NSW Education Standards Authority

2022 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number on the Question 12 Writing Booklet attached

Total marks: 70

Section I – 10 marks (pages 2–9)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 10–18)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 It is given that $\cos\left(\frac{23\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$.

Which of the following is the value of $\cos^{-1}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$?

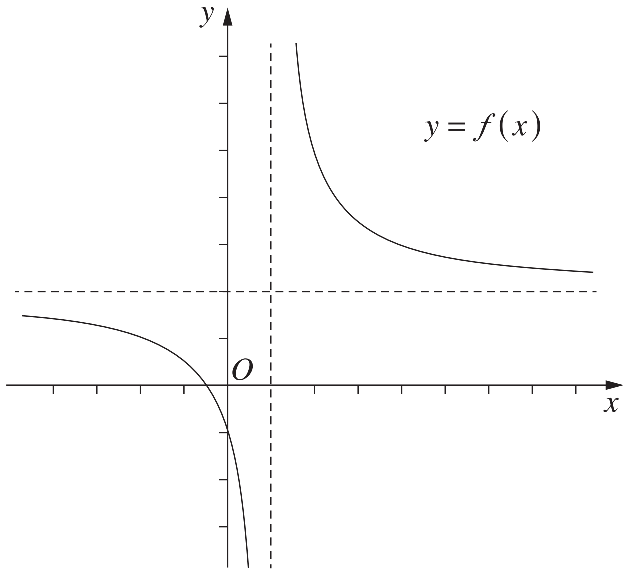
A. $\frac{23\pi}{12}$

B. $\frac{11\pi}{12}$

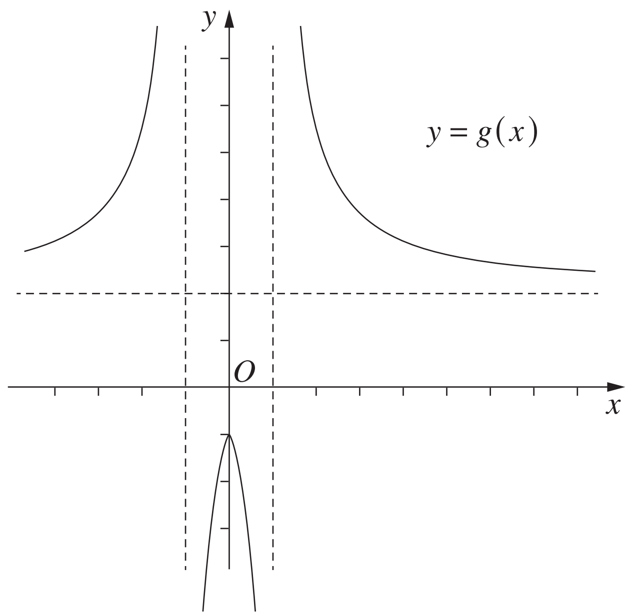
C. $\frac{\pi}{12}$

D. $-\frac{11\pi}{12}$

2 The graph of $f(x) = \dfrac{3}{x-1} + 2$ is shown.



The graph of $f(x)$ was transformed to get the graph of $g(x)$ as shown.



What transformation was applied?

A. $g(x) = f(|x|)$

B. $g(x) = \sqrt{f(x)}$

C. $g(x) = f(-x)$

D. $g(x) = \dfrac{1}{f(x)}$

3 Let $P(x)$ be a polynomial of degree 5. When $P(x)$ is divided by the polynomial $Q(x)$, the remainder is $2x + 5$.

Which of the following is true about the degree of Q?

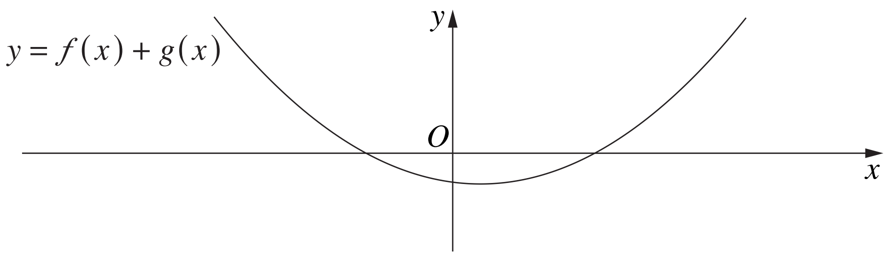
A. The degree must be 1.

B. The degree could be 1.

C. The degree must be 2.

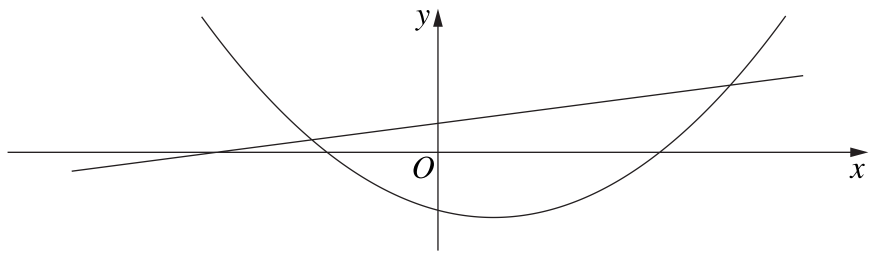
D. The degree could be 2.

4 The diagram shows the graph of the sum of the functions $f(x)$ and $g(x)$.

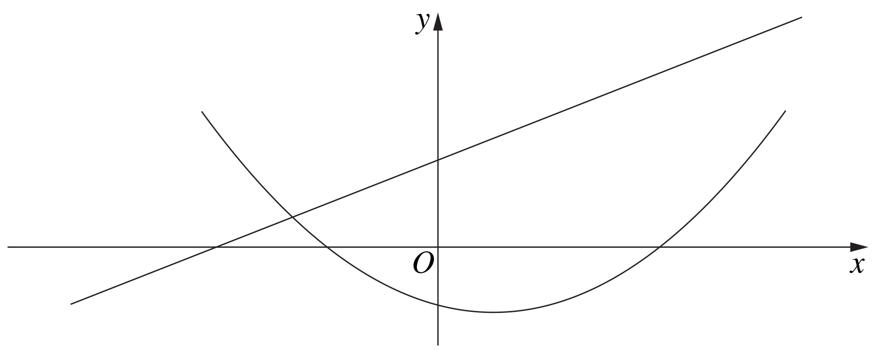


Which of the following best represents the graphs of both $f(x)$ and $g(x)$?

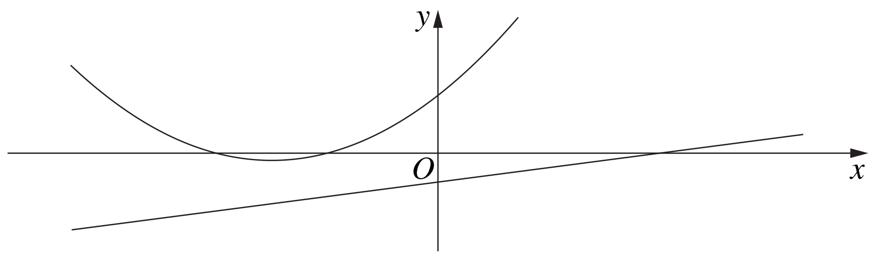
A.



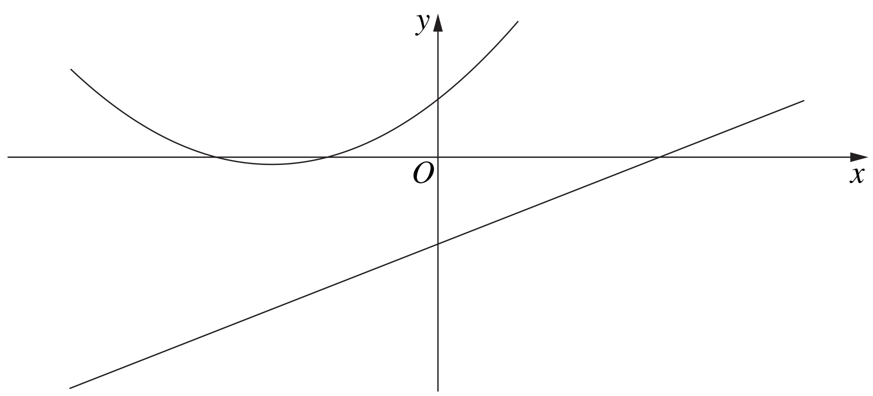
B.



C.



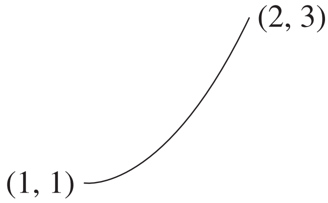
D.



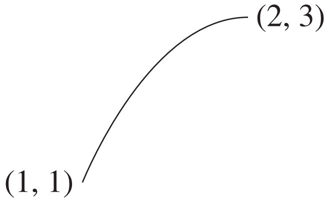
5 A curve is defined in parametric form by $x = 2 + t$ and $y = 3 - 2t^2$ for $-1 \le t \le 0$.

Which diagram best represents this curve?

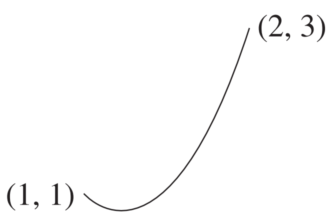
A.



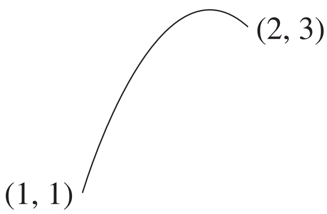
B.



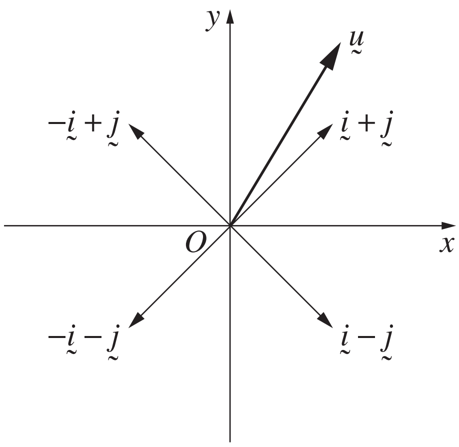
C.



D.



6 The following diagram shows the vector $\underset{\sim}{u}$ and the vectors $\underset{\sim}{i} + \underset{\sim}{j}$, $-\underset{\sim}{i} + \underset{\sim}{j}$, $-\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{i} - \underset{\sim}{j}$.



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TO SCALE

Which statement regarding this diagram could be true?

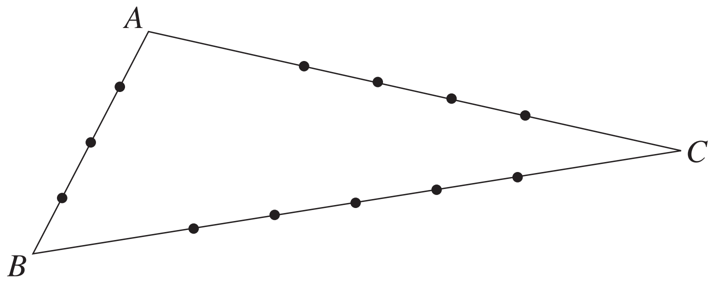
A. The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{i} + \underset{\sim}{j}$ is the vector $1.1\underset{\sim}{i} + 1.8\underset{\sim}{j}$.

B. The projection of $\underset{\sim}{u}$ onto $-\underset{\sim}{i} + \underset{\sim}{j}$ is the vector $-0.4\underset{\sim}{i} + 0.4\underset{\sim}{j}$.

C. The projection of $\underset{\sim}{u}$ onto $-\underset{\sim}{i} - \underset{\sim}{j}$ is the vector $3.2\underset{\sim}{i} + 3.2\underset{\sim}{j}$.

D. The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{i} - \underset{\sim}{j}$ is the vector $0.5\underset{\sim}{i} - 0.5\underset{\sim}{j}$.

7 The diagram shows triangle ABC with points chosen on each of the sides. On side AB, 3 points are chosen. On side AC, 4 points are chosen. On side BC, 5 points are chosen.



How many triangles can be formed using the chosen points as vertices?

A. 60

B. 145

C. 205

D. 220

8 The angle between two unit vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is θ and $\left| \underset{\sim}{a} + \underset{\sim}{b} \right| < 1$.

Which of the following best describes the possible range of values of θ?

A. $0 \le \theta < \dfrac{\pi}{3}$

B. $0 \le \theta < \dfrac{2\pi}{3}$

C. $\dfrac{\pi}{3} < \theta \le \pi$

D. $\dfrac{2\pi}{3} < \theta \le \pi$

9 A given function $f(x)$ has an inverse $f^{-1}(x)$.

The derivatives of $f(x)$ and $f^{-1}(x)$ exist for all real numbers x.

The graphs $y = f(x)$ and $y = f^{-1}(x)$ have at least one point of intersection.

Which statement is true for all points of intersection of these graphs?

A. All points of intersection lie on the line $y = x$.

B. None of the points of intersection lie on the line $y = x$.

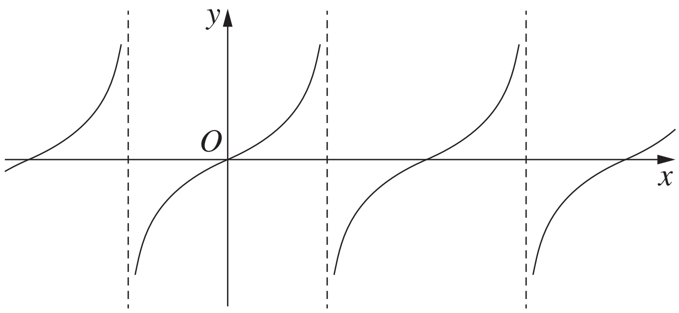
C. At no point of intersection are the tangents to the graphs parallel.

D. At no point of intersection are the tangents to the graphs perpendicular.

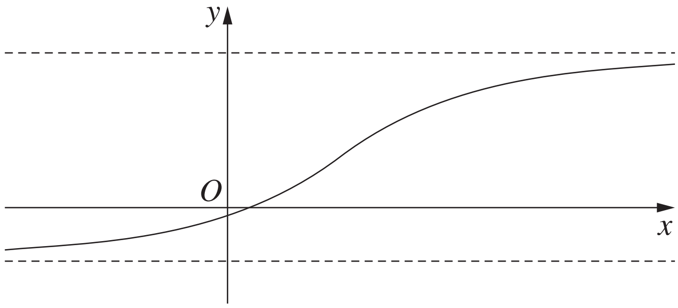
10 Which of the following could be the graph of a solution to the differential equation

$$\frac{dy}{dx} = \sin y + 1?$$

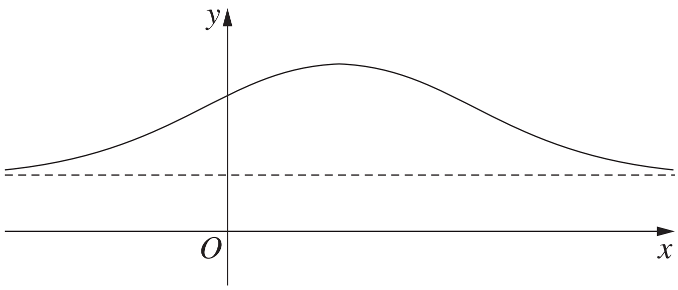
A.



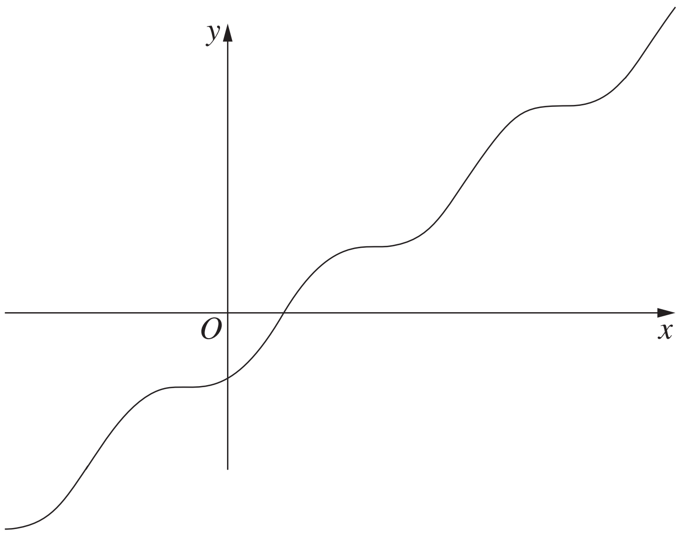
B.



C.



D.



Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) For the vectors $\underset{\sim}{u} = \underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + \underset{\sim}{j}$, evaluate each of the following.

(i) $\underset{\sim}{u} + 3\underset{\sim}{v}$ **1**

(ii) $\underset{\sim}{u} \cdot \underset{\sim}{v}$ **1**

(b) Find the exact value of $\displaystyle\int_0^1 \frac{x}{\sqrt{x^2+4}}\,dx$ using the substitution $u = x^2 + 4$. **3**

(c) Find the coefficients of x^2 and x^3 in the expansion of $\left(1 - \dfrac{x}{2}\right)^8$. **2**

(d) The vectors $\underset{\sim}{u} = \begin{pmatrix} a \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix}$ are perpendicular. **2**

What are the possible values of a?

(e) Express $\sqrt{3}\sin(x) - 3\cos(x)$ in the form $R\sin(x + \alpha)$. **3**

(f) Solve $\dfrac{x}{2-x} \geq 5$. **3**

Question 12 (16 marks) Use the Question 12 Writing Booklet

(a) A direction field is to be drawn for the differential equation **2**

$$\frac{dy}{dx} = \frac{x - 2y}{x^2 + y^2}.$$

On the diagram on page 1 of the Question 12 Writing Booklet, clearly draw the correct slopes of the direction field at the points P, Q and R.

(b) A sports association manages 13 junior teams. It decides to check the age of all players. Any team that has more than 3 players above the age limit will be penalised. **2**

A total of 41 players are found to be above the age limit.

Will any team be penalised? Justify your answer.

(c) Find the equation of the tangent to the curve $y = x\arctan(x)$ at the point with coordinates $\left(1, \frac{\pi}{4}\right)$. Give your answer in the form $y = mx + c$. **3**

Question 12 continues on page 12

(d) In a room with temperature 12°C, coffee is poured into a cup. The temperature of the coffee when it is poured into the cup is 92°C, and it is far too hot to drink.

The temperature, T, in degrees Celsius, of the coffee, t minutes after it is made, can be modelled using the differential equation $\dfrac{dT}{dt} = k\left(T - T_1\right)$, where k is the constant of proportionality and T_1 is a constant.

(i) It takes 5 minutes for the coffee to cool to a temperature of 76°C. **3**

Using separation of variables, solve the given differential equation to show that $T = 12 + 80e^{\frac{t}{5}\ln\left(\frac{4}{5}\right)}$.

(ii) The optimal drinking temperature for a hot beverage is 57°C. **1**

Find the value of t when the coffee reaches this temperature, giving your answer to the nearest minute.

(e) A game consists of randomly selecting 4 balls from a bag. After each ball is selected it is replaced in the bag. The bag contains 3 red balls and 7 green balls. For each red ball selected, 10 points are earned and for each green ball selected, 5 points are deducted. For instance, if a player picks 3 red balls and 1 green ball, the score will be $3 \times 10 - 1 \times 5 = 25$ points. **2**

What is the expected score in the game?

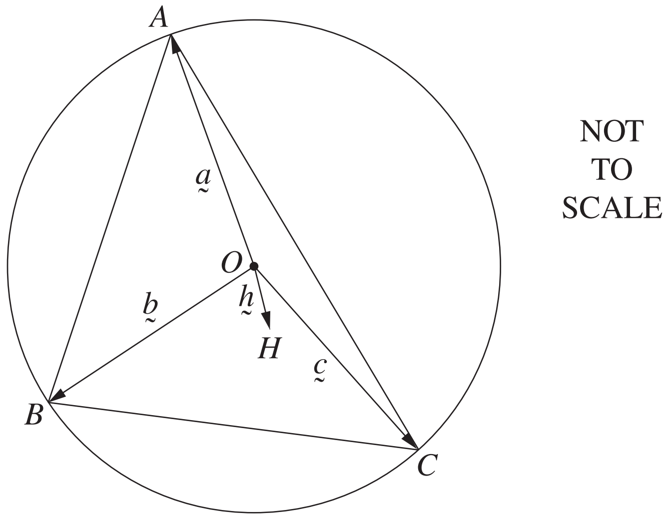
(f) Use mathematical induction to prove that $15^n + 6^{2n+1}$ is divisible by 7 for all integers $n \geq 0$. **3**

End of Question 12

Question 13 (14 marks) Use the Question 13 Writing Booklet

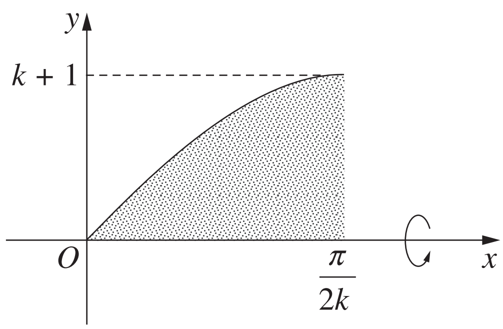
(a) Three different points A, B and C are chosen on a circle centred at O. **3**

Let $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{b} = \overrightarrow{OB}$ and $\underset{\sim}{c} = \overrightarrow{OC}$. Let $\underset{\sim}{h} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ and let H be the point such that $\overrightarrow{OH} = \underset{\sim}{h}$, as shown in the diagram.



Show that $\overrightarrow{BH}$ and $\overrightarrow{CA}$ are perpendicular.

(b) A solid of revolution is to be found by rotating the region bounded by the x-axis and the curve $y = (k+1)\sin(kx)$, where $k > 0$, between $x = 0$ and $x = \dfrac{\pi}{2k}$ about the x-axis. **3**



Find the value of k for which the volume is π^2.

Question 13 continues on page 14

(c) The function f is defined by $f(x) = \sin(x)$ for all real numbers x. Let g be the function defined on $[-1, 1]$ by $g(x) = \arcsin(x)$. **2**

Is g the inverse of f? Justify your answer.

(d) The monic polynomial, P, has degree 3 and roots α, β, γ. **3**

It is given that

$$\alpha^2 + \beta^2 + \gamma^2 = 85 \text{ and}$$
$$P'(\alpha) + P'(\beta) + P'(\gamma) = 87.$$

Find $\alpha\beta + \beta\gamma + \gamma\alpha$.

(e) *You may use the information on page 18 to answer this question.*

A chocolate factory sells 150-gram chocolate bars. There has been a complaint that the bars actually weigh less than 150 grams, so a team of inspectors was sent to the factory to check. They randomly selected 16 bars, weighed them and noted that 8 bars weighed less than 150 grams.

The factory manager claims 80% of the chocolate bars produced by the factory weigh 150 grams or more.

(i) The inspectors used the normal approximation to the binomial distribution to calculate the probability, $\mathcal{P}$, of having at least 8 bars weighing less than 150 grams in a random sample of 16, assuming the factory manager's claim is correct. **2**

Calculate the value of $\mathcal{P}$.

(ii) The factory manager disagrees with the method used by the inspectors as described in part (i). **1**

Explain why the method used by the inspectors might not be valid.

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Find the particular solution to the differential equation $(x-2)\dfrac{dy}{dx} = xy$ that passes through the point $(0, 1)$. **4**

(b) The vectors $\vec{u}$ and $\vec{v}$ are not parallel. The vector $\vec{p}$ is the projection of $\vec{u}$ onto the vector $\vec{v}$. **3**

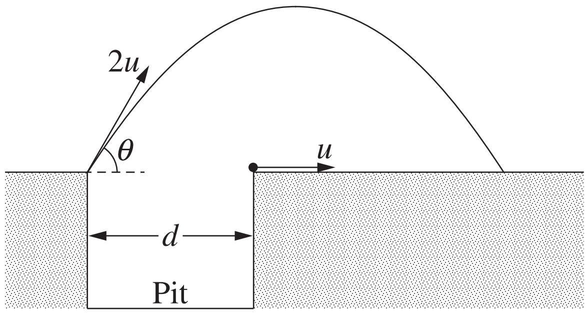
The vector $\vec{p}$ is parallel to $\vec{v}$ so it can be written $\lambda_0 \vec{v}$ for some real number λ_0. (Do NOT prove this.)

Prove that $|\vec{u} - \lambda \vec{v}|$ is smallest when $\lambda = \lambda_0$ by showing that, for all real numbers λ, $|\vec{u} - \lambda_0 \vec{v}| \le |\vec{u} - \lambda \vec{v}|$.

Question 14 continues on page 16

(c) A video game designer wants to include an obstacle in the game they are developing. The player will reach one side of a pit and must shoot a projectile to hit a target on the other side of the pit in order to be able to cross. However, the instant the player shoots, the target begins to move away from the player at a constant speed that is half the initial speed of the projectile shot by the player, as shown in the diagram below. **4**

The initial distance between the player and the target is d, the initial speed of the projectile is $2u$ and it is launched at an angle of θ to the horizontal. The acceleration due to gravity is g. The launch angle is the ONLY parameter that the player can change.



Taking the position of the player when the projectile is launched as the origin, the positions of the projectile and target at time t after the projectile is launched are as follows.

$$\vec{r}_P = \begin{pmatrix} 2ut\cos\theta \\ 2ut\sin\theta - \dfrac{g}{2}t^2 \end{pmatrix} \quad \text{Projectile}$$

$$\vec{r}_T = \begin{pmatrix} d + ut \\ 0 \end{pmatrix} \quad \text{Target}$$

(Do NOT prove these.)

Show that, for the player to have a chance of hitting the target, d must be less than 37% of the maximum possible range of the projectile (to 2 significant figures).

Question 14 continues on page 17

(d) *You may use the information on page 18 to answer this question.* **4**

An airline company that has empty seats on a flight is not maximising its profit.

An airline company has found that there is a probability of 5% that a passenger books a flight but misses it. The management of the airline company decides to allow for overbooking, which means selling more tickets than the number of seats available on each flight.

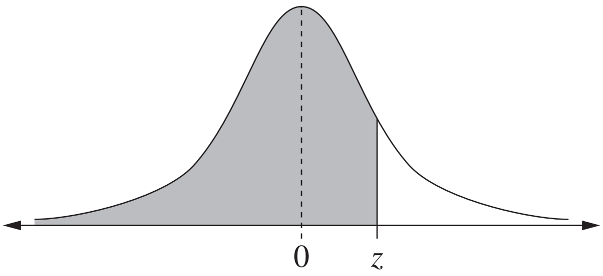
To protect their reputation, management makes the decision that no more than 1% of their flights should have more passengers showing up for the flight than available seats.

Given management's decision and using a suitable approximation, find the maximum number of tickets that can be sold for a flight which has 350 seats.

End of paper

You may use the information below to answer Question 13 (e) and Question 14 (d).

Table of values $P(Z \leq z)$ for the normal distribution $N(0, 1)$



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995

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NSW GOVERNMENT

NSW Education Standards Authority

Centre Number

Student Number

2022 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Writing Booklet

Question 12

Instructions

- Use this Writing Booklet to answer Question 12.
- Write the number of this booklet and the total number of booklets that you have used for this question (eg: 1 of 3).
- Write your Centre Number and Student Number at the top of this page.
- Write using black pen.
- You may ask for an extra writing booklet if you need more space.
- If you have not attempted the question(s), you must still hand in the writing booklet, with 'NOT ATTEMPTED' written clearly on the front cover.
- You may NOT take any writing booklets, used or unused, from the examination room.

12

of

this booklet

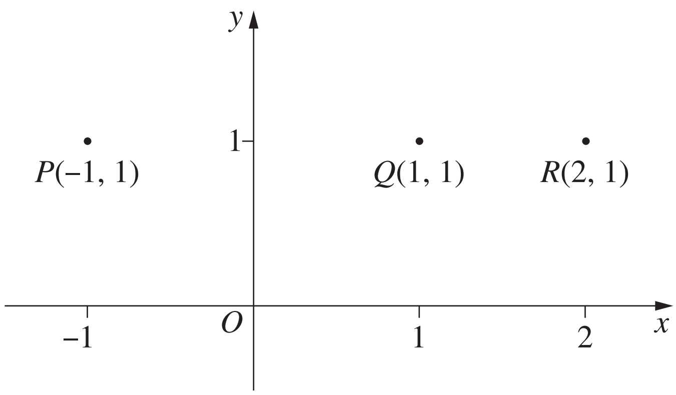
number of booklets for this question

Start here for Question Number: **12**

(a) A direction field is to be drawn for the differential equation

$$\frac{dy}{dx} = \frac{x - 2y}{x^2 + y^2}.$$

Clearly draw the correct slopes of the direction field at the points P, Q and R shown below.



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NSW GOVERNMENT

NSW Education Standards Authority

2022 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \; r \neq 1$$

$$S = \frac{a}{1-r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

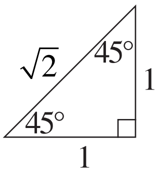
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

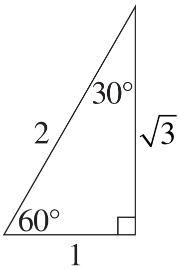
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

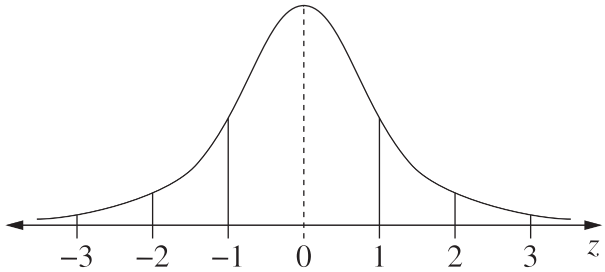
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

2022 HSC Mathematics Extension 1
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	A
3	D
4	A
5	B
6	B
7	C
8	D
9	D
10	B

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$\underset{\sim}{u} = \underset{\sim}{i} - \underset{\sim}{j}, \qquad \underset{\sim}{v} = 2\underset{\sim}{i} + \underset{\sim}{j}$

$$\begin{aligned} \underset{\sim}{u} + 3\underset{\sim}{v} &= \underset{\sim}{i} - \underset{\sim}{j} + 3\left(2\underset{\sim}{i} + \underset{\sim}{j}\right) \\ &= \underset{\sim}{i} - \underset{\sim}{j} + 6\underset{\sim}{i} + 3\underset{\sim}{j} \\ &= 7\underset{\sim}{i} + 2\underset{\sim}{j} \end{aligned}$$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{v} &= 1 \times 2 + (-1) \times 1 \\ &= 1 \end{aligned}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains correct integrand, or equivalent merit	2
• Attempts to use the substitution, or equivalent merit	1

Sample answer:

$$u = x^2 + 4$$
$$du = 2x\,dx$$
$$\text{when } x = 0, \quad u = 4$$
$$\text{when } x = 1, \quad u = 5$$

$$\int_0^1 \frac{x}{\sqrt{x^2+4}}\,dx$$

$$= \int_0^1 \frac{2x}{2\sqrt{x^2+4}}\,dx$$

$$= \int_4^5 \frac{1}{2\sqrt{u}}\,du$$

$$= \left[\sqrt{u}\,\right]_4^5$$

$$= \sqrt{5} - 2$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	2
• Obtains a correct expression for one coefficient, or equivalent merit	1

Sample answer:

$$\left(1 - \frac{x}{2}\right)^8 = \cdots + \binom{8}{2}(1)^6\left(-\frac{x}{2}\right)^2 + \binom{8}{3}(1)^5\left(-\frac{x}{2}\right)^3 + \cdots$$

$$= \cdots + 28 \times \frac{x^2}{4} + 56 \times \frac{-x^3}{8} + \cdots$$

$$= \cdots + 7x^2 - 7x^3 + \cdots$$

$\therefore$ Coefficient of x^2 is 7.

And coefficient of x^3 is -7.

Question 11 (d)

Criteria	Marks
• Provides correct solution	2
• Observes that $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$, or equivalent merit	1

Sample answer:

$$\underset{\sim}{u} = \begin{pmatrix} a \\ 2 \end{pmatrix}, \qquad \underset{\sim}{v} = \begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix}$$

$\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular $\therefore\ \underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

That is $\quad a(a-7)+2(4a-1)=0$

$$a^2-7a+8a-2=0$$
$$a^2+a-2=0$$
$$(a+2)(a-1)=0$$
$$\therefore \quad a=-2 \quad \text{or} \quad 1$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	3
• Finds correct argument, or equivalent merit	2
• Finds the correct value of R, or equivalent merit	1

Sample answer:

$$\sqrt{3}\sin(x) - 3\cos(x) = R\sin(x+\alpha)$$
$$= R\sin x\cos\alpha + R\cos x\sin\alpha$$

$\therefore \quad R\sin\alpha = -3 \qquad (1)$

$R\cos\alpha = \sqrt{3} \qquad (2)$

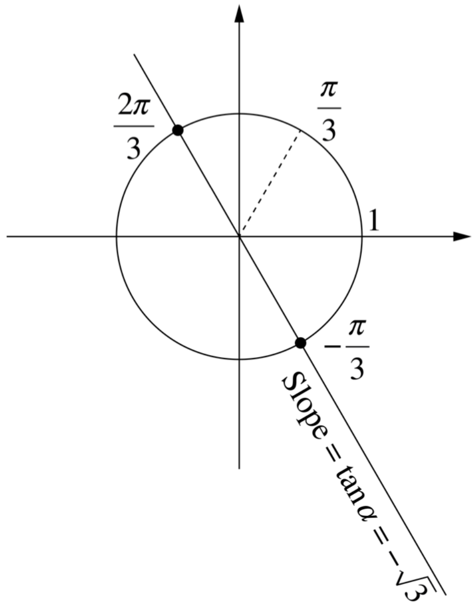
(1) ÷ (2): $\tan\alpha = \dfrac{-3}{\sqrt{3}}$

$= -\sqrt{3}$

$\therefore \quad \alpha = -\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$ (See unit circle)

Since $\cos\alpha > 0$ and $\sin\alpha < 0$

$\alpha = -\dfrac{\pi}{3}$



From (1) $\quad R^2\sin^2\alpha = 9 \qquad (3)$

From (2) $\quad R^2\cos^2\alpha = 3 \qquad (4)$

$(3)+(4): \quad R^2\left(\sin^2\alpha + \cos^2\alpha\right) = 12$

$R^2 = 12$

$R = 2\sqrt{3}$

$$\therefore \sqrt{3}\sin(x) - 3\cos(x) = 2\sqrt{3}\sin\left(x - \frac{\pi}{3}\right)$$

Question 11 (f)

Criteria	Marks
• Provides correct solution	3
• Identifies the correct critical values, or equivalent merit	2
• Excludes $x = 2$ OR attempts to deal with the denominator, or equivalent merit	1

Sample answer:

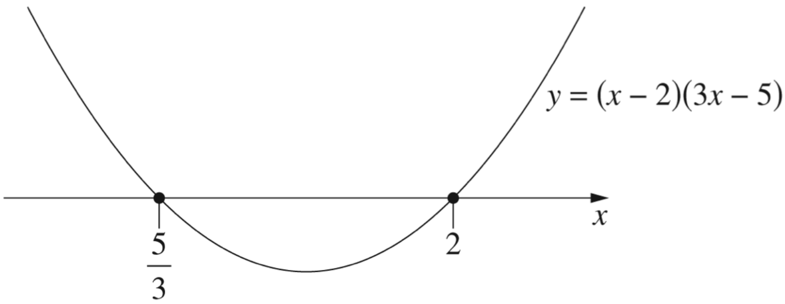
$$\frac{x}{2-x} \geq 5 \qquad x \neq 2$$

$$x(2-x) \geq 5(2-x)^2$$

$$(2-x)[5(2-x)-x] \leq 0$$

$$(2-x)(-6x+10) \leq 0$$

$$(x-2)(3x-5) \leq 0$$

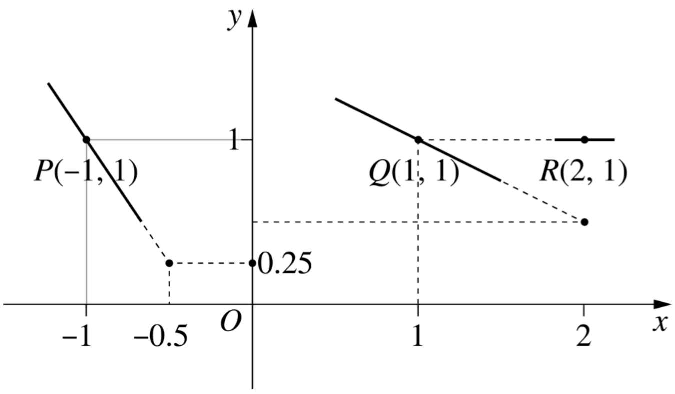


Solutions $= \left[\frac{5}{3}, 2\right)$

Question 12 (a)

Criteria	Marks
• Provides correct drawing	2
• Shows one correct slope	1

Sample answer:



At $P(-1, 1)$ $\qquad \dfrac{dy}{dx} = \dfrac{-1-2}{1+1} = -\dfrac{3}{2} = -\dfrac{0.75}{0.5}$

At $Q(1, 1)$ $\qquad \dfrac{dy}{dx} = \dfrac{1-2}{1+1} = -\dfrac{1}{2} = -\dfrac{0.5}{1}$

At $R(2, 1)$ $\qquad \dfrac{dy}{dx} = \dfrac{2-2}{4+1} = 0$

Question 12 (b)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the pigeonhole principle, or equivalent merit	1

Sample answer:

$$\frac{\text{Number of players above limit}}{\text{Number of teams}} = \frac{41}{13} > 3$$

$\therefore$ Using the Pigeonhole principle, at least one team has more than 3 players above limit. At least one team will be penalised.

Question 12 (c)

Criteria	Marks
• Provides correct solution	3
• Finds the slope of the tangent, or equivalent merit	2
• Attempts to find the derivative of $x \arctan x$, or equivalent merit	1

Sample answer:

$$y = x\tan^{-1}(x)$$

$$y' = x \times \frac{1}{1+x^2} + \tan^{-1}(x) \times 1$$

$$= \frac{x}{1+x^2} + \tan^{-1}(x)$$

At $\left(1, \frac{\pi}{4}\right)$ $\quad y' = \frac{1}{2} + \tan^{-1}(1)$

$$= \frac{1}{2} + \frac{\pi}{4}$$

$$= \frac{\pi + 2}{4}$$

Equation of tangent

$$y - \frac{\pi}{4} = \left(\frac{\pi+2}{4}\right)\left(x - 1\right)$$

$$4y - \pi = (\pi + 2)(x - 1)$$

$$4y = (\pi + 2)x - \pi - 2 + \pi$$

$$4y = (\pi + 2)x - 2$$

$$\therefore \quad y = \frac{(\pi + 2)}{4}x - \frac{1}{2}$$

Question 12 (d) (i)

Criteria	Marks
• Provides correct solution	3
• Integrates to obtain T as a function involving an exponential function, or equivalent merit	2
• Separates the variables in the differential equation, or equivalent merit	1

Sample answer:

$$\frac{dT}{dt} = k\left(T - T_1\right)$$

Note that as $t \to +\infty$, $\dfrac{dT}{dt} \to 0$ and T approaches room temperature so T_1 = room temperature = 12.

Also T decreases with time, so $T - T_1 > 0$.

$\dfrac{dT}{T-12} = k\,dt$ (separate variables)

$\ln(T - 12) = kt + c$ where c is a constant

$T - 12 = \mathrm{A}e^{kt}$ where $A = e^c$

$T = 12 + \mathrm{A}e^{kt}$

When $t = 0$: $92 = 12 + A$ so $A = 80$

When $t = 5$: $76 = 12 + 80e^{5k}$

$$e^{5k} = \frac{4}{5}$$

$$5k = \ln\left(\frac{4}{5}\right)$$

$$k = \frac{1}{5}\ln\left(\frac{4}{5}\right)$$

Finally $T = 12 + 80e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t}$ for $t \geq 0$.

Question 12 (d) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

For $T = 57$

$$57 = 12 + 80e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t}$$

$$e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t} = \frac{57-12}{80} = \frac{9}{16}$$

$$\frac{1}{5}\ln\left(\frac{4}{5}\right)t = \ln\left(\frac{9}{16}\right)$$

$$t = \frac{5\ln\left(\frac{9}{16}\right)}{\ln\left(\frac{4}{5}\right)} = 12.89\ldots \qquad \approx 13 \text{ minutes}$$

The coffee will reach a temperature of 57°C after about 13 minutes.

Question 12 (e)

Criteria	Marks
• Provides correct solution	2
• Attempts to use result associated with the binomial distribution, or equivalent merit	1

Sample answer:

Let p be the proportion of red balls and n the number of trials.

Expected number of red balls $= np = 4 \times \dfrac{3}{10}$

Expected number of green balls $= n(1-p) = 4 \times \dfrac{7}{10}$

Expected score $= 10 \times 4 \times \dfrac{3}{10} - 5 \times 4 \times \dfrac{7}{10} = 12 - 14 = -2$.

Answers could include:

- The expected score if you pick one ball is $\dfrac{3 \times 10 + 7 \times (-5)}{10} = -0.5$.
- It is the same for every one of the 4 balls since the balls are replaced so the expected score for the game is $4 \times (-0.5) = -2$.

Question 12 (f)

Criteria	Marks
• Provides correct proof	3
• Proves the inductive step, or equivalent merit	2
• Establishes the base case, or equivalent merit	1

Sample answer:

METHOD 1

Let $a_n = 15^n + 6^{2n+1}$

Base case: $n = 0$

$a_0 = 1 + 6 = 7$ is divisible by 7

so the property holds for $n = 0$

Assume a_k is divisible by 7

$a_k = 15^k + 6^{2k+1} = 7M$, for some integer M.

Prove true for a_{k+1}.

$a_{k+1} = 15^{k+1} + 6^{2k+3} = 7Q$, for some integer Q.

$$\begin{aligned}
LHS &= 15\left(15^k\right) + 6^{2k+3} \\
&= 15\left(7M - 6^{2k+1}\right) + 6^2 \times 6^{2k+1} \qquad \text{(from assumption)} \\
&= 105M - 15 \times 6^{2k+1} + 36 \times 6^{2k+1} \\
&= 105M + 21 \times 6^{2k+1} \\
&= 7\left(15M + 3 \times 6^{2k+1}\right) \\
&= 7Q \\
&= RHS
\end{aligned}$$

This proves, using mathematical induction, that for all integers $n \geq 0$, $15^n + 6^{2n+1}$ is divisible by 7.

Answers could include:

<u>METHOD 2</u>
Let $a_n = 15^n + 6^{2n+1}$

Base case: $n = 0$

$a_0 = 1 + 6 = 7$ is divisible by 7

so the property holds for $n = 0$

Assume a_k is divisible by 7

$$a_{k+1} = 15^{k+1} + 6^{2k+3}$$
$$= 15\left(15^k\right) + 6^{2k+3}$$

Substituting $15^k = a_k - 6^{2k+1}$

$$a_{k+1} = 15\left(a_k - 6^{2k+1}\right) + 6^{2k+3}$$
$$= 15a_k + 6^{2k+1}(-15 + 36)$$
$$= 15a_k + 21 \times 6^{2k+1}$$

a_k and 21 are both divisible by 7

so a_{k+1} is also divisible by 7

This proves, using mathematical induction, that for all integers $n \geq 0$, $15^n + 6^{2n+1}$ is divisible by 7.

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Attempts to find the dot product of $\overrightarrow{BH}$ and $\overrightarrow{CA}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, or equivalent merit	2
• Write $\overrightarrow{BH}$ or $\overrightarrow{CA}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, or equivalent merit	1

Sample answer:

We have $\overrightarrow{BH} = \underset{\sim}{h} - \underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{c}$, while $\overrightarrow{CA} = \underset{\sim}{a} - \underset{\sim}{c}$. Hence

$$\begin{aligned}
\overrightarrow{BH} \cdot \overrightarrow{CA} &= \left(\underset{\sim}{a} + \underset{\sim}{c}\right) \cdot \left(\underset{\sim}{a} - \underset{\sim}{c}\right) \\
&= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{c} \\
&= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{c} \\
&= |\underset{\sim}{a}|^2 - |\underset{\sim}{c}|^2 \\
&= 0 \qquad \text{(as the lengths of } \underset{\sim}{a} \text{ and } \underset{\sim}{c} \text{ are equal, as they are radii of a circle)}
\end{aligned}$$

Hence, the two vectors are perpendicular.

Answers could include:

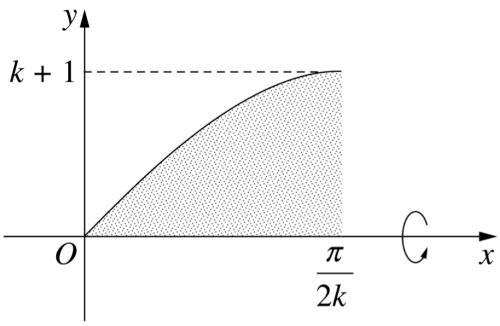
We may observe that $\underset{\sim}{a} + \underset{\sim}{c}$ and $\underset{\sim}{a} - \underset{\sim}{c}$ form the diagonals of a rhombus, as $\underset{\sim}{a}$ and $\underset{\sim}{c}$ have equal length. The diagonals of a rhombus are perpendicular and so $\overrightarrow{BH}$ and $\overrightarrow{CA}$ are perpendicular.

Question 13 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains primitive in terms of $\sin(2kx)$, or equivalent merit	2
• Finds an integral expression for the volume of the solid in terms of $\sin(kx)$, or equivalent merit	1

Sample answer:

$y = (k+1)\sin(kx)$



$$V = \pi\int_0^{\frac{\pi}{2k}} (k+1)^2 \sin^2(kx)\cdot dx$$

$$= \pi(k+1)^2\int_0^{\frac{\pi}{2k}} \frac{1-\cos 2kx}{2}\cdot dx$$

$$= \frac{\pi(k+1)^2}{2}\int_0^{\frac{\pi}{2k}} 1-\cos 2kx\cdot dx$$

$$= \frac{\pi(k+1)^2}{2}\left[x-\frac{\sin 2kx}{2k}\right]_0^{\frac{\pi}{2k}}$$

$$= \frac{\pi(k+1)^2}{2}\left(\frac{\pi}{2k}-\frac{\sin\pi}{2k}\right)$$

$$= \frac{\pi^2(k+1)^2}{4k}$$

$$\cos 2\theta = 1-2\sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\therefore\ \sin^2(kx) = \frac{1}{2}(1-\cos 2kx)$$

For $V = \pi^2$,

$$\frac{(k+1)^2}{4k} = 1$$

$\therefore\quad k = 1\quad$ (by inspection)

Or solve $k^2 + 2k + 1 = 4k$

$$(k-1)^2 = 0$$

$$k = 1$$

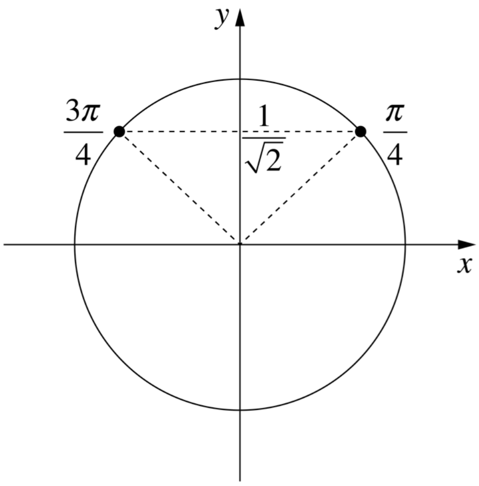
Question 13 (c)

Criteria	Marks
• Provides correct solution	2
• Finds a value of x where $g(f(x)) \neq x$	1

Sample answer:

The domain of $g(f(x))$ is $\mathbb{R}$.
$g(f(x)) = \arcsin(\sin x)$

If g is the inverse function of f, then $g(f(x))$ is equal to x for all x in the domain of $g(f(x))$ by definition of an inverse function.



Let $x = \dfrac{3\pi}{4}$

$\arcsin\left(\sin\dfrac{3\pi}{4}\right) = \dfrac{\pi}{4}$, see unit circle above

So for $x = \dfrac{3\pi}{4}$, $\arcsin(\sin x) \neq x$

So it is not true that $g(f(x)) = x$ for all x in the domain $g(f(x))$. So, g is NOT the inverse of f.

Question 13 (d)

Criteria	Marks
• Provides correct solution	3
• Evaluates $P'(\alpha)+P'(\beta)+P'(\gamma)$, or equivalent merit	2
• Writes $\alpha^2+\beta^2+\gamma^2$ in term of $\alpha+\beta+\gamma$ and so on OR Defines $P(x)$ and evaluates $P'(\alpha)$ OR Equivalent merit	1

Sample answer:

$$\alpha^2+\beta^2+\gamma^2=85$$
$$P'(\alpha)+P'(\beta)+P'(\gamma)=87$$

Let $P(x)=x^3+ax^2+bx+c$

Then $a=-(\alpha+\beta+\gamma)$

$b=\alpha\beta+\beta\gamma+\gamma\alpha$

$c=-\alpha\beta\gamma$

$$P'(x)=3x^2+2ax+b$$

$$\begin{aligned}
&P'(\alpha)+P'(\beta)+P'(\gamma)\\
&=3\left(\alpha^2+\beta^2+\gamma^2\right)+2a(\alpha+\beta+\gamma)+3b\\
&=3\left(\alpha^2+\beta^2+\gamma^2\right)-2(\alpha+\beta+\gamma)^2+3(\alpha\beta+\beta\gamma+\gamma\alpha)\\
&=3\left(\alpha^2+\beta^2+\gamma^2\right)-2\left(\alpha^2+\beta^2+\gamma^2\right)-4(\alpha\beta+\beta\gamma+\gamma\alpha)+3(\alpha\beta+\beta\gamma+\gamma\alpha)\\
&=\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\gamma\alpha)
\end{aligned}$$

$$\begin{aligned}
87&=85-(\alpha\beta+\beta\gamma+\gamma\alpha)\\
\alpha\beta+\beta\gamma+\gamma\alpha&=85-87\\
&=-2
\end{aligned}$$

Question 13 (e) (i)

Criteria	Marks
• Provides correct solution	2
• Introduces a suitable normal approximation, or equivalent merit	1

Sample answer:

METHOD 1

Let $\hat{p}$ be the proportion of chocolate bars which weigh <u>less</u> than 150 g.

$\mathcal{P} = P(\hat{p} \geq 0.5)$

Approximate $\hat{p}$ by a normal distribution with the same mean as $\hat{p}$ and the same standard deviation as $\hat{p}$.

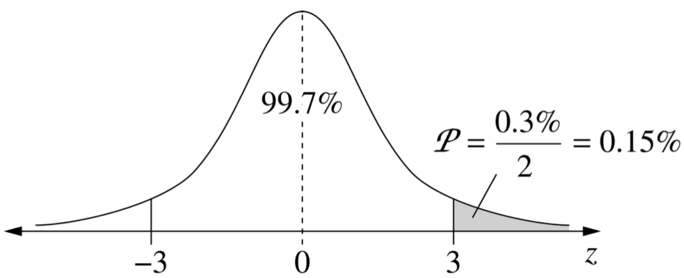
$$E(\hat{p}) = 0.2$$

$$\sigma(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.2 \times 0.8}{16}} = 0.1$$

$$\begin{aligned}\mathcal{P} &= P\left(\frac{\hat{p}-0.2}{0.1} \geq \frac{0.5-0.2}{0.1}\right)\\ &\approx P(Z \geq 3)\end{aligned}$$

where Z follows a standard normal distribution.

Using known values



So $\mathcal{P} = P(Z \geq 3) = 0.15\% = 0.0015$

Alternatively using the table of values for the normal distribution, we get the more precise value of $\mathcal{P} = 0.13\% = 0.0013$

Answers could include:

METHOD 2

Let $\hat{p}$ be the proportion of bars in a sample of 16 bars which weigh more than 150 g.

$$\mathcal{P} = P(\hat{p} < 0.5)$$

(If 50% or more of the bars weigh less than 150 g, the remaining ones weigh 150 g or more.)

$$E(\hat{p}) = 0.8$$

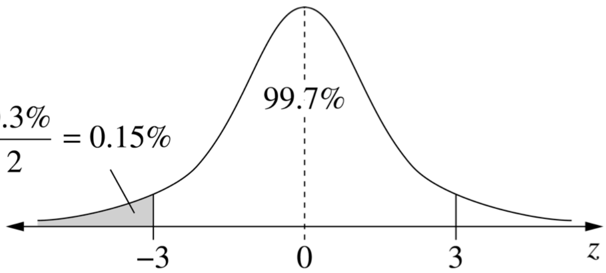
$$\sigma(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.2 \times 0.8}{16}} = 0.1$$

$$\begin{aligned} \mathcal{P} &= P(\hat{p} < 0.5) \\ &= P\left(\frac{\hat{p} - 0.8}{0.1} < \frac{0.5 - 0.8}{0.1}\right) \\ &\approx P(Z < -3) \end{aligned}$$

When we approximate $\hat{p}$ by a normal distribution.

$$\mathcal{P} \simeq P(Z < -3) \approx \frac{0.3\%}{2} = 0.15\% \text{ using known values}$$

$\mathcal{P} = \dfrac{0.3\%}{2} = 0.15\%$



(and $\mathcal{P} = 0.13\%$ using the table of normal values)

METHOD 3: (Using counts rather than proportion, without continuity correction)

Let X be the number of chocolate bars in a sample of 16 which weigh less than 150 g.

$\mathcal{P} = P(X \geq 8)$

X follows a binomial distribution $\text{Bin}(n, p) = \text{Bin}(16, 0.2)$, assuming the factory manager's claim is correct.

$$E(X) = np = 16 \times 0.2 = 3.2$$
$$\sigma(X) = \sqrt{np(1-p)} = \sqrt{16 \times 0.2 \times 0.8} = 1.6$$

$$\begin{aligned}\mathcal{P} &= P(X \geq 8) \\ &= P\left(\frac{X - 3.2}{1.6} \geq \frac{8 - 3.2}{1.6}\right) \\ &\approx P(Z \geq 3)\end{aligned}$$

If we approximate X by a normal distribution (without the continuity correction) we get

$\mathcal{P} \approx P(Z \geq 3)$ where Z follows a standard normal distribution

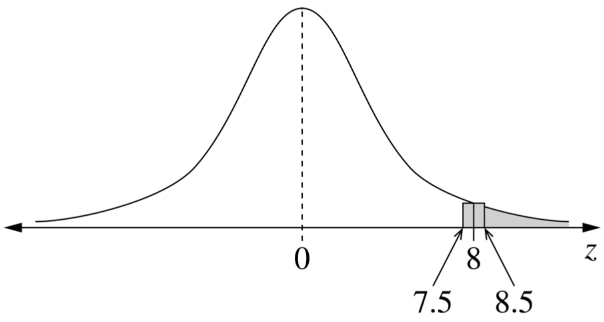
So $\mathcal{P} \approx 0.15\%$ using known values

METHOD 4: (Using counts and the continuity correction.)

Same X as method 3.

We approximate X by a normal distribution Y with the same mean and standard deviation as X.

$$\mathcal{P} = P(X \geq 8)$$
$$= P(Y > 7.5)$$



$$\mathcal{P} = P\left(\frac{Y-3.2}{1.6} > \frac{7.5-3.2}{1.6}\right)$$

$\approx P(Z > 2.6875)$ where Z is a standard normal

$= 1 - P(Z \leq 2.6875)$ using the nearest value from the table

$\approx 1 - 0.9964$

$= 0.0036$

$= 0.36\%$

Question 13 (e) (ii)

Criteria	Marks
• Provides correct explanation	1

Sample answer:

The number of items checked may not be large enough to assume the data is normally distributed.

Question 14 (a)

Criteria	Marks
• Provides correct solution	4
• Evaluates the constant of integration to find a correct equation for $\ln\lvert y \rvert$	3
• Obtains correct primitive, or equivalent merit	2
• Separates the variables in the differential equation, or equivalent merit	1

Sample answer:

$(x-2)\dfrac{dy}{dx} = xy$ and $(0,\ 1)$

$$\int \frac{1}{y}\,dy = \int \frac{x}{x-2}\,dx$$

$$\begin{aligned}\ln\lvert y\rvert &= \int 1 + \frac{2}{x-2}\,dx \\ &= x + 2\ln\lvert x-2\rvert + C \\ &= \ln\lvert x-2\rvert^2 + x + C\end{aligned}$$

$$\ln\left(\frac{\lvert y\rvert}{(x-2)^2}\right) = x + C$$

$$\frac{\lvert y\rvert}{(x-2)^2} = e^{x+C}$$

$$\lvert y\rvert = e^{x+C}(x-2)^2$$

$$y = ke^x(x-2)^2 \qquad \text{where} \quad k = \pm e^C$$

When $x = 0,\ y = 1 \qquad \therefore\quad k = \dfrac{1}{4}$

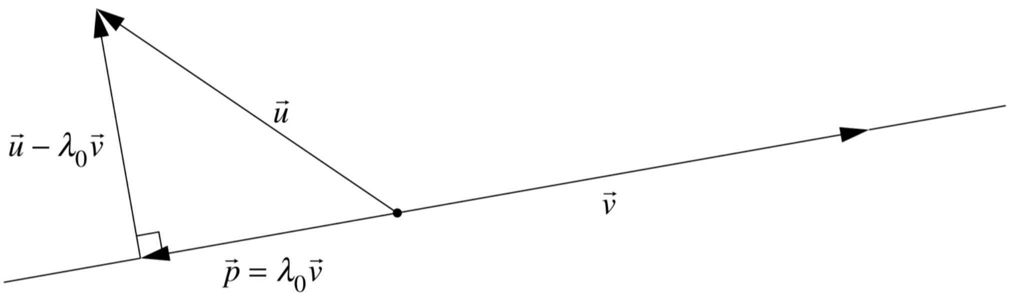
$$y = \frac{1}{4}e^x(x-2)^2$$

Question 14 (b)

Criteria	Marks
• Provides correct solution	3
• Uses an appropriate vector method to obtain a suitable inequality, or equivalent merit	2
• Writes $\underset{\sim}{u} - \lambda \underset{\sim}{v}$ as a sum of vectors parallel and perpendicular to $\underset{\sim}{v}$, or equivalent merit	1

Sample answer:

METHOD 1



Let λ be a real number.

$$
\begin{aligned}
|\vec{u} - \lambda\vec{v}|^2 &= |\vec{u} - \lambda_0\vec{v} + \lambda_0\vec{v} - \lambda\vec{v}|^2 \\
&= \Big|\underbrace{(\vec{u} - \lambda_0\vec{v})}_{\substack{\text{Perpendicular} \\ \text{to } \vec{v}}} + \underbrace{(\lambda_0 - \lambda)}_{\substack{\text{Parallel} \\ \text{to } \vec{v}}}\vec{v}\Big|^2 \\
&= \left[(\vec{u} - \lambda_0\vec{v}) + (\lambda_0 - \lambda)\vec{v}\right] \cdot \left[(\vec{u} - \lambda_0\vec{v}) + (\lambda_0 - \lambda)\vec{v}\right] \\
&= |\vec{u} - \lambda_0\vec{v}|^2 + |(\lambda_0 - \lambda)\vec{v}|^2 + 0 + 0
\end{aligned}
$$

$$|(\lambda_0 - \lambda)\vec{v}|^2 \geq 0$$

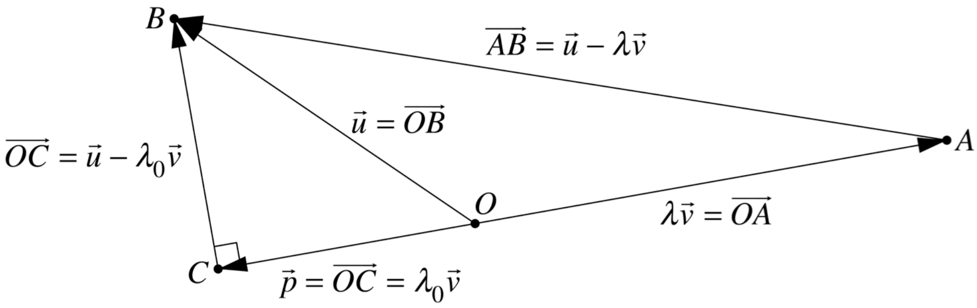
Therefore $|\vec{u} - \lambda\vec{v}|^2 \geq |\vec{u} - \lambda_0\vec{v}|^2$,

So $|\vec{u} - \lambda\vec{v}| \geq |\vec{u} - \lambda_0\vec{v}|$.

Hence $|\vec{u} - \lambda\vec{v}|$ is smallest when it equals $|\vec{u} - \lambda_0\vec{v}|$, and so is smallest when $\lambda = \lambda_0$.

Answers could include:

METHOD 2



Let $\lambda \in \mathbb{R}$.

Let O be a point in the plane.

Let A, B and C be the points defined by

$\overrightarrow{OA} = \lambda\vec{v}$

$\overrightarrow{OB} = \vec{u}$

$\overrightarrow{OC} = \vec{p}$

Because $\vec{p}$ is the projection of $\vec{u}$ onto $\vec{v}$, the triangle ABC has a right angle at C.

In any right-angled triangle the length of the hypotenuse is greater than or equal to the length of any of the sides

so $\left|\overrightarrow{AB}\right| \geq \left|\overrightarrow{BC}\right|$

That is $\left|\vec{u} - \lambda\vec{v}\right| \geq \left|\vec{u} - \lambda_0\vec{v}\right|$.

Hence $\left|\vec{u} - \lambda\vec{v}\right|$ is smallest when it equals $\left|\vec{u} - \lambda_0\vec{v}\right|$, and so is smallest when $\lambda = \lambda_0$.

Question 14 (c)

Criteria	Marks
• Provides correct solution	4
• Finds an expression for d in terms of the maximum range, or equivalent merit	3
• Finds maximum range, or equivalent merit	2
• Attempts to find the time of flight of projectile, or equivalent merit	1

Sample answer:

For the player to hit the target there must be a time t_1 where the positions of the target and of the projectile coincide:

$$\begin{cases} 2ut_1\cos\theta = d + ut_1 & \text{(i)} \\ 2ut_1\sin\theta = \dfrac{g}{2}t_1^{\,2} & \text{(ii)} \end{cases}$$

By (i) $\quad t_1 = \dfrac{d}{2u\cos\theta - u}$

t_1 is not 0, so dividing (ii) by t_1 yields

$$2u\sin\theta = \frac{g}{2}t_1 \qquad \text{(iii)}$$

Substitute in t_1 in (iii):

$$2u\sin\theta = \frac{g}{2}\left(\frac{d}{2u\cos\theta - u}\right)$$

$$d = \frac{2u\sin\theta\big(u\times(2\cos\theta - 1)\big)\times 2}{g}$$

$$d = \frac{4u^2}{g}\sin\theta(2\cos\theta - 1)$$

<u>Maximum range</u>:

It occurs at time t_2 and corresponds to $2ut_2\sin\theta - \frac{g}{2}t_2^{\,2} = 0$.

$t_2 \neq 0 \quad$ so $\quad 2u\sin\theta - \dfrac{g}{2}t_2 = 0$

$$t_2 = \frac{4u\sin\theta}{g}$$

The range is $\quad 2ut_2\cos\theta = \dfrac{8u^2}{g}\sin\theta\cos\theta$

$$= \frac{4u^2}{g}\sin 2\theta$$

This is maximum when $\sin 2\theta$ is maximum, that is when $\sin 2\theta = 1$.

The maximum range is $\dfrac{4u^2}{g}$.

Find the maximum value of $\sin\theta(2\cos\theta - 1)$ for θ in $\left(0, \dfrac{\pi}{2}\right)$.

$\sin\theta(2\cos\theta - 1) = \sin 2\theta - \sin\theta$

Let f be the function defined on $\left(0, \dfrac{\pi}{2}\right)$

by $\quad f(\theta) = \sin 2\theta - \sin\theta$

$$f'(\theta) = 2\cos 2\theta - \cos\theta$$

$$= 2\left(2\cos^2\theta - 1\right) - \cos\theta$$

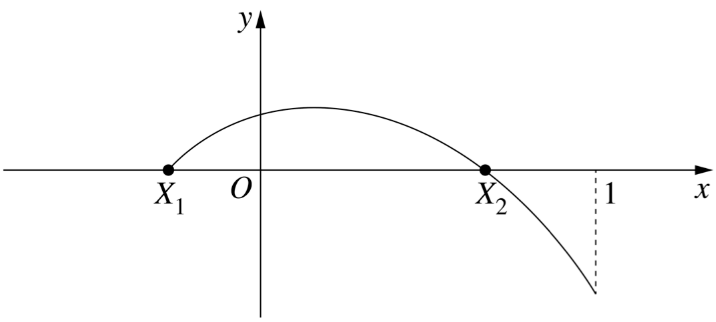
Let $X = \cos\theta$

$f'(\theta) = 2\left(2X^2 - 1\right) - X = 4X^2 - X - 2$

$\Delta = b^2 - 4ac = 1 - 4(-8) = 33$

$X_1 = \dfrac{1 - \sqrt{33}}{8}$ and $X_2 = \dfrac{1 + \sqrt{33}}{8}$

$\approx -0.59 \qquad\qquad \approx 0.84$



$f'(\theta) \ge 0$ if $0 \le X \le X_2$

$f'(\theta) \le 0$ if $X_2 \le X \le 1$

So $f(\theta)$ is maximum when $X = X_2$

That is $\cos\theta = \dfrac{1 + \sqrt{33}}{8}$

With $0 < \theta < \dfrac{\pi}{2}$, that means $\theta \approx 0.5678$ which corresponds to $\sin 2\theta - \sin\theta = 0.3690\ldots \simeq 37\%$.

$$d = \underbrace{\frac{4u^2}{g}}_{\text{maximum range}} \times \underbrace{\sin\theta(2\cos\theta - 1)}_{\le 37\%}$$

So d must be less than 37% of the maximum range.

Question 14 (d)

Criteria	Marks
• Provides correct solution	4
• Obtains a quadratic inequality in N, where N is the number of tickets sold, or equivalent merit	3
• Uses a normal approximation to find an expression for the probability, or equivalent merit	2
• Writes a probability statement similar to $P(X > 350) \leq 0.01$, where X is the number of passengers who turn up, or equivalent merit	1

Sample answer:

METHOD 1:

Let n be the number of tickets sold for a 350-seat flight ($n \geq 350$).

Let X be the number of passengers showing up on a 350-seat flight $X \sim \text{Bin}(n,\ 0.95)$.

The management's decision can be written as:
$P(X > 350) \leq 0.01$
or
$P(X \leq 350) \geq 0.99$

Find n such that $P(X \leq 350) = 0.99$.

Approximating X by the appropriate normal random variable Y, without continuity correction, the condition $P(X \leq 350) \geq 0.99$ can be rewritten.

$P(Y \leq 350) \geq 0.99$.

Need to find n such that $P(Y \geq 350) = 0.99$

This becomes

$$P\left(\frac{Y - 0.95n}{\sqrt{n \times 0.95 \times 0.05}} < \frac{350 - 0.95n}{\sqrt{n \times 0.95 \times 0.05}}\right) = 0.99$$

This happens if

$$\frac{350 - 0.95n}{\sqrt{0.95 \times 0.05n}} \simeq 2.33$$

$$350 - 0.95n = 2.33\sqrt{0.95 \times 0.05n}$$

Let $x = \sqrt{n}$

$$0.95x^2 + 2.33\sqrt{0.95 \times 0.05}x - 350 = 0$$

$$x = \sqrt{n} = \frac{-2.33\sqrt{0.95 \times 0.05} \pm \sqrt{(2.33)^2 \times 0.95 \times 0.05 + 4 \times 0.95 \times 350}}{2 \times 0.95}$$

$\sqrt{n} \approx 18.93$

$n \simeq 358.30$

So $n = 358$ (359 would result in getting too many passengers more than 1% of the time)

Given the management's decision, the optimal number of tickets sold on a 350-seat flight is 358.

Answers could include:

<u>METHOD 2</u>: (with continuity correction)

X, Y and n defined as before.

Approximate the binomial X by the appropriate normal random variable Y, with continuity correction.

$P(Y < 350.5) \geq 0.99$ (i)

$E(Y) = E(X) = n \times 0.95$

$\text{Var}(Y) = n \times 0.95 \times 0.05$

(i) becomes

$$P\left(\frac{Y - 0.95n}{\sqrt{n \times 0.95 \times 0.05}} < \frac{350.5 - 0.95n}{\sqrt{n \times 0.95 \times 0.05}}\right) = 0.99$$

Using the normal distribution table

$$\frac{350.5 - 0.95n}{\sqrt{0.95 \times 0.05n}} \simeq 2.33$$

$$350.5 - 0.95n = 2.33\sqrt{0.95 \times 0.05} \times \sqrt{n}$$

Let $x = \sqrt{n}$

$$0.95x^2 + 2.33\sqrt{0.95 \times 0.05}x - 350.5 = 0$$

$$\Delta = b^2 - 4ac = 0.95 \times 0.05 \times (2.33)^2 - 4 \times 0.95 \times (-350.5)$$

$$\simeq 1332.158$$

$$x = \sqrt{n} = \frac{-2.33\sqrt{0.95 \times 0.05} + \sqrt{1332.158}}{2 \times 0.95}$$

(Can ignore the negative solution since $x = \sqrt{n} > 0$.)

$\sqrt{n} \approx 18.94$

$n \simeq 358.8$

$n = 358$ (359 would result in getting too many passengers more than 1% of the time)

2022 HSC Mathematics Extension 1 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	ME-T1 Inverse Trigonometric Functions	ME11-3
2	1	ME-F1 Further Work with Functions	ME11-1
3	1	ME-F2 Polynomials	ME11-2
4	1	ME-F1 Further Work with Functions	ME11-1
5	1	ME-F1 Further Work with Functions	ME11-2
6	1	ME-V1 Introduction to Vectors	ME12-2
7	1	ME-A1 Working with Combinatorics	ME11-5
8	1	ME-V1 Introduction to Vectors	ME12-2
9	1	ME-C2 Further Calculus Skills	ME12-1
10	1	ME-C3 Applications of Calculus	ME12-4

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	1	ME-V1 Introduction to Vectors	ME12-2
11 (a) (ii)	1	ME-V1 Introduction to Vectors	ME12-2
11 (b)	3	ME-C2 Further Calculus Skills	ME12-1
11 (c)	2	ME-A1 Working with Combinatorics	ME11-5
11 (d)	2	ME-V1 Introduction to Vectors	ME12-2
11 (e)	3	ME-T3 Trigonometric Equations	ME12-3
11 (f)	3	ME-F1 Further Work with Functions	ME11-1
12 (a)	2	ME-C3 Applications of Calculus	ME12-1
12 (b)	2	ME-A1 Working with Combinatorics	ME11-5
12 (c)	3	ME-C2 Further Calculus Skills	ME12-1
12 (d) (i)	3	ME-C3 Applications of Calculus	ME11-4, ME12-4
12 (d) (ii)	1	ME-C1 Rate of Change	ME11-4
12 (e)	2	ME-S1 The Binomial Distribution	ME12-5
12 (f)	3	ME-P1 Proof by Mathematical Induction	ME12-1
13 (a)	3	ME-V1 Introduction to Vectors	ME12-2

Question	Marks	Content	Syllabus outcomes
13 (b)	3	ME-C3 Applications of Calculus	ME12-4
13 (c)	2	ME-T1 Inverse Trigonometric Functions	ME11-1
13 (d)	3	ME-F2 Polynomials	ME11-1
13 (e) (i)	2	ME-S1 The Binomial Distribution	ME12-5, ME12-7
13 (e) (ii)	1	ME-S1 The Binomial Distribution	ME12-5, ME12-7
14 (a)	4	ME-C3 Applications of Calculus	ME12-4
14 (b)	3	ME-V1 Introduction to Vectors	ME12-2, ME12-7
14 (c)	4	ME-V1 Introduction to Vectors ME-C3 Applications of Calculus	ME12-2, ME12-3
14 (d)	4	ME-S1 The Binomial Distribution ME-S1 The Binomial Distribution	ME12-5, ME12-7

Mathematics Extension 1

HSC Marking Feedback 2022

Question 11

Part (a) (i)

Students should:

- know that when a vector is multiplied by a scalar the result is a vector
- be able to add vectors in two dimensions.

In better responses, students were able to:

- find the scalar multiple of a vector before adding it to the other vector
- work separately with the components of each vector.

Areas for students to improve include:

- using the correct notation for vectors
- avoiding arithmetic errors.

Part (a) (ii)

Students should:

- understand that the dot product of two vectors is a scalar
- select and use the most appropriate form of the equation for dot product of vectors from the Reference Sheet.

In better responses, students were able to:

- show full working before arriving at the numerical answer.

Areas for students to improve include:

- using the most appropriate form of the equation for the dot product of vectors from the Reference Sheet
- understanding that the dot product of two vectors is a scalar
- completing a variety of examples of the application of the dot product, with emphasis on the fact that the result is a number
- avoiding arithmetic errors when adding and subtracting numbers.

Part (b)

Students should:

- realise that integration by substitution is used to simplify the integration of a more complex expression
- be able to arrive at an integrand involving only the substituted variable
- change the limits of the integral to match the boundaries of the substituted variable.

In better responses, students were able to:

- clearly set out the substitution and find dx in terms of du
- transform the integrand and its limits in terms of the substituted variable
- find the correct primitive and apply the limits correctly.

Areas for students to improve include:

- understanding how the derivative of the substituted variable is used to facilitate the formation of the transformed integral
- ensuring that the correct limits are applied to the transformed integrand
- evaluating integrals involving variables with negative fractional indices.

Part (c)

Students should:

- be able to use the binomial theorem to expand the given binomial expression raised to a given power
- know how to use the general term formula to find the required terms in the expansion of a binomial.

In better responses, students were able to:

- correctly expand the given binomial to include only the terms required
- use the general term formula to correctly identify the coefficients of the required terms.

Areas for students to improve include:

- expanding binomials in which one of the terms is negative
- using the general term formula to find individual terms without expanding the binomial entirely
- using index laws with fractions (both positive and negative) raised to odd and even powers.

Part (d)

Students should:

- understand that the dot product of 2 perpendicular vectors is zero or that the product of their gradients is equal to –1
- be able to apply the dot product.

In better responses, students were able to:

- state that the dot product of 2 perpendicular vectors is zero

- apply the dot product rule to arrive at the correct quadratic equation
- correctly factorise the quadratic to arrive at the required values of a.

Areas for students to improve include:

- expanding bracketed terms
- factorising and solving quadratic equations.

Part (e)

Students should:

- understand that the sum or difference of a sine or cosine function with the same frequency can be written as a single sine or cosine function with the same frequency
- be able to find the amplitude and horizontal translation of the resulting single trigonometric function.

In better responses, students were able to:

- use the coefficients of the given functions to find the amplitude of the resulting trigonometric function
- use the coefficients of the given functions to find the auxiliary angle and the relevant quadrant.

Areas for students to improve include:

- using the Reference Sheet to expand compound angle functions such as $R\sin(x+\alpha)$
- equating coefficients to find the correct values of R and α for different types of auxiliary angle problems, observing restrictions on the quadrant of the argument.

Part (f)

Students should:

- be able to solve an inequality with a variable in the denominator.

In better responses, students were able to:

- multiply both sides of the inequality by the square of the denominator, solve the quadratic inequality and provide the correct solution
- find the critical points and test the inequality in each region to obtain the correct solution.

Areas for students to improve include:

- understanding and using different methods for solving inequalities with a variable in the denominator, such as multiplying by the square of the denominator, finding critical values and testing regions, considering options involving the polarity of the denominator and sketching curves to find the solutions
- not using the method of multiplying both sides by the denominator when solving an inequality
- understanding that the solution cannot include the value of the variable that makes the denominator zero
- showing clear working

- writing the solution of a quadratic inequation correctly.

Question 12

Part (a)

Students should:

- be able to draw all 3 slopes at the given points clearly
- read the question carefully.

In better responses, students were able to:

- draw the 3 slopes correctly
- demonstrate that the slope at P was steeper than that at Q.

Areas for students to improve include:

- showing working for their substitution into the required derivative
- drawing relative slopes correctly with respect to their steepness.

Part (b)

Students should:

- be able to apply the Pigeonhole Principle.

In better responses, students were able to:

- demonstrate the Pigeonhole Principle correctly and provide a correct conclusion.

Areas for students to improve include:

- applying the Pigeonhole Principle in a variety of situations
- indicating a remainder after division and, hence, concluding the need of an extra 'hole' (ceiling function) or stating the minimum number of 'holes' required.

Part (c)

Students should:

- be able to differentiate a given function using the product rule
- be able to substitute the given x – coordinate into the derived equation to find the slope of the tangent
- be able to use the given x – and y – coordinates to find the equation of the tangent.

In better responses, students were able to:

- differentiate the given function, find the slope and the correct equation of the tangent.

Areas for students to improve include:

- using the Reference Sheet to differentiate $\tan^{-1} x$ correctly
- evaluating $\tan^{-1} x$ when $x = 1$
- reading the question carefully.

Part (d) (i)

Students should:

- be able to separate the variables in the differential equation
- be able to integrate to obtain T as a function involving an exponential equation
- be able to use the given conditions to find A and k.

In better responses, students were able to:

- recognise $T_1 = 12$ and substitute early
- separate the variables and integrate to obtain an equation in exponential form
- use the given conditions to find A and k.

Areas for students to improve include:

- separating the variables in a differential equation
- integrating to obtain T as a function involving an exponential equation with $A = e^c$
- using the given conditions to evaluate the relevant equations to find A and k.

Part (d) (ii)

Students should:

- substitute for T in the given exponential equation to arrive at the correct answer.

In better responses, students were able to:

- substitute for T and arrived at the correct time to the nearest minute.

Areas for students to improve include:

- using their calculator to obtain an answer correct to the nearest minute with respect to time.

Part (e)

Students should:

- be able to use the binomial distribution result to obtain the expected score of the game
- be able to find the expected score of one ball and then multiply that to obtain the result of 4 balls
- read the question carefully.

In better responses, students were able to:

- use the binomial distribution result and multiply that by the points either earned or deducted to arrive at the expected score
- find the expected score of one ball.

Areas for students to improve include:

- identifying the correct formula from the Reference Sheet for the binomial distribution.

Part (f)

Students should:

- establish the base case
- write a statement assuming that k is true
- use the assumption statement to prove the inductive step.

In better responses, students were able to:

- prove the base case was true, assume that k was true and prove the inductive step with a conclusion
- substitute and establish the base case without numerical errors
- manipulate the algebra in the inductive step without error.

Areas for students to improve include:

- showing the statement is true for the base case
- providing a proof by mathematical induction including all necessary steps.

Question 13

Part (a)

Students should:

- know that two vectors are perpendicular if their dot product is equal to zero
- use the correct notation for vectors and magnitude of vectors
- complete a proof with logical steps and reasoning as required.

In better responses, students were able to:

- rewrite direction vectors, for example $\overrightarrow{BH} = \underset{\sim}{h} - \underset{\sim}{b}$
- state the property of perpendicular vectors
- provide reasons for their steps (as required)
- recognise $\underset{\sim}{a}.\underset{\sim}{a} = \left|\underset{\sim}{a}\right|^2$ and apply this knowledge to prove the given statement.

Areas for students to improve include:

- knowing how to find and use the dot product of two vectors
- showing all the steps in a proof with reasons (as required)
- understanding and using vector notation.

Part (b)

Students should:

- write an integral expression for the required volume with the correct limits
- use the double angle formula for $\cos 2\theta$ correctly

- be able to integrate trigonometric functions
- substitute the limits of integration into the primitive of the volume
- solve an equation to find the value of the unknown.

In better responses, students were able to:

- provide the correct integral for the volume
- correctly use the double angle formula
- integrate the function and correctly substitute the limits of integration
- solve the resultant equation for k.

Areas for students to improve include:

- knowing and using the volume of solids of revolution formula
- using the Reference Sheet to change $sin^2 kx$ to $\cos 2kx$
- simplifying an integral by moving a constant to the front of the integral sign
- knowing how to integrate trigonometric functions.

Part (c)

Students should:

- understand the conditions on domain and range of functions and their inverses
- use the property $g\big(f(x)\big) = x$
- test the possibility of $f(x)$ having an inverse.

In better responses, students were able to:

- use graphs as a form of explanation connecting functions and their inverses
- find one value of x for which $g\big(f(x)\big) \neq x$
- recognise $f(x)$ as failing the horizontal line test or not being a one-to-one function, hence no inverse in the given domain of $f(x)$
- show the domain and range of $f(x)$ does not match the range and domain of $g(x)$ respectively, hence no inverse in the given domain of $f(x)$.

Areas for students to improve include:

- reading the question carefully and extracting the appropriate information
- providing clear logical explanations
- understanding the tests required for an inverse to exist in a given domain.

Part (d)

Students should:

- write an expression for a monic cubic polynomial and find its derivative
- state the relationships between roots and the coefficients of a polynomial of degree 3.

In better responses, students were able to:

- find an expression for the monic cubic polynomial $P(x)$ and its derivative $P'(x)$
- expand $(\alpha + \beta + \gamma)^2$ to obtain an expression for $\alpha^2 + \beta^2 + \gamma^2$
- obtain an expression for $P'(\alpha)$ + $P'(\beta)$ + $P'(\gamma)$
- use these expressions to find the required value.

Areas for students to improve include:

- reading the question carefully to determine what equations or expressions are required
- building the answer step by step from the provided statements
- stating known facts about the polynomial.

Part (e) (i)

Students should:

- understand the parameters associated with binomial and normal distributions
- use the Reference Sheet formulae to find the mean, standard deviation and z - score
- use either the normal distribution graph or the z - score table to find the probability.

In better responses, students were able to:

- find the mean and standard deviation of the sample
- calculate the required z - score
- use the table of z - scores provided or use the normal distribution graph on the Reference Sheet to find the required probability.

Areas for students to improve include:

- familiarising themselves with the information provided on the Reference Sheet
- determining the values to be substituted into the z - score formula
- understanding the significance of the score obtained from the z - score table
- knowing how to read values from the z - score table.

Part (e) (ii)

Students should:

- understand normal and binomial distributions
- understand the significance of sample size.

In better responses, students were able to:

- relate the size of the sample to the validity of the method used.

Areas for students to improve include:

- understanding the importance of sample size
- providing specific reasons rather than general comments.

Question 14

Part (a)

Students should:

- use integration to solve a differential equation.

In better responses, students were able to:

- separate the variables correctly
- find the primitives of each integrand
- evaluate the constant of integration
- determine whether the solution involved the positive or negative case of $|y|$
- simplify the resulting expression.

Areas for students to improve include:

- developing the algebraic skills necessary to successfully separate variables and simplify expressions
- correctly integrating $\frac{1}{y}$ to obtain $ln|y|$
- understanding how to determine the solution from an expression involving $|y|$ and justifying the use of the positive or negative case.

Part (b)

Students should:

- use their knowledge of vectors to prove the required result.

In better responses, students were able to:

- draw a large, clearly labelled diagram which displayed each of the vectors mentioned in the question
- use the concept of the perpendicular distance from a point to a line or Pythagoras' Theorem to explain clearly why $|\vec{u} - \lambda_0\vec{v}| \leq |\vec{u} - \lambda\vec{v}|$.

Areas for students to improve include:

- drawing large, clearly labelled diagrams
- understanding where the projection lies on the diagram
- understanding what $\vec{u} - \lambda\vec{v}$ and $|\vec{u} - \lambda\vec{v}|$ represent geometrically
- providing clear explanations.

Part (c)

Students should:

- use knowledge of projectile motion to solve a problem.

In better responses, students were able to:

- demonstrate an orderly progression through the stages of the problem

- find the time of flight of the projectile
- find the horizontal distance travelled in this time by both the projectile and the target
- equate the distances to find an expression for the location of the point of impact
- determine the maximum range of the projectile
- use calculus to show that the point of impact was within 37% of the maximum range.

Areas for students to improve include:

- planning their approach to this type of question
- developing the algebraic skills to simplify the different trigonometric expressions
- understanding which of the given equations represented horizontal and which represented vertical motion
- understanding how to find the time of flight
- recognising that the maximum range for the point of impact was not necessarily the maximum range of the projectile
- recognising that calculus was needed to demonstrate progress to the supplied solution.

Part (d)

Students should:

- use a normal distribution to approximate a binomial distribution.

In better responses, students were able to:

- recognise the values needed and use them to find the relevant expressions for $E(x)$ and $Var(x)$
- find the relevant z - score to represent the probability
- put these into the formula for z – score to generate a quadratic equation
- solve the quadratic equation to determine the number of tickets which should be sold.

Areas for students to improve include:

- understanding how to use each of the pieces of supplied information and what they represented with respect to $E(x)$ and $Var(x)$
- reading the question carefully, and using all columns of the supplied z - score table
- developing algebraic skills to reorganise an expression for z - score into a quadratic equation which can be solved.