Student Number: ____________

Parramatta Marist High School

2020 **YEAR 12 TRIAL 1**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- Show relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (page 5–19)

- Attempt Questions 11–23
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the contrapositive of $P \implies \neg Q$?

A. $Q \implies P$

B. $\neg Q \implies P$

C. $Q \implies \neg P$

D. $\neg Q \implies \neg P$

2 Let $z = 2 - 7i$ and $w = 5 + 3i$.

What is the value of $\bar{z} - 2w$?

A. $-8 - 13i$

B. $-8 + i$

C. $12 - i$

D. $12 + 13i$

3 What is the Cartesian form of $\underset{\sim}{r} = \underset{\sim}{i}\sec\theta + \underset{\sim}{j}\tan\theta$?

A. $x^2 - y^2 = 1$

B. $x^2 + y^2 = 1$

C. $y^2 - x^2 = 1$

D. $x^2 - y^2 = -1$

4 What are the roots of the polynomial $P(x) = x^3 + 3x^2 + 4x + 2$?

A. $1,\ 1+i,\ 1-i$

B. $-1,\ 1+i,\ 1-i$

C. $1,\ -1+i,\ -1-i$

D. $-1,\ -1+i,\ -1-i$

5 What is the negation of $\exists x \in \mathbb{Z} : x^2 = -1$?

A. $\exists x \notin \mathbb{Z} : x^2 = -1$

B. $\exists x \in \mathbb{Z} : x^2 \neq -1$

C. $\forall x \in \mathbb{Z} : x^2 = -1$

D. $\forall x \in \mathbb{Z} : x^2 \neq -1$

6 What is the angle between the vectors $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{j} - \underset{\sim}{k}$?

A. $\frac{\pi}{3}$

B. $\frac{\pi}{2}$

C. $\frac{2\pi}{3}$

D. π

7 Let $z = 1 - i$.

What is z^3 in exponential form?

A. $e^{\frac{3\pi i}{4}}$

B. $e^{-\frac{3\pi i}{4}}$

C. $2^{\frac{3}{2}} e^{\frac{3\pi i}{4}}$

D. $2^{\frac{3}{2}} e^{-\frac{3\pi i}{4}}$

8 The point $(0, 1, -1)$ lies on which line?

A. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + \lambda(\underset{\sim}{i} + \underset{\sim}{k})$

B. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + \lambda(\underset{\sim}{j} + \underset{\sim}{k})$

C. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{k} + \lambda(\underset{\sim}{i} + \underset{\sim}{j})$

D. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{k} + \lambda(\underset{\sim}{j} + \underset{\sim}{k})$

9 The probability function $f(x) = \begin{cases} \pi x \sin \pi x, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$.

What is $P(x \leq \frac{1}{2})$?

A. $\frac{1}{\pi^2}$

B. $\frac{1}{\pi}$

C. $\frac{1}{2}$

D. $\frac{\pi}{2}$

10 A particle, initially at rest at the origin, moves with equation of motion $a = 1 + v^2$.

What is the equation of motion for v in terms of x?

A. $v = \tan x$

B. $v = e^x - 1$

C. $v = x + \frac{1}{3}x^3$

D. $v = \sqrt{e^{2x} - 1}$

Section II

90 marks
Attempt Questions 11–23
Allow about 2 hours and 45 minutes for this section

Answer the questions in the spaces provided. Sufficient spaces are provided for typical responses.

Your responses should include relevant mathematical reasoning and/or calculations.

Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate which question you are answering.

Question 11 (3 marks)

Prove that $2^{n+1} + 3^{2n-1}$ is divisible by 7 for $n \in \mathbb{Z}^+$. **3**

Question 12 (4 marks)

(a) Express $1 + i\sqrt{3}$ in exponential form. **2**

(b) Hence find the two values of $\sqrt{1 + i\sqrt{3}}$ in Cartesian form. **2**

Question 13 (4 marks)

In the box below, shade the region in the complex plane that simultaneously satisfies $|z| < |z - 2 - 2i|$ and $\frac{\pi}{12} \leq \arg z \leq \frac{5\pi}{12}$.

Question 14 (7 marks)

(a) Let $u_1 = 1$, $u_n = u_{n-1} + n$ for $n \geq 2$. **3**
Prove that $u_n = \frac{1}{2}n(n+1)$ for $n \in \mathbb{Z}^+$.

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(b) Hence, prove that $\sum_{k=0}^{n} k^3 = u_n^2$ for $n \in \mathbb{Z}^+$. **4**

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Question 15 (15 marks)

(a) Find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$. **3**

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(b) Find $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+2}}$. **4**

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Question 15 continues on page 9

Question 15 (continued)

(c) Find $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^4 x \, dx$. **4**

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(d) Find $\int \frac{4 \, dx}{(x^2+1)(x-1)}$. **4**

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End of Question 15

Question 16 (5 marks)

The displacement x at time t of a particle moving on the x-axis is given by

$$x = 3 + \sqrt{3}\sin 3t + \cos 3t.$$

(a) Show that the motion of the particle is simple harmonic. **2**

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(b) Find the amplitude and phase of the motion. **3**

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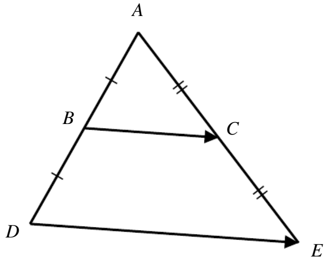
Question 17 (4 marks)

In the diagram below, B is the midpoint of AD and C is the midpoint of AE. **4**



Using vectors, prove that $\overrightarrow{BC}$ is half the magnitude of, and parallel to, $\overrightarrow{DE}$.

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Questions 11–17 are worth 42 marks in total.

Question 18 (4 marks)

(a) Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$ for $x, y \in \mathbb{R}^+$. **2**

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(b) Hence prove that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$ for $a, b \in \mathbb{R}^+$. **2**

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Question 19 (8 marks)

(a) Consider the sphere given by the Cartesian equation $x^2+y^2+z^2+2x-4z-4=0$. **2**

Show that the vector equation of the sphere is $\left| \underset{\sim}{r} - \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right| = 3$.

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(b) Find the points of intersection between the sphere and the line $\underset{\sim}{r} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. **3**

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(c) Show that the line $\underset{\sim}{r} = -\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} + \mu \underset{\sim}{j}$ is tangent to the sphere. **3**

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Question 20 (7 marks)

(a) Let $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, dx$. **3**

Show that for $n \geq 2$, $I_n = \frac{n-1}{n} I_{n-2}$.

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(b) Hence find the volume of revolution for $y = \cos^3 x$ about the x-axis for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **4**

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Question 21 (7 marks)

(a) Using De Moivre, show that $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$. **3**

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(b) Hence, show that $\cos 18^\circ = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$. **4**

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Question 22 (10 marks)

A projectile is launched from the ground with an initial velocity of u m/s at an angle of θ to the horizontal. The projectile experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions.

The equations of motion are given by $\underset{\sim}{a} = -0.1\underset{\sim}{v} - 10\underset{\sim}{j}$, where $\underset{\sim}{a}$ is the acceleration vector.

(a) Show that the velocity vector $\underset{\sim}{v} = e^{-0.1t}\left(\underset{\sim}{u} + \underset{\sim}{j}\right) - \underset{\sim}{j}$, where $\underset{\sim}{u} = u\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$, satisfies the equations of motion and initial conditions. **3**

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(b) Show that the projectile reaches its peak (maximum height) after $10\ln(u\sin\theta + 1)$ seconds. **3**

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Question 22 continues on page 17

Question 22 (continued)

(c) Let the speed of the projectile at its peak be w m/s. **4**

Show that $\lim_{u \to \infty} w = \cot\theta$.

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End of Question 22

Question 23 (12 marks)

Recall that $x \in \mathbb{Q} \iff \exists p \in \mathbb{Z}, q \in \mathbb{Z}^+ : x = \frac{p}{q}$.

(a) Show that $\forall r \in \mathbb{Z}^+ \exists n \in \mathbb{Z}^+ : 0 < \int_0^1 x^n e^x \, dx < \frac{1}{r}$. **3**

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(b) Prove, using induction, that for all integers $n \geq 0$, $\exists \alpha, \beta \in \mathbb{Z} : \int_0^1 x^n e^x \, dx = \alpha + \beta e$. **4**

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Question 23 continues on page 19

Question 23 (continued)

(c) Let $p \in \mathbb{Z}, q \in \mathbb{Z}^+, x = \frac{p}{q} \in \mathbb{Q}$.

Show that $\forall \alpha, \beta \in \mathbb{Z} : \left\{ \alpha + \beta x \neq 0 \implies |\alpha + \beta x| \geq \frac{1}{q} \right\}$. **2**

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(d) Hence, prove that $e \notin \mathbb{Q}$. **3**

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End of Question 23

End of Paper

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \ne 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

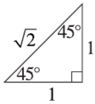
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

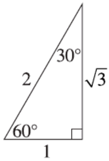
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

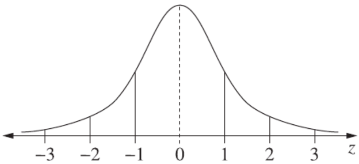
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

Student Number: ________________

Parramatta Marist High School

2020 **YEAR 12 TRIAL 1**

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Section I

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1 What is the contrapositive of $P \implies \neg Q$?

$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

A. $Q \implies P$
B. $\neg Q \implies P$
C. $Q \implies \neg P$
D. $\neg Q \implies \neg P$

2 Let $z = 2 - 7i$ and $w = 5 + 3i$.

What is the value of $\bar{z} - 2w$?

$\bar{z} - 2w = 2 + 7i - 2(5+3i)$
$= 2 - 10 + (7 - 6)i$
$\therefore B$

A. $-8 - 13i$
B. $-8 + i$
C. $12 - i$
D. $12 + 13i$

3 What is the Cartesian form of $\underset{\sim}{r} = \underset{\sim}{i}\sec\theta + \underset{\sim}{j}\tan\theta$?

$1 + \tan^2\theta = \sec^2\theta$
$\sec^2\theta - \tan^2\theta = 1$

A. $x^2 - y^2 = 1$
B. $x^2 + y^2 = 1$
C. $y^2 - x^2 = 1$
D. $x^2 - y^2 = -1$

4 What are the roots of the polynomial $P(x) = x^3 + 3x^2 + 4x + 2$?

sum $= -3$, $\therefore$ D.

A. $1,\ 1+i,\ 1-i$
B. $-1,\ 1+i,\ 1-i$
C. $1,\ -1+i,\ -1-i$
D. $-1,\ -1+i,\ -1-i$

5 What is the negation of $\exists x \in \mathbb{Z} : x^2 = -1$?

It's not the case that there exists $x \in \mathbb{Z}$ such that $x^2 = -1$

$\therefore$ for any $x \in \mathbb{Z}$, $x^2 \neq -1$.

A. $\exists x \notin \mathbb{Z} : x^2 = -1$

B. $\exists x \in \mathbb{Z} : x^2 \neq -1$

C. $\forall x \in \mathbb{Z} : x^2 = -1$

D. $\forall x \in \mathbb{Z} : x^2 \neq -1$

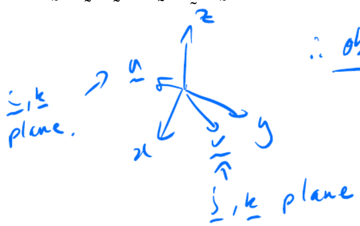
6 What is the angle between the vectors $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{j} - \underset{\sim}{k}$?

A. $\frac{\pi}{3}$

B. $\frac{\pi}{2}$

C. $\frac{2\pi}{3}$

D. π



$\therefore$ obtuse angle.

7 Let $z = 1 - i$.

What is z^3 in exponential form?

A. $e^{\frac{3\pi i}{4}}$

B. $e^{-\frac{3\pi i}{4}}$

C. $2^{\frac{3}{2}} e^{\frac{3\pi i}{4}}$

D. $2^{\frac{3}{2}} e^{-\frac{3\pi i}{4}}$

$|z| = \sqrt{2}$

$\mathrm{Arg}(z^3) < 0$

$\therefore$ D

8 The point $(0, 1, -1)$ lies on which line?

A. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + \lambda(\underset{\sim}{i} + \underset{\sim}{k})$

B. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + \lambda(\underset{\sim}{j} + \underset{\sim}{k})$

C. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{k} + \lambda(\underset{\sim}{i} + \underset{\sim}{j})$

D. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{k} + \lambda(\underset{\sim}{j} + \underset{\sim}{k})$

$x = 0 \Rightarrow$ A or C, with $\lambda = -1$.

$y = 1$ when $\lambda = -1 \Rightarrow$ A oof!

9 The probability function $f(x) = \begin{cases} \pi x \sin \pi x, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$.

$$\int_0^{1/2} \pi x \sin \pi x \, dx$$
$$= \left[-\frac{\pi}{\pi} x \cos \pi x \right]_0^{1/2} + \int_0^{1/2} \frac{\pi}{\pi} \cos \pi x \, dx$$
$$= \left[-\frac{1}{2} \cos\left(\frac{\pi}{2}\right) - 0 \right] + \left[\frac{\sin \pi x}{\pi} \right]_0^{1/2}$$
$$= 0 + \frac{\sin(\pi/2)}{\pi} - \frac{\sin(0)}{\pi}$$
$$= \frac{1}{\pi}$$

What is $P(x \leq \frac{1}{2})$?

A. $\frac{1}{\pi^2}$

B. $\frac{1}{\pi}$

C. $\frac{1}{2}$

D. $\frac{\pi}{2}$

10 A particle, initially at rest at the origin, moves with equation of motion $a = 1 + v^2$.
What is the equation of motion for v in terms of x?

A. $v = \tan x$

B. $v = e^x - 1$

C. $v = x + \frac{1}{3}x^3$

D. $v = \sqrt{e^{2x} - 1}$

$$v \frac{dv}{dx} = 1 + v^2$$
$$\frac{1}{2} \int \frac{2v \, dv}{1+v^2} = \int dx$$
$$\frac{1}{2} \ln(1+v^2) = x + C$$

when $t = 0$, $x = 0$, $v = 0$:

$$\frac{1}{2} \ln(1 + 0^2) = 0 + C$$
$$\therefore C = 0$$
$$\ln(1+v^2) = 2x$$
$$1 + v^2 = e^{2x}$$
$$v^2 = e^{2x} - 1 \Rightarrow v = \sqrt{e^{2x}-1}$$

As this is M.C., looking at $\int \frac{v\,dv}{1+v^2}$ should indicate the solution is the square root of an exponential, hence D.

Section II

90 marks
Attempt Questions 11–23
Allow about 2 hours and 45 minutes for this section

Answer the questions in the spaces provided. Sufficient spaces are provided for typical responses.

Your responses should include relevant mathematical reasoning and/or calculations.

Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate which question you are answering.

Question 11 (3 marks)

Prove that $2^{n+1} + 3^{2n-1}$ is divisible by 7 for $n \in \mathbb{Z}^+$. **3**

Base case $(n=1)$: $2^{1+1} + 3^{2\times1-1} = 4+3$

$= 7$. $\therefore$ true for $n=1$.

Assume $\exists k \in \mathbb{Z}^+ : \exists P \in \mathbb{Z} : 2^{k+1} + 3^{2k-1} = 7P \Rightarrow 2^{k+1} = 7P - 3^{2k-1}$

consider: $2^{k+2} + 3^{2k+1} = 2(2^{k+1}) + 3^{2k+1}$

$= 2(7P - 3^{2k-1}) + 3^{2k+1}$

$= 7(2P) + 3^{2k-1}(9-2)$

$= 7(2P + 3^{2k-1})$

$= 7Q$, for $Q = 2P + 3^{2k-1}$,

and Q clearly in $\mathbb{Z}$.

$\therefore$ true by mathematical induction.

Question 12 (4 marks)

(a) Express $1+i\sqrt{3}$ in exponential form. **2**

modulus $= 2$, argument $= \pi/3$

$\therefore\ 1 + i\sqrt{3} = 2e^{i\pi/3}$

(b) Hence find the two values of $\sqrt{1+i\sqrt{3}}$ in Cartesian form. **2**

roots are: $\sqrt{2}\, e^{i\frac{\pi}{6}}$ or $-\sqrt{2}\, e^{i\frac{\pi}{6}}$

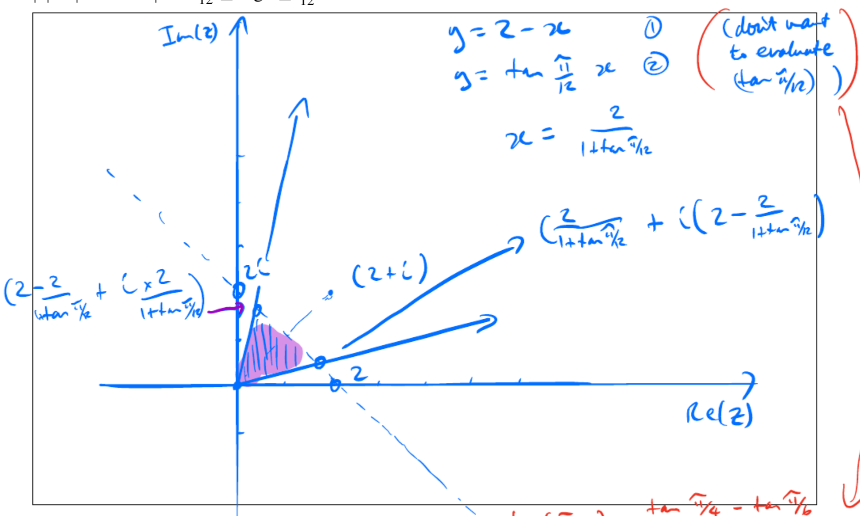
$= \sqrt{2}\left(\sqrt{3}/2 + \tfrac{1}{2}i\right)$ or $\sqrt{2}\left(-\sqrt{3}/2 - \tfrac{1}{2}i\right)$

$= \pm\left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

Question 13 (4 marks)

$|z-0| < |z-(2+2i)| \to z$'s closer to 0 than $(2+2i)$

In the box below, shade the region in the complex plane that simultaneously satisfies $|z| < |z-2-2i|$ and $\frac{\pi}{12} \le \arg z \le \frac{5\pi}{12}$.



(don't want to evaluate $(\tan\pi/12)$)

$\tan(\pi/12) = \frac{\tan\pi/4 - \tan\pi/6}{1 + \tan\pi/4\tan\pi/6}$

$= \frac{1 - 1/\sqrt{3}}{1 + 1/\sqrt{3}} = 2 - \sqrt{3}$

Question 14 (7 marks) triangle numbers! Yay!

(a) Let $u_1 = 1$, $u_n = u_{n-1} + n$ for $n \geq 2$. **3**

Prove that $u_n = \frac{1}{2}n(n+1)$ for $n \in \mathbb{Z}^+$.

when $n=1$, $u_1 = \frac{1}{2}\times 1(1+1)$

$= 1$ ✓

$\therefore$ true for $n=1$.

Assume $\exists k \in \mathbb{Z}^+ : u_k = \frac{k(k+1)}{2}$

Consider $u_{k+1} = \frac{1}{2}k(k+1) + (k+1)$

$= \frac{1}{2}k(k+1) + \frac{2}{2}(k+1)$

$= \frac{1}{2}(k+1)[k+2]$

$= \frac{1}{2}(k+1)(k+2)$

$\therefore$ true by mathematical induction.

sum of cubes Yay!

(b) Hence, prove that $\sum_{k=0}^{n} k^3 = u_n^2$ for $n \in \mathbb{Z}^+$. **4**

Base case, $n=1$: $\sum_{k=0}^{1} k^3 = 0^3 + 1^3$, $u_1^2 = 1^2$

$= 1$ $= 1$ $\therefore$ true ✓

Assume $\exists j \in \mathbb{Z}^+ : \sum_{k=0}^{j} k^3 = u_j^2$

Consider $\sum_{k=0}^{j+1} k^3 = \sum_{k=0}^{j} k^3 + (j+1)^3$

$= \frac{1}{4}j^2(j+1)^2 + (j+1)^3 \times \frac{4}{4}$

$= (j+1)^2[j^2 + 4j + 4] \times \frac{1}{4}$

$= \frac{1}{4}(j+1)^2(j+2)^2$

$= u_{j+1}^2$

$\therefore$ true by mathematical induction.

Question 15 (15 marks) Very tricky for 3 marks!

(a) Find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$. $= I$ **3**

$I = \displaystyle\int_0^{\pi/2} \frac{1-\cos x}{\sin^2 x}\,dx$

$= \displaystyle\int_0^{\pi/2} \operatorname{cosec}^2 x\,dx - \int_0^{\pi/2} (\sin x)^{-2} \cos x\,dx$

$= \left[-\cot x\right]_0^{\pi/2} - \left[\frac{1}{-1}(\sin x)^{-1}\right]_0^{\pi/2}$

$= \left[\left(\frac{-\cos x + 1}{\sin x}\right) \times \frac{(1+\cos x)}{(1+\cos x)}\right]_0^{\pi/2}$

$= \left[\frac{\sin^2 x}{\sin x(1+\cos x)}\right]_0^{\pi/2}$

$= \left[\frac{\sin x}{1+\cos x}\right]_0^{\pi/2}$

$= 1 - 0 = 1$

(b) Find $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+2}}$. **4**

$\displaystyle\int \frac{dx}{\sqrt{(x+1)^2+1}} = \ln\left((x+1)^2 + \sqrt{(x+1)^2+1}\right) + C$

$= \ln\left(x^2+2x+1+\sqrt{x^2+2x+2}\right) + C$

What was the intention here?
Prove a standard integral?

Question 15 continues on page 9

Question 15 (continued) $1 + \tan^2 x = \sec^2 x$

(c) Find $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^4 x \, dx$. **4**

$= \int_0^{\pi/4} \sec^2 x \left[\tan^5 x + \tan^3 x \right] dx$

$= \int_0^{\pi/4} \tan^5 x \sec^2 x \, dx + \int_0^{\pi/4} \tan^3 x \sec^2 x \, dx$

$= \left[\frac{1}{6} \tan^6 x \right]_0^{\pi/4} + \left[\frac{1}{4} \tan^4 x \right]_0^{\pi/4}$

$= 1/6 + 1/4$

$= 5/12.$

(d) Find $\int \frac{4\,dx}{(x^2+1)(x-1)}$. $= I$ **4**

$4 \equiv (Ax + B)(x-1) + (C)(x^2+1)$

$x = 1 \Rightarrow 4 = 2C \Rightarrow C = 2.$

$x = i \Rightarrow 4 = -A - B, \quad 0 = -A + B \Rightarrow B = A$

$4 = -2A$

$A = -2 = B$

$\therefore I = -2 \int \frac{dx}{(x^2+1)} + 2 \int \frac{dx}{x-1}$

$= -2 \tan^{-1} x + 2 \ln|x-1| + C$

End of Question 15

Question 16 (5 marks)

The displacement x at time t of a particle moving on the x-axis is given by

$$x = 3 + \sqrt{3}\sin 3t + \cos 3t.$$

(a) Show that the motion of the particle is simple harmonic. **2**

$\dot{x} = 3\sqrt{3}\cos 3t - 3\sin 3t$

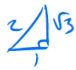
$\ddot{x} = -3^2\sqrt{3}\sin 3t - 3^2\cos 3t$

$= -9[\sqrt{3}\sin 3t + \cos 3t]$

$= -9[x]$

$\therefore$ SHM

(b) Find the amplitude and phase of the motion. **3**



$x = 3 + 2\left(\frac{\sqrt{3}}{2}\sin 3t + \frac{1}{2}\cos 3t\right)$

$= 3 + 2(\sin \pi/3 \sin 3t + \cos \pi/3 \cos 3t)$

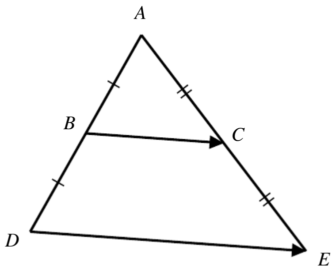
$= 3 + 2\cos(3t - \pi/3)$

$= 3 + 2\cos(3(t - \pi/9))$

$\therefore$ amplitude $= 2$, phase $= \pi/9$

Question 17 (4 marks)

In the diagram below, B is the midpoint of AD and C is the midpoint of AE. **4**



Yay! Vectors!

Using vectors, prove that $\overrightarrow{BC}$ is half the magnitude of, and parallel to, $\overrightarrow{DE}$.

$\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE}$

$= \overrightarrow{DB} + \overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CE}$

$= 2\overrightarrow{BA} + 2\overrightarrow{AC}$ (as $\overrightarrow{DB} = \overrightarrow{BA}$ and $\overrightarrow{AC} = \overrightarrow{CE}$)

$= 2(\overrightarrow{BC})$

Questions 11–17 are worth 42 marks in total.

Question 18 (4 marks)

(a) Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$ for $x, y \in \mathbb{R}^+$. **2**

let $x, y \in \mathbb{R}^+$. $(\sqrt{x} - \sqrt{y})^2 \geqslant 0$

$x + y - 2\sqrt{x}\sqrt{y} \geqslant 0$

$x + y \geqslant 2\sqrt{xy}$

$\dfrac{x+y}{2} \geqslant \sqrt{xy}$

(b) Hence prove that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$ for $a, b \in \mathbb{R}^+$. **2**

hence: let $x = \frac{a}{b}$, $y = \frac{b}{a}$

then $\dfrac{x+y}{2} \geqslant \sqrt{xy}$

$\Rightarrow \frac{a}{b} + \frac{b}{a} \geqslant 2\sqrt{\frac{a}{b}\frac{b}{a}}$

$\frac{a}{b} + \frac{b}{a} \geqslant 2$

otherwise: let $a, b \in \mathbb{R}^+$.

$\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 \geqslant 0$

$\frac{a}{b} + \frac{b}{a} - 2 \geqslant 0$

$\frac{a}{b} + \frac{b}{a} \geqslant 2$

Question 19 (8 marks)

C_1

(a) Consider the sphere given by the Cartesian equation $x^2+y^2+z^2+2x-4z-4=0$. **2**

Show that the vector equation of the sphere is $\left| \underset{\sim}{r} - \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right| = 3$.

$C_1: (x^2+2x+1) - 1 + y^2 + (z^2-4z+4) - 4 - 4 = 0$

$(x+1)^2 + y^2 + (z-2)^2 - 9 = 0$

$\therefore (x+1)^2 + y^2 + (z-2)^2 = 3^2$

and this is the equation of a sphere

centered at $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ with radius 3.

(b) Find the points of intersection between the sphere and the line $\underset{\sim}{r} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. **3**

$\left| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right| = 3$

$\begin{pmatrix} 1+\lambda \\ 1+\lambda \\ -1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1+\lambda \\ 1+\lambda \\ -1+2\lambda \end{pmatrix} = 9$

$(1+\lambda)^2 + (1+\lambda)^2 + (2\lambda-1)^2 = 9$

$6\lambda^2 + 3 = 9$

$6\lambda^2 = 6$

$\lambda = \pm 1$

$\therefore \begin{pmatrix} 0+1 \\ 1+1 \\ 1+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ AND $\begin{pmatrix} 0-1 \\ 1-1 \\ 1-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

(c) Show that the line $\underset{\sim}{r} = -\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} + \mu \underset{\sim}{j}$ is tangent to the sphere. **3**

$\begin{pmatrix} -1+1 \\ 1+\mu \\ -1-2 \end{pmatrix} \cdot \begin{pmatrix} -1+1 \\ 1+\mu \\ -1-2 \end{pmatrix} = 9$

$(1+\mu)^2 + (-3)^2 = 9$

$(1+\mu)^2 = 0$

has one solution ($\mu = 0$)

$\therefore$ this line is a tangent to the sphere.

Question 20 (7 marks)

(a) Let $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, dx$. **3**

Show that for $n \geq 2$, $I_n = \frac{n-1}{n} I_{n-2}$.

$I_n = \int_{-\pi/2}^{\pi/2} \cos x \cos^{n-1} x \, dx$

$= \left[\sin x \cos^{n-1} x \right]_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \sin x \times (n-1) \sin x \cos^{n-2} dx$

$= 0 + \int_{-\pi/2}^{\pi/2} (n-1)(1-\cos^2 x) \cos^{n-2} x \, dx$

$= (n-1) I_{n-2} - (n-1) I_n$

$\therefore I_n (1 + (n-1)) = (n-1) I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

(b) Hence find the volume of revolution for $y = \cos^3 x$ about the x-axis for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **4**

$V = \pi \int_{-\pi/2}^{\pi/2} y^2 \, dx$

$= \pi \int_{-\pi/2}^{\pi/2} \cos^6 x \, dx$

$= \pi \left[\frac{6-1}{6} I_4 \right]$

$= \pi (5/6) \left(\frac{4-1}{4}\right) I_2$

$= \pi (5/6) \left(\frac{3}{4}\right) \left(\frac{2-1}{2}\right) \times \int_{-\frac{\pi}{2}}^{\pi/2} dx$

$= \pi^2 \times \frac{15}{48}$

$= \frac{15\pi^2}{48}$ u^3

Question 21 (7 marks)

(a) Using De Moivre, show that $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$. **3**

$(\cos 5x + i\sin 5x) = (\cos x + i\sin x)^5$

$= \cos^5 x + 5i\cos^4 x\sin x - 10\cos^3 x\sin^2 x - 10i\cos^2 x\sin^3 x$

$+ 5\cos x\sin^4 x + i\sin^5 x$

Equating real and imaginary:

$\cos 5x = \cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x$

$= \cos^5 x - 10\cos^3 x(1-\cos^2 x) + 5\cos x(1 - 2\cos^2 x + \cos^4 x)$

$= \cos^5 x - 10\cos^3 x + 10\cos^5 x + 5\cos x - 10\cos^3 x + 5\cos^5 x$

$= 16\cos^5 x - 20\cos^3 x + 5\cos x$

(b) Hence, show that $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$. **4**

$\cos(5\times 18°) = 16\cos^5 18° - 20\cos^3 18° + 5\cos 18°$

$0 = 16\cos^5 18° - 20\cos^3 18° + 5\cos 18°$

Now $\cos 18° \neq 0$, so

$0 = 16\cos^4 18° - 20\cos^2 18° + 5$

$\cos^2 18° = \dfrac{20 \pm \sqrt{400-320}}{32}$ by calculator methods, deduce to take +ve square root.

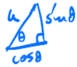
$= \dfrac{20 + \sqrt{16}\sqrt{5}}{32}$

$= \dfrac{10 + 2\sqrt{5}}{16}$

$\therefore \cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$ now $\cos 18° > 0$, so, take positive option.

Question 22 (10 marks)

A projectile is launched from the ground with an initial velocity of u m/s at an angle of θ to the horizontal. The projectile experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions.



The equations of motion are given by $\underset{\sim}{a} = -0.1\underset{\sim}{v} - 10\underset{\sim}{j}$, where $\underset{\sim}{a}$ is the acceleration vector.

(a) Show that the velocity vector $\underset{\sim}{v} = e^{-0.1t}\left(\underset{\sim}{u} + \underset{\sim}{j}\right) - \underset{\sim}{j}$, where $\underset{\sim}{u} = u\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$, satisfies the equations of motion and initial conditions. **3**

When $t = 0$,

$\underset{\sim}{v} = u\cos\theta\,\underset{\sim}{i} + u(\sin\theta + 1 - 1)\underset{\sim}{j}$

$= u\cos\theta\,\underset{\sim}{i} + u\sin\theta\,\underset{\sim}{j}$

which satisfies the initial condition on $\underset{\sim}{v}$.

$\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} = -0.1\,e^{-0.1t}\left(\underset{\sim}{u} + \underset{\sim}{j}\right)$

$= -0.1\left[e^{-0.1t}\left(\underset{\sim}{u} + \underset{\sim}{j}\right) - \underset{\sim}{j} + \underset{\sim}{j}\right]$

$\Rightarrow \underset{\sim}{a} = -0.1\,\underset{\sim}{v} - 0.1\,\underset{\sim}{j}$

either I misread or misunderstood the question, or this should say $-100\underset{\sim}{j}$

(b) Show that the projectile reaches its peak (maximum height) after $10\ln(u\sin\theta + 1)$ seconds. **3**

Peak reached when $v_y = e^{-0.1t}(u\sin\theta + 1) - 1 = 0$

Consider v_y when $t = 10\ln(u\sin\theta + 1)$:

$v_y = (u\sin\theta + 1)^{-\frac{10}{10}}(u\sin\theta + 1) - 1$

$= (u\sin\theta + 1)^0 - 1$

$= 0.$

$\therefore$ peak after $10\ln(u\sin\theta + 1)$ seconds.

Question 22 continues on page 17

Question 22 (continued)

(c) Let the speed of the projectile at its peak be w m/s. **4**
Show that $\lim_{u \to \infty} w = \cot\theta$.

$V_x = u\cos\theta\, e^{-0.1t}$

When $t = 10\ln(u\sin\theta + 1)$,

$V_x = u\cos\theta\,(u\sin\theta + 1)^{-1}$

$= \dfrac{u\cos\theta}{u\sin\theta + 1} = w$

$\lim_{u \to \infty} w = \lim_{u \to \infty} \dfrac{\cos\theta}{\sin\theta + 1/u}$

$= \dfrac{\cos\theta}{\sin\theta}$

$= \cot\theta$.

End of Question 22

Question 23 (12 marks)

Recall that $x \in \mathbb{Q} \iff \exists p \in \mathbb{Z}, q \in \mathbb{Z}^+ : x = \frac{p}{q}$.

(a) Show that $\forall r \in \mathbb{Z}^+ \exists n \in \mathbb{Z}^+ : 0 < \displaystyle\int_0^1 x^n e^x \, dx < \frac{1}{r}$. **3**

$x^n e^x$ is concave up, monotonic increasing in $x \in [0,1]$.

clearly $\int_0^1 x^n e^x dx > 0$ as $x^n e^x > 0$ in $x \in (0,1]$

as $e < 3$, $e^x < 3$ for $x \in [0,1)$

hence: $0 < \int_0^1 x^n e^x dx < \int_0^1 3x^n dx$

$0 < \int_0^1 x^n e^x dx < \frac{3}{n+1}$

now choose n such that $\frac{3}{n+1} < \frac{1}{r}$: $r > \frac{n+1}{3}$

$n < 3r - 1$

Hence $\forall r \in \mathbb{Z}^+$ $0 < \int_0^1 x^n e^x dx < \frac{1}{r}$

(b) Prove, using induction, that for all integers $n \geq 0$, $\exists \alpha, \beta \in \mathbb{Z} : \displaystyle\int_0^1 x^n e^x \, dx = \alpha + \beta e$. **4**

Base case: $n=0$: $I_0 = \int_0^1 e^x dx = e - 1 \Rightarrow \alpha = -1, \beta = 1$ ✓

Assume $\exists k \in \mathbb{N} : \exists \alpha, \beta \in \mathbb{Z}^+ : I_k = \int_0^1 x^k e^x dx = \alpha + \beta e$.

Consider $I_{k+1} = \int_0^1 x^{k+1} e^x dx$

$= [x^{k+1} e^x]_0^1 - \int_0^1 (k+1) x^k e^x dx$

$= e - (k+1) I_k$

$= e - (k+1)(\alpha + \beta e)$

$= -(k+1)\alpha + (1 - (k+1)\beta)e$

Now $-(k+1)\alpha \in \mathbb{Z}$ as $k, \alpha \in \mathbb{Z}$

and $(1 - (k+1)\beta) \in \mathbb{Z}$ as $k, \beta \in \mathbb{Z}$.

$\therefore$ true by mathematical induction

Question 23 continues on page 19

Question 23 (continued)

(c) Let $p \in \mathbb{Z}$, $q \in \mathbb{Z}^+$, $x = \frac{p}{q} \in \mathbb{Q}$.

Show that $\forall \alpha, \beta \in \mathbb{Z} : \left\{ \alpha + \beta x \neq 0 \implies |\alpha + \beta x| \geq \frac{1}{q} \right\}$. **2**

Consider $q^2(\alpha+\beta x)^2$. Note $q \neq 0$. Suppose $\alpha + \beta x \neq 0$. Then $q^2(\alpha+\beta x)^2 \neq 0$. Now $q^2(\alpha+\beta x)^2 = (\alpha q + \beta p)^2 \in \mathbb{Z}^+$, as $\alpha, q, \beta, p \in \mathbb{Z}$, and $q^2(\alpha+\beta x)^2 \neq 0$.

So $q^2(\alpha+\beta x)^2 \geq 1 \Rightarrow (\alpha+\beta x)^2 \geq \frac{1}{q^2}$

$\Rightarrow |\alpha + \beta x| \geq \frac{1}{q}$ as required.

(d) Hence, prove that $e \notin \mathbb{Q}$. **3**

from a) $0 < \int_0^1 x^n e^x \, dx < \frac{1}{r}$

from b) $\int_0^1 x^n e^x \, dx = \alpha + \beta e$ for some $\alpha, \beta \in \mathbb{Z}$.

from c), suppose e is rational; $e = p/q$, as defined previously.

then $|\alpha + \beta e| \geq \frac{1}{q}$

Using a) and b): $0 < \alpha + \beta e < \frac{1}{q}$

which contradicts c): $\alpha + \beta e \geq \frac{1}{q}$ or $\alpha + \beta e \leq \frac{-1}{q}$.

$\therefore$ e is not rational, i.e. $e \notin \mathbb{Q}$.

End of Question 23

End of Paper

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

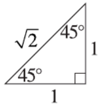
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

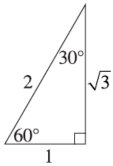


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

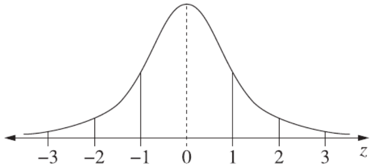
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$