NESA No. ....

Teacher: JH/MN AF SE

2024

Year 12 TRIAL EXAMINATION

ASCHAM SCHOOL

Mathematics Extension 1

Monday 29th July 2024

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided

Total marks: 70

SECTION I - (10 marks)

- Use the multiple-choice answer sheet for Questions 1 – 10
- Allow about 15 minutes for this section

SECTION II - (60 marks)

- This section consists of 4 questions
- Start a new booklet for each question
- All necessary working should be shown
- Allow about 1 hour 45 minutes for this section

SECTION I (10 marks)

Attempt Questions 1 - 10
Use the multiple-choice answer sheet for Questions 1 - 10

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1. Four boys and four girls are randomly assigned a seat around a circular table. Which of the following is the probability that the boys and girls alternate?

A. $\dfrac{4!\,3!}{7!}$

B. $\dfrac{4!\,4!}{7!}$

C. $\dfrac{4!\,3!}{8!}$

D. $\dfrac{4!\,4!}{8!}$

2. Given $f(x) = \sin^{-1}\left(\frac{a}{x}\right)$, which of the following is the correct expression for $f'(x)$?

A. $\dfrac{-a}{x\sqrt{x^2-a^2}}$

B. $\dfrac{a}{x\sqrt{x^2-a^2}}$

C. $\dfrac{-ax}{\sqrt{x^2-a^2}}$

D. $\dfrac{ax}{\sqrt{x^2-a^2}}$

3. What is the value of β such that $\sqrt{3}\sin x - \cos x = 2\cos(x-\beta)$?

A. $\frac{\pi}{6}$

B. $\frac{\pi}{3}$

C. $\frac{2\pi}{3}$

D. $\frac{5\pi}{6}$

4. Which of the following is the equation of the tangent to $y = \cos^{-1} x$ at $x = 0$?

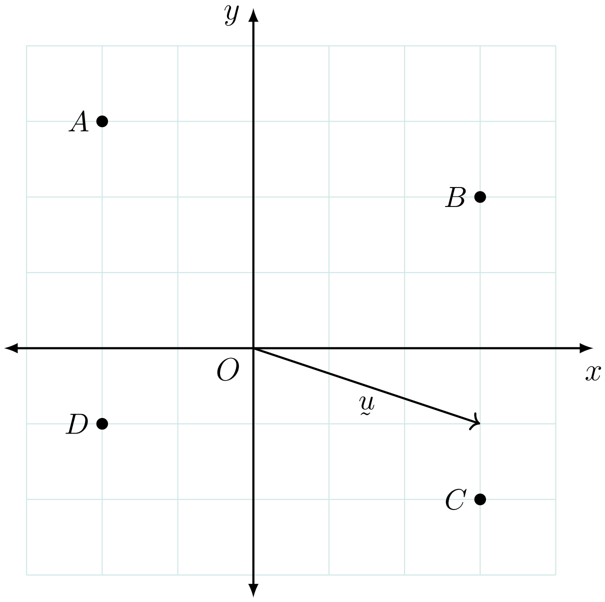
A. $2x + 2y + \pi = 0$

B. $2x + 2y - \pi = 0$

C. $2x - 2y + \pi = 0$

D. $2x - 2y - \pi = 0$

5. Which of the following vectors has the greatest magnitude?



A. $\text{proj}_{\underset{\sim}{u}} \overrightarrow{OA}$

B. $\text{proj}_{\underset{\sim}{u}} \overrightarrow{OB}$

C. $\text{proj}_{\underset{\sim}{u}} \overrightarrow{OC}$

D. $\text{proj}_{\underset{\sim}{u}} \overrightarrow{OD}$

6. What is the range of the function $y = \tan^{-1}(\sin x)$?

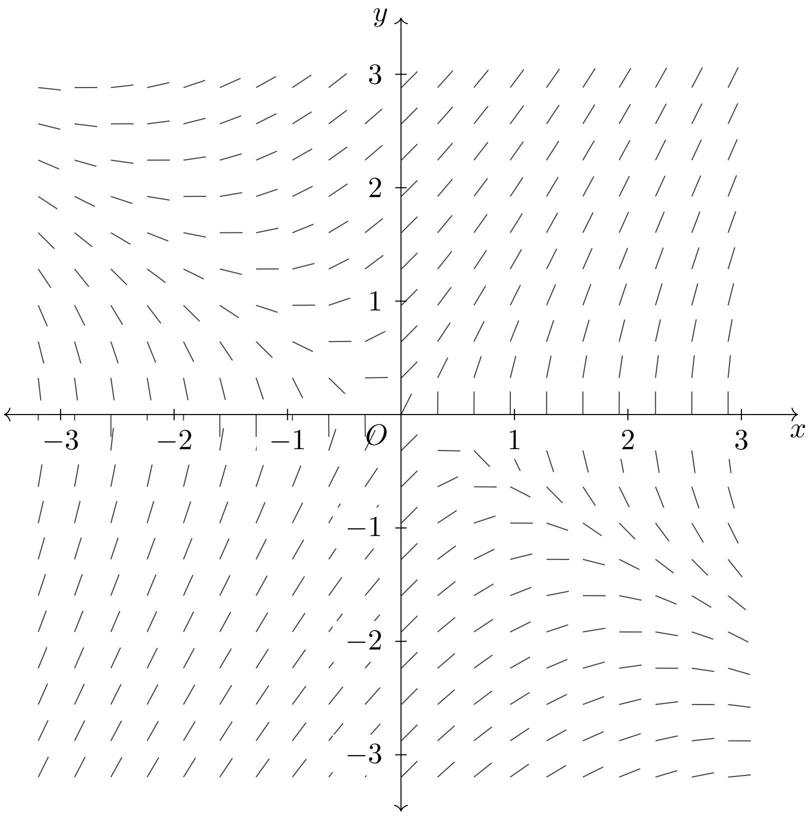
A. $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

B. $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

C. $y \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

D. $y \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

7. The diagram below shows the direction field for a differential equation.



Which of the following differential equations could be represented by this direction field?

A. $\dfrac{dy}{dx} = \dfrac{y - x}{y}$

B. $\dfrac{dy}{dx} = \dfrac{y + x}{y}$

C. $\dfrac{dy}{dx} = \dfrac{x - y}{x}$

D. $\dfrac{dy}{dx} = \dfrac{x + y}{x}$

8. The polynomial $P(x) = x^4 - 5x^2 + x + 2$ has four real zeros. If $P(x)$ is translated to the left by 1 unit, what is the product of the zeros of the transformed polynomial?

A. -3

B. -1

C. 1

D. 7

9. Given that $\tan\alpha = \dfrac{1}{2}$, which of the following is the exact value of $\tan\left(\alpha + \dfrac{\pi}{3}\right)$?

A. $\dfrac{\sqrt{3}-2}{2\sqrt{3}+1}$

B. $\dfrac{\sqrt{3}+2}{2\sqrt{3}-1}$

C. $\dfrac{1-2\sqrt{3}}{2+\sqrt{3}}$

D. $\dfrac{1+2\sqrt{3}}{2-\sqrt{3}}$

10. The sum of two unit vectors is a unit vector. That is, $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{a} + \underset{\sim}{b}$ all have magnitude 1. What is the value of $|\underset{\sim}{a} - \underset{\sim}{b}|$?

A. 1

B. $\sqrt{2}$

C. $\sqrt{3}$

D. 2

SECTION II (60 marks)

Attempt Questions 11 - 14
Answer each question in a new booklet. Extra booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new booklet

(a) Find $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,dx$. **[2]**

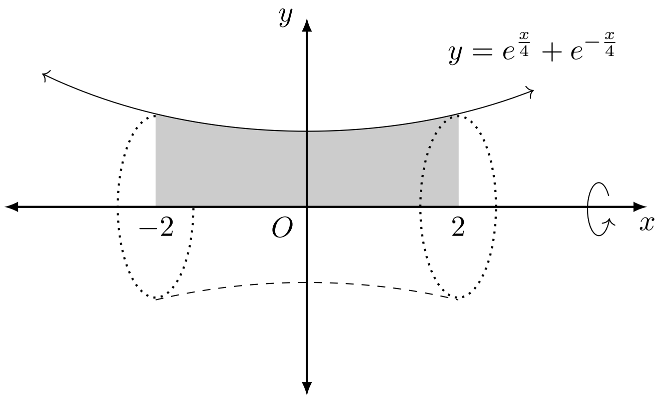
(b) Given the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, find $\mathbf{c}$ if $\mathbf{a} + 2\mathbf{b} + \mathbf{c} = \underset{\sim}{0}$. **[2]**

(c) Five regular dice are rolled. Calculate the probability that exactly two of the dice show a 6 on the uppermost face. **[2]**

(d) Find the exact value of $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x\,dx$. **[3]**

(e) The percentage of left-handed people in a population is 10%. If one hundred people are selected at random from the population what is the probability that less than 5% are left-handed? [You may refer to the table on page 11.] **[3]**

(f) To make a solid rim for a wheel, the region between the curve $y = e^{\frac{x}{4}} + e^{-\frac{x}{4}}$, the lines $x = -2$, $x = 2$ and the x-axis is rotated around the x-axis as shown below. **[3]**



Find the volume of the solid formed.

End of Question 11

Question 12 (15 marks) Start a new booklet

(a) The polynomial $P(x) = x^3 + bx^2 + cx + 3$ has a root of multiplicity 2 at $x = -1$. Find the values of b and c. **[3]**

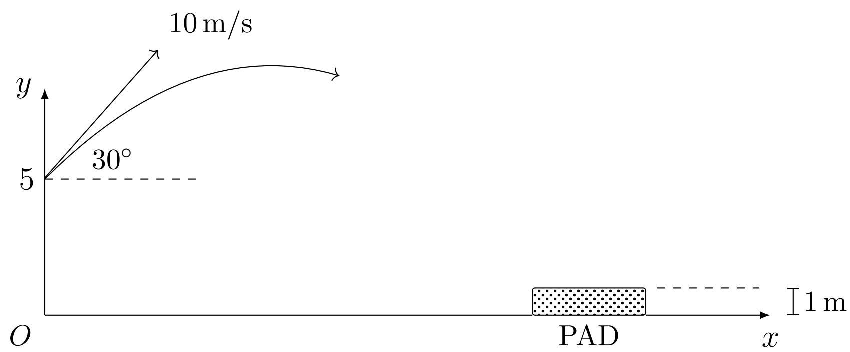
(b) Use mathematical induction to prove that **[3]**

$$(-1)^n + 2^{n+1}$$

is a multiple of 3 for all integers, $n \geq 1$.

(c) Use the substitution $u = \sqrt{2-x}$ to find the exact value of $\displaystyle\int_0^1 \frac{x}{\sqrt{2-x}}\,dx$. **[3]**

(d) From a point five metres above level ground a stuntman is projected at an angle of elevation of 30° and at an initial speed of 10 m/s. A one metre high pad is placed on the ground to cushion his impact.



The displacement vector function is

$$\mathbf{r} = 5\sqrt{3}\,t\ \mathbf{i} + 5(1 + t - t^2)\,\mathbf{j}. \qquad \text{(DO NOT SHOW THIS)}$$

(i) Show that the time of flight for the stuntman to land on the pad is approximately 1.52 seconds. **[1]**

(ii) Determine the distance from O, correct to the nearest centimetre, where the pad should be centred, to hopefully cushion his impact. **[1]**

(iii) Calculate his impact speed, to the nearest m/s. **[2]**

(iv) This is to be performed at an indoor stadium with a ceiling height of 8 metres. Calculate the distance the stuntman will miss the ceiling by at his highest point. **[2]**

End of Question 12

Question 13 (15 marks) Start a new booklet

(a) (i) Show that $y = e^{-\frac{x^2}{2}}$ is is a solution to the differential equation $y'' = y(x^2 - 1)$. **[1]**

(ii) Hence, or otherwise, find the coordinates of the inflection points of $y = e^{-\frac{x^2}{2}}$. **[2]**

(b) Find $\displaystyle\int_0^{\frac{\pi}{6}} \cos 3x \cos x \, dx$. **[3]**

(c) The function $f(x) = \dfrac{2}{x+1}$ passes through the point $\left(\frac{1}{2}, \frac{4}{3}\right)$. **[2]**

Find the gradient of $y = f^{-1}(x)$ at $\left(\frac{4}{3}, \frac{1}{2}\right)$.

(d) The differential equation for the repopulation of koalas in a particular habitat is given by

$$\frac{dP}{dt} = \frac{P(300 - P)}{300}$$

where P is the population of koalas and t is the number of years.

Initially 120 koalas are released.

(i) Using the identity $\dfrac{300}{P(300-P)} = \dfrac{1}{P} + \dfrac{1}{300-P}$ (without proof) show that **[3]**

$$P = \frac{600}{2 + 3e^{-t}}.$$

(ii) This population of koalas reaches its maximum growth rate within the first year. After how many complete months will the growth rate start decreasing? Justify your answer. **[2]**

(e) Consider the function $f(x) = x^2 + c$, where c is a constant. **[2]**

By considering the graph of $y = \dfrac{1}{f(x)}$ or otherwise, find all values for c such that $f(x) = \dfrac{1}{f(x)}$ has exactly two real solutions.

End of Question 13

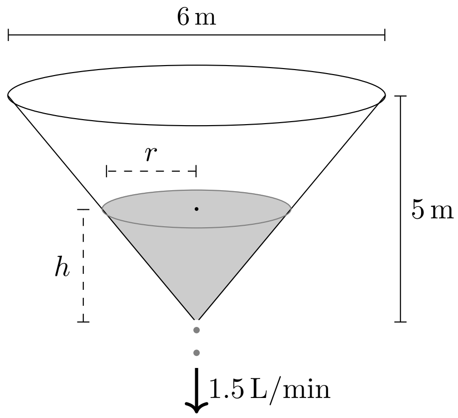
Question 14 (15 marks) Start a new booklet

(a) Solve $\dfrac{2}{|x-1|} \geq 3$. **[3]**

(b) Sketch the function represented by the parametric equations **[3]**

$$x = 2t - 1 \qquad y = t^2 + t \qquad -1 \leq t \leq 1.$$

(c) A tank in the shape of a cone is filled with water, which is leaking out from the bottom at a rate of 1.5 litres per minute. **[3]**



The volume of water in the tank is given by $V = \frac{1}{3}\pi r^2 h$, where h is the height in metres and r is the radius in metres.

Find the rate at which the water level is falling when the height of the water is 1 metre.
[Note: $1\,\text{m}^3 = 1000$ L]

(d) For two distinct values of k, the position vectors $\overrightarrow{OA} = 3\mathbf{i} - \mathbf{j}$ and $\overrightarrow{OB} = 2\mathbf{i} + k\mathbf{j}$ form the adjacent sides of a rhombus. Find the two possible values for the area of each rhombus. **[4]**

(e) Two teams of four players and an umpire are to be formed from nine people. If Amelia and Chloe cannot be on the same team, although they can be an umpire, find how many different ways the nine people can be grouped into two teams and an umpire. **[2]**

End of Examination

NORMAL CUMULATIVE DISTRIBUTION FUNCTION

Entries represent $P(Z \leq z)$. The value of z to the first decimal place is given in the left column. The second decimal place is given in the top row.

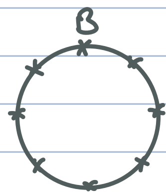
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Extension 1 Trial 2024 · Solutions

Multiple Choice

1. Total no. of arrangements (without restriction) is $7!$

Place Boy



Place remaining Boys $3!$

Place Girls $4!$

$\therefore \quad P(B, G \text{ alternate}) = \dfrac{4!\,3!}{7!}$ A

2. $f'(x) = \dfrac{-a/x^2}{\sqrt{1-\left(\frac{a}{x}\right)^2}}$

$= \dfrac{-a}{x^2\sqrt{\frac{x^2-a^2}{x^2}}}$

$= \dfrac{-a}{x\sqrt{x^2-a^2}}$ A

3. $2\cos(x-\beta) = 2\left[\cos x \cos\beta + \sin x \sin\beta\right]$

$\therefore \quad \cos\beta = \dfrac{-\sqrt{3}}{2}$ and $\sin\beta = +\dfrac{1}{2}$

So $\beta = \dfrac{2\pi}{3}$ C

4.

$y' = -\dfrac{1}{\sqrt{1-x^2}}$

when $x=0$, $y = \dfrac{\pi}{2}$, $y' = -1$

Eq'n of tangent $y - \dfrac{\pi}{2} = -1(x-0)$

$\therefore 2y - \pi = -2x$

$2x + 2y - \pi = 0$

B

5. By extending $\underset{\sim}{a}$ in both directions and dropping a perpendicular.

$\text{proj}_{\underset{\sim}{a}} \overrightarrow{OC}$ has the greatest length C

6. $y = \tan^{-1}(\sin x)$ D: $-1 \leq x \leq 1$

$\therefore$ R: $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$ C

7. By inspection B

8. Transformed function: $(x+1)^4 - 5(x+1)^2 + (x+1) + 2$

$x^4 + \ldots + 1^4 - 5x^2 - \ldots - 5(1)^2 + x + 1 + 2 = x^4 + \ldots - 1$

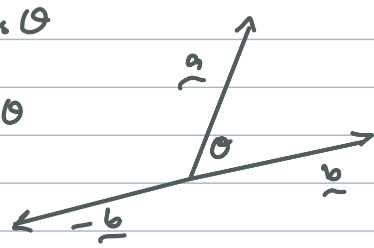
Product of zeros is $\frac{\text{constant term}}{\text{coeff. of } x^4} = -1$ B

9. $$\tan(\alpha + \tfrac{\pi}{3}) = \frac{\tan\alpha + \tan\pi/3}{1 - \tan\alpha \tan\pi/3}$$

$$= \frac{\frac{1}{2} + \sqrt{3}}{1 - \frac{\sqrt{3}}{2}}$$

$$= \frac{1 + 2\sqrt{3}}{2 - \sqrt{3}}$$ D

10.



$$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |-\underset{\sim}{b}|^2 + 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$1 = 1 + 1 + 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$\therefore |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta = -\frac{1}{2}$$

So $$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$= 1 + 1 - 2(-\tfrac{1}{2})$$

$$= 3$$

$$\therefore |\underset{\sim}{a} - \underset{\sim}{b}| = \sqrt{3}$$ C

Question 11

(a) $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{3\,dx}{\sqrt{4-9x^2}}$ ✓

$\displaystyle = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + c$ ✓

or

$\left.\begin{aligned} a &= 2 \\ f(x) &= 3x \\ f'(x) &= 3 \end{aligned}\right\}$ ✓

(b) $\underset{\sim}{c} = -\underset{\sim}{a} - 2\underset{\sim}{b}$

$= -\begin{pmatrix} 1 \\ -2 \end{pmatrix} - 2\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ ✓

$= \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -6 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ ✓

(c) $P(\text{success}) = P(6) = \frac{1}{6}$ $\quad n = 5$

let X be number of 6's

$\therefore \quad P(X=2) = {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$ ✓✓

$= \frac{625}{3888} \quad \doteqdot 0.16075$ (5 sf)

(d) $\displaystyle\int_{\pi/6}^{\pi/3} \cos^2 x\,dx = \int_{\pi/6}^{\pi/3} \frac{1}{2}(1+\cos 2x)\,dx$ ✓

$\displaystyle = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{\pi/6}^{\pi/3}$ ✓

$\displaystyle = \frac{1}{2}\left[\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right]$

$= \frac{\pi}{12}$ ✓

(e) $p = 0.10$

$$\sigma^2 = \frac{pq}{n} = \frac{(0.10)(0.90)}{100}$$

$$= \frac{9}{10000}$$

$$\therefore \quad \sigma = \frac{3}{100}$$

$$= 0.03$$

$$P(\hat{p} < 0.05) \approx P\left(z < \frac{0.05 - 0.10}{0.03}\right)$$

$$= P(z < -1.67)$$

$$= 1 - P(z < 1.67)$$

$$= 1 - 0.9525$$

$$= 0.0475$$

Alternate solutions:

1. Normal approximation to binomial (no continuity correction)

$$P(x < 5) = P\left(z < \frac{5-10}{3}\right) \qquad \mu = 10$$

$$= P(z < -1.67) \qquad \sigma = 3$$

(as above)

2. Normal approximation to binomial (with continuity correction)

$$P(X < 5) = P(X \leq 4.5)$$

$$= P\left(Z < \frac{4.5 - 10}{3}\right)$$

$$= P(Z < -1.83)$$

$$= 1 - P(Z < 1.83)$$

$$= 1 - 0.9664$$

$$= 0.0336$$

3. $P(X<5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$$= {}^{100}C_0\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{100} + {}^{100}C_1\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^{99} + {}^{100}C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{98}$$

$$+ {}^{100}C_3\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^{97} + {}^{100}C_4\left(\frac{1}{10}\right)^4\left(\frac{9}{10}\right)^{96}$$

$$\doteqdot 0.0237$$

(f) $V = \pi \int_{-2}^{2} (e^{x/4} + e^{-x/4})^2 \, dx$

$$= 2\pi \int_0^2 e^{x/2} + 2 + e^{-x/2} \, dx \quad ✓$$

$$= 2\pi \left[2e^{x/2} + 2x - 2e^{-x/2}\right]_0^2 \quad ✓$$

$$= 2\pi \left[(e + 4 - e^{-1}) - (2 + 0 - 2)\right]$$

$$= 4\pi\left(e + 2 - \frac{1}{e}\right) \quad ✓ \text{ (or equivalent)}$$

Question 12

(a) $P(x) = x^3 + bx^2 + cx + 3$

Let β be the other root

Product : $(-1)(-1)\beta = -3$ $\quad\therefore\ \beta = -3$ ✓

Sum : $(-1) + (-1) + (-3) = -b$

$-5 = -b$

$\therefore\ b = 5$ ✓

Pairs : $(-1)(-1) + (-1)(-3) + (-1)(-3) = c$

$1 + 3 + 3 = 7$

$\therefore\ c = 7$ ✓

(Other methods possible)

(b) Step 1. $n = 1$ $\quad (-1)^1 + 2^{1+1} = 1 + 2$

$= 3$

which is a multiple of 3. ✓

Step 2 Assume the result is true for $n = k$ (k is an integer)

i.e. $(-1)^k + 2^{k+1} = 3M$ (M is an integer)

$(-1)^k = 3M - 2^{k+1}$ $\quad *$

We need to show that it is also true for $n = k+1$

i.e. $(-1)^{k+1} + 2^{k+2}$ is also a multiple of 3 ✓

Now $(-1)^{k+1} + 2^{k+2} = (-1)\left[3M - 2^{k+1}\right] + 2^{k+2}$ (by $*$)

$= -3M + 2^{k+1} + 2^{k+2}$

$= 2^{k+1}(1+2) - 3M$

✓

$= 3 \times 2^{k+1} - 3M$

$= 3(2^{k+1} - M)$

which is also a multiple of 3 since $2^{k+1} - M$ is also an integer

<u>Step 3</u> By the principle of mathematical induction the result is true for all integers $n \geq 1$.

(c) $\int_0^1 \frac{x}{\sqrt{2-x}}\,dx = \int_{\sqrt{2}}^{1} (2-u^2)(-2)\,du$

$u = \sqrt{2-x}$ $\quad x = 2-u^2$

$du = \frac{-1}{2\sqrt{2-x}}\,dx$ $\quad x = 0,\ u = \sqrt{2}$; $\quad x = 1,\ u = 1$

$-2\,du = \frac{dx}{\sqrt{2-x}}$

$= \int_1^{\sqrt{2}} 4 - 2u^2\,du$

$= \left[4u - \frac{2u^3}{3}\right]_1^{\sqrt{2}}$

$= \left(4\sqrt{2} - \frac{4\sqrt{2}}{3}\right) - \left(4 - \frac{2}{3}\right)$

$= \frac{8\sqrt{2} - 10}{3}$

(e) $\underset{\sim}{r} = 5\sqrt{3}\,t\,\underset{\sim}{i} + 5(1+t-t^2)\,\underset{\sim}{j}$

(i) Time of flight ($y = 1$) $\therefore$ $1 = 5(1+t-t^2)$

$5t^2 - 5t - 4 = 0$

$t = \frac{--5 \pm \sqrt{5^2 + 4(5)4}}{10}$

$= \frac{5 + \sqrt{105}}{10}$, $t > 0$

$\doteqdot 1.52$ s. (as required)

(ii) $x = 5\sqrt{3}(1.52\ldots)$

$= 13.204\ldots$

$\doteqdot 13.20$ m (nearest cm)

$\doteqdot 13.16$ m (using 1.52 s)

(iii) $\underset{\sim}{v} = 5\sqrt{3}\,\underset{\sim}{i} + 5(1-2t)\,\underset{\sim}{j}$

$\therefore$ Impact speed $= \sqrt{(5\sqrt{3})^2 + 5(1-2\times 1.52)^2}$ ✓

$= 13.38\ldots$

$\doteqdot 13$ m/s. ✓

(iv) max. height when $5(1-2t) = 0$

$\therefore\ t = \frac{1}{2}$ s. $\frac{1}{2}$

when $t = \frac{1}{2}$, $y = 5\left(1 + \frac{1}{2} - \left(\frac{1}{2}\right)^2\right)$

$= 5\left(\frac{5}{4}\right)$ $\frac{1}{2}$

$= \frac{25}{4}$

$= 6\frac{1}{4}$ m.

$\therefore$ misses ceiling by 1.75 m ✓

Question 13

(a) (i) $y = e^{-x^2/2}$

$y' = -x e^{-x^2/2}$

$y'' = -e^{-x^2/2} + -x(-x)e^{-x^2/2}$

$= x^2 e^{-x^2/2} - e^{-x^2/2}$

$= e^{-x^2/2}(x^2-1)$

$= y(x^2-1)$ as required

✓

(ii) Inflection points when $y''=0 \Rightarrow x=\pm 1$

$x=1$, $y=e^{-1/2}$

$x=-1$, $y=e^{-1/2}$

x	0	1	2
y''	$-e^{-1/2}$	0	$3e^{-1/2}$
conc.	$\frown$	$\cdot$	$\smile$

By even symmetry of $y=e^{-x^2/2}$, the coordinates of the inflection points are

$\left(-1, \frac{1}{\sqrt{e}}\right)$ ✓ and $\left(1, \frac{1}{\sqrt{e}}\right)$ ✓

(b) $$\int_0^{\pi/6} \cos 3x \cos x \, dx = \frac{1}{2}\int_0^{\pi/6} \cos 2x + \cos 4x \, dx \quad ✓$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right]_0^{\pi/6} \quad ✓$$

$$= \frac{1}{2}\left[\left(\frac{\sin \pi/3}{2} + \frac{\sin 2\pi/3}{4}\right) - (0+0)\right]$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8}\right) \quad ✓$$

$$= \frac{3\sqrt{3}}{16}$$

(c) Let $y = \frac{2}{x+1}$

Inverse function $x = \frac{2}{y+1}$

$$\frac{dx}{dy} = \frac{-2}{(y+1)^2} \quad ✓ \text{ or equivalent}$$

At $\left(\frac{4}{3}, \frac{1}{2}\right)$ $$\frac{dx}{dy} = \frac{-2}{\left(\frac{1}{2}+1\right)^2}$$

$$= \frac{-2}{9/4}$$

$$= -8/9$$

$\therefore$ gradient of inverse function is $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = -\frac{9}{8}$ ✓

(d) (i) $$\frac{dP}{dt} = \frac{P(300-P)}{300}$$

$$\frac{300\,dP}{P(300-P)} = dt$$

$$\frac{dP}{P} + \frac{dP}{300-P} = dt$$

Integrating both sides.

$$\ln|P| - \ln|300-P| = t + c$$

$$\ln\left|\frac{P}{300-P}\right| = t + c$$

$$\frac{P}{300-P} = Ae^{t}, \quad A \in \mathbb{R}.$$

Initial conditions $t=0, \quad P = 120$

$$\frac{120}{180} = A \qquad \therefore A = \frac{2}{3}$$

$$\therefore \quad \frac{P}{300-P} = \frac{2}{3}e^{t}$$

$$3P = 600e^{t} - 2Pe^{t}$$

$$P(3+2e^{t}) = 600e^{t}$$

$$P = \frac{600e^{t}}{3+2e^{t}}$$

$$= \frac{600}{2+3e^{-t}} \quad \text{(as required)}$$

(ii) Max growth rate is when $\frac{dP}{dt}$ is a maximum.

$\frac{dP}{dt} = \frac{P(300-P)}{300}$ is a concave down parabola

$\therefore$ Vertex is when $P = 150$.

Putting $P = 150$ into $P = \frac{600}{2+3e^{-t}}$ or $\frac{P}{300-P} = \frac{2}{3}e^{t}$

and solving for t $\quad \frac{3}{2} = e^{t}$

$$t = \ln \frac{3}{2}$$

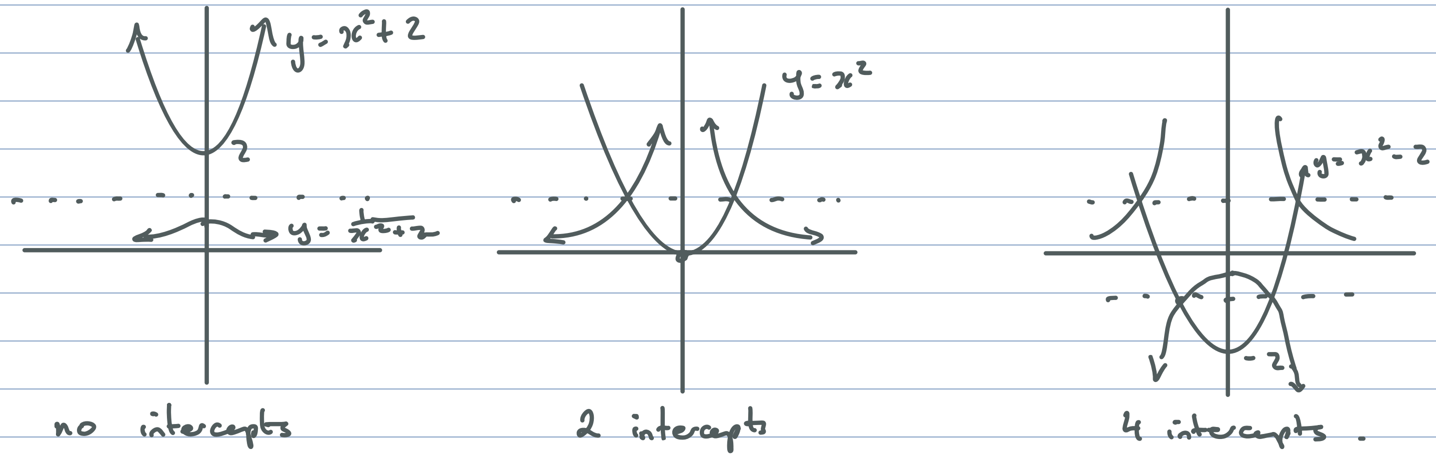
$$= 0.40546\ldots \text{ years}$$

$$\doteq 4.8 \text{ months}$$

$\therefore$ After 5 months the growth rate starts to decrease.

(e) Since $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect when $y = \pm 1$

Examine the cases when $c = 2$, $c = 0$, $c = -2$



We can also infer that when $c = 1$ there is 1 intercept and when $c = -1$ there is 3 intercepts.

$\therefore$ The values of c such that the curves intersect exactly twice is

$$-1 < c < 1$$

correct justification or reasoning

Question 14

(a)

$$\frac{2}{|x-1|} \geq 3 \qquad x \neq 1$$

$$2 \geq 3|x-1| \qquad |x-1| > 0 \text{ for all } x,\ x \neq 1$$

$$|x-1| \leq \frac{2}{3}$$

$$\therefore \quad x - 1 \leq \frac{2}{3} \quad \text{and} \quad -(x-1) \leq \frac{2}{3}$$

$$x \leq \frac{5}{3} \qquad\qquad x - 1 \geq -\frac{2}{3}$$

$$x \geq \frac{1}{3}$$

$$\therefore \quad \frac{1}{3} \leq x \leq \frac{5}{3}, \quad x \neq 1$$

(b)

$$\left.\begin{aligned} x &= 2t-1 \\ y &= t^2+t \end{aligned}\right\} \ -1 \leq t \leq 1$$

$$t = \frac{x+1}{2} \qquad \therefore \quad y = \left(\frac{x+1}{2}\right)^2 + \frac{x+1}{2}$$

$$= \frac{1}{4}\left(x^2 + 2x + 1 + 2x + 2\right)$$

$$= \frac{1}{4}\left(x^2 + 4x + 3\right)$$

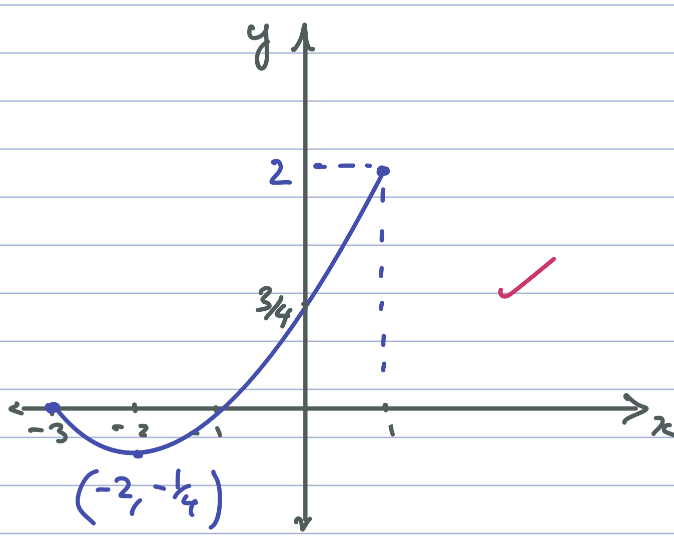
$$= \frac{1}{4}(x+3)(x+1)$$

which is a parabola with x-intercepts -3, -1

when $t = -1$, $x = -3$

$t = 1$, $x = 1$

$\therefore$ D: $-3 \leq x \leq 1$



(c) $\frac{dh}{dt} = ?$ $\quad\quad \frac{dV}{dt} = -1.5$ L/min

$$V = \frac{1}{3}\pi r^2 h$$

By similarity $\frac{r}{h} = \frac{3}{5}$

$r = \frac{3h}{5}$

$$= \frac{1}{3}\pi \frac{9h^2}{25} h$$

$$= \frac{3\pi h^3}{25}$$

$$\therefore \frac{dV}{dh} = \frac{9\pi h^2}{25}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$\frac{dV}{dt} = -1.5$ L/min (given)

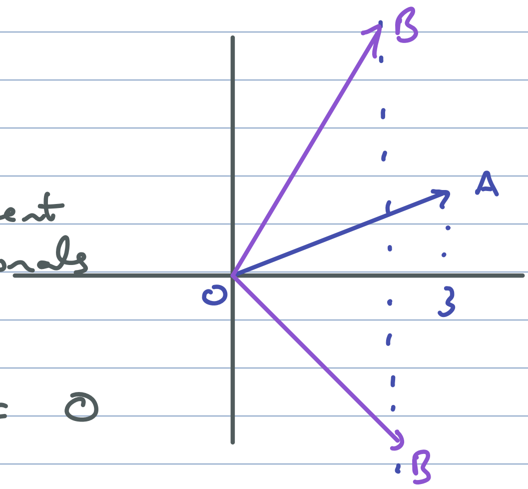
$= -0.0015$ m^3/min

$$= \frac{25}{9\pi h^2} \times -0.0015 \ m^3/min$$

when $h = 1$ $\quad = -0.0013$ m/min

$\therefore$ the height is falling by 1.3 mm/min.

(d) $\overrightarrow{OA} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $\quad \overrightarrow{OB} = \begin{bmatrix} 2 \\ k \end{bmatrix}$



Since $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are adjacent sides of a rhombus, the diagonals are perpendicular.

$$\therefore (\overrightarrow{OA} + \overrightarrow{OB}) \cdot (\overrightarrow{OA} - \overrightarrow{OB}) = 0$$

$$\begin{bmatrix} 5 \\ k-1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ k+1 \end{bmatrix} = 0$$

~~OA~~ $\quad -5 + k^2 - 1 = 0$

$$k^2 = 6$$

$$k = \pm\sqrt{6}$$

Note: $k = \pm\sqrt{6}$ can also be found by solving $|\overrightarrow{OA}| = |\overrightarrow{OB}|$

If $k = \sqrt{6}$

$$\vec{OA} \cdot \vec{OB} = |\vec{OA}||\vec{OB}| \cos\theta = 3 \times 2 - \sqrt{6}$$

$$= \sqrt{10}\sqrt{10} \cos\theta = 6 - \sqrt{6}$$ ✓

So $\cos\theta = \dfrac{6-\sqrt{6}}{10}$

Area of rhombus $= 2 \times \frac{1}{2} |\vec{OA}||\vec{OB}| \sin\theta$

$= \sqrt{10}\sqrt{10} \sin\left[\cos^{-1}\left(\dfrac{6-\sqrt{6}}{10}\right)\right]$

$= 9.348\ldots$

$\doteq 9.4 \ u^2$ (2 sig. figs) ✓

If $k = -\sqrt{6}$

Area $= \sqrt{10}\sqrt{10} \sin\left[\cos^{-1}\left(\dfrac{6+\sqrt{6}}{10}\right)\right]$

$= 5.348\ldots$

$\doteq 5.3 \ u^2$ (2 sig. figs) ✓

(e) case (i) Amelia is an umpire

- Place Amelia 1
- Place 4 into a team ${}^{8}C_{4}$ (Red Team)
- Remainder form other team ${}^{4}C_{4}$ (Blue Team)

Note we have overcounted by a factor of 2, since the four people chosen in Blue is the same situation when chosen in Red.

$\therefore$ No. of ways $= \dfrac{{}^{8}C_{4}}{2}$

$= 35$ ✓

case (ii) Chloe is an umpire.

Similar to case (i) $\therefore$ No. of ways $= 35$

<u>case (iii)</u> Someone other than A or C is an umpire. 7

Place A in a team 1

Place C in other team 1

Place 3 remaining in A's team $^6C_3$

Place last 3 $^3C_3$

$\therefore$ Total no. of ways $= 7 \times {^6C_3}$

$= 140$

$\therefore$ Total number of ways $= 210$.

<u>Alternate method</u>: Without restriction: $\dfrac{^9C_4 \times {^5C_4}}{2}$

Amelia and Chloe on same team $^7C_2 \times {^5C_4}$

$\therefore$ Total number of ways $= \dfrac{^9C_4 \times {^5C_4}}{2} - {^7C_2} \times {^5C_4}$

$= 315 - 105 = 210$ (as above)