p)$

$\Rightarrow P(X = x)$

$= \dbinom{n}{x} p^x (1 - p)^{n-x}, x = 0, 1, \ldots, n$

$E(X) = np$

$\text{Var}(X) = np(1 - p)$

Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)\, f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)\, f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1 - \left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1 - \left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1 + \left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$
where $n \neq -1$

$$\int f'(x)\sin f(x)\, dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\, dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\, dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\, dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\, dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\, dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\, dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\, dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx$$

$$\int_a^b f(x)\, dx$$
$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \ldots + f\left(x_{n-1}\right)\right]\right\}$$
where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \ldots + \binom{n}{r}x^{n-r}a^{r} + \ldots + a^{n}$$

Vectors

$$|\underset{\sim}{u}| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2} + y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta = x_1 x_2 + y_1 y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^{n} = r^{n}(\cos n\theta + i\sin n\theta)$$
$$= r^{n}e^{in\theta}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^{2}(x - c)$$