CSSA

2022
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

DO NOT REMOVE PAPER FROM EXAMINATION ROOM

Centre Number

Student Number

Mathematics Extension 2

Morning Session
Monday, 8 August 2022

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- Use the Multiple-Choice Answer Sheet provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 100

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I
10 marks

Attempt Questions 1–10
Allow about 15 minutes for this section
Use the Multiple-Choice Answer Sheet for Questions 1–10

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1 What is the smallest positive value for n so that $(\sqrt{3}+i)^n$ is real?

A. 0

B. 3

C. 6

D. 12

2 The displacement x metres of a particle undergoing simple harmonic motion at time t seconds is given by $x = 3\sin\left(2t + \frac{\pi}{3}\right) + 1$. Which of the following statements is true?

A. The period is π and the amplitude is 3.

B. The period is π and the amplitude is 4.

C. The period is $\frac{\pi}{3}$ and the amplitude is 3.

D. The period is $\frac{\pi}{3}$ and the amplitude is 4.

3 What is the remainder when $17z^4 - 5z + 2$ is divided by $z + i$?

A. $-15 - 5i$

B. $-15 + 5i$

C. $19 - 5i$

D. $19 + 5i$

4 Consider the statement:

'If it is sunny, then Jamie wears a hat'.

Which of the following is the converse of this statement?

A. If Jamie wears a hat, then it is sunny.

B. If Jamie wears a hat, then it is not sunny.

C. If Jamie does not wear a hat, then it is sunny.

D. If Jamie does not wear a hat, then it is not sunny.

5 Given that $z = 2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$, which expression is equal to $(\bar{z})^{-1}$?

A. $\frac{1}{2}(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5})$

B. $2(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5})$

C. $\frac{1}{2}(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$

D. $2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$

6 Which expression is equal to $\int \frac{2x+4}{x^2+16}\,dx$?

A. $2\ln|x^2+16| + 4\tan^{-1}\left(\frac{x}{4}\right) + c$

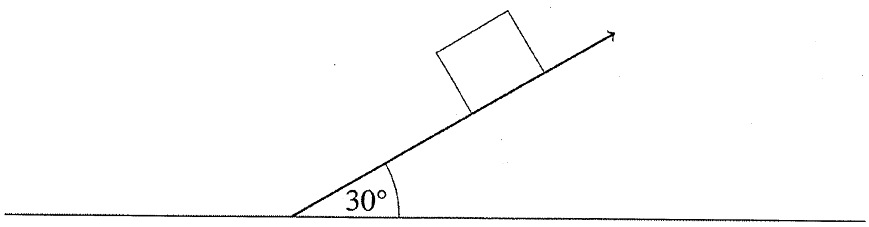
B. $\ln|x^2+16| + \tan^{-1}\left(\frac{x}{4}\right) + c$

C. $\ln|x^2+16| + 4\tan^{-1}\left(\frac{x}{4}\right) + c$

D. $2\ln|x^2+16| + \tan^{-1}\left(\frac{x}{4}\right) + c$

7 A 10 kg box on a plane inclined at an angle of 30° to the horizontal is undergoing uniform acceleration of $1\cdot5$ m/s^2.

Take the acceleration g due to gravity to be $9\cdot8$ m/s^2.



What is the magnitude of the frictional force resisting the motion of the box?

A. 34 N

B. 64 N

C. 70 N

D. 100 N

8 Consider the lines $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ a \end{pmatrix}$ and $\underset{\sim}{s} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$, where $\lambda, \mu \in \mathbb{R}$.

For what value of a will the lines $\underset{\sim}{r}$ and $\underset{\sim}{s}$ intersect at a point?

A. $a = -6$

B. $a = -1$

C. $a = 1$

D. $a = 6$

9 A particle of mass m moves horizontally through a medium with velocity v at time t. Initially, the particle is at the origin O moving with speed v_0. The resistance on the particle due to the medium is proportional to the square of the speed.

If k is a constant of proportionality, which expression gives the correct velocity of the particle?

A. $v = \dfrac{k}{m}t + \dfrac{1}{v_0}$

B. $v = \dfrac{mv_0}{ktv_0 + m}$

C. $v = v_0 e^{-\frac{k}{m}t}$

D. $v = -\dfrac{k}{m}t + \ln v_0$

10 The position vector of the point P is given by $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ where $\lambda \in \mathbb{R}$.

The point Q has coordinates $(2, -2, -5)$.
Which of the following gives the correct expression for $|\overrightarrow{QP}|$ in terms of λ?

A. $\sqrt{5\lambda^2 + 18\lambda + 18}$

B. $\sqrt{5\lambda^2 + 10\lambda + 66}$

C. $\sqrt{5\lambda^2 + 8\lambda + 9}$

D. $\sqrt{5\lambda^2 + 6\lambda + 18}$

Section II
90 marks

Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

Your responses for Questions 11-16 should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Write the contrapositive of the following statement. **1**

'If you have measured your size correctly then your clothes fit you well'.

(b) Find $\displaystyle\int \frac{7x-11}{(x-1)(x-3)}\,dx$. **3**

(c) The complex numbers $z = 2+3i$ and $w = 3-2i$ are given.

(i) Find the value of $z + 2\overline{w}$ in the form $x+iy$. **1**

(ii) Find the value of $\dfrac{w}{z}$ in the form $x+iy$. **2**

(d) A particle moves in one dimension such that its acceleration a ms^{-2} is inversely proportional to its velocity v ms^{-1} as given by the equation $a = \dfrac{72}{v}$. When the time t seconds is $t = 1$ its displacement x metres will be $x = 8$ and also $v = 12$. **3**

Given that $t > 0$ show that $x = 8t^{3/2}$.

(e) Find $\displaystyle\int \frac{1}{4x^2+8x+13}\,dx$. **3**

(f) Prove by contradiction that $\log_{10} 7$ is an irrational number. **2**

Question 12 (14 marks) Use a SEPARATE writing booklet.

(a) Consider the equation $z^3 + 15z^2 + cz + 34 = 0$ where c is a real number. One of the roots of the equation is $1 + i$.

(i) Find the real root of the equation. **1**

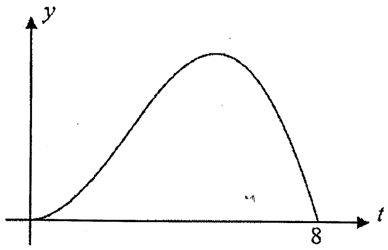
(ii) Determine the value of c. **1**

(b) A complex number z satisfies the inequation $|z - 4i| \leq 2$.

(i) Sketch the region of z on an Argand diagram. **2**

(ii) Find the range of possible values for the principal argument of z. **2**

(c) The instantaneous rate of energy production of a solar panel, y megajoules per hour, during an 8 hour period is given by the equation $y = t\sin\left(\frac{\pi t}{8}\right)$ as shown in the diagram below. **3**



By finding the area under the curve, calculate the number of megajoules produced by the solar panel over the 8 hour period. Give your answer correct to 2 decimal places.

(d) Consider the line $\underset{\sim}{l} = \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 7 \\ 8 \end{pmatrix}$ where $\lambda \in \mathbb{R}$, and the line

$\underset{\sim}{m} = \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ where $\mu \in \mathbb{R}$.

(i) Show that $\underset{\sim}{l}$ and $\underset{\sim}{m}$ intersect at right angles. **2**

(ii) Find the equation of a line that intersects both $\underset{\sim}{l}$ and $\underset{\sim}{m}$ at right angles. **3**

Question 13 (16 marks) Use a SEPARATE writing booklet.

(a) The nth term T_n of a sequence is defined such that $T_n = 2T_{n-1} - n^2$, and $T_1 = 10$. Prove by mathematical induction that $T_n = n^2 + 4n + 6 - 2^{n-1}$ for all positive integers n. **3**

(b) (i) Given $z = e^{i\theta}$, show that $2\cos(k\theta) = z^k + z^{-k}$. **1**

(ii) Expand $\left(z - z^{-1}\right)^4$. Hence, or otherwise, show that **3**

$$\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3).$$

(c) (i) Show that $\frac{d}{dx}\sec x = \sec x \tan x$. **1**

(ii) A constant k satisfies $\int_0^{\frac{\pi}{3}} \left(k\cos^2 x - \sec^2 x\right)\sin x\,dx = \frac{11}{24}$. Evaluate k. **3**

(d) A particle moving in one dimension has position x m and its velocity v m/s is given by

$$\tfrac{1}{2}v^2 = 2 - 4x - 2x^2.$$

(i) Show that the motion of the particle is simple harmonic. **2**

(ii) Given the range of motion is $x_1 \le x \le x_2$, determine the values of x_1 and x_2. **2**

(iii) At time $t = 0$, $x = 0$ and $v > 0$. Find when the particle is next at the origin. **1**

Question 14 (16 marks) Use a SEPARATE writing booklet.

(a) (i) If a and b are real numbers, and $\underset{\sim}{p} = 3a\underset{\sim}{i} + b\underset{\sim}{j}$ show that $|\underset{\sim}{p}| = \sqrt{9a^2 + b^2}$. **1**

(ii) By choosing an appropriate vector $\underset{\sim}{q}$, use the triangle inequality, or otherwise, to prove for all real numbers a and b, that **3**

$$\sqrt{a^2+b^2} \leq \frac{\sqrt{9a^2+b^2}+\sqrt{a^2+9b^2}}{4}.$$

(b) Let $I_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}\,dx$.

(i) Show when $n \geq 2$, that $I_n = \frac{\sqrt{2}}{n} - \frac{n-1}{n} I_{n-2}$. **3**

(ii) Hence, or otherwise, evaluate $\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx$. **2**

(c) Prove that the double of the sum of the squares of two distinct positive integers can be written as the sum of two distinct non-zero square integers. **2**

(d) Let $z = a + ib$, where $a > 0$ and $b > 0$, be represented by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

(i) Find the vector representation for $\frac{1}{z}$. **1**

(ii) Let the angle between the two vectors represented by z and $\frac{1}{z}$ be θ. **2**

By using the dot product, show $\theta = \cos^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right)$.

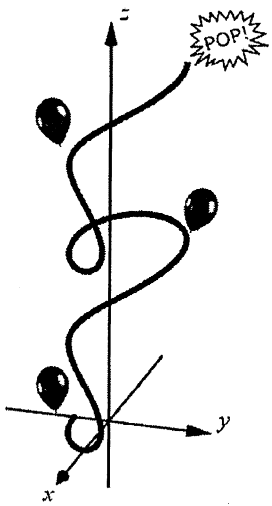
(iii) Hence show that $\cos^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$. **2**

Question 15 (13 marks) Use a SEPARATE writing booklet.

(a) By considering the roots of the equation $z^9+1=0$, or otherwise, show that **3**

$$\cos\left(\frac{\pi}{9}\right)=\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{4\pi}{9}\right).$$

(b) A helium balloon is released from the ground and floats upwards for 10 seconds before bursting as shown in the diagram below.



The position in metres of the balloon after t seconds is given by the vector

$$\underset{\sim}{r}=\begin{pmatrix}4\sin t\\ -\cos 2t\\ 2t-\sin 2t\end{pmatrix}.$$

(i) Find an expression for the velocity $\underset{\sim}{v}$ of the balloon at time t. **2**

(ii) Show that the speed of the balloon $|\underset{\sim}{v}|$ is a constant 4 m/s. **2**

(iii) Hence find the length of the path the balloon took from when it was released to when it burst at $t=10$. **1**

(c) (i) Show that $\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\operatorname{Re}\left(e^{i\theta}\frac{1-e^{in\theta}}{1-e^{i\theta}}\right)$. **2**

(ii) Hence, or otherwise, show that **3**

$$\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\cos\left((n+1)\tfrac{\theta}{2}\right)\times\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

Question 16 (16 marks) Use a SEPARATE writing booklet.

(a) (i) Given that p and q are two positive integers, show that **2**

$$\int_0^1 x^p (1-x)^q \, dx = \frac{q}{p+1} \int_0^1 x^{p+1} (1-x)^{q-1} dx.$$

(ii) Hence, show that $\displaystyle\int_0^1 x^p (1-x)^q \, dx = \frac{p!q!}{(p+q+1)!}$. **4**

(b) By considering the concavity of $y = \sqrt[3]{x}$, prove that if $a > b > 0$, then **3**

$$\sqrt[3]{a-b} + \sqrt[3]{a+b} < 2\sqrt[3]{a}.$$

(c) A falling object of mass m kg experiences acceleration due to gravity of g m/s^2 and air resistance of magnitude kv^2 newtons where v is the object's velocity in m/s at time t seconds.

(i) Assuming that the upwards direction is positive, show that the velocity v of a dropped object is given by **4**

$$v = \sqrt{\frac{mg}{k}} \left(\frac{e^{-t\sqrt{gk/m}} - e^{t\sqrt{gk/m}}}{e^{-t\sqrt{gk/m}} + e^{t\sqrt{gk/m}}} \right).$$

(ii) Andre steps from a plane at an altitude of 5000 metres and must open his parachute at an altitude of 1500 metres to land safely. His coefficient k of air resistance is $0 \cdot 25$, his mass is 100 kg, and the acceleration due to gravity is 10 m/s^2. After how many seconds must Andre open his parachute? **3**

End of Examination

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EXAMINERS

David Houghton (Convenor)	Oxley College, Burradoo
Geoff Carroll	Sydney Grammar School, Darlinghurst
Rebekah Johnson	Loreto Kirribilli, Kirribilli
Svetlana Onisczenko	Meriden School, Strathfield
Gerry Sozio	Edmund Rice College, West Wollongong

2022

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

CSSA

MARKING GUIDELINES

Mathematics Extension 2

Section I
10 marks

Multiple Choice Answer Key

Question	Answer	Outcomes Assessed	Targeted Performance Bands
1	C	MEX12-1, MEX12-4	E2
2	A	MEX12-6	E2
3	D	MEX12-4	E2
4	A	MEX12-2	E2
5	C	MEX12-4	E2
6	B	MEX12-5	E2-E3
7	A	MEX12-6	E3
8	D	MEX12-3	E3-E4
9	B	MEX12-6	E4
10	D	MEX12-3	E4

Question 1 (1 mark)
Outcomes Assessed: *MEX12-1, MEX12-4*
Targeted Performance Bands: *E2*

Solution	Mark
$(\sqrt{3}+i)^n = \left(2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right)^n$ $= 2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right)$ by de Moivre's Theorem For this to be real, we need $0=\sin\frac{n\pi}{6}$, which is true if n is a multiple of 6. Since 0 is not positive, $n=6$ is the smallest value to satisfy this condition. Hence C	1

Question 2 (1 mark)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E2*

Solution	Mark
$3\sin\left(2t+\frac{\pi}{3}\right)+1=3\sin\left(2\left(t+\frac{\pi}{6}\right)\right)+1$ So the period is $\frac{2\pi}{2}=\pi$ and the amplitude is 3. Hence A	1

Question 3 (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E2*

Solution	Mark
Let $P(z)=17z^4-5z+2$ $P(-i)=17\times(-i)^4-5\times(-i)+2$ $=17+5i+2$ $=19+5i$ Hence D	1

Question 4 (1 mark)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E2*

Solution	Mark
The converse of $p\Rightarrow q$ is $q\Rightarrow p$. So the converse of the statement is "If Jamie wears a hat, then it is sunny". Hence A	1

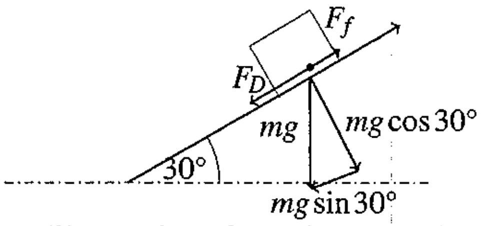
Question 5 (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E2*

Solution	Mark
$\lvert\bar{z}\rvert^{-1}=\left(2\left(\overline{\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}}\right)\right)^{-1}$ $=\frac{1}{2}(\cos\frac{\pi}{5}-i\sin\frac{\pi}{5})^{-1}$ $=\frac{1}{2}(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5})$ Hence C	1

Question 6 (1 mark)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E2-E3*

Solution	Mark
$\int \frac{2x+4}{x^2+16}\,dx = \int \frac{2x}{x^2+16}\,dx + \int \frac{4}{x^2+16}\,dx$ $= \ln\left(x^2+16\right) + \tan^{-1}\left(\frac{x}{4}\right) + c$ Hence B	1

Question 7 (1 mark)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E3*

Solution	Mark
 The resultant force down the plane is $F_R = F_D - F_f$ $ma = mg\sin 30^\circ - F_f$ $F_f = mg\sin 30^\circ - ma$ $= 98\sin 30^\circ - 10 \times 1.5$ $= 34N$ Hence A	1

Question 8 (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3-E4*

Solution	Mark
The two lines have different gradients and hence are not parallel. To find intersection consider $3+\lambda = 2+\mu$ and $-5-3\lambda = 2-5\mu$. Solving simultaneously gives $\lambda = 1$ and $\mu = 2$. $4+\lambda a = 2+\mu \times 4$ $4+a = 2+2\times 4$ $a = 6$ If $a = 6$ then the two lines intersect. Hence D	1

Question 9 (1 mark)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E4*

Solution	Mark
$F \propto -v^2$ $m\ddot{x} = -kv^2$ $\frac{dv}{dt} = -\frac{kv^2}{m}$ $\frac{dt}{dv} = -\frac{m}{kv^2}$ $-\frac{k}{m}\frac{dt}{dv} = v^{-2}$ $\int -\frac{k}{m}\,dt = \int v^{-2}\,dv$ $-\frac{k}{m}t = -\frac{1}{v} + c$ when $t = 0$, $v = v_0$, giving $c = \frac{1}{v_0}$ $\frac{1}{v} = \frac{kt}{m} + \frac{1}{v_0}$ $v = \frac{mv_0}{ktv_0 + m}$ Hence B	1

Question 10 (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E4*

Solution	Mark
$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$ $= \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -5 \end{pmatrix}$ $= \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ $\left\lvert\overrightarrow{QP}\right\rvert = \sqrt{(2\lambda + 3)^2 + (3 - \lambda)^2}$ $= \sqrt{4\lambda^2 + 12\lambda + 9 + 9 - 6\lambda + \lambda^2}$ $= \sqrt{5\lambda^2 + 6\lambda + 18}$ Hence D	1

Section II

90 marks

Question 11 (15 marks)

11(a) (1 mark)

Outcomes Assessed: *MEX12-2*

Targeted Performance Bands: *E2*

Criteria	Mark
• Provides correct solution	1

Sample Answer:

"If your clothes do not fit you well, then you have not measured your size correctly."

11(b) (3 marks)

Outcomes Assessed: *MEX12-5*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct solution	3
• Equates coefficients to give equations in A and B	2
• Attempts partial fractions	1

Sample Answer:

$$\text{Let } \frac{7x-11}{(x-1)(x-3)} \equiv \frac{A}{x-1} + \frac{B}{x-3}$$

$$7x-11 \equiv A(x-3)+B(x-1)$$

$$7x-11 \equiv x(A+B)-3A-B$$

Equating coefficients:
$$A+B=7$$
$$-3A-B=-11$$

Solving simultaneously $A=2$ and $B=5$, so

$$\int \left(\frac{2}{x-1}+\frac{5}{x-3}\right) dx = 2\ln|x-1|+5\ln|x-3|+c$$

11(c) (i) (1 mark)

Outcomes Assessed: *MEX12-4*

Targeted Performance Bands: *E2-E3*

Criteria	Mark
• Provides correct solution	1

Sample Answer:

$z+2\overline{w} = 2+3i+2(3+2i) = 8+7i$

11(c) (ii) (2 marks)

Outcomes Assessed: *MEX12-4*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct solution	2
• Attempts to use complex conjugate of denominator	1

Sample Answer:

$$\begin{aligned}\frac{w}{z} &= \frac{3-2i}{2+3i} \times \frac{2-3i}{2-3i} \\ &= \frac{6-9i-4i-6}{4+9} \\ &= -\frac{13i}{13} = -i\end{aligned}$$

11(d) (3 marks)

Outcomes Assessed: *MEX12-6, MEX12-7*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct solution	3
• Integrates correctly twice while ignoring constant of integration OR Integrates correctly once and uses conditions at $t = 1$ to evaluate constant OR derives a from v, having correctly tested conditions at $t = 1$ for both x and v	2
• Integrates correctly at least once OR derives v from x, testing conditions at $t = 1$ for at least one of them	1

Sample Answer:

$$a = v\frac{dv}{dx} = \frac{72}{v}$$

$$\frac{dx}{dv} = \frac{v^2}{72}$$

$$x = \frac{v^3}{216} + c_1$$

When $x = 8$, $v = 12$ so $c_1 = 0$.

$$v = 6x^{1/3}$$

$$\frac{dt}{dx} = \frac{1}{6}x^{-1/3}$$

$$t = \frac{1}{6}x^{2/3} \times \frac{3}{2} + c_2$$

When $x = 8$, $t = 1$ so $c_2 = 0$.

$$t = \frac{1}{4}x^{2/3}$$

$$x = 8t^{3/2}$$

OR (an alternative method)

test if $x = 8t^{3/2}$ solves $a = \dfrac{72}{v}$

$$x = 8t^{3/2}$$

When $t = 1$, $x = 8$.

$$\begin{aligned}v &= \frac{3}{2}8t^{1/2} \\ &= 12t^{1/2}\end{aligned}$$

When $t = 1$, $v = 12$.

$$\begin{aligned}a &= \frac{1}{2} \times 12t^{-1/2} \\ &= \frac{6}{\sqrt{t}}\end{aligned}$$

Now, verifying by substituting into the equation $a = \dfrac{72}{v}$, gives:

$$\text{LHS} = \frac{6}{\sqrt{t}}$$

$$\text{RHS} = \frac{72}{12t^{1/2}} = \text{LHS}$$

11(e) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct solution	3
• Rearranges integrand into a form recognisable as chain rule with inverse tan OR correctly substitutes e.g. $u = 2x + 2$	2
• Completes the square in the denominator, either non-monic or by factoring 4	1

Sample Answer:

$$\begin{aligned}\int \frac{1}{4x^2+8x+13}\,dx &= \frac{1}{4}\int \frac{1}{x^2+2x+\frac{13}{4}}\,dx \\ &= \frac{1}{4}\int \frac{2}{(x+1)^2+\left(\frac{3}{2}\right)^2}\,dx \\ &= \frac{1}{4}\times\frac{2}{3}\tan^{-1}\left(\frac{x+1}{\frac{3}{2}}\right)+c \\ &= \frac{1}{6}\tan^{-1}\left(\frac{2(x+1)}{3}\right)+c\end{aligned}$$

OR

$$\begin{aligned}\int \frac{1}{4x^2+8x+13}\,dx &= \int \frac{1}{(2x+2)^2+9}\,dx \\ &= \frac{1}{2}\int \frac{2}{(2x+2)^2+3^2}\,dx \\ &= \frac{1}{6}\tan^{-1}\left(\frac{2x+2}{3}\right)+c\end{aligned}$$

11(f) (2 marks)
Outcomes Assessed: *MEX12-2, MEX12-8*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Correctly arrives at contradiction	2
• Expresses $\log_{10}7$ as a fraction noting p and q are integers and also positive. (NOTE if p and q are not assumed positive, the "odd $\neq$ even" or "only factors of 10 $\neq$ only factors of 7" contradiction would not be reached.)	1

Sample Answer:

If we assume the result is false, we assume that $\log_{10}7$ is rational, that is:

$$\begin{aligned}\log_{10}7 &= \frac{p}{q} \quad \text{where } p,q\in\mathbb{Z}.\ \text{Also since } 0<\log_{10}7<1,\ 0<p<q.\ \text{Hence } p,q\in\mathbb{N}. \\ 10^{p/q} &= 7 \\ 10^p &= 7^q\end{aligned}$$

But for $p,q\in\mathbb{N}$, 10^p is even, while 7^q is odd, which is a contradiction. Therefore the assumption is false, and and $\log_{10}7$ is an irrational number.

Question 12 (14 marks)

12(a) (i) (1 mark)
Outcomes Assessed: *MEX12-1, MEX12-4*
Targeted Performance Bands: *E3*

Criteria	Mark
• Provides correct solution	1

Sample Answer:

Since the coefficients are real, $1-i$ is also a solution, by the conjugate root theorem.

$$\begin{aligned} \alpha\beta\gamma &= -\frac{d}{a} \\ (1+i)(1-i)\gamma &= -34 \\ 2\gamma &= -34 \\ \gamma &= -17 \end{aligned}$$

The real root is -17.

12(a) (ii) (1 mark)
Outcomes Assessed: *MEX12-1, MEX12-4*
Targeted Performance Bands: *E3*

Criteria	Mark
• Correctly evaluates c	1

Sample Answer:

$$\begin{aligned} &(z+17)(z-(1+i))(z-(1-i)) \\ &= (z+17)(z^2-2z+2) \\ &= z^3-2z^2+2z+17z^2-34z+34 \\ &= z^2+15z^2-32z+34 \end{aligned}$$

So, $c=-32$

OR

$$\begin{aligned} c &= \alpha\beta+\alpha\gamma+\beta\gamma \\ &= (1+i)(1-i)-17(1+i)-17(1-i) \\ &= 2-17-17i-17+17i \\ &= -32 \end{aligned}$$

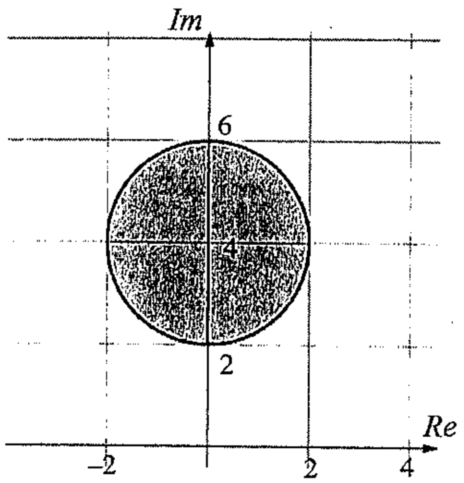
12(b) (i) (2 marks)

Outcomes Assessed: *MEX12-1, MEX12-4*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct graph	2
• Find correct radius or correct centre or work with equivalent progress	1

Sample Answer:



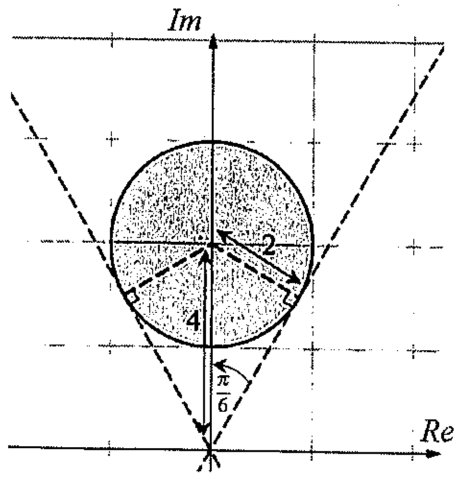
12(b) (ii) (2 marks)

Outcomes Assessed: *MEX12-1, MEX12-4*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct solution	2
• Notices the exact value triangle in the geometry of the tangents OR attempts to solve $x^2+(y-4)^2=4$ and $y=mx$.	1

Sample Answer:



From the sketch the max and min of $\operatorname{Arg} z$ will be given by tangents to the circle passing through origin. Triangle formed by radius, tangent, and y-axis has angle $\frac{\pi}{6}$. Hence $\frac{\pi}{3} \le \operatorname{Arg}(z) \le \frac{2\pi}{3}$.

OR solve $y = mx$ with $x^2+(y-4)^2=4$:

$$x^2+(mx-4)^2=4$$

$$\left(1+m^2\right)x^2-8mx+12=0$$

$$\text{ONE soln means: } \Delta = 0 = 64m^2-48(1+m^2)$$

$$m^2=3$$

Therefore gradients of tangents are $m=\pm\sqrt{3}$ which gives $\frac{\pi}{3} \le \operatorname{Arg}(z) \le \frac{2\pi}{3}$.

12(c) (3 marks)
Outcomes Assessed: *MEX12-5, MEX12-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution	3
• Correctly uses integration by parts for one step	2
• Writes integral correctly	1

Sample Answer:

$$\text{Total production } = \int_0^8 t\sin\left(\frac{\pi t}{8}\right)dt$$

$$\text{Now, let } dv = \sin\left(\frac{\pi t}{8}\right)dt \quad \text{and } u = t$$

$$\text{Hence } v = -\frac{8}{\pi}\cos\left(\frac{\pi t}{8}\right) \quad \text{and } du = dt$$

$$\text{Now } \int u\,dv = uv - \int v\,du$$

$$\begin{aligned}\text{Total production } &= \left[t \times \frac{-8}{\pi}\cos\left(\frac{\pi t}{8}\right)\right]_0^8 - \int_0^8 -\frac{8}{\pi}\cos\left(\frac{\pi t}{8}\right)dt \\ &= \left(-\frac{64}{\pi}\cos\pi\right) - (0) + \frac{8}{\pi}\left[\frac{8}{\pi}\sin\left(\frac{\pi t}{8}\right)\right]_0^8 \\ &= \frac{64}{\pi} + [0-0] \\ &= \frac{64}{\pi} \approx 20{\cdot}37\end{aligned}$$

Therefore the total production is approximately 20·37 megajoules.

12(d) (i) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Criteria	Marks
• Provides correct solution	2
• Notes that lines intersect at $(-2,1,5)$ OR finds dot product of direction vectors.	1

Sample Answer:

When $\lambda = \mu = 0$ both lines pass through $(-2,1,5)$, so they intersect.

Finding the dot product of their direction vectors: $\underset{\sim}{l} = \begin{pmatrix}1\\7\\8\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} = 1+7-8 = 0.$

Since the dot product of their direction vectors is zero, the lines are perpendicular.

12(d) (ii) (3 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides the equation of a line with correct intercept AND correct direction	3
• Provides the equation of a line with correct intercept OR correct direction	2
• Notes that the direction vector must have dot product = 0 with both $\underset{\sim}{l}$ and $\underset{\sim}{m}$	1

Sample Answer:

The third line $\underset{\sim}{n}$ will be of the form: $\begin{pmatrix}-2\\1\\-5\end{pmatrix}+\alpha\begin{pmatrix}a\\b\\c\end{pmatrix}$, with the direction vector having dot product of zero with the other two.

Without loss of generality, let $a=1$, and so $\begin{pmatrix}1\\b\\c\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=0$ and $\begin{pmatrix}1\\b\\c\end{pmatrix}\cdot\begin{pmatrix}1\\7\\8\end{pmatrix}=0$.

$$\begin{aligned}b-c&=-1 \qquad \Rightarrow b=c-1\\ 7b+8c&=-1\\ \text{By substitution: } 7(c-1)+8c&=-1\\ 15c&=6\\ c&=\tfrac{2}{5}\\ b&=-\tfrac{3}{5}\end{aligned}$$

Now, the direction vector will be in simplest terms if $a=5$, so a simple vector equation of the required line is $\underset{\sim}{n}=\begin{pmatrix}-2\\1\\-5\end{pmatrix}+\alpha\begin{pmatrix}5\\-3\\2\end{pmatrix}$, where $\alpha\in\mathbb{R}$.

A Cartesian equation for the line is $\dfrac{x+2}{5}=\dfrac{y-1}{-3}=\dfrac{z+5}{2}$.

Note for markers: while there is only one line $\underset{\sim}{n}$ which satisfies the conditions, there are infinitely many vector and Cartesian representations of $\underset{\sim}{n}$. The direction vector must be a scalar multiple of $\begin{pmatrix}5\\-3\\2\end{pmatrix}$, and the line must pass through $(-2,1,-5)$.

Question 13 (16 marks)

13(a) (3 marks)
Outcomes Assessed: *MEX12-2, MEX12-8*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct solution	3
• Assumes true for a positive integer (e.g. k) and makes attempt to show true for next integer (e.g. $k+1$)	2
• Shows result is true for $n=1$.	1

Sample Answer:

RTP: If $T_n = 2T_{n-1} - n^2$ and $T_1 = 10$, then $T_n = n^2 + 4n + 6 - 2^{n-1}$ for $n \in \mathbb{Z}^+$.

Proof: If $n = 1$, LHS $= T_1 = 10$, and RHS $= 1 + 4 + 6 - 2^0 = 10 =$ LHS.

Therefore the result is true for $n = 1$.

Now, let's assume the result is true for some positive integer k, that is:

IF $\quad T_k = k^2 + 4k + 6 - 2^{k-1}$, where k $\in \mathbb{Z}^+$,

$$\begin{aligned}\text{THEN} \quad T_{k+1} &= 2 \times T_k - (k+1)^2 \\ &= 2\left(k^2 + 4k + 6 - 2^{k-1}\right) - (k+1)^2 \\ &= 2k^2 + 8k + 12 - 2^k - k^2 - 2k - 1 \\ &= k^2 + 6k + 11 - 2^k \\ &= k^2 + 2k + 1 + 4k + 4 + 6 - 2^k \\ &= (k+1)^2 + 4(k+1) + 6 - 2^{(k+1)-1}.\end{aligned}$$

By the principle of Mathematical Induction, the result is true for all positive integers n.

13(b) (i) (1 mark)
Outcomes Assessed: *MEX12-4, MEX12-8*
Targeted Performance Bands: *E3*

Criteria	Mark
• Provides correct solution	1

Sample Answer:

$$\begin{aligned} z^k &= \left(e^{i\theta}\right)^k = (\cos\theta + i\sin\theta)^k \\ &= \cos(k\theta) + i\sin(k\theta) \text{ by de Moivre's theorem}\end{aligned}$$

$$\begin{aligned}\text{Similarly } z^{-k} &= \cos(-k\theta) + i\sin(-k\theta) \\ &= \cos(k\theta) - i\sin(k\theta), \quad \text{since cosine is an even function and sine is odd}\end{aligned}$$

$$\begin{aligned}\text{RHS } &= z^k + z^{-k} \\ &= \cos(k\theta) + i\sin(k\theta) + \cos(k\theta) - i\sin(k\theta) \\ &= 2\cos(k\theta) = \text{ LHS as required.}\end{aligned}$$

13(b) (ii) (3 marks)
Outcomes Assessed: *MEX12-4, MEX12-8*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution	3
• Expands the quartic	2
• Makes some attempt to use (i)	1

Sample Answer: From above we see: $z - z^{-1} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$
$= 2i\sin\theta$

Using binomial expansion we have: $\left(z - z^{-1}\right)^4 = z^4 - 4z^3z^{-1} + 6z^2z^{-2} - 4zz^{-3} + z^{-4}$

$$(2i\sin\theta)^4 = z^4 + z^{-4} - 4(z^2 + z^{-2}) + 6$$

$$16\sin^4\theta = 2\cos 4\theta - 4 \times 2\cos 2\theta + 6 \text{ (from part (i))}$$

Hence $\sin^4\theta = \dfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$, as required.

13(c) (i) (1 mark)
Outcomes Assessed: *MEX12-5, MEX12-7*
Targeted Performance Bands: *E2-E3*

Criteria	Mark
• Correctly differentiates using chain rule	1

Sample Answer:

$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}(\cos x)^{-1}\\ &= (-1)(\cos x)^{-2} \times (-\sin x)\\ &= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x}\\ &= \sec x \tan x\end{aligned}$$

13(c) (ii) (3 marks)
Outcomes Assessed: *MEX12-5, MEX12-7*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Correctly solves for k	3
• Integrates and substitutes limits into one term OR integrates both terms	2
• Correctly integrates one term	1

Sample Answer:

$$\begin{aligned}\frac{11}{24} &= \int_0^{\frac{\pi}{3}} \left(k\cos^2 x - \sec^2 x\right)\sin x\,dx\\ &= k\left[-\frac{1}{3}\cos^3 x\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \sec x \tan x\,dx\\ &= -\tfrac{k}{3}\left(\tfrac{1}{8} - 1\right) - [\sec x]_0^{\pi/3}\end{aligned}$$

$$\begin{aligned}\frac{11}{24} &= \frac{7k}{24} - (2-1)\\ 11 &= 7k - 24\\ k &= 5.\end{aligned}$$

13(d) (i) (2 marks)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Provides correct solution	2
• Makes use of $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	1

Sample Answer:

$$\tfrac{1}{2}v^2 = 2 - 4x - 2x^2$$

$$\begin{aligned} \text{So, } a = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) &= \frac{d}{dx}\left(2 - 4x - 2x^2\right) \\ &= -4 - 4x \\ &= -2^2\left(x - (-1)\right) \end{aligned}$$

Which is SHM where $x = -1$ is the centre of motion. Also note $n = 2$.

13(d) (ii) (2 marks)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct solution	2
• Attempts to solve for $v = 0$	1

Sample Answer:

Ends of motion will be when $v = 0$:

$$\tfrac{1}{2}v^2 = 2 - 4x - 2x^2$$

$$0 = -2\left(x^2 + 2x - 1\right)$$

$$(x+1)^2 = 2$$

$$x = -1 \pm \sqrt{2}$$

So, $x_1 = -1 - \sqrt{2}$, and $x_2 = -1 + \sqrt{2}$.

13(d) (iii) (1 mark)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E2-E3*

Criteria	Mark
• Provides correct solution	1

Sample Answer:

From (i), we have $n = 2$, $c = -1$, and
from (ii) we have $a = \sqrt{2}$.

$$\text{So, } x = \sqrt{2}\sin(2t + \alpha) - 1$$

Substituting $t = 0, x = 0$ gives:

$$\sin\alpha = \frac{1}{\sqrt{2}}$$

$$\alpha = \frac{\pi}{4}$$

Solving for $x = 0$ gives

$$0 = \sqrt{2}\sin\left(2t + \frac{\pi}{4}\right) - 1$$

$$2t + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \cdots$$

$$t = 0, \frac{\pi}{2}, \cdots$$

Hence the particle is at the origin again after $\frac{\pi}{2}$ seconds.

Question 14 (16 marks)

14(a) (i) (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Criteria	Mark
• Provides correct solution.	1

Sample Answer:

$$|\underset{\sim}{p}| = \sqrt{(3a)^2 + b^2} = \sqrt{9a^2 + b^2}$$

14(a) (ii) (3 marks)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Correct solution	3
• Uses triangle inequality properly	2
• Chooses $\underset{\sim}{q} = a\underset{\sim}{i} + 3b\underset{\sim}{j}$	1

Sample Answer:

$$\text{Choose } \underset{\sim}{q} = a\underset{\sim}{i} + 3b\underset{\sim}{j}$$

$$\text{Hence } |\underset{\sim}{q}| = \sqrt{a^2 + 9b^2}$$

Now, the triangle inequality gives: $|\underset{\sim}{p} + \underset{\sim}{q}| \leq |\underset{\sim}{p}| + |\underset{\sim}{q}|$

$$|4a\underset{\sim}{i} + 4b\underset{\sim}{j}| \leq \sqrt{9a^2 + b^2} + \sqrt{a^2 + 9b^2}$$

$$\sqrt{16a^2 + 16b^2} \leq \sqrt{9a^2 + b^2} + \sqrt{a^2 + 9b^2}$$

$$\text{so } \sqrt{a^2 + b^2} \leq \frac{\sqrt{9a^2 + b^2} + \sqrt{a^2 + 9b^2}}{4} \text{ as required.}$$

14(b) (i) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Provides correct solution	3
• Uses integration by parts correctly OR recognises $\sqrt{1+x^2} = \frac{1+x^2}{\sqrt{1+x^2}}$	2
• Recognises $\frac{d}{dx}\sqrt{1+x^2} = \frac{x}{\sqrt{1+x^2}}$ OR makes some valid progress in integral	1

Sample Answer:

$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}\,dx \qquad \text{Let } v = \sqrt{1+x^2}$$

$$\frac{dv}{dx} = \tfrac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}} \times 2x \qquad \text{So, } u = x^{n-1}$$

$$dv = \frac{x}{\sqrt{1+x^2}}\,dx \qquad du = (n-1)x^{n-2}\,dx$$

$$\begin{aligned}
\text{Hence, } I_n &= \int_0^1 x^{n-1} \times \frac{x}{\sqrt{1+x^2}}\,dx \\
&= \left[x^{n-1} \times \sqrt{1+x^2}\right]_0^1 - \int_0^1 \sqrt{1+x^2} \times (n-1)x^{n-2}\,dx \\
&= (1 \times \sqrt{2}) - (0) - (n-1)\int_0^1 \frac{1+x^2}{\sqrt{1+x^2}} \times x^{n-2}\,dx \\
&= \sqrt{2} - (n-1)\left(\int_0^1 \frac{x^{n-2}}{\sqrt{1+x^2}}\,dx + \int_0^1 \frac{x^n}{\sqrt{1+x^2}}\,dx\right) \\
&= \sqrt{2} - (n-1)\left(I_{n-2} + I_n\right) \\
I_n(1+n-1) &= \sqrt{2} - (n-1)I_{n-2} \\
I_n &= \tfrac{\sqrt{2}}{n} - \tfrac{n-1}{n}I_{n-2}, \qquad \text{as required.}
\end{aligned}$$

14(b) (ii) (2 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution	2
• Correctly calculates I_1 OR Substitutes into reduction formula correctly	1

Sample Answer:

$$\begin{aligned}
I_n &= \tfrac{\sqrt{2}}{n} - \tfrac{n-1}{n}I_{n-2} \\
I_3 &= \int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx \\
&= \tfrac{\sqrt{2}}{3} - \tfrac{t}{3}I_1
\end{aligned}$$

$$\begin{aligned}
\text{now } I_1 &= \int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx \\
&= \left[\sqrt{1+x^2}\right]_0^1 \\
&= \sqrt{2} - 1 \\
\text{So } I_3 &= \tfrac{\sqrt{2}}{3} - \tfrac{2}{3}\left(\sqrt{2}-1\right) \\
&= \tfrac{2-\sqrt{2}}{3}
\end{aligned}$$

14(c) (2 marks)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct proof	2
• Expresses the RTP in some correct algebraic fashion	1

Sample Answer:
Let the two distinct positive integers be a and b:

$$\begin{aligned} \text{Consider } 2\left(a^2+b^2\right) &= 2a^2+2b^2 \\ &= a^2+2ab+b^2+a^2-2ab+b^2 \\ &= (a+b)^2+(a-b)^2 \end{aligned}$$

Which is the sum of two distinct non-zero integers.

14(d) (i) (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3*

Criteria	Mark
• Provides correct solution	1

Sample Answer:

We know $z = a+ib = \begin{pmatrix} a \\ b \end{pmatrix}$. $\quad \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2} = \dfrac{1}{a^2+b^2}\begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} \frac{a}{a^2+b^2} \\ \frac{-b}{a^2+b^2} \end{pmatrix}$

14(d) (ii) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution	2
• Uses dot product and substitutes results from (i) correctly	1

Sample Answer: $\quad \underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$

$$\frac{a^2}{a^2+b^2} - \frac{b^2}{a^2+b^2} = \sqrt{a^2+b^2} \times \sqrt{\frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2}} \times \cos\theta$$

$$\frac{a^2-b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} \times \cos\theta$$

$$\text{Hence } \cos\theta = \frac{a^2-b^2}{a^2+b^2}$$

$$\theta = \cos^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right) \text{ as required.}$$

14(d) (iii) (2 marks)
Outcomes Assessed: *MEX12-3, MEX12-7*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Provides correct solution	2
• Finds $\arg z$ and $\arg\left(\frac{1}{z}\right)$	1

Sample Answer:

$$\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\arg\left(\frac{1}{z}\right) = \arg\left(z^{-1}\right) \text{ by de Moivre's Theorem}$$

$$= -\tan^{-1}\left(\frac{b}{a}\right)$$

So the angle between z and $\frac{1}{z}$ is $2\tan^{-1}\left(\frac{b}{a}\right)$
Hence $\cos^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$, as required.

Question 15 (13 marks)

15(a) (3 marks)
Outcomes Assessed: *MEX12-4, MEX12-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• Uses properties of cosine function to arrive at equality	3
• Uses sum of roots and evaluates the rational terms	2
• Give nine roots of equation in arg form	1

Sample Answer:

$$z^9 + 1 = 0 \qquad\qquad 9\theta = \pm\pi, \pm 3\pi, \pm 5\pi, \pm 7\pi, 9\pi$$

$$(\operatorname{cis}\theta)^9 = -1 \qquad\qquad \theta = \pm\frac{\pi}{9}, \pm\frac{3\pi}{9}, \pm\frac{5\pi}{9}, \pm\frac{7\pi}{9}, \frac{9\pi}{9}$$

Therefore $z = \operatorname{cis}\left(\pm\frac{\pi}{9}\right), \operatorname{cis}\left(\pm\frac{\pi}{3}\right), \operatorname{cis}\left(\pm\frac{5\pi}{9}\right), \operatorname{cis}\left(\pm\frac{7\pi}{9}\right), \operatorname{cis}\pi$
Now, the sum of roots of this polynomial will give
$0 = \operatorname{cis}\frac{\pi}{9} + \operatorname{cis}\frac{-\pi}{9} + \operatorname{cis}\frac{\pi}{3} + \operatorname{cis}\frac{-\pi}{3} + \operatorname{cis}\frac{5\pi}{9} + \operatorname{cis}\frac{-5\pi}{9} + \operatorname{cis}\frac{7\pi}{9} + \operatorname{cis}\frac{-7\pi}{9} + \operatorname{cis}\pi$
Further, using $\operatorname{cis}(\alpha) + \operatorname{cis}(-\alpha) = 2\cos\alpha$ will give:
$0 = 2\cos\frac{\pi}{9} + 2\times\frac{1}{2} + 2\cos\frac{5\pi}{9} + 2\cos\frac{7\pi}{9} - 1$
Also, $\cos(\pi-\alpha) = -\cos\alpha$, so

$$0 = \cos\left(\frac{\pi}{9}\right) - \cos\left(\frac{4\pi}{9}\right) - \cos\left(\frac{2\pi}{9}\right)$$

$$\cos\left(\frac{\pi}{9}\right) = \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right)$$

15(b) (i) (2 marks)
Outcomes Assessed: *MEX12-3, MEX12-6*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct solution	2
• Differentiates correctly with respect to t at least two of the three dimensions	1

Sample Answer:

$$\underset{\sim}{v} = \begin{pmatrix} 4\cos t \\ 2\sin 2t \\ 2 - 2\cos 2t \end{pmatrix}.$$

15(b) (ii) (2 marks)
Outcomes Assessed: *MEX12-3, MEX12-6*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Simplifies correctly to arrive at $\lvert\underset{\sim}{v}\rvert = 4$	2
• Provides correct expression for $\lvert\underset{\sim}{v}\rvert^2$	1

Sample Answer:

$$\begin{aligned} \lvert\underset{\sim}{v}\rvert^2 &= (4\cos t)^2 + (2\sin 2t)^2 + (2 - 2\cos 2t)^2 \\ &= 16\cos^2 t + 4\sin^2(2t) + 4 - 8\cos 2t + 4\cos^2(2t) \\ &\qquad \text{Note, } 4\sin^2(2t) + 4\cos^2(2t) = 4 \\ &= 16\cos^2 t - 8\left(2\cos^2 t - 1\right) + 8 \\ &= 16 \end{aligned}$$

So, $\lvert\underset{\sim}{v}\rvert = 4$.

15(b) (iii) (1 mark)
Outcomes Assessed: *MEX12-3, MEX12-6*
Targeted Performance Bands: *E3*

Criteria	Mark
• Provides correct length	1

Sample Answer:

The length of the path of the balloon is the constant speed of 4 m/s times the 10 seconds it was inflated, which is 40 metres.

15(c) (i) (2 marks)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Provides correct solution	2
• Makes significant progress towards identity	1

Sample Answer:

$$\begin{aligned}
\text{RHS} &= \text{Re}\left(e^{i\theta}\frac{1-e^{in\theta}}{1-e^{i\theta}}\right)\\
&= \text{Re}\left(e^{i\theta}\frac{\left(1-e^{i\theta}\right)\left(1+e^{i\theta}+e^{i2\theta}+\cdots+e^{i(n-1)\theta}\right)}{1-e^{i\theta}}\right)\\
&= \text{Re}\left(e^{i\theta}+e^{i2\theta}+e^{i3\theta}+\cdots+e^{in\theta}\right)\\
&= \cos\theta+\cos 2\theta+\cdots+\cos n\theta = \text{LHS}
\end{aligned}$$

15(c) (ii) (3 marks)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Provides correct solution	3
• Uses $e^{ik\theta}-e^{-ik\theta}=2\sin k\theta$ or similar progress	2
• Manipulates RHS to approach a factor of $\left(e^{i(n+1)\frac{\theta}{2}}\right)$ or similar progress	1

Sample Answer:

$$\begin{aligned}
\text{RHS} &= \text{Re}\left(e^{i\theta}\frac{1-e^{in\theta}}{1-e^{i\theta}}\right)\\
&= \text{Re}\left(e^{i\theta}\times\frac{e^{i\frac{n}{2}\theta}\left(e^{-i\frac{n}{2}\theta}-e^{i\frac{n}{2}\theta}\right)}{e^{i\frac{1}{2}\theta}\left(e^{-i\frac{1}{2}\theta}-e^{i\frac{1}{2}\theta}\right)}\right)\\
&= \text{Re}\left(e^{i\theta\left(1+\frac{n}{2}-\frac{1}{2}\right)}\times\frac{-2i\sin\left(\frac{n\theta}{2}\right)}{-2i\sin\left(\frac{\theta}{2}\right)}\right)\\
&= \text{Re}\left(e^{i(n+1)\frac{\theta}{2}}\right)\times\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\\
&= \cos\left((n+1)\frac{\theta}{2}\right)\times\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}
\end{aligned}$$

Question 16 (16 marks)

16(a) (i) (2 marks)
Outcomes Assessed: *MEX12-5, MEX12-8*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Integrates and simplifies to show result	2
• Integrates correctly OR simplifies an incorrect integral correctly	1

Sample Answer:

$$\text{LHS} = \int_0^1 x^p(1-x)^q\,dx$$

Now, let $dv = x^p\,dx$, and $u = (1-x)^q$

Hence $v = \frac{1}{p+1}x^{p+1}$, and $du = -q(1-x)^{q-1}\,dx$

$$\text{So, } \int_0^1 x^p(1-x)^q\,dx = \left[(1-x)^q\tfrac{1}{p+1}x^{p+1}\right]_0^1 - \int_0^1 \tfrac{1}{p+1}x^{p+1}\times -q(1-x)^{q-1}\,dx$$

$$= [0\times 1 - 1\times 0] + \frac{q}{p+1}\int_0^1 x^{p+1}(1-x)^{q-1}\,dx = \text{RHS}.$$

16(a) (ii) (4 marks)
Outcomes Assessed: *MEX12-5, MEX12-8*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Uses factorial notation to simplify the numerator and denominator sequences to arrive at result	4
• Evaluates integral and is left with only algebraic terms	3
• Arrives at the $\frac{1}{p+q}$ term and final integral	2
• Uses formula in (i) to begin a product of a sequence	1

Sample Answer:

$$\begin{aligned}
\text{LHS} &= \int_0^1 x^p(1-x)^q.dx = \tfrac{q}{p+1}\int_0^1 x^{p+1}(1-x)^{q-1}\,dx \\
&= \tfrac{q}{p+1}\times\tfrac{q-1}{p+2}\times\tfrac{q-2}{p+3}\times\cdots\times\tfrac{q-(q-1)}{p+q}\times\int_0^1 x^{p+q}(1-x)^{q-q}\,dx \\
&= \tfrac{q}{p+1}\times\tfrac{q-1}{p+2}\times\tfrac{q-2}{p+3}\times\cdots\times\tfrac{1}{p+q}\times\left[\tfrac{1}{p+q+1}x^{p+q+1}\right]_0^1 \\
&= \frac{q(q-1)(q-2)\times\cdots\times 1}{(p+1)(p+2)(p+3)\times\cdots\times(p+q)(p+q+1)} \\
&= \frac{q!}{\frac{(p+q+1)!}{p!}} = \text{RHS}.
\end{aligned}$$

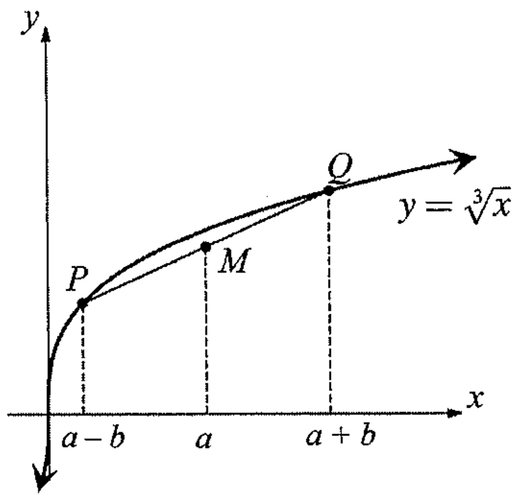
16(b) (3 marks)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E4*

Criteria	Marks
• Uses concavity to show inequality	3
• Uses a graph or similar algebraic argument to analyse $\sqrt[3]{a}$, $\sqrt[3]{a+b}$, and $\sqrt[3]{a-b}$	2
• Differentiates correctly twice	1

Sample Answer:

$$y = x^{1/3}$$
$$\frac{dy}{dx} = \tfrac{1}{3}x^{-2/3}$$
$$\frac{d^2y}{dx^2} = -\tfrac{2}{9}x^{-5/3}$$

Considering this function for $x > 0$ gives $\dfrac{d^2y}{dx^2} < 0$, and so $y = \sqrt[3]{x}$ is concave down in the first quadrant.



Consider interval PQ as shown above with midpoint M. The y-value of M is $\dfrac{\sqrt[3]{a-b}+\sqrt[3]{a+b}}{2}$. Since $y = \sqrt[3]{x}$ is concave down,

$$\frac{\sqrt[3]{a-b}+\sqrt[3]{a+b}}{2} < \sqrt[3]{a}$$

$$\text{So, } \sqrt[3]{a-b}+\sqrt[3]{a+b} < 2\sqrt[3]{a}.$$

16(c) (i) (4 marks)
Outcomes Assessed: *MEX12-5, MEX12-6, MEX12-7*
Targeted Performance Bands: *E4*

Criteria	Marks
• Provides correct solution	4
• Provides a correct expression for v but not in required form	3
• Provides correct expression for t including the constant of integration	2
• Recognises need for partial fraction decomposition and some progress toward	1

Sample Answer:

$$ma = kv^2 - mg$$
$$\frac{dv}{dt} = \frac{kv^2 - mg}{m}$$
$$\frac{dt}{dv} = \frac{m}{kv^2 - mg}$$

$$\frac{dt}{dv} = \frac{m/k}{v^2 - mg/k}$$
$$= \frac{m}{k} \times \frac{1}{(v - \sqrt{mg/k})(v + \sqrt{mg/k})}$$

Using partial fractions:

$$\frac{1}{v - \sqrt{mg/k}} - \frac{1}{v + \sqrt{mg/k}} = \frac{2\sqrt{mg/k}}{(v - \sqrt{mg/k})(v + \sqrt{mg/k})}$$

$$\text{So, } \frac{dt}{dv} = \frac{m}{k} \times \frac{\sqrt{k}}{2\sqrt{mg}} \left(\frac{1}{v - \sqrt{mg/k}} - \frac{1}{v + \sqrt{mg/k}} \right)$$

$$t = \frac{1}{2}\sqrt{\frac{m}{kg}} \ln\left(\frac{|v - \sqrt{mg/k}|}{|v + \sqrt{mg/k}|} \right) + c$$

Substituting $t = 0$ and $v = 0$ gives $c = 0$. So $\dfrac{|v - \sqrt{mg/k}|}{|v + \sqrt{mg/k}|} = e^{2t\sqrt{kg/m}}$

Since downwards is negative, $kv^2 - mg < 0$, so $v - \sqrt{mg/k} < 0$ and $v + \sqrt{mg/k} > 0$.

$$\text{So:} \quad -(v - \sqrt{mg/k}) = (v + \sqrt{mg/k})e^{2t\sqrt{kg/m}}$$

Collecting like terms in v gives: $\quad v(e^{2t\sqrt{kg/m}} + 1) = \sqrt{mg/k}(1 - e^{2t\sqrt{kg/m}})$.

$$\text{Thus:} \quad v = \sqrt{\frac{mg}{k}} \left(\frac{1 - e^{2t\sqrt{gk/m}}}{1 + e^{2t\sqrt{gk/m}}} \right)$$

Dividing top and bottom by $e^{t\sqrt{gk/m}}$ gives

$$v = \sqrt{\frac{mg}{k}} \left(\frac{e^{-t\sqrt{gk/m}} - e^{t\sqrt{gk/m}}}{e^{-t\sqrt{gk/m}} + e^{t\sqrt{gk/m}}} \right)$$

16(c) (ii) (3 marks)
Outcomes Assessed: *MEX12-6, MEX12-7*
Targeted Performance Bands: *E4*

Criteria	Marks
• Provides correct solution	3
• Establishes correct quadratic equation in t	2
• Integrates to give correct expression for x	1

Sample Answer:

Setting $k = 0{\cdot}25$, $m = 100$ and $g = 10$ gives

$$v = 20\sqrt{10}\left(\frac{e^{-t\sqrt{10}/20} - e^{t\sqrt{10}/20}}{e^{-t\sqrt{10}/20} + e^{t\sqrt{10}/20}}\right)$$

$$= -20\sqrt{10} \times \frac{20}{\sqrt{10}} \times \left(\frac{\frac{\sqrt{10}}{20}\left(-e^{-t\sqrt{10}/20} + e^{t\sqrt{10}/20}\right)}{e^{-t\sqrt{10}/20} + e^{t\sqrt{10}/20}}\right)$$

Integrating both sides with respect to t gives

$$x = -400\ln\left|e^{-t\sqrt{10}/20} + e^{t\sqrt{10}/20}\right| + c$$

When $t = 0$ and $x = 5000$, $c = 5000 + 400\ln 2$. So

$$x = 5000 - 400\ln\left|\frac{e^{-t\sqrt{10}/20} + e^{t\sqrt{10}/20}}{2}\right|$$

Substituting $x = 1500$ and rearranging gives

$$\ln\left|\frac{e^{-t\sqrt{10}/20} + e^{t\sqrt{10}/20}}{2}\right| = \frac{3500}{400}$$

$$e^{-t\sqrt{10}/20} + e^{t\sqrt{10}/20} = 2e^{35/4}$$

$$e^{2t\sqrt{10}/20} - 2e^{35/4}e^{t\sqrt{10}/20} + 1 = 0$$

which is a quadratic in $e^{t\sqrt{10}/20}$. Solving for $e^{t\sqrt{10}/20}$ using the quadratic formula gives

$$e^{t\sqrt{10}/20} = \frac{2e^{35/4} \pm \sqrt{4e^{2\times(35/4)} - 4}}{2}$$

$$= e^{35/4} \pm \sqrt{e^{35/2} - 1}$$

We may eliminate the 'minus' solution since for $t > 0$, $e^{t\sqrt{10}/20} > 1$ but $e^{35/4} - \sqrt{e^{35/2} - 1} < 1$.

Finally, solving for t gives $t = \ln\left(e^{35/4} + \sqrt{e^{35/2} - 1}\right) \div \left(\frac{\sqrt{10}}{20}\right) \approx 59{\cdot}72$ seconds