Neap

Trial Examination 2021

Year 12 Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 3 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • For questions in Section II, show relevant mathematical reasoning and/or calculations
Total Marks: 100	**Section I – 10 marks (pages 2–5)** • Attempt Questions 1– 10 • Allow about 15 minutes for this section **Section II – 90 marks (pages 6–13)** • Attempt Questions 11– 16 • Allow about 2 hours and 45 minutes for this section

SECTION I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Let $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$.
Which of the following is equal to $2\underset{\sim}{a} - \underset{\sim}{b}$?

A. $7\underset{\sim}{i} - 10\underset{\sim}{j} - \underset{\sim}{k}$

B. $7\underset{\sim}{i} - 10\underset{\sim}{j} + 5\underset{\sim}{k}$

C. $5\underset{\sim}{i} - 10\underset{\sim}{j} + 5\underset{\sim}{k}$

D. $5\underset{\sim}{i} - 10\underset{\sim}{j} - \underset{\sim}{k}$

2 Let $z = \dfrac{\sqrt{3} - i}{1 + i}$.
What is the modulus and argument of z?

A. $\dfrac{1}{\sqrt{2}}$ and $-\dfrac{5\pi}{12}$

B. $\dfrac{1}{\sqrt{2}}$ and $-\dfrac{\pi}{12}$

C. $\sqrt{2}$ and $-\dfrac{5\pi}{12}$

D. $\sqrt{2}$ and $-\dfrac{\pi}{12}$

3 What is the Cartesian equation of the line $\underset{\sim}{r} = \begin{pmatrix} -5 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \end{pmatrix}$?

A. $2y + 3x = -3$

B. $2x - 3y = -8$

C. $3y + 2x = 8$

D. $3x - 2y = 28$

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4 In the complex plane, a circle C has diameter AB, where the points A and B represent the complex numbers $1-5i$ and $3-i$ respectively.
What is the equation of C?

A. $|z-2+3i| = 2\sqrt{5}$

B. $|z-2+3i| = \sqrt{5}$

C. $|z+2-3i| = 2\sqrt{5}$

D. $|z+2-3i| = \sqrt{5}$

5 A particle moves in a straight line with simple harmonic motion about a centre O. The period of the motion is π seconds. When the particle is 0.50 metres from O, its speed is 2.40 m/s.
What is the particle's maximum speed?

A. 2.40 m/s

B. 2.60 m/s

C. 3.38 m/s

D. 5.20 m/s

6 Which of the following expressions is equal to $\int \frac{8x+1}{x^2+9} dx$?

A. $8\ln\left(x^2+9\right)+3\tan^{-1}\frac{x}{3}+c$

B. $8\ln\left(x^2+9\right)+\frac{1}{3}\tan^{-1}\frac{x}{3}+c$

C. $4\ln\left(x^2+9\right)+3\tan^{-1}\frac{x}{3}+c$

D. $4\ln\left(x^2+9\right)+\frac{1}{3}\tan^{-1}\frac{x}{3}+c$

7 Given that x and y are natural numbers, which of the following is a FALSE statement?

A. $\forall x \exists y \left(x-y=0\right)$

B. $\forall x \exists y \left(3x-y=0\right)$

C. $\forall x \exists y \left(x-3y=0\right)$

D. $\exists x \exists y \left(x+y=8\right)$

8 Consider the statement, where p and q are odd primes and r and s are positive non-primes.

‘For all odd primes $p < q$ there exists positive non-primes $r < s$ such that $p^2 + q^2 = r^2 + s^2$.’

Which of the following is the negation of the statement?

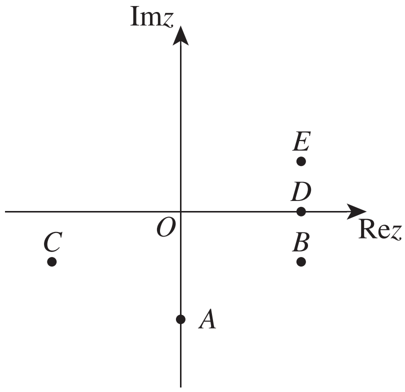
A. For all odd primes $p < q$ there exists positive non-primes $r < s$ such that $p^2 + q^2 = r^2 + s^2$.

B. For all odd primes $p < q$ and for all positive non-primes $r < s, p^2 + q^2 \neq r^2 + s^2$.

C. There exists odd primes $p < q$ such that for all positive non-primes $r < s, p^2 + q^2 = r^2 + s^2$.

D. There exists odd primes $p < q$ such that for all positive non-primes $r < s, p^2 + q^2 \neq r^2 + s^2$.

9 Consider the Argand diagram, where $z = a + ib$.



Which of the following pairs of points in the complex plane could represent the square roots of z?

A. A and D

B. B and C

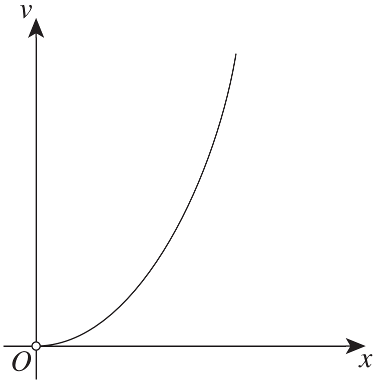
C. B and E

D. C and E

10 A particle moves in a straight line such that its acceleration a is given by $a = vx$, where v is the particle's velocity, x is the particle's position and $v, x > 0$.

Which of the following diagrams best shows the relationship between v and x?

A.



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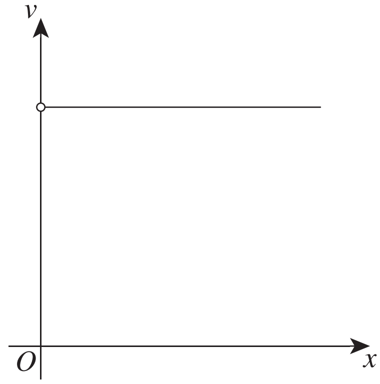
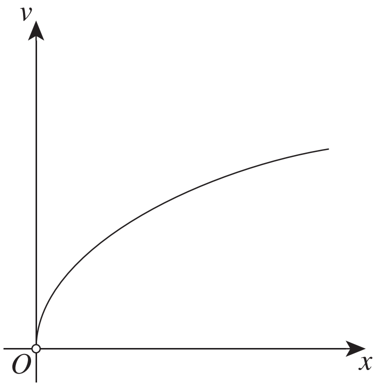
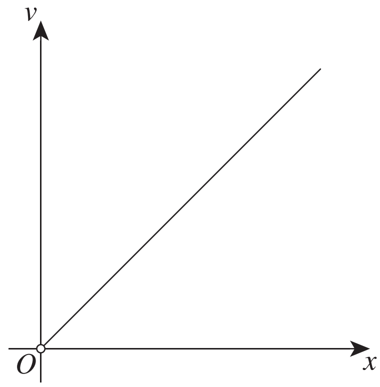


C.



D.



SECTION II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The quadratic equation $z^2 + pz + q = 0$, where p and q are real, has a root $\sqrt{5} - i$. **2**

Find the values of p and q.

(b) Use integration by parts to evaluate $\int_0^1 \sin^{-1} x dx$. **3**

(c) A particle moves in a straight line. At time t seconds, its displacement from a fixed origin is x metres and its velocity is v m/s. The acceleration of the particle is given by $\ddot{x} = x + 3$. At $t = 0$, the particle is at the origin and moving with velocity 3 m/s.

(i) Show that $v = x + 3$. **2**

(ii) Find an expression for x, the displacement of the particle, in terms of t. **2**

(d) Consider the vectors $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}, \underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{c} = 4\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}$, where $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{c}$. Let $\hat{\underset{\sim}{u}} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, where $x > 0$, be a unit vector perpendicular to both $\underset{\sim}{b}$ and $\underset{\sim}{c}$. **3**

Find $\hat{\underset{\sim}{u}}$ and hence show that $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are mutually perpendicular.

(e) Consider $z = \dfrac{2}{1 - e^{2i\theta}}$. **3**

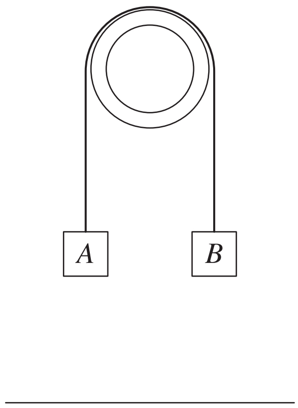
Show that $z = 1 + i \cot \theta$.

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is undergoing simple harmonic motion about a fixed origin with period $\frac{\pi}{3}$ seconds. Initially, the particle is at rest 4 metres to the right of the origin. **3**

Find the first TWO times when the particle's speed is half its maximum speed.

(b) Two particles, A and B, have masses $5m$ kg and km kg respectively, where $k < 5$. The particles are connected by a light, inextensible string that passes over a smooth light fixed pulley. The system is held at rest with the string taut, the hanging parts of the string are vertical and A and B are at the same height above a horizontal floor as shown.



The system is released from rest and particle A descends with acceleration $\frac{g}{4}$ m/s^2.

(i) Show that the tension in the string as particle A descends is $\frac{15mg}{4}$ newtons. **2**

(ii) Find the value of k. **2**

(iii) After descending for 1 second, particle A impacts the floor and is immediately brought to rest. In its subsequent motion, particle B does not reach the pulley. **3**

Show that the greatest height reached by particle B above the floor is $\frac{9g}{32}$ metres.

(c) A subset of the complex plane is described by the relation $\left|z-\left(2\sqrt{2}+2\sqrt{2}i\right)\right| \leq 2$.

(i) Draw a sketch of this relation. **1**

(ii) Given that z is a complex number that satisfies the relation, find the minimum and maximum values of $|z|$. **2**

(iii) Given that z is a complex number that satisfies the relation, find the minimum and maximum values of $\operatorname{Arg} z$. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet.

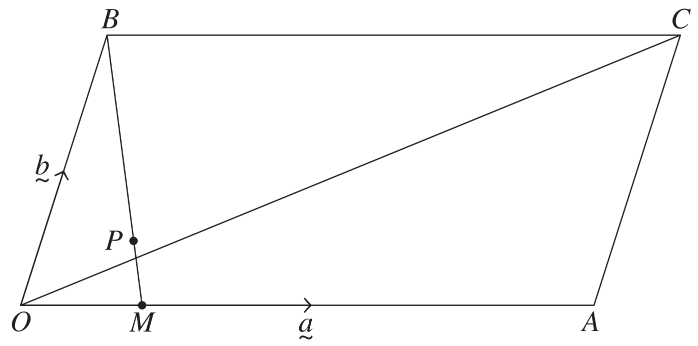
(a) Consider the statement, where x is an integer.

'If $x^2 - 6x + 5$ is even, then x is odd.'

(i) Write down the contrapositive of the statement. **1**

(ii) Prove the statement by proving the contrapositive. **2**

(b) Let $OACB$ be a parallelogram with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. M is a point on OA such that $\left|\overrightarrow{OM}\right| = \frac{1}{5}\left|\overrightarrow{OA}\right|$. P is a point on MB such that $\left|\overrightarrow{MP}\right| = \frac{1}{6}\left|\overrightarrow{MB}\right|$, as shown in the diagram.



(i) Show that P lies on OC. **3**

(ii) State the ratio of lengths $OP : PC$. **1**

(c) (i) Show that, for any integer n, $e^{in\theta} + e^{-in\theta} = 2\cos n\theta$. **1**

(ii) By expanding $\left(e^{i\theta} + e^{-i\theta}\right)^5$, show that $\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$. **3**

(iii) Hence, or otherwise, find $\int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta$. **2**

(iv) Using the result of part (ii), solve the equation $\cos 5\theta + 5\cos 3\theta + 9\cos\theta = 0$ for $0 \le \theta \le \pi$. **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove that $\dfrac{a+b}{2} \geq \sqrt{ab}$ for all positive real numbers a and b. **1**

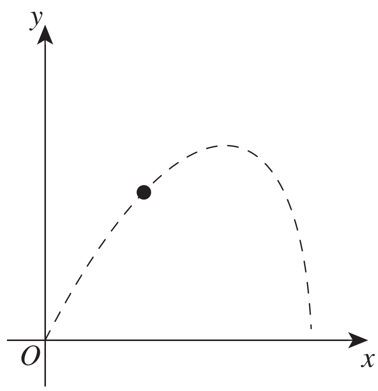
(ii) Prove that $(a+b)(b+c)(c+a) \geq 8abc$ for all positive real numbers a, b and c. **1**

(iii) Suppose that x, y and z are the sides of a triangle. **2**

Using the result from part (ii), deduce that $xyz \geq (y+z-x)(z+x-y)(x+y-z)$.

(b) Prove by contradiction that $\sin\theta + \cos\theta \geq 1$ for $0 \leq \theta \leq \dfrac{\pi}{2}$. **3**

(c) A particle is projected from a fixed origin, as shown in the diagram. The forces acting on the particle are its weight and air resistance. Its initial horizontal component of velocity is v_1 and its subsequent horizontal velocity $\dot{x}$ is modelled by $\dfrac{d\dot{x}}{dt} = -k\dot{x}$. The particle's displacement is measured in metres and time is measured in seconds.



(i) Show that the particle's horizontal displacement, x, from the origin is given by **2**

$$x = \frac{v_1}{k}\left(1 - e^{-kt}\right).$$

The particle's initial vertical component of velocity is v_2 and its subsequent vertical velocity $\dot{y}$ is modelled by $\dfrac{d\dot{y}}{dt} = -k\dot{y} - g$, where g is the acceleration due to gravity.

(ii) Show that the particle's vertical displacement, y, from the origin is given by **3**

$$y = \frac{kv_2 + g}{k^2}\left(1 - e^{-kt}\right) - \frac{g}{k}t.$$

Question 14 continues on page 10

Question 14 (continued)

(iii) Show that the Cartesian equation of the particle's path of flight is given by **2**

$$y = \left(\frac{kv_2 + g}{kv_1}\right)x + \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_1}\right).$$

(iv) In the case $v_1 = v_2 = 10, k = 0.1$ and $g = 9.8,$ determine whether the particle will pass over a wall of height 4 metres at a horizontal distance of 6 metres from the origin. **1**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The line l_1 has vector equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$ and the line l_2 has vector equation $\underset{\sim}{r} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

(i) Show that l_1 and l_2 do NOT intersect. **3**

(ii) Show that the distance d between a point on l_1 and a point on l_2 is given by **4**

$d = \sqrt{\left(3\lambda_2 - 4\lambda_1 - 5\right)^2 + \left(\lambda_1 - 1\right)^2 + 36}$.

(iii) Hence, determine the minimum distance between l_1 and l_2. **1**

(iv) Find the coordinates of the points on the two lines that are the minimum distance apart. **1**

(b) A particle is projected from the origin on a horizontal plane with initial velocity v m/s

at an angle θ to the horizontal. The position vector $\underset{\sim}{r}(t)$ of the particle is given by

$\underset{\sim}{r}(t) = \begin{pmatrix} vt\cos\theta \\ vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$, where g is the acceleration due to gravity. (Do NOT prove this.)

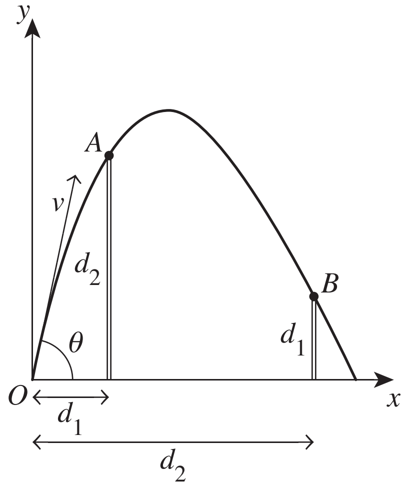
(i) Show that the Cartesian equation of the path of flight is given **2**

by $y = x\tan\theta - \dfrac{gx^2}{2v^2\cos^2\theta}$.

Question 15 continues on page 12

Question 15 (continued)

(ii) The particle just passes over two walls at points A and B. The two walls are at horizontal distance d_1 and d_2 metres from the point of projection and are of heights d_2 and d_1 metres respectively, as shown in the diagram. **4**



Show that $\theta = \tan^{-1}\left(\dfrac{d_1{}^2 + d_1 d_2 + d_2{}^2}{d_1 d_2}\right)$. You may use the result $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction that a convex polygon with n vertices has $\frac{1}{2}n(n-3)$ diagonals for $n \geq 4$. **4**

(b) Let $t = \tan\frac{x}{2}$.

(i) Show that $\frac{dx}{dt} = \frac{2}{1+t^2}$. **1**

(ii) Show that $\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \sin x$. **1**

(iii) Hence, show that $\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x}dx = \frac{2}{\sqrt{1-k^2}}\tan^{-1}\sqrt{\frac{1-k}{1+k}}$, where $0 < k < 1$. **4**

Let $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{2+\sin x}dx$, where $n = 0, 1, 2, \ldots$.

(iv) Show that $I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$. **1**

(v) Hence, or otherwise, find the value of I_2. Give your answer in the form $m\pi + 1$, where m is irrational. **4**

End of paper

MATHEMATICS ADVANCED
MATHEMATICS EXTENSION 1
MATHEMATICS EXTENSION 2
REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2}ab\sin C$

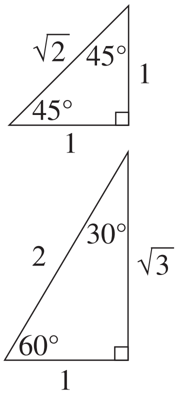
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$\cos A = \dfrac{1-t^2}{1+t^2}$

$\tan A = \dfrac{2t}{1-t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

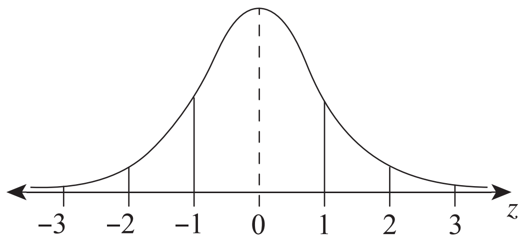
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 - 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X-\mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq r) = \displaystyle\int_a^r f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r\, p^r (1-p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow P(X = x)$

$= \dbinom{n}{x} p^x (1-p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1-p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1+\left[f(x)\right]^2}}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$
where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 - \left[f(x)\right]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$
$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \ldots + f\left(x_{n-1}\right)\right]\right\}$$
where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \ldots + \binom{n}{r}x^{n-r}a^{r} + \ldots + a^{n}$$

Vectors

$$|\underset{\sim}{u}| = \left|x\underset{\sim}{i} + x\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^{2}(x-c)$$

Neap

Trial Examination 2021

HSC Year 12 Mathematics Extension 2

Solutions and marking guidelines

SECTION I

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 1 **B** $2\underset{\sim}{a}-\underset{\sim}{b}=2\left(3\underset{\sim}{i}-4\underset{\sim}{j}+\underset{\sim}{k}\right)-\left(-\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}\right)$ $=6\underset{\sim}{i}-8\underset{\sim}{j}+2\underset{\sim}{k}+\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k}$ $=7\underset{\sim}{i}-10\underset{\sim}{j}+5\underset{\sim}{k}$	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E3
Question 2 **C** $z=\dfrac{\sqrt{3}-i}{1+i}$ $\lvert z\rvert=\dfrac{\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}}{\sqrt{1^2+(-1)^2}}$ $=\dfrac{2}{\sqrt{2}}$ $=\sqrt{2}$ $\arg\left(\dfrac{\sqrt{3}-i}{1+i}\right)=\arg\left(\sqrt{3}-i\right)-\arg\left(1+i\right)$ $=-\dfrac{\pi}{6}-\dfrac{\pi}{4}$ $=-\dfrac{5\pi}{12}$	MEX–N1 Introduction to Complex Numbers MEX12–4 Bands E2–E3
Question 3 **A** The parametric equations are $x=-5+2\lambda$ and $y=6-3\lambda$. $x=-5+2\lambda \Rightarrow \lambda=\dfrac{x+5}{2}$ Substituting $\lambda=\dfrac{x+5}{2}$ into $y=6-3\lambda$ gives $y=6-3\left(\dfrac{x+5}{2}\right)$. Multiply all terms by 2: $2y=12-3\left(x+5\right)$ $2y=12-3x-15$ Rearranging gives $2y+3x=-3$.	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E3

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 4 B The centre of C is the midpoint of AB, which is given by $\dfrac{1-5i+3-i}{2}=2-3i.$ $AB=\sqrt{(3-1)^2+(-1+5)^2}$ $=\sqrt{2^2+4^2}$ $=\sqrt{20}$ $=2\sqrt{5}$ So the radius of C is $\sqrt{5}$. The equation of C is $\lvert z-(2-3i)\rvert=\sqrt{5}$, and so $\lvert z-2+3i\rvert=\sqrt{5}$.	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E3
Question 5 B $\dfrac{2\pi}{n}=\pi \Rightarrow n=2$ Using $v^2=n^2\left(a^2-x^2\right)$ with $v=2.40$, $n=2$ and $x=0.50$: $2.40^2=4\left(a^2-0.50^2\right)$ $a^2=\dfrac{2.40^2}{4}+0.50^2$ $a=\sqrt{1.69}$ $=1.3\text{ (m)}$ Let the maximum speed be $v_{\max}$, which occurs at the centre of motion $\left(\text{where } \ddot{x}=0\right)$. Using $v_{\max}=na$ with $n=2$ and $a=1.30$: $v_{\max}=2\times 1.30$ $=2.60\text{ (m/s)}$	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4
Question 6 D $\displaystyle\int\frac{8x+1}{x^2+9}dx=4\int\frac{2x}{x^2+9}dx+\int\frac{1}{x^2+9}dx$ $=4\ln\left(x^2+9\right)+\dfrac{1}{3}\tan^{-1}\dfrac{x}{3}+c$	MEX–C1 Further Integration MEX12–5 Bands E2–E3

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 7 **C** **C** is correct. When $x = 1, y = \frac{1}{3}$, which is not a natural number, and so this counterexample demonstrates that **C** is a false statement. **A** is incorrect. For all *x* there exists a *y* such that $x - y = 0$. Consider $y = x$. **B** is incorrect. For all *x* there exists a *y* such that $3x - y = 0$. Consider $y = 3x$. **D** is incorrect. For example, $x = 2$ and $y = 6$.	MEX–P1 The Nature of Proof MEX12–2 Bands E3–E4
Question 8 **D** **D** is correct. In a negation of the statement, the phrases 'for all' and 'there exists' must be interchanged, and '=' must change to '$\neq$'. The negation of the given statement will therefore be: 'There exists odd primes $p < q$ such that for all positive non-primes $r < s, p^2 + q^2 \neq r^2 + s^2$'. **A**, **B** and **C** are incorrect. These options do not show an accurate negation of the statement.	MEX–V1 The Nature of Proof MEX12–2 Bands E3–E4
Question 9 **D** **D** is correct. Let the square roots of z be z_1 and z_2. $z = r(\cos\theta + i\sin\theta)$ and so $\sqrt{z} = \pm\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$. Hence, $z_1 = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ and $z_2 = -\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$. If z_1 has coordinates (x_1, y_1), then z_2 has coordinates $(-x_1, -y_1)$, where $x_1 = \sqrt{r}\cos\frac{\theta}{2}$ and $y_1 = \sqrt{r}\sin\frac{\theta}{2}$. Points *C* and *E* satisfy this condition. **A**, **B** and **C** are incorrect. These options do not satisfy the above condition.	MEX–N1 Introduction to Complex Numbers MEX12–4 Bands E3–E4

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 10 **A** **A** is correct. As $a = vx$ and $v\dfrac{dv}{dx} = a$, we have $\dfrac{dv}{dx} = x$. Thus, $\displaystyle\int \frac{dv}{dx}dx = \int x\,dx$ and so $v = \dfrac{x^2}{2} + c$. The graph of v against x must be a parabola. **B**, **C** and **D** are incorrect. These options do not show a parabola as required.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4

SECTION II

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 11	
(a) The other root is $\sqrt{5}+i$. The sum of roots is $\left(\sqrt{5}-i\right)+\left(\sqrt{5}+i\right)=2\sqrt{5}$. The product of roots is $\left(\sqrt{5}-i\right)\left(\sqrt{5}+i\right)=6$. So, $p=-2\sqrt{5}$ and $q=6$.	MEX–N1 Introduction to Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution . . . . 2 • Attempts to find the sum and product of roots OR equivalent merit 1
(b) Integration by parts takes the form $\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$. Let $u=\sin^{-1}x$ and $\frac{dv}{dx}=1$. So, $\frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$ and $v=x$. $\int_0^1 \sin^{-1}x\,dx=\left[x\sin^{-1}x\right]_0^1-\int_0^1\frac{x}{\sqrt{1-x^2}}dx$ $=\left(\frac{\pi}{2}-0\right)+\frac{1}{2}\int_0^1\frac{-2x}{\sqrt{1-x^2}}dx$ $=\frac{\pi}{2}+\frac{1}{2}\left[2\sqrt{1-x^2}\right]_0^1$ $=\frac{\pi}{2}+\left(0-1\right)$ $=\frac{\pi}{2}-1$	MEX–C1 Further Integration MEX12–5 Bands E2–E3 • Gives the correct solution . . . . 3 • Correctly applies integration by parts OR equivalent merit. . . . . . 2 • Correctly identifies the two functions to be used in integration by parts OR equivalent merit 1

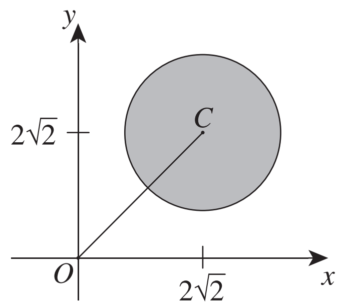
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $= x + 3$ with $t = 0, x = 0$ and $v = 3$ $\frac{1}{2}v^2 = \int (x+3)\,dx$ $v^2 = 2\left(\frac{x^2}{2} + 3x\right) + c$ $= x^2 + 6x + c$ Applying the initial condition $x = 0$ and $v = 3$ gives $c = 9$. $v^2 = x^2 + 6x + 9$ $= (x+3)^2$ $v = \pm(x+3)$ As $v = 3$ when $t = 0$, $v = x + 3$ (taking the positive root as $v > 0$).	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E3 • Gives the correct solution . . . . 2 • Attempts to use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to find an expression for v^2 OR equivalent merit. 1
(ii) $v = \frac{dx}{dt} = x + 3$ with $t = 0, x = 0$ $\frac{dt}{dx} = \frac{1}{x+3}$ $\Rightarrow t = \int \frac{1}{x+3}\,dx$ $t = \ln(x+3) + d$ $e^{t-d} = x + 3$ $x = Ae^t - 3$, where $A = e^{-d}$ Applying the condition $t = 0, x = 0$ gives $A = 3$. So $x = 3e^t - 3$. *Note: The value of the constant can be determined immediately after integrating.*	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E3 • Gives the correct solution . . . . 2 • Attempts to integrate a correct expression for $\frac{dt}{dx}$ OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(d) Given $\hat{\underset{\sim}{u}} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, form three equations involving x, y and z. $\underset{\sim}{b} \cdot \hat{\underset{\sim}{u}} = 0 \Rightarrow 2x + 3y - z = 0 \quad (1)$ $\underset{\sim}{c} \cdot \hat{\underset{\sim}{u}} = 0 \Rightarrow 4x - y + 5z = 0 \quad (2)$ $\lvert\hat{\underset{\sim}{u}}\rvert = 1 \Rightarrow x^2 + y^2 + z^2 = 1 \quad (3)$ For example, $2 \times (1) - (2)$ gives $y = z$. Substituting $z = y$ into (1) and solving gives $x = -y$. Substituting $y = z$ and $x = -y$ into (3) and solving $3x^2 = 1$ for x gives $x = \frac{1}{\sqrt{3}}$ since $x > 0$. So, $y = -\frac{1}{\sqrt{3}}$ and $z = -\frac{1}{\sqrt{3}}$. Hence, $\hat{\underset{\sim}{u}} = \frac{1}{\sqrt{3}}\left(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)$. As $\underset{\sim}{a} = \sqrt{3}\,\hat{\underset{\sim}{u}}, \underset{\sim}{a}$ is perpendicular to both $\underset{\sim}{b}$ and $\underset{\sim}{c}$. Hence, $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are mutually perpendicular.	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . 3 • Finds $\hat{\underset{\sim}{u}} = \frac{1}{\sqrt{3}}\left(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)$ . . . . 2 • Attempts to form three equations in x, y and z 1

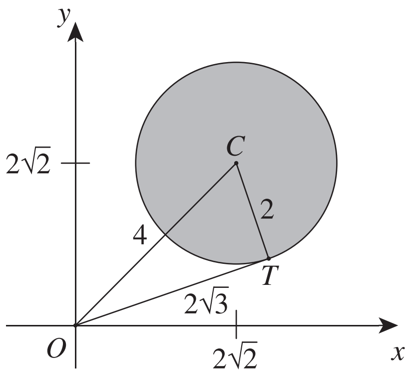
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(e) **Method 1:** $1-e^{2i\theta}=(1-\cos 2\theta)-i\sin 2\theta.$ $\overline{1-e^{2i\theta}}=(1-\cos 2\theta)+i\sin 2\theta$ $\dfrac{2}{1-e^{2i\theta}}=\dfrac{2\big((1-\cos 2\theta)+i\sin 2\theta\big)}{(1-\cos 2\theta)^2+\sin^2 2\theta}$ $=\dfrac{2\big((1-\cos 2\theta)+i\sin 2\theta\big)}{2(1-\cos 2\theta)}$ $\left(\text{since } \sin^2 2\theta+\cos^2 2\theta=1\right)$ $=1+\dfrac{i\sin 2\theta}{1-\cos 2\theta}$ $=1+\dfrac{2i\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)}$ (use of double-angle formula) $=1+\dfrac{i\cos\theta}{\sin\theta}$ So $z=1+i\cot\theta$. **Method 2:** $\dfrac{2}{1-e^{2i\theta}}=\dfrac{2}{e^{i\theta}\left(e^{-i\theta}-e^{i\theta}\right)}$ $=\dfrac{2e^{-i\theta}}{e^{-i\theta}-e^{i\theta}}$ $=\dfrac{2(\cos\theta-i\sin\theta)}{(\cos\theta-i\sin\theta)-(\cos\theta+i\sin\theta)}$ $=\dfrac{2(\cos\theta-i\sin\theta)}{-2i\sin\theta}$ $=1-\dfrac{1}{i}\cot\theta$ $=1+i\cot\theta$	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E4 • Gives the correct solution . . . . 3 • Uses $\sin^2 2\theta+\cos^2 2\theta=1$ and appropriate double-angle formulae OR equivalent merit . . . . . 2 • Multiplies and divides by $\overline{1-e^{2i\theta}}=(1-\cos 2\theta)+i\sin 2\theta$ OR equivalent merit. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 12	
(a) The equation of motion has the form $x = a\cos nt$. $a = 4$ and $\dfrac{2\pi}{n} = \dfrac{\pi}{3}$, so $n = 6$. $x = 4\cos 6t$ and so $v = -24\sin 6t$. The maximum speed is 24 m/s and so half the maximum speed is 12 m/s. Find the first two values of t such that $12 = 24\sin 6t$. $\sin 6t = \dfrac{1}{2}$ $6t = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ So, $t = \dfrac{\pi}{36}, \dfrac{5\pi}{36}$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 3 • Attempts to formulate the required equation $24\sin 6t = 12$ . . . . 2 • Finds $a = 4$ and $n = 6$ 1
(b) (i) The equation of motion for particle A is: $5mg - T = \dfrac{5mg}{4}$ $T = 5mg - \dfrac{5mg}{4}$ $= \dfrac{15mg}{4}$ (newtons)	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 2 • Attempts to apply Newton's second law OR equivalent merit. 1
(ii) The equation of motion for particle B is: $T - kmg = \dfrac{kmg}{4}$ $\dfrac{15mg}{4} = kmg + \dfrac{kmg}{4}$ $\dfrac{15mg}{4} = \dfrac{5kmg}{4}$ Therefore, $k = 3$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 2 • Attempts to apply Newton's second law OR equivalent merit. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) Firstly consider the descending motion of particle A: $\ddot{x} = \frac{g}{4}$ $\dot{x} = \frac{gt}{4} + c_1$ When $t = 0, \dot{x} = 0$ and so $c_1 = 0$. Hence, $\dot{x} = \frac{gt}{4}$. When $t = 1, \dot{x} = \frac{g}{4}$ and so particle A impacts the floor at $\frac{g}{4}$ m/s. $x = \frac{gt^2}{8} + c_2$ When $t = 0, x = 0$ and so $c_2 = 0$. Hence, $x = \frac{gt^2}{8}$. When $t = 1, x = \frac{g}{8}$ and so particle A descends $\frac{g}{8}$ metres before impacting the floor. Let s_1 be the initial height of particle A above the floor. So, $s_1 = \frac{g}{8}$ (m). Let h be the greatest height reached by particle B above the floor. $h = 2s_1 + s_2$, where s_2 is the distance particle B travels under gravity. (continues on page 12)	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4 • Gives the correct solution . . . . 3 • Attempts to find the distance particle B travels under gravity . . . . 2 • Attempts to find the velocity of particle A when it impacts the floor AND its initial height 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) (continued) Now consider the ascending motion of particle B under gravity: $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -g$ $\frac{1}{2}v^2 = -gx + d$ When $x = 0, v = \frac{g}{4}$ and so $d = \frac{g^2}{32}$. So, $\frac{1}{2}v^2 = -gx + \frac{g^2}{32}$. When $v = 0, x = s_2$ and so $s_2 = \frac{g}{32}$ (m). Substituting $s_1 = \frac{g}{8}$ and $s_2 = \frac{g}{32}$ into $h = 2s_1 + s_2$ gives: $h = 2\left(\frac{g}{8}\right) + \frac{g}{32}$ $= \frac{9g}{32}$ So, the greatest height reached by particle B is $\frac{9g}{32}$ m.	
(c) (i) 	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Correctly sketches the relation 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Let d be the distance from O to the centre of the circle C. $d = \sqrt{\left(2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2}$ $= 4$ The radius of the circle is 2. So, the minimum value of $\lvert z \rvert$ is $4 - 2 = 2$ and the maximum value of $\lvert z \rvert$ is $4 + 2 = 6$.	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Finds the minimum AND maximum value of $\lvert z \rvert$ . . . . 2 • Finds the minimum OR maximum value of $\lvert z \rvert$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) OC makes an angle of $\frac{\pi}{4}$ with the x-axis. **Either:** $\sin\angle COT = \frac{2}{4}$ $= \frac{1}{2}$ $\Rightarrow \angle COT = \frac{\pi}{6}$ **Or:** Form a right-angled triangle with OC, the radius of the circle, and T, the point of tangency to the circle. $OT = \sqrt{4^2 - 2^2}$ $= 2\sqrt{3}$ $\tan\angle COT = \frac{2}{2\sqrt{3}}$ $= \frac{1}{\sqrt{3}}$ $\Rightarrow \angle COT = \frac{\pi}{6}$ Then:  So, the minimum value of $\operatorname{Arg} z$ is $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$. By similar considerations, the maximum value of $\operatorname{Arg} z$ is $\frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$.	MEX–N2 Using Complex Numbers MEX12–4 Bands E3–E4 • Finds the minimum AND maximum value of Arg z . . . . 2 ——— • Finds the minimum OR maximum value of Arg z 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 13	
(a) (i) If x is even, then $x^2 - 6x + 5$ is odd.	MEX–P1 The Nature of Proof MEX12–2 Bands E2–E3 • Gives the correct contrapositive . . . 1
(ii) Suppose that x is even. Let $x = 2k$ for integer k. Substituting $x = 2k$ into $x^2 - 6x + 5$ gives $(2k)^2 - 6(2k) + 5$. $(2k)^2 - 6(2k) + 5 = 4k^2 - 12k + 5$ $= 2\left(2k^2 - 6k + 2\right) + 1$ Therefore, $x^2 - 6x + 5 = 2b + 1$, where b is the integer $2k^2 - 6k + 2$. Hence, $x^2 - 6x + 5$ is odd and the statement is proven by proving the contrapositive.	MEX–P1 The Nature of Proof MEX12–2 Bands E2–E3 • Proves the contrapositive. . . . . 2 • Substitutes $x = 2k$ into $x^2 - 6x + 5$ AND attempts to express it in the form $2b + 1$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) (i) Let $\overrightarrow{OP}$ be the point of intersection of $\overrightarrow{OC}$ and $\overrightarrow{MB}$. $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$ $= \underset{\sim}{a} + \underset{\sim}{b}$ The equation of line OC is $\underset{\sim}{r} = \lambda\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$. The equation of line MB is $\underset{\sim}{r} = \overrightarrow{OM} + \mu\left(\overrightarrow{OB} - \overrightarrow{OM}\right)$. $\underset{\sim}{r} = \frac{1}{5}\underset{\sim}{a} + \mu\left(-\frac{1}{5}\underset{\sim}{a} + \underset{\sim}{b}\right)$ Equating: $\lambda\underset{\sim}{a} + \lambda\underset{\sim}{b} = \frac{1}{5}\underset{\sim}{a} - \frac{\mu}{5}\underset{\sim}{a} + \mu\underset{\sim}{b}$ $\lambda = \frac{1-\mu}{5}$ and $\lambda = \mu \Rightarrow \lambda = \frac{1}{6}$ The point of intersection is $\overrightarrow{OP} = \frac{1}{6}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$. So, $\overrightarrow{OP} = \frac{1}{6}\overrightarrow{OC}$ and P is a common point of OC and MB. Hence, P lies on OC.	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E4 • Gives the correct solution . . . . 3 • Equates equations of lines OC and MB. AND • Attempts to solve. OR • Attempts to write $\overrightarrow{OP}$ as a multiple of $\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b}$. OR • Equivalent merit . . . . 2 • Finds equation of line OC. OR • Writes $\overrightarrow{OP}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{MB}$. OR • Equivalent merit 1
(ii) $\overrightarrow{OP} = \frac{1}{6}\overrightarrow{OC}$ $OP : PC = 1 : 5$	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E4 • Gives the correct solution 1
(c) (i) $e^{in\theta} + e^{-in\theta} = \left(\cos n\theta + i\sin n\theta\right) + \left(\cos n\theta - i\sin n\theta\right)$ $= 2\cos n\theta$	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $\left(e^{i\theta}+e^{-i\theta}\right)^5=e^{5i\theta}+5\left(e^{4i\theta}\right)\left(e^{-i\theta}\right)$ $+10\left(e^{3i\theta}\right)\left(e^{-2i\theta}\right)$ $+10\left(e^{2i\theta}\right)\left(e^{-3i\theta}\right)$ $+5\left(e^{i\theta}\right)\left(e^{-4i\theta}\right)+e^{-5i\theta}$ $=\left(e^{5i\theta}+e^{-5i\theta}\right)+5\left(e^{3i\theta}+e^{-3i\theta}\right)$ $+10\left(e^{i\theta}+e^{-i\theta}\right)$ $=2\cos 5\theta+10\cos 3\theta+20\cos\theta$ Also, $\left(e^{i\theta}+e^{-i\theta}\right)^5=2^5\cos^5\theta.$ So, $\cos^5\theta=\frac{1}{16}\left(\cos 5\theta+5\cos 3\theta+10\cos\theta\right).$	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E4 • Gives the correct solution . . . . 3 • Correctly uses binomial theorem and groups conjugate pairs OR equivalent merit . . . . 2 • Obtains $\left(e^{i\theta}+e^{-i\theta}\right)^5=32\cos^5\theta$ OR attempts to use binomial theorem OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) **Method 1:** $\int_0^{\frac{\pi}{2}} \cos^5 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)d\theta$ $= \frac{1}{16}\left[\frac{1}{5}\sin 5\theta + \frac{5}{3}\sin 3\theta + 10\sin\theta\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{16}\left(\frac{1}{5}\sin\frac{5\pi}{2} + \frac{5}{3}\sin\frac{3\pi}{2} + 10\sin\frac{\pi}{2} - (0+0+0)\right)$ $= \frac{1}{16}\left(\frac{1}{5} - \frac{5}{3} + 10\right)$ $= \frac{8}{15}$ *Note: Consequential on answer to Question 13 part (c)(ii).* **Method 2:** $\int_0^{\frac{\pi}{2}} \cos^5 \theta d\theta = \int_0^{\frac{\pi}{2}} \left(\cos^2\theta\right)^2 \cos\theta d\theta$ $= \int_0^{\frac{\pi}{2}} \left(1-\sin^2\theta\right)^2 \cos\theta d\theta$ Let $u = \sin\theta$ and so $\frac{du}{d\theta} = \cos\theta.$ When $\theta = 0, u = 0$ and when $\theta = \frac{\pi}{2}, u = 1.$ $\int_0^1 \left(1-u^2\right)^2 du = \int_0^1 \left(1 - 2u^2 + u^4\right)du$ $= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1$ $= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0 - 0 + 0)$ $= \frac{8}{15}$	MEX–N2 Using Complex Numbers MEX12–7 Bands E2–E4 • Gives the correct solution . . . . 2 • Uses part (c)(ii) and attempts to integrate OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iv) $16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$ $16\cos^5\theta - \cos\theta = \cos 5\theta + 5\cos 3\theta + 9\cos\theta$ Hence, to solve the given equation we solve the equation $16\cos^5\theta - \cos\theta = 0$. $\cos\theta = 0, \pm\frac{1}{2}$ $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ and $\cos\theta = \pm\frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}$ $\theta = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$	MEX–N2 Using Complex Numbers MEX12–7 Bands E2–E4 • Gives the correct solution . . . . 2 • Determines that $16\cos^5\theta - \cos\theta = 0$ 1
Question 14	
(a) (i) **Method 1:** Since squares cannot be negative, $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$. Expanding the LHS gives: $a - 2\sqrt{ab} + b \geq 0$ $a + b \geq 2\sqrt{ab}$ So $\frac{a+b}{2} \geq \sqrt{ab}$. **Method 2:** Consider: $\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(a+b-2\sqrt{ab}\right)$ $= \frac{1}{2}\left(\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{ab}\right)$ $= \frac{1}{2}\left(\sqrt{a}-\sqrt{b}\right)^2$ ≥ 0 So $\frac{a+b}{2} \geq \sqrt{ab}$.	MEX–P1 The Nature of Proof MEX12–2 Bands E2–E3 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) From the AM–GM inequality: $\frac{a+b}{2} \geq \sqrt{ab}, \frac{b+c}{2} \geq \sqrt{bc}$ and $\frac{c+a}{2} \geq \sqrt{ca}$. Multiply together: $\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right) \geq \left(\sqrt{ab}\right)\left(\sqrt{bc}\right)\left(\sqrt{ca}\right)$ $\frac{1}{8}(a+b)(b+c)(c+a) \geq \left(\sqrt{a^2b^2c^2}\right)$ So, $(a+b)(b+c)(c+a) \geq 8abc$.	MEX–P1 The Nature of Proof MEX12–2 Bands E2–E3 • Gives the correct solution 1
(iii) Let $a+b=z, b+c=x$ and $c+a=y$. $a=\frac{y+z-x}{2}, b=\frac{z+x-y}{2}$ and $c=\frac{x+y-z}{2}$. Using the result $(a+b)(b+c)(c+a) \geq 8abc$ with the above substitutions: $zxy \geq 8\left(\frac{y+z-x}{2}\right)\left(\frac{z+x-y}{2}\right)\left(\frac{x+y-z}{2}\right)$ So, $xyz \geq (y+z-x)(z+x-y)(x+y-z)$.	MEX–P1 The Nature of Proof MEX12–2 Bands E3–E4 • Gives the correct solution . . . . 2 • Correctly uses the result $(a+b)(b+c)(c+a) \geq 8abc$ 1
(b) Assume there exists an $x \in \left[0, \frac{\pi}{2}\right]$ for which $\sin\theta + \cos\theta < 1$. Since $x \in \left[0, \frac{\pi}{2}\right]$, neither $\sin\theta$ nor $\cos\theta$ is negative, so $0 \leq \sin\theta + \cos\theta < 1$. $0^2 \leq (\sin\theta + \cos\theta)^2 < 1^2$ $0^2 \leq \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta < 1^2$ As $\sin^2\theta + \cos^2\theta = 1$, this becomes $0 \leq 1 + 2\sin\theta\cos\theta < 1$. So, $1 + 2\sin\theta\cos\theta < 1$. Hence, $2\sin\theta\cos\theta < 0$. This contradicts the fact that neither $\sin\theta$ nor $\cos\theta$ is negative. So, $\sin\theta + \cos\theta \geq 1$ for $0 \leq \theta \leq \frac{\pi}{2}$.	MEX–P1 The Nature of Proof MEX12–2 Bands E3–E4 • Gives the correct solution . . . . 3 • Attempts to establish that the assumption leads to $2\sin\theta\cos\theta < 0$ OR equivalent merit . . . . 2 • Assumes there exists an $x \in \left[0, \frac{\pi}{2}\right]$ for which $\sin\theta + \cos\theta < 1$ OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) $\dfrac{d\dot{x}}{dt} = -k\dot{x}$ Separating variables gives: $\displaystyle\int \frac{1}{\dot{x}} d\dot{x} = -k\int dt$ $\ln\dot{x} = -kt + c_1$ $\dot{x} = Ae^{-kt}$, where $A = e^{c_1}$ When $t = 0, \dot{x} = v_1$ and so $A = v_1$. $\dot{x} = v_1 e^{-kt}$ $\displaystyle x = \int v_1 e^{-kt} dt$ $\displaystyle = -\frac{v_1}{k}e^{-kt} + d_1$ When $t = 0, x = 0$ and so $d_1 = \dfrac{v_1}{k}$. So, $x = \dfrac{v_1}{k}\left(1 - e^{-kt}\right)$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 2 • Correctly separates variables, attempts to integrate to find $\dot{x}$ as a function of t and evaluates the constant OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $\dfrac{d\dot{y}}{dt} = -k\dot{y} - g$ Separating variables gives: $\displaystyle\int \frac{1}{\dot{y} + \frac{g}{k}} d\dot{y} = -k\int dt$ $\ln\left(\dot{y} + \dfrac{g}{k}\right) = -kt + c_2$ $\dot{y} + \dfrac{g}{k} = Be^{-kt}$, where $B = e^{c_2}$ When $t = 0, \dot{y} = v_2$ and so $B = v_2 + \dfrac{g}{k}$. $\dot{y} = \left(v_2 + \dfrac{g}{k}\right)e^{-kt} - \dfrac{g}{k}$ $y = \displaystyle\int\left(\left(v_2 + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}\right)dt$ $= -\dfrac{1}{k}\left(v_2 + \dfrac{g}{k}\right)e^{-kt} - \dfrac{g}{k}t + d_2$ When $t = 0, y = 0$ and so $d_2 = \dfrac{1}{k}\left(v_2 + \dfrac{g}{k}\right)$. $y = -\dfrac{1}{k}\left(v_2 + \dfrac{g}{k}\right)e^{-kt} - \dfrac{g}{k}t + \dfrac{1}{k}\left(v_2 + \dfrac{g}{k}\right)$ $= \dfrac{1}{k}\left(v_2 + \dfrac{g}{k}\right)\left(1 - e^{-kt}\right) - \dfrac{g}{k}t$ So, $y = \dfrac{kv_2 + g}{k^2}\left(1 - e^{-kt}\right) - \dfrac{g}{k}t$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 3 • Attempts to integrate to find y as a function of t and evaluates the constant OR equivalent merit. . . . . 2 • Correctly separates variables attempts to integrate to find $\dot{y}$ as a function of t and evaluates the constant OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) Rearranging $x = \frac{v_1}{k}\left(1 - e^{-kt}\right)$ gives: $1 - e^{-kt} = \frac{kx}{v_1}$ Solving for t gives: $t = -\frac{1}{k}\ln\left(1 - \frac{kx}{v_1}\right)$ Substituting $t = -\frac{1}{k}\ln\left(1 - \frac{kx}{v_1}\right)$ into y gives: $y = \frac{kv_2 + g}{k^2}\left(1 - \left(1 - \frac{kx}{v_1}\right)\right) + \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_1}\right)$ So, $y = \left(\frac{kv_2 + g}{kv_1}\right)x + \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_1}\right)$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 2 • Attempts to solve $x = \frac{v_1}{k}\left(1 - e^{-kt}\right)$ for t 1
(iv) Substituting $x = 6, v_1 = v_2 = 10, k = 0.1$ and $g = 9.8$ into y gives: $y = 4.1621...$ (m) Hence, the particle will clear the wall.	MEX–M1 Applications of Calculus to Mechanics MEX12–7 Bands E2–E3 • Gives the correct solution 1
Question 15	
(a) (i) Equating components: $1 + 2\lambda_1 = 4 + \lambda_2$ (1) $2\lambda_1 = -2 + 2\lambda_2$ (2) $2 - 3\lambda_1 = 9 - 2\lambda_2$ (3) (1) – (2) gives: $1 = 6 - \lambda_2 \Rightarrow \lambda_2 = 5$ Substituting $\lambda_2 = 5$ into (1) and solving gives: $1 + 2\lambda_1 = 9 \Rightarrow \lambda_1 = 4$ Substituting $\lambda_1 = 4$ and $\lambda_2 = 5$ into (3) gives: $-10 \neq -1$ Since the equations are inconsistent, the lines l_1 and l_2 do not intersect.	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . 3 • Correctly finds λ_1 and λ_2 OR equivalent merit. . . . . 2 • Finds at least one equation linking λ_1 and λ_2 OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $d=\sqrt{\left(3-2\lambda_1+\lambda_2\right)^2+\left(-2-2\lambda_1+2\lambda_2\right)^2+\left(7+3\lambda_1-2\lambda_2\right)^2}$ $=\sqrt{17\lambda_1^2+9\lambda_2^2-24\lambda_1\lambda_2+38\lambda_1-30\lambda_2+62}$ $=\sqrt{\left(9\lambda_2^2-24\lambda_1\lambda_2+16\lambda_1^2+40\lambda_1-30\lambda_2+25\right)+\left(\lambda_1^2-2\lambda_1+1\right)+36}$ $9\lambda_2^2-24\lambda_1\lambda_2+16\lambda_1^2+40\lambda_1-30\lambda_2+25$ $=\left(3\lambda_2\right)^2-2\left(3\lambda_2\right)\left(4\lambda_1+5\right)+\left(4\lambda_1+5\right)^2$ $=\left(3\lambda_2-4\lambda_1-5\right)^2$ So $d=\sqrt{\left(3\lambda_2-4\lambda_1-5\right)^2+\left(\lambda_1-1\right)^2+36}$. *Note: The result* $a^2+b^2+c^2=\left(a+b+c\right)^2-2\left(ab+ac+bc\right)$ *can be used where* $a=3-2\lambda_1+\lambda_2$, $b=-2-2\lambda_1+2\lambda_2$ *and* $c=7+3\lambda_1-2\lambda_2$.	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E4 • Gives the correct solution . . . . 4 • Attempts to factorise an expression for *d* into the required form . . . . 3 • Attempts to find an expression for *d* in expanded form . . . . 2 • Finds a correct expression for *d* in terms of λ_1 and λ_2 1
(iii) $\left(3\lambda_2-4\lambda_1-5\right)^2\geq 0$ and $\left(\lambda_1-1\right)^2\geq 0$ for all λ_1,λ_2. The minimum value of *d* is obtained by setting $3\lambda_2-4\lambda_1-5=0$ and $\lambda_1-1=0$. $\lambda_1-1=0$ $\lambda_1=1$ $3\lambda_2-4\times 1-5=0$ $3\lambda_2=9$ $\lambda_2=3$ So, the minimum value of *d* is 6.	MEX–V1 Further Work with Vectors MEX12–3 Bands E3–E4 • Gives the correct solution 1

	Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iv)	When $\lambda_1 = 1, \underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$ and so the corresponding point is (3, 2, –1). When $\lambda_2 = 3, \underset{\sim}{r} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and so the corresponding point is (7, 4, 3). The points that are the minimum distance apart are (3, 2, –1) and (7, 4, 3).	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution 1
(b) (i)	The parametric equations are: $x = vt\cos\theta$ (1) $y = vt\sin\theta - \frac{1}{2}gt^2$ (2) From (1), $t = \frac{x}{v\cos\theta}$. Substituting $t = \frac{x}{v\cos\theta}$ into (2) gives: $y = \frac{vx\sin\theta}{v\cos\theta} - \frac{1}{2}g\left(\frac{x^2}{v^2\cos^2\theta}\right)$ So, $y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E3 • Gives the correct solution . . . . 2 • Attempts to eliminate t OR equivalent merit. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $y = x\tan\theta - \dfrac{gx^2}{2v^2\cos^2\theta}$ (1) Substituting $x = d_1$ and $y = d_2$ into (1) gives: $d_2 = d_1\tan\theta - \dfrac{gd_1^{\,2}}{2v^2\cos^2\theta}$ (2) Substituting $x = d_2$ and $y = d_1$ into (1) gives: $d_1 = d_2\tan\theta - \dfrac{gd_2^{\,2}}{2v^2\cos^2\theta}$ (3) Rearranging (2) and (3) gives: $\dfrac{gd_1^{\,2}}{2v^2\cos^2\theta} = d_1\tan\theta - d_2$ (4) $\dfrac{gd_2^{\,2}}{2v^2\cos^2\theta} = d_2\tan\theta - d_1$ (5) Multiplying (4) by $d_2^{\,2}$, multiplying (5) by $d_1^{\,2}$ and equating the RHSs gives: $(d_1\tan\theta - d_2)d_2^{\,2} = (d_2\tan\theta - d_1)d_1^{\,2}$ Expanding gives: $d_1d_2^{\,2}\tan\theta - d_2^{\,3} = d_1^{\,2}d_2\tan\theta - d_1^{\,3}$ $\left(d_1d_2^{\,2} - d_1^{\,2}d_2\right)\tan\theta = d_2^{\,3} - d_1^{\,3}$ (continues on page 27)	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4 • Gives the correct solution . . . . 4 • Factorises $d_2^{\,3} - d_1^{\,3}$ . . . . 3 • Attempts to form an equation involving d_1, d_2 and $\tan\theta$ only OR equivalent merit. . . . . 2 • Substitutes into the Cartesian equation to form two equations OR equivalent merit. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) (continued) Factorising the LHS and applying the result $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$ to the RHS gives: $d_1 d_2 (d_2 - d_1)\tan\theta = (d_2 - d_1)\left({d_2}^2 + d_1 d_2 + {d_1}^2\right)$ Dividing by $d_1 d_2 (d_2 - d_1)(\neq 0)$ gives: $\tan\theta = \dfrac{{d_1}^2 + d_1 d_2 + {d_2}^2}{d_1 d_2}$ So, $\theta = \tan^{-1}\left(\dfrac{{d_1}^2 + d_1 d_2 + {d_2}^2}{d_1 d_2}\right)$.	

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 16	
(a) Let $P(n)$ be the given proposition. Consider $P(4)$: If $n = 4$, the polygon is a quadrilateral, which has two diagonals. Also, for $n = 4, \frac{1}{2}n(n-3) = \frac{1}{2}(4)(1) = 2$. So, $P(4)$ is true. Suppose $P(n)$ is true for $n = k$. A convex polygon with k vertices has $\frac{1}{2}k(k-3)$ diagonals for $k \geq 4$. It is required to show that $P(k+1)$ is true. That is, a convex polygon with $(k + 1)$ vertices has $\frac{1}{2}(k+1)((k+1)-3) = \frac{1}{2}(k+1)(k-2)$ diagonals. When another vertex is added, we have $\frac{1}{2}k(k-3)$ existing diagonals + $(k - 2)$ extra diagonals (formed from the added vertex to all other vertices except the two adjacent vertices) + 1 diagonal (formerly a side of the polygon that is now a diagonal). Hence, we have $\frac{1}{2}k(k-3)+(k-2)+1$ diagonals. $$\begin{aligned}\frac{1}{2}k(k-3)+(k-2)+1 &= \frac{1}{2}\left(k(k-3)+2k-2\right)\\ &= \frac{1}{2}\left(k^2-k-2\right)\\ &= \frac{1}{2}(k+1)(k-2)\end{aligned}$$ Hence, $P(k+1)$ is true. So, $P(n)$ is true for (all integers) $n \geq 4$ by induction.	MEX–P2 Further Proof by Mathematical Induction MEX12–2, 12–8 Bands E2–E4 • Gives the correct solution . . . . 4 • Determines that there are $\frac{1}{2}k(k-3)+(k-2)+1$ diagonals . . . . 3 • Assumes true for $n = k$ and attempts to prove true for $n = k+1$ . . . . 2 • Establishes the initial case $(n = 4)$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) (i) **Method 1:** $t=\tan\frac{x}{2}\Rightarrow\frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}$ The identity $\sec^2\frac{x}{2}=1+\tan^2\frac{x}{2}$. $t=\tan\frac{x}{2}$ gives $\frac{dt}{dx}=\frac{1}{2}\left(1+t^2\right)$. So, $\frac{dx}{dt}=\frac{2}{1+t^2}$. **Method 2:** $t=\tan\frac{x}{2}\Rightarrow x=2\tan^{-1}t$ So, $\frac{dx}{dt}=\frac{2}{1+t^2}$.	MEX– C1 Further Integration MEX12–5 Bands E2–E3 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) **Method 1 (starting on the LHS):** $\text{LHS} = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ $= \dfrac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}}$ $= \dfrac{2\sin\frac{x}{2}\cos^2\frac{x}{2}}{\cos\frac{x}{2}}$ $= 2\sin\frac{x}{2}\cos\frac{x}{2}$ $= \sin x$ $= \text{RHS}$ **Method 2 (starting on the RHS):** $\text{RHS} = \sin x$ $= 2\sin\frac{x}{2}\cos\frac{x}{2}$ $= 2\left(\dfrac{t}{\sqrt{1+t^2}}\right)\left(\dfrac{1}{\sqrt{1+t^2}}\right)$ $= \dfrac{2t}{1+t^2}$ As $t = \tan\frac{x}{2}$, $\dfrac{2t}{1+t^2} = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \text{LHS}.$	MEX– C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) When $x = \frac{\pi}{2}, t = 1$ and when $x = 0$, $t = 0$.	MEX– C1 Further Integration MEX12–5 Bands E3–E4 • Gives the correct solution 4
$\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x}dx = \int_0^1 \frac{\frac{2}{1+t^2}}{1+\frac{2kt}{1+t^2}}dt$	• Applies $A - B = \tan^{-1}\left(\frac{\tan A - \tan B}{1+\tan A \tan B}\right)$ 3
$= \int_0^1 \frac{2}{t^2+2kt+1}dt$ $= \int_0^1 \frac{2}{\left(1-k^2\right)+\left(k+t\right)^2}dt$	• Obtains $\frac{2}{\sqrt{1-k^2}}\left[\tan^{-1}\left(\frac{k+t}{\sqrt{1-k^2}}\right)\right]_0^1$ OR equivalent merit. 2
$= \frac{2}{\sqrt{1-k^2}}\left[\tan^{-1}\left(\frac{k+t}{\sqrt{1-k^2}}\right)\right]_0^1$ $= \frac{2}{\sqrt{1-k^2}}\left(\tan^{-1}\left(\frac{k+t}{\sqrt{1-k^2}}\right) - \tan^{-1}\left(\frac{k}{\sqrt{1-k^2}}\right)\right)$ $\tan\left(A-B\right) = \frac{\tan A - \tan B}{1+\tan A \tan B}$ $\Rightarrow A - B = \tan^{-1}\left(\frac{\tan A - \tan B}{1+\tan A \tan B}\right)$: (continues on page 32)	• Makes a substitution to convert to a definite integral involving t OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) (continued) $$\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x}dx = \frac{2}{\sqrt{1-k^2}} \times \tan^{-1}\left(\frac{\frac{k+1-k}{\sqrt{1-k^2}}}{1+\frac{k(k+1)}{1-k^2}}\right)$$ $$= \frac{2}{\sqrt{1-k^2}} \times \tan^{-1}\left(\frac{\sqrt{1-k^2}}{1-k^2+k+k^2}\right)$$ $$= \frac{2}{\sqrt{1-k^2}} \times \tan^{-1}\left(\frac{\sqrt{1-k}\sqrt{1+k}}{1+k}\right)$$ So, $\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x} = \frac{2}{\sqrt{1-k^2}}\tan^{-1}\sqrt{\frac{1-k}{1+k}}$.	
(iv) $$I_{n+1}+2I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1} x}{2+\sin x}dx + 2\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{2+\sin x}dx$$ $$= \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1} x + 2\sin^n x}{2+\sin x}dx$$ $$= \int_0^{\frac{\pi}{2}} \frac{\sin^n x(\sin x + 2)}{2+\sin x}dx$$ So, $I_{n+1}+2I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$.	MEX–C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(v) Using $I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ with $n = 0$	MEX– C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution 4
gives: $I_1 + 2I_0 = \int_0^{\frac{\pi}{2}} 1dx$ $= \frac{\pi}{2}$	• Correctly uses $I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \sin^n xdx$ with n = 1 OR equivalent merit 3
$I_0 = \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x} dx$	• Correctly applies the part (b)(iii) result to find I_0 OR equivalent merit. 2
$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{2}\sin x} dx$ $= \frac{1}{2} \frac{2}{\sqrt{1-\left(\frac{1}{2}\right)^2}} \tan^{-1}\sqrt{\frac{1-\frac{1}{2}}{1+\frac{1}{2}}}$ $\left(\begin{array}{l}\text{using result from} \\ \text{part (b)(iii) with } k = \frac{1}{2}\end{array}\right)$ $= \frac{2}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}}$ So, $I_0 = \frac{\pi}{3\sqrt{3}}$. Using $I_1 = \frac{\pi}{2} - 2I_0$ with $I_0 = \frac{\pi}{3\sqrt{3}}$ gives $I_1 = \frac{\pi}{2} - 2\left(\frac{\pi}{3\sqrt{3}}\right)$. (continues on page 34)	• Forms $I_1 + 2I_0 = \frac{\pi}{2}$ OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(v) (continued) Using $I_{n+1}+2I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$ with $n=1$ gives: $I_2+2I_1=\int_0^{\frac{\pi}{2}}\sin x\,dx$ $I_2=-\left[\cos x\right]_0^{\frac{\pi}{2}}-2I_1$ $=-(0-1)-2\left(\frac{\pi}{2}-2\left(\frac{\pi}{3\sqrt{3}}\right)\right)$ So, $I_2=\pi\left(\frac{4\sqrt{3}}{9}-1\right)+1.$	

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