ASSESSMENT TASK – HSC COURSE – 2020

MATHEMATICS EXTENSION 2

Student's Name:______________________________

Task No: 4 **Date to be worked: 10 August 2020**

Topics: Complex Numbers, Applying Complex Numbers, Mathematical Proof, Further Mathematical Induction, Vectors, Further Integration, Mechanics, Harder Mathematics Advanced and Extension 1 topics could be assessed.

Marks: _____/ 100 **Weighting: 30 %**

Outcomes to be assessed:

- MEX 12-1
- MEX 12-2
- MEX 12-3
- MEX 12-4
- MEX 12-5
- MEX 12-6
- MEX 12-7
- MEX 12-8

Task: Examination

Assessment Criteria

For this task students will be assessed according to their ability to:

- use concepts, skills and techniques to solve mathematical problems in a wide range of theoretical and practical contexts
- apply reasoning and communication in appropriate forms to construct mathematical arguments and proofs and to interpret and use mathematical models

Mathematics Extension 2 Outcomes

Year 12

Objectives	Year 12 Extension 2 Outcomes
Students will:	A student:
• develop efficient strategies to solve complex problems using pattern recognition, generalisation, proof and modelling techniques	**MEX12-1** understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts
• develop their knowledge, skills and understanding to model and solve complex and interconnected problems in the areas of proof, vectors and mechanics, calculus and complex numbers	**MEX12-2** chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings **MEX12-3** uses vectors to model and solve problems in two and three dimensions **MEX12-4** uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems **MEX12-5** applies techniques of integration to structured and unstructured problems **MEX12-6** uses mechanics to model and solve practical problems
• develop their problem-solving and reasoning skills to create appropriate mathematical models in a variety of forms and apply these to difficult unstructured problems	**MEX12-7** applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems
• use mathematics as an effective means of communication and justification in complex situations	**MEX12-8** communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

NSW Department of Education

Sydney Distance Education High School

2020
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION (TASK 4)

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- NESA-approved calculators may be used.
- A Reference Sheet for Mathematics Advanced, Mathematics Extension 1, Mathematics Extension 2 is given at the end of the paper.
- For Questions 11-16 show relevant mathematical reasoning and/or calculations.

Total Marks: 100

Section I – 10 Marks (Pages 6-9)

- Attempt Questions 1-10.
- Allow about 15 minutes for this section.
- Record answers on multiple answer sheet provided.

Section II – 90 Marks (Pages 10-16)

- Attempt Questions 11-16.
- Allow about 2 hours 45 minutes for this section
- Record your solutions in the writing booklets provided.
- Show all necessary working.

Mathematics

HSC

Extension 2

Assessment Task 4

2020

Answer Sheet to Section I

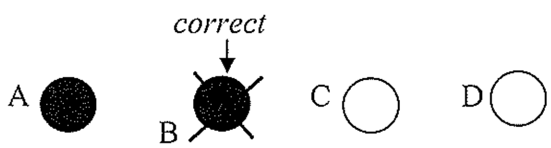
Use the multiple-choice answer sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:



Mathematics

HSC

Extension 2

Assessment Task 4

2020

Answer Sheet to Section I

Name: ___________________________________

1	A	B	C	D
2	A	B	C	D
3	A	B	C	D
4	A	B	C	D
5	A	B	C	D
6	A	B	C	D
7	A	B	C	D
8	A	B	C	D
9	A	B	C	D
10	A	B	C	D

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. Given that $a > b > 0$, which of the following is not always true?

   (A) $a^2 + b^2 > 2ab$

   (B) $a^2 + b^2 > 4ab$

   (C) $(a+b)^2 > 4ab$

   (D) $a + b > 2\sqrt{ab}$

2. If $f(x) = x^5 - 3x^4 - 5x^3 + 15x^2 + 5x - 12$ then $f(x) = x$ for x an integer.
   A counter example is provided by $x =$?

   (A) -2

   (B) 1

   (C) 3

   (D) 4

3. Which of the following is a primitive of $\dfrac{e^{\sqrt{x}}}{\sqrt{x}}$?

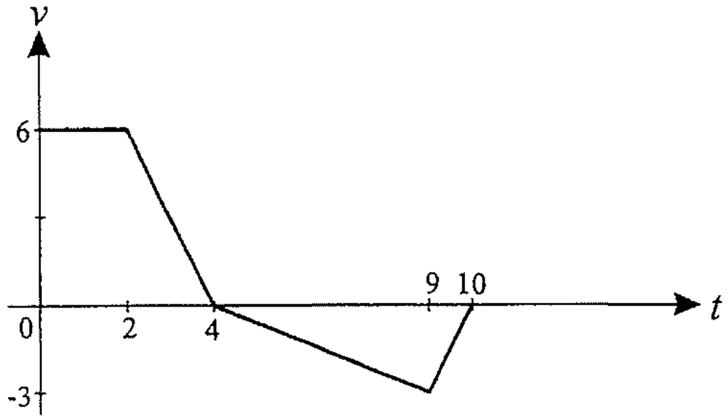
   (A) $2e^{\sqrt{x}}$

   (B) $\left(e^{\sqrt{x}}\right)^2$

   (C) $\ln\left(e^{-\sqrt{x}}\right)$

   (D) $\sqrt{x}e^{\sqrt{x}}$

4.



The diagram above shows the velocity-time graph of an object that moves over a 10 second time interval. For what percentage of the time is the speed of the object decreasing?

(A) 30%

(B) 60%

(C) 70%

(D) It cannot be determined from the graph.

5. If $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 6\underset{\sim}{j} + 3\underset{\sim}{k}$, then the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is:

(A) -1

(B) -7

(C) $-\dfrac{1}{\sqrt{26}}$

(D) $-\dfrac{7}{\sqrt{26}}$

6. By considering the scalar product $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ and $2\underset{\sim}{i} + 4\underset{\sim}{j} - 5\underset{\sim}{k}$, the maximum value of $2x + 4y - 5z$ subject to the constraint $x^2 + y^2 + z^2 = 20$ is:

(A) 29

(B) 30

(C) 31

(D) 32

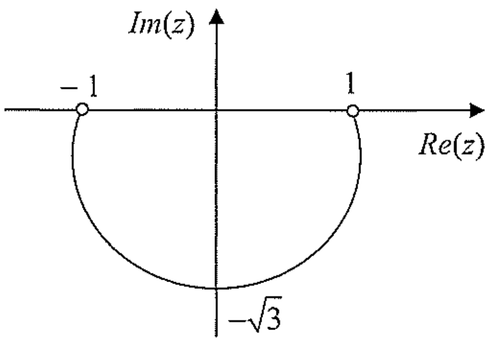
7. If $\int_1^4 f(x)\,dx = 6$, what is the value of $\int_1^4 f(5-x)\,dx$?

(A) 6
(B) 3
(C) -1
(D) -6

8. Consider the integral $I = \int_{-2}^{4} x^3\sqrt{16-x^2}\,dx$. Which is a true statement?

(A) $I = \int_2^4 x^3\sqrt{16-x^2}\,dx$
(B) $I = 2\int_0^2 x^3\sqrt{16-x^2}\,dx + \int_2^4 x^3\sqrt{16-x^2}\,dx$
(C) $I = \int_{-4}^{-2} x^3\sqrt{16-x^2}\,dx$
(D) $I = \int_{-4}^{-2} x^3\sqrt{16-x^2}\,dx + 2\int_{-2}^{0} x^3\sqrt{16-x^2}\,dx$

9.

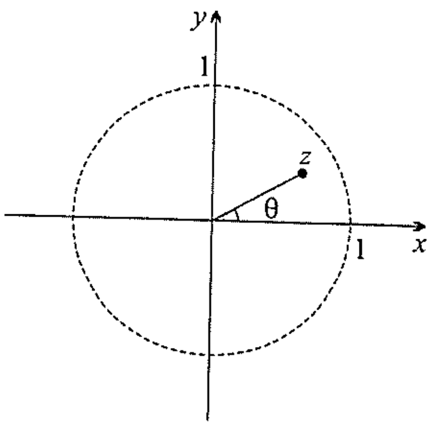


The equation, in complex form, of the circular arc shown above, not including the points $(-1,0)$ and $(1,0)$, is :

(A) $\arg\left(\dfrac{z+1}{z-1}\right) = \dfrac{\pi}{6}$ (B) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{6}$

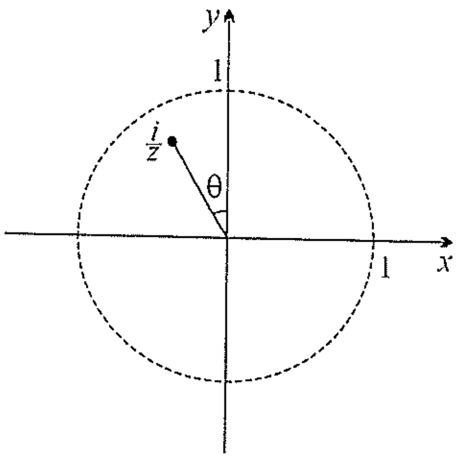
(C) $\arg\left(\dfrac{z+1}{z-1}\right) = \dfrac{\pi}{3}$ (D) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{3}$

10.

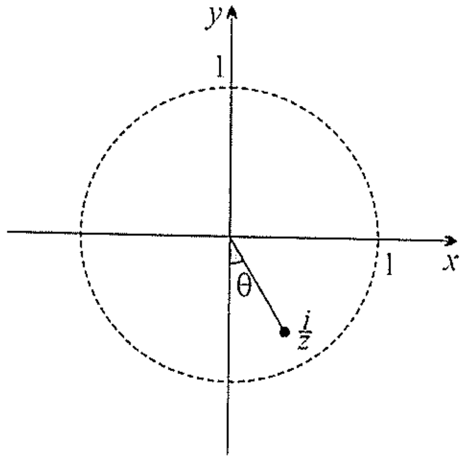


The Argand diagram above shows the complex number z. By considering the modulus and argument, which diagram below best represents the complex number $\frac{i}{z}$?

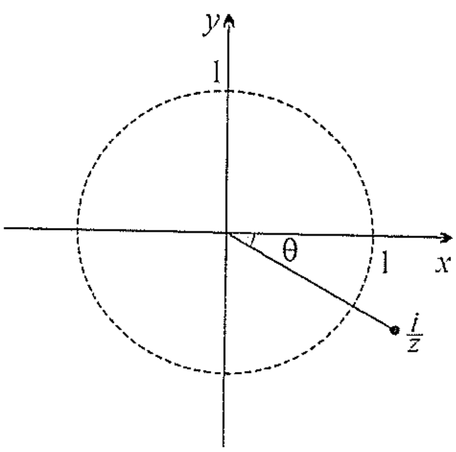
(A)



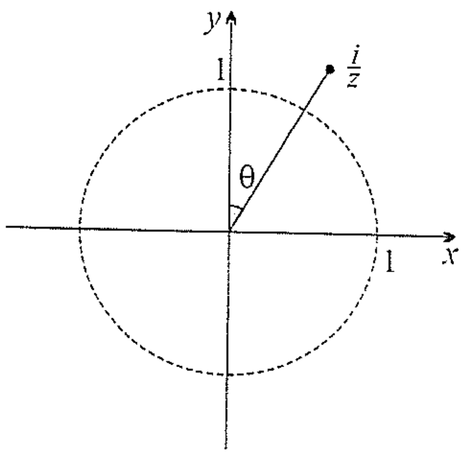
(B)



(C)



(D)



End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Start a new page for each question.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Use the provided booklets to write your solutions. Extra booklets will be provided if necessary.

Question 11 (15 marks) **Marks**

a) If $z = ai$ is a solution to the equation

$$z^2 + (1-i)z + (2-2i) = 0$$

find the real value of a. **2**

b) If $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$, convert z_1 and z_2 into the exponential form of a complex number and hence find $\frac{z_1}{z_2}$. **2**

c) Prove that, if n is an odd integer, then $n^3 + n^2 - n - 1$ is divisible by 8. **2**

d) Find a vector of length 4 units which is parallel to $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$. **2**

e) $ABCD$ is a parallelogram. The position vectors of A, B and D are $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ respectively. Find the position vector of C. **3**

f) (i) Find the real numbers a, b and c such that

$$\frac{x^3 + 5x^2 + x + 2}{x^2(x^2+1)} \equiv \frac{x+a}{x^2} + \frac{bx+c}{x^2+1}.$$ **2**

(ii) Find $\int \frac{x^3 + 5x^2 + x + 2}{x^2(x^2+1)}dx$. **2**

Question 12 (15 marks) Please start a new answer booklet to record your solutions. **Marks**

a) (i) Use the arithmetic-geometric means inequality to prove that

$$e^a + e^b \geq 2e^{\frac{a+b}{2}} \text{ for all real } a, b. \quad \textbf{1}$$

(ii) Hence find the minimum value of $e^{-2x} + e^{-x} + 1 + e^x + e^{2x}$ for all real x. **2**

b) The point P on Argand diagram represents the complex number z, where z satisfies

$$\frac{1}{z} + \frac{1}{\overline{z}} = 1.$$

Give a geometrical description of the locus of P as z varies. **3**

c) The polynomial equation $2z^3 + az^2 + bz + 5 = 0$ where a and b are real, has $z = 2 - i$ as one of its roots.

Find a quadratic factor of $P(z)$ and hence calculate the constants a and b. **4**

d) i. Show that the derivative of $\sec\theta$ is $\sec\theta\tan\theta$. **1**

ii. Use the substitution $x = \sec\theta - 1$ to find the exact value of

$$\int_{\sqrt{2}-1}^{1} \frac{2}{(x+1)\sqrt{x^2+2x}}\,dx. \quad \textbf{4}$$

Question 13 (15 marks) Please start a new answer booklet to record your solutions. **Marks**

a) Let $t = \tan\frac{1}{2}x$.

i) Show that $dx = \frac{2}{1+t^2}dt$. **1**

ii) Hence find $\int \frac{1}{3\sin x + 4\cos x + 5}dx$. **2**

b) Prove that $\frac{1}{3}a + \frac{3}{4}b \geq \sqrt{ab}$ where a and b are positive real numbers. **3**

c) The angle between the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\frac{\pi}{3}$.

If $|\underset{\sim}{a}| = 1$ and $|\underset{\sim}{b}| = 2$, find the angle between $2\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a}$. **4**

d) In the air polluted city of Thaams a resident asthma sufferer is told by his doctor that he can only go outside his home when the pollution reading is less than 60 parts per million (ppm). For each day in Thaams the pollution is lowest at 5 am with a reading of 10 ppm and highest at 2 pm with a reading of 90 ppm.

Assuming that the level of pollution is simple harmonic in nature determine the earliest time interval after 5 am that the asthma sufferer is NOT to go outside his home. (to the nearest minute) **5**

Question 14 (15 marks) Please start a new answer booklet to record your solutions. **Marks**

a) i) Use the substitution $x=\theta+\frac{\pi}{4}$ to show that

$$\int \frac{\cos x}{\cos x+\sin x}\,dx=\frac{1}{2}\int \frac{\cos\theta-\sin\theta}{\cos\theta}\,d\theta. \qquad \mathbf{2}$$

ii) Hence show that $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{1+\tan x}\,dx=\frac{1}{8}\left(\pi+\log_e 4\right)$. **2**

b) Let $\underset{\sim}{a}=\begin{pmatrix}1\\0\\2\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}1\\1\\3\end{pmatrix}$ and $\underset{\sim}{c}=\begin{pmatrix}4\\-2\\6\end{pmatrix}$ be the position vectors of the points A, B and C respectively.

i) Find an equation of the line L which passes through A and B. **2**

ii) Hence or otherwise find the shortest distance from C to the line L. **3**

c) A particle moving along the x – axis starts at the origin with an initial velocity v_0 and acceleration is given by $\frac{d^2x}{dt^2}=4x^3-28x$.

i) Given that the initial velocity is $v_0=3\sqrt{10}$, express v^2 in terms of x and prove that the particle remains at all times in the region $-\sqrt{5}\le x\le\sqrt{5}$. **4**

ii) If the initial velocity is $v_0=12$, discuss the nature of the subsequent motion. **2**

Question 15 (15 marks) Please start a new answer booklet to record your solutions. **Marks**

a) Prove by contradiction that if a, b are integers then

$$a^2 - 4b - 3 \neq 0.$$ **4**

b) A function $f(x)$ is such that $f(x) > 0$ for all real numbers x and $f(a+b) = f(a) \times f(b)$ for any real numbers a and b.

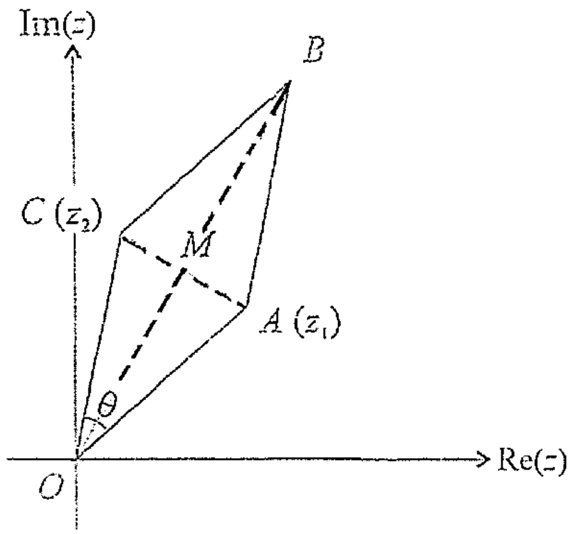
(i) Show that $f(0) = 1$ and deduce that $f(-x) = \dfrac{1}{f(x)}$. **3**

(ii) Use the method of mathematical induction to show that $f(nx) = [f(x)]^n$ for all positive integers n. **2**

(iii) Without using mathematical induction again, deduce that

$$f(-nx) = [f(x)]^{-n} \text{ for all positive integers } n.$$ **1**

c)



The diagram shows a parallelogram $OABC$ in the complex plane. Point M is the point of intersection of diagonals OB and AC. The vertices A and C correspond to complex numbers z_1 and z_2 respectively, with $\frac{z_1+z_2}{z_1-z_2}=3i$.

(i) Write down, in terms of z_1 and z_2, the complex numbers represented by the vertex B and the vector $\overrightarrow{CA}$. **1**

(ii) Explain why $|z_1|=|z_2|$. **2**

(iii) If the angle between vectors $\overrightarrow{OC}$ and $\overrightarrow{OA}$ is θ, show that $\tan\theta=\frac{3}{4}$. **2**

Question 16 (15 marks) Please start a new answer booklet to record your solutions. **Marks**

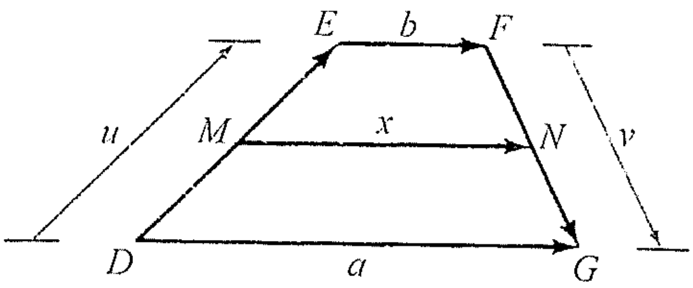
a) Let $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}}\,dx, \quad n = 0, 1, 2, 3, \ldots\ldots\ldots$

i) Show that $(2n+1)I_n = 2\sqrt{2} - 2n\,I_{n-1}$ for $n = 1, 2, 3, \ldots\ldots\ldots$. **3**

ii) Evaluate $\int_0^1 \frac{x^3}{\sqrt{x+1}}\,dx$. **2**

b) In the trapezium $DGFE$ shown in the diagram, M is the midpoint of DE and N is the midpoint of FG.

Let $\overrightarrow{DG}, \overrightarrow{MN}, \overrightarrow{EF}, \overrightarrow{DE}$ and $\overrightarrow{FG}$ be $\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}, \underset{\sim}{u}$ and $\underset{\sim}{v}$ respectively.



i) Write $\underset{\sim}{a}$ in terms of $\underset{\sim}{u}, \underset{\sim}{b}$ and $\underset{\sim}{v}$ only. **1**

ii) Write $\underset{\sim}{a}$ in terms of $\underset{\sim}{u}, \underset{\sim}{x}$ and $\underset{\sim}{v}$ only. **1**

iii) Hence prove that MN (called the median of the trapezium) is parallel to DG and EF (called bases) and its length is one-half the sum of the lengths of the bases. **3**

c) Let $w = cos\frac{2\pi}{9} + i\,sin\frac{2\pi}{9}$.

(i) Show that w^k is a solution of $z^9 - 1 = 0$, where k is an integer. **1**

(ii) Prove that $w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 = -1$ **1**

(ii) Hence show that $\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right) = \frac{1}{8}$. **3**

(You may use the rule: $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$.)

End of Section II and the examination

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\text{and } \alpha\beta\gamma = -\frac{d}{a}$$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

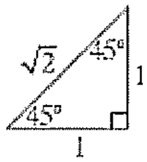
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

$A = \frac{1}{2}ab\sin C$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

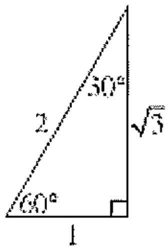


$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \frac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$\cos A = \frac{1-t^2}{1+t^2}$

$\tan A = \frac{2t}{1-t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

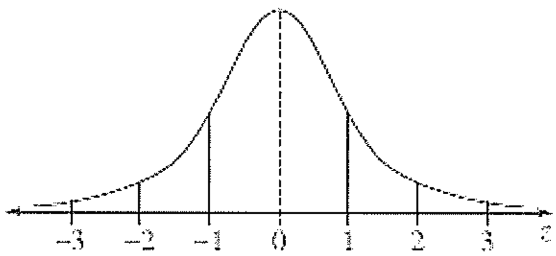
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X-\mu)^2\right] = E(X^2) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1-p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \binom{n}{x}p^x(1-p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1-p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$= r^n e^{in\theta}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

NSW Department of Education

Sydney Distance Education High School

HSC Course

Mathematics

Extension 2

Assessment Task 4

Term 3 2020

Time Allowed: 3 hours + 10 minutes reading.

Solutions

Mathematics

HSC

Extension 2

Assessment Task 4

Term 3 2020

Answers Section I

	A	B	C	D
1	A	**B**	C	D
2	A	B	C	**D**
3	**A**	B	C	D
4	**A**	B	C	D
5	**A**	B	C	D
6	A	**B**	C	D
7	A	B	C	**D**
8	**A**	B	C	D
9	A	B	**C**	D
10	A	B	C	**D**

Section I

10 marks **Answer**

1. Consider B ie. $a^2 + b^2 > 4ab$.

$$\text{Difference} = a^2 + b^2 - 4ab$$
$$= (a-b)^2 - 2ab$$

The difference is not always positive. **B**

2. $f(x) = x^5 - 3x^4 - 5x^3 + 15x^2 + 5x - 12$

$f(4) = 1024 - 768 - 320 + 240 + 20 - 12$
$= 184$
$\neq 4$ **D**

3. $\dfrac{d2e^{\sqrt{x}}}{dx} = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$ **A**

4. Speed is decreasing for $2 < t < 4$ and $9 < t < 10$.

$\dfrac{3}{10} = 30\%$ **A**

5. Scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$
$$= \frac{(4\underset{\sim}{i} + 3\underset{\sim}{i} + \underset{\sim}{k}) \cdot (2\underset{\sim}{i} - 6\underset{\sim}{j} + 3\underset{\sim}{k})}{|2\underset{\sim}{i} - 6\underset{\sim}{j} + 3\underset{\sim}{k}|}$$
$$= \frac{8 - 18 + 3}{\sqrt{4 + 36 + 9}}$$
$$= \frac{-7}{7}$$
$$= -1.$$ **A**

6. The scalar product of the two given vectors is given by

$$\left(x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}\right)\cdot\left(2\underset{\sim}{i}+4\underset{\sim}{j}-5\underset{\sim}{k}\right)=2x+4y-5z.$$

If θ is the angle between the two given vectors, then

$$\begin{aligned}2x+4y-5z&=\sqrt{x^2+y^2+z^2}\sqrt{2^2+4^2+5^2}\cos\theta\\&=\sqrt{20}\sqrt{45}\cos\theta\\&=30\cos\theta.\end{aligned}$$

Since $-1\le\cos\theta\le 1$, the maximum value of $2x+4y-5z$ is 30 **B**

7. Let $x=5-u$ $\quad$ $x=4\rightarrow u=1$

$dx=-du$ $\quad$ $x=1\rightarrow u=4$

$$\int_1^4 f(x)\,dx=6$$

$$\int_4^1 f(5-u)\times -du=6$$

$$-\int_4^1 f(5-u)\,du=6$$

$$\int_1^4 f(5-u)\,du=-6$$

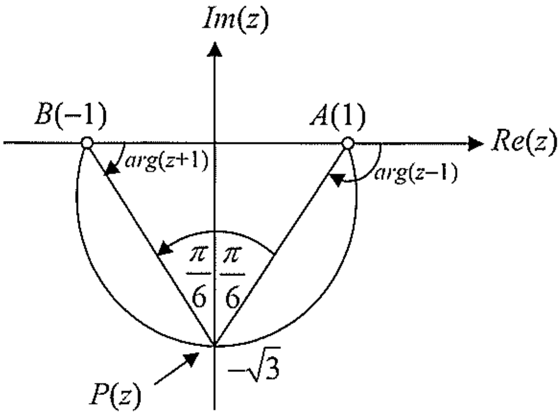
D

8. $I=\int_{-2}^{4}x^3\sqrt{16-x^2}\,dx$ $\quad$ $f(x)=x^3\sqrt{16-x^2}$ is an odd function.

$$\begin{aligned}&=\int_{-2}^{2}x^3\sqrt{16-x^2}\,dx+\int_{2}^{4}x^3\sqrt{16-x^2}\,dx\\&=0+\int_{2}^{4}x^3\sqrt{16-x^2}\,dx\end{aligned}$$

A

9.



Let $P(z)$ be as shown.

$\angle APB = \frac{\pi}{3}$

This excludes **A** and **B**.

Consider **D**.

$$\arg(z-1) - \arg(z+1) = \frac{\pi}{3}$$

$$\underbrace{\arg(z-1)}_{\text{negative}} = \frac{\pi}{3} + \underbrace{\arg(z+1)}_{\text{negative}}$$

e.g. $-110^\circ = 60^\circ - 50^\circ$?

Reject **D**.

Consider **C**.

$$\arg(z+1) - \arg(z-1) = \frac{\pi}{3}$$

$$\underbrace{-\arg(z-1)}_{\text{positive}} = \frac{\pi}{3} \underbrace{-\arg(z+1)}_{\text{positive}}$$ **C**

10. $\left|\frac{i}{z}\right| = \frac{|i|}{|z|} = \frac{1}{|z|}$

Since $|z| < 1$

$$\frac{1}{|z|} > 1$$

$\therefore \left|\frac{i}{2}\right| > 1 \quad \therefore C$ or D.

$$\arg\frac{i}{z} = \arg i - \arg z$$

$$= \frac{\pi}{2} - \theta$$

Since $\quad 0 < \theta < \frac{\pi}{2}$

$$0 > -\theta > -\frac{\pi}{2}$$

$$-\frac{\pi}{2} < \theta < 0$$

$$0 < \frac{\pi}{2} - \theta < \frac{\pi}{2}$$

$$0 < \arg\frac{i}{z} < \frac{\pi}{2}$$

Note: $(A) \quad iz \qquad (B) \quad -iz \qquad (C) \quad \frac{1}{z}$ **D**

Section II

90 marks

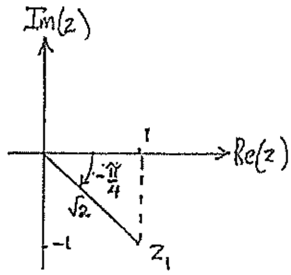
Question 11 (15 marks) **Marks**

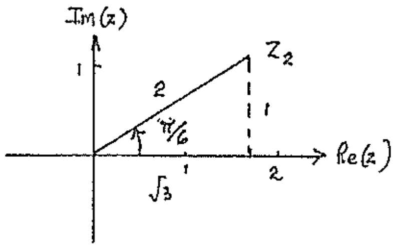
a)
$$(ai)^2+(1-i)ai+(2-2i)=0$$
$$-a^2+ai+a+2-2i=0$$
$$(-a^2+a+2)+i(a-2)=0 \quad \checkmark$$
$$-a^2+a+2=0 \text{ and } a-2=0$$
$$(1+a)(2-a)=0 \qquad a=2$$
$$a=2 \text{ or } a=-1$$

The result $a=2$ holds for both equations. $\checkmark$

$\therefore a=2$ **2**

b)





$r=\sqrt{2}$ $\qquad$ $r=2$

$\theta=\dfrac{-\pi}{4}$ $\qquad$ $\theta=\dfrac{\pi}{6}$

$$z_1=1-i = \sqrt{2}e^{\frac{-i\pi}{4}} \qquad z_2=\sqrt{3}+i = 2e^{\frac{i\pi}{6}} \quad \checkmark$$

$$\frac{z_1}{z_2}=\frac{\sqrt{2}e^{\frac{-i\pi}{4}}}{2e^{\frac{i\pi}{6}}} = \frac{\sqrt{2}}{2}e^{\frac{-5i\pi}{12}} \quad \checkmark$$

2

Question 11 (continued) **Marks**

c) Let $n = 2k + 1$.

$$\begin{aligned} n^3 + n^2 - n - 1 &= n^2(n+1) - 1(n+1) \\ &= (n+1)(n^2 - 1) \\ &= (2k+1+1)(4k^2 + 4k + 1 - 1) \quad \checkmark \\ &= (2k+2)(4k^2 + 4k) \\ &= 8(k+1)(k^2 + k) \\ &= 8M \text{ where } M \text{ is an integer.} \quad \checkmark \end{aligned}$$

$n^3 + n^2 - n - 1$ is divisible by 8. **2**

d) $\left|\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}\right| = \sqrt{9+4+1}$

$= \sqrt{14}$

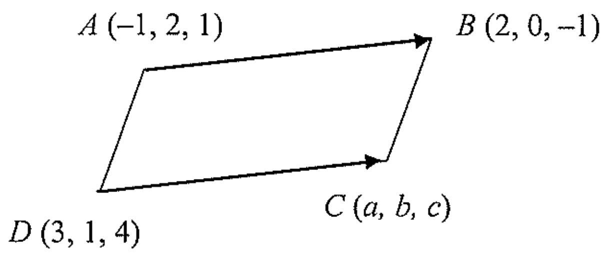
$\dfrac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ is a unit vector. ✓

$\dfrac{\pm 4}{\sqrt{14}}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ are two (only one is required) vectors

of length 4 units parallel to $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$. ✓ **2**

Question 11 (continued) **Marks**

e) Let C be $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.



$$\overrightarrow{AB} = \overrightarrow{DC}$$

$$\begin{pmatrix} a-3 \\ b-1 \\ c-4 \end{pmatrix} = \begin{pmatrix} 2+1 \\ 0-2 \\ -1-1 \end{pmatrix}$$

$$\begin{aligned} a-3 &= 3 \\ a &= 6 \end{aligned} \qquad \begin{aligned} b-1 &= -2 \\ b &= -1 \end{aligned} \qquad \begin{aligned} c-4 &= -2 \\ c &= 2 \end{aligned}$$

C is $\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$. **3**

f) (i)
$$\frac{x^3+5x^2+x+2}{x^2\left(x^2+1\right)} = \frac{x+a}{x^2} + \frac{bx+c}{x^2+1}$$

$$= \frac{(x+a)\left(x^2+1\right)+x^2(bx+c)}{x^2\left(x^2+1\right)}$$

$$x^3+5x^2+x+2 = (b+1)x^3+(a+c)x^2+x+a.$$

Method 1. Equate coefficients.

$$\begin{aligned} b+1 &= 1 \\ b &= 0 \end{aligned} \qquad \begin{aligned} a+c &= 5 \\ c &= 3 \end{aligned} \qquad a = 2$$

Method 2.

Let $x=0$. $2 = a$

Let $x=1$. $9 = b+1+2+c+1+2$

$3 = b+c$

Let $x=-1$. $5 = -b-1+2+c-1+2$

$3 = -b+c$

Add. $6 = 2c$

$c = 3$

$b = 0$

$a=2,\ b=0,\ c=3$. **2**

Question 11 (f) (continued) **Marks**

(ii) $$\int \frac{x^3+5x^2+x+2}{x^2(x^2+1)}\,dx = \int\left(\frac{x+2}{x^2}+\frac{3}{x^2+1}\right)dx$$

$$= \int\left(\frac{1}{x}+2x^{-2}+\frac{3}{1+x^2}\right)dx \quad \checkmark$$

$$= \ell n|x| - \frac{2}{x} + 3\tan^{-1}x + c \quad \checkmark$$ **2**

Question 12 (15 marks) **Marks**

a) (i) Since e^a and e^b are both positive for all real a and b, then

$$\frac{e^a+e^b}{2} \geq \sqrt{e^a e^b}$$

$$e^a+e^b \geq 2\left(e^{a+b}\right)^{\frac{1}{2}}$$

$$e^a+e^b \geq 2e^{\frac{a+b}{2}}$$

1

(ii) $e^{-2x}+e^{-x}+1+e^x+e^{2x}$

$$=\left(e^{2x}+e^{-2x}\right)+\left(e^x+e^{-x}\right)+1$$

$$\geq 2e^{\frac{2x-2x}{2}}+2e^{\frac{x-x}{2}}+1 \qquad \text{(using (i))}$$

$$=2+2+1$$

$$=5$$

The minimum value is 5. 2

b) Let $z=x+iy$.

$$\frac{1}{z}+\frac{1}{z}=1$$

$$\frac{1}{x+iy}+\frac{1}{x-iy}=1$$

$$\frac{x-iy+x+iy}{x^2+y^2}=1$$

$$\frac{2x}{x^2+y^2}=1$$

$$2x=x^2+y^2$$

$$x^2-2x+y^2=0$$

$$(x-1)^2+y^2=1$$

The locus is the circle of centre (1,0), radius 1, but excluding the origin as $z \neq 0$. 3

Question 12 (continued) **Marks**

c) $z = 2 - i$ is a root implies $z = 2 + i$ is also a root (coefficients are real). ✓

$\{z-(2-i)\}\{z-(2+i)\}$ is a factor.

$z^2 - 4z + 5$ is the quadratic factor. ✓

$2z^3 + az^2 + bz + 5 = (z^2 - 4z + 5)(2z + 1)$ ✓

$2z^3 + az^2 + bz + 5 = 2z^2 - 7z^2 + 6z + 5.$ ✓

$a = -7,\ b = 6$ **4**

d) (i) $\dfrac{d\sec\theta}{d\theta} = \dfrac{d(\cos\theta)^{-1}}{d\theta}$

$= -(\cos\theta)^{-2} \times -\sin\theta$

$= \dfrac{1}{\cos\theta} \times \dfrac{\sin\theta}{\cos\theta}$ ✓

$= \sec\theta\tan\theta$ **1**

(ii) $x = \sec\theta - 1$

$dx = \sec\theta\tan\theta\, d\theta$

$$\int_{\sqrt{2}-1}^{1} \frac{2}{(x+1)\sqrt{x^2+2x}}\,dx$$ ✓

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sec\theta\tan\theta d\theta}{(\sec\theta)\sqrt{(\sec\theta-1)^2 + 2(\sec\theta-1)}}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\tan\theta\, d\theta}{\sqrt{\sec^2\theta - 2\sec\theta + 1 + 2\sec\theta - 2}}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\tan\theta\, d\theta}{\sqrt{\tan^2\theta}}$$ ✓

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\, d\theta$$

$$= 2[\theta]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{6}$$ ✓ **4**

Limits: $x = 1 \rightarrow 1 = \sec\theta - 1$

$\sec\theta = 2$

$\cos\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{3}$

$x = \sqrt{2} - 1 \rightarrow \sqrt{2} - 1 = \sec\theta - 1$

$\sec\theta = \sqrt{2}$

$\cos\theta = \dfrac{1}{\sqrt{2}}$

$\theta = \dfrac{\pi}{4}$ ✓

Question 13 (15 marks) **Marks**

a) (i) **Method 1:** $t = \tan\frac{1}{2}x$

$$\frac{dt}{dx} = \sec^2\frac{1}{2}x \times \frac{1}{2}$$

$$\frac{dx}{dt} = \frac{2}{\sec^2\frac{1}{2}x}$$

$$= \frac{2}{1+\tan^2\frac{1}{2}x}$$

$$\frac{dx}{dt} = \frac{2}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2} \quad \checkmark$$

Method 2: $t = \tan\frac{1}{2}x$

$$\tan^{-1}t = \frac{1}{2}x$$

$$x = 2\tan^{-1}t$$

$$\frac{dx}{dt} = \frac{2}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2} \quad \checkmark$$

1

(ii) $$\int\frac{1}{3\sin x+4\cos x+5}dx$$

$$= \int\frac{1}{\frac{6t}{1+t^2}+\frac{4\left(1-t^2\right)}{1+t^2}+5}\times\frac{2}{1+t^2}dt$$

$$= \int\frac{2dt}{6t+4-4t^2+5+5t^2}$$

$$= \int\frac{2dt}{t^2+6t+9} \quad \checkmark$$

$$= 2\int\left(t+3\right)^{-2}dt$$

$$= \frac{2\left(t+3\right)^{-1}}{-1}+c$$

$$= \frac{-2}{3+\tan\frac{1}{2}x}+c \quad \checkmark$$

2

b) **Method 1:**

$$\text{Difference} = \frac{1}{3}a + \frac{3}{4}b - \sqrt{ab}$$
$$= \frac{4a - 12\sqrt{ab} + 9b}{12}$$
$$= \frac{\left(2\sqrt{a} - 3\sqrt{b}\right)^2}{12}$$
$$\geq 0$$

✓ ✓ ✓

Method 2:

$$\frac{1}{3}a + \frac{3}{4}b \geq \sqrt{ab}$$
$$4a + 9b \geq 12\sqrt{ab}$$
$$16a^2 + 72ab + 81b^2 \geq 144ab$$
$$16a^2 - 72ab + 81b^2 \geq 0$$
$$(4a - 9b)^2 \geq 0 \quad (1)$$

✓

This statement is true (squares can't be negative) ✓

Now cross out all of the above and reverse steps starting with (1).

$$(4a - 9b)^2 \geq 0$$
$$16a^2 - 72ab + 81b^2 \geq 0$$
$$16a^2 + 72ab + 81b^2 \geq 144ab$$
$$\frac{(4a - 9b)^2}{144} \geq ab$$
$$\frac{4a - 9b}{144} \geq \sqrt{ab}$$
$$\frac{1}{3}a + \frac{3}{4}b \geq \sqrt{ab}$$

✓ 3

c) **Method 1:**

Aim: Find $\cos\theta = \dfrac{(2\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{a}}{|2\underset{\sim}{a}+\underset{\sim}{b}||\underset{\sim}{a}|}$

$$\underset{\sim}{a}\cdot\underset{\sim}{a}=|\underset{\sim}{a}|^2=1 \qquad \underset{\sim}{b}\cdot\underset{\sim}{b}=|\underset{\sim}{b}|^2=4$$

$$\begin{aligned}\underset{\sim}{a}\cdot\underset{\sim}{b}&=|\underset{\sim}{a}||\underset{\sim}{b}|\cos\frac{\pi}{3}\\&=1\times2\times\frac{1}{2}\\&=1 \quad \checkmark\end{aligned}$$

$$\begin{aligned}|2\underset{\sim}{a}+\underset{\sim}{b}|^2&=(2\underset{\sim}{a}+\underset{\sim}{b})\cdot(2\underset{\sim}{a}+\underset{\sim}{b})\\&=4\underset{\sim}{a}\cdot\underset{\sim}{a}+4\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{b}\\&=4+4+4\\&=12\end{aligned}$$

$$|2\underset{\sim}{a}+\underset{\sim}{b}|=2\sqrt{3} \quad \checkmark$$

$$\begin{aligned}(2\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{a}&=2\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{a}\cdot\underset{\sim}{b}\\&=2+1\\&=3 \quad \checkmark\end{aligned}$$

$$\begin{aligned}\cos\theta&=\frac{3}{2\sqrt{3}\times1}\\&=\frac{\sqrt{3}}{2}\end{aligned}$$

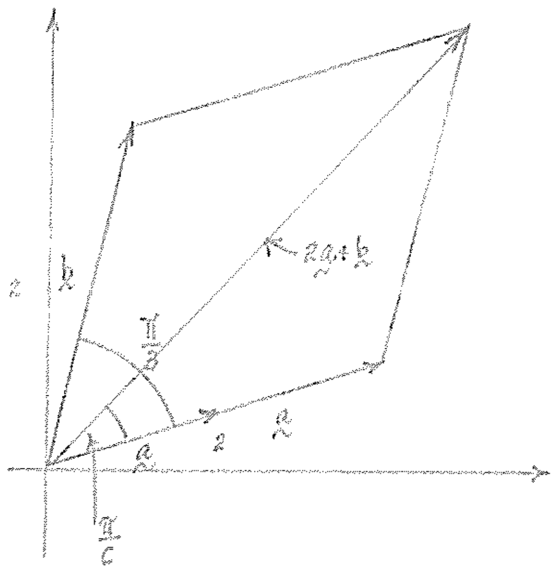
$$\theta=\frac{\pi}{6} \quad \checkmark$$

Method 2: Michael Seberry

The parallelogram is a rhombus (sides equal). ✓

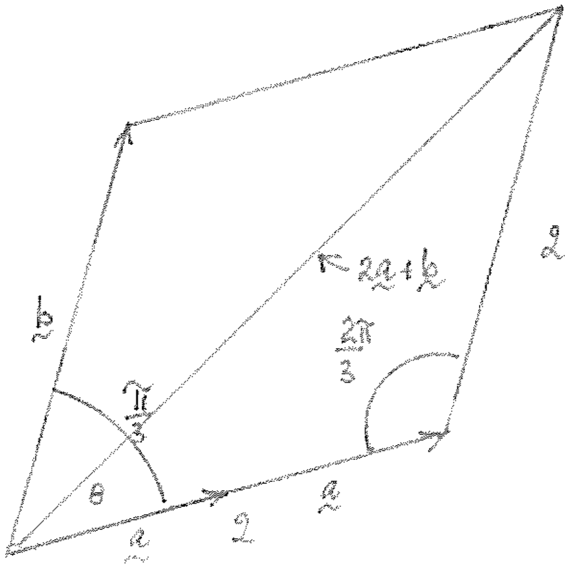
The angle between $2\underset{\sim}{a}+\underset{\sim}{b}$ and $\underset{\sim}{a}$ is $\dfrac{\pi}{6}$ ✓

(diagonal of a rhombus bisects each vertex angle). ✓



✓

Method 3: Nathan Lai, Ky Howie



Let θ = angle between $2\underset{\sim}{a}+\underset{\sim}{b}$ and $\underset{\sim}{a}$.

$$\left|2\underset{\sim}{a}+\underset{\sim}{b}\right|^2 = 2^2+2^2-2(2)(2)\cos\frac{2\pi}{3} \quad \text{(Cosine Rule)}$$

$$\left|2\underset{\sim}{a}+\underset{\sim}{b}\right|^2 = \sqrt{12}$$

$$\frac{2}{\sin\theta} = \frac{\sqrt{12}}{\sin\frac{2\pi}{3}} \quad \text{(Sine Rule)}$$

$$\frac{2}{\sin\theta} = \frac{2\sqrt{2}}{\frac{\sqrt{3}}{2}}$$

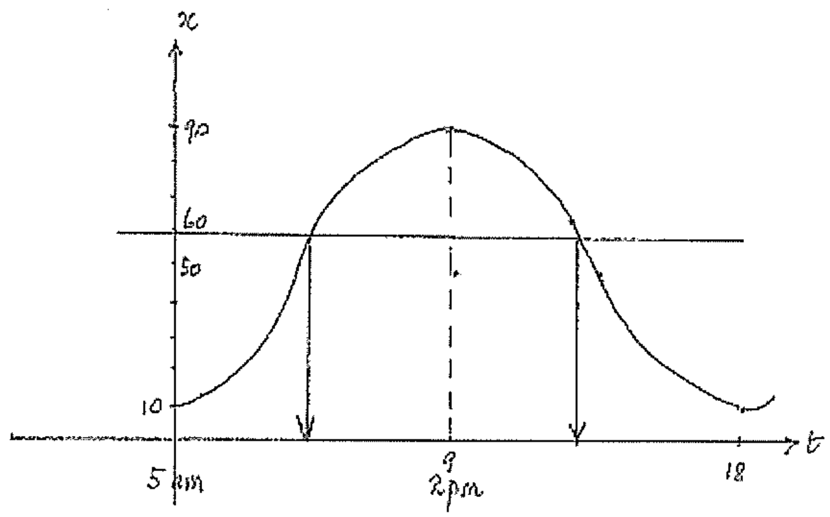
$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

The angle between $2\underset{\sim}{a}+\underset{\sim}{b}$ and $\underset{\sim}{a}$ is $\frac{\pi}{6}$. **4**

Question 13 (continued)

d)



Let t represent the time after 5 am.

Period $= 18$ hours $\quad\quad$ Amplitude $= \dfrac{90-10}{2} = 40$

$$\frac{2\pi}{n} = 18 \quad\quad c = 50$$

$$n = \frac{\pi}{9}$$

$$x = 50 - 40\cos\frac{\pi}{9}t$$

Find t when $x = 60$.

$$60 = 50 - 40\cos\frac{\pi}{9}t$$

$$\cos\frac{\pi t}{9} = -\frac{1}{4}$$

$$\frac{\pi}{9}t = \ldots\ldots\ldots, \quad \pi - \cos^{-1}\frac{1}{4}, \quad \pi + \cos^{-1}\frac{1}{4}, \quad \ldots\ldots\ldots$$

$$t = \ldots\ldots\ldots, \quad \frac{9}{\pi}\left(\pi - \cos^{-1}\frac{1}{4}\right), \quad \frac{9}{\pi}\left(\pi + \cos^{-1}\frac{1}{4}\right), \quad \ldots\ldots\ldots$$

$$= \ldots\ldots\ldots, \quad 5.2238, \quad 12.7761, \quad \ldots\ldots\ldots$$

$\uparrow$ 5 hours 13.428 min $\quad\quad$ $\uparrow$ 12 hours 46.566 min

✓ ✓ ✓ ✓

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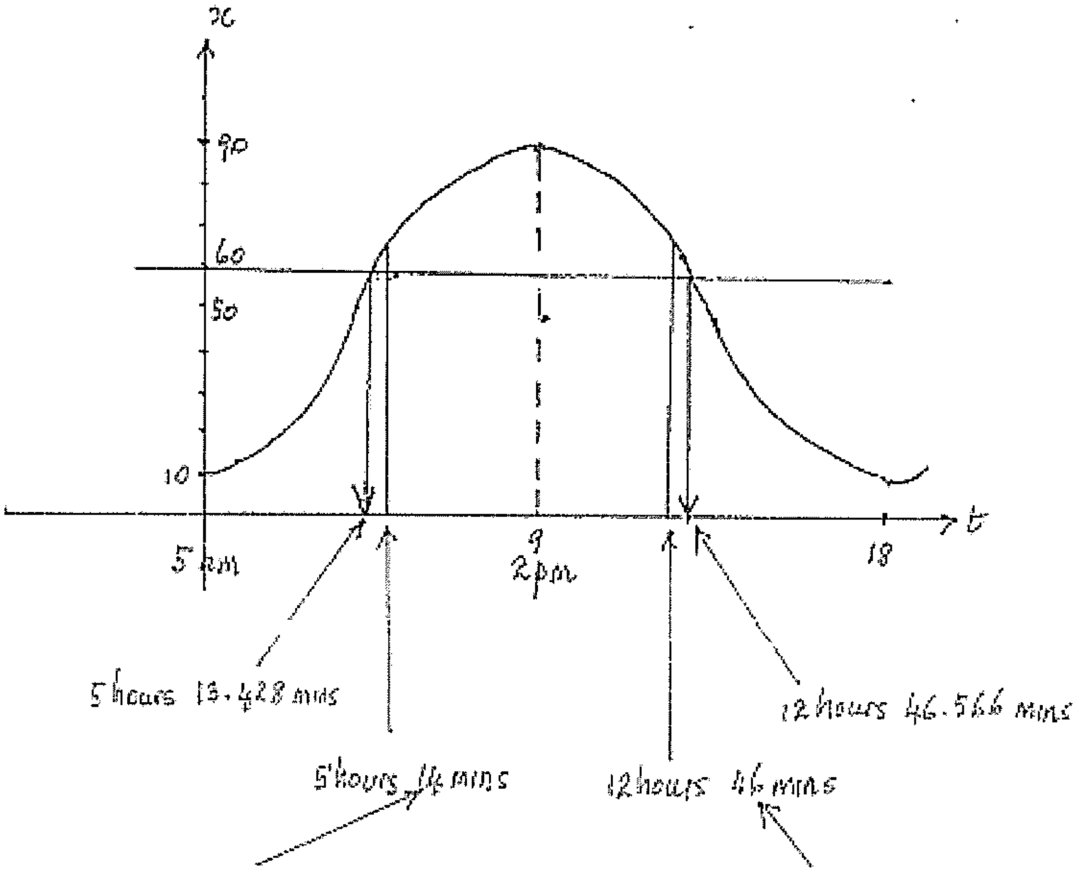
Question 13 d) (continued) **Marks**



The lower value is rounded up (instead of being rounded down to the nearest minute) because the range must not include time before 5 hours 13.428 min.

The upper value is rounded down (instead of being rounded up to the nearest minute) because the range must not include time after 12 hours 46.566 min.

He is not allowed out for: 10.14 am $\leq$ time $\leq$ 5.46 pm . ✓ **5**

Question 14 (15 marks) **Marks**

a) i) $\displaystyle\int \frac{\cos x}{\cos x+\sin x}\,dx$

Let $x=\theta+\frac{\pi}{4}$

$dx=d\theta$

$$=\int \frac{\cos\left(\theta+\frac{\pi}{4}\right)}{\cos\left(\theta+\frac{\pi}{4}\right)+\sin\left(\theta+\frac{\pi}{4}\right)}d\theta \quad \checkmark$$

$$=\int \frac{\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta}{\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta}d\theta$$

$$=\frac{1}{2}\int \frac{\cos\theta-\sin\theta}{\cos\theta}d\theta \quad \checkmark$$

2

ii) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{1+\tan x}\,dx$

$$=\int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos x+\sin x}dx$$

Let $x=\theta+\frac{\pi}{4}$.

x	0	$\frac{\pi}{4}$
θ	$-\frac{\pi}{4}$	0

$$=\frac{1}{2}\int_{-\frac{\pi}{4}}^{0} \frac{\cos\theta-\sin\theta}{\cos\theta}d\theta \quad \text{(From (i))}$$

$$=\frac{1}{2}\int_{-\frac{\pi}{4}}^{0} 1-\tan\theta\,d\theta \quad \checkmark$$

$$=\frac{1}{2}\left[\theta+\log_e|\cos\theta|\right]_{-\frac{\pi}{4}}^{0}$$

$$=\frac{1}{2}\left(0+\log_e 1-\left(-\frac{\pi}{4}+\log_e\frac{1}{\sqrt{2}}\right)\right)$$

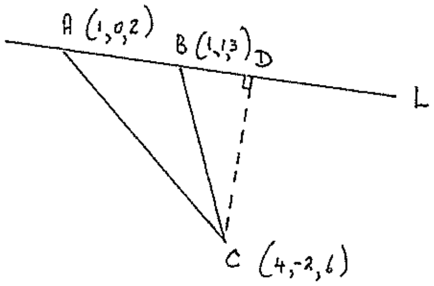
$$=\frac{1}{8}\left(\pi+2\log_e 2\right)$$

$$=\frac{1}{8}\left(\pi+\log_e 4\right) \quad \checkmark$$

2

Question 14 (continued) **Marks**

b)



(i) $\overrightarrow{AB} = \begin{pmatrix} 1-1 \\ 1-0 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$ Note: $\underset{\sim}{b}$ here in the formula is not the same as $\underset{\sim}{b}$ in the question.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is one possible equation of L. **2**

(ii) **Method 1.**

Let D be the foot of the perpendicular from C to L.

$\overrightarrow{OD} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ for the some value of λ.

$\overrightarrow{DC} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ \lambda \\ 2+\lambda \end{pmatrix} = \begin{pmatrix} 3 \\ -2-\lambda \\ 4-\lambda \end{pmatrix}$

$\overrightarrow{DC}$ and $\overrightarrow{AB}$ are perpendicular.

$$\begin{pmatrix} 3 \\ -2-\lambda \\ 4-\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$0 \times 3 + 1(-2-\lambda) + 1(4-\lambda) = 0$$
$$2 - 2\lambda = 0$$
$$\lambda = 1$$

If $\lambda = 1$, $\overrightarrow{DC} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$

$$\left|\overrightarrow{DC}\right| = \sqrt{9+9+9}$$
$$= 3\sqrt{3}$$

The shortest distance is $3\sqrt{3}$ units.

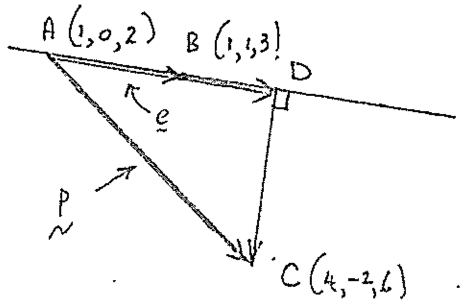
Question 14 b) (ii) (continued) **Marks**

Method 2

Let D be the foot of the perpendicular from C to L.

Let $\overrightarrow{AC} = \underset{\sim}{p}$. $\underset{\sim}{p} = \begin{pmatrix} 4-1 \\ -2-0 \\ 6-2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$

Let $\overrightarrow{AB} = \underset{\sim}{e}$. $\underset{\sim}{e} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$



$$\text{proj}_{\underset{\sim}{e}}\, \underset{\sim}{p} = \frac{\underset{\sim}{p} \cdot \underset{\sim}{e}}{|\underset{\sim}{e}||\underset{\sim}{e}|}\underset{\sim}{e}$$

$$= \frac{\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right|\left|\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right|}\left|\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right|$$

$$= \frac{3(0)-2(1)+4(1)}{1^2+1^2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

D is actually B. $\overrightarrow{AD} = \overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\overrightarrow{DC} = \overrightarrow{AC} - \overrightarrow{AD}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$

$$\left|\overrightarrow{DC}\right| = \sqrt{9+9+9} = 3\sqrt{3}$$

The shortest distance from C to L is $3\sqrt{3}$ units. 3

c) (i)

$$\frac{d^2x}{dt^2} = 4x^3 - 28x$$

$$\frac{d\,\frac{1}{2}v^2}{dx} = 4x^3 - 28x$$

$$\frac{1}{2}v^2 = \int\left(4x^3 - 28x\right)dx$$

$$\frac{1}{2}v^2 = x^4 - 14x^2 + c$$

$$\boxed{\begin{array}{l} v = 3\sqrt{10} \\ x = 0 \end{array}}$$

$$45 = c$$

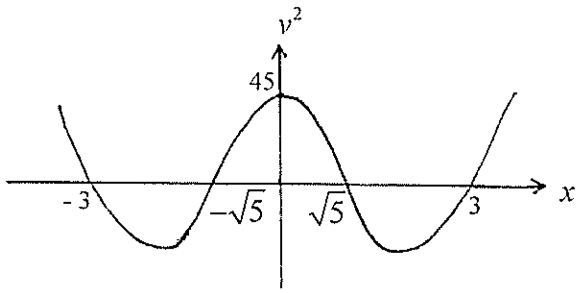
$$\frac{1}{2}v^2 = x^4 - 14x^2 + 45$$

$$v^2 = 2\left(x^4 - 14x^2 + 45\right)$$

$$v^2 = 2\left(x^2 - 5\right)\left(x^2 - 9\right)$$

Solve $v^2 \geq 0$.

Sketch $v^2 = 2\left(x^2 - 5\right)\left(x^2 - 9\right)$



Where is $v^2 \geq 0$?

$x \leq -3$ or $-\sqrt{5} \leq x \leq \sqrt{5}$ or $x \geq 3$.

It starts at $x = 0$.

$\therefore$ Motion is restricted to $-\sqrt{5} \leq x \leq \sqrt{5}$.

i.e. the particle oscillates between $x = -\sqrt{5}$ and $x = \sqrt{5}$. **4**

(ii)

$$\frac{1}{2}v^2 = x^4 - 14x^2 + c$$

$$\boxed{\begin{array}{l} v = 12 \\ x = 0 \end{array}}$$

$$72 = c$$

$$v^2 = 2\left(x^2 - 14x^2 + 72\right)$$

$$v^2 = 2\left\{\left(x^2 - 14x^2 + 49\right) + 23\right\}$$

$$v^2 = 2\left\{\left(x^2 - 7\right)^2 + 23\right\}$$

$\left(x^2 - 7\right)^2 + 23 > 0$ for all x.

$\therefore v^2 > 0$

v can only be positive (since initially $v = 12 > 0$) i.e. v can never be zero and, therefore, the particle can never come to rest (and reverse the motion). The particle must always move to the right. **2**

Question 15 (15 marks) **Marks**

a) Assume there exists integers a, b such that $a^2 = 4b+3$.

$$a^2 = 4b+3$$
$$= 2(2\lambda+1)+1$$

a^2 must be odd.
a must be odd.
Let $a = 2k+1$ for some integer k.

$$(2k+1)^2 = 4b+3$$
$$4k^2+4k+1 = 4b+3$$
$$4k^2+4k-4b = 2$$
$$4(k^2+k-b) = 2$$
$$k^2+k-b = \frac{1}{2}$$

The L.H.S. of this equation is an integer, while the R.H.S. is not.
This is a contradiction.
Therefore, there cannot possibly exist integers a and b such that $a^2 = 4b+3$. **4**

b) (i) $f(a+b) = f(a)f(b)$ for any real numbers a and b (identity)

This is a hint to let $a = b = 0$.

$$f(0) = f(0)f(0)$$
$$f(0) = [f(0)]^2$$
$$0 = [f(0)]^2 - f(0)$$
$$0 = f(0)\{f(0)-1\}$$
$$f(0) = 0 \quad \text{or} \quad f(0) = 1$$

But $f(x) > 0$ for all x.

$$\therefore f(0) = 1$$

Let $\left.\begin{array}{l} a = x \\ b = -x \end{array}\right\}$

$f(a+b) = f(a)f(b)$ becomes $f(0) = f(x)f(-x)$

$$1 = f(x)f(-x)$$
$$\frac{1}{f(x)} = f(-x)$$

3

(ii) **Step 1.** Prove true for $n=1$

L.H.S. $= f(kx)$ R.H.S $= [f(x)]^1$

$= f(x)$

$\therefore$ true for $n=1$

Step 2. Assume true for $n=k$ i.e. $f(kx)=[f(x)]^k$

Prove true for $n=k+1$ i.e. $f\{(k+1)x\}=[f(x)]^{k+1}$ ✓

L.H.S. $= f(kx+x)$

$= f(kx)f(x)$ using $f(a+b)=f(a)f(b)$

$=[f(x)]^k f(x)$

$=[f(x)]^{k+1}$

$=$ R.H.S ✓

Step 3. True for $n=1$ by Step 1.

true for $n=1$ $\rightarrow$ true for $n=2$ by Step 2.
true for $n=2$ $\rightarrow$ true for $n=3$ by Step 2.
etc. True for all positive integral values on n. **2**

(iii) L.H.S. $= f(-nx)$ Use $f(-x)=\dfrac{1}{f(x)}$

$=\dfrac{1}{f(nx)}$ $f(-X)=\dfrac{1}{f(X)}$

$=\dfrac{1}{[f(x)]^n}$ e.g. $f(-nx)=\dfrac{1}{f(nx)}$

$=[f(x)]^{-n}$

$=$ R.H.S. ✓ **1**

Question 15 (continued) **Marks**

c) (i) $\overrightarrow{OB} = z_1 + z_2$ $\quad\quad$ $\overrightarrow{CA} = z_1 - z_2$ ✓ **1**

(ii) **Method 1.** $z_1 + z_2 = 3i\left(z_1 - z_2\right)$

$$\overrightarrow{OB} = 3i\overrightarrow{CA}$$

$\angle AMB = \dfrac{\pi}{2}$ (multiplication by i means anticlockwise rotation through $\dfrac{\pi}{2}$). ✓

$OABC$ is a parallelogram with perpendicular diagonals.
Hence it is a rhombus.

$$OA = OC$$

$$\left|z_1\right| = \left|z_2\right|$$ ✓

Method 2. $z_1 + z_2 = 3i\left(z_1 - z_2\right)$

$$\left(1-3i\right)z_1 = \left(-1-3i\right)z_2$$

$$\frac{z_1}{z_2} = -\frac{1+3i}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= \frac{-\left(1+6i-9\right)}{1+9}$$

$$= \frac{-\left(-8+6i\right)}{10}$$

$$= \frac{4}{5} - \frac{3i}{5}$$ ✓

Take mods of both sides.

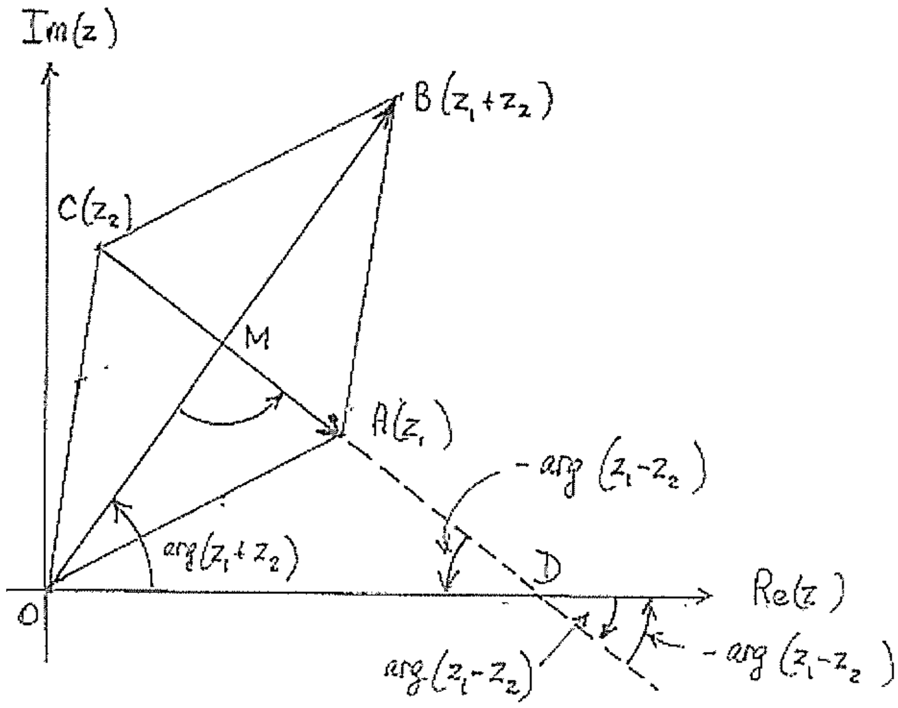
$$\left|\frac{z_1}{z_2}\right| = \left|\frac{4}{5} - \frac{3}{5}i\right|$$

$$\frac{\left|z_1\right|}{\left|z_2\right|} = \sqrt{\frac{16}{25} + \frac{9}{25}}$$

$$= 1$$

$$\left|z_1\right| = \left|z_2\right|$$ ✓

Method 3.



Aim: Prove $OABC$ is a rhombus.
Let D be as shown on the diagram.

$$\frac{z_1+z_2}{z_1-z_2}=3i$$

Take arguments of both sides.

$$\arg\frac{z_1+z_2}{z_1-z_2}=\arg 3i$$

$$\arg(z_1+z_2)-\arg(z_1-z_2)=\frac{\pi}{2}$$

$$\arg(z_1+z_2)+\{-\arg(z_1-z_2)\}=\frac{\pi}{2}$$

Consider ΔODM.

$\angle OMD=\frac{\pi}{2}$ (angle sum of Δ)

$OABC$ is a rhombus (diagonals of parallelogram intersect at $90°$)

$\therefore |z_1|=|z_2|$ **2**

Question 15 c) (continued) **Marks**

(iii) Diagonal OB bisects $\angle AOC$ (rhombus).

$$\angle AOM = \frac{\theta}{2}$$

$$OM = \frac{1}{2}OB \quad \text{and} \quad MA = \frac{1}{2}CA$$

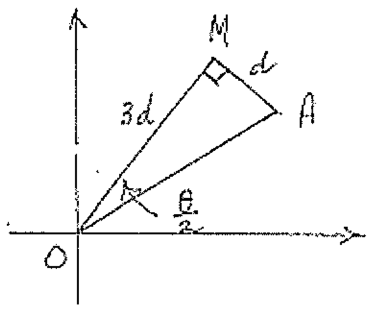
Also $OB = 3CA$.

$$\frac{1}{2}OB = 3 \times \frac{1}{2}CA$$

$$OM = 3MA$$

Let $MA = d$.

$$\therefore OM = 3d$$



$$\tan\frac{\theta}{2} = \frac{d}{3d} = \frac{1}{3}$$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{2 \times \frac{1}{3}}{1-\frac{1}{9}}$$

$$= \frac{3}{4}.$$ 2

Question 16 (15 marks) **Marks**

a) (i)

$$u = x^n \qquad \frac{dv}{dx} = \frac{1}{\sqrt{x+1}}$$
$$\frac{du}{dx} = nx^{n-1} \qquad v = 2\sqrt{x+1}$$

$$\begin{aligned}
I_n &= \int_0^1 \frac{x^n}{\sqrt{x+1}}\,dx \\
&= \left[x^n \times 2\sqrt{x+1}\right]_0^1 - n\int_0^1 x^{n-1} \times 2\sqrt{x+1}\,dx \quad \checkmark \\
&= 2\sqrt{2} - 2n\int_0^1 x^{n-1}\sqrt{x+1}\,dx \\
&= 2\sqrt{2} - 2n\int_0^1 x^{n-1} \times \frac{(x+1)}{\sqrt{x+1}}\,dx \quad \left(\begin{array}{l}\text{rationalising} \\ \text{the numerator}\end{array}\right) \quad \checkmark \\
&= 2\sqrt{2} - 2n\int_0^1 \frac{x^n}{\sqrt{x+1}} + \frac{x^{n-1}}{\sqrt{x+1}}\,dx \\
&= 2\sqrt{2} - 2n\,I_n - 2n\,I_{n-1}
\end{aligned}$$

$$(2n+1)I_n = 2\sqrt{2} - 2n\,I_{n-1} \quad \checkmark$$

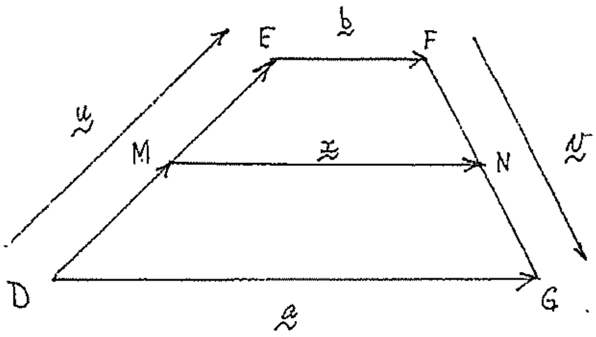
3

(ii)

$$\begin{aligned}
I_3 &= \frac{2\sqrt{2}}{7} - \frac{6}{7}I_2 \\
&= \frac{2\sqrt{2}}{7} - \frac{6}{7}\left[\frac{2\sqrt{5}}{5} - \frac{4}{5}I_1\right] \\
&= \frac{2\sqrt{2}}{7} - \frac{12\sqrt{2}}{35} + \frac{24}{35}\left[\frac{2\sqrt{2}}{3} - \frac{2}{3}I_0\right] \\
&= -\frac{2\sqrt{2}}{35} + \frac{48\sqrt{2}}{105} - \frac{48}{105}I_0 \quad \checkmark \\
&= \frac{42\sqrt{2}}{105} - \frac{48}{105}\int_0^1 \frac{1}{\sqrt{x+1}}\,dx \\
&= \frac{42\sqrt{2}}{105} - \frac{48}{105}\left[2\sqrt{x+1}\right]_0^1 \\
&= \frac{42\sqrt{2}}{105} - \frac{48}{105}\left(2\sqrt{2} - 2\right) \\
&= \frac{32}{35} - \frac{18}{35}\sqrt{2} \quad \checkmark
\end{aligned}$$

2

b)



(i) $\underset{\sim}{a} = \underset{\sim}{u} + \underset{\sim}{b} + \underset{\sim}{v}$ **1**

(ii) $\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{u} + \underset{\sim}{x} + \frac{1}{2}\underset{\sim}{v}$ **1**

(iii) $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{u} + \underset{\sim}{v}$ 1.

$\underset{\sim}{a} - \underset{\sim}{x} = \frac{1}{2}\left(\underset{\sim}{u} + \underset{\sim}{v}\right)$ 2.

Eliminate $\underset{\sim}{u}$ and $\underset{\sim}{v}$ by substituting 1. into 2.

$$\underset{\sim}{a} - \underset{\sim}{x} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$$

$$2\underset{\sim}{a} - 2\underset{\sim}{x} = \underset{\sim}{a} - \underset{\sim}{b}$$

$$\underset{\sim}{x} = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$$

Let $\underset{\sim}{b} = k\underset{\sim}{a}$.

Then $\underset{\sim}{x} = \frac{1}{2}\left(\underset{\sim}{a} + k\underset{\sim}{a}\right)$

$$= \frac{1}{2}(k+1)\underset{\sim}{a}$$

$\underset{\sim}{x}$ is parallel to $\underset{\sim}{a}$ (and similarly to $\underset{\sim}{b}$)

Since $\underset{\sim}{x} = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$ and $\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}$ are parallel,

$$MN = \frac{1}{2}(DG + EF)$$

MN is parallel to DG and EF and its length is one-half of $DG + EF$. **3**

c) (i) $w = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$

$w^k = \cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}$

Show that $z = w^k = \cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}$ is a solution of

$z^9 - 1 = 0$

$$\begin{aligned}\text{L.H.S.} &= \left(\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}\right)^9 - 1\\ &= \cos 2k\pi + i\sin 2k\pi \quad -1\\ &= 1 \quad + \quad 0 \quad -1\\ &= 0\end{aligned}$$

$\therefore$ w^k is a solution of $z^9 - 1 = 0$. ✓ **1**

(ii) **Method 1.** Prove $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 = 0$

$\text{L.H.S.} = \frac{1(w^9 - 1)}{w - 1}$ using $S_N = \frac{a(r^N - 1)}{r - 1}$

$= \frac{1 - 1}{w - 1}$ $z = w$ is a solution of $z^9 = 1$

i.e. $w^9 = 1$ ✓

$= 0$

Method 2. $1, w, w^2, w^3, w^4, w^5, w^6, w^7, w^8$ are roots of

$z^9 + 0z^8 + 0z^7 + \ldots\ldots\ldots\ldots - 1 = 0$

$\therefore$ Sum of the roots $= 1 + w + w^2 + \ldots\ldots\ldots\ldots + w^8 = 0$ ✓ **1**

Question 16 c) (continued)

(iii) **Method 1.**

$$w = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$$

$$w^2 = \cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}$$

$$w^3 = \cos\frac{6\pi}{9} + i\sin\frac{6\pi}{9}$$

$$w^4 = \cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}$$

$$w^5 = \cos\frac{10\pi}{9} + i\sin\frac{10\pi}{9} = \cos\left(2\pi - \frac{8\pi}{9}\right) + i\sin\left(2\pi - \frac{8\pi}{9}\right)$$

$$= \cos\frac{8\pi}{9} - i\sin\frac{8\pi}{9}$$

$$w^6 = \cos\frac{12\pi}{9} + i\sin\frac{12\pi}{9} = \cos\left(2\pi - \frac{6\pi}{9}\right) + i\sin\left(2\pi - \frac{6\pi}{9}\right)$$

$$= \cos\frac{6\pi}{9} - i\sin\frac{6\pi}{9}$$

$$w^7 = \cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9} = \cos\left(2\pi - \frac{4\pi}{9}\right) + i\sin\left(2\pi - \frac{4\pi}{9}\right)$$

$$= \cos\frac{4\pi}{9} - i\sin\frac{4\pi}{9}$$

$$w^8 = \cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9} = \cos\left(2\pi - \frac{2\pi}{9}\right) + i\sin\left(2\pi - \frac{2\pi}{9}\right)$$

$$= \cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9} \qquad \checkmark$$

Adding and using (ii)

$$\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right) + \left(\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}\right) + \left(\cos\frac{6\pi}{9} + i\sin\frac{6\pi}{9}\right) + \left(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\right)$$

$$+ \left(\cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9}\right) + \left(\cos\frac{4\pi}{9} - i\sin\frac{4\pi}{9}\right) + \left(\cos\frac{6\pi}{9} - i\sin\frac{6\pi}{9}\right) + \left(\cos\frac{8\pi}{9} - i\sin\frac{8\pi}{9}\right)$$

$$= -1$$

$$2\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9} + 2\cos\frac{6\pi}{9} + 2\cos\frac{8\pi}{9} = -1$$

$$\left(\cos\frac{8\pi}{9} + \cos\frac{4\pi}{9}\right) + \left(\cos\frac{6\pi}{9} + \cos\frac{2\pi}{9}\right) = -\frac{1}{2}$$ See Note below

$$2\cos\frac{6\pi}{9}\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9}\cos\frac{2\pi}{9} = -\frac{1}{2} \qquad \checkmark$$

(using rule $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$)

Question 16 c) iii) (continued) **Marks**

$$\cos\frac{2\pi}{9}\left(\cos\frac{6\pi}{9}+\cos\frac{4\pi}{9}\right)=-\frac{1}{4}$$

$$\cos\frac{2\pi}{9}\times 2\cos\frac{5\pi}{9}\cos\frac{\pi}{9}=-\frac{1}{4}$$

$$\cos\frac{2\pi}{9}\cos\left(\pi-\frac{4\pi}{9}\right)\cos\frac{\pi}{9}=-\frac{1}{8}$$

$$\cos\frac{2\pi}{9}\times-\cos\frac{4\pi}{9}\times\cos\frac{\pi}{9}=-\frac{1}{8}$$

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{8} \quad \checkmark$$

Note: The grouping could have been

$$\left(\cos\frac{4\pi}{9}+\cos\frac{2\pi}{9}\right)+\left(\cos\frac{8\pi}{9}+\cos\frac{6\pi}{9}\right)=-\frac{1}{2}$$

$$2\cos\frac{3\pi}{9}\cos\frac{\pi}{9} \quad + \quad 2\cos\frac{7\pi}{9}\cos\frac{\pi}{9}=-\frac{1}{2}$$

$$\cos\frac{\pi}{9}\left(\cos\frac{7\pi}{9}+\cos\frac{3\pi}{9}\right)=-\frac{1}{4}$$

$$\cos\frac{\pi}{9}\times 2\cos\frac{5\pi}{9}\cos\frac{2\pi}{9}=-\frac{1}{4}$$

etc.

Method 2.

$$z^9-1=(z-1)(z-w)(z-w^2)(z-w^3)(z-w^4)(z-z^5)(z-z^6)(z-z^7)(z-z^8)$$

$$=(z-1)\left\{(z-w)(z-w^8)\right\}\left\{(z-w^2)(z-w^7)\right\}\left\{(z-w^3)(z-w^6)\right\}\left\{(z-w^4)(z-w^5)\right\}$$

Note: w^1 and w^8, w^2 and w^7, w^3 and w^6, w^4 and w^5 are conjugates as seen in Method 1.

$$z^9-1=(z-1)\left\{z^2-(w+w^8)z+1\right\}\left\{z^2-(w^2+w^7)z+1\right\}\left\{z^2-(w^3+w^6)z+1\right\}\left\{z^2-(w^4+w^5)z+1\right\} \quad \checkmark$$

Let $z=i$ so that $z^2+1=0$.

$$i-1=(i-1)\left\{-i(w+w^8)\right\}\left\{-i(w^2+w^7)\right\}\left\{-i(w^3+w^6)\right\}\left\{-i(w^4+w^5)\right\} \quad \checkmark$$

Divide by $(i-1)$.

$$1=(w+w^8)(w^2+w^7)(w^3+w^6)(w^4+w^5)$$

$$1=\left(2\cos\frac{2\pi}{9}\right)\left(2\cos\frac{4\pi}{9}\right)\left(2\cos\frac{6\pi}{9}\right)\left(2\cos\frac{8\pi}{9}\right)$$

$$\frac{1}{8}=\cos\frac{2\pi}{9}\times\cos\frac{4\pi}{9}\times-\frac{1}{2}\times-\cos\frac{\pi}{9}$$

$$\cos\frac{6\pi}{9}=\cos\frac{2\pi}{3} =\cos\left(\pi-\frac{\pi}{3}\right) =-\cos\frac{\pi}{3} =-\frac{1}{2}$$

$$\cos\frac{8\pi}{9}=\cos\left(\pi-\frac{\pi}{9}\right) =-\cos\frac{\pi}{9}$$

$$\frac{1}{8}=\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} \quad \checkmark$$

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