HIGHER SCHOOL CERTIFICATE EXAMINATION

2024

Mathematics Extension 2

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General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A separate reference sheet is provided at the back of this paper.
- All necessary working should be shown in Questions 11 – 16.

Total marks – 100

Section I: Multiple Choice

Questions 1 – 10   10 marks

- Attempt all questions
- Allow about 15 minutes for this section

Section II: Extended Response

Questions 11 – 16   90 marks

- Attempt all questions
- Allow about 2 hours 45 minutes for this section

Section I
Questions 1 – 10 (1 mark for each question)

Read each question and choose an answer A, B, C or D.
Record your answer on the Answer Sheet provided.
Allow about 15 minutes for this section.

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1 Let $\alpha = 2 - i$ and $\beta = 1 - 3i$.

What is the value of $(\alpha - \beta)^4$?

A) $7 - 24i$

B) $-7 - 24i$

C) $-7 - 40i$

D) $41 - 24i$

2 Consider the statement.

"If a quadrilateral is a rectangle, then its diagonals are equal in length."

Which of the following is true about this statement and its converse?

A) Both the statement and its converse are true.

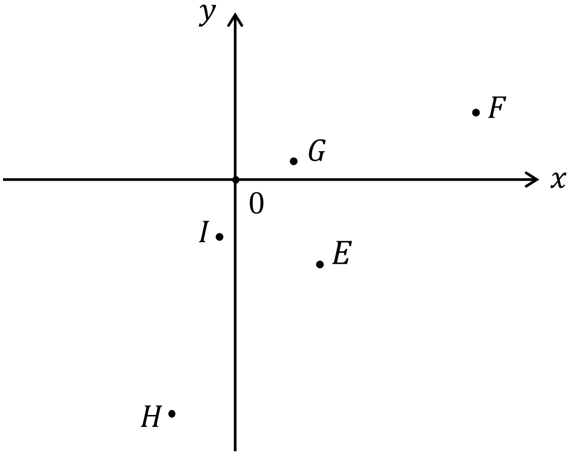
B) Both the statement and its converse are false.

C) The statement is false and its converse is true.

D) The statement is true and its converse is false.

3 In the Argand diagram the point E represents a complex number.

When this number is multiplied by $\left(\sqrt{3}+i\right)^4\left(\frac{1}{4}-\frac{\sqrt{3}}{4}i\right)^3$ it gave a new complex number.



Which of the following points represents the new complex number?

A) F

B) G

C) H

D) I

4 A particle is moving in simple harmonic motion. The displacement x of the particle is given by

$x = 1 + 4\sin\left(2\text{t} - \frac{\pi}{5}\right)$.

Which of the following is the first time the velocity of the particle is at a minimum?

A) $\frac{3\pi}{5}$

B) $\frac{\pi}{10}$

C) $\frac{7\pi}{10}$

D) $\frac{2\pi}{5}$

5 Let the points A and B represent the complex numbers $\lambda = ln^3\alpha + i\ ln^2\ \alpha$ and $\phi = -\,2\ ln\,\alpha + 8i$ respectively.

When the point B is rotated about the origin by $\frac{\pi}{2}$ in a clockwise direction we get the point C.

For which values of α do A and C coincide?

A) $\alpha = 1$

B) $\alpha = e^2$

C) $\alpha = 1\ or\ \alpha = e^2$

D) $\alpha = 0\ \ or\ \ \alpha = e^2$

6 Consider the position vector of a particle $r = -3\sin t\ \ i + 3\cos t\, j + t\ k.$

$\sim$

Which of the following statements best describes the motion of the particle?

A) A spiral about the z axis in an anticlockwise direction.

B) A spiral about the z axis in a clockwise direction.

C) A spiral about the x axis in a anticlockwise direction.

D) A spiral about the x axis in a clockwise direction.

7 Consider the following statement:

"$\forall\ n \in \mathbb{N}$, if n^2 is odd then n is odd."

Which of the following is the negation of the statement?

A) $\forall\ n \in \mathbb{N}$, if n^2 is not odd then n is also not odd.

B) $\exists\ n \in \mathbb{N}$, if n is not odd then n^2 is also not odd.

C) $\forall\ n \in \mathbb{N}$, if n is not odd then n^2 is not odd.

D) $\exists\ n \in \mathbb{N}$, if n^2 is not odd then n is also not odd.

8 The equations of two lines ℓ_1 and ℓ_2 are

$$\overrightarrow{r_1} = \begin{pmatrix} -1 + t\,(2^{\lambda}) \\ -5 + t\,(6 - 2^{\lambda}) \\ -2 + t\,(7 - 2^{\lambda}) \end{pmatrix} \text{ and } \overrightarrow{r_2} = \begin{pmatrix} 1 + t\,(8 + 2^{\lambda}) \\ 2 + 6t \\ -3 + t\,(5 + 2^{\lambda}) \end{pmatrix},$$

where t is a real number and λ is a constant.

For what value of λ are the two lines ℓ_1 and ℓ_2 parallel?

A) 4

B) 3

C) 2

D) 1

9 A particle initially at rest at the origin starts to move along the x axis.
Its velocity at any time t in seconds is $v\ m\ s^{-1}$.
The acceleration of the particle is given by $a = (1 + v^2)^{\frac{3}{2}}$.

Which of the following is the correct expression of the velocity v ?

A) $v = \tan\,(\cos^{-1} t)$

B) $v = \tan\,(\sin^{-1} t)$

C) $v = \sin^{-1}\,(\tan t)$

D) $v = \cos\,(\sin^{-1} t)$

10 The complex number z satisfies $|\, z + a| = a$, where a is a positive real number.

The greatest distance that z can be from the point P representing the complex number $ka + a\ (k+1)i$, where k is a positive real number, is $\left(3\sqrt{2}+1\right)a$.

What is the value of k ?

A) 2

B) 3

C) 1

D) 4

MATHEMATICS EXTENSION 2

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section.

INSTRUCTIONS

- Answer the questions in the appropriate writing booklet.
- Extra writing booklets are available.
- Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

a) Consider the complex numbers $\alpha = 2i$ and $\beta = \sqrt{3} + i$. **3**

Show that $(\alpha - \beta)^6$ is purely real.

b) Consider the line ℓ with equation $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$. **2**

Show that the point $A\ (2, -5, 3)$ lies on the line ℓ.

c) Solve $z^2 - z + (4 - 2i) = 0$. Give your answers in the form $x + yi$, where x and y are real. **3**

d) PQRS is a parallelogram with vertices $P\ (2, 4, 3)$, $Q\ (4, 3, 1)$, $R\ (3, 2, 5)$ and $S\ (x, y, z)$. **3**

Find the coordinates of S and the length of the diagonal $|\overrightarrow{QS}|$.

e) A particle is moving along the x axis in simple harmonic motion centred at $x = 2$ and with amplitude 3. **4**

The velocity of the particle is given by $v^2 = p + qx - 6x^2$, where p and q are constants.

Find the period of the motion.

Question 12 (15 marks)

a) Prove that $x^3 + \frac{1}{x^3} \geq 2$, where $x > 0$. **2**

b) Consider the statement P in the set of integers. **3**

"If $5n^4 + 8n$ is multiple of 16, then n is even."

Prove that the converse of P is true.

c) Find all the 4th roots of $1 + \sqrt{3}\, i$. Give your answer in exponential form. **3**

d) A particle starts moving from $x = ln\,(e^4 + 1)$ with velocity e and acceleration given by $v^2\,(e^4\ + \ln v\,)$, where v is the velocity of the particle. **3**

Find an expression for x, the displacement of the particle, in terms of v.

e) The polynomial $Q(z) = z^4 - 8z^3 + pz^2 + qz - 80$ has root $3 + i$, where p and q are real numbers.

i) Find all the roots of $Q(z)$. **3**

ii) Write $Q(z)$ as a product of two real quadratic factors. **1**

Question 13 (15 marks)

a) The cubic equation $2x^3 - 7x + 4 = 0$ has roots α, β and γ. **3**

Find a cubic equation with roots $\dfrac{1}{\alpha\,\beta}$, $\dfrac{1}{\alpha\,\gamma}$ and $\dfrac{1}{\beta\,\gamma}$.

b) A boat needs to travel from port A to port B on a true bearing of $333°$ at a speed of $35\ \text{kmh}^{-1}$. **3**

However, a constant current of $15\ \text{kmh}^{-1}$ is flowing from the southwest.

Find the speed and the direction to which the boat should head to compensate for the current and maintain its speed of $35\ \text{kmh}^{-1}$ in the desired direction.

c) Use contrapositive proof to prove that " If p is divisible by 5 then p can be expressed as a sum of five consecutive integers". **2**

d) Find $\displaystyle\int \frac{(2\tan\theta + 3)\ \sec^2\theta}{\sec^2\theta + \tan\theta}\ d\theta$. **3**

e) i) Show $\dfrac{2k}{k+2} < \dfrac{2k+2}{k+3}$ for k > 0. **1**

ii) Use mathematical induction to prove that **3**

$$\frac{1}{3!} + \frac{2}{4!} + \frac{3}{5!} + \cdots + \frac{n}{(n+2)!} < \frac{2n}{n+2} - \frac{1}{(n+2)!} \text{ for } n \geq 1.$$

Question 14 (14 marks)

a) Consider the complex equation $z^5 = i$.

i) Find the roots of this equation. **3**

ii) Show that $\cos\frac{\pi}{5}\sin\frac{\pi}{10} = \frac{1}{4}$. **3**

b) A particle of mass $1\ kg$ is projected from the origin with initial speed $V\ ms^{-1}$ at an angle α to the horizontal plane.

The forces acting on the particle are gravity $\overrightarrow{1g}$ and air resistance $5\vec{v}$, where $\vec{v}$ is the velocity vector of the particle.

Let the acceleration due to gravity be $10\ ms^{-2}$.

The position vector of the particle, at time t seconds after the particle is projected, is $\vec{r}(t)$ and the velocity vector is $\vec{v}(t)$.

i) Show that $\vec{v}(t) = \begin{pmatrix} Ve^{-5t}\cos\alpha \\ (V\sin\alpha + 2)\,e^{-5t} - 2 \end{pmatrix}$. **3**

ii) Given that $\vec{v}(1) = \begin{pmatrix} 250\,e^{-5} \\ (250\sqrt{3} + 2)\,e^{-5} - 2 \end{pmatrix}$, find the initial speed V and the angle of projection α. **2**

iii) Show that the ratio of the horizontal velocity at the origin to the horizontal velocity at the maximum height is $(1 + 125\sqrt{3}) : 1$. **3**

Question 15 (15 marks)

a) A particle of mass m kg is moving along the x axis under the action of a resisting force $m\,(pv + v^2)$, where v is its velocity and p is a positive constant.

Initially, the particle is at $x = \ln 2$ and is travelling with velocity p.

i) Show that the displacement x of the particle in terms of v is **2**

$$x = \ln\left(\frac{4\mathrm{p}}{p + v}\right).$$

ii) Show that the time t, which has elapsed when the particle is travelling with velocity v, is **3**

$$t = \frac{1}{p}\ln\left(\frac{p + v}{2v}\right).$$

iii) It took the particle $\frac{1}{2}\ln 2$ seconds to reach the point where $x = \ln 3$. Find the value of p. **2**

b) Two lines, L_1 and L_2, are represented by position vectors $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ respectively.

The vector equations of these two lines are

$$\underset{\sim}{r_1} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{r_2} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

where λ and μ are real numbers.

i) Show that L_1 and L_2 are skew lines. **4**

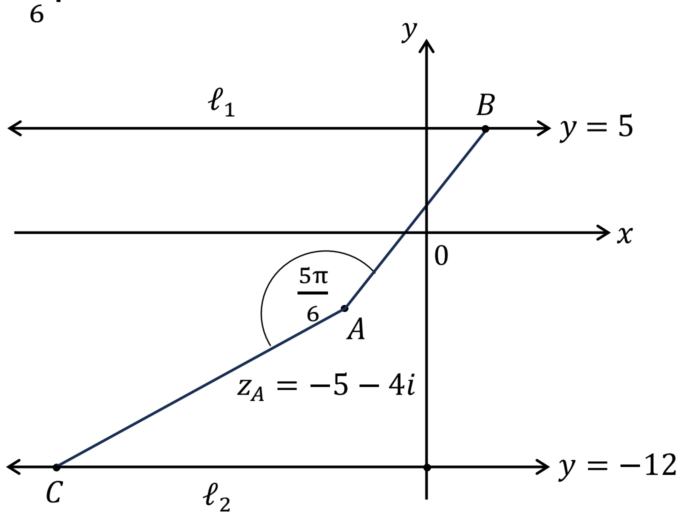
ii) Find the shortest distance between the two skew lines L_1 and L_2. **4**

Question 16 (16 marks)

a) The point A, which represents the complex number $z_A = -5 - 4i$, lies between two parallel lines ℓ_1 and ℓ_2.

The equations of ℓ_1 and ℓ_2 are $y = 5$ and $y = -12$ respectively.

The point B, which represents the complex number z_B, lies on ℓ_1.The point C, which represents the complex number z_C, lies on ℓ_2 such that $AC = 2AB$ and $\angle \text{BAC} = \frac{5\pi}{6}$.



Find the exact value of the complex number z_C. **4**

b) Let $I_n = \int_0^{\frac{\pi}{2}} sin^{2n+3}x\, cos^5 x\, dx,\ n = 0, 1, 2, \ ...$

i) Prove that $I_n = \frac{n+1}{n+4} I_{n-1},\ \ for\ n \geq 1.$ **3**

...

ii) Deduce that $I_n = \frac{1}{(\text{n} + 4)\ (\text{n} + 3)(\text{n} + 2)}$. **3**

iii) Let $J_n = \int_0^1 x^{4n+7}\, (1 - x^4)^2\ dx,\ n = 0, 1, 2, ...$

Show that $J_n = \frac{1}{2} I_n$. **3**

c) Prove by contradiction that

$(\sin x)^{\frac{2}{n}} + (\cos x)^{\frac{2}{n}} \geq 1, for\ 0 \leq x \leq \frac{\pi}{2}$ and $n = 3, 4, 5, ...$ **3**

END OF PAPER

NSW Education Standards Authority

2024 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

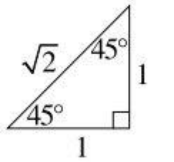
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

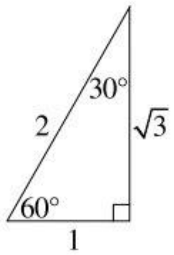
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

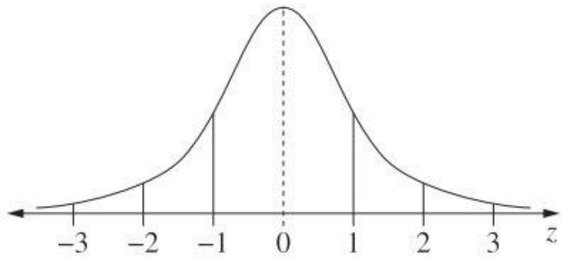
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^{2}(x - c)$$

2024 Year 12 Mathematics Extension 2 -Worked Solutions

1. **B**

$(\alpha-\beta)^4 = (2-i-1+3i)^4 = (1+2i)^4$.
Also, $(1+2i)^4 = 1+4(2i)+6(2i)^2 + 4(2i)^3+(2i)^4$

$= 1+8i + 6\times 4\, i^2 + 4\times 8\, i^3 + 16\, i^4$

$= 1+8i-24-32i+16$

Therefore, $(\alpha-\beta)^4 = -7-24i$.

Hence, the correct option is **B**.

2. **D**

The diagonals of a rectangle are equal in length and must bisect each other. So, if the diagonals of a quadrilateral are equal but they do not bisect each other this indicates that the quadrilateral is not a rectangle. Hence, the statement that
" If a quadrilateral is a rectangle, then its diagonals are equal in length" is true, but its converse, "if the diagonals of a quadrilateral are equal it must be a rectangle" is false.

An example if the quadrilateral is an isosceles trapezium, then its diagonals are equal but they do not bisect each other.

Hence, the correct option is **D**.

3. **C**

$\sqrt{3}+i = 2\, cis\,\frac{\pi}{6}$ this means

$(\sqrt{3}+i)^4 = (2)^4 cis\,\frac{4\pi}{6} = 16\, cis\,\frac{2\pi}{3}$

$\frac{1}{4}-\frac{\sqrt{3}}{4}i = \sqrt{\left(\frac{1}{4}\right)^2+\left(-\frac{\sqrt{3}}{4}\right)^2}\; cis\left(-\frac{\pi}{3}\right)$

$= \frac{1}{2} cis\left(-\frac{\pi}{3}\right)$ this means

$\left(\frac{1}{4}-\frac{\sqrt{3}}{4}i\right)^3 = \left(\frac{1}{2}\right)^3 cis\left(3\times-\frac{\pi}{3}\right) = \frac{1}{8}\; cis(-\pi)$

Hence, $(\sqrt{3}+i)^4\left(\frac{1}{4}-\frac{\sqrt{3}}{4}i\right)^3$

$16\, cis\,\frac{2\pi}{3}\times\frac{1}{8}\; cis(-\pi) = 2\, cis\left(-\frac{\pi}{3}\right)$

So, the new complex number will be twice as far from the origin as E. Also, the new complex number will have an argument which is rotated $-\frac{\pi}{3}$ *radians* from E. The point H is in this position.

Hence, the correct option is **C.**

4. **A**

As $x = 1+4\sin\left(2t-\frac{\pi}{5}\right)$ then

$v = 8\cos\left(2t-\frac{\pi}{5}\right) = 8\cos 2\left(t-\frac{\pi}{10}\right)$

A cos curve without any horizontal translation when $t=0$ it has maximum and its first minimum will be at $t=\frac{1}{2}T$, where T is the period. As this curve is translated by $\frac{\pi}{10}$ to the right then the first maximum is at $t=\frac{\pi}{10}$ and as its period is $T=\frac{2\pi}{2}=\pi$, then the first minimum is at

$t=\frac{\pi}{10}+\frac{1}{2}\times\pi = \frac{3\pi}{5}$.

Hence, the correct option is **A**.

Alternative method

As the minimum value of the velocity is -8 then to find the time the velocity of the particle is minimum, we let $v=-8$. We get $8\cos\left(2t-\frac{\pi}{5}\right)=-8$, that is $\cos\left(2t-\frac{\pi}{5}\right)=-1=\cos\pi$.

$2t-\frac{\pi}{5}=\pi+2n\pi$ or $2t-\frac{\pi}{5}=-\pi+2n\pi$

$2t=\frac{6\pi}{5}+2n\pi \qquad 2t=-\frac{4\pi}{5}+2n\pi$

$t=\frac{3\pi}{5}+n\pi \qquad t=-\frac{2\pi}{5}+n\pi$

When $n=1, t=\frac{3\pi}{5}$ is the smallest positive value for t for which velocity is at a minimum.

Hence, the correct option is **A**.

5. **B**

We are given $\lambda = ln^3\alpha + i\,ln^2\alpha$ and $\phi = -2\,ln\,\alpha+8i$. This indicates $\overrightarrow{OA} = ln^3\alpha+i\,ln^2\alpha$ and $\overrightarrow{OB} = -2\,ln\,\alpha+8i$.

By rotating vector $\overrightarrow{OB}$ by $\frac{\pi}{2}$ in a clockwise direction, we get vector $\overrightarrow{OC}$, this means

$\overrightarrow{OC} = -i\,(-2\,ln\,\alpha+8i) = 8+2i\,ln\,\alpha.$

Now, A and C coincide if their real and imaginary parts are equal. This means

$ln^3\alpha = 8 \;\; and \;\; ln^2\alpha = 2\,ln\,\alpha$

$ln\,\alpha = 2 \;\; (1) \;\; and \;\; ln\,\alpha = 0 \; or \; ln\,\alpha = 2 \;\; (2)$

From the above we can see that (1) and (2) are both valid if only $ln\,\alpha=2$, that is, $\alpha = e^2$.

Hence, the correct option is **B.**

6. **A**

The table below shows the position of the particle for certain values of t.

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$x = -3\sin t$	0	-3	0	3	0
$y = 3\cos t$	3	0	-3	0	3
$z = t$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

From the table we can see that in the xy plane the particle is rotating in anticlockwise direction starting from $(0, 3)$ to $(-3, 0)$ to $(0, -3)$ to $(3, 0)$ and then it repeats the same moves. Also, it should be noted that at the same time its z coordinate is steadily increasing.
So, the motion of the particle is a spiral about the z axis in an anticlockwise direction
Hence, the correct option is **A**.

7. **D**

For a statement "$\forall\, n \in \mathbb{N}$, if P then Q" its negation statement is "$\exists\, n \in \mathbb{N}$ if not P then not Q".

So, as the statement is "$\forall\, n \in \mathbb{N}$, if n^2 is odd then n is odd", the negation statement would be "$\exists\, n \in \mathbb{N}$, if n^2 is not odd then n is also not odd."

Hence, the correct option is **D**.

8. **C**

ℓ_1 and ℓ_2 are parallel if the components of their directional vectors are in the same ratio.

This means $\begin{pmatrix} 2^{\lambda} \\ 6-2^{\lambda} \\ 7-2^{\lambda} \end{pmatrix} = k\begin{pmatrix} 8+2^{\lambda} \\ 6 \\ 5+2^{\lambda} \end{pmatrix}$

So, $2^{\lambda} = k\left(8+2^{\lambda}\right)$ that is $k = \dfrac{2^{\lambda}}{8+2^{\lambda}}$ (1)

Also, $6-2^{\lambda} = 6k$ that is $k = \dfrac{6-2^{\lambda}}{6}$ (2)

From (1) and (2), we get

$$\frac{2^{\lambda}}{8+2^{\lambda}} = \frac{6-2^{\lambda}}{6}$$

$6\times 2^{\lambda} = 48 - 8\times 2^{\lambda} + 6\times 2^{\lambda} - 2^{2\lambda}$
$2^{2\lambda} + 8\times 2^{\lambda} - 48 = 0$
$\left(2^{\lambda}+12\right)\left(2^{\lambda}-4\right) = 0$
$2^{\lambda} = -12$ or $2^{\lambda} = 4$
Note: $2^{\lambda} = -12$ is invalid this indicates that $2^{\lambda} = 4$, that is $\lambda = 2$ is the only solution.
Note: To check solutions, we substitute $\lambda = 2$ into the directional vectors of ℓ_1 and ℓ_2, we get

$\begin{pmatrix} 2^{\lambda} \\ 6-2^{\lambda} \\ 7-2^{\lambda} \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$ and
$\begin{pmatrix} 8+2^{\lambda} \\ 6 \\ 5+2^{\lambda} \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$.

This indicates that ℓ_1 and ℓ_2 are parallel.
Hence, the correct option is **C**.

9. **B**

$$\frac{dv}{dt} = \left(1+v^2\right)^{\frac{3}{2}}$$
$$\frac{dt}{dv} = \frac{1}{\left(1+v^2\right)^{\frac{3}{2}}}$$
$$\int dt = \int \frac{1}{\left(1+v^2\right)^{\frac{3}{2}}}\,dv$$
$$t = \int \frac{1}{\left(1+v^2\right)^{\frac{3}{2}}}\,dv$$

Let $v = \tan\theta$, so $dv = \sec^2\theta\, d\theta$
and $1+v^2 = 1+\tan^2\theta = \sec^2\theta$

$$t = \int \frac{1}{\left(\sec^2\theta\right)^{\frac{3}{2}}}\sec^2\theta\, d\theta$$
$$t = \int \frac{1}{\sec^3\theta}\sec^2\theta\, d\theta$$
$$t = \int \frac{1}{\sec\theta}\, d\theta$$
$$t = \int \cos\theta\, d\theta$$
$$t = \sin\theta + c$$

So, $t = \sin\left(\tan^{-1} v\right) + c$
When $t = 0, v = 0$
so $0 = \sin\left(\tan^{-1} 0\right) + c$, that is
$c = 0$. This indicates that

$t = \sin\left(\tan^{-1} v\right)$

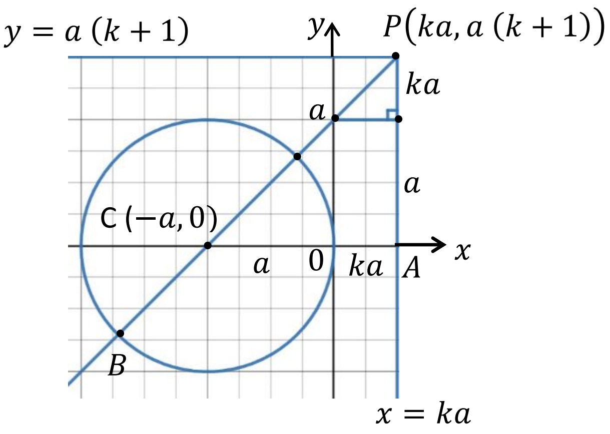
$\sin^{-1} t = \tan^{-1} v$

$v = \tan\left(\sin^{-1} t\right)$

Hence, the correct option is **B.**

10. **A**

As the complex number z satisfies $|z + a| = a$, then it lies on a circle with centre $C(-a, 0)$ and radius a. The diagram shows the circle with centre C and the point P.



From the diagram above we can see that PB is the greatest distance that z can be from the point P.
Using Pythagoras' theorem in triangle APC, we get

$PC = \sqrt{a^2(k+1)^2 + a^2(k+1)^2}$

$PC = \sqrt{2a^2(k+1)^2}$, that is $PC = a(k+1)\sqrt{2}$.

So, $PB = PC + CB = a(k+1)\sqrt{2} + a$

But $PB = \left(3\sqrt{2} + 1\right)a$, this means

$a(k+1)\sqrt{2} + a = \left(3\sqrt{2} + 1\right)a$

Dividing by a, which is a positive real number, we get

$(k+1)\sqrt{2} + 1 = 3\sqrt{2} + 1$

$(k+1)\sqrt{2} = 3\sqrt{2}$ that is $k + 1 = 3$

Therefore, $k = 2$.

Hence, the correct option is **A**.

Question 11

a) Note: $\alpha - \beta = 2i - (\sqrt{3} + i) = -\sqrt{3} + i$.

Now, $|\alpha - \beta| = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$

and $Arg(\alpha - \beta) = \frac{5\pi}{6}$.

So, $(\alpha - \beta)^6 = \left(2\, cis\left(\frac{5\pi}{6}\right)\right)^6 = 2^6\, cis\,(5\pi)$

$= 64\, cis\,(5\pi) = 64(-1) = -64$.

b) To show that $A(2, -5, 3)$ lies on the line ℓ we substitute its coordinates into the equation of ℓ.
We get

$$\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$$

Now, we need to verify that there is only one value for λ in the solutions of the following three equations.

$2 = 3 + 2\lambda$ and $-5 = -2 + 6\lambda$ and $3 = 5 + 4\lambda$

$-1 = 2\lambda \qquad -3 = 6\lambda \qquad -2 = 4\lambda$

$\lambda = -\frac{1}{2} \qquad \lambda = -\frac{1}{2} \qquad \lambda = -\frac{1}{2}$

As $\lambda = -\frac{1}{2}$ is the common solution for the above equations then A lies on the line ℓ.

c) We can solve $z^2 - z + (4 - 2i) = 0$
Using the quadratic formula.

$z = \dfrac{1 \pm \sqrt{1 - 4(4 - 2i)}}{2}$

$z = \dfrac{1 \pm \sqrt{1 - 16 + 8i}}{2}$

$z = \dfrac{1 \pm \sqrt{-15 + 8i}}{2}$ (1)

Now, let $(a + ib)^2 = -15 + 8i$

$a^2 - b^2 + 2ab\,i = -15 + 8i$

Equating reals, we get

$a^2 - b^2 = -15$ (2)

Equating imaginaries, we get

$2ab = 8$ (3)

Equating moduli, we get

$a^2 + b^2 = 17$ (4)

(2) + (4) gives

$2a^2 = 2$, that is $a^2 = 1$.

So, $a = \pm 1$.

Now, when $a = 1, b = 4$ and
when $a = -1, b = -4$,

So, $\sqrt{-15 + 8i} = \pm(1 + 4i)$ (5)

By substituting (5) in (1), we get

$z = \dfrac{1 \pm (1 + 4i)}{2}$

$z = \dfrac{1 + (1 + 4i)}{2}$ or $z = \dfrac{1 - (1 + 4i)}{2}$

$z = \dfrac{2 + 4i}{2}$ or $z = \dfrac{-4i}{2}$

$z = 1 + 2i$ or $z = -2i$.

d) As PQRS is a parallelogram then $\overrightarrow{QP} = \overrightarrow{RS}$.

$$\overrightarrow{QP} = \begin{pmatrix} 2-4 \\ 4-3 \\ 3\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$\overrightarrow{RS} = \begin{pmatrix} x-3 \\ y-2 \\ z\ -5 \end{pmatrix}$$

Now, we equate the components of the two vectors, we get

$x-3=-2$ and $y-2=1$ and $z-5=2$
$x=1$ $\quad y=3$ $\quad z=7$

So, $S\ (1\ ,3\ ,7)$.

Also, $\overrightarrow{QS} = \begin{pmatrix} 1-4 \\ 3-3 \\ 7\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix}$

$\left|\overrightarrow{QS}\right| = \sqrt{(-3)^2+\ 0^2\ +6^2} = 3\sqrt{5}$ units.

e) As the motion is simple harmonic motion with centre at $x=2$ and amplitude 3 this indicates that $v=0$ when $x=5$ and when $x=-1$.

By substituting into $v^2=p+qx-6x^2$, we get
$0=p+5q-150$ (1) and $0=p-q-6$ (2)
By subtracting (2) from (1), we get
$0=6q-144$, that is $q=24$.
By substituting into (2), we get
$0=p-24-6$, that is $p=30$.
So, $v^2=30+24x-6x^2$

$\frac{1}{2}v^2 = 15+12x-3x^2$

$a = \frac{\mathrm{d}\left(\frac{1}{2}v^2\right)}{\mathrm{d}x} = 12-6x$

$a=-6(\ x-2)$ this means
$n^2=\ 6$, that is $n=\sqrt{6}$ as $n>0$.

Hence, the period is $T=\frac{2\pi}{\sqrt{6}}\ =\ \frac{\pi\sqrt{6}}{3}$.

Question 12

a) $\left(x\sqrt{x}-\frac{1}{x\sqrt{x}}\right)^2 \geq 0$, where $x>0$.

This means $x^3-2+\frac{1}{x^3}\geq 0$.

Hence, $x^3+\frac{1}{x^3}\geq 2$.

b) The converse of P would be " If n is even, then $5n^4+8n$ will be a multiple of 16."

Proof: As n is even, let $n=2m$, where m is a positive integer.

$$\begin{aligned} \text{So } 5n^4+8n\ &=\ 5(2m)^4+8\times 2m \\ &= 5\times 16\ m^4+16m \\ &= 16\ (5m^4+m) \end{aligned}$$

Let $5m^4+m=k$, where k is a positive integer.
So, $5n^4+8n=16k$
Hence, the converse of P is true.

c) Let $z=re^{i\theta}$ be a fourth root of $1+\sqrt{3}i$.
This means $z^4=1+\sqrt{3}i$.

Now, as $1+\sqrt{3}i=2e^{\frac{i\pi}{3}}$ then

$z^4=1+\sqrt{3}\,i$ can be written as $r^4e^{4i\theta}=2e^{\frac{i\pi}{3}}$.

This indicates that $r^4=2$, that is $r=2^{\frac{1}{4}}$ $as\ r>0$, and $cos\ 4\theta = cos\ \frac{\pi}{3}$.

So, $4\theta=\frac{\pi}{3}+2n\pi$, that is $\theta=\frac{\pi}{12}+\frac{2n\pi}{4}$.

Therefore, $\theta=\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}\ or\ \frac{19\,\pi}{12}$.

Hence, the four roots of $z^4=1+\sqrt{3}\,i$ are
$2^{\frac{1}{4}}e^{\frac{i\pi}{12}},\ 2^{\frac{1}{4}}e^{\frac{i7\pi}{12}},\ 2^{\frac{1}{4}}e^{\frac{i13\pi}{12}}=2^{\frac{1}{4}}e^{\frac{-i11\pi}{12}}$

$or\ 2^{\frac{1}{4}}e^{\frac{i19\pi}{12}}=2^{\frac{1}{4}}e^{\frac{-i5\pi}{12}}$.

d) Note: the initial velocity is e which is positive. So, the particle will start moving in the positive direction and as its acceleration a at the that time is $e^2(e^4+\ln e)=e^2(e^4+1)$,which is also positive, then the velocity will increase and a will always be positive.

$a=v\frac{dv}{dx}=v^2\ (e^4+\ln v)$, that is

$\frac{dv}{dx}=v\ (e^4+\ln v)$, so $\frac{dx}{dv}=\frac{1}{v\ (e^4+\ln v)}$.

Integrating both sides, we get

$$\int dx = \int \frac{1}{v\ (\ e^4\ +\ ln\,v)}\ dv$$

$$x = \int \frac{1}{v\ \ (e^4\ +\ ln\,v)}\ dv$$

Let $u=e^4+ln\,v$ this indicates that $du=\frac{1}{v}dv$

$x=\int\frac{1}{u}\ du = ln\ u\ +c$ this means

$x=ln\ (e^4\ +\ ln\,v)\ +c$.

When $t=0$, $x=ln\left(\ e^4\ +\ 1\right)$ and $v=e$.
This indicates $ln\ (\ e^4\ +\ 1)=ln\ (\ e^4\ +\ ln\,e\ )+c$
So, $c=0$.
Hence, $x=ln\ (e^4+ln\,v)$.

e) i) $Q(z)=z^4-8z^3+pz^2+qz-80$
As the coefficients of this polynomial are real, the conjugate of the root $3+i$, which is $3-i$, is also a root. Now, let the roots be $3+i\ , 3-i\ , \alpha, \beta$.
The sum of the roots is
$3+i+3-i\ +\ \alpha+\ \beta=8$ that is $\alpha+\ \beta=2$.

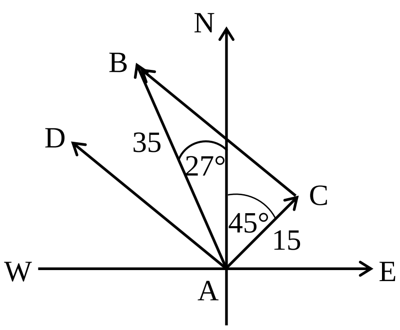
The product of the roots is
$\alpha\,\beta\,(3+i)(3-i) = -80$
$10\,\alpha\,\beta = -80$, this means $\alpha\,\beta = -8$.
The quadratic equation with α and β is
$z^2 - 2z - 8 = 0$
$(z-4)(z+2) = 0$. So, $z = 4$ *or* -2.
Hence, the four roots are $3+i,\ 3-i,\ 4, -2$.
ii) Using part (i), we can write

$$\begin{aligned} Q(z) &= (z-3-i)(z-3+i)(z-4)(z+2) \\ &= ((z-3)^2 - i^2)(z^2-2z-8) \\ &= (z^2-6z+9+1)(z^2-2z-8) \\ &= (z^2-6z+10)(z^2-2z-8). \end{aligned}$$

Question 13

a) The roots of the cubic equation
$2x^3 - 7x + 4 = 0$ are $\alpha,\ \beta$ and γ.
Product of the roots is $\alpha\beta\gamma = -\frac{4}{2} = -2$.
Now, consider the roots of the new cubic equation that is $\frac{1}{\alpha\beta}$, $\frac{1}{\alpha\,\gamma}$ and $\frac{1}{\beta\gamma}$. These roots can written as $\frac{\gamma}{\alpha\beta\gamma}$, $\frac{\beta}{\alpha\beta\gamma}$ and $\frac{\alpha}{\alpha\beta\gamma}$. By Substituting $\alpha\beta\gamma = -2$, these roots will be in fact the same as
$-\frac{\gamma}{2}, -\frac{\beta}{2}$ and $-\frac{\alpha}{2}$.
This indicates the equation needed can be found by substituting $y = -\frac{x}{2}$ that is $x = -2y$ in the original cubic equation.
So, the new equation is $2(-2y)^3 - 7(-2y) + 4 = 0$
$-16y^3 + 14y + 4 = 0$, that is $8y^3 - 7y - 2 = 0$.
Now, as the variable is immaterial, we can write the equation as $8x^3 - 7x - 2 = 0$.

b)



$\overrightarrow{AB}$ represents the course from port A to port B.
$\overrightarrow{AC}$ represents the flow of the current.
$\overrightarrow{CB} = \overrightarrow{AD}$ represents the course and velocity that must be steered to achieve the resultant course $\overrightarrow{AB}$

$\angle BAC = 27° + 45° = 72°$.
Using the cosine rule in triangle ABC, we get
$BC^2 = 35^2 + 15^2 - 2 \times 35 \times 15 \times \cos 72°$
$BC = 33.5489\ldots$
$\angle DAB = \angle ABC$ $(alternate\ angles, AD \parallel CB)$
Now, using the cosine rule, we get
$\cos\angle ABC = \dfrac{35^2 + 33.548\ldots^2 - 15^2}{2\times 35\times 33.5489\ldots}$ that is
$\angle ABC = 25.16°$.
This means $\angle DAB = 25.16°$.
Hence, the boat should head on a bearing of $360° - 27° - 25.16° = 307.84°$ and at a speed of 33.5489 ...km h^{-1} to reach its destination at the required speed.

c) To prove the given statement by contrapositive we need to prove "If p can not be expressed as a sum of five consecutive integers then p is not divisible by 5".
Let us assume that p can not be expressed as a sum of five consecutive integers, this indicates there are no integer q such that $p = q + (q+1) + (q+2) + (q+3) + (q+4)$.
This means that $p \neq 5q + 10$, that is $p \neq 5(q+2)$.
So, $p \neq 5r$, where $r = q + 2$ is an integer and this shows that p is not divisible by 5.
Hence, by contrapositive the statement is true.

d) $I = \displaystyle\int \frac{(2\tan\theta + 3)\sec^2\theta}{\sec^2\theta + \tan\theta}\,d\theta$

$I = \displaystyle\int \frac{(2\tan\theta + 3)\sec^2\theta}{1 + \tan^2\theta + \tan\theta}\,d\theta$

Let $u = \tan\theta$ so $du = \sec^2\theta\ d\theta$

So, $I = \displaystyle\int \frac{2u+3}{1+u+u^2}\,du = \int \frac{2u+1+2}{1+u+u^2}\,du$

$= \displaystyle\int \frac{2u+1}{1+u+u^2}\,du + \int \frac{2}{1+u+u^2}\,du$

$= \ln|1+u+u^2| + \displaystyle\int \frac{2}{1+u+u^2}\,du$

Note: $1+u+u^2 = \left(u+\frac{1}{2}\right)^2 + \frac{3}{4}$ which is positive.

$I = \ln(1+u+u^2) + \displaystyle\int \frac{2}{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}\,du$

$I = \ln(1+u+u^2) + \dfrac{4}{\sqrt{3}}\tan^{-1}\dfrac{2}{\sqrt{3}}\left(u+\dfrac{1}{2}\right) + k$

$I = \ln(1+\tan\theta+\tan^2\theta) + \dfrac{4}{\sqrt{3}}\tan^{-1}\left(\dfrac{2\tan\theta+1}{\sqrt{3}}\right) + k$

e) i) Consider $\dfrac{2k}{k+2} - \dfrac{2k+2}{k+3}$

$= \dfrac{2k}{k+2} \times \dfrac{k+3}{k+3} - \dfrac{2k+2}{k+3} \times \dfrac{k+2}{k+2}$

$= \dfrac{2k^2+6k-(2k^2+6k+4)}{(k+2)(k+3)}$

$= \dfrac{-4}{(k+2)(k+3)} < 0$ as $k > 0$.

So, $\dfrac{2k}{k+2} - \dfrac{2k+2}{k+3} < 0$

Hence, $\dfrac{2k}{k+2} < \dfrac{2k+2}{k+3}$ for $k > 0$.

ii) $\frac{1}{3!} + \frac{2}{4!} + \cdots + \frac{n}{(n+2)!} < \frac{2n}{n+2} - \frac{1}{(n+2)!}$ for $n \geq 1$

For $n = 1$

LHS $= \frac{1}{3!} = \frac{1}{6}$ RHS $= \frac{2}{3} - \frac{1}{3!} = \frac{1}{2}$

As $\frac{1}{6} < \frac{1}{2}$, the statement is true for n = 1.

Assume the statement is true for $n = k$, that is

$\frac{1}{3!} + \frac{2}{4!} + \cdots + \frac{k}{(k+2)!} < \frac{2k}{k+2} - \frac{1}{(k+2)!}$ (1)

Our aim is to prove the statement is true for $n = k+1$, that is

$\frac{1}{3!} + \frac{2}{4!} + \cdots + \frac{k}{(k+2)!} + \frac{k+1}{(k+3)!} < \frac{2k+2}{k+3} - \frac{1}{(k+3)!}$

Starting from the assumption (1) and adding $\frac{k+1}{(k+3)!}$ on both sides, we get:

$$\frac{1}{3!} + \frac{2}{4!} + \cdots + \frac{k}{(k+2)!} + \frac{k+1}{(k+3)!} < \frac{2k}{k+2} - \frac{1}{(k+2)!} + \frac{k+1}{(k+3)!}$$

$$< \frac{2k}{k+2} - \frac{1(k+3)}{(k+3)!} + \frac{k+1}{(k+3)!}$$

$$< \frac{2k}{k+2} - \frac{(k+3)-(k+1)}{(k+3)!}$$

$$< \frac{2k}{k+2} - \frac{2}{(k+3)!}$$

$$< \frac{2k}{k+2} - \frac{1}{(k+3)!}$$

Note: The above inequality is valid as we are subtracting a smaller term.

Now, from part i) $\frac{2k}{k+2} < \frac{2k+2}{k+3}$, by subtracting $\frac{1}{(k+3)!}$ on both sides, we get:

$$\frac{2k}{k+2} - \frac{1}{(k+3)!} < \frac{2k+2}{k+3} - \frac{1}{(k+3)!}$$

Now, from the above, we can state that

$\frac{1}{3!} + \frac{2}{4!} + \cdots + \frac{k}{(k+2)!} + \frac{k+1}{(k+3)!} < \frac{2k+2}{k+3} - \frac{1}{(k+3)!}$

Hence, if the statement is true for $n = k$, it is also true for $n = k+1$.

The statement is true for $n = 1$, and by mathematical induction it is also true for ` $n = 2, 3, 4$ and so on.

Hence, the statement is true for all integers $n \geq 1$.

Question 14

a) i) Let $z = re^{i\theta}$ be a root of the complex equation $z^5 = i$. This means $z^5 = r^5 e^{5i\theta}$ and $i = e^{\frac{i\pi}{2}}$. Then $z^5 = i$ can be written as $r^5 e^{5i\theta} = e^{\frac{i\pi}{2}}$.

This indicates that $r^5 = 1$, that is $r = 1$ as $r > 0$.

and $e^{5i\theta} = e^{\frac{i\pi}{2}}$, that is $cis\ 5\theta = cis\ \frac{\pi}{2}$.

So, $cos\ 5\theta = cos\ \frac{\pi}{2}$, this means

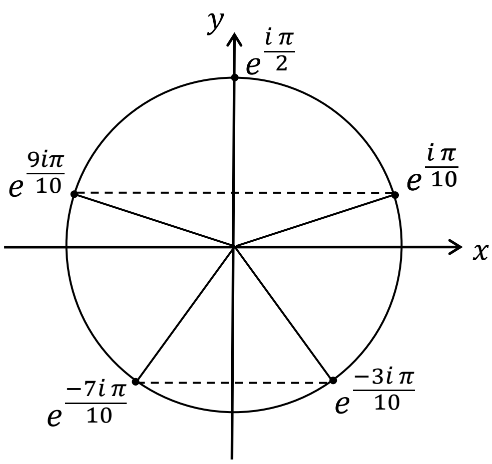
$5\theta = \dfrac{\pi}{2} + 2n\pi$, that is $\theta = \dfrac{\pi}{10} + \dfrac{2n\pi}{5}$.

Therefore, $\theta = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, -\frac{3\pi}{10}$ or $-\frac{7\pi}{10}$.

Hence, the five roots of $z^5 = i$ are

$e^{\frac{i\pi}{10}}$, $e^{\frac{i\pi}{2}}$, $e^{\frac{9i\pi}{10}}$, $e^{\frac{-7i\pi}{10}} = e^{\frac{13i\pi}{10}}$ or $e^{\frac{-3i\pi}{10}} = e^{\frac{17i\pi}{10}}$.

ii) The Argand diagram below shows the five roots of of the equation $z^5 = i$ or $z^5 - i = 0$.



The sum of the roots of the equation $z^5 - i = 0$ is

$e^{\frac{i\pi}{10}} + e^{\frac{i\pi}{2}} + e^{\frac{9i\pi}{10}} + e^{\frac{-7i\pi}{10}} + e^{\frac{-3i\pi}{10}} = 0$

Also, from the diagram $e^{\frac{i\pi}{10}} + e^{\frac{9i\pi}{10}} = 2i\ sin\frac{\pi}{10}$,

and $e^{\frac{-7i\pi}{10}} + e^{\frac{-3i\pi}{10}} = 2i\sin\frac{-3\pi}{10} = -2i\sin\frac{3\pi}{10}$

and $e^{\frac{i\pi}{2}} = i$.

This indicates that $2i\sin\frac{\pi}{10} + i - 2i\sin\frac{3\pi}{10} = 0$.

This means $2\sin\frac{\pi}{10} + 1 - 2\sin\frac{3\pi}{10} = 0$, that is

$\sin\dfrac{3\pi}{10} - \sin\dfrac{\pi}{10} = \dfrac{1}{2}$

From the Reference Sheet,

$sin(A+B) - sin(A-B) = 2\ cosA\ sinB$

Let $A + B = \frac{3\pi}{10}$ (1) and $A - B = \frac{\pi}{10}$ (2)

By adding (1) and (2), we get

$2A = \frac{4\pi}{10}$ that is $A = \frac{\pi}{5}$.

By substituting in (1), we get $B = \frac{\pi}{10}$.

From the above, we can state that

$\sin\dfrac{3\pi}{10} - \sin\dfrac{\pi}{10} = 2\cos\dfrac{\pi}{5}\sin\dfrac{\pi}{10} = \dfrac{1}{2}$

Hence, $\cos\frac{\pi}{5}\ \sin\frac{\pi}{10} = \frac{1}{4}$.

Alternative method

Using $\sin\frac{3\pi}{10} - \sin\frac{\pi}{10} = \frac{1}{2}$, we get

$$2\cos\left(\frac{\frac{3\pi}{10}+\frac{\pi}{10}}{2}\right)\sin\left(\frac{\frac{3\pi}{10}-\frac{\pi}{10}}{2}\right) = \frac{1}{2}$$

$2\cos\frac{\pi}{5}\ \sin\frac{\pi}{10} = \frac{1}{2}$

Hence, $\cos\frac{\pi}{5}\ \sin\frac{\pi}{10} = \frac{1}{4}$.

b) i) $\overrightarrow{ma} = \overrightarrow{10m} + 5\vec{v}$

As the mass is $m = 1$, then

$\vec{a} = \overrightarrow{10} + 5\vec{v}$

$\vec{a}(t) = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -5\dot{x} \\ -5\dot{y} - 10 \end{pmatrix}$

$\ddot{x} = \dfrac{d\dot{x}}{dt} = -5\dot{x}$

$\displaystyle\int \frac{d\dot{x}}{\dot{x}} = \int -5\ dt$

$\ln \dot{x} = -5t + A$, where A is a constant.

$\dot{x} = e^{-5t+A} = e^{A}e^{-5t}$

Let $e^{A} = B$. So, $\dot{x} = Be^{-5t}$

Now, when $t = 0, \dot{x} = V\cos\alpha$, this means

$B = V\cos\alpha$.

Hence, $\dot{x} = Ve^{-5t}\cos\alpha$.

Also, $\ddot{y} = \frac{d\dot{y}}{dt} = -5\dot{y} - 10$

$\displaystyle\int \frac{d\dot{y}}{-5\dot{y}-10} = \int dt$

$\displaystyle -\frac{1}{5}\int \frac{d\dot{y}}{\dot{y}+2} = t$

$\displaystyle\int \frac{d\dot{y}}{\dot{y}+2} = -\,5t$

$\ln\ (\dot{y}+2) = -5t + C$, where C is a constant.

$\dot{y} + 2 = e^{-5t\,+\,C} = e^{c}e^{-5t}$

Let $e^{c} = D$. So, $\dot{y} + 2 = De^{-5t}$

So, $\dot{y} = De^{-5t} - 2$

Now, when $t = 0, \dot{y} = V\sin\alpha$ this means

$V\sin\alpha = D - 2$ that is $D = V\sin\alpha + 2$.

So, $\dot{y} = (V\sin\alpha + 2)\ e^{-5t} - 2$.

Hence, $\vec{v}(t) = \begin{pmatrix} Ve^{-5t}\cos\alpha \\ (\ V\sin\alpha + 2\ )\ e^{-5t} - 2 \end{pmatrix}$.

ii) $\vec{v}(1) = \begin{pmatrix} Ve^{-5}\cos\alpha \\ (\ V\sin\alpha + 2\ )\ e^{-5} - 2 \end{pmatrix}$, but we are given $\vec{v}\ (1) = \begin{pmatrix} 250\ e^{-5} \\ \left(250\sqrt{3}+2\right)\ e^{-5} - 2 \end{pmatrix}$.

This means that $V\cos\alpha = 250$ and

$V\sin\alpha + 2 = 250\sqrt{3} + 2$. So, $V\sin\alpha = 250\sqrt{3}$.

Dividing these two equations, we get

$\tan\alpha = \sqrt{3}$, that is

$\alpha = 60°$ ($as\ \alpha$ is an acute angle).

Now, by substituting $\alpha = 60°$, we get

$V\cos 60° = 250$ that is $V = 500\ ms^{-1}$.

iii) By substituting $\alpha = 60°$ and $V = 500$ in the vector velocity , we get

$\vec{v}(t) = \begin{pmatrix} 500e^{-5t}\cos 60° \\ (500\sin 60° + 2\ )\ e^{-5t} - 2 \end{pmatrix}$

So, $\vec{v}(t) = \begin{pmatrix} 250\ e^{-5t} \\ (\ 250\sqrt{3} + 2\ )\ e^{-5t} - 2 \end{pmatrix}$

Now, to find the horizontal velocity at the origin we let $t = 0$ in $\dot{x} = 250\ e^{-5t}$, we get $\dot{x} = 250$.

Also, we let T be the time taken by the particle to reach its maximum height. This means the horizontal velocity at that position is

$\dot{x} = 250\ e^{-5T}$.

Therefore, the ratio of the horizontal velocity at the origin to the horizontal velocity at the maximum height is

$250 : 250\ e^{-5T} = 1 : e^{-5T}$ (1)

Now, to find T we let $\dot{y} = 0$, we get

$\left(\ 250\sqrt{3} + 2\ \right)\ e^{-5T} - 2 = 0$

$e^{-5T} = \dfrac{2}{250\sqrt{3}+2} = \dfrac{1}{125\sqrt{3}+1}$

By substituting in (1), we get

Ratio $= 1 : \dfrac{1}{125\sqrt{3}+1}$

$= \left(\ 1 + 125\sqrt{3}\ \right) : 1.$

Question 15

a) i) $F = ma = -m\ (\ pv + v^2)$

Dividing by m, we get

$a = -\ (\ pv + v^2)$, that is $a = -\ v\ (\ p + v\ )$.

This means $a = v\frac{dv}{dx} = -\ v\ (\ p + v\ )$

$\dfrac{dv}{dx} = -(\ p + v\ )$

$\displaystyle\int \frac{-1}{p+v}\ dv = \int dx$

$x = -\ln\ |\ p + v\ | + c$

As the particle's initial velocity is p, which is positive this means $p + v > 0$. Then

$x = -\ln(p+v) + c$ (1)

Now, when $t = 0, v = p$ and $x = \ln 2$. This means $\ln 2 = -\ln(p+p) + c$, that is $\ln 2 = -\ln 2p + c$. So, $c = \ln 4p$.

By substituting the value of c in (1), we get

$x = -\ln(p+v) + \ln 4p$

$x = \ln\left(\dfrac{4p}{p+v}\right)$.

ii) $a = \dfrac{dv}{dt} = -v(p+v)$

$\int dt = \int -\dfrac{dv}{v(p+v)}$

$t = -\int \dfrac{dv}{v(p+v)}$ (2)

Now, using partial fractions, we let

$\dfrac{1}{v(p+v)} = \dfrac{a}{v} + \dfrac{b}{p+v}$

Multiplying both sides by $v(p+v)$, we get

$1 = a(p+v) + bv$

For $v = -p$, we get $1 = -b\,p$, that is $b = -\frac{1}{p}$.

For $v = 0$, we get $1 = a\,p$, that is $a = \frac{1}{p}$.

Therefore, $\dfrac{1}{v(p+v)} = \dfrac{1}{p}\left(\dfrac{1}{v} - \dfrac{1}{p+v}\right)$

Now, by substituting in (2), we get

$t = -\dfrac{1}{p}\int\left(\dfrac{1}{v} - \dfrac{1}{p+v}\right)dv$

$t = -\dfrac{1}{p}(\ln|v| - \ln|p+v|) + c$

$t = \dfrac{1}{p}\ln\left|\dfrac{p+v}{v}\right| + c$

As $v > 0$ and $p + v > 0$, this indicates

$t = \dfrac{1}{p}\ln\left(\dfrac{p+v}{v}\right) + c$ (3)

Now, when $t = 0, v = p$, this means

$0 = \frac{1}{p}\ln\left(\frac{p+p}{p}\right) + c$, that is $0 = \frac{1}{p}\ln 2 + c$

So, $c = -\frac{1}{p}\ln 2$.

By substituting in(3), we get

$t = \dfrac{1}{p}\ln\left(\dfrac{p+v}{v}\right) - \dfrac{1}{p}\ln 2$

$t = \dfrac{1}{p}\left(\ln\left(\dfrac{p+v}{v}\right) - \ln 2\right)$

Hence, $t = \frac{1}{p}\ln\left(\frac{p+v}{2v}\right)$.

iii) By substituting $x = \ln 3$ into $x = \ln\left(\dfrac{4p}{p+v}\right)$, we get $\ln 3 = \ln\left(\frac{4p}{p+v}\right)$ that is $\dfrac{4p}{p+v} = 3$.

This means $4p = 3p + 3v$ So, $p = 3v$.

By substituting $t = \frac{1}{2}\ln 2$ and $p = 3v$ into $t = \frac{1}{p}\ln\left(\frac{p+v}{2v}\right)$, we get $\frac{1}{2}\ln 2 = \frac{1}{p}\ln\left(\frac{3v+v}{2v}\right)$.

that is $\frac{1}{2}\ln 2 = \frac{1}{p}\ln 2$.

Dividing both sides by $\ln 2$, we get $\frac{1}{2} = \frac{1}{p}$

Hence, $p = 2$.

Alternative method

By substituting $t = \frac{1}{2}\ln 2$ and $p = 3v$ into $t = \frac{1}{p}\ln\left(\frac{p+v}{v}\right)$, we get $\frac{1}{2}\ln 2 = \frac{1}{3v}\ln\left(\frac{3v+v}{v}\right)$,

that is $\frac{1}{2}\ln 2 = \frac{1}{3v}\ln 2$.

Dividing both sides by $\ln 2$, we get $\frac{1}{2} = \frac{1}{3v}$.

Hence, $3v = 2$ and as $p = 3v$, then $p = 2$.

b) i) L_1 and L_2 are skew lines if they are not parallel and not intersecting.

First, let us check if they are parallel by considering the components of their directional vectors and suppose they are in the same ratio.

This means $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

So, $-1 = k$ (1)

Also, $2 = -k$, that is $k = -2$. (2)

From (1) and (2), we can see that k is not the same So, their directional vectors are not in the same ratio and this indicates that they are not parallel.

Now, we check if they are intersecting by equating their parametric equations.

The parametric equation of the line L_1 or the coordinates of any point on L_1 are

$A(-1-\lambda, -1+2\lambda, 1+\lambda)$ (A)

The parametric equation of the line L_2 or the coordinates of any point on L_2 are

$B(-1+\mu, 1-\mu, -1+\mu)$ (B)

Now, we equate the coordinates of the above two points, we get

$-1-\lambda = -1+\mu$, that is $\mu = -\lambda$. (3)

Also, $-1+2\lambda = 1-\mu$, that is $\mu = 2-2\lambda$. (4)

By equating (3) in (4), we get

$-\lambda = 2-2\lambda$. So, $\lambda = 2$. Also, as $\mu = -\lambda = -2$.

Now, by substituting $\lambda = 2$ into (A) the parametric equations of L_1, that is $(-1-\lambda, -1+2\lambda, 1+\lambda)$.

We get $(-3, 3, 3)$ and then we substitute $\mu = -2$ into (B) the parametric equations of L_2 that is

$(-1+\mu, 1-\mu, -1+\mu)$, we get $(-3, 3, -3)$.
As the points obtained are not the same, then there is no common point between the lines L_1 and L_2. Hence, the lines L_1 and L_2 do not intersect. From the above, we can conclude that L_1 and L_2 are skew lines.

ii) From part i), we use the points A and B that are on L_1 and L_2 respectively to find the vector $\overrightarrow{AB}$. That is

$$\overrightarrow{AB} = \begin{pmatrix} -1+\mu+1+\lambda \\ 1-\mu+1-2\lambda \\ -1+\mu-1-\lambda \end{pmatrix} = \begin{pmatrix} \mu+\lambda \\ 2-\mu-2\lambda \\ -2+\mu-\lambda \end{pmatrix}$$

The shortest distance occurs when $\overrightarrow{AB}$ is perpendicular to both L_1 and L_2.

So, $\overrightarrow{AB}.\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$ and $\overrightarrow{AB}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$.

First, we find $\overrightarrow{AB}.\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$. This means

$$\begin{pmatrix} \mu+\lambda \\ 2-\mu-2\lambda \\ -2+\mu-\lambda \end{pmatrix}.\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$$

$-\mu-\lambda+4-2\mu-4\lambda-2+\mu-\lambda=0$

$2-2\mu-6\lambda=0$ that is $1-\mu-3\lambda=0$

$\mu = 1-3\lambda$ (1)

Also, $\overrightarrow{AB}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$, this means

$$\begin{pmatrix} \mu+\lambda \\ 2-\mu-2\lambda \\ -2+\mu-\lambda \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$\mu+\lambda-2+\mu+2\lambda-2+\mu-\lambda=0$

$-4+3\mu+2\lambda=0$, that is $3\mu = 4-2\lambda$ (2)

Substitute (1) in (2), we get

$3-9\lambda = 4-2\lambda$, that is $\lambda - \frac{1}{7}$.

By substituting in (1), we get $\mu = 1+\frac{3}{7} = \frac{10}{7}$.

Now, the vector $\overrightarrow{AB}$ with the minimum magnitude is

$$\overrightarrow{AB} = \begin{pmatrix} \mu+\lambda \\ 2-\mu-2\lambda \\ -2+\mu-\lambda \end{pmatrix} = \begin{pmatrix} \frac{10}{7}-\frac{1}{7} \\ 2-\frac{10}{7}+\frac{2}{7} \\ -2+\frac{10}{7}+\frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{9}{7} \\ \frac{6}{7} \\ -\frac{3}{7} \end{pmatrix}$$

Alternative method

We substitute $\lambda = -\frac{1}{7}$ and $\mu = \frac{10}{7}$ to find the coordinates of the points A and B when they are the closest to each other, and we get

$A\left(-1+\frac{1}{7}, -1-\frac{2}{7}, 1-\frac{1}{7}\right) = \left(-\frac{6}{7}, -\frac{9}{7}, \frac{6}{7}\right)$

and $B\left(-1+\frac{10}{7}, 1-\frac{10}{7}, -1+\frac{10}{7}\right) = \left(\frac{3}{7}, -\frac{3}{7}, \frac{3}{7}\right)$.

So, $\overrightarrow{AB} = \begin{pmatrix} \frac{3}{7}+\frac{6}{7} \\ -\frac{3}{7}+\frac{9}{7} \\ \frac{3}{7}-\frac{6}{7} \end{pmatrix} = \begin{pmatrix} \frac{9}{7} \\ \frac{6}{7} \\ -\frac{3}{7} \end{pmatrix}$.

Now, to find the shortest distance between the two skew lines L_1 and L_2, we calculate $\left|\overrightarrow{AB}\right|$.

Hence, the shortest distance between the two lines is

$$\sqrt{\left(\frac{9}{7}\right)^2+\left(\frac{6}{7}\right)^2+\left(\frac{-3}{7}\right)^2} = \frac{3\sqrt{14}}{7} \text{ units.}$$

Question 16

a) We are given $A\ (-5,-4)$ and let B be $(b, 5)$, and $C\ (c,-12)$.

This indicates that

$\overrightarrow{AB} = (b+5)+(5+4)i = (b+5)+9i$ and

$\overrightarrow{AC} = (c+5)+(-12+4)i = (c+5)-8i$

By rotating vector $\overrightarrow{AB}$ by $\frac{5\pi}{6}$ in an anticlockwise direction and multiply its magnitude by 2, we get vector $\overrightarrow{AC}$. This means

$\overrightarrow{AC} = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\overrightarrow{AB}$

By substituting $\overrightarrow{AC}$ and $\overrightarrow{AB}$, we get

$(c+5)-8i = 2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left((b+5)+9i\right)$

$(c+5)-8i = \left(-\sqrt{3}+i\right)\left((b+5)+9i\right)$

$(c+5)-8i = -\sqrt{3}(b+5)-9\sqrt{3}i+(b+5)i-9$

$(c+5)-8i = -\sqrt{3}b-5\sqrt{3}-9\sqrt{3}i+bi+5i-9$

$(c+5)-8i = -\sqrt{3}b-5\sqrt{3}-9+\left(-9\sqrt{3}+b+5\right)i$ (1)

Equating imaginary parts in (1), $we\ get$

$-8 = -9\sqrt{3}+b+5$, that is

$-13 = -9\sqrt{3}+b$

Therefore, $b = 9\sqrt{3}-13$.

Now, equating real parts in (1), $we\ get$

$c+5 = -\sqrt{3}b-5\sqrt{3}-9$

By substituting, $b = 9\sqrt{3}-13$, we get

$c+5 = -\sqrt{3}\left(9\sqrt{3}-13\right)-5\sqrt{3}-9$

$c+5 = -27+13\sqrt{3}-5\sqrt{3}-9$

Therefore, $c = 8\sqrt{3}-41$.

Hence, $z_C = 8\sqrt{3}-41-12i$.

b) i) $I_n = \int_0^{\frac{\pi}{2}} sin^{2n+3}x\ \ cos^5 x\, dx,\ \ n = 0, 1, 2, \ldots$

$$= \int_0^{\frac{\pi}{2}} sin^{2n+2}x\,.\ sin\,x\ \ .cos^5 x\, dx$$

Using integration by parts,

Let $u = sin^{2n+2}x$

$du = (2n+2)\, sin^{2n+1}x\ \cos x\ dx$

and let $dv = sin\,x\ \ .cos^5 x\, dx$

$v = \int sin\,x\,.cos^5 x\ \ dx = -\frac{1}{6}cos^6 x$

$$I_n = \left[sin^{2n+2}x \times -\frac{1}{6}cos^6 x \right]_0^{\frac{\pi}{2}} + \frac{2n\ +\ 2}{6}\int_0^{\frac{\pi}{2}} sin^{2n+1}x\,.\,cos^7 x\, dx$$

Note: $cos^6 \frac{\pi}{2} = \left(\cos\frac{\pi}{2}\right)^6 = 0$

and $sin^{2n+2}0 = (\sin 0)^{2n+2} = 0$

$$I_n = 0 + \frac{n\ +\ 1}{3}\int_0^{\frac{\pi}{2}} sin^{2n+1}x\,.\,cos^7 x\, dx$$

$$I_n = \frac{n\ +\ 1}{3}\int_0^{\frac{\pi}{2}} sin^{2n+1}x\,.\,cos^5 x\,(1 - sin^2 x)\ \ dx$$

$$I_n = \frac{n\ +\ 1}{3}\left[\int_0^{\frac{\pi}{2}} sin^{2n+1}x\,.\,cos^5 x\, dx - \int_0^{\frac{\pi}{2}} sin^{2n+3}x\,.\,cos^5 x\, dx\right]$$

$$I_n = \frac{n\ +\ 1}{3}(\,I_{n-1} - I_n\,)$$

$3I_n = (n+1)I_{n-1} - (n+1)I_n$

$(3 + n + 1)I_n = (n+1)I_{n-1}$

$(n+4)I_n = (n+1)I_{n-1}$

Hence, $I_n = \left(\frac{n+1}{n+4}\right)\ I_{n-1}$ for $n \geq 1$.

ii) $I_n = \left(\frac{n+1}{n+4}\right)I_{n-1}$

$I_{n-1} = \left(\frac{n}{n+3}\right)I_{n-2}$

$I_{n-2} = \left(\frac{n-1}{n+2}\right)I_{n-3}$

$I_{n-3} = \left(\frac{n-2}{n+1}\right)I_{n-4}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$I_2 = \left(\frac{3}{6}\right)I_1$

$I_1 = \left(\frac{2}{5}\right)I_0$

So, $I_n = \left(\frac{n+1}{n+4}\right) \times \left(\frac{n}{n+3}\right)I_{n-2}$

$I_n = \left(\frac{n+1}{n+4}\right) \times \left(\frac{n}{n+3}\right) \times \left(\frac{n-1}{n+2}\right)I_{n-3}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$$I_n = \frac{(\,n+1\,)\,(\,n\,)\,(\,n-1\,)\,(n-2)\ldots.\ 5 \times 4 \times\ 3 \times 2}{(n+4)(n+3)\,(n+2)\,(n+1)\ldots 8 \times 7 \times 6 \times 5} \times I_0 \quad (1)$$

Now, to find I_0 *we* let $n = 0$ in

$I_n = \int_0^{\frac{\pi}{2}}\ sin^{2n+3}x\ \ cos^5 x\, dx$, we get

$I_0 = \int_0^{\frac{\pi}{2}} sin^3 x\ \ cos^5 x\, dx$

$$I_0 = \int_0^{\frac{\pi}{2}} sin\,x\,(1 - cos^2 x)\, cos^5 x\, dx$$

Let $u = cos\,x$ When $x = 0$, $u = 1$

So, $du = -\,sin\,x\,dx$ and When $x = \frac{\pi}{2}$, $u = 0$

Now, $I_0 = -\int_1^0 (1 - u^2)\, u^5\, du$

$I_0 = \int_0^1 (\,u^5 - u^7)\ du$

$I_0 = \left[\frac{u^6}{6} - \frac{u^8}{8}\right]_0^1$

$I_0 = \left(\frac{1}{6} - \frac{1}{8}\right) - (0 - 0)$

$I_0 = \frac{1}{24}$

By substituting the value of I_0 in (1), we get

$$I_n = \frac{(\,n+1\,)\,(\,n\,)\,(\,n-1\,)\ldots.\ \times 3 \times 2}{(n+4)\,(n+3)\,(n+2)\ldots \times 6 \times 5} \times \frac{1}{24}$$

$$I_n = \frac{(\,n+1\,)\,(\,n\,)(n-1\,)\ldots.\ \times 3 \times 2}{(n+4)(n+3)(n+2)\ldots \times 6 \times 5 \times 4 \times 3 \times 2}$$

Hence, $I_n = \frac{1}{(\mathrm{n}+4)\,(\mathrm{n}+3)\,(\mathrm{n}+2)}$.

iii) $J_n = \int_0^1\ x^{4n\,+\,7}\,(1 - x^4)^{\,2}\ \ dx$

Let $x^2 = sin\theta$ and when $x = 1, \theta = \frac{\pi}{2}$

$2x\, dx = cos\theta\ d\theta$ when $x = 0$, $\theta = 0$

$x\, dx = \frac{1}{2}cos\theta\ d\theta$

$$J_n = \int_0^1 (\,x^2)^{\,2n\,+3}\,(1 - (\,x^2)^{\,2})^{\,2}.\ \ x\, dx$$

$$J_n = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin^{2n+3}\theta\ (1 - sin^2\theta)^{\,2}\, cos\theta\ d\theta$$

$$J_n = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin^{2n+3}\theta\ (cos^2\theta)^{\,2}\, cos\theta\ d\theta$$

$$J_n = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin^{2n+3}\theta\ \ cos^5\theta\ d\theta$$

As the variable is immaterial,

$$J_n = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin^{2n+3}x\ \ \ cos^5 x\ \ dx$$

Hence, $J_n = \frac{1}{2}I_n$.

c) Assume there is an x in the domain $0 \le x \le \frac{\pi}{2}$ for which $(\sin x)^{\frac{2}{n}} + (\cos x)^{\frac{2}{n}} < 1$, and $n = 3, 4, 5, \ldots$.
Also, as $0 \le x \le \frac{\pi}{2}$, then $\sin x \ge 0$ and $\cos x \ge 0$. This indicates that

$$0 \le (\sin x)^{\frac{2}{n}} + (\cos x)^{\frac{2}{n}} < 1.$$

Now, by raising the inequality to the power of n, we get

$$0 \le \left((\sin x)^{\frac{2}{n}} + (\cos x)^{\frac{2}{n}} \right)^n < 1 \qquad (1)$$

Note: $\left((\sin x)^{\frac{2}{n}} + (\cos x)^{\frac{2}{n}}\right)^n$

$$= \binom{n}{0}\left((\sin x)^{\frac{2}{n}}\right)^n + \binom{n}{1}\left((\sin x)^{\frac{2}{n}}\right)^{n-1}\left((\cos x)^{\frac{2}{n}}\right)^1 + \cdots + \binom{n}{n-2}\left((\sin x)^{\frac{2}{n}}\right)^2\left((\cos x)^{\frac{2}{n}}\right)^{n-2} + \binom{n}{n-1}\left((\sin x)^{\frac{2}{n}}\right)^1\left((\cos x)^{\frac{2}{n}}\right)^{n-1} + \binom{n}{n}\left((\cos x)^{\frac{2}{n}}\right)^n$$

But as $\binom{n}{0}\left((\sin x)^{\frac{2}{n}}\right)^n = 1 \times (\sin x)^{\frac{2\times n}{n}} = sin^2 x$

and $\binom{n}{n}\left((\cos x)^{\frac{2}{n}}\right)^n = 1 \times (\cos x)^{\frac{2\times n}{n}} = cos^2 x$,

this means $\left((\sin x)^{\frac{2}{n}} + (\cos x)^{\frac{2}{n}}\right)^n$

$$= sin^2 x + \binom{n}{1}\left((\sin x)^{\frac{2}{n}}\right)^{n-1}\left((\cos x)^{\frac{2}{n}}\right)^1 + \binom{n}{2}\left((\sin x)^{\frac{2}{n}}\right)^{n-2}\left((\cos x)^{\frac{2}{n}}\right)^2 + \cdots + \binom{n}{n-1}\left((\sin x)^{\frac{2}{n}}\right)^1\left((\cos x)^{\frac{2}{n}}\right)^{n-1} + cos^2 x$$

$$= 1 + \binom{n}{1}\left((\sin x)^{\frac{2}{n}}\right)^{n-1}\left((\cos x)^{\frac{2}{n}}\right)^1 + \binom{n}{2}\left((\sin x)^{\frac{2}{n}}\right)^{n-2}\left((\cos x)^{\frac{2}{n}}\right)^2 + \cdots + \binom{n}{n-1}\left((\sin x)^{\frac{2}{n}}\right)^1\left((\cos x)^{\frac{2}{n}}\right)^{n-1} \qquad (2)$$

By substituting (2) into (1), we get

$$0 \le 1 + \binom{n}{1}\left((\sin x)^{\frac{2}{n}}\right)^{n-1}\left((\cos x)^{\frac{2}{n}}\right)^1 + \binom{n}{2}\left((\sin x)^{\frac{2}{n}}\right)^{n-2}\left((\cos x)^{\frac{2}{n}}\right)^2 + \cdots + \binom{n}{n-1}\left((\sin x)^{\frac{2}{n}}\right)^1\left((\cos x)^{\frac{2}{n}}\right)^{n-1} < 1$$

By subtracting 1, we get

$$-1 \le \binom{n}{1}\left((\sin x)^{\frac{2}{n}}\right)^{n-1}\left((\cos x)^{\frac{2}{n}}\right)^1 + \binom{n}{2}\left((\sin x)^{\frac{2}{n}}\right)^{n-2}\left((\cos x)^{\frac{2}{n}}\right)^2 + \cdots + \binom{n}{n-1}\left((\sin x)^{\frac{2}{n}}\right)^1\left((\cos x)^{\frac{2}{n}}\right)^{n-1} < 0 \qquad (3)$$

But the sum $\binom{n}{1}\left((\sin x)^{\frac{2}{n}}\right)^{n-1}\left((\cos x)^{\frac{2}{n}}\right)^1 + \binom{n}{2}\left((\sin x)^{\frac{2}{n}}\right)^{n-2}\left((\cos x)^{\frac{2}{n}}\right)^2 + \cdots + \binom{n}{n-1}\left((\sin x)^{\frac{2}{n}}\right)^1\left((\cos x)^{\frac{2}{n}}\right)^{n-1}$ is positive as all its terms are positive. This means it cannot have a value between -1 and 0 as shown in (3).
This causes a contradiction which means the assumption

$(\sin x)^{\frac{2}{n}} + (\cos x)^{\frac{2}{n}} < 1$ is not valid.

Hence, $(\sin x)^{\frac{2}{n}} + (\cos x)^{\frac{2}{n}} \ge 1$,
$for\ 0 \le x \le \frac{\pi}{2}$ and $n = 3, 4, 5, \ldots$