Section I

10 marks

Attempt Questions 1-10.

Allow approximately 15 minutes for this section.

Write your answers on the multiple-choice answer sheet provided.

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1. A complex number z is defined such that $|z-1+2i|=1$. Which of the following is the maximum modulus of z?

   (A) $2\sqrt{2}$

   (B) $\sqrt{5}+1$

   (C) $\sqrt{5}$

   (D) $2\sqrt{2}+1$

2. Which of the following is the magnitude of the vector $\cos\theta\,\underset{\sim}{i}+\sin\theta\,\underset{\sim}{j}+\tan\theta\,\underset{\sim}{k}$, where $0<\theta<\frac{\pi}{2}$?

   (A) 1

   (B) $\operatorname{cosec}\theta$

   (C) $\cot\theta$

   (D) $\sec\theta$

3. Suppose John found a raven, and it was black. Which of the following **must** be false?

   (A) "There exist non-black ravens as well as black ravens."

   (B) "There exist non-black ravens."

   (C) "All ravens are black."

   (D) None of the above.

4. The points A, B and C are collinear where

$$\overrightarrow{OA}=\underset{\sim}{i}+\underset{\sim}{j},\qquad \overrightarrow{OB}=2\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k},\qquad \overrightarrow{OC}=3\underset{\sim}{i}+a\underset{\sim}{j}+b\underset{\sim}{k}$$

   Which of the following are the values of a and b?

   (A) $a=-3,\ b=-2$

   (B) $a=3,\ b=-2$

   (C) $a=-3,\ b=2$

   (D) $a=3,\ b=2$

5. Let $z = \sqrt{3} + i$. Which of the following gives the geometric effect of multiplying the complex number w by $\frac{\bar{z}}{z}$?

(A) w is rotated anticlockwise by an angle of $\frac{\pi}{3}$.

(B) w is rotated anticlockwise by an angle of $\frac{2\pi}{3}$.

(C) w is rotated clockwise by an angle of $\frac{\pi}{3}$.

(D) w is rotated clockwise by an angle of $\frac{2\pi}{3}$.

6. Let A and B be true statements such that $A \Rightarrow B$. Which of the following statements is necessarily false?

(A) $\sim B \Rightarrow \sim A$

(B) A and $\sim B$

(C) A or $\sim B$

(D) None of the above.

7. It is given that $z = 1 + i$ is a root of $z^3 + bz^2 + 6z - 4 = 0$, where b is a real number. Which of the following is the value of b?

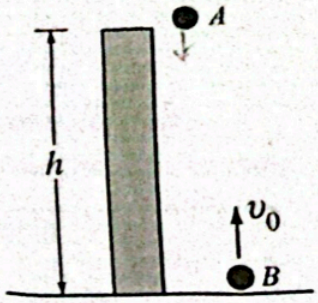
(A) -4

(B) 4

(C) 2

(D) -2

8. The diagram shows ball A being dropped from rest at time $t = 0$ seconds from a tower of height h metres. At the same instant, ball B is launched upward from the ground with initial speed v_0. If air resistance is negligible, and assuming all measurements are with respect to the centre of mass of each object, which of the following gives the best approximation for the time at which the balls pass each other?



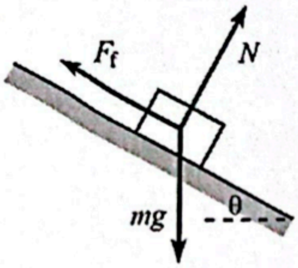
(A) $\frac{2h}{v_0}$

(B) $\frac{h}{v_0}$

(C) $\frac{h}{2v_0}$

(D) $\frac{h}{4v_0}$

9. The diagram below shows a block at rest on a plane that is inclined at an angle of measure θ. The forces acting on the mass are gravitational, normal and frictional forces, as indicated. The forces are NOT depicted to scale. Which of the following statements is always true?



(A) $N \leq mg,\ F_f \leq mg$

(B) $N \geq mg,\ F_f \leq mg$

(C) $N \leq mg,\ F_f \geq mg$

(D) None of the above.

10. Consider the integral

$$\int_{-3}^{x^2-3x} e^{t^2}\,dt$$

where x is a variable limit. Which of the following values of x minimises the integral?

(A) $\frac{1}{2}$

(B) $\frac{3}{2}$

(C) $\frac{5}{2}$

(D) None of the above.

Section II begins on the next page.

90 marks
Attempt Questions 11-15.
Allow approximately 2 hours and 45 minutes for this section.
Start each question on a new page. Extra paper is available.
In Questions 11-15, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (18 marks) **Start a new page.**

(a) Consider the complex numbers $a = 3 - 5i$ and $b = 2 + 3i$. Evaluate $a - \bar{b}$. **1**

(b) Solve for $z \in \mathbb{C}$, **3**

$$z^2 + 2\bar{z} + 3 = 0$$

(c) Use the substitution $u = \tan x$ to evaluate **3**

$$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x \, dx$$

(d) By using an appropriate substitution, find **2**

$$\int x^3\sqrt{1 + x^2}\, dx$$

(e) Let $n \in \mathbb{Z}$ with $n \geq 0$. Define

$$I_n = \int_0^{\pi} x^n \sin x \, dx$$

(i) Show that **3**

$$I_n = \pi^n - n(n-1)I_{n-2}$$

(ii) Hence evaluate **2**

$$\int_0^{\pi} x^6 \sin x \, dx$$

leaving your answer in exact form.

(f) A particle has velocity equation

$$\dot{x}^2 = 4x - x^2$$

(i) Show that the particle is undergoing simple harmonic motion. **1**
(ii) Find the centre of motion, amplitude and period of the motion. **2**
(iii) If the displacement is $x = 4$ at time $t = 0$, find the displacement x as a function of time t. **1**

Question 12 (18 marks) **Start a new page.**

(a) On an Argand diagram, sketch the region representing the set of all z satisfying $2 < |z| \leq 4$ and $\frac{\pi}{6} \leq \arg(z) < \frac{\pi}{3}$, showing the coordinates of any vertices. **3**

(b) If $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are unit vectors such that $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$, find the value of $\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{a}$. **2**

(c) A sphere S_1, with centre $C(2, 2, 2)$ passes through the point $A(4, 4, 4)$.

(i) Find the Cartesian equation of S_1. **2**

(ii) A second sphere S_2 has equation $(x-2)^2 + (y-2)^2 + (z-5)^2 = 1$. Find the equation of the circle in which S_1 and S_2 intersect. **2**

(d) (i) Show that $a^2 + b^2 \geq 2ab$. **1**

(ii) Hence or otherwise, show that for all positive numbers a, b and c, **2**

$$2(a^3 + b^3 + c^3) \geq ab(a+b) + bc(b+c) + ca(c+a)$$

(e) Suppose a particle is moving horizontally, measured by a coordinate system. Initially, the particle is at the origin O and moving with velocity 2 ms^{-1}. The acceleration of the particle is given by $\ddot{x} = x - 2$ where x is its displacement in metres at time t, measured in seconds.

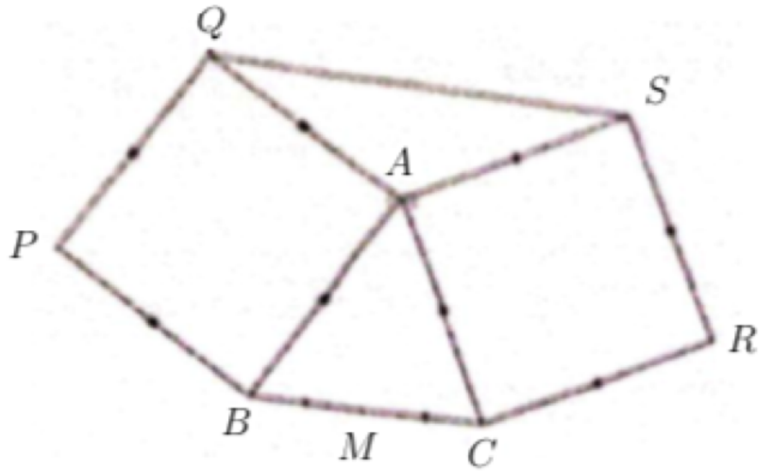
(i) Show that the velocity of the particle is given by $\dot{x}^2 = (x-2)^2$. **2**

(ii) Explain why $0 \leq x < 2$ for all $t \in [0, \infty)$ and hence show that $\dot{x} = 2 - x$. **2**

(iii) Find an expression for x as a function of t. **2**

Question 13 (18 marks) **Start a new page.**

(a) Let ABC be a triangle and let M be the midpoint of BC. **3**
Squares $ABPQ$ and $ACRS$ are erected on sides AB and AC as shown in the diagram below. Prove, using vector methods, that $|\overrightarrow{QS}| = 2|\overrightarrow{AM}|$.



(b) Consider the two perpendicular vectors $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(i) Find the unit vectors $\hat{\underset{\sim}{u}}$ and $\hat{\underset{\sim}{v}}$. **2**

(ii) Find a unit vector $\hat{\underset{\sim}{w}}$ that is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **2**

(iii) Let the vector $\underset{\sim}{x} = a\hat{\underset{\sim}{u}} + b\hat{\underset{\sim}{v}} + c\hat{\underset{\sim}{w}}$ where a, b, c are scalar constants.
If $|\underset{\sim}{x}| = 1$, prove that $a^2 + b^2 + c^2 = 1$. **2**

(iv) Let α, β and γ be the measure of the angles between $\hat{\underset{\sim}{u}}$ and $\underset{\sim}{x}$, $\hat{\underset{\sim}{v}}$ and $\underset{\sim}{x}$, $\hat{\underset{\sim}{w}}$ and $\underset{\sim}{x}$ respectively. Prove that $\cos\alpha + \cos\beta + \cos\gamma = a + b + c$. **2**

(c) (i) Given ω is a non-real root of $x^3 - 1 = 0$, show that ω is also a root of $x^2 + x + 1 = 0$. **1**

(ii) Hence prove by contradiction that $(x+1)^{2n} + x^{2n} + 1$ is not divisible by $(x^2 + x + 1)$ if n is divisible by 3, where x is non-real. **2**

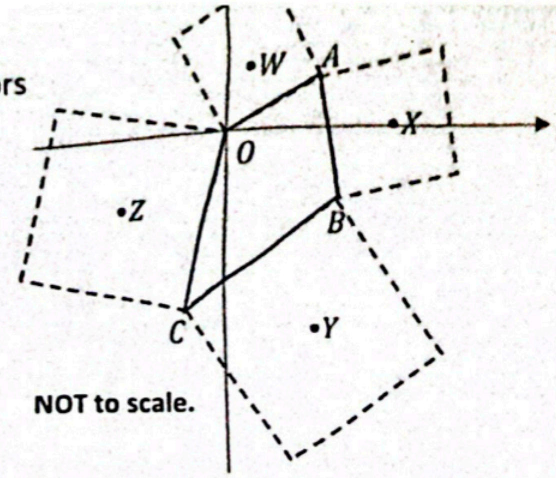
(d) Prove by mathematical induction that **4**

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta = \frac{\sin(\frac{2n-1}{2})\theta}{2\sin\frac{\theta}{2}}$$

for all $n \geq 2, n \in \mathbb{Z}$ and $\theta \neq 2k\pi$ for all $k \in \mathbb{Z}$.

(a) Let $OABC$ be a convex quadrilateral where the vectors $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CO}$ correspond to $2a$, $2b$, $2c$ and $2d$ respectively, where $a, b, c, d \in \mathbb{C}$.

The centres of the squares erected on each side of $OABC$ are W, X, Y and Z as shown in the diagram.



(i) Show that X is represented by the complex number $2a + b + ib$. **2**

(ii) Show that the line segments WY and XZ are equal in length and perpendicular. **3**

(b) Consider the following number sequence given by: **4**

$$a_1 = 0$$

$$a_n = \frac{1 + a_{n-1}}{2 + a_{n-1}} \text{ for } n \in \mathbb{Z}, n \geq 2$$

Prove, by mathematical induction, that $a_{n-1} < a_n$ for all integers $n \geq 2$.

(c) (i) Prove, by the method of partial fractions, that **2**

$$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$$

(ii) Hence, or otherwise, use integration by parts to find **2**

$$\int \frac{\tan^{-1} x}{x^2} dx$$

(d) (i) Show that, for $x \in \mathbb{R}$ such that $|x| < 1$, **1**

$$\sum_{k=0}^{n} (-1)^k x^k = \frac{1}{1+x} - \frac{(-1)^{n+1} x^{n+1}}{1+x}$$

(ii) Prove that **2**

$$0 \leq \int_0^1 \frac{x^{n+1}}{1+x} dx \leq \frac{1}{n+2}$$

(iii) By integrating part (i) and using the result in part (ii), show that **2**

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k \cancel{x^k}}{1+k} = \ln 2$$

Question 15 (18 marks) **Start a new page.**

(a) (i) The equation $(z+i)^7 + (z-i)^7 = 0$ can be shown to have solutions satisfying $\frac{z+i}{z-i} = e^{\frac{2m+1}{7}\pi i}$ (do NOT prove this). Write down the set of values for m such that the equation has unique solutions where $\left(\frac{2m+1}{7}\right)\pi \in (-\pi, \pi]$. **1**

(ii) Given that $\frac{e^{2i\theta}+1}{e^{2i\theta}-1} = \frac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}-e^{-i\theta}}$, show that the solutions of $(z+i)^7 + (z-i)^7 = 0$ can be written as $z = \cot\left(\frac{2m+1}{14}\pi\right)$ for these values of m. **3**

(iii) Using the binomial theorem, or otherwise, show that the non-zero roots of $(z+i)^7 + (z-i)^7 = 0$ are the roots of $z^6 - 21z^4 + 35z^2 - 7 = 0$. **2**

(iv) Hence show that **2**

$$\sum_{k=0}^{2} \cot^2\left(\frac{2k+1}{14}\pi\right) = 21$$

(b) A projectile of mass m is fired vertically upwards into a resistive medium, under the effect of gravity, with an initial velocity u. After reaching its maximum height H, it freefalls back to the ground, vertically. The object experiences a resistive force kv^2, when travelling in both directions, where v is the speed of the object and k is a positive constant.

The equation of motion for the object in freefall is given by $m\dot{v} = mg - kv^2$.

(i) Show that the terminal velocity v_τ that the object experiences when falling is **1**

$$v_\tau = \sqrt{\frac{mg}{k}}$$

(ii) Show that the time T at which the **maximum** height is reached is **3**

$$T = \frac{v_\tau}{g}\tan^{-1}\left(\frac{u}{v_\tau}\right)$$

(iii) Show that the **maximum** height H is given by **3**

$$H = \frac{v_\tau^2}{2g}\ln\left(1 + \frac{u^2}{v_\tau^2}\right)$$

(iv) Hence show that the speed W on impact with the ground is **3**

$$W = \frac{u}{\sqrt{1 + \frac{u^2}{v_\tau^2}}}$$

END OF EXAMINATION