2024

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

CSSA

DO NOT REMOVE PAPER FROM EXAMINATION ROOM

Centre Number

Student Number

Mathematics Extension 2

Morning Session
Monday, 12 August 2024

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- Use the Multiple-Choice Answer Sheet provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 100

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–14)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1–10

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1 What is the negation of the statement "Violet likes sleeping and crawling"?

A. Violet likes sleeping but does not like crawling.

B. Violet does not like sleeping but likes crawling.

C. Violet does not like sleeping or does not like crawling.

D. Violet does not like sleeping and does not like crawling.

2 A particle moves in simple harmonic motion such that the relationship between its velocity v ms^{-1} and its displacement x m is given by the equation

$$v^2 = -n^2\left((x-c)^2 - a^2\right),$$

where n, c, and a are constants and $n > 1$.

Which of the following is true at the endpoints of its oscillation?

A. $n(x-c)^2 = a$

B. $n(x-c)^2 = a^2$

C. $(x-c)^2 = a$

D. $(x-c)^2 = a^2$

3 Points A, B and C are represented by the vectors $\underset{\sim}{a}=\begin{pmatrix}5\\1\\-3\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}4\\m\\-1\end{pmatrix}$ and $\underset{\sim}{c}=\begin{pmatrix}1\\-7\\n\end{pmatrix}$ respectively.

Which values of m and n will ensure that A, B and C are collinear?

A. $m=1$ and $n=-5$

B. $m=-1$ and $n=-5$

C. $m=1$ and $n=5$

D. $m=-1$ and $n=5$

4 Which of the following expressions is equivalent to $3\sqrt{3}-3i$?

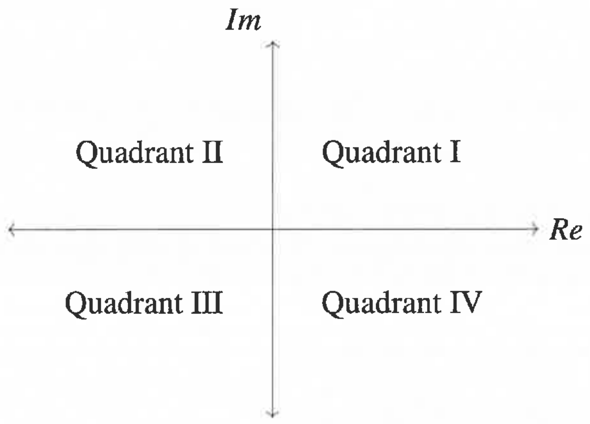
A. $6e^{\frac{5i\pi}{6}}$

B. $6e^{-\frac{i\pi}{6}}$

C. $3\sqrt{2}e^{\frac{i5\pi}{6}}$

D. $3\sqrt{2}e^{-\frac{i\pi}{6}}$

5 For this question we define the quadrants of the Argand diagram to be as follows:



In which quadrants of the Argand diagram do the square roots of $-3-4i$ lie?

A. Quadrants I and III

B. Quadrants I and IV

C. Quadrants II and III

D. Quadrants II and IV

6 A projectile is launched from horizontal ground in the direction $-\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$ at a speed of 10 ms^{-1} where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are horizontal and $\underset{\sim}{k}$ is vertical.

Assuming no air resistance, at what angle, to the nearest degree, does the projectile hit the ground?

A. 35 degrees

B. 45 degrees

C. 60 degrees

D. 63 degrees

7 Which of the following expressions is equivalent to $(1+i\tan\theta)^n+(1-i\tan\theta)^n$ for all integers n?

A. $\dfrac{2\cos n\theta}{(\sin\theta)^n}$

B. $\dfrac{2\cos n\theta}{(\cos\theta)^n}$

C. $\dfrac{2\sin n\theta}{(\sin\theta)^n}$

D. $\dfrac{2\sin n\theta}{(\cos\theta)^n}$

8 Given $P(z)$ is a polynomial function with real coefficients, which of the following is true?

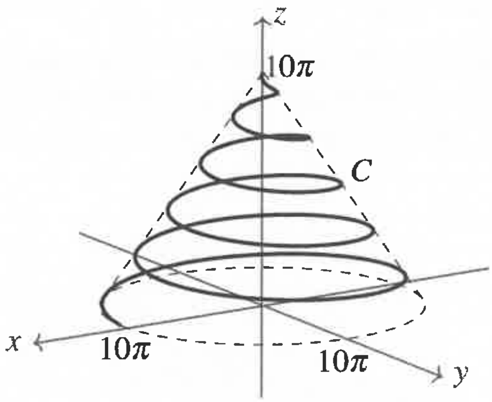
A. $\forall\alpha\in\mathbb{C}, P(\alpha)=0 \implies P(\overline{\alpha})=0$

B. $(P(\alpha)=0 \implies P(\overline{\alpha})=0)$ iff $\alpha\neq 0$

C. $(P(\alpha)=0 \implies P(\overline{\alpha})=0)$ iff $\alpha\in\mathbb{R}$

D. $(P(\alpha)=0 \implies P(\overline{\alpha})=0)$ iff $\alpha\notin\mathbb{R}$

9 A curve C spirals around a cone as shown in the diagram.



A particle moves along C, starting at the point $(0,0,10\pi)$ and ending at the point $(10\pi,0,0)$.

Which of the following vector equations best describes the path of the particle for $0 \leq t \leq 10\pi$?

A. $\underset{\sim}{r}_A = t\cos(t)\underset{\sim}{i} + t\sin(t)\underset{\sim}{j} + t\underset{\sim}{k}$

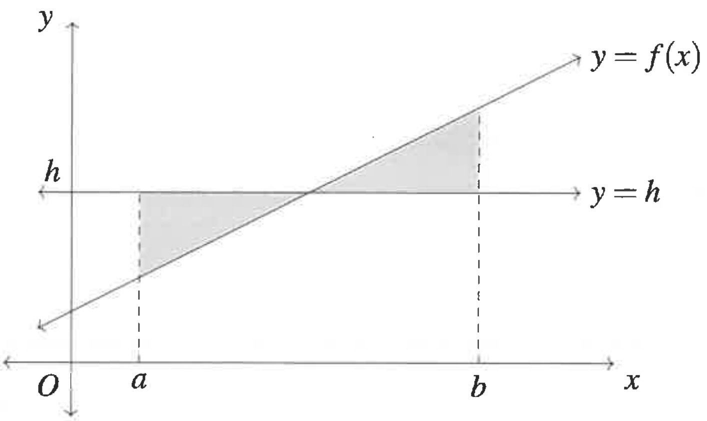
B. $\underset{\sim}{r}_B = t\sin(t)\underset{\sim}{i} + t\cos(t)\underset{\sim}{j} + t\underset{\sim}{k}$

C. $\underset{\sim}{r}_C = t\cos(t)\underset{\sim}{i} + t\sin(t)\underset{\sim}{j} + (10\pi - t)\underset{\sim}{k}$

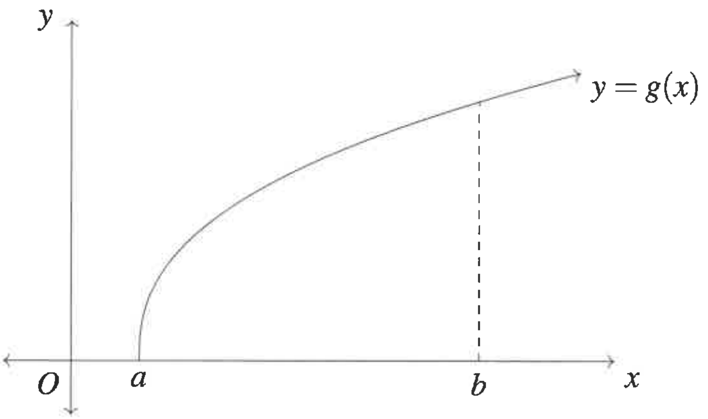
D. $\underset{\sim}{r}_D = t\sin(t)\underset{\sim}{i} + t\cos(t)\underset{\sim}{j} + (10\pi - t)\underset{\sim}{k}$

10 The average height h of a continuous function $y = f(x)$ on the interval $[a,b]$ is given by the formula $h = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

Consider the areas between the function $y = f(x)$ and the line $y = h$ on the interval $[a,b]$ as shown in the diagram. Here, h is the average height of $y = f(x)$ on $[a,b]$ if the sum of areas above $y = h$ is equal to the sum of the areas below $y = h$.



Now consider the function $y = g(x)$ in the diagram below which has the property that $g'(x) > 0$ and $g''(x) < 0$ for all $x > a$.



Let h be the average height of $y = g(x)$ on the interval $[a,b]$ where $0 < a < b$. You may want to sketch $y = h$ on the diagram above. This will not be marked.

Which of the following statements is true?

A. $h < g\left(\frac{a+b}{2}\right)$ and $g^{-1}(h) < \frac{a+b}{2}$

B. $h < g\left(\frac{a+b}{2}\right)$ and $g^{-1}(h) > \frac{a+b}{2}$

C. $h > g\left(\frac{a+b}{2}\right)$ and $g^{-1}(h) < \frac{a+b}{2}$

D. $h > g\left(\frac{a+b}{2}\right)$ and $g^{-1}(h) > \frac{a+b}{2}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

Your responses for Questions 11–16 should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Consider $z = 2 - 5i$ and $w = -3 - i$.

Find simplified expressions for the following.

(i) $z - w$ **1**

(ii) $\dfrac{z}{w}$ **1**

(iii) $z\bar{z}$ **1**

(b) Prove algebraically that the difference between two consecutive square numbers is odd. **2**

(c) Find $\displaystyle\int_{-1}^{1} \frac{12}{x^2 - 9}\,dx$. **3**

(d) Given $z = 2e^{i\frac{2\pi}{3}}$, express z^4 in the form $a + ib$. **2**

(e) Find the acute angle between the vectors $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$. Give your answer correct to the nearest degree. **2**

(f) An 80 kg solid metal ball is placed on the surface of a deep lake and released. **3**

The ball experiences acceleration due to gravity g m/s^2. It also experiences a resistive force R newtons proportional to velocity v m/s such that $R = -1000v$, and an upwards buoyancy force B of magnitude $10g$ newtons.

Assuming $g = 10$, calculate the terminal velocity of the ball in the liquid.

Question 12 (16 marks)

(a) Prove by contradiction that if a is rational and b is irrational, then $a+b$ is irrational. **2**

(b) A sequence is defined by $u_1 = 4$, and $u_n = 5u_{n-1} - 3$ for integers $n \geq 1$. Prove by mathematical induction that $u_n = \frac{1}{20}(13 \times 5^n + 15)$, for all positive integers n. **3**

(c) Consider $z = \cos\theta + i\sin\theta$.

(i) Show that $z^n + z^{-n} = 2\cos n\theta$. **1**

(ii) Hence solve $3(z^2 + z^{-2}) - (z + z^{-1}) + 2 = 0$. **3**

(d) The positions of two particles A and B in metres after t seconds are given by the vector equations below.

$$\underset{\sim}{r}_A(t) = t^2\underset{\sim}{i} + (6-t)\underset{\sim}{j} + (2t-4)\underset{\sim}{k}$$

$$\underset{\sim}{r}_B(t) = (t+2)\underset{\sim}{i} + t^2\underset{\sim}{j} + (t-2)\underset{\sim}{k}$$

(i) Find the time when the particles collide. **2**

(ii) Find the speed of particle B at the time the particles collide. **2**

(e) One tonne of sand is being carried around in a truck when the back gate falls open, spilling sand onto the street. The rate at which the mass M tonnes of sand is spilling out of the truck at time t minutes after the gate opens is given by $M'(t) = te^{-t}$ tonnes per minute. **3**

How much sand is left in the truck after 5 minutes? Give your answer correct to the nearest kilogram.

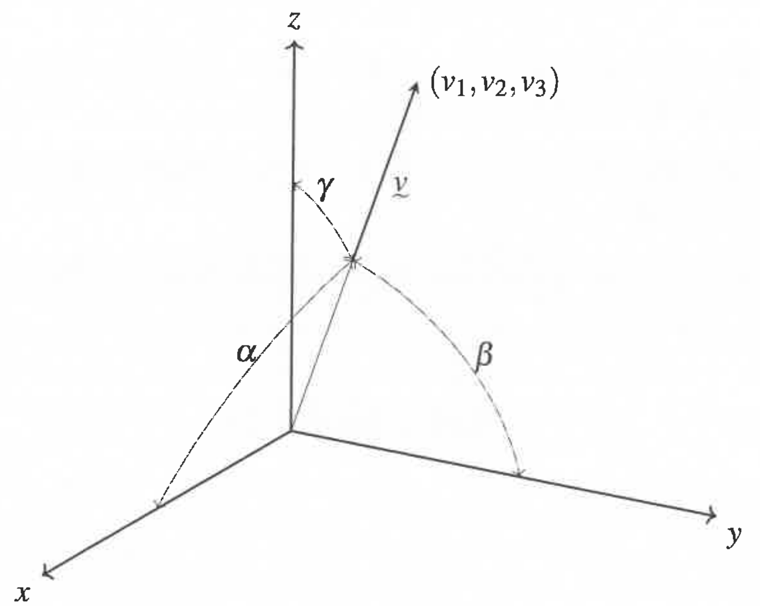
Question 13 (15 marks)

(a) Find the vector equation of the sphere $x^2+y^2+z^2+2x-14y+25=0$. **2**

(b) Given the complex number $z=x+iy$, sketch and shade the subset on the Argand diagram such that $|z| \leq \sqrt{10}$ and $\text{Im}\left(z^2\right) \geq 6$. **3**

(c) Find $\displaystyle\int \frac{x^2+4x}{x^2+4x+13}\,dx$. **3**

(d) The vector $\underset{\sim}{v}$ is represented by the point with coordinates (v_1, v_2, v_3) as shown in the diagram.



Let α, β and γ be the angles between $\underset{\sim}{v}$ and the positive x, y and z axes, respectively. The direction cosines of a vector are the cosines of the angles between the vector and the three positive coordinate axes. That is, the direction cosines of $\underset{\sim}{v}$ are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$.

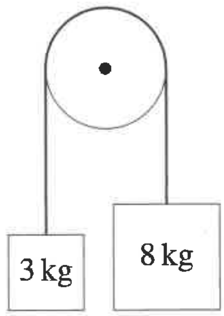
(i) Show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. **2**

(ii) A vector $\underset{\sim}{r}$ makes an angle of $72°$ with the positive x-axis and $36°$ with the positive y-axis. **1**

Find the angle of elevation of $\underset{\sim}{r}$ from the xy plane.

Question 13 continues on page 10

(e) A light inextensible string passes over a smooth pulley. Particles of mass 3 kg and 8 kg are attached to each end of the string as shown in the diagram below.



The two particles are both held at rest 1.62 metres above the floor. The system is released from rest and each mass now experiences constant air resistance of 2.5 newtons.

The 8 kg particle reaches the floor before the 3 kg particle reaches the pulley. Let the acceleration due to gravity be $9.8\ \text{m/s}^2$.

(i) Show that the tension in the string before the 8 kg mass reaches the floor is 43.9 newtons. **2**

(ii) Determine the speed at which the 8 kg particle hits the floor. **2**

End of Question 13

Question 14 (14 marks)

(a) A stationary object suspended in a vacuum is moved by two forces F_A and F_B where the magnitude of F_A is twice the magnitude of F_B. **2**

If the direction of F_A is $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and the direction of F_B is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, calculate the direction the object moves. Give your answer in vector form.

(b) The complex number $z \neq -1$ lies on the unit circle in the Argand diagram and $w = \dfrac{1}{z+1}$. **2**

Show that $\text{Re}(w) = \frac{1}{2}$.

(c) Find the two square roots of $z = e^{i\frac{\pi}{3}} + 1$. Express your answer in exponential form. **3**

(d) Consider $a, b \in \mathbb{R}$.

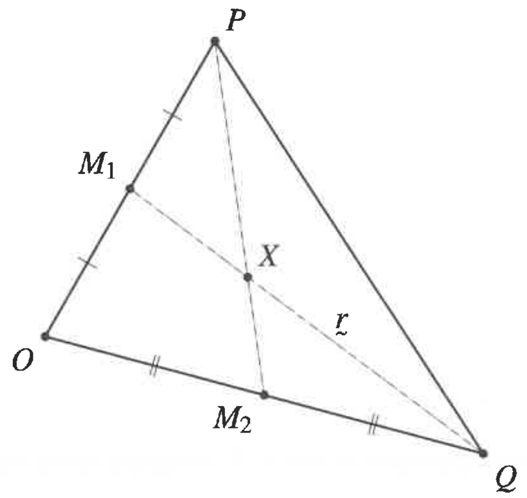
(i) Show that $a^2 + 9b^2 \geq 6ab$. **1**

(ii) Hence show that $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$. **2**

Question 14 continues on page 12

Question 14 (continued)

(e) The points P and Q have position vectors $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OQ} = \underset{\sim}{q}$ respectively, where $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are non-zero and non-parallel.



As shown in the diagram above, OPQ forms a triangle. The midpoints of OP and OQ are M_1 and M_2, respectively.

(i) Show that the vector equation of the line $\underset{\sim}{r}$ that passes through Q and M_1 is given by **1**

$$\underset{\sim}{r} = \underset{\sim}{q} + \lambda\left(\frac{1}{2}\underset{\sim}{p} - \underset{\sim}{q}\right), \text{ for } \lambda \in \mathbb{R}.$$

(ii) The point of intersection of QM_1 and PM_2 is X. Find $\underset{\sim}{x}$, the position vector of X, in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$. **3**

End of Question 14

Question 15 (15 marks)

(a) A particle moves along the x-axis according to the differential equation **4**

$$\frac{d^2x}{dt^2} = 2 - x.$$

Initially the particle is at rest at $x = 5$. Find the displacement of the particle as a function of time.

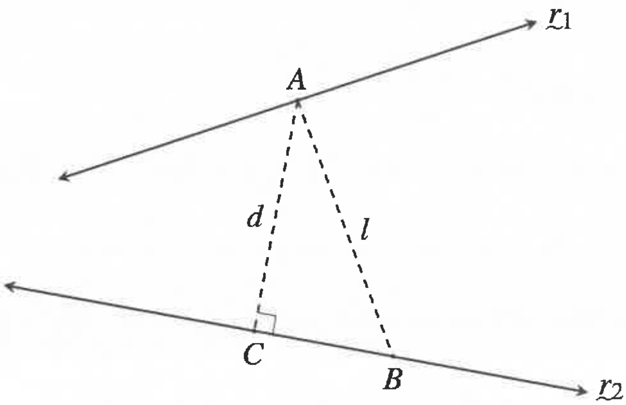
(b) Let $I_n = \int_1^e \frac{1}{x^2} (\log_e x)^n \, dx$ for all integers $n \geq 0$.

(i) Show that $I_n = nI_{n-1} - \frac{1}{e}$ for all integers $n \geq 1$. **2**

(ii) Hence show that $I_n = n! - \frac{1}{e} \left({}^nP_0 + {}^nP_1 + {}^nP_2 + \cdots + {}^nP_n \right)$. **3**

(c) Consider the lines $\underset{\sim}{r_1} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\underset{\sim}{r_2} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda, \mu \in \mathbb{R}$.

Let l represent the distance from the general point $A(-1+\lambda, 1+\lambda, 4-\lambda)$ on $\underset{\sim}{r_1}$ to the point $B(5, 3, -3)$ on $\underset{\sim}{r_2}$ as shown in the diagram.



Let d be the perpendicular distance between the point A and the line $\underset{\sim}{r_2}$.

(i) Show $l^2 = 3\lambda^2 - 30\lambda + 89$. **2**

(ii) Using a projection, or otherwise, show $d^2 = \frac{6}{5}(\lambda^2 - 6\lambda + 14)$. **3**

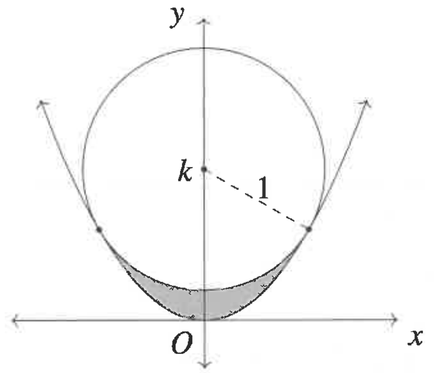
(iii) Hence find the minimum distance between the lines $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$. **1**

Question 16 (15 marks)

(a) Prove $m!n! < (m+n)!$ for positive integers m and n. **2**

(b) Find $\displaystyle\int \cos\sqrt{x}\,dx$. **3**

(c) The diagram shows a circle of radius 1, with its centre on the y-axis, that is tangent to the parabola $y = x^2$ at two distinct points. This circle has centre $(0,k)$ for some value $k > 0$ and therefore has equation $x^2 + (y-k)^2 = 1$. (Do NOT prove this.) **5**



By showing the x-values of the tangent points are $x = \pm\dfrac{\sqrt{3}}{2}$, find the exact area between the circle and the parabola.

(d) Let $f(x) = 1 + \dfrac{1}{x}$ and $\varphi = \dfrac{1+\sqrt{5}}{2}$.

Define $f \circ f(x)$ to be the composition of $f(x)$ with itself twice, that is $f \circ f(x) = f\big(f(x)\big)$.

Define $f^n(x)$ to be the function $f(x)$ composed with itself n times, for integers $n \geq 0$, that is

$$f^n(x) = \begin{cases} 1 & \text{if } n = 0 \\ \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}(x) & \text{if } n \geq 1\,. \end{cases}$$

(i) Show that $1 + \dfrac{1}{\varphi} = \varphi$ and $1 + \dfrac{1}{1-\varphi} = 1 - \varphi$. **1**

(ii) Show using mathematical induction that for all integers $n \geq 0$, **3**

$$f^n(1) = \frac{\varphi^{n+2} - (1-\varphi)^{n+2}}{\varphi^{n+1} - (1-\varphi)^{n+1}}\,.$$

(iii) Hence, or otherwise, find the value of the infinite composition $\displaystyle\lim_{n\to\infty} f^n(1)$. **1**

End of Examination

EXAMINERS

Geoff Carroll (Convenor)	Sydney Grammar School, Darlinghurst
Stephen Ewington	Ascham School, Edgecliff
Catheryn Gray	School of Mathematics and Statistics, UNSW
David Houghton	Oxley College, Burradoo
Rebekah Johnson	Loreto Kirribilli, Kirribilli
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2024

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MARKING GUIDELINES

Mathematics Extension 2

Section I
10 marks

Multiple Choice Answer Key

Question	Answer	Outcomes Assessed	Targeted Performance Bands
1	C	MEX12-2	E1-E2
2	D	MEX12-6	E1-E2
3	D	MEX12-3	E1-E2
4	B	MEX12-4	E3
5	D	MEX12-4	E3
6	A	MEX12-6	E2-E3
7	B	MEX12-4	E3-E4
8	A	MEX12-2, MEX12-4	E2-E3
9	C	MEX12-3	E3-E4
10	A	MEX12-5	E4

Question 1 (1 mark)
***Outcomes Assessed:** MEX12-2*
***Targeted Performance Bands:** E1-E2*

Solution	Mark
X: Violet likes sleeping Y: Violet likes crawling $$\neg(X \wedge Y) \Leftrightarrow (\neg X \vee \neg Y)$$ Thus the negation would be *Violet does not like sleeping or does not like crawling*. Hence C	1

Question 2 (1 mark)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E1-E2*

Solution	Mark
At endpoints of its oscillation $v = 0$. That is $n^2\left((x-c)^2 - a^2\right) = 0$. So $(x-c)^2 = a^2$. Hence D	1

Question 3 (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E1-E2*

Solution	Mark
A, B and C are collinear iff $\underset{\sim}{c} - \underset{\sim}{a} = \lambda(\underset{\sim}{b} - \underset{\sim}{a})$ for some $\lambda \in \mathbb{R}$. $\underset{\sim}{b} - \underset{\sim}{a} = -\underset{\sim}{i} + (m-1)\underset{\sim}{j} + 2\underset{\sim}{k}$ $\underset{\sim}{c} - \underset{\sim}{a} = -4\underset{\sim}{i} - 8\underset{\sim}{j} + (n+3)\underset{\sim}{k}$ Now, to equate $\underset{\sim}{i}$ components set $\lambda = 4$: $4(\underset{\sim}{b} - \underset{\sim}{a}) = -4\underset{\sim}{i} + (4m-1)\underset{\sim}{j} + 8\underset{\sim}{k}$ Equating $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components: $4m - 4 = -8 \Rightarrow m = -1$ $n + 3 = 8 \Rightarrow n = 5$ Hence D	1

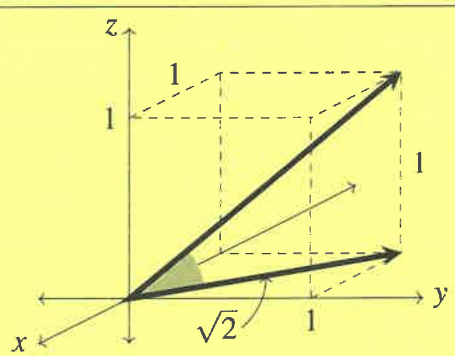
Question 4 (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E3*

Solution	Mark
Let $z = 3\sqrt{3} - 3i$. $\text{Arg}(z) = \tan^{-1}\frac{-3}{3\sqrt{3}} = -\frac{\pi}{6}$. $\lvert z\rvert = \sqrt{\left(3\sqrt{3}\right)^2 + (-3)^2} = 6$. So $3\sqrt{3} - 3i = 6e^{-\frac{i\pi}{6}}$ Hence B	1

Question 5 (1 mark)
***Outcomes Assessed:** MEX12-4*
***Targeted Performance Bands:** E3*

Solution	Mark
The complex number $-3-4i$ is in Quadrant III. By De Moivre's Theorem, halving the principal argument of $-3-4i$ indicates one of the roots of $-3-4i$ is in Quadrant IV. Hence the other root must be diagonally opposite in Quadrant II. Alternatively: $\sqrt{-3-4i} = a+ib$ $-3-4i = a^2-b^2+2abi$ Equating Re and Im gives $a^2-b^2=-3$ and $ab=-2 \Rightarrow a=\pm1$ and $b=\mp2$. So the square roots are $1-2i$ (Quadrant IV) and $-1+2i$ (Quadrant II). Hence D	1

Question 6 (1 mark)
***Outcomes Assessed:** MEX12-6*
***Targeted Performance Bands:** E2-E3*

Solution	Mark
Without air resistance, the path of flight is a parabola. By symmetry, the angle of impact equals the angle of projection. With respect to the direction vector, the vertical displacement is 1 unit against a horizontal displacement of $\sqrt{1^2+(-1)^2}=\sqrt{2}$ units. So the angle of impact is $\tan^{-1}\frac{1}{\sqrt{2}} \approx 35$ degrees.  Hence A	1

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Question 7 (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E3-E4*

Solution	Mark
$(1+i\tan\theta)^n+(1-i\tan\theta)^n = \left(\dfrac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n+\left(\dfrac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n$ $= \dfrac{\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta}{(\cos\theta)^n}$ by De Moivre's Thm $= \dfrac{2\cos n\theta}{(\cos\theta)^n}$ Hence B	1

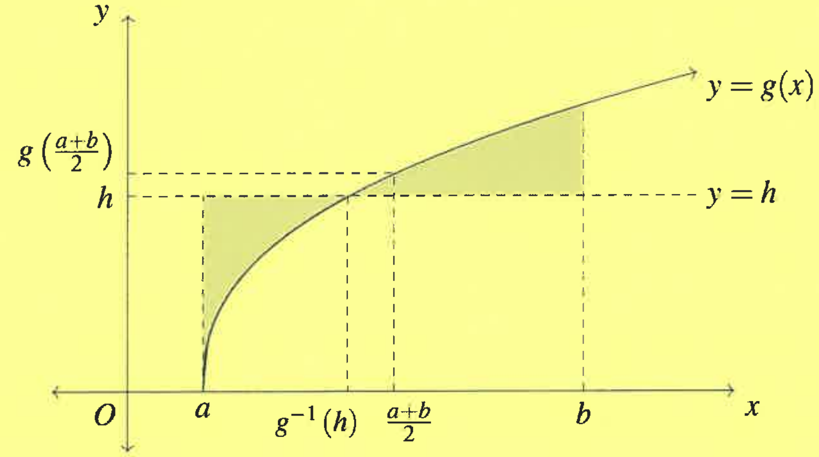
Question 8 (1 mark)
Outcomes Assessed: *MEX12-2, MEX12-4*
Targeted Performance Bands: *E2-E3*

Solution	Mark
For polynomials with real coefficients, roots occur in complex conjugate pairs. To be clear the set of complex numbers includes the set of real numbers. Further, the complex conjugate of a real number is itself. A claims the theorem applies to all complex numbers (including reals), which is correct. B claims the theorem only applies to non-zero numbers. OR B claims that if $P(0)=0$ then $P(\overline{0})\neq 0$, which is false. $(\overline{0}=0)$ C claims the theorem only applies to real numbers, which is false. D claims the theorem only applies to non-real numbers. OR D claims that if $\alpha\in\mathbb{R}$ then if $P(\alpha)=0$ then $P(\overline{\alpha})\neq 0$, which is false. $(\overline{\alpha}=\alpha)$ Hence A	1

Question 9 (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3-E4*

Solution	Mark
When $t=0$, $\underset{\sim}{r}_A=\underset{\sim}{r}_B=0\underset{\sim}{i}+0\underset{\sim}{j}+0\underset{\sim}{k}$. So only C or D works. When $t=10\pi$, $\underset{\sim}{r}_C=10\pi\cos(10\pi)\underset{\sim}{i}+10\pi\sin(10\pi)\underset{\sim}{j}+0\underset{\sim}{k}=10\pi\underset{\sim}{i}+0\underset{\sim}{j}+0\underset{\sim}{k}$. So C works. When $t=10\pi$, $\underset{\sim}{r}_D=10\pi\sin(10\pi)\underset{\sim}{i}+10\pi\cos(10\pi)\underset{\sim}{j}+0\underset{\sim}{k}=0\underset{\sim}{i}+10\pi\underset{\sim}{j}+0\underset{\sim}{k}$. So D doesn't work. Hence C	1

Question 10 (1 mark)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E4*

Solution	Mark
Consider the increasing concave down function $y = g(x)$ on the domain $[a,b]$ below with $\frac{a+b}{2}$ and $g\left(\frac{a+b}{2}\right)$ marked on the x- and y-axes respectively.  Let $h = \frac{1}{b-a}\int_a^b g(x)\,dx$ be the average height of the function over the interval $a \le x < b$. Set $y = h$ to be the line where the 'positive' and 'negative' areas between $y = g(x)$ and $y = h$ are equal. From the graph, it is clear that for an increasing concave down function $y = h$ is below $y = g\left(\frac{a+b}{2}\right)$. Dropping a line from the point of intersection of $y = g(x)$ and $y = h$ to the x-axis gives us $x = g^{-1}(h)$. From the graph, it is clear that $g^{-1}(h) < \frac{a+b}{2}$. Hence A	1

Section II

90 marks

Question 11 (15 marks)

Question 11(a) (i) (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1*

Criteria	Mark
• correct solution	1

Sample Answer:

$z - w = 2 - 5i - (-3 - i) = 5 - 4i$

Question 11(a) (ii) (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1-E2*

Criteria	Mark
• correct solution	1

Sample Answer:

$$\frac{z}{w} = \frac{2-5i}{-3-i} \times \frac{-3+i}{-3+i} = \frac{-6+2i+15i+5}{9+1} = -\frac{1}{10} + \frac{17}{10}i$$

Question 11(a) (iii) (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1-E2*

Criteria	Mark
• correct solution	1

Sample Answer:
$z\bar{z} = (2-5i)(2+5i) = 4 + 10i - 10i + 25 = 29$
OR $z\bar{z} = |z|^2 = 2^2 + 5^2 = 29$

Question 11(b) (2 marks)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	2
• setting up the proof w.r.t. a pair of consecutive numbers such as n and $n+1$	1

Sample Answer:

Let the two square numbers be n^2 and $(n+1)^2$, where $n \in \mathbb{Z}$.
$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$, which is odd.

Question 11(c) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E2*

Criteria	Marks
• substitutes limits to evaluate integral	3
• correctly integrates the expression	2
• finds correct partial fraction expression to integrate, or equivalent merit	1

Sample Answer:

$$\begin{aligned}\int_{-1}^{1}\frac{12}{x^2-9}\,dx &= \int_{-1}^{1}\frac{12}{(x-3)(x+3)}\,dx\\ &= \int_{-1}^{1}\left(\frac{2}{x+3}-\frac{2}{x-3}\right)dx\\ &= 2\Big[\ln|x+3|-\ln|x-3|\Big]_{-1}^{1}\\ &= 2\left[\ln\left|\frac{x-3}{x+3}\right|\right]_{-1}^{1}\\ &= 2\left(\ln\left|\frac{-2}{4}\right|-\ln\left|\frac{4}{2}\right|\right)\\ &= 2\ln\left(\frac{1}{4}\right)\\ &= -4\ln 2\end{aligned}$$

If $\dfrac{12}{(x-3)(x+3)} \equiv \dfrac{A}{x-3}+\dfrac{B}{x+3}$
Then $12 \equiv A(x+3)+B(x-3)$
Hence $A=2$, $B=-2$.

Question 11(d) (2 marks)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1*

Criteria	Marks
• provides correct solution	2
• applies De Moivre's theorem, or equivalent merit	1

Sample Answer:

$$\begin{aligned}\left(2e^{i\frac{2\pi}{3}}\right)^4 &= 16e^{i\frac{8\pi}{3}}\\ &= 16e^{i\frac{2\pi}{3}}\\ &= 16\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\\ &= -8+8\sqrt{3}i\end{aligned}$$

Question 11(e) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Criteria	Marks
• provides correct solution	2
• finds the dot product of $\underset{\sim}{a} \cdot \underset{\sim}{b}$, or equivalent merit	1

Sample Answer:

$$\begin{aligned}
\cos\theta &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} \\
&= \frac{(-1\times 2)+(1\times 4)+(1\times 1)}{\sqrt{1+1+1}\sqrt{4+16+1}} \\
&= \frac{3}{\sqrt{3}\times\sqrt{21}} \\
\text{So } \theta &= \cos^{-1}\left(\frac{3}{\sqrt{63}}\right) \\
&\approx 67.7923\ldots \\
&\approx 68^\circ \qquad \text{(to the nearest degree)}
\end{aligned}$$

Question 11(f) (3 marks)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	3
• state and use $\ddot{x} = 0$, or equivalent merit	2
• sets up equation resolving forces	1

Sample Answer:

Assuming down to be positive, resolving forces gives

$$\begin{aligned}
F = m\ddot{x} &= mg - 10g - 1000v \\
80\ddot{x} &= 800 - 100 - 1000v \\
&= 700 - 1000v.
\end{aligned}$$

Terminal velocity occurs when $\ddot{x} = 0$, that is when $700 - 1000v = 0$.
So $v = 0.7$ metres per second.

Question 12 (16 marks)

Question 12(a) (2 marks)
***Outcomes Assessed:** MEX12-2*
***Targeted Performance Bands:** E2-E3*

Criteria	Marks
• provides correct proof	2
• set up the proof by contradiction	1

Sample Answer:

Let's assume the result is false, that is a is rational, b is irrational but $a+b$ is rational.
Let $a = \dfrac{m}{n}$ and $a+b = \dfrac{p}{q}$ where m, n, p, and $q \in \mathbb{Z}$, also n and $q \neq 0$.
So $b = \dfrac{p}{q} - \dfrac{m}{n} = \dfrac{pn-qm}{qn}$.
Since p, q, m and n are integers, then $pn-qm$ and qn are integers.
Therefore b is rational, which is a contradiction.
Therefore the sum of a rational number and an irrational number must be irrational.

Question 12(b) (3 marks)
***Outcomes Assessed:** MEX12-2*
***Targeted Performance Bands:** E2-E3*

Criteria	Marks
• provides correct proof	3
• proves the inductive step, or equivalent merit	2
• establishes the base case, or equivalent merit	1

Sample Answer:

Step 1: By definition $u_1 = 4$, and by the formula $u_1 = \frac{1}{20}\left(13 \times 5^1 + 15\right) = 4$.
So the result is true for $n = 1$.

Step 2: Suppose the statement is true for $n = k$, that is $u_k = \frac{1}{20}\left(13 \times 5^k + 15\right)$.
Prove the statement is true for $n = k+1$. Now, by definition,

$$\begin{aligned} u_{k+1} &= 5u_k - 3 \\ &= 5 \times \tfrac{1}{20}\left(13 \times 5^k + 15\right) - 3 \\ &= \tfrac{1}{20}\left(13 \times 5 \times 5^k + 75\right) - \tfrac{60}{20} \\ &= \tfrac{1}{20}\left(13 \times 5^{k+1} + 15\right) \end{aligned}$$

Step 3: We have shown that if the result is true for $n = k$ then it is also true for $n = k+1$.
Since the result is true for $n = 1$, using the principle of mathematical induction, the result is true for all positive integers n.

Question 12(c) (i) (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1-E2*

Criteria	Mark
• correct proof	1

Sample Answer:

$$\begin{aligned} z^n + z^{-n} &= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) && \text{by De Moivre's Theorem} \\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta && \text{since sine is odd and cosine even} \\ &= 2\cos n\theta \end{aligned}$$

Question 12(c) (ii) (3 marks)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E2*

Criteria	Marks
• correct solution	3
• finds correct values of $\cos\theta$	2
• correctly substitutes result from (i), or equivalent merit	1

Sample Answer:

$$\begin{aligned} 3\left(z^2 + z^{-2}\right) - \left(z + z^{-1}\right) + 2 &= 0 \\ 3 \times 2\cos 2\theta - 2\cos\theta + 2 &= 0 && \text{from part (i)} \\ 3\cos 2\theta - \cos\theta + 1 &= 0 \\ 3(2\cos^2\theta - 1) - \cos\theta + 1 &= 0 \\ 6\cos^2\theta - \cos\theta - 2 &= 0 \\ (3\cos\theta - 2)(2\cos\theta + 1) &= 0 \end{aligned}$$

Now $\cos\theta = \dfrac{2}{3} \Rightarrow \sin\theta = \pm\dfrac{\sqrt{5}}{3}$ and $\cos\theta = -\dfrac{1}{2} \Rightarrow \sin\theta = \pm\dfrac{\sqrt{3}}{2}$.

So $z = \dfrac{2 \pm i\sqrt{5}}{3}$ or $z = \dfrac{-1 \pm i\sqrt{3}}{2}$.

Question 12(d) (i) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Criteria	Marks
• explicitly shows that $\underset{\sim}{r}_A(2) = \underset{\sim}{r}_B(2)$	2
• finds $t = 2$	1

Sample Answer:

It's simplest to equate k components of the vectors, since they are linear expressions in t.
So $2t - 4 = t - 2 \Rightarrow t = 2$ seconds.
Substituting into $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components confirms $\underset{\sim}{r}_A(2) = \underset{\sim}{r}_B(2) = 4\underset{\sim}{i} + 4\underset{\sim}{j} + 0\underset{\sim}{k}$.

Question 12(d) (ii) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	2
• correctly differentiates $\underset{\sim}{r}_B(t)$	1

Sample Answer:

Differentiating $\underset{\sim}{r}_B(t)$ gives $\underset{\sim}{v}_B(t) = \underset{\sim}{i} + 2t\underset{\sim}{j} + \underset{\sim}{k}$. When $t = 2$, $\underset{\sim}{v}_B(2) = \underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}$.

So the speed of particle B at $t = 2$ is $|\underset{\sim}{v}_B(2)| = \sqrt{1^2 + 4^2 + 1^2} = 3\sqrt{2}$ metres per second.

Question 12(e) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	3
• correct evaluation of integral	2
• sets up integral and attempts integration by parts	1

Sample Answer:

$$M'(t) = te^{-t}$$
$$u = t \Rightarrow du = dt$$
$$dv = e^{-t}dt \Rightarrow v = -e^{-t}$$

Total mass of sand spilled is given by:

$$\begin{aligned} M(5) &= \int_0^5 te^{-t}\,dt \\ &= \left[-te^{-t}\right]_0^5 - \int_0^5 -e^{-t}\,dt \\ &= -5e^{-5} - \left[e^{-t}\right]_0^5 \\ &= -5e^{-5} - e^{-5} + 1 \\ &\approx 0.9596 \end{aligned}$$

So approximately 40 kg of sand remains.

Question 13 (15 marks)

Question 13(a) (2 marks)

Outcomes Assessed: *MEX12-3*

Targeted Performance Bands: *E2*

Criteria	Marks
• correct solution	2
• completes the square on x and y, or equivalent merit	1

Sample Answer:

$$x^2+y^2+z^2+2x-14y+25=0$$
$$x^2+2x+1+y^2-14y+49+z^2=-25+1+49$$
$$(x+1)^2+(y-7)^2+z^2=25$$

Therefore $\left|\underset{\sim}{r}-\begin{pmatrix}-1\\7\\0\end{pmatrix}\right|=5$

Question 13(b) (3 marks)

Outcomes Assessed: *MEX12-4, MEX12-8*

Targeted Performance Bands: *E3*

Criteria	Marks
• correct region shaded	3
• correctly sketched both shapes and at least 1 point of intersection, or correctly sketched region without points of intersection	2
• correctly sketched circle or hyperbola	1

Sample Answer:

$|z|\le\sqrt{10}\Rightarrow x^2+y^2\le 10.$

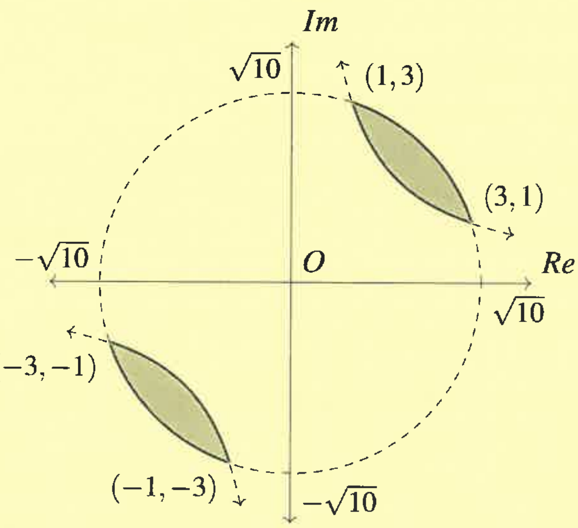
$\mathrm{Im}(z^2)\ge 6\Rightarrow \mathrm{Im}(x^2-y^2+2ixy)\ge 6$

$\Rightarrow xy\ge 3.$

Solving $x^2+y^2=10$ and $xy=3$ gives

$$x^2+\left(\tfrac{3}{x}\right)^2=10$$
$$\left(x^2\right)^2-10x^2+9=0$$
$$\left(x^2-9\right)\left(x^2-1\right)=0$$

So $x=\pm3$ or $x=\pm1$.



So the points of intersection are $3+i$, $-3-i$, $1+3i$, and $-1-3i$.
Note that $xy\ge 3$ is equivalent to $y\ge\frac{3}{x}$ when $x\ge 0$, but $y\le\frac{3}{x}$ when $x\le 0$.

Question 13(c) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	3
• completing the square in the quadratic denominator	2
• progress using a correct algebraic manipulation or a division	1

Sample Answer:

$$\begin{aligned}\int \frac{x^2+4x}{x^2+4x+13}\,dx &= \int \frac{x^2+4x+13-13}{x^2+4x+13}\,dx \\ &= \int \left(1-\frac{13}{x^2+4x+13}\right) dx \\ &= \int \left(1-\frac{13}{(x+2)^2+3^2}\right) dx \\ &= x-\frac{13}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c\end{aligned}$$

Question 13(d) (i) (2 marks)
Outcomes Assessed: *MEX12-3, MEX12-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	2
• finding an expression for $\cos\alpha$ or similar.	1

Sample Answer:

Consider the basis vectors $\underset{\sim}{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\underset{\sim}{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\underset{\sim}{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$.

For $\underset{\sim}{v}\neq\underset{\sim}{0}$, $\quad \cos\alpha=\dfrac{\underset{\sim}{v}\cdot\underset{\sim}{i}}{|\underset{\sim}{v}||\underset{\sim}{i}|}=\dfrac{v_1}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}$

Similarly, $\cos\beta=\dfrac{v_2}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}$, and $\cos\gamma=\dfrac{v_3}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}$.

$$\begin{aligned}\text{So, } \cos^2\alpha+\cos^2\beta+\cos^2\gamma &= \left(\frac{v_1}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}\right)^2+\left(\frac{v_2}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}\right)^2 \\ &\quad+\left(\frac{v_3}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}\right)^2 \\ &= \frac{{v_1}^2}{{v_1}^2+{v_2}^2+{v_3}^2}+\frac{{v_2}^2}{{v_1}^2+{v_2}^2+{v_3}^2}+\frac{{v_3}^2}{{v_1}^2+{v_2}^2+{v_3}^2} \\ &= 1\end{aligned}$$

Question 13(d) (ii) (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2-E3*

Criteria	Mark
• correct solution	1

Sample Answer:

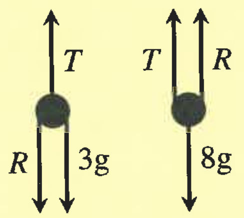
$\cos^2 72° + \cos^2 36° + \cos^2\gamma = 1$. Hence, by calculator, $\cos^2\gamma = \frac{1}{4} \Rightarrow \cos\gamma = \pm\frac{1}{2}$.
Choosing the positive acute solution gives $\gamma = 60°$.
So the angle of elevation is $90° - 60° = 30°$.

Question 13(e) (i) (2 marks)
Outcomes Assessed: *MEX12-6, MEX12-7*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	2
• resolves at least one set of forces correctly	1

Sample Answer:

Consider the forces on each mass where $g = 9.8\,\text{m/s}^2$ and air resistance $R = 2.5\,\text{N}$.



Resolving forces where $F = ma$ gives:

$8a = 8 \times 9.8 - T - 2.5$, and
$3a = -3 \times 9.8 + T - 2.5$

Adding these gives:
$11a = 5 \times 9.8 - 5 \Rightarrow a = 4$
Substituting back gives:
$32 = 8 \times 9.8 - T - 2.5$
$\therefore T = 43.9\text{N}$

Question 13(e) (ii) (2 marks)
Outcomes Assessed: *MEX12-6, MEX12-5*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	2
• integrates correctly to find an equation for displacement	1

Sample Answer:

From part (i), $a = 4$, and working w.r.t. time gives $v = 4t + c_1$.
When $t = 0$, $v = 0$ and hence $c_1 = 0$.
So $x = 2t^2 + c_2$
When $t = 0$, $x = 0$ and hence $c_2 = 0$.

Hence impact at $x = 1.62 \Rightarrow 2t^2 = 1.62$
So $t = \sqrt{0.81} = 0.9\,\text{s}$
Substituting into v gives $v = 4 \times 0.9 = 3.6$ m/s.

Question 14 (14 marks)

Question 14(a) (2 marks)

Outcomes Assessed: *MEX12-6, MEX12-3*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	2
• finds the unit direction vectors of F_A and F_B	1

Sample Answer:

The unit direction vectors of F_A and F_B are $\widehat{F_A} = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix}$ and $\widehat{F_B} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$.

Given the magnitude of F_A is twice that of F_B, the direction of the forces acting together is

$$2\widehat{F_A} + \widehat{F_B} = \frac{2}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{\sqrt{2}}{\sqrt{3}}\begin{pmatrix}-1\\2\\1\end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1-\sqrt{2}\\1+2\sqrt{2}\\1+\sqrt{2}\end{pmatrix}.$$

Note that any multiple of $\begin{pmatrix}1-\sqrt{2}\\1+2\sqrt{2}\\1+\sqrt{2}\end{pmatrix}$ is valid.

Question 14(b) (2 marks)

Outcomes Assessed: *MEX12-4*

Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	2
• correctly realises denominator or equivalent merit	1

Sample Answer:

Write z in Cartesian form as $z = x + iy$.
Since z lies on the unit circle, $x^2 + y^2 = 1$.

$$\text{Then } w = \frac{1}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$
$$= \frac{(x+1)-iy}{(x+1)^2 - i^2y^2}$$

$$\text{Thus } \mathrm{Re}(w) = \frac{x+1}{x^2+2x+1+y^2}$$
$$= \frac{x+1}{2x+2}, \quad (\text{since } x^2+y^2=1)$$
$$= \frac{1}{2}$$

Question 14(c) (3 marks)
***Outcomes Assessed:** MEX12-4, MEX12-8*
***Targeted Performance Bands:** E3*

Criteria	Marks
• correct solution	3
• finding two arguments for the square root of z	2
• finds $\operatorname{Arg}(z)$ and $\|z\|$	1

Sample Answer:

$$z = e^{i\frac{\pi}{3}} + 1 = \cos\tfrac{\pi}{3} + 1 + i\sin\tfrac{\pi}{3}$$
$$= \tfrac{3}{2} + i\tfrac{\sqrt{3}}{2}$$

So $\operatorname{Arg}(z) = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ and

$|z| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$

Let $w^2 = z$ and by De Moivre's theorem,
$2\operatorname{Arg}(w) = \frac{\pi}{6}$ or $-\frac{11\pi}{6}$ and $|w| = 3^{1/4}$.

So the two roots are
$3^{1/4}e^{i\pi/12}$ and $3^{1/4}e^{-11i\pi/12}$.

Question 14(d) (i) (1 mark)
***Outcomes Assessed:** MEX12-2*
***Targeted Performance Bands:** E2-E3*

Criteria	Mark
• provides correct proof	1

Sample Answer:

$$(a-3b) \in \mathbb{R} \Rightarrow (a-3b)^2 \geq 0$$
$$a^2 - 6ab + 9b^2 \geq 0$$
$$a^2 + 9b^2 \geq 6ab$$

Question 14(d) (ii) (2 marks)
***Outcomes Assessed:** MEX12-2*
***Targeted Performance Bands:** E3*

Criteria	Marks
• provides correct proof	2
• some progress towards correct result	1

Sample Answer:

From (i): $a^2 + 9b^2 \geq 6ab$
$a^2 + 9c^2 \geq 6ac$
$b^2 + 9c^2 \geq 6bc$

Adding these inequalities gives

$$2a^2 + 10b^2 + 18c^2 \geq 6(ab + ac + bc)$$
$$a^2 + 5b^2 + 9c^2 \geq 3(ab + ac + bc),$$

as required.

Question 14(e) (i) (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Criteria	Mark
• correct solution	1

Sample Answer:

$\overrightarrow{OM_1} = \frac{1}{2}\underset{\sim}{p}$. The direction vector of $\underset{\sim}{r}$ is $\overrightarrow{QM_1} = \overrightarrow{OM_1} - \overrightarrow{OQ} = \frac{1}{2}\underset{\sim}{p} - \underset{\sim}{q}$.

Since $\underset{\sim}{r}$ passes through the point Q, the equation is $\underset{\sim}{r} = \underset{\sim}{q} + \lambda\left(\frac{1}{2}\underset{\sim}{p} - \underset{\sim}{q}\right)$ as required.

Question 14(e) (ii) (3 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correct solution	3
• finds μ and λ, or similar progress	2
• finds equation of line through PM_2, or similar progress	1

Sample Answer:

As in part (i), it can be shown that $\underset{\sim}{s}$, the equation for the line through PM_2, is given by

$$\underset{\sim}{s} = \underset{\sim}{p} + \mu\left(\frac{1}{2}\underset{\sim}{q} - \underset{\sim}{p}\right) \text{ for some scalar } \mu.$$

To find X, we equate $\underset{\sim}{r}$ and $\underset{\sim}{s}$. Thus $\underset{\sim}{p} + \mu\left(\frac{1}{2}\underset{\sim}{q} - \underset{\sim}{p}\right) = \underset{\sim}{q} + \lambda\left(\frac{1}{2}\underset{\sim}{p} - \underset{\sim}{q}\right)$

Rearranging, we have $\left(1 - \mu - \frac{\lambda}{2}\right)\underset{\sim}{p} - \left(1 - \lambda - \frac{\mu}{2}\right)\underset{\sim}{q} = \underset{\sim}{0}$

However, since the vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are not parallel, this is only possible if both

$1 - \mu - \frac{\lambda}{2} = 0$, AND $1 - \lambda - \frac{\mu}{2} = 0$,
which has the solution $\lambda = \mu = \frac{2}{3}$. Hence $\underset{\sim}{x} = \underset{\sim}{p} + \frac{2}{3}\left(\frac{1}{2}\underset{\sim}{q} - \underset{\sim}{p}\right) = \frac{1}{3}\left(\underset{\sim}{p} + \underset{\sim}{q}\right)$.

Question 15 (15 marks)

Question 15(a) (4 marks)

Outcomes Assessed: *MEX12-6, MEX12-7*

Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correct solution	4
• finds correct expression for t as an intergral dx or equivalent merit	3
• makes a justification for their choice of $v < 0$ or $v > 0$ for dx/dt	2
• finds correct expression for v^2 or equivalent merit	1

Sample Answer:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - x$$

$$\frac{1}{2}v^2 = \int (2-x)\,dx$$

$$\frac{1}{2}v^2 = 2x - \frac{x^2}{2} + c_1$$

$$v^2 = 4x - x^2 + c_2$$

When $x = 5$, $v = 0$, so $0 = 20 - 25 + c_2$.

$\therefore c_2 = 5 \Rightarrow v^2 = 4x - x^2 + 5$.

When $x = 5$, $v = 0$ and $a = 2 - x = -3$, so the particle's subsequent motion will be towards the origin, i.e. $v < 0$.

$$v = \frac{dx}{dt} = -\sqrt{4x - x^2 + 5}$$

$$\text{So } \frac{dt}{dx} = \frac{-1}{\sqrt{4x - x^2 + 5}}$$

$$= \frac{-1}{\sqrt{3^2 - (x-2)^2}}$$

$$\text{So } t = \int \frac{-1}{\sqrt{3^2 - (x-2)^2}}\,dx$$

$$= -\sin^{-1}\left(\frac{x-2}{3}\right) + c_3$$

When $t = 0$, $x = 5$, so $0 = -\sin^{-1} 1 + c_3$.

$\therefore c_3 = \frac{\pi}{2} \Rightarrow t = -\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{\pi}{2}$.

Rearranging gives $x = 2 - 3\sin\left(t - \frac{\pi}{2}\right)$.

Note, $x = 3\cos t + 2$, $x = 2 - 3\cos(t + \pi)$ and $x = 2 + 3\sin\left(t + \frac{\pi}{2}\right)$ are acceptable also.

Question 15(b) (i) (2 marks)

Outcomes Assessed: *MEX12-5*

Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	2
• progress towards integration by parts	1

Sample Answer:

Applying integration by parts to

$$I_n = \int_1^e \frac{1}{x^2}(\log_e x)^n\,dx$$

$$u = (\log_e x)^n \Rightarrow du = n(\log_e x)^{n-1} \times \frac{1}{x}dx$$

$$dv = x^{-2}\,dx \Rightarrow v = -x^{-1}$$

$$\text{So } I_n = \left[-\frac{1}{x}(\log_e x)^n\right]_1^e - \int_1^e -\frac{n(\log_e x)^{n-1}}{x^2}\,dx$$

$$= \left(-\frac{1}{e} + 0\right) + n\int_1^e \frac{(\log_e x)^{n-1}}{x^2}\,dx$$

$$= nI_{n-1} - \frac{1}{e} \text{ as required.}$$

Question 15(b) (ii) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correct solution	3
• correctly iterates to I_0 and evaluates I_0 or equivalent merit	2
• correctly iterates to I_{n-3}	1

Sample Answer:

$$I_n = \int_1^e \frac{1}{x^2}(\log_e x)^n\,dx$$
$$= nI_{n-1} - \frac{1}{e}$$
$$= n\left[(n-1)I_{n-2} - \frac{1}{e}\right] - \frac{1}{e}$$
$$= n(n-1)I_{n-2} - \frac{1}{e}(1+n)$$
$$= n(n-1)\left[(n-2)I_{n-3} - \frac{1}{e}\right] - \frac{1}{e}(1+n)$$
$$= n(n-1)(n-2)I_{n-3} - \frac{1}{e}(1+n+n(n-1))$$

Iterating n times gives

$$I_n = n(n-1)(n-2)\times\cdots\times 1\times I_0 - \frac{1}{e}\left[1+n+n(n-1)+\cdots+\Big(n(n-1)(n-2)\times\cdots\times 2\Big)\right]$$
$$= n!I_0 - \frac{1}{e}\left({}^nP_0 + {}^nP_1 + {}^nP_2 + \cdots + {}^nP_{n-1}\right)$$

But $I_0 = \displaystyle\int_1^e \frac{1}{x^2}\,dx = -\left[\frac{1}{x}\right]_1^e = 1 - \frac{1}{e}$. Then substituting this in gives:

$$I_n = n!\left(1-\frac{1}{e}\right) - \frac{1}{e}\left({}^nP_0 + {}^nP_1 + {}^nP_2 + \cdots + {}^nP_{n-1}\right) \qquad \text{and since } n! = {}^nP_n$$
$$= n! - \frac{1}{e}\left({}^nP_0 + {}^nP_1 + {}^nP_2 + \cdots + {}^nP_{n-1} + {}^nP_n\right) \text{ as required.}$$

Question 15(c) (i) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	2
• finds an expression for l^2 in terms of λ	1

Sample Answer:

Applying vector distance formula gives

$$l^2 = \left|\overrightarrow{AB}\right|^2 = (5-(-1+\lambda))^2 + (3-(1+\lambda))^2 + (-3-(4-\lambda))^2$$
$$= (6-\lambda)^2 + (2-\lambda)^2 + (-7+\lambda)^2$$
$$= 36 - 12\lambda + \lambda^2 + 4 - 4\lambda + \lambda^2 + 49 - 14\lambda + \lambda^2$$
$$= 3\lambda^2 - 30\lambda + 89 \text{ as required.}$$

Question 15(c) (ii) (3 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	3
• determining the magnitude of $\overrightarrow{BC}$, or equivalent merit	2
• evaluation of $\overrightarrow{BC}$ using a projection, or equivalent merit	1

Sample Answer:

Project $\overrightarrow{BA} = \begin{pmatrix} -1+\lambda \\ 1+\lambda \\ 4-\lambda \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -6+\lambda \\ -2+\lambda \\ 7-\lambda \end{pmatrix}$ onto the direction vector of $\underset{\sim}{r_2}$: $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

$$\overrightarrow{BC} = \text{proj}_{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} -6+\lambda \\ -2+\lambda \\ 7-\lambda \end{pmatrix} = \frac{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -6+\lambda \\ -2+\lambda \\ 7-\lambda \end{pmatrix}}{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{12-2\lambda+7-\lambda}{4+1} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \frac{19-3\lambda}{5} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$d^2 = l^2 - \left|\overrightarrow{BC}\right|^2 = 3\lambda^2 - 30\lambda + 89 - \frac{(19-3\lambda)^2}{25}\left((-2)^2+1^2\right)$$

$$= 3\lambda^2 - 30\lambda + 89 - \frac{361-114\lambda+9\lambda^2}{5} = \frac{6}{5}(\lambda^2 - 6\lambda + 14) \text{ as required.}$$

Question 15(c) (iii) (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3-E4*

Criteria	Mark
• correct solution	1

Sample Answer:

Completing the square on the expression for d^2 gives

$d^2 = \frac{6}{5}(\lambda^2 - 6\lambda + 14) = \frac{6}{5}\left((\lambda-3)^2+5\right)$, which is minimum when $\lambda = 3$.

So the minimum distance between $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ is $d = \sqrt{\frac{6}{5}(5)} = \sqrt{6}$ units.

Question 16 (15 marks)

Question 16(a) (2 marks)

Outcomes Assessed: *MEX12-2*

Targeted Performance Bands: *E3-E4*

Criteria	Marks
• provides correct proof	2
• deduces that $m < m+n$, or equivalent merit	1

Sample Answer:

For $n > 1$, the following inequalities are true

$$1 < n+1$$
$$2 < n+2$$
$$\vdots$$
$$m < n+m$$

Multiplying these inequalities gives

$$m! < (n+1)(n+2)(n+3)\times\cdots\times(n+m)$$

Multiplying both sides by $n!$ gives

$$m!n! < n!(n+1)(n+2)(n+3)\times\cdots\times(n+m) = (n+m)! \text{ as required.}$$

Question 16(b) (3 marks)

Outcomes Assessed: *MEX12-6, MEX12-7*

Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correct solution	3
• correct reformulation of integral using parts	2
• correct substitution $\theta = \sqrt{x}$ or $\theta^2 = x$, or equivalent merit	1

Sample Answer:

Let $\theta^2 = x \Rightarrow 2\theta\, d\theta = dx$.

So $\displaystyle\int \cos\sqrt{x}\,dx = 2\int \theta\cos\theta\, d\theta$

Now, let $u = \theta \Rightarrow du = d\theta$

and $dv = \cos\theta\, d\theta \Rightarrow v = \sin\theta$

$$= 2\left(\theta\sin\theta - \int \sin\theta\, d\theta\right)$$
$$= 2(\theta\sin\theta + \cos\theta) + c$$
$$= 2\left(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) + c$$

Question 16(c) (5 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E4*

Criteria	Marks
• correct solution	5
• makes correct substitution into integral, or equivalent merit	4
• finds correct expression for the area, or equivalent merit	3
• finds x-values of the tangent points	2
• sets up simultaneous equation, or equivalent merit	1

Sample Answer:

Solving simultaneously gives $y+(y-k)^2=1 \quad \Rightarrow \quad y^2+(1-2k)y+k^2-1=0$.
This is a quadratic in y with one solution so $\Delta=0$
$\Rightarrow (1-2k)^2-4(k^2-1)=0 \quad \Rightarrow \quad 1-4k+4k^2-4k^2+4=0 \quad \Rightarrow \quad k=\frac{5}{4}$
Substituting this back in gives: $y^2+\left(1-\frac{5}{2}\right)y+\frac{25}{16}-1=0 \quad \Rightarrow \quad y^2-\frac{3}{2}y+\frac{9}{16}=0$
$\Rightarrow \quad \left(y-\frac{3}{4}\right)^2=0$. So $y=\frac{3}{4}$ and $x=\pm\frac{\sqrt{3}}{2}$.
For eqn of semicircle: $x^2+(y-\frac{5}{4})^2=1 \Rightarrow y-\frac{5}{4}=-\sqrt{1-x^2} \Rightarrow y=\frac{5}{4}-\sqrt{1-x^2}$.

$$\text{So Area } = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(\tfrac{5}{4}-\sqrt{1-x^2}-x^2\right) dx$$

$$= 2\int_0^{\frac{\sqrt{3}}{2}} \left(\tfrac{5}{4}-\sqrt{1-x^2}-x^2\right) dx$$

$$= 2\left[\tfrac{5}{4}x-\tfrac{1}{3}x^3\right]_0^{\frac{\sqrt{3}}{2}} - 2\int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\,dx$$

Now, let $x=\sin\theta$, $dx=\cos\theta\,d\theta$

$$= 2\left[\tfrac{5\sqrt{3}}{8}-\tfrac{\sqrt{3}}{8}\right] - 2\int_0^{\frac{\pi}{3}} \sqrt{1-\sin^2\theta}\cos\theta\,d\theta$$

$$\therefore \text{Area } = \sqrt{3}-\int_0^{\frac{\pi}{3}} 2\cos^2\theta\,d\theta$$

$$= \sqrt{3}-\int_0^{\frac{\pi}{3}} (\cos 2\theta+1)\,d\theta$$

$$= \sqrt{3}-\left[\tfrac{1}{2}\sin 2\theta+\theta\right]_0^{\frac{\pi}{3}}$$

$$= \sqrt{3}-\left(\tfrac{\sqrt{3}}{4}+\tfrac{\pi}{3}\right)$$

$$= \tfrac{3\sqrt{3}}{4}-\tfrac{\pi}{3} \text{ units}^2 \approx 0.25 \text{ units}^2$$

Question 16(d) (i) (1 mark)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E2-E3*

Criteria	Mark
• correct solution	1

Sample Answer:

$$1+\tfrac{1}{\varphi} = 1+\tfrac{2}{1+\sqrt{5}} = \tfrac{3+\sqrt{5}}{1+\sqrt{5}}\times\tfrac{1-\sqrt{5}}{1-\sqrt{5}} = \tfrac{-2-2\sqrt{5}}{-4} = \tfrac{1+\sqrt{5}}{2} = \varphi \text{ as required.}$$

$$1+\tfrac{1}{1-\varphi} = 1+\tfrac{2}{1-\sqrt{5}} = \tfrac{3-\sqrt{5}}{1-\sqrt{5}}\times\tfrac{1+\sqrt{5}}{1+\sqrt{5}} = \tfrac{-2+2\sqrt{5}}{-4} = \tfrac{1-\sqrt{5}}{2} = 1-\varphi \text{ as required.}$$

Question 16(d) (ii) (3 marks)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E4*

Criteria	Marks
• correct solution	3
• correctly using assumption	2
• correctly showing $f^0(1) = 1$	1

Sample Answer:

Step 1: Show $f^0(1) = 1$.

$$f^0(1) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}$$

$$= \frac{\left(1+2\sqrt{5}+5\right) - \left(1-2\sqrt{5}+5\right)}{2\left(1+\sqrt{5}-1+\sqrt{5}\right)}$$

$$= \frac{4\sqrt{5}}{4\sqrt{5}} = 1 \text{ as required. So true for } n = 0.$$

OR $f^0(1) = \dfrac{\varphi^2 - (1-\varphi)^2}{\varphi^1 - (1-\varphi)^1}$

$$= \frac{\varphi^2 - 1 + 2\varphi - \varphi^2}{2\varphi - 1}$$

$$= \frac{2\varphi - 1}{2\varphi - 1} = 1, \text{ as required.}$$

So true for $n = 0$.

Step 2: Suppose the statement is true for $n = k$, that is $f^k(1) = \dfrac{\varphi^{k+2} - (1-\varphi)^{k+2}}{\varphi^{k+1} - (1-\varphi)^{k+1}}$.

We are now r.t.p. the statement is true for $n = k+1$, that is $f^{k+1}(1) = \dfrac{\varphi^{k+3} - (1-\varphi)^{k+3}}{\varphi^{k+2} - (1-\varphi)^{k+2}}$.

$$f^{k+1}(1) = f \circ f^k(1)$$

$$= 1 + \frac{1}{f^k(1)}$$

$$= 1 + \frac{\varphi^{k+1} - (1-\varphi)^{k+1}}{\varphi^{k+2} - (1-\varphi)^{k+2}} \qquad \text{by assumption}$$

$$= \frac{\varphi^{k+2} - (1-\varphi)^{k+2} + \varphi^{k+1} - (1-\varphi)^{k+1}}{\varphi^{k+2} - (1-\varphi)^{k+2}}$$

$$= \frac{\varphi^{k+2} - (1-\varphi)^{k+2} + \varphi^{k+2}\left(\frac{1}{\varphi}\right) - (1-\varphi)^{k+2}\left(\frac{1}{1-\varphi}\right)}{\varphi^{k+2} - (1-\varphi)^{k+2}}$$

$$= \frac{\varphi^{k+2}\left(1 + \frac{1}{\varphi}\right) - (1-\varphi)^{k+2}\left(1 + \frac{1}{1-\varphi}\right)}{\varphi^{k+2} - (1-\varphi)^{k+2}}$$

$$= \frac{\varphi^{k+3} - (1-\varphi)^{k+3}}{\varphi^{k+2} - (1-\varphi)^{k+2}} \text{ from part (i) as required.}$$

Step 3: We have shown that if the result is true for $n = k$ then it is also true for $n = k+1$.
Since the result is true for $n = 0$, using the principle of mathematical induction, the result is true for all integers $n \geq 0$.

Question 16(d) (iii) (1 mark)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E3-E4*

Criteria	Mark
• correct solution	1

Sample Answer:

$$\begin{aligned}
\lim_{n\to\infty} f^n(1) &= \lim_{n\to\infty} \frac{\varphi^{n+2}-(1-\varphi)^{n+2}}{\varphi^{n+1}-(1-\varphi)^{n+1}} \\
&= \lim_{n\to\infty} \frac{\varphi^{n+2}}{\varphi^{n+1}} \qquad \text{as } |1-\varphi|<1,\ \lim_{k\to\infty}(1-\varphi)^k = 0 \\
&= \lim_{n\to\infty} \frac{\varphi}{1} \qquad \text{dividing top and bottom by } \varphi^{n+1} \\
&= \varphi
\end{aligned}$$