**2020**
**Higher School Certificate**
# Trial Examination

# Mathematics Extension 2

*General Instructions*

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your student name and/or number at the top of every page

**Total marks – 100**

**Section I – 10 marks** (pages 3 - 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

**Section II – 90 marks** (pages 6 - 11)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

**This paper MUST NOT be removed from the examination room.**

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STUDENT NAME/NUMBER……………………………………

**Section I**

**10 Marks**
**Attempt Questions 1-10.**
**Allow about 15 minutes for this section.**

Select the alternative A, B, C, D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

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| | **A** | **B** | **C** | **D** |
|---|---|---|---|---|
| **1** | | | | |
| **2** | | | | |
| **3** | | | | |
| **4** | | | | |
| **5** | | | | |
| **6** | | | | |
| **7** | | | | |
| **8** | | | | |
| **9** | | | | |
| **10** | | | | |

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Marks

**Section I**

**10 Marks**
**Attempt Questions 1-10.**
**Allow about 15 minutes for this section.**

**Use the multiple-choice answer sheet for questions 1-10.**

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**1** Given that $x$ and $y$ are real numbers, which of the following is a true statement ? **1**

(A) $\forall y\left(\exists x : |x| = y\right)$

(B) $\forall y\left(\exists x : |x| < y\right)$

(C) $\forall y(\exists x : |x| > y)$

(D) $\forall y\left(\exists x : |x| = -y\right)$

**2** What is the magnitude of the vector $\cos\theta\,\underset{\sim}{i} + \sin\theta\,\underset{\sim}{j} + \tan\theta\,\underset{\sim}{k}$ where $0 < \theta < \frac{\pi}{2}$ ? **1**

(A) 1

(B) $\operatorname{cosec}\theta$

(C) $\cot\theta$

(D) $\sec\theta$

**3** Consider the statement $x^2 = 9 \Rightarrow x = 3$. Which of the following statements is logically equivalent to the given statement ? **1**

(A) $x \neq 3 \Rightarrow x^2 \neq 9$

(B) $x^2 \neq 9 \Rightarrow x \neq 3$

(C) $x = 3 \Rightarrow x^2 = 9$

(D) $x \neq 3 \Leftrightarrow x^2 \neq 9$

**4** The points $A$, $B$, $C$ are collinear where $\overrightarrow{OA} = \underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OB} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, $\overrightarrow{OC} = 3\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}$. **1**
What are the values of $a$ and $b$ ?

(A) $a = -3,\ b = -2$

(B) $a = 3,\ b = -2$

(C) $a = -3,\ b = 2$

(D) $a = 3,\ b = 2$

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Marks

5 Which of the following expressions is obtained from $\int \frac{1}{2+\cos x}\,dx$ after the substitution $t=\tan\frac{x}{2}$ ? 1

(A) $\int \frac{2}{3+t^2}\,dt$

(B) $\int \frac{1}{3+t^2}\,dt$

(C) $\int \frac{1}{3-t^2}\,dt$

(D) $\int \frac{2}{3-t^2}\,dt$

6 Which of the following curves represents the solution of the equation $|z-1| = \text{Re}(z)+1$ ? 1

(A) a straight line

(B) a parabola

(C) a circle

(D) a semi-circle

7 A particle is moving in Simple Harmonic Motion about a fixed point $O$ on a line. At time $t$ seconds it has displacement $x = 2\cos \pi t$ metres from $O$. What is the time taken by the particle to travel the first 100 metres of its motion ? 1

(A) 20 seconds

(B) 25 seconds

(C) 50 seconds

(D) 100 seconds

8 What is the value of $\int_1^e x^4 \ln x\,dx$ ? 1

(A) $\frac{1}{25}\left(5e^5-1\right)$

(B) $\frac{1}{25}\left(5e^5+1\right)$

(C) $\frac{1}{25}\left(4e^5-1\right)$

(D) $\frac{1}{25}\left(4e^5+1\right)$

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Marks

**9** A particle is projected horizontally with speed $\sqrt{gh}$ ms$^{-1}$ from the top of a tower of height $h$ metres. It moves under gravity where the acceleration due to gravity is $g$ ms$^{-2}$. At what angle to the horizontal will the particle hit the ground ? **1**

(A) $\tan^{-1}\frac{1}{2}$

(B) $\tan^{-1}\frac{1}{\sqrt{2}}$

(C) $\tan^{-1}\sqrt{2}$

(D) $\tan^{-1}2$

**10** The equation $z^5 = 1$ has roots $1,\ \omega,\ \omega^2,\ \omega^3,\ \omega^4$ where $\omega = e^{\frac{2\pi}{5}i}$. What is the value of $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$ ? **1**

(A) $-5$

(B) $-4$

(C) $4$

(D) $5$

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Marks

**Section II**

**90 Marks**
**Attempt Questions 11-16**
**Allow about 2 hours and 45 minutes for this section.**

Answer the questions in writing booklets provided. Use a separate writing booklet for each question. In Questions 11-16 your responses should include relevant mathematical reasoning and/or calculations.

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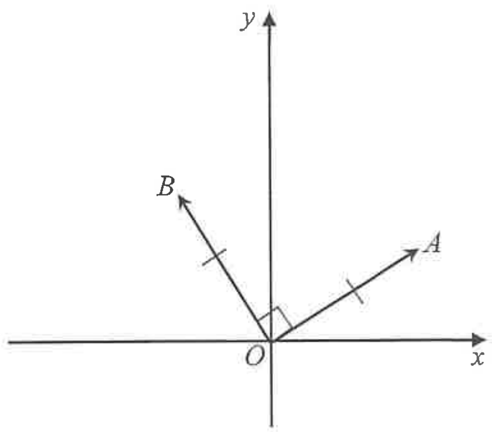
**Question 11 (15 marks)** **Use a separate writing booklet.**

(a)




In the Argand diagram above, the points $A$ and $B$ represent $z_1$ and $z_2$ respectively. $\angle AOB = 90°$ and $OA = OB$.

(i) Express $z_2$ in terms of $z_1$. **1**

(ii) Copy the diagram and on it draw the locus $L_1$ of points satisfying $|z - z_2| = |z - z_1|$. **1**

(iii) On your diagram draw the locus $L_2$ of points satisfying $\arg(z - z_2) = \arg z_1$. **1**

(iv) Find in terms of $z_1$ the complex number representing the point of intersection of $L_1$ and $L_2$. **1**

(b)(i) Find numbers $A$, $B$, $C$ such that $\dfrac{1-x}{(1+x)(1+x^2)} \equiv \dfrac{A}{1+x} + \dfrac{Bx+C}{1+x^2}$. **2**

(ii) Hence find in simplest exact form $\displaystyle\int_0^1 \frac{1-x}{(1+x)(1+x^2)}\,dx$. **3**

(c) It is given that $z = e^{\frac{\pi}{12}i}$ is a root of the equation $z^4 = a\left(1+\sqrt{3}\,i\right)$ where $a$ is real.

(i) Express $1+\sqrt{3}\,i$ in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$. **2**

(ii) Find the value of $a$. **1**

(iii) Find the other three roots of the equation in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$. **3**

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Marks

**Question 12 (15 marks)** **Use a separate writing booklet.**

(a) Use the substitution $u = \tan\theta$ to evaluate in simplest form $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan\theta \sec^4\theta \, d\theta$ . **4**

(b) In an Argand diagram the point $P$ represents $z_1 = 3 + 2i$, the point $Q$ represents $z_2 = \dfrac{12 - 5i}{z_1}$ and $O$ is the origin. The centre $C$ of the circle passing through $P$, $Q$ and $O$ represents $z_3$.

(i) Express $z_2$ in the form $a + i\,b$ where $a$ and $b$ are real. **2**

(ii) Show that $\angle POQ = \frac{\pi}{2}$ . **1**

(iii) Express $z_3$ in the form $a + i\,b$ where $a$ and $b$ are real. **2**

(c) $A(2, 3, 0)$ and $B(3, 1, 2)$ are two points. The line $l$ passes through $A$ and is perpendicular to the line $AB$. When the line $l$ is rotated about $AB$ it traces out a plane which is perpendicular to $AB$ and passes through the point $A$. Let $l$ have vector equation $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ for some real $u$, $v$, $w$ and parameter $\lambda$ .

(i) Show that $u - 2v + 2w = 0$. **2**

(ii) If $(x, y, z)$ is a point on $l$, find the relationship between $x$, $y$ and $z$ in the form $ax + by + cz = d$ for real numbers $a$, $b$, $c$, $d$. **2**

(iii) This relationship is the Cartesian equation of the plane through $A$ that is perpendicular to $AB$. Find the form of the Cartesian equation of any plane that is perpendicular to $AB$. **2**

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Marks

**Question 13 (15 marks)** **Use a separate writing booklet.**

(a) It is given that $a>0$ and $b>0$ are real numbers. Consider the statement $\forall a\left(\forall b, \log_{\frac{1}{a}}\frac{1}{b}=\log_a b\right)$. Either prove that the statement is true or give a counter example. **2**

(b) Use proof by contradiction to show that $\log_2 5$ is irrational. **2**

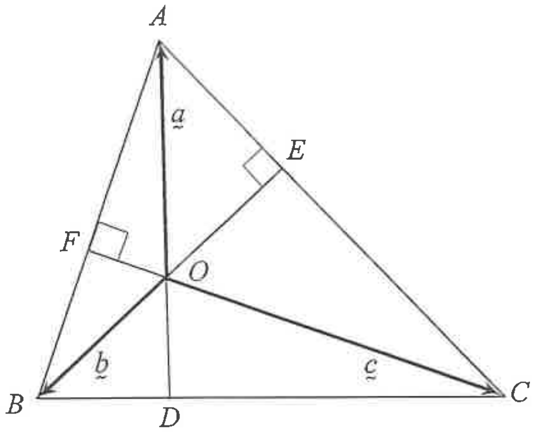
(c) Let $I_n=\displaystyle\int_0^1 \frac{1}{(1+x^2)^n}dx$ for $n=1,2,3,\ldots$ .

(i) Evaluate $I_1$ in simplest exact form. **1**

(ii) Show that $2nI_{n+1}=2^{-n}+(2n-1)I_n$ for $n=1,2,3,\ldots$ . **3**

(iii) Evaluate $I_3$ in simplest exact form. **2**

(d)




$ABC$ is an acute angled triangle. The altitudes $BE$ and $CF$ intersect at $O$. The line $AO$ produced meets $BC$ at $D$. Relative to $O$ the position vectors of $A$, $B$, $C$ are $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ respectively.

(i) Show that $\underset{\sim}{b}.(\underset{\sim}{c}-\underset{\sim}{a})=0$ and $\underset{\sim}{c}.(\underset{\sim}{b}-\underset{\sim}{a})=0$. **2**

(ii) Hence show that $AD\perp BC$. **2**

(iii) What geometrical property of the triangle has been proved? **1**

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Marks

**Question 14 (15 marks)** **Use a separate writing booklet.**

(a) Recall that $x+\frac{1}{x}\geq 2$ for any real number $x>0$. (DO NOT PROVE THIS RESULT)

(i) Prove that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$ for any real numbers $a>0,\ b>0,\ c>0$. **2**

(ii) Prove that $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$ for any real numbers $a>0,\ b>0,\ c>0$. **3**

(b) A particle is moving in a straight line. At time $t$ seconds it has displacement $x$ metres from a fixed point $O$ on the line, where $x$ is given by $x=1+\cos 2t+\sin 2t$.

(i) Express $x$ in the form $x=1+a\cos(2t-\alpha)$ for constants $a>0$ and $0<\alpha<\frac{\pi}{2}$. **2**

(ii) Find correct to 2 decimal places the average speed of the particle during the time it takes to first reach $O$. **3**

(c) $a_n$, $n=1,2,3,\ldots$ is a sequence of real numbers defined by $a_1=\sqrt{2}$ and for $n\geq 1$, $a_{n+1}=\sqrt{2+a_n}$ .

(i) Prove by Mathematical Induction that $a_n=2\cos\left(\frac{\pi}{2^{n+1}}\right)$ for $n\geq 1$. **3**

(ii) Hence evaluate $\lim\limits_{n\to\infty}\left\{4^n\left(2-a_n\right)\right\}$. **2**

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Marks

**Question 15 (15 marks)** **Use a separate writing booklet.**

(a) Prove that if $a$, $b$, $c$, $d$ are distinct integers, then the only possible integer solution of the equation $(x-a)(x-b)(x-c)(x-d)-4=0$ is $x=\dfrac{a+b+c+d}{4}$. **3**

(b) A particle is moving in a straight line so that at time $t$ seconds it has displacement $x$ metres from a fixed point $O$ on the line, velocity $v$ ms$^{-1}$ and acceleration $a$ ms$^{-2}$ given by $a=e^{\frac{1}{2}x}$. Initially the particle is at $O$ and moving with a speed of 2 ms$^{-1}$ while slowing down.

(i) Show that $v=-2e^{\frac{1}{4}x}$. **2**

(ii) Find expressions for $x$, $v$ and $a$ in terms of $t$. **3**

(iii) Describe the subsequent motion of the particle. **1**

(c)(i) Show that $(1+i\tan\theta)^4+(1-i\tan\theta)^4=\dfrac{2\cos 4\theta}{\cos^4\theta}$. **2**

(ii) Hence find the four roots of the equation $(1+z)^4+(1-z)^4=0$ in the form $z=i\tan\theta$ where $0\le\theta\le\pi$. **2**

(iii) Hence show that $\tan^2\frac{\pi}{8}\ \tan^2\frac{3\pi}{8}=1$. **2**

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Marks

**Question 16 (15 marks)** **Use a separate writing booklet.**

(a)(i) If $\dfrac{du}{dt}+u\,f(t)=g(t)$ and $F'(t)=f(t)$, by considering $\dfrac{d}{dt}\left\{u\,e^{F(t)}\right\}$, show that **2**

$u=e^{-F(t)}\displaystyle\int e^{F(t)}g(t)\,dt$ .

(ii) A projectile is fired with velocity $\underset{\sim}{v}=\left(16\underset{\sim}{i}+12\underset{\sim}{j}\right)$ $\text{ms}^{-1}$ from a point $O$ and travels in a vertical plane with position vector $\underset{\sim}{r}(t)=\left(x\,\underset{\sim}{i}+y\,\underset{\sim}{j}\right)$ metres at time $t$ seconds where $\underset{\sim}{i}$, $\underset{\sim}{j}$ respectively are unit vectors horizontally and vertically upwards from $O$. The acceleration due to gravity is $10\ \text{ms}^{-2}$ and a resistance force acts against motion so that $\ddot{\underset{\sim}{r}}(t)=\left\{-\tfrac{1}{10}\dot{x}\,\underset{\sim}{i}-\left(\tfrac{1}{10}\dot{y}+10\right)\underset{\sim}{j}\right\}\text{ms}^{-2}$. **4**

By writing $\dfrac{d}{dt}(\dot{y})+\tfrac{1}{10}\dot{y}=-10$ and using part (i) or otherwise, find $\dot{y}$ as a function of $t$ and hence find the maximum height attained by the projectile.

(iii) Assuming the projectile is free to fall indefinitely after reaching its maximum height, find the limiting motion of the projectile as $t\to\infty$ by considering the expressions for $\underset{\sim}{r}(t)$ and $\dot{\underset{\sim}{r}}(t)$ as functions of $t$. **4**

(b)(i) Show that $\dfrac{k}{(k+1)!}=\dfrac{1}{k!}-\dfrac{1}{(k+1)!}$ and hence that $\displaystyle\sum_{k=1}^{n-1}\frac{k}{(k+1)!}=1-\frac{1}{n!}$ . **2**

(ii) Hence show that $1=\dfrac{1}{m_1}+\dfrac{1}{m_2}+\ldots+\dfrac{1}{m_{n-1}}+\dfrac{1}{m_n}$ where $m_n=n!$ and **1**

$m_k=\dfrac{(k+1)!}{k}$ for $k=1,2,\ldots,n-1$ .

(iii) Deduce that for $n\ge 3$, $n!$ can be expressed as the sum of $n$ distinct divisors of itself. **2**

**Independent Trial HSC 2020 Mathematics Extension 2 Marking Guidelines**

**Section 1 Questions 1-10 (1 mark each)**

| Question | Answer | Solution | Outcomes |
|---|---|---|---|
| 1 | C | No real $x$ satisfies $\lvert x\rvert \le -2$. Hence none of A, B, D is a true statement.<br>If $y \le 0$, $\lvert 1\rvert > y$, and if $y > 0$, $\lvert y+1\rvert > y$. Hence C is a true statement. | MEX12-2 |
| 2 | D | $\cos^2\theta + \sin^2\theta + \tan^2\theta = 1+\tan^2\theta = \sec^2\theta$. Magnitude is $\sec\theta$ $\left(0<\theta<\frac{\pi}{2}\right)$ | MEX12-3 |
| 3 | A | A is the contra-positive which is logically equivalent (though neither is true) | MEX12-2 |
| 4 | C | $\overrightarrow{AB} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{AC} = 2\underset{\sim}{i} + (a-1)\underset{\sim}{j} + b\underset{\sim}{k}$ and $\overrightarrow{AC} = \lambda\,\overrightarrow{AB}$ for some real $\lambda$.<br>Hence $\frac{2}{1} = \frac{a-1}{-2} = \frac{b}{1}$ $\therefore a=-3$ and $b=2$ | MEX12-3 |
| 5 | A | $dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$ $\quad \frac{2}{1+t^2}\,dt = dx$<br>$2+\cos x = \frac{2(1+t^2)+(1-t^2)}{1+t^2}$ $\quad \frac{1}{2+\cos x} = \frac{1+t^2}{3+t^2}$<br>$\int \frac{1}{2+\cos x}\,dx = \int \frac{2}{3+t^2}\,dt$ | MEX12-5 |
| 6 | B | $\lvert z-1\rvert$ is the distance from $z$ to the point (1, 0). $\mathrm{Re}(z)+1$ is the distance from $z$ to the line $x=-1$ (since $\mathrm{Re}(z)\ge -1$). Hence curve is a parabola with focus (1, 0) and directrix $x=-1$. | MEX12-4 |
| 7 | B | Period is 2 s and amplitude is 2 m. Hence particle travels 8 m in one oscillation. $100 = 12\times 8+4$ and $12\frac{1}{2}$ oscillations takes 25 s. | MEX12-6 |
| 8 | D | $\int_1^e x^4 \ln x\,dx = \frac{1}{5}\left[x^5 \ln x\right]_1^e - \frac{1}{5}\int_1^e x^5 \frac{1}{x}\,dx$<br>$\int_1^e x^4 \ln x\,dx = \frac{1}{5}e^5 - \frac{1}{25}\left[x^5\right]_1^e = \frac{1}{25}\left(4e^5+1\right)$ | MEX12-5 |
| 9 | C | $\ddot{x}=0$ $\quad \dot{x}=\sqrt{gh}$ $\quad x=\sqrt{gh}\,t$<br>$\ddot{y}=-g$ $\quad \dot{y}=-gt$ $\quad y=-\frac{1}{2}gt^2$<br>$y=-h \Rightarrow t=\sqrt{\frac{2h}{g}}$ $\quad \therefore \frac{\dot{y}}{\dot{x}} = \frac{-\sqrt{2gh}}{\sqrt{gh}} = -\sqrt{2}$<br>$\therefore$ hits ground at angle $\tan^{-1}\sqrt{2}$ | MEX12-6 |
| 10 | D | $z^5-1 \equiv (z-1)(z^4+z^3+z^2+z+1) \equiv (z-1)(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$<br>$\therefore z^4+z^3+z^2+z+1 \equiv (z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$<br>Then putting $z=1$ gives $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)=5$ | MEX12-4 |

**Section II**

**Question 11**

**a.i Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Correct expression | 1 |

**Answer** $\quad z_2 = i\,z_1$

(Q11 cont.)

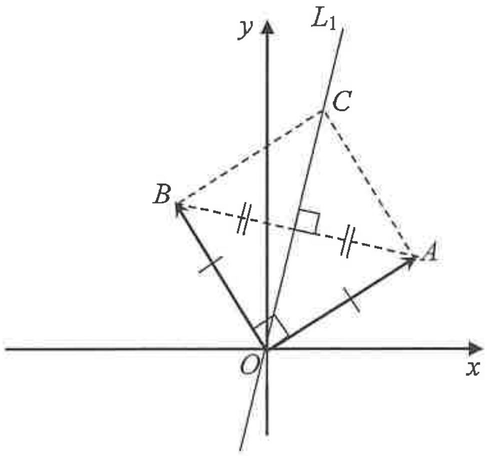
**a.ii Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Sketches the perpendicular bisector of $AB$ | 1 |

**Answer**

$L_1$ is the perpendicular bisector of $AB$.
If C is such that $OACB$ is a square, then $L_1$ is the line $OC$.



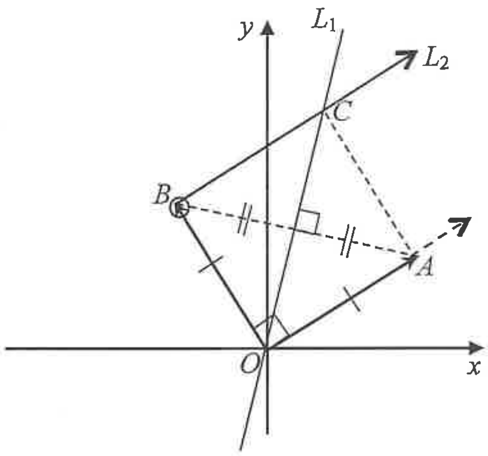

**a.iii Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Sketches a ray from $B$ parallel to vector $\overline{OA}$ | 1 |

**Answer**

$L_2$ is a ray from $B$ (excluding the point $B$) that is parallel to the vector $\overrightarrow{OA}$ . This ray will pass through the vertex C of the square $OACB$.




**a.iv Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Correct answer | 1 |

**Answer**

The intersection of $L_1$ and $L_2$ is the vertex C of the square OACB and C represents $z_1 + z_2 = (1+i)z_1$ .

(Q11 cont.)

**b.i. Outcomes assessed: MEX12-5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses properties of a quadratic identity to find $A$, $B$, C | 2 |
| Some progress eg. correct procedure but one error | 1 |

**Answer**

$$\frac{1-x}{(1+x)(1+x^2)} \equiv \frac{A}{(1+x)} + \frac{Bx+C}{(1+x^2)}$$

$$1-x \equiv A(1+x^2)+(1+x)(Bx+C)$$

$x=-1 \Rightarrow 2=2A$ $\quad\therefore A=1$

$x=0 \Rightarrow 1=A+C$ $\quad\therefore C=0$

Equate coeff. of $x^2$ : $0=A+B$ $\quad\therefore B=-1$

$$\therefore \frac{1-x}{(1+x)(1+x^2)} = \frac{1}{1+x} - \frac{x}{1+x^2}$$

**b.ii. Outcomes assessed: MEX12-5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Writes primitive function and evaluates in simplest exact form | 3 |
| Substantial progress eg. correct process but minor error made in primitive function or evaluation | 2 |
| Some progress eg. rearranges integrand with part of primitive function correct | 1 |

**Answer**

$$\int_0^1 \frac{1-x}{(1+x)(1+x^2)}\,dx = \int_0^1 \left(\frac{1}{1+x} - \frac{x}{1+x^2}\right)dx = \left[\ln(1+x) - \tfrac{1}{2}\ln(1+x^2)\right]_0^1 = \tfrac{1}{2}\ln 2$$

**c.i. Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Correct expression | 2 |
| Some progress eg. finds the modulus | 1 |

**Answer**

$$1+\sqrt{3}\,i = 2\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) = 2\left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right) = 2e^{\frac{\pi}{3}i}$$

**c.ii. Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Correct value of $a$ | 1 |

**Answer**

$$\left(2e^{\frac{\pi}{12}i}\right)^4 = 2ae^{\frac{\pi}{3}i}$$

$$16e^{\frac{\pi}{3}i} = 2ae^{\frac{\pi}{3}i}$$

$$\therefore a = 8$$

(Q11 cont.)

**c.iii. Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| States other 3 roots in required form | 3 |
| Substantial progress eg. correct except that $\theta$ is not in specified domain | 2 |
| Some progress eg. finds one other root | 1 |

**Answer**

$$\arg z^4 = \tfrac{\pi}{3} + 2k\pi, \qquad k = 0,\ \pm 1,\ \pm 2, \ldots$$

$$\arg z = \tfrac{\pi}{12} + k\tfrac{\pi}{2}$$

$$|z| = 2$$

Other roots are

$$2e^{\left(\frac{\pi}{12}+\frac{\pi}{2}\right)i} = 2e^{\frac{7\pi}{12}i}$$

$$2e^{\left(\frac{\pi}{12}-\frac{\pi}{2}\right)i} = 2e^{-\frac{5\pi}{12}i}$$

$$2e^{\left(\frac{\pi}{12}-\pi\right)i} = 2e^{-\frac{11\pi}{12}i}$$

## Question 12

**a. Outcomes assessed: MEX12-5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Makes required substitution then evaluates | 4 |
| Substantial progress eg. obtains transformed definite integral, but makes one error in evaluation | 3 |
| Moderate progress eg. substitution correct but makes two or more subsequent errors | 2 |
| Some progress eg. Finds new limits and writes $du$ in terms of $d\theta$ | 1 |

**Answer**

$$u = \tan\theta, \quad 0 < \theta < \tfrac{\pi}{2}$$

$$du = \sec^2\theta\, d\theta$$

$$\theta = \tfrac{\pi}{4} \Rightarrow u = 1$$

$$\theta = \tfrac{\pi}{3} \Rightarrow u = \sqrt{3}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan\theta \sec^4\theta\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan\theta\left(1+\tan^2\theta\right)\sec^2\theta\, d\theta$$

$$= \int_1^{\sqrt{3}} u\left(1+u^2\right)du$$

$$= \tfrac{1}{4}\left[\left(1+u^2\right)^2\right]_1^{\sqrt{3}}$$

$$= 3$$

**b.i. Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Writes $z_2$ in correct form | 2 |
| Some progress eg. attempts to realize denominator, but makes one error | 1 |

**Answer**

$$\frac{12-5i}{3+2i} = \frac{(12-5i)(3-2i)}{3^2+2^2} = \frac{26-39i}{13} \qquad \therefore z_2 = 2-3i$$

**b.ii. Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Shows required result. | 1 |

**Answer**

$i z_2 = i(2-3i) = 3+2i = z_1$ Hence rotation of $\overrightarrow{OQ}$ anti-clockwise by $\frac{\pi}{2}$ gives $\overrightarrow{OP}$. $\therefore \angle POQ = \frac{\pi}{2}$.

(Q12 cont)
**b.iii. Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Correct value of $z_3$ | 2 |
| Some progress eg. one of real, imaginary parts stated correctly | 1 |

**Answer**

Interval $PQ$ must be the diameter of the circle. Hence $C$ is the midpoint of $PQ$. $\therefore z_3 = \frac{1}{2}(z_1 + z_2) = \frac{5}{2} - \frac{1}{2}i$

**c.i. Outcomes assessed: MEX12-3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Shows the dot product of the direction vectors of $l$ and line $AB$ is zero to deduce result | 2 |
| Some progress eg. correct procedure but makes one error | 1 |

**Answer** A line through $A$ and $B$ has the vector equation $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ for a real parameter $\gamma$.

$$l \perp AB \Rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 0 . \quad \text{Hence} \quad u - 2v + 2w = 0$$

**c.ii Outcomes assessed: MEX12-3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Eliminates $\lambda$ from parametric equations for $x, y, z$ using the relationship between $u, v, w$ | 2 |
| Some progress eg. writes parametric equations for $x, y, z$ but makes an error in eliminating $\lambda$ | 1 |

**Answer**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{aligned} x &= 2 + \lambda u \quad (1) \\ y &= 3 + \lambda v \quad (2) \\ z &= \quad \lambda w \quad (3) \end{aligned}$$

$(1) - 2\times(2) + 2\times(3)$ gives

$$x - 2y + 2z = (2 - 6 + 0) + \lambda(u - 2v + 2w)$$

$$\therefore x - 2y + 2z = -4$$

**c.iii. Outcomes assessed: MEX12-3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses vector form for the line $AB$ to produce required Cartesian equation | 2 |
| Some progress eg. writes the Cartesian equation but without adequate explanation | 1 |

**Answer** Any point on $AB$ has the vector representation $\begin{pmatrix} 2+\gamma \\ 3-2\gamma \\ 2\gamma \end{pmatrix}$ for some real $\gamma$.

If the line $l$ passes through such a point and is perpendicular to $AB$, from parts (I) and (ii) with this vector replacing $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$, the equations for $x, y, z$ become

$$\begin{aligned} x &= (2+\gamma) + \lambda u \quad (1) \\ y &= (3-2\gamma) + \lambda v \quad (2) \\ z &= \quad 2\gamma + \lambda w \quad (3) \end{aligned}$$

where $u - 2v + 2w = 0$.

Then $(1) - 2\times(2) + 2\times(3)$ gives $\quad x - 2y + 2z = (2+\gamma) - 2(3-2\gamma) + 4\gamma = -4 + 9\gamma$

Hence the Cartesian equation of any plane $\perp AB$ has equation $x - 2y + 2z = k$ for some constant $k$.

**Question 13**

**a. Outcomes assessed: MEX12-2**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses log laws to prove statement is true | 2 |
| Some progress eg. applies log laws but one error or incomplete explanation | 1 |

**Answer**

Using log laws, for $\forall a>0, \forall b>0$, $\log_{\frac{1}{a}}\frac{1}{b}=\dfrac{\log_a\frac{1}{b}}{\log_a\frac{1}{a}}=\dfrac{-\log_a b}{-1}=\log_a b$

**b. Outcomes assessed: MEX12-2**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses the definition of a rational number to construct a proof by contradiction | 2 |
| Some progress eg. quotes the condition for $\log_2 5$ to be rational | 1 |

**Answer**

$5>1 \therefore \log_2 5>0$. Hence $\log_2 5$ is rational $\Rightarrow \exists$ positive integers $p$, $q$ with no common factor

such that $\log_2 5=\dfrac{p}{q}$. Then

$$q\log_2 5=p$$

$$\therefore \log_2 5^q=p$$

$$\therefore 5^q=2^p$$

But 5 and 2 are prime numbers, so this last statement cannot be true for any positive integers $p$, $q$.
Hence by contradiction $\log_2 5$ is irrational.

**c.i. Outcomes assessed: MEX12-5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Correct answer | 1 |

**Answer**

$$I_1=\int_0^1\frac{1}{1+x^2}dx=\left[\tan^{-1}x\right]_0^1=\frac{\pi}{4}$$

(Q13 cont.)

**c.ii. Outcomes assessed: MEX12-5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Applies integration by parts and rearranges result into required form | 3 |
| Substantial progress eg. correct procedure but with one error in execution | 2 |
| Some progress eg. applies integration by parts | 1 |

**Answer**

$$I_n = \int_0^1 1 \cdot \frac{1}{(1+x^2)^n}\,dx$$

$$= \left[\frac{x}{(1+x^2)^n}\right]_0^1 - \int_0^1 x \cdot \frac{-2nx}{(1+x^2)^{n+1}}\,dx$$

$$= 2^{-n} + 2n\int_0^1 \frac{(1+x^2)-1}{(1+x^2)^{n+1}}\,dx$$

$$I_n = 2^{-n} + 2n\{I_n - I_{n+1}\}$$

$$2nI_{n+1} = 2^{-n} + (2n-1)I_n$$

**c.iii. Outcomes assessed: MEX12-5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Applies recurrence formula to evaluate as required | 2 |
| Some progress eg. one error in application of recurrence formula | 1 |

**Answer**

$$4I_3 = 2^{-2} + 3I_2 = 2^{-2} + \tfrac{3}{2}\left(2^{-1} + I_1\right) = 1 + \tfrac{3\pi}{8} \qquad \therefore I_3 = \frac{8+3\pi}{32}$$

**d.i. Outcomes assessed: MEX12-3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses the given perpendicular lines to deduce the results | 2 |
| Some progress eg. correct procedure but poorly explained | 1 |

**Answer**

$\overrightarrow{EB} \perp \overrightarrow{AC}$ and $O$ lies on $EB$. Hence $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$. $\therefore \underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = 0$.

Similarly, since $\overrightarrow{FC} \perp \overrightarrow{AB}$ and $O$ lies on $FC$, $\underset{\sim}{c} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0$.

**d.ii. Outcomes assessed: MEX12-3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses the result from (i) and the properties of dot products to prove the result | 2 |
| Some progress eg. use of dot product properties is partially correct | 1 |

**Answer**

$0 = \underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{a} \qquad \therefore \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot \underset{\sim}{a}$

$0 = \underset{\sim}{c} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = \underset{\sim}{c} \cdot \underset{\sim}{b} - \underset{\sim}{c} \cdot \underset{\sim}{a} \qquad \therefore \underset{\sim}{c} \cdot \underset{\sim}{b} = \underset{\sim}{c} \cdot \underset{\sim}{a}$

Then $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \underset{\sim}{c}$

$\therefore \underset{\sim}{a} \cdot (\underset{\sim}{b} - \underset{\sim}{c}) = 0$

Hence $AD \perp CB$

(Q13 cont.)

**d.iii. Outcomes assessed: MEX12-3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| States geometric property | 1 |

**Answer**

The altitudes of a triangle are concurrent

**Question 14**

**a.i. Outcomes assessed: MEX12-2**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Expands and regroups to establish result | 2 |
| Some progress eg. expands | 1 |

**Answer**

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)$$
$$\geq 3+2+2+2$$
$$=9$$

**a.ii Outcomes assessed: MEX12-2**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Makes an appropriate transformation to establish required result | 3 |
| Substantial progress eg. appropriate transformation, and almost completes proof | 2 |
| Some progress eg. appropriate transformation with some attempt to expand | 1 |

**Answer**

$a \to a+b$

$b \to b+c$ gives

$c \to c+a$

$$2(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\geq 9$$
$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\left(\frac{a}{a+b}+\frac{b}{a+b}\right)+\left(\frac{b}{b+c}+\frac{c}{b+c}\right)+\left(\frac{c}{c+a}+\frac{a}{c+a}\right)\geq\frac{9}{2}$$
$$\therefore \left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+1+1+1\geq\frac{9}{2}$$
$$\therefore \quad \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

**b.i Outcomes assessed: MEX12-6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses compound angle trigonometric identities to establish result | 2 |
| Some progress eg. correct procedure with one error | 1 |

**Answer**

$$\cos 2t+\sin 2t=\sqrt{2}\left(\tfrac{1}{\sqrt{2}}\cos 2t+\tfrac{1}{\sqrt{2}}\sin 2t\right)$$
$$=\sqrt{2}\left(\cos 2t\cos\tfrac{\pi}{4}+\sin 2t\sin\tfrac{\pi}{4}\right)$$
$$=\sqrt{2}\cos\left(2t-\tfrac{\pi}{4}\right)$$

$$\therefore x=1+\sqrt{2}\cos\left(2t-\tfrac{\pi}{4}\right)$$

(Q14 cont.)

**b.ii. Outcomes assessed: MEX12-6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Finds distance travelled and time taken to first reach $O$ then calculates average speed | 3 |
| Substantial progress eg. finds time to reach $O$ and initial position | 2 |
| Some progress eg. finds time to reach $O$ | 1 |

**Answer**

$$x=0 \Rightarrow \sqrt{2}\cos\left(2t-\tfrac{\pi}{4}\right)=-1$$
$$\cos\left(2t-\tfrac{\pi}{4}\right)=-\tfrac{1}{\sqrt{2}}$$
$$2t-\tfrac{\pi}{4}=\tfrac{3\pi}{4},\ \tfrac{5\pi}{4},\ldots$$
$$t=\tfrac{\pi}{2},\ \tfrac{3\pi}{4},\ldots$$

$$x=1+\sqrt{2}\cos\left(2t-\tfrac{\pi}{4}\right)$$
$$v=-2\sqrt{2}\sin\left(2t-\tfrac{\pi}{4}\right)$$
$$t=0 \Rightarrow x=2,\ v=2$$

In first $\frac{\pi}{2}$ seconds particle moves right from $x=2$ to $x=1+\sqrt{2}$ , then left to $x=0$ .

Hence average speed during time it takes to first reach $O$ is $\dfrac{\left(1+\sqrt{2}\right)-2+\left(1+\sqrt{2}\right)}{\left(\frac{\pi}{2}\right)}=\dfrac{4\sqrt{2}}{\pi}\approx 1\cdot 80\ \text{ms}^{-1}$

**c.i. Outcomes assessed: MEX12-2**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses process of Mathematical Induction to prove required result | 3 |
| Substantial progress eg. shows proposition true for $a_1$ and writes $a_{k+1}$ in terms of the appropriate cosine function, conditional on the truth of the proposition for $a_k$ | 2 |
| Some progress eg. shows proposition true for $a_1$ | 1 |

**Answer**

Let P($n$), $n$=1, 2, 3, … be the sequence of propositions $a_n=2\cos\left(\dfrac{\pi}{2^{n+1}}\right)$ .

Consider P(1): $a_1=\sqrt{2}=2\times\frac{1}{\sqrt{2}}=2\cos\left(\dfrac{\pi}{2^2}\right)$ $\therefore$ P(1) is true.

If P($k$) is true : $a_k=2\cos\left(\dfrac{\pi}{2^{k+1}}\right)$ *

Consider P($k$+1) : $a_{k+1}=\sqrt{2+a_k}$

$$=\sqrt{2+2\cos\left(\frac{\pi}{2^{k+1}}\right)} \quad \text{if P}(k)\text{ is true, using *}$$
$$=\sqrt{2\left\{1+\cos\left(2\frac{\pi}{2^{k+2}}\right)\right\}}$$
$$=\sqrt{4\cos^2\left(\frac{\pi}{2^{k+2}}\right)}$$
$$=2\cos\left(\frac{\pi}{2^{k+2}}\right)$$

Hence if P($k$) is true, then P($k$+1) is true. But P(1) is true. Hence by Mathematical induction P($n$) is true for all positive integers $n$.

(Q14 cont.)

**c.ii. Outcomes assessed: MEX12-1**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses a known trigonometric limit to deduce the limiting value | 2 |
| Some progress eg. correct procedure with one error in algebraic manipulation | 1 |

**Answer**

$$\lim_{n\to\infty}\left\{4^n\left(2-a_n\right)\right\}=\lim_{n\to\infty}\left\{4^n\times 2\left(1-\cos\left(2\frac{\pi}{2^{n+2}}\right)\right)\right\}$$

$$=\lim_{n\to\infty}\left\{4^n\times 4\sin^2\left(\frac{\pi}{2^{n+2}}\right)\right\}$$

$$=\lim_{n\to\infty}\left\{\left[\frac{\sin\left(\frac{\pi}{2^{n+2}}\right)}{\left(\frac{\pi}{2^{n+2}}\right)}\right]^2\times\frac{\pi^2}{4}\right\}$$

$$=\frac{\pi^2}{4}$$

## Question 15

**a. Outcomes assessed: MEX12-2**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses factors of 4 to deduce the possible values of each bracket to deduce result | 3 |
| Substantial progress eg. determines the possible values of the brackets with explanation | 2 |
| Some progress eg. realises each bracket is a factor of 4 | 1 |

**Answer**

If $(x-a)(x-b)(x-c)(x-d)=4$ has an integer solution $x$, then the values of the brackets are also integers, each a factor of 4, with magnitudes $1, 1, 1, 4$ or $1, 1, 2, 2$. Since these values must be distinct, the brackets must have the values $1, -1, 2, -2$ in some order.

$$(x-a)+(x-b)+(x-c)+(x-d)=0$$

Hence $$4x-(a+b+c+d)=0$$

$$x=\frac{a+b+c+d}{4}$$

(Note this does not imply this is an integer solution, only that no other integer solution is possible)

(Q15 cont.)
**b.i. Outcomes assessed: MEX12-6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses appropriate expression for $a$ and integration to produce required expression for $v$ | 2 |
| Some progress eg. correct procedure but neglects to explain –ve sign | 1 |

**Answer**

Initially $v<0$ since $a>0$ and particle is slowing down.

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=e^{\frac{1}{2}x}$$

$$\tfrac{1}{2}v^2=2e^{\frac{1}{2}x}+c$$

$$\left.\begin{array}{l} t=0 \\ x=0 \\ v=-2 \end{array}\right\} \Rightarrow \begin{array}{l} c=0 \\ v^2=4e^{\frac{1}{2}x} \\ v=-2e^{\frac{1}{4}x} \end{array}$$

**b.ii. Outcomes assessed: MEX12-6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Finds $x$ in terms of $t$ by integration, then states $v$ and $a$ in terms of $t$ | 3 |
| Substantial progress eg. finds $x$ in terms of $t$ by integration | 2 |
| Some progress eg. uses integration to find $x$ in terms of $t$, but makes one error | 1 |

**Answer**

$$\frac{dx}{dt}=-2e^{\frac{1}{4}x}$$

$$\int e^{-\frac{1}{4}x}\,dx=\int -2\,dt$$

$$-4e^{-\frac{1}{4}x}=-2t+c$$

$$\left.\begin{array}{l} t=0 \\ x=0 \end{array}\right\} \Rightarrow \begin{array}{l} c=-4 \\ e^{-\frac{1}{4}x}=\dfrac{t+2}{2} \end{array}$$

$$x=4\ln\left(\frac{2}{2+t}\right)$$

$$v=-2e^{\frac{1}{4}x}$$

$$\therefore v=\frac{-4}{2+t}$$

$$a=e^{\frac{1}{2}x}$$

$$=\left(e^{\frac{1}{4}x}\right)^2$$

$$\therefore a=\frac{4}{(2+t)^2}$$

**b.iii. Outcomes assessed: MEX12-6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Correct description | 1 |

**Answer**

Particle continues to move in the initial direction of travel without bound, but slowing down at an ever decreasing rate with velocity and acceleration both approaching 0.

(Q15 cont.)

**c.i. Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Uses de'Moivre's theorem to deduce result | 2 |
| Some progress eg. rearranges in terms of $\sin\theta$ and $\cos\theta$ | 1 |

**Answer**

$$\begin{aligned}(1+i\tan\theta)^4+(1-i\tan\theta)^4 &= \frac{(\cos\theta+i\sin\theta)^4+(\cos\theta-i\sin\theta)^4}{\cos^4\theta}\\ &= \frac{(\cos4\theta+i\sin4\theta)+(\cos4\theta-i\sin4\theta)}{\cos^4\theta}\\ &= \frac{2\cos4\theta}{\cos^4\theta}\end{aligned}$$

**c.ii. Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Transforms the equation and solves for $\theta$ and hence for $z$ | 2 |
| Some progress eg. correct procedure with an error in execution | 1 |

**Answer**

Let $z=i\tan\theta$. Then $(1+z)^4+(1-z)^4=0$

becomes $\dfrac{2\cos4\theta}{\cos^4\theta}=0$

$$\cos4\theta=0$$

$$4\theta=n\tfrac{\pi}{2},\quad n\ odd$$

$$\theta=n\tfrac{\pi}{8},\quad n\ odd$$

$\therefore$ the 4 roots are

$i\tan\frac{\pi}{8},\ \ i\tan\frac{3\pi}{8},\ \ i\tan\frac{5\pi}{8},\ \ i\tan\frac{7\pi}{8}$

**c.iii. Outcomes assessed: MEX12-4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Deduces result from relationship between roots and coefficients using trig identities | 2 |
| Some progress eg. evaluates the product of the roots from the coefficients of the polynomial | 1 |

**Answer**

$4^{\text{th}}$ degree polynomial $(1+z)^4+(1-z)^4=0$ has leading term $2z^4$ and constant term 2.

Hence the product of the roots is 1. $\quad\therefore\left(i\tan\frac{\pi}{8}\right)\left(i\tan\frac{3\pi}{8}\right)\left(i\tan\frac{5\pi}{8}\right)\left(i\tan\frac{7\pi}{8}\right)=1$

But $\begin{cases}\tan\frac{5\pi}{8}=\tan\left(\pi-\frac{3\pi}{8}\right)=-\tan\frac{3\pi}{8}\\ \tan\frac{7\pi}{8}=\tan\left(\pi-\frac{\pi}{8}\right)=-\tan\frac{\pi}{8}\end{cases}$

$\therefore\ \left(-i^2\tan^2\frac{\pi}{8}\right)\left(-i^2\tan^2\frac{3\pi}{8}\right)=1$

$\therefore\ \tan^2\frac{\pi}{8}\tan^2\frac{3\pi}{8}=1$

**Question 16**

**a.i. Outcomes assessed: MEX12-6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Applies product rule for differentiation then deduces required result | 2 |
| Some progress eg. applies product rule for differentiation | 1 |

**Answer**

$$\frac{du}{dt}+u\,f(t)=g(t)$$

$$\frac{d}{dt}\left(u\,e^{F(t)}\right)=\left(\frac{du}{dt}\right)e^{F(t)}+u\,f(t)e^{F(t)}$$

$$=e^{F(t)}\left\{\frac{du}{dt}+u\,f(t)\right\}$$

$$=e^{F(t)}\,g(t)$$

$$\therefore u\,e^{F(t)}=\int e^{F(t)}g(t)\,dt$$

$$\therefore \quad u=e^{-F(t)}\int e^{F(t)}g(t)\,dt$$

**a.ii. Outcomes assessed: MEX12-6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Finds $\dot{y}$ and $y$ as functions of $t$ by integration, then finds $y$ when $\dot{y}=0$ | 4 |
| Substantial progress eg. correct procedure but with one error in evaluation | 3 |
| Moderate progress eg. finds $\dot{y}$ as a function of $t$ by integration, then $t$ when $\dot{y}=0$ | 2 |
| Some progress eg. finds $\dot{y}$ as a function of $t$ | 1 |

**Answer**

$$\frac{d}{dt}(\dot{y})+\tfrac{1}{10}\dot{y}=-10 \qquad \text{Applying (i) with } f(t)=\tfrac{1}{10},\ F(t)=\tfrac{1}{10}t,\ g(t)=-10$$

$$\dot{y}=e^{-\frac{1}{10}t}\int e^{\frac{1}{10}t}(-10)\,dt$$

$$\dot{y}=e^{-\frac{1}{10}t}\left\{-100e^{\frac{1}{10}t}+c\right\}$$

$$\dot{y}=ce^{-\frac{1}{10}t}-100$$

$$\left.\begin{array}{l} t=0 \\ \dot{y}=12 \end{array}\right\} \Rightarrow \begin{array}{l} c=112 \\ \dot{y}=112e^{-\frac{1}{10}t}-100 \end{array}$$

$$\dot{y}=0 \Rightarrow e^{-\frac{1}{10}t}=\tfrac{100}{112}=\tfrac{25}{28}$$

$$t=10\ln\tfrac{28}{25}$$

$$y=-1120e^{-\frac{1}{10}t}-100t+c_1$$

$$\left.\begin{array}{l} t=0 \\ y=0 \end{array}\right\} \Rightarrow \begin{array}{l} c_1=1120 \\ y=1120-1120e^{-\frac{1}{10}t}-100t \end{array}$$

Hence at greatest height,

$$\dot{y}=0 \Rightarrow y=1120-1120\times\tfrac{100}{112}-1000\ln\tfrac{28}{25}$$

$$=120-1000\ln\tfrac{28}{25}$$

$\therefore$ Maximum height attained is $120-1000\ln\frac{28}{25}\approx 109$ m (to nearest metre)

(Q16 cont.)

**a.iii. Outcomes assessed: MEX12-6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Finds $\dot{x}$ and x as functions of $t$ by integration then considers limiting position and velocity vectors as $t \to \infty$ to describe limiting motion | 4 |
| Substantial progress eg. correct procedure but one error, or one feature of limiting motion omitted | 3 |
| Moderate progress eg. finds $\dot{x}$ and $x$ in terms of $t$ by integration | 2 |
| Some progress eg. attempts integration to find $\dot{x}$ and $x$ by integration but makes an error | 1 |

**Answer**

$$\frac{d}{dt}(\dot{x})+\tfrac{1}{10}\dot{x}=0 \qquad \text{Applying (i) with } f(t)=\tfrac{1}{10},\ F(t)=\tfrac{1}{10}t,\ g(t)=0$$

$$\dot{x}=e^{-\frac{1}{10}t}\int 0\,dt=ce^{-\frac{1}{10}t}$$

$$\left.\begin{aligned}t&=0\\ \dot{x}&=16\end{aligned}\right\}\Rightarrow \begin{aligned}c&=16\\ \dot{x}&=16e^{-\frac{1}{10}t}\end{aligned}$$

$$x=-160e^{-\frac{1}{10}t}+c_1$$

$$\left.\begin{aligned}t&=0\\ x&=0\end{aligned}\right\}\Rightarrow \begin{aligned}c_1&=160\\ x&=160\left(1-e^{-\frac{1}{10}t}\right)\end{aligned}$$

$$\underset{\sim}{r}(t)=160\left(1-e^{-\frac{1}{10}t}\right)\underset{\sim}{i}+\left\{1120\left(1-e^{-\frac{1}{10}t}\right)-100t\right\}\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}}(t)=16e^{-\frac{1}{10}t}\,\underset{\sim}{i}+\left(112e^{-\frac{1}{10}t}-100\right)\underset{\sim}{j}$$

As $t\to\infty$, $\dot{\underset{\sim}{r}}(t)\to -100\underset{\sim}{j}$

$\underset{\sim}{r}(t)\,.\,\underset{\sim}{i}\to 160$

Hence the particle will eventually be travelling vertically downwards with terminal velocity $100\ \text{ms}^{-1}$ at a horizontal displacement 160 m from $O$.

**b.i. Outcomes assessed: MEX12-1**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Manipulates factorials, and interprets or manipulates sigma notation to prove the results | 2 |
| Some progress eg. manipulates factorials to show first result | 1 |

**Answer**

$$\frac{1}{k!}-\frac{1}{(k+1)!}=\frac{(k+1)-1}{(k+1)!}=\frac{k}{(k+1)!}$$

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{k}{(k+1)!}&=\sum_{k=1}^{n-1}\left\{\frac{1}{k!}-\frac{1}{(k+1)!}\right\}\\ &=\sum_{k=1}^{n-1}\frac{1}{k!}-\sum_{k=2}^{n}\frac{1}{k!}\\ &=\frac{1}{1!}-\frac{1}{n!}\\ &=1-\frac{1}{n!}\end{aligned}$$

(Q16 cont.)
**b.ii. Outcomes assessed: MEX12-1**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Establishes result | 1 |

**Answer**

$$1=\sum_{k=1}^{n-1}\frac{k}{(k+1)!}+\frac{1}{n!}$$

$$=\sum_{k=1}^{n-1}\frac{1}{m_k}+\frac{1}{m_n} \qquad \text{where } m_k=\frac{(k+1)!}{k},\ k=1,2,\ldots,n-1 \quad \text{and} \quad m_n=n!$$

Then $1=\dfrac{1}{m_1}+\dfrac{1}{m_2}+\ldots+\dfrac{1}{m_{n-1}}+\dfrac{1}{m_n}$ .

**b.iii. Outcomes assessed: MEX12-1**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| Writes $n!$ as a sum of $n$ divisors of itself and determines that these are distinct | 2 |
| Some progress eg. expresses $n!$ as a sum of $n$ divisors of itself | 1 |

**Answer**

From (ii) $\quad n!=\sum_{k=1}^{n}\dfrac{n!}{m_k} \qquad$ For $k=1,\ 2,\ \ldots,n-1$ , $\quad \dfrac{n!}{m_k}=\dfrac{n!}{(k+1)(k-1)!}=n(n-1)\ldots(k+2)k$

and $\dfrac{n!}{m_n}=1$ . Hence $n!$ has been expressed as a sum of $n$ divisors of $n!$

Also $\dfrac{m_k}{m_{k-1}}=\dfrac{(k+1)!}{k}\times\dfrac{k-1}{k!}=\dfrac{(k+1)(k-1)}{k}=k-\dfrac{1}{k}>1 \quad$ for $k=2,3,\ldots,n-1$

$\therefore m_1<m_2<\ldots<m_{n-1}$ and $m_n=n!>\dfrac{n!}{n-1}=m_{n-1}$ for $n\geq 3$.

Hence $m_1, m_2,\ldots,m_n$, and hence $\dfrac{n!}{m_1}, \dfrac{n!}{m_2},\ldots,\dfrac{n!}{m_n}$ , are distinct for $n\geq 3$.

**Independent HSC Trial Examinations 2020 Mathematics Extension 2 Mapping Grid**

| Question | Marks | Content | Syllabus Outcomes | Targeted Performance Bands |
|---|---|---|---|---|
| 1 | 1 | MEX-P1 The nature of proof | MEX12-2 | 2-3 |
| 2 | 1 | MEX-V1 Further work with vectors | MEX12-3 | 2-3 |
| 3 | 1 | MEX-P1 The nature of proof | MEX12-2 | 2-3 |
| 4 | 1 | MEX-V1 Further work with vectors | MEX12-3 | 2-3 |
| 5 | 1 | MEX-C1 Further integration | MEX12-5 | 2-3 |
| 6 | 1 | MEX-N1 Introduction to complex numbers | MEX12-4 | 3-4 |
| 7 | 1 | MEX-M1 Applications of calculus to mechanics | MEX12-6 | 3-4 |
| 8 | 1 | MEX-C1 Further integration | MEX12-5 | 3-4 |
| 9 | 1 | MEX-M1 Applications of calculus to mechanics | MEX12-6 | 3-4 |
| 10 | 1 | MEX-N2 Using complex numbers | MEX12-4 | 3-4 |
| | | | | |
| 11 a i | 1 | MEX-N1 Introduction to complex numbers | MEX12-4 | 2-3 |
| ii | 1 | MEX-N1 Introduction to complex numbers | MEX12-4 | 2-3 |
| iii | 1 | MEX-N1 Introduction to complex numbers | MEX12-4 | 2-3 |
| iv | 1 | MEX-N1 Introduction to complex numbers | MEX12-4 | 2-3 |
| b i | 2 | MEX-C1 Further integration | MEX12-5 | 2-3 |
| ii | 3 | MEX-C1 Further integration | MEX12-5 | 2-3 |
| c i | 2 | MEX-N2 Using complex numbers | MEX12-4 | 2-3 |
| ii | 1 | MEX-N2 Using complex numbers | MEX12-4 | 2-3 |
| iii | 3 | MEX-N2 Using complex numbers | MEX12-4 | 2-3 |
| 12 a | 4 | MEX-C1 Further integration | MEX12-5 | 2-3 |
| b i | 2 | MEX-N1 Introduction to complex numbers | MEX12-4 | 2-3 |
| ii | 1 | MEX-N1 Introduction to complex numbers | MEX12-4 | 2-3 |
| iii | 2 | MEX-N1 Introduction to complex numbers | MEX12-4 | 2-3 |
| c i | 2 | MEX-V1 Further work with vectors | MEX12-3 | 3-4 |
| ii | 2 | MEX-V1 Further work with vectors | MEX12-3 | 3-4 |
| iii | 2 | MEX-V1 Further work with vectors | MEX12-3 | 3-4 |
| 13 a | 2 | MEX-P1 The nature of proof | MEX12-2 | 2-3 |
| b | 2 | MEX-P1 The nature of proof | MEX12-2 | 2-3 |
| c i | 1 | MEX-C1 Further integration | MEX12-5 | 2-3 |
| ii | 3 | MEX-C1 Further integration | MEX12-5 | 3-4 |
| iii | 2 | MEX-C1 Further integration | MEX12-5 | 2-3 |
| d i | 2 | MEX-V1 Further work with vectors | MEX12-3 | 2-3 |
| ii | 2 | MEX-V1 Further work with vectors | MEX12-3 | 2-3 |
| iii | 1 | MEX-V1 Further work with vectors | MEX12-3 | 2-3 |
| 14 a i | 2 | MEX-P1 The nature of proof | MEX12-2 | 2-3 |
| ii | 3 | MEX-P1 The nature of proof | MEX12-2 | 3-4 |
| b i | 2 | MEX-M1 Applications of calculus to mechanics | MEX12-6 | 2-3 |
| ii | 3 | MEX-M1 Applications of calculus to mechanics | MEX12-6 | 2-3 |
| c i | 3 | MEX-P2 Further proof by mathematical induction | MEX12-2 | 3-4 |
| ii | 2 | MEX-P2 Further proof by mathematical induction | MEX12-1 | 3-4 |
| 15 a | 3 | MEX-P1 The nature of proof | MEX12-2 | 3-4 |
| b i | 2 | MEX-M1 Applications of calculus to mechanics | MEX12-6 | 3-4 |
| ii | 3 | MEX-M1 Applications of calculus to mechanics | MEX12-6 | 3-4 |
| iii | 1 | MEX-M1 Applications of calculus to mechanics | MEX12-6 | 3-4 |
| c i | 2 | MEX-N2 Using complex numbers | MEX12-4 | 3-4 |
| ii | 2 | MEX-N2 Using complex numbers | MEX12-4 | 2-3 |
| iii | 2 | MEX-N2 Using complex numbers | MEX12-4 | 3-4 |
| | | | | |

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|---|---|---|---|---|
| 16 a i | 2 | MEX-M1 Applications of calculus to mechanics | MEX12-6 | 3-4 |
| ii | 4 | MEX-M1 Applications of calculus to mechanics | MEX12-6 | 3-4 |
| iii | 4 | MEX-M1 Applications of calculus to mechanics | MEX12-6 | 3-4 |
| b i | 2 | MEX-P1 The nature of proof | MEX12-1 | 3-4 |
| ii | 1 | MEX-P1 The nature of proof | MEX12-1 | 3-4 |
| iii | 2 | MEX-P1 The nature of proof | MEX12-1 | 3-4 |