Sydney Girls High School
2024
Trial Higher School Certificate Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2-5)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6-13)

- Attempt Questions 11-16
- Allow about 2 hour and 45 minutes for this section

Name: …………………………………………………………… Teacher: ……………………………………………………………	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2024 HSC Examination Paper in this subject.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Consider the following statement:

"If n is a perfect square, then n is not 2 more than a multiple of 3."

Which of the following is the negation of this statement?

A. If n is not 2 more than a multiple of 3, then n is a perfect square.

B. n is a perfect square and n is 2 more than a multiple of 3.

C. If n is 2 more than a multiple of 3, then n is not a perfect square.

D. n is not a perfect square and n is not 2 more than a multiple of 3.

2 Which of the following gives the two solutions of $z^2 - (2+6i)z - 5 + 2i = 0$?

A. $z = 2$ or $z = 2 + 5i$

B. $z = 2$ or $z = 6i$

C. $z = i$ or $z = 2 + 5i$

D. $z = i$ or $z = 6i$

3 Consider the statement:

"If $n = 2^k$ then n has exactly $k + 1$ factors."

Which of the following is the contrapositive of this statement?

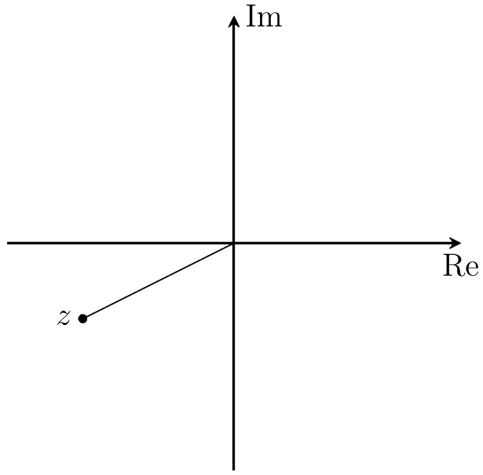
A. If n has $k + 1$ factors then $n = 2^k$.

B. If n does not have exactly $k + 1$ factors, then $n = 2^k$.

C. If n has exactly $k + 1$ factors, then $n \neq 2^k$.

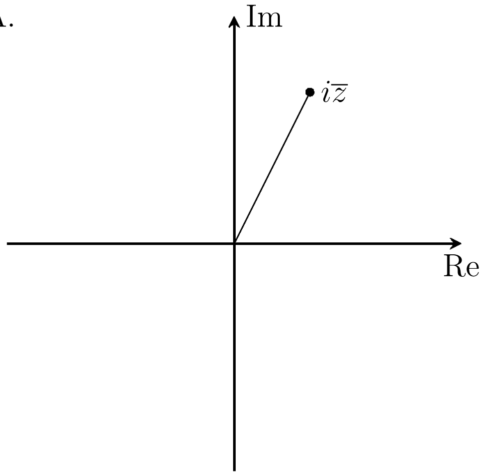
D. If n does not have exactly $k + 1$ factors, then $n \neq 2^k$.

4 The Argand diagram below shows the location of the complex number z.

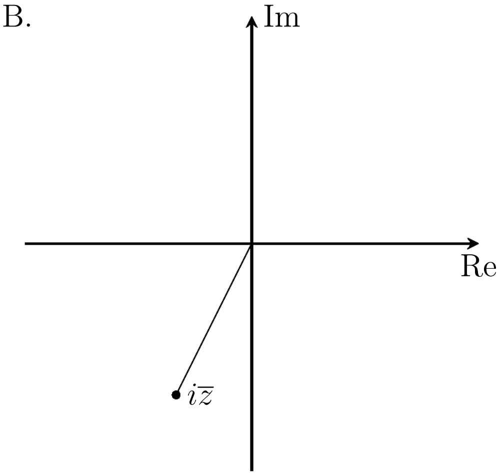


Which of the following shows the location of $i\overline{z}$?

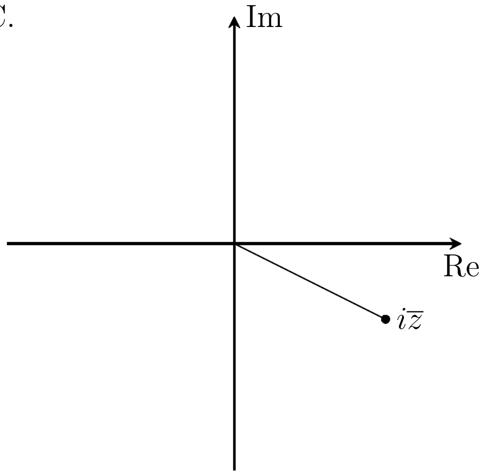
A.



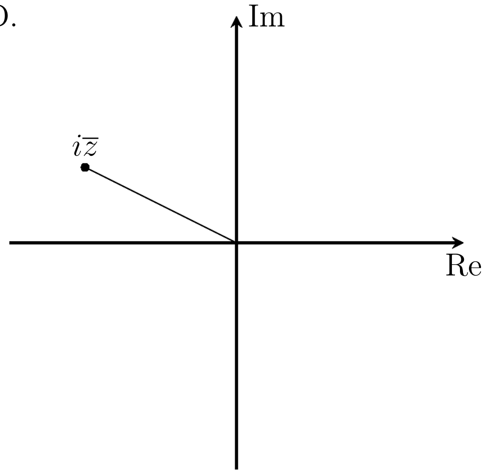
B.



C.



D.



5 What is the value of $(-\sqrt{3}+i)^{2024}$?

A. $2^{2024}\left(-1+\sqrt{3}i\right)$

B. $2^{2024}\left(1+\sqrt{3}i\right)$

C. $2^{2023}\left(-1+\sqrt{3}i\right)$

D. $2^{2023}\left(1+\sqrt{3}i\right)$

6 Which of the following is equal to $\displaystyle\int \frac{x^3-2x^2+3x}{x-2}\,dx$?

A. $\frac{1}{3}x^3+3x+6\ln|x-2|+C$

B. $\frac{1}{3}x^3+3x-6\ln|x-2|+C$

C. $\frac{1}{4}x^4+3x+6\ln|x-2|+C$

D. $\frac{1}{4}x^4+3x-6\ln|x-2|+C$

7 Which of the following is equal to $\displaystyle\int_{\frac{9\pi^2}{4}}^{\frac{25\pi^2}{4}} \sin\sqrt{x}\;dx$?

A. 2

B. -2

C. 4

D. -4

8 Let z be a complex number satisfying $|z + i| = 3$.

Which of the following shows the minimum and maximum values of $|4z - 3|$?

A. $5 \leqslant |4z - 3| \leqslant 15$

B. $5 \leqslant |4z - 3| \leqslant 17$

C. $7 \leqslant |4z - 3| \leqslant 15$

D. $7 \leqslant |4z - 3| \leqslant 17$

9 Consider the sketch of $y = \tan x$ for x satisfying $\frac{\pi}{4} < x < \frac{\pi}{2}$.

Which of the following holds for all $t \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$?

A. $\sec t < e^{\frac{1}{2}\left(t - \frac{\pi}{4}\right)(1+\tan t)}$

B. $\sec t > e^{\frac{1}{2}\left(t - \frac{\pi}{4}\right)(1+\tan t)}$

C. $\sec t < \sqrt{2}e^{\frac{1}{2}\left(t - \frac{\pi}{4}\right)(1+\tan t)}$

D. $\sec t > \sqrt{2}e^{\frac{1}{2}\left(t - \frac{\pi}{4}\right)(1+\tan t)}$

10 Suppose there exist vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{r}$ such that $\underset{\sim}{r} = \underset{\sim}{a} + (\underset{\sim}{a} \cdot \underset{\sim}{b})\,\underset{\sim}{b}$.

If $\underset{\sim}{r}$ is perpendicular to $\underset{\sim}{a} - \underset{\sim}{b}$, what is the minimum value of $1 + |\underset{\sim}{b}|^2$?

A. $|\underset{\sim}{a}| + 1$

B. $\dfrac{|\underset{\sim}{a}| + 1}{2}$

C. $2\,|\underset{\sim}{a}|$

D. $|\underset{\sim}{a}|$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Start each question on a new sheet of paper.

Question 11 (16 marks) Use a new sheet of paper.

(a) Express $\overline{7+6i} + \dfrac{2}{3-i}$ in the form $x + iy$, where x and y are real. **2**

(b) Find $\displaystyle\int \frac{\tan^2 x + 2}{x + \tan x}\, dx$. **2**

(c) (i) If a and b are divisible by d, show that $ma + nb$ is divisible by d. **1**

(ii) Prove by counterexample that the converse to (i) is false. **1**

(d) (i) Expand and simplify $(\cos\theta + i\sin\theta)^4$. **2**

(ii) Hence, show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$. **2**

(e) Let $\underset{\sim}{u} = \begin{pmatrix} 2\lambda \\ 2 \\ 3 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 6\lambda \\ -12\lambda \\ 6 \end{pmatrix}$, where λ is a real number.

(i) Show that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are not parallel. **2**

(ii) Show that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are not perpendicular. **2**

(f) A particle moves in simple harmonic motion about the origin. **2**
It is initially at the origin. If the period of the particle is $\frac{\pi}{3}$ seconds,
find the first two times when the particle's speed is half the maximum speed.

End of Question 11

Question 12 (14 marks) Use a new sheet of paper.

(a) Let $\omega = e^{i\frac{2\pi}{5}}$ be a fifth root of unity.

(i) Prove that $\omega + \omega^2 + \omega^3 + \omega^4 = -1$. **1**

(ii) Let $\alpha = \omega + \omega^4$ and $\beta = \omega^2 + \omega^3$. Suppose that α and β are the roots of the quadratic equation $x^2 + bx + c$. **2**

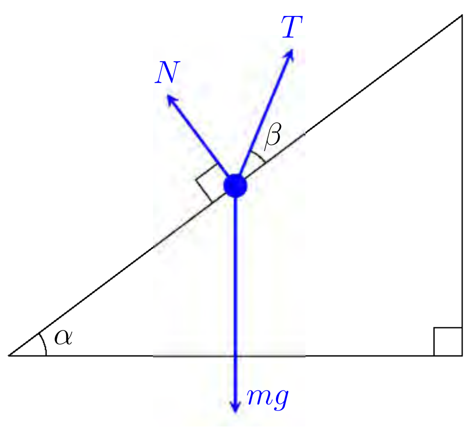
Determine the values of b and c.

(iii) Hence, determine the exact value of $\cos\frac{4\pi}{5}$. **2**

(b) (i) Find $\displaystyle\int \frac{x^2 + x + 3}{x^2 + x - 2}\,dx$. **3**

(ii) Evaluate $\displaystyle\int_0^1 \frac{1}{1 + \sqrt[3]{x}}\,dx$. **3**

(c) A bowling ball of mass m kg on a frictionless ramp is held stationary by a light inextensible string with tension of magnitude T newtons. The normal force of the bowling ball has a magnitude of N newtons. The ramp is inclined at an angle of α to the horizontal, and the string is held at an angle of β to the ramp, as shown: **3**



Let g be the magnitude of the acceleration due to gravity.

Prove that $N = mg(\cos\alpha - \tan\beta\sin\alpha)$.

End of Question 12

Question 13 (15 marks) Use a new sheet of paper.

(a) Let $P(z) = z^3 + kz^2 + 6$, where z is complex and k is real. **2**

When $P(z)$ is divided by $z^2 + 4$, the remainder is $-4z - 6$.

Find the value of k.

(b) (i) Prove that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. **2**

(ii) Hence, prove for all real A and B that **3**

$$\sin 2A + \sin 2B + \sin(2(A+B)) = 4\cos A \cos B \sin(A+B).$$

(c) Find $\displaystyle\int \sqrt{\frac{2+x}{3-x}}\,dx$. **2**

(d) Two lines are given by $\underset{\sim}{r_1} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\underset{\sim}{r_2} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ p \\ 4 \end{pmatrix}$, **3**

where p is a real number. These lines intersect and meet at an angle of θ.

Show that $\theta = \arccos\left(\frac{1}{5}\right)$.

(e) Prove that $\sqrt{29}$ is irrational. **3**

End of Question 13

Question 14 (14 marks) Use a new sheet of paper.

(a) Evaluate $\displaystyle\int_0^1 x^2\sqrt{1-x^2}\,dx$. **3**

(b) A particle undergoing simple harmonic motion satisfies

$$\ddot{x} = -n^2 x.$$

(i) Prove that $v^2 = n^2\left(a^2 - x^2\right)$. **2**

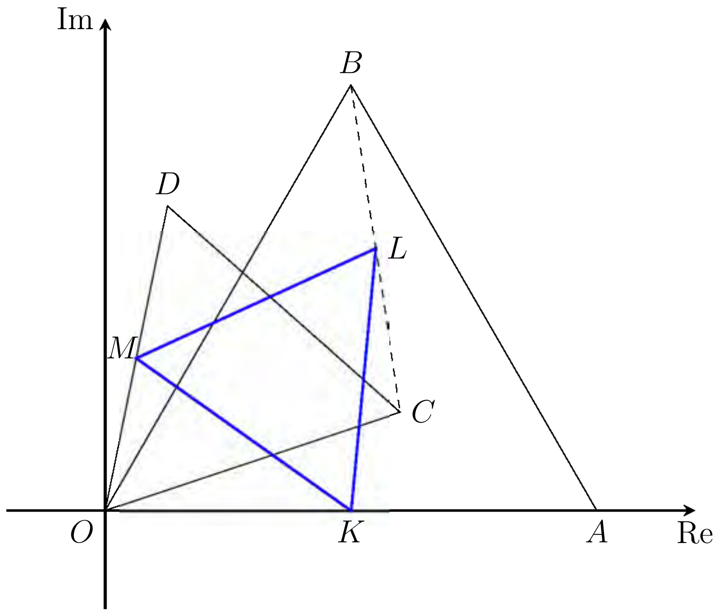
(ii) Let M be the maximum speed of the particle. Find the speed of the particle when it is halfway between the origin and an endpoint in terms of M only. **2**

(c) Find the closest point on the sphere $(x-1)^2 + (y+1)^2 + (z+1)^2 = 5$ to the line $\underset{\sim}{r} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, given that the line and the sphere do not intersect. **3**

Question 14 continues on the next page

(d) In the diagram below, $\triangle OAB$ and $\triangle OCD$ are equilateral triangles.

The points K, L and M are the midpoints of OA, BC and OD, respectively.



Let the complex numbers a, b, c and d represent the points A, B, C and D, respectively. Let $\omega = e^{i\frac{\pi}{3}}$.

(i) Show that $\overrightarrow{KL} = \frac{1}{2}\left(a(\omega - 1) + c\right)$ and $\overrightarrow{KM} = \frac{1}{2}(c\omega - a)$. **2**

(ii) Deduce that $\triangle KLM$ is also equilateral. **2**

End of Question 14

Question 15 (15 marks) Use a new sheet of paper.

(a) A particle of mass 3 kg moves in a straight line subject to a force

$$F = 18x^5 + 24x^3 + 6x,$$

where x is the displacement of the particle at time t.

Let the velocity of the particle be v. Initially, $x = 1$ and $v = -2\sqrt{2}$.

The particle always moves to the left and never passes through the origin.

(i) Show that $v = -\sqrt{2}x(1 + x^2)$. **2**

(ii) Hence, show that **3**

$$x = \frac{1}{\sqrt{2e^{2\sqrt{2}t} - 1}}.$$

(b) It is given that for all positive real numbers $x_1, x_2, \dots, x_n$, **2**

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \dots x_n}.$$

Suppose that positive real numbers $x_1, x_2, \dots, x_n$, satisfy $x_1 + x_2 + \cdots + x_n = n^2$. Show that

$$\frac{1}{x_1^n} + \frac{1}{x_2^n} + \cdots + \frac{1}{x_n^n} \geqslant n^{1-n}.$$

(c) Let $I_n = \displaystyle\int_0^1 x(1 - x^3)^n \, dx$, where n is a non-negative integer.

(i) Show that $I_n = \dfrac{3n}{3n+2} I_{n-1}$ for $n \geqslant 1$. **3**

(ii) Deduce that **2**

$$\sum_{r=0}^{n} {}^nC_r (-1)^r \frac{1}{3r+2} = \frac{3^n (n!)}{(3n+2) \times (3n-1) \times \cdots \times 2}.$$

(d) Prove by mathematical induction that for all positive integers n, **3**

$$\frac{1^2}{\sqrt{2}} + \frac{2^2}{\sqrt{3}} + \cdots + \frac{n^2}{\sqrt{n+1}} < n^2 \sqrt{n}.$$

End of Question 15

Question 16 (16 marks) Use a new sheet of paper.

(a) Let n be a positive integer. Consider the function

$$P(x) = \frac{(1+ix)^{4n+1} - (1-ix)^{4n+1}}{2ix}.$$

(i) Show that $x = \tan\left(\frac{\pi}{4n+1}\right)$ is a zero of $P(x)$. **2**

(ii) Show that $P(x) = {}^{4n+1}C_1 - {}^{4n+1}C_3\, x^2 + {}^{4n+1}C_5\, x^4 - \cdots + x^{4n}$. **2**

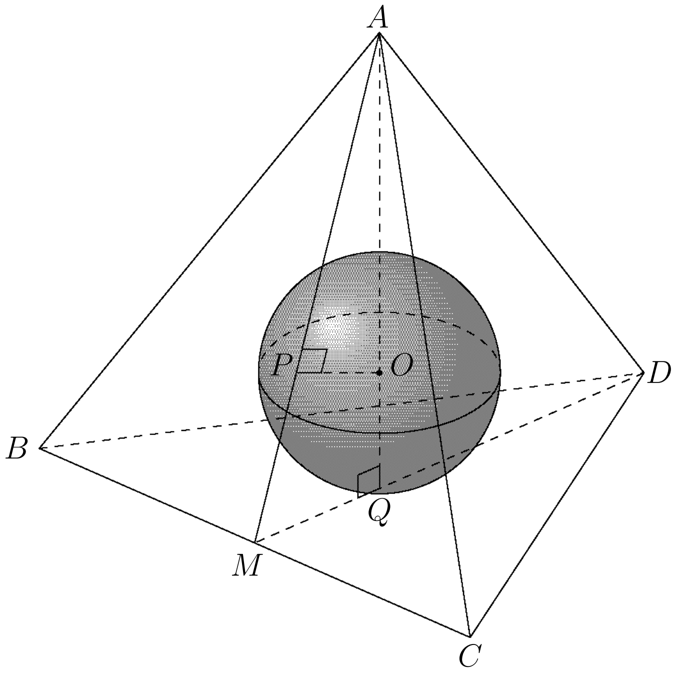
(iii) Prove that if r is a rational zero of $P(x)$, then r is an integer. **2**

(iv) Deduce that $\tan\left(\frac{\pi}{4n+1}\right)$ is irrational. **1**

Question 16 continues on the next page

Question 16 (continued)

(b) The diagram below shows a sphere with centre O and radius r inscribed in a regular tetrahedron $ABCD$. The surface of the sphere touches $\triangle ABC$ and $\triangle BDC$ at P and Q respectively. Let M be the midpoint of BC. It is given that OP is perpendicular to AM and OQ is perpendicular to DM.



Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$, and $\overrightarrow{OD} = \underset{\sim}{d}$.

Let $|\underset{\sim}{a}| = |\underset{\sim}{b}| = |\underset{\sim}{c}| = |\underset{\sim}{d}| = R$.

(i) Use the fact that $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right| = \left|\overrightarrow{AC}\right|$ to show $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c}$. **2**

(ii) Hence, show that $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AM}$. **3**

(iii) Similarly, the point Q satisfies $\overrightarrow{DQ} = \frac{2}{3}\overrightarrow{DM}$. **2**

Furthermore, A, O and Q are collinear.

Deduce that $R = 3r$.

(c) Sketch the set of all complex numbers z satisfying $\text{Arg}(z-1) = \text{Arg}(\bar{z}+i)$ on the Argand diagram. **2**

End of paper

Sydney Girls High School
2024 Extension 2 Mathematics Trial
Marking Guidelines

Multiple choice Answer Key

Question	Answer
1	B
2	C
3	D
4	B
5	C
6	A
7	C
8	D
9	C
10	C

Question 1	1 mark

Solution

The negation of "If A then B" is "A and not B".
Answer: "n is a perfect square and n is two more than a multiple of 3."

Hence (B).

Question 2	1 mark

Solution

We are looking for two complex numbers whose sum is

$$-\frac{b}{a} = 2 + 6i$$

and whose product is

$$\frac{c}{a} = -5 + 2i$$

We find that $z = i$ and $z = 2 + 5i$ work, because

$$\begin{aligned} i + (2 + 5i) &= 2 + 6i \\ i(2 + 5i) &= -5 + 2i \end{aligned}$$

Hence (C).

Question 3	1 mark

Solution

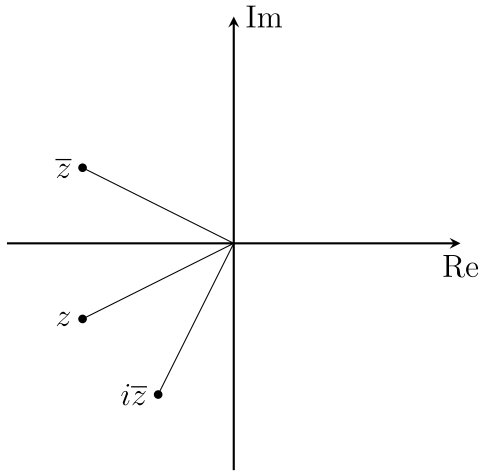
The contrapositive of "If A then B" is "If not B then not A".
Answer: "If n does not have exactly $k + 1$ factors, then $n \neq 2^k$."

Hence (D).

Question 4	1 mark

Solution

The conjugate of z is a reflection of z in the real axis. Multiplying by i rotates a complex number $\frac{\pi}{2}$ radians anti-clockwise about the origin. This gives the following diagram:



Hence (B). ✓

Question 5	1 mark

Solution

Write $-\sqrt{3}+i$ in mod-arg form, then simplify using De Moivre's Theorem:

$$\begin{aligned}\left(-\sqrt{3}+i\right)^{2024} &= \left(2\operatorname{cis}\tfrac{5\pi}{6}\right)^{2024}\\ &= 2^{2024}\operatorname{cis}\left(2024\times\tfrac{5\pi}{6}\right)\\ &= 2^{2024}\operatorname{cis}\left(1686\pi+\tfrac{2\pi}{3}\right)\\ &= 2^{2024}\operatorname{cis}\tfrac{2\pi}{3}\\ &= 2^{2024}\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\right)\\ &= 2^{2023}(-1+\sqrt{3})\end{aligned}$$

Hence (C). ✓

Question 6	1 mark

Solution

$$\begin{aligned}\int \frac{x^3-2x^2+3x}{x-2}\,dx &= \int \frac{x^2(x-2)+3x}{x-2}\,dx\\ &= \int \left(x^2+\frac{3x}{x-2}\right)dx\\ &= \int \left(x^2+\frac{3(x-2)+6}{x-2}\right)dx\\ &= \int \left(x^2+3+\frac{6}{x-2}\right)dx\\ &= \tfrac{1}{3}x^3+3x+6\ln|x-2|+C\end{aligned}$$

Hence (A). ✓

Question 7	1 mark

Solution
Use a substitution followed by integration by parts.
Let $x = u^2$.

$$\begin{aligned}\int_{\frac{9\pi^2}{4}}^{\frac{25\pi^2}{4}} \sin\sqrt{x}\ dx &= \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \sin\sqrt{u^2}\times 2u\,du\\ &= 2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} u\sin u\,du\\ &= 2\Big[-u\cos u\Big]_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} - 2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}(-\cos u)\,du\\ &= 0+2\Big[\sin u\Big]_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}\\ &= 2(1-(-1))\\ &= 4\end{aligned}$$

Hence (C). ✓

Question 8	1 mark

Solution

The triangle inequality states that, for any complex numbers u and v, that

$$\Big||u|-|v|\Big| \leqslant |u \pm v| \leqslant |u|+|v|$$

We have

$$\begin{aligned}|4z-3| &= |4(z+i)-(3+4i)| \\ &\leqslant |4(z+i)|+|3+4i| \\ &= 4|z+i|+5 \\ &= 4\times 3+5 \\ &= 17.\end{aligned}$$

Similarly,

$$\begin{aligned}|4z-3| &= |4(z+i)-(3+4i)| \\ &\geqslant \Big||4(z+i)|-|3+4i|\Big| \\ &= \Big|4|z+i|-5\Big| \\ &= |4\times 3-5| \\ &= 7.\end{aligned}$$

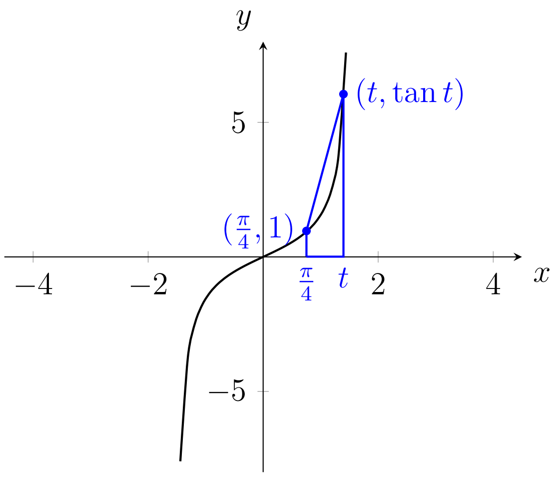
Hence $7 \leqslant |4z-3| \leqslant 17$.

Hence (D). ✓

Question 9	1 mark

Solution

Let t be a real number in the interval $(\frac{\pi}{4}, \frac{\pi}{2})$.
Consider the sketch of $y = \tan x$ for $\frac{\pi}{4} < x < \frac{\pi}{2}$.



The area of the blue trapezium shown above is greater than the area under the curve from $\frac{\pi}{4}$ to t. Therefore

$$\begin{aligned}
\text{Area of trapezium} &> \text{Area under curve} \\
\tfrac{1}{2}(t - \tfrac{\pi}{4})(1 + \tan t) &> \int_{\frac{\pi}{4}}^{t} \tan x \, dx \\
&= \Big[\ln|\sec x| \Big]_{\frac{\pi}{4}}^{t} \\
&= \ln|\sec t| - \ln|\sec \tfrac{\pi}{4}| \\
&= \ln\left(\frac{\sec t}{\sqrt{2}}\right)
\end{aligned}$$

Note that we can remove the absolute value signs because cos is positive between
$\frac{\pi}{4}$ and $\frac{\pi}{2}$. Now exponentiate both sides:

$$\begin{aligned}
\frac{\sec t}{\sqrt{2}} &< e^{\frac{1}{2}(t-\frac{\pi}{4})(1+\tan t)} \\
\sec t &< \sqrt{2} e^{\frac{1}{2}(t-\frac{\pi}{4})(1+\tan t)}
\end{aligned}$$

Hence (C). ✓

Question 10	1 mark

Solution

Since $\underset{\sim}{r}$ is perpendicular to $\underset{\sim}{a} - \underset{\sim}{b}$, we have

$$\begin{aligned} \underset{\sim}{r} \cdot (\underset{\sim}{a} - \underset{\sim}{b}) &= 0 \\ \underset{\sim}{r} \cdot \underset{\sim}{a} - \underset{\sim}{r} \cdot \underset{\sim}{b} &= 0 \\ \underset{\sim}{r} \cdot \underset{\sim}{a} &= \underset{\sim}{r} \cdot \underset{\sim}{b} \end{aligned}$$

Now substitute $\underset{\sim}{r} = \underset{\sim}{a} + (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{b}$ and expand:

$$\begin{aligned} (\underset{\sim}{a} + (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{b}) \cdot \underset{\sim}{a} &= (\underset{\sim}{a} + (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{b}) \cdot \underset{\sim}{b} \\ \underset{\sim}{a} \cdot \underset{\sim}{a} + (\underset{\sim}{a} \cdot \underset{\sim}{b})^2 &= \underset{\sim}{a} \cdot \underset{\sim}{b} + (\underset{\sim}{a} \cdot \underset{\sim}{b})(\underset{\sim}{b} \cdot \underset{\sim}{b}) \end{aligned}$$

Let $\lambda = \underset{\sim}{a} \cdot \underset{\sim}{b}$.

$$\begin{aligned} |\underset{\sim}{a}|^2 + \lambda^2 &= \lambda + \lambda|\underset{\sim}{b}|^2 \\ \lambda^2 - (1 + |\underset{\sim}{b}|^2)\lambda + |\underset{\sim}{a}|^2 &= 0 \end{aligned}$$

This is a quadratic equation in λ. Since the vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{r}$ exist, this quadratic must have real solutions. Therefore, the discriminant is non-negative, so

$$\begin{aligned} \Delta &\geqslant 0 \\ b^2 - 4ac &\geqslant 0 \\ (1 + |\underset{\sim}{b}|^2)^2 - 4(1)(|\underset{\sim}{a}|^2) &\geqslant 0 \\ (1 + |\underset{\sim}{b}|^2)^2 &\geqslant 4|\underset{\sim}{a}|^2 \\ 1 + |\underset{\sim}{b}|^2 &\geqslant 2|\underset{\sim}{a}|. \end{aligned}$$

Hence the minimum value of $1 + |\underset{\sim}{b}|^2$ is $2|\underset{\sim}{a}|$.

Hence (C).

Question 11

a)

$$\overline{7+6i}+\frac{2}{3-i}=7-6i+\frac{2(3+i)}{3^2+1^2}$$ ✓

$$=7-6i+\frac{3+i}{5}$$

$$=\frac{38}{5}-\frac{29i}{5}$$ ✓

b)

$$\int\frac{\tan^2 x+2}{x+\tan x}dx=\int\frac{\sec^2 x-1+2}{x+\tan x}dx$$ ✓

$$=\int\frac{\sec^2 x+1}{x+\tan x}dx$$ ✓

$$=\ln|x+\tan x|+c$$

AW 1 for subs $\tan^2 x = \sec^2 x - 1$
Integration by subs can also be used.

c) i) Let $a = pd$
$b = qd$

$ma + nb = mpd + nqd$
$= d(mp + nq)$ ✓

which is divisible by d

Most students did well.

ii) If $ma + nb$ is divisible by d then a and b are divisible by d.

Let $a = b = m = n = 1$

$ma + nb = 1\times1 + 1\times1 = 2$ which is divisible by 2 but $a = 1$, $b = 1$ which are not divisible by 2. The converse statement is false. ✓

Just write the converse statement without proof ∴ No marks

d) i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ ①

$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\sin^2\theta\, i^2 +$ ✓

$4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$ ✓

②

Most students did well.

ii) Equating ① and ② from part (i)

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$ ✓

divide by $\cos^4\theta$

$\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ ✓

AW 1 mark for expressing $\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta}$ in terms of $\sin\theta$ and $\cos\theta$

e)

i) Suppose there existed a real number μ such that $\underset{\sim}{v} = \mu\underset{\sim}{u}$

$$\begin{pmatrix} 6\lambda \\ -12\lambda \\ 6 \end{pmatrix} = \mu\begin{pmatrix} 2\lambda \\ 2 \\ 3 \end{pmatrix}$$

z-component $3\mu = 6 \quad \therefore \quad \mu = 2$

x - component $\quad 6\lambda = \mu \times 2\lambda$

$6\lambda = 4\lambda \quad \therefore \lambda = 0$

Subs $\lambda = 0$, $\mu = 2$ into the y-component

$-12\lambda = 2\mu$

$-12 \times 0 = 2 \times 2$ Not true

$\therefore \quad \underset{\sim}{v} \neq \mu \underset{\sim}{u}$

$\underset{\sim}{u}$ is not parallel to $\underset{\sim}{v}$

AW 1 for solving for λ

State $\underset{\sim}{v} \neq \lambda \underset{\sim}{u}$ without proper proving : No marks.

ii) Suppose $\underset{\sim}{u} . \underset{\sim}{v} = 0$

$2\lambda . 6\lambda + 2(-12\lambda) + 3(6) = 0$

$12\lambda^2 - 24\lambda + 18 = 0$

$2\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 1)^2 = 1 - \frac{3}{2}$

$(\lambda - 1)^2 = -\frac{1}{2} < 0$

There is No solutions for real λ

$\therefore \quad \underset{\sim}{u}$ and $\underset{\sim}{v}$ are Not perpendicular.

AW 1 for having correct quadratic equation for λ.

The direction vectors can also be applied to prove this.

f) $T = \frac{2\pi}{n} = \frac{\pi}{3} \longrightarrow n = 6$

$x = a\sin nt$

$x = a\sin 6t$ ✓

$\dot{x} = 6a\cos 6t$

Max speed $|\dot{x}| = |6a|$ when $\cos 6t = 1$

$6a\cos 6t = |3a|$

$\cos 6t = \pm\frac{1}{2}$

$6t = \frac{\pi}{3}, \frac{2\pi}{3}$

$t = \frac{\pi}{18}, \frac{\pi}{9}$ sec ✓

The first two times : $t = \frac{\pi}{18}$ and $t = \frac{\pi}{9}$

Not $6t = \pi + \frac{\pi}{3}$

$t = \frac{2\pi}{9}$

Question 12 (a)(i) (1 mark)

$$\omega^5 = 1$$
$$(\omega - 1)(1 + \omega + \omega^2 + \omega^3 + \omega^4) = 0$$
$$\text{As } \omega \neq 1, \ \therefore 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$$
$$\therefore \omega + \omega^2 + \omega^3 + \omega^4 = -1 \quad \checkmark$$

Marking Scheme

✓ [1] for correctly showing the result

Marker's Comments:

- Generally well done.
- Although it was not penalised for not writing "As $\omega \neq 1$", students are advised to write it in their HSC, as it is important to reason why the alternative $\omega = 1$ is ruled out.

Question 12 (a)(ii) (2 marks)

$$\begin{aligned}\alpha + \beta &= \omega + \omega^4 + \omega^2 + \omega^3 \\ &= -1, \quad \text{by (i)} \quad \checkmark\end{aligned}$$

$$\begin{aligned}\alpha\beta &= (\omega + \omega^4)(\omega^2 + \omega^3) \\ &= \omega^3 + \omega^4 + \omega^6 + \omega^7 \\ &= \omega^3 + \omega^4 + \omega + \omega^2, \quad \text{since } \omega^5 = 1 \\ &= -1\end{aligned}$$

$$\therefore \alpha \text{ and } \beta \text{ are roots of } x^2 + x - 1 = 0$$
$$\therefore b = 1, c = -1 \quad \checkmark$$

Marking Scheme

✓ [1] for obtaining the value of $\alpha + \beta$, or equivalent merit

✓ [1] for obtaining the correct values of b ***and*** c

Marker's Comments:

- Generally well done.
- Some students however did not know that sum of roots $= -\dfrac{b}{a}$, and instead wrote $\dfrac{b}{a}$, which resulted in deducing that $b = -1$.
- Some students also left their answers unsimplified. E.g. writing $b = -(\omega + \omega^2 + \omega^3 + \omega^4)$ and then not using part (i).

Question 12 (a)(iii) (2 marks)

$$\begin{aligned}\beta &= \omega^2 + \omega^3 \\ &= e^{i\frac{4\pi}{5}} + e^{i\frac{6\pi}{5}} \\ &= e^{i\frac{4\pi}{5}} + e^{-i\frac{4\pi}{5}} \\ &= 2\cos\frac{4\pi}{5} \quad \checkmark\end{aligned}$$

$$\text{Also, } \beta = \frac{-1 \pm \sqrt{1+4}}{2} \quad \text{from (ii)}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

$$\text{As } \beta < 0, \ \therefore \beta = \frac{-1-\sqrt{5}}{2}$$

$$\therefore \cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4} \quad \checkmark$$

Marking Scheme

✓ [1] for deducing that $\beta = 2\cos\frac{4\pi}{5}$, or equivalent merit

✓ [1] for correctly deducing (i.e. with proper working) the exact value of $\cos\frac{4\pi}{5}$

Marker's Comments:

- Generally not done well. Many students were unable to see the connection between parts (ii) and (iii).
- Students who deduced that $\beta = 2\cos\frac{4\pi}{5}$ were generally successful in obtaining full marks.

Question 12 (b)(i) (3 marks)

$$\int \frac{x^2+x+3}{x^2+x-2}\,dx = \int \frac{x^2+x-2}{x^2+x-2} + \frac{5}{x^2+x-2}\,dx \quad ✓$$

$$= \int 1 + \frac{5}{(x+2)(x-1)}\,dx \quad ✓$$

$$= \int 1 + \frac{-\frac{5}{3}}{x+2} + \frac{\frac{5}{3}}{x-1}\,dx \quad \text{(by the cover-up rule)}$$

$$= x - \frac{5}{3}\ln|x+2| + \frac{5}{3}\ln|x-1| + C$$

$$= x + \frac{5}{3}\ln\left|\frac{x-1}{x+2}\right| + C \quad ✓$$

Marking Scheme

- ✓ [1] for using algebraic manipulation to rewrite the integrand
- ✓ [1] for factorising x^2+x-2 correctly
- ✓ [1] for correct partial fraction decomposition and correct final answer

Marker's Comments:

- Generally well done. Students who were able to recognise that x^2+x-2 can be factorised were generally successful – being able to identify whether the denominator can be factorised is an important technique in integration.
- However, some students did not factorise x^2+x-2, and instead completed the square, leading to erroneous answers most of the time. Only a handful of students who completed the square and used a trigonometric substitution were successful, but this was not the best approach.
- Some students also incorrectly performed partial decomposition on $\dfrac{x^2+x+3}{x^2+x-2}$. Students should remember that partial fraction decomposition only works when the degree of the numerator is less than the degree of the denominator.

Question 12 (b)(ii) (3 marks)

Let $u^3 = x$

$$3u^2\,du = dx$$
$$x = 0,\ u = 0$$
$$x = 1,\ u = 1 \quad ✓$$

$$\begin{aligned}\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx &= \int_0^1 \frac{1}{1+u}3u^2\,du\\ &= 3\int_0^1 \frac{u^2+u}{1+u} - \frac{u}{1+u}\,du\\ &= 3\int_0^1 u - \left(\frac{1+u}{1+u} - \frac{1}{1+u}\right)\,dx\\ &= 3\int_0^1 u - 1 + \frac{1}{1+u}\,dx \quad ✓\\ &= 3\left[\frac{u^2}{2} - u + \ln(1+u)\right]_0^1\\ &= 3\left(\frac{1}{2} - 1 + \ln 2\right)\\ &= -\frac{3}{2} + 3\ln 2 \quad ✓\end{aligned}$$

Marking Scheme

✓ [1] for correct substitution, correctly differentiating it, and correct new limits

✓ [1] for correct algebraic manipulation to express the integrand as $u - 1 + \dfrac{1}{1+u}$

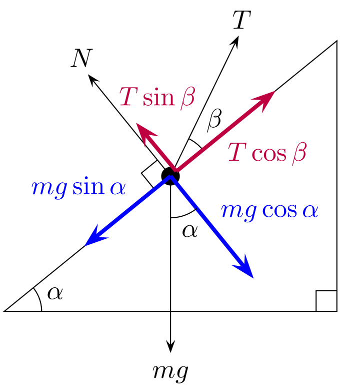
✓ [1] for correct final answer

Marker's Comments:

- Students who had the correct substitution were generally successful.
- Some students incorrectly thought that $\dfrac{1}{1+\sqrt[3]{x}} = \dfrac{x^3}{(1+\sqrt[3]{x})x^3} = \dfrac{x^3}{x^3+1}$ or $\dfrac{x^3}{x^3+x}$.
- A nice alternative method after obtaining $\displaystyle\int_0^1 \frac{1}{1+u}3u^2\,du$ is to write it as

$$3\int_0^1 \frac{u^2-1}{1+u} + \frac{1}{1+u}\,du = 3\int_0^1 u - 1 + \frac{1}{1+u}\,du.$$

Question 12 (c) (3 marks)



Normally to the plane,

$$N + T\sin\beta = mg\cos\alpha$$ ✓

$$N = mg\cos\alpha - T\sin\beta$$

Along the plane,

$$mg\sin\alpha = T\cos\beta$$ ✓

$$T = \frac{mg\sin\alpha}{\cos\beta}$$

$$\therefore N = mg\cos\alpha - \frac{mg\sin\alpha}{\cos\beta}\sin\beta$$

$$= mg\cos\alpha - mg\sin\alpha\tan\beta$$

$$= mg(\cos\alpha - \tan\beta\sin\alpha)$$ ✓

Marking Scheme
✓ [1] for correctly balancing the magnitudes of the forces that are normal to the plane
✓ [1] for correctly balancing the magnitudes of the forces that are along the plane
✓ [1] for correctly showing the required result

Marker's Comments:

- Generally well done.
- Students who did not resolve the forces correctly, or had resolved the forces into vertical and horizontal components were not successful.

Question 13 a)

(a) Let $P(z) = z^3 + kz^2 + 6$, where z is complex and k is real. **2**

When $P(z)$ is divided by $z^2 + 4$, the remainder is $-4z - 6$.

Find the value of k.

Method 1

$$\begin{array}{r|l} & z + k \\ z^2+4 & z^3 + kz^2 + 0z + 6 \\ & z^3 \quad\quad + 4z \\ \hline & kz^2 - 4z + 6 \\ & kz^2 \quad\quad + 4k \\ \hline & -4z + (6-4k) \end{array}$$

$$\therefore R(z) = -4z - 6$$
$$\equiv -4z + (6-4k)$$

$$\therefore 6 - 4k = -6$$
$$-4k = -12$$
$$k = 3$$

Alternatively: Method 2.

$$z^2 + 4 = (z+2i)(z-2i)$$

By division transformation:

$$P(z) = (z^2+4)\,Q(z) + R(z)$$
$$\therefore P(2i) = R(2i)$$

$$(2i)^3 + k(2i)^2 + 6 = -4(2i) - 6$$
$$8i^3 + 4ki^2 + 6 = -8i - 6$$
$$\cancel{-8i} - 4k + 6 = \cancel{-8i} - 6$$

$$-4k = -12$$
$$k = 3.$$

* Some students incorrectly assumed that $z = \pm 2i$ were the roots of the polynomial.

* Most students attempted this question successfully, using Method 1.

Question 13 b)

(b) (i) Prove that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. **2**

$$\frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta))$$

$$= \frac{1}{2}(\cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta})$$

$$= \cos\theta$$

$$\therefore \cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$$

$$\frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}(\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta))$$

$$= \frac{1}{2i}(\cancel{\cos\theta} + i\sin\theta - \cancel{\cos\theta} + i\sin\theta)$$

$$= \frac{1}{\cancel{2i}}(\cancel{2i}\sin\theta)$$

$$= \sin\theta$$

$$\therefore \sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta})$$

* Well-answered question.

* Students lost marks if not enough working was shown.

Question 13 b)

(ii) Hence, prove for all real A, B that : **3**

$$\sin 2A + \sin 2B + \sin(2(A + B)) = 4\cos A \cos B \sin(A + B).$$

$RHS = 4\cos A \cos B \sin(A+B)$

$= \cancel{4}\cdot\frac{1}{\cancel{2}}\left(e^{iA}+e^{-iA}\right)\frac{1}{\cancel{2}}\left(e^{iB}+e^{-iB}\right)\frac{1}{2i}\left(e^{i(A+B)}-e^{-i(A+B)}\right)$

$=\frac{1}{2i}\left(e^{i(A+B)}+e^{i(A-B)}+e^{i(B-A)}+e^{-i(A+B)}\right)\left(e^{i(A+B)}-e^{-i(A+B)}\right)$ ✓ formulae + expanding

$=\frac{1}{2i}\left(e^{2i(A+B)}+e^{2iA}+e^{2iB}+e^{0}\right)$

$-\frac{1}{2i}\left(e^{0}+e^{-2iB}+e^{-2iA}+e^{-2i(A+B)}\right)$ ✓ expanding more

$=\frac{1}{2i}\left(e^{2i(A+B)}-e^{-2i(A+B)}\right)+\frac{1}{2i}\left(e^{2iA}-e^{-2iA}\right)$

$+\frac{1}{2i}\left(e^{2iB}-e^{2iB}\right)+\frac{1}{2i}\left(e^{0}-e^{0}\right)$ ✓ rewriting

$= \sin(2(A+B)) + \sin 2A + \sin 2B + 0$

$= \sin 2A + \sin 2B + \sin(2(A+B))$

$= LHS.$

* This was a challenging question for students.
* It is a skill-based question which needs to be proven using part i) for full marks.
* Work had to be clearly shown. Some writing was difficult to understand and marks were not awarded as a consequence.

Question 13c)

(c) Find $\int \sqrt{\frac{2+x}{3-x}}\,dx$. **2**

$$\Rightarrow \int \frac{(2+x)\,dx}{\sqrt{(3-x)(2+x)}}$$

$$= \int \frac{(2+x)\,dx}{\sqrt{6+x-x^2}}$$

$$= -\int \left(\frac{-x+\frac{1}{2}}{\sqrt{6+x-x^2}} - \frac{\frac{5}{2}}{\sqrt{6+x-x^2}} \right) dx$$

$$\frac{d}{dx}\left(\sqrt{6+x-x^2}\right) = \frac{\frac{1}{2}(-2x+1)}{\sqrt{6+x-x^2}} = \frac{\left(-x+\frac{1}{2}\right)}{\sqrt{6+x-x^2}}$$

$$= -\sqrt{6+x-x^2} + \frac{5}{2}\int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}}$$

$$= -\sqrt{6+x-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{2\left(x-\frac{1}{2}\right)}{5}\right) + C$$

$$= -\sqrt{6+x-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{2x-1}{5}\right) + C.$$

* Rationalising the numerator then splitting up into 2 fractions was one mark.
* Correct answer was given the second mark.
* minor arithmetic errors were ignored, but students need to take care with their work in future.

Question 13 d)

(d) Two lines are given by $\underset{\sim}{r_1} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\underset{\sim}{r_2} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ p \\ 4 \end{pmatrix}$, **3**

where p is a real number. These lines intersect and meet at an angle of θ.

Show that $\theta = \arccos\left(\frac{1}{5}\right)$.

$$\underset{\sim}{r_1} = \underset{\sim}{r_2}$$

$$\therefore \quad 3 + 2\lambda = -3 \quad \Rightarrow 2\lambda = -6 \quad \therefore \underline{\lambda = -3}.$$
$$-1 = 1 + \mu p$$
$$4 + \lambda = 2 + 4\mu$$

$$\therefore \quad -3 = -3$$
$$-1 = 1 + \mu p \quad \Rightarrow \quad \mu p = -2$$
$$1 = 2 + 4\mu \quad \Rightarrow \quad 4\mu = -1$$
$$\underline{\mu = -\frac{1}{4}}.$$

$$\therefore \quad -\frac{1}{4}p = -2$$
$$\underline{p = 8}$$

$$\cos\theta = \frac{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 8 \\ 4 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 0 \\ 8 \\ 4 \end{pmatrix}\right|}$$

← Students needed to <u>show</u> this step.

$$\cos\theta = \frac{4}{\sqrt{5}\sqrt{80}}$$

$$= \frac{\cancel{4}}{\sqrt{5} \cdot \cancel{4}\sqrt{5}}$$

$$\therefore \cos\theta = \frac{1}{5}$$

$$\therefore \theta = \arccos\left(\frac{1}{5}\right)$$

* Most students completed this successfully.

Question 13e)

(e) Prove that $\sqrt{29}$ is irrational. 3

Assume $\sqrt{29}$ is rational (by way of contradiction)

Let $\sqrt{29} = \frac{p}{q}$, $p, q \in \mathbb{Z}$ and are coprime.

then $29 = \frac{p^2}{q^2}$

$p^2 = 29q^2$

hence p^2 is divisible by 29

so p is divisible by 29

Let $p = 29m$, $m \in \mathbb{Z}$

then $(29m)^2 = 29q^2$

$29m^2 = q^2$

hence q^2 is divisible by 29

so q is divisible by 29

Since p and q are both divisible by 29 this is a contradiction to the original assumption that p and q are coprime.

Hence the original assumption that $\sqrt{29}$ is rational is false.

$\therefore \sqrt{29}$ is irrational.

* This is a textbook proof.
* Marks were deducted if the proof was insufficient.

Question 14(a)	Marks
· Provides correct solution.	3 marks
· Uses $\sin^2\theta\cos^2\theta = \frac{1}{4}\sin^2(2\theta)$ OR uses double angle formulae for $\cos 2\theta$.	2 marks
· Substitutes $x = \sin\theta$ and obtains integral in terms of θ only.	1 mark

Solution

$$I = \int_0^1 x^2\sqrt{1-x^2}\,dx$$

$$x = \sin\theta \qquad\qquad x = 0 \implies \theta = 0$$

$$dx = \cos\theta\,d\theta \qquad\qquad x = 1 \implies \theta = \frac{\pi}{2}$$

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta \checkmark \\
&= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos\theta\sqrt{\cos^2\theta}\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^2\,d\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2\,d\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2(2\theta)\,d\theta \checkmark \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1-\cos(4\theta)\right)\,d\theta \\
&= \frac{1}{8}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{8}\left(\frac{\pi}{2} - \frac{1}{4}\sin(2\pi) - \left(0 - \frac{1}{4}\sin 0\right)\right) \\
&= \frac{\pi}{16} \checkmark
\end{aligned}$$

Comments

· There were many unsuccessful attempts including $u = 1 - x^2$ and integration by parts.
· To make progress, the trigonometric substitution $x = \sin\theta$ was necessary.
· Clues for trig substitutions are factors of $1 - x^2, 1 + x^2$ or $x^2 - 1$ appearing.
· One interesting approach involved algebraic manipulation and integration by parts.
· The area of a quarter circle was then used to relate I to itself.

Question 14(b)(i)	**Marks**
· Provides correct solution.	2 marks
· Integrates $\ddot{x} = -n^2x$ to show $\frac{1}{2}v^2 = -\frac{1}{2}n^2x^2 + C$, OR lets $x = a\sin(nt)$ to show $\dot{x} = na\cos(nt)$ and $v^2(a^2 - x^2) = n^2(a^2 - a^2\sin^2(nt))$.	1 mark

Solution

Use the formula $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ and integrate:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x$$

$$\frac{1}{2}v^2 = -\frac{1}{2}n^2x^2 + C \checkmark$$

When $x = a, v = 0$. Therefore

$$0 = -\frac{1}{2}n^2a^2 + C$$

$$C = \frac{1}{2}n^2a^2$$

Therefore

$$\frac{1}{2}v^2 = -\frac{1}{2}n^2x^2 + \frac{1}{2}n^2a^2$$

$$v^2 = -n^2x^2 + n^2a^2$$

$$v^2 = n^2(a^2 - x^2). \checkmark$$

Comments

· Students were often unable to find C. They had to use $v = 0$ when $x = a$.
· Similarly, limits of integration needed to reflect this boundary condition.
· Alternatively, $x = a\sin(nt + \alpha)$, differentiate, then use Pythagorean identity.

Question 14(b)(ii)	Marks
· Provides correct solution.	2 marks
· Finds $M = na$.	1 mark

Solution

The maximum speed is obtained when $x = 0$. Sub this into $v^2 = n^2(a^2 - x^2)$:

$$\begin{aligned} M^2 &= n^2(a^2 - 0^2) \\ &= n^2a^2 \\ M &= na \quad (M > 0) \end{aligned}$$

When $x = \frac{a}{2}$, the particle is halfway between an endpoint and the origin.
Sub this into $v^2 = n^2(a^2 - x^2)$:

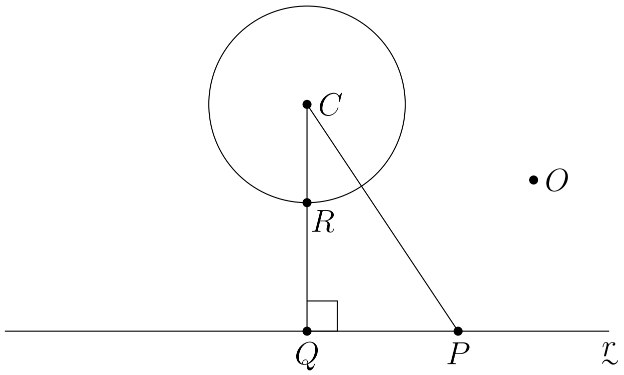
$$\begin{aligned} v^2 &= n^2\left(a^2 - \left(\frac{a}{2}\right)^2\right) \\ &= \frac{3}{4}n^2a^2 \\ \therefore v &= \frac{\sqrt{3}}{2}na \\ &= \frac{\sqrt{3}}{2}M \end{aligned}$$

Comments

· Quite well done.
· Note that maximum speed is positive, so $v = \pm\frac{\sqrt{3}}{2}M$ is incorrect.

Question 14(c)	**Marks**
· Provides correct solution.	3 marks
· Finds $\overrightarrow{CQ}$.	2 marks
· Obtains an expression for $\overrightarrow{CQ} \cdot (1, 0, -2) = 0$ in terms of λ OR finds a suitable projection onto the line $\underset{\sim}{r}$.	1 mark

Solution
Let the centre of the sphere be C, the foot of the perpendicular from C to $\underset{\sim}{r}$ be P, and let Q be the intersection between the line CQ and the sphere, as shown below:



Since the shortest distance between a point and a line is the perpendicular distance, Q is the closest point from C to the line $\underset{\sim}{r}$. It follows that R is the desired closest point on the sphere to the line $\underset{\sim}{r}$.

Firstly, note that $\overrightarrow{PC} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$. Therefore

$$\text{proj}_{\overrightarrow{PQ}}\overrightarrow{PC} = \frac{\begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$= \frac{7}{5}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \checkmark$$

Hence

$$\begin{aligned} \overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ} \\ &= \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \frac{7}{5}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} -3 \\ 5 \\ -9 \end{pmatrix} \end{aligned}$$

Now

$$\overrightarrow{CQ} = \frac{1}{5}\begin{pmatrix}-3\\5\\-9\end{pmatrix} - \begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}-3-5\\5+5\\-9+5\end{pmatrix}$$

$$= \frac{2}{5}\begin{pmatrix}-4\\5\\-2\end{pmatrix} \checkmark$$

$$|\overrightarrow{CQ}| = \frac{2}{5}\sqrt{4^2+5^2+2^2}$$

$$= \frac{6\sqrt{5}}{5}$$

Therefore, a unit vector in the same direction as $\overrightarrow{CQ}$ is

$$\frac{5}{6\sqrt{5}} \times \frac{2}{5}\begin{pmatrix}-4\\5\\-2\end{pmatrix} = \frac{1}{3\sqrt{5}}\begin{pmatrix}-4\\5\\-2\end{pmatrix}.$$

The length of $\overrightarrow{CR}$ is $\sqrt{5}$, since that is the radius of the sphere.

$$\overrightarrow{CR} = \sqrt{5} \times \frac{1}{3\sqrt{5}}\begin{pmatrix}-4\\5\\-2\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}-4\\5\\-2\end{pmatrix}$$

$$\therefore \overrightarrow{OR} = \overrightarrow{OC} + \overrightarrow{CR}$$

$$= \begin{pmatrix}1\\-1\\-1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-4\\5\\-2\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}-1\\2\\-5\end{pmatrix}$$

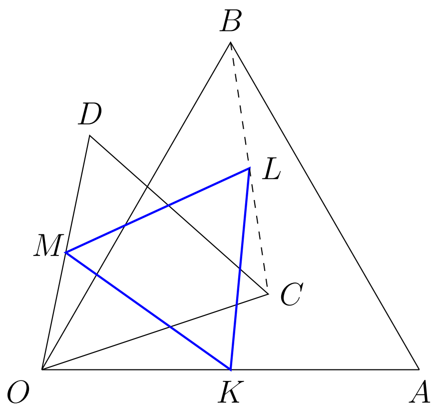
Thus the desired point is $R = (-\frac{1}{3}, \frac{2}{3}, -\frac{5}{3})$. $\checkmark$

Comments

- This question was challenging, and many students did not know where to begin.
- Some tried erroneously to find points of intersection between the sphere and the line.
- A common mistake was to say the radius of the sphere was 5, when it is really $\sqrt{5}$.

Question 14(d)(i)	Marks
· Provides correct solution.	2 marks
· Finds the complex numbers representing K, L and M OR finds one of $\overrightarrow{KL}$ and $\overrightarrow{KM}$.	1 mark

Solution



Since $\triangle OAB$ and $\triangle OCD$ are equilateral, $\overrightarrow{OB}$ and $\overrightarrow{OD}$ are anti-clockwise rotations of $\overrightarrow{OA}$ and $\overrightarrow{OC}$ respectively by $\frac{\pi}{3}$ about O. Therefore $b = \omega a$ and $d = \omega c$.

Since K is the midpoint of OA, we have $\overrightarrow{OK} = \frac{1}{2}a$.

Since L is the midpoint BC, we have $\overrightarrow{OL} = \frac{1}{2}(c + \omega a)$

Since M is the midpoint of OD, we have $\overrightarrow{OM} = \frac{1}{2}\omega c$. ✓

Hence

$$\begin{aligned}\overrightarrow{KL} &= \overrightarrow{OL} - \overrightarrow{OK}\\ &= \frac{1}{2}(c + \omega a) - \frac{1}{2}a\\ &= \frac{1}{2}(c + \omega a - a)\\ &= \frac{1}{2}(a(\omega - 1) + c)\,.\end{aligned}$$

$$\begin{aligned}\overrightarrow{KM} &= \overrightarrow{OM} - \overrightarrow{OK}\\ &= \frac{1}{2}\omega c - \frac{1}{2}a\\ &= \frac{1}{2}(c\omega - a)\,. \checkmark\end{aligned}$$

Comments

- There were many inaccuracies with notation that could be improved.
- Vector arrows were forgotten, e.g. KL instead of $\overrightarrow{KL}$.
- Tildes were placed under complex numbers, such as $\underset{\sim}{\omega}$, which is not correct.
- One misconception was that $\overrightarrow{OL} = \frac{1}{2}\overrightarrow{BC}$. In truth, $\overrightarrow{OL} = \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OC}\right)$.
- Students used circuitous paths to find $\overrightarrow{KL}$.
- For example, $\overrightarrow{KL} = \overrightarrow{KO} + \overrightarrow{OC} + \overrightarrow{CB} + \overrightarrow{BL}$.
- The simplest path is just $\overrightarrow{KL} = \overrightarrow{KO} + \overrightarrow{OL}$.

Question 14(d)(ii)	Marks
· Provides correct solution.	2 marks
· Obtains $\omega^2 - \omega = -1$ or attempts to prove $\omega\overrightarrow{KL} = \overrightarrow{KM}$.	1 mark

Solution
First note that

$$\begin{aligned}\omega^3 &= \left(e^{i\frac{\pi}{3}}\right)^3 \\ &= e^{i\pi} \\ &= -1.\end{aligned}$$

It follows that

$$\begin{aligned}\omega^3 + 1 &= 0 \\ (\omega+1)(\omega^2 - \omega + 1) &= 0 \\ \omega^2 - \omega + 1 &= 0 \quad (\omega \neq -1) \\ \omega^2 - \omega &= -1. \;\; (*)\end{aligned}$$

It suffices to prove that $\overrightarrow{KM}$ is an anti-clockwise rotation of $\overrightarrow{KL}$ by $\frac{\pi}{3}$ about O.
That is, we wish to prove that $\omega\overrightarrow{KL} = \overrightarrow{KM}$.
Use part (i) to obtain:

$$\begin{aligned}\omega\overrightarrow{KL} &= \frac{\omega}{2}\left(a(\omega-1)+c\right) \\ &= \frac{1}{2}\left(a(\omega^2-\omega)+c\omega\right) \\ &= \frac{1}{2}\left(a(-1)+c\omega\right) \quad (\text{from } (*)) \\ &= \frac{1}{2}(c\omega - a) \\ &= \overrightarrow{KM}\end{aligned}$$

Comments
· The dot product cannot be used to answer this question.
· The dot product only applies to vectors with real components.
· Thus expressions such as $(c\omega + a)\cdot(c\omega + a)$ are not well-defined.

Question 15 (a)(i) (2 marks)

$$
\begin{aligned}
3v\frac{dv}{dx} &= 18x^5 + 24x^3 + 6x, \quad \text{using } F = ma = 3v\frac{dv}{dx} \\
v\frac{dv}{dx} &= 6x^5 + 8x^3 + 2x \\
\int_{-2\sqrt{2}}^{v} v\,dv &= \int_1^x 6x^5 + 8x^3 + 2x\,dx \quad \checkmark \\
\frac{1}{2}\Big[v^2\Big]_{-2\sqrt{2}}^{v} &= \Big[x^6 + 2x^4 + x^2\Big]_1^x \\
\frac{1}{2}(v^2 - 8) &= x^6 + 2x^4 + x^2 - 4 \\
v^2 - 8 &= 2x^6 + 4x^4 + 2x^2 - 8 \\
\therefore v^2 &= 2x^6 + 4x^4 + 2x^2 \\
&= 2x^2(x^4 + 2x^2 + 1) \\
&= 2x^2(x^2+1)^2 \\
\therefore v &= -\sqrt{2}x(1+x^2), \quad \text{since } v < 0 \text{ and } x > 0 \quad \checkmark
\end{aligned}
$$

Marking Scheme
✓ [1] for correct separation of variables and integrating (with or without limits)
✓ [1] for correctly showing the required result

Marker's Comments:

- Generally well done.
- Some students who used indefinite integrals did not write the constant of integration, or did not evaluate it using the initial conditions.
- Students who did not convert a into $v\dfrac{dv}{dx}$ were not successful. Converting a into $v\dfrac{dv}{dx}$ is a key step in Extension 2 Mechanics.

Question 15 (a)(ii) (3 marks)

$$\frac{dx}{dt} = -\sqrt{2}x(1+x^2)$$
$$\frac{dx}{x(1+x^2)} = -\sqrt{2}\,dt$$
$$\int_1^x \frac{dx}{x(1+x^2)} = -\sqrt{2}\int_0^t dt$$

Consider $\dfrac{1}{x(1+x^2)} = \dfrac{A}{x} + \dfrac{Bx+C}{1+x^2}$

$$1 = A(1+x^2) + (Bx+C)x$$

Let $x=0$ $\quad \therefore A = 1$ ✓

Equate coefficients of x $\quad 0 = C$

Equate coefficients of x^2 $\quad 0 = A + B \quad \therefore B = -1$

$$\therefore \int_1^x \frac{1}{x} - \frac{x}{1+x^2}\,dx = -\sqrt{2}t$$
$$\left[\ln x - \frac{1}{2}\ln(1+x^2)\right]_1^x = -\sqrt{2}t \quad ✓$$
$$\frac{1}{2}\left[2\ln x - \ln(1+x^2)\right]_1^x = -\sqrt{2}t$$
$$\frac{1}{2}\left[\ln\frac{x^2}{1+x^2}\right]_1^x = -\sqrt{2}t$$
$$\ln\frac{x^2}{1+x^2} - \ln\frac{1}{2} = -2\sqrt{2}t$$
$$\ln\frac{2x^2}{1+x^2} = -2\sqrt{2}t$$
$$\frac{2x^2}{1+x^2} = e^{-2\sqrt{2}t}$$
$$\frac{1+x^2}{2x^2} = e^{2\sqrt{2}t}$$
$$1+x^2 = 2x^2e^{2\sqrt{2}t}$$
$$1 = x^2\left(2e^{2\sqrt{2}t} - 1\right)$$
$$\therefore x = \frac{1}{\sqrt{2e^{2\sqrt{2}t} - 1}}, \quad \text{since } x > 0 \quad ✓$$

Marking Scheme
✓ [1] for correct partial fraction decomposition and correct value of one of A, B or C
✓ [1] for correctly integrating $\dfrac{1}{x} - \dfrac{x}{1+x^2}$
✓ [1] for correctly showing the required result

Question 15 (a)(ii) (3 marks) continued...

Marker's Comments:

- Generally well done. However, some students were unsuccessful in making x the subject after they had obtained $\ln \dfrac{2x^2}{1+x^2} = -2\sqrt{2}t$ (or similar variants).
- Some students who used indefinite integrals did not write the constant of integration, or did not evaluate it using the initial conditions.
- Alternatively, to evaluate $\displaystyle\int_1^x \frac{dx}{x(1+x^2)}$, the substitution $x = \tan\theta$ can be used.

Question 15 (b) (2 marks)

Given $x_1 + x_2 + \cdots + x_n = n^2$,

$\therefore \dfrac{n^2}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$, (using the given AM-GM inequality)

$\therefore n^n \geq x_1 x_2 \cdots x_n$ ✓

$$\begin{aligned} \text{Now } \frac{1}{x_1^n} + \frac{1}{x_2^n} + \cdots + \frac{1}{x_n^n} &\geq n\sqrt[n]{\frac{1}{x_1^n}\frac{1}{x_2^n}\cdots\frac{1}{x_n^n}}, \quad \text{(using the given AM-GM inequality)} \\ &= \frac{n}{x_1 x_2 \cdots x_n} \\ &\geq \frac{n}{n^n} \\ &= n^{1-n} \end{aligned}$$ ✓

Marking Scheme

✓ [1]	for correctly deducing that $n^n \geq x_1 x_2 \cdots x_n$ (or $n \geq \sqrt[n]{x_1 x_2 \cdots x_n}$)
✓ [1]	for correctly showing the required result

Marker's Comments:

- This proved to be a challenging question for most students. Some students were able to utilise what was given in the question to deduce that $n^n \geq x_1 x_2 \cdots x_n$.
- Many, however, were unable to deduce the required result. Students should keep in mind that certain results/techniques may need to be applied more than once, such as applying the given AM-GM inequality twice in this question, or applying integration by parts more than once in other questions.

Question 15 (c)(i) (3 marks)

$$\left.\begin{aligned} &\text{Let } u = (1-x^3)^n && v' = x \\ &u' = n(1-x^3)^{n-1}(-3x^2) && v = \frac{x^2}{2} \end{aligned}\right\} \checkmark$$

$$I_n = \left[\frac{x^2}{2}(1-x^3)^n\right]_0^1 + \int_0^1 \frac{3nx^4}{2}(1-x^3)^{n-1}\,dx \quad \checkmark$$

$$= \frac{3n}{2}\int_0^1 x\left(1-(1-x^3)\right)(1-x^3)^{n-1}\,dx$$

$$= \frac{3n}{2}\int_0^1 x(1-x^3)^{n-1} - x(1-x^3)^n\,dx$$

$$= \frac{3n}{2}\left(I_{n-1} - I_n\right)$$

$$2I_n + 3nI_n = 3nI_{n-1}$$

$$\therefore I_n = \frac{3n}{3n+2}I_{n-1} \quad \checkmark$$

Marking Scheme

✓ [1]	for correct expressions for u, u', v' and v
✓ [1]	for correct application of integration by parts
✓ [1]	for correctly showing the required result

Marker's Comments:

- Generally well done. However some careless errors such as writing $u' = n(1-x^3)^{n-1}(3x^2)$ (no minus sign on $3x^2$), or incorrect application of integration by parts were made by a handful of students.
- Care must also be taken when algebraically manipulating x^4 into $\big(1-(1-x^3)\big)$. Students are advised to check whether their new expression is equal to the previous expression.
- A repeated error seen in a number of responses included letting $v' = (1-x^3)^n$, and incorrectly integrating to obtain $v = \dfrac{(1-x^3)^{n+1}}{-3x^2(n+1)}$.

Question 15 (c)(ii) (2 marks)

$$\begin{aligned}
I_n &= \int_0^1 x(1-x^3)^n \, dx \\
&= \int_0^1 x \sum_{r=0}^{n} \binom{n}{r} (-x^3)^r \, dx \\
&= \sum_{r=0}^{n} \binom{n}{r} (-1)^r \int_0^1 x^{3r+1} \, dx \\
&= \sum_{r=0}^{n} \binom{n}{r} (-1)^r \left[\frac{x^{3r+2}}{3r+2} \right]_0^1 \\
&= \sum_{r=0}^{n} \binom{n}{r} (-1)^r \frac{1}{3r+2} \quad \checkmark
\end{aligned}$$

$$\begin{aligned}
\text{From (i), } I_n &= \frac{3n}{3n+2} I_{n-1} \\
&= \frac{3n}{3n+2} \times \frac{3(n-1)}{3n-1} I_{n-2} \\
&= \frac{3n}{3n+2} \times \frac{3(n-1)}{3n-1} \times \frac{3(n-1)}{3n-1} \times \cdots \times \frac{3}{5} \times I_0 \\
&= \frac{3^n n!}{(3n+2)(3n-1)(3n-4)\cdots 5} I_0, \\
&= \frac{3^n n!}{(3n+2)(3n-1)(3n-4)\cdots 5 \times 2}
\end{aligned}$$

$$\begin{aligned}
\text{where } I_0 &= \int_0^1 x \, dx \\
&= \frac{1}{2} \left[x^2 \right]_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

$$\therefore \sum_{r=0}^{n} \binom{n}{r} (-1)^r \frac{1}{3r+2} = \frac{3^n n!}{(3n+2)(3n-1)(3n-4)\cdots 5 \times 2} \quad \checkmark$$

Marking Scheme

✓ [1] for correctly showing that $I_n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r \frac{1}{3r+2}$, or equivalent merit

✓ [1] for correctly showing the required result

Marker's Comments:

- Generally not well done. Many students were unable to show that $I_n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r \frac{1}{3r+2}$. Students should remember that content from Mathematics Advanced and Extension 1 are assumed knowledge for Mathematics Extension 2 (in this case, The Binomial Theorem from Y11 Extension 1).
- When showing $I_n = \frac{3^n n!}{(3n+2)(3n-1)(3n-4)\cdots 5 \times 2}$, students need to write the last term, I_0, as well as the previous term, $\frac{3}{5}$, i.e. students need to clearly show that $I_n = \frac{3n}{3n+2} \times \frac{3(n-1)}{3n-1} \times \frac{3(n-1)}{3n-1} \times \cdots \times \frac{3}{5} \times I_0$, and evaluate I_0.

Question 15 (d) (3 marks)

For $n = 1$

$$\text{LHS} = \frac{1^2}{\sqrt{2}} \approx 0.71 \qquad \text{RHS} = 1^2\sqrt{1} = 1$$

$\therefore$ LHS < RHS $\quad \therefore$ true for $n = 1$ ✓

Assume true for $n = k$

i.e. assume $\dfrac{1^2}{\sqrt{2}} + \dfrac{2^2}{\sqrt{3}} + \cdots + \dfrac{k^2}{\sqrt{k+1}} < k^2\sqrt{k}$

Prove true for $n = k + 1$

i.e. RTP $\dfrac{1^2}{\sqrt{2}} + \dfrac{2^2}{\sqrt{3}} + \cdots + \dfrac{k^2}{\sqrt{k+1}} + \dfrac{(k+1)^2}{\sqrt{k+2}} < (k+1)^2\sqrt{k+1}$

$$\begin{aligned}
\text{LHS} - \text{RHS} &= \frac{1^2}{\sqrt{2}} + \frac{2^2}{\sqrt{3}} + \cdots + \frac{k^2}{\sqrt{k+1}} + \frac{(k+1)^2}{\sqrt{k+2}} - (k+1)^2\sqrt{k+1} \\
&< k^2\sqrt{k} + \frac{(k+1)^2}{\sqrt{k+2}} - (k+1)^2\sqrt{k+1} \\
&< k^2\sqrt{k} + \frac{(k+1)^2}{\sqrt{k+1}} - (k+1)^2\sqrt{k+1} \\
&= k^2\sqrt{k} + (k+1)\sqrt{k+1} - (k+1)^2\sqrt{k+1} \quad ✓ \\
&< k^2\sqrt{k+1} + (k+1)\sqrt{k+1} - (k+1)^2\sqrt{k+1} \\
&= (k^2 + k + 1 - k^2 - 2k - 1)\sqrt{k+1} \\
&= -k\sqrt{k+1} \\
&< 0
\end{aligned}$$

$\therefore$ LHS < RHS $\quad \therefore$ true for $n = k + 1 \quad \therefore$ true for all integers $n \geq 1$ by mathematical induction ✓

Marking Scheme
✓ [1] for correctly showing the base case is true
✓ [1] for correct and substantial progress in the inductive step
✓ [1] for correctly completing the induction

Marker's Comments:

- This was a challenging question for many students. Students should carefully examine what is needed to be proven for the inductive step to gain insight on what steps are required to complete the proof.
- Students should read the above solutions and incorporate the techniques into their toolbox of problem solving strategies.

Question 16

a/i)

$$p(x) = \frac{(1+ix)^{4n+1} - (1-ix)^{4n+1}}{2ix}$$

$$p\left(\tan\frac{\pi}{4n+1}\right) = \frac{\left(1 + \frac{i\sin\frac{\pi}{4n+1}}{\cos\frac{\pi}{4n+1}}\right)^{4n+1} - \left(1 - \frac{i\sin\frac{\pi}{4n+1}}{\cos\frac{\pi}{4n+1}}\right)^{4n+1}}{2i\tan\frac{\pi}{4n+1}}$$

$$= \frac{\left(\sec\frac{\pi}{4n+1}\right)^{4n+1}}{2i\tan\frac{\pi}{4n+1}}\left[\left(\operatorname{cis}\frac{\pi}{4n+1}\right)^{4n+1} - \left(\operatorname{cis}\left(\frac{-\pi}{4n+1}\right)\right)^{4n+1}\right] \checkmark$$

$$= \frac{\left(\sec\frac{\pi}{4n+1}\right)^{4n+1}}{2i\tan\frac{\pi}{4n+1}}\left[\operatorname{cis}\pi - \operatorname{cis}(-\pi)\right]$$

$$= \frac{\left(\sec\frac{\pi}{4n+1}\right)^{4n+1}}{2i\tan\frac{\pi}{4n+1}}\left[-1 - (-1)\right] \checkmark$$

$$= 0$$

$x = \tan\frac{\pi}{4n+1}$ is a zero of $p(x)$.

AW 1 for express $\tan\frac{\pi}{4n+1} = \frac{\sin\frac{\pi}{4n+1}}{\cos\frac{\pi}{4n+1}}$

and convert into $\operatorname{cis}\left(\frac{\pi}{4n+1}\right)$.

a/ii) Note: $z - \bar{z} = 2i\,\mathrm{Im}(z)$

$$LHS = p(x) = \frac{(1+ix)^{4n+1} - (1-ix)^{4n+1}}{2ix}$$

$$= \frac{2i\,\mathrm{Im}(1+ix)^{4n+1}}{2ix}$$ ✓

$$= \frac{1}{x}\,\mathrm{Im}(1+ix)^{4n+1}$$

$$= \frac{1}{x}\,\mathrm{Im}\left[{}^{4n+1}C_0 + {}^{4n+1}C_1\, ix + {}^{4n+1}C_2 (ix)^2 + {}^{4n+1}C_3 (ix)^3 + \cdots + {}^{4n+1}C_{4n+1} (ix)^{4n+1} \right]$$

$$= \frac{1}{x}\,\mathrm{Im}\left[{}^{4n+1}C_0 + {}^{4n+1}C_1\, xi - {}^{4n+1}C_2\, x^2 - {}^{4n+1}C_3\, ix^3 + \cdots + {}^{4n+1}C_{4n+1}\, ix^{4n+1} \right]$$

$$= \frac{1}{x}\left[{}^{4n+1}C_1\, x - {}^{4n+1}C_3\, x^3 + {}^{4n+1}C_5\, x^5 - {}^{4n+1}C_7\, x^7 + \cdots + {}^{4n+1}C_{4n+1}\, x^{4n+1} \right]$$ ✓

$$p(x) = {}^{4n+1}C_1 - {}^{4n+1}C_3\, x^2 + {}^{4n+1}C_5\, x^4 - {}^{4n+1}C_7\, x^6 + \cdots + x^{4n}$$

a/iii) Assume where p, q have no common factors.

let $r = \frac{p}{q}$, $p\left(\frac{p}{q}\right) = 0$

$$p\left(\frac{p}{q}\right) = {}^{4n+1}C_1 - {}^{4n+1}C_3 \frac{p^2}{q^2} + {}^{4n+1}C_5 \frac{p^4}{q^4} - \cdots + \frac{p^{4n}}{q^{4n}} = 0$$ ✓

$$p\left(\frac{p}{q}\right) = {}^{4n+1}C_1\, q^{4n} - {}^{4n+1}C_3\, p^2 q^{4n-2} + {}^{4n+1}C_5\, p^4 q^{4n-4} = -p^{4n}$$

$$q\left[{}^{4n+1}C_1\, q^{4n-1} - {}^{4n+1}C_3\, p^2 q^{4n-3} + {}^{4n+1}C_5\, p^5 q^{4n-5} \right] = -p^{4n}$$

p^{4n} is divisible by q however p, q have no common factors

AW 1 for subs (p/q) into $p(x) = 0$

from assumption, so $q=1$ is the only possibility
$r = \frac{p}{q} = \frac{p}{1} = p$ is an integer. ✓

a/iv) If $\tan\left(\frac{\pi}{4n+1}\right)$ was rational, it would be an integer by part (i) and (iii)

However $0 < \tan\frac{\pi}{4n+1} < \tan\frac{\pi}{4}$

$0 < \tan\frac{\pi}{4n+1} < 1$ ✓

which means $\tan\left(\frac{\pi}{4n+1}\right)$ can't be an integer, a contradiction. Hence $\tan\left(\frac{\pi}{4n+1}\right)$ is an irrational.

Some students stated $\tan\frac{\pi}{4n+1}$ can't be an integer without proving it. No marks.

b/i) $\overrightarrow{AB} = (\underset{\sim}{b} - \underset{\sim}{a})$, $\overrightarrow{BC} = (\underset{\sim}{c} - \underset{\sim}{b})$, $\overrightarrow{AC} = (\underset{\sim}{c} - \underset{\sim}{a})$

$|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{AC}|$

$(\underset{\sim}{b}-\underset{\sim}{a})(\underset{\sim}{b}-\underset{\sim}{a}) = (\underset{\sim}{c}-\underset{\sim}{b})(\underset{\sim}{c}-\underset{\sim}{b}) = (\underset{\sim}{c}-\underset{\sim}{a})(\underset{\sim}{c}-\underset{\sim}{a})$ ✓

$\underset{\sim}{b}\cdot\underset{\sim}{b} - 2\underset{\sim}{a}\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{a} = \underset{\sim}{c}\cdot\underset{\sim}{c} - 2\underset{\sim}{c}\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} = \underset{\sim}{c}\underset{\sim}{c} - 2\underset{\sim}{a}\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{a}$

$|\underset{\sim}{b}| - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{a}| = |\underset{\sim}{c}| - 2\underset{\sim}{c}\cdot\underset{\sim}{b} + |\underset{\sim}{b}| = |\underset{\sim}{c}| - 2\underset{\sim}{a}\cdot\underset{\sim}{c} + |\underset{\sim}{a}|$

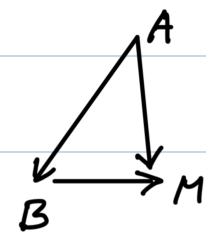
$\cancel{2}R - 2\underset{\sim}{a}\cdot\underset{\sim}{b} = \cancel{2R} - 2\underset{\sim}{b}\cdot\underset{\sim}{c} = \cancel{2R} - 2\underset{\sim}{a}\cdot\underset{\sim}{c}$

$-2\underset{\sim}{a}\cdot\underset{\sim}{b} = -2\underset{\sim}{b}\cdot\underset{\sim}{c} = -2\underset{\sim}{a}\cdot\underset{\sim}{c}$

$\underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{b}\cdot\underset{\sim}{c} = \underset{\sim}{a}\cdot\underset{\sim}{c}$ ✓ $|\overrightarrow{AC}|$

AW 1 for having $|\overrightarrow{AB}| = (\underset{\sim}{b}-\underset{\sim}{a})(\underset{\sim}{b}-\underset{\sim}{a})$ or similar for $|\overrightarrow{BC}|$ or b/ii) Show $\vec{AP} = \frac{2}{3}\vec{AM}$

$$\vec{AM} = \vec{BM} + \vec{AB}$$
$$= \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b}) + \underset{\sim}{b} - \underset{\sim}{a}$$
$$= \frac{1}{2}\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}$$



$$\vec{AP} = \text{Vector Proj}_{\vec{AM}}\vec{AO}$$

$$\vec{AP} = \frac{\vec{AO} \cdot \vec{AM}}{\vec{AM} \cdot \vec{AM}} \cdot \vec{AM}$$

$$\vec{AP} = \frac{-\underset{\sim}{a}\left(\frac{1}{2}\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}\right)}{\left(\frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c} - \underset{\sim}{a}\right)\left(\frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c} - \underset{\sim}{a}\right)} \cdot \vec{AM} \quad ✓$$

$$\vec{AP} = \frac{-\frac{\underset{\sim}{a}\underset{\sim}{b}}{2} - \frac{\underset{\sim}{a}\underset{\sim}{c}}{2} + |\underset{\sim}{a}|^2}{\frac{1}{4}|\underset{\sim}{b}|^2 + \frac{1}{4}\underset{\sim}{b}\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}\underset{\sim}{b} + \frac{1}{4}\underset{\sim}{b}\underset{\sim}{c} + \frac{1}{4}|\underset{\sim}{c}|^2 - \frac{1}{2}\underset{\sim}{a} \cdot \underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}\underset{\sim}{c} + |\underset{\sim}{a}|^2} \cdot \vec{AM}$$

$$\vec{AP} = \frac{R^2 - \frac{1}{2}(\underset{\sim}{a}\underset{\sim}{b} + \underset{\sim}{a}\underset{\sim}{c})}{\frac{3}{2}R^2 + \frac{1}{2}\underset{\sim}{b}\underset{\sim}{c} - \underset{\sim}{a}\underset{\sim}{b} - \underset{\sim}{a}\underset{\sim}{c}} \vec{AM}$$

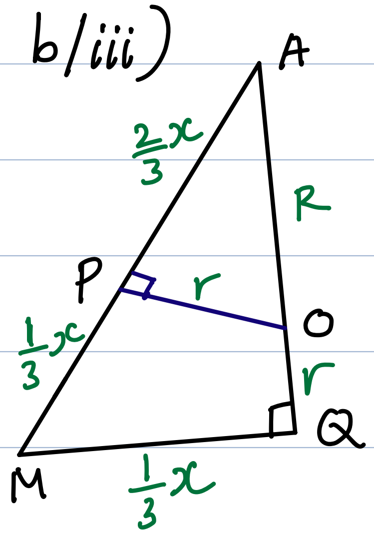
$$\vec{AP} = \frac{\cancel{(R^2 - \underset{\sim}{a}\underset{\sim}{b})} \cdot \vec{AM}}{\frac{3}{2}\cancel{(R^2 - \underset{\sim}{a}\underset{\sim}{b})}} \quad ✓ \qquad (\underset{\sim}{a}\underset{\sim}{b} = \underset{\sim}{b}\underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c})$$

$$\vec{AP} = \frac{\vec{AM}}{\frac{3}{2}}$$

$$\vec{AP} = \frac{2}{3}\vec{AM}$$

AW 1 for recognise $\vec{AP} = \text{Vector Proj}_{\vec{AM}}\vec{AO}$ then simplify in terms of $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$.

b/iii)



$$\left(\tfrac{2}{3}x\right)^2 + r^2 = R^2$$
$$R^2 - r^2 = \tfrac{4}{9}x^2 \quad (1)$$
$$\left(\tfrac{1}{3}x\right)^2 + (R+r)^2 = x^2$$
$$(R+r)^2 = \tfrac{8}{9}x^2 \quad (2)$$
$$(1)\times 2: \quad 2(R^2 - r^2) = \tfrac{8}{9}x^2 \quad (3)$$

from (2) and (3)

$$2(R^2 - r^2) = (R+r)^2$$
$$2R^2 - 2r^2 = R^2 + 2Rr + r^2$$
$$R^2 - 2Rr - 3r^2 = 0$$
$$(R-3r)(R+r) = 0$$
$$R - 3r = 0 \qquad R \neq r$$
$$R = 3r$$

Only a few students achieved full marks for this part.

c) $$\mathrm{Arg}(z-1) = \mathrm{Arg}(\bar{z}+i)$$
$$\mathrm{Arg}(z-1) - \mathrm{Arg}(\bar{z}+i) = 0$$
$$\mathrm{Arg}\left(\frac{z-1}{\bar{z}+i}\right) = 0$$

$\frac{z-1}{\bar{z}+i}$ is real OR $\mathrm{Im}\left(\frac{z-1}{\bar{z}+i}\right) = 0$

Let $z = x + iy$

$$\frac{z-1}{\bar{z}+i} = \frac{x+iy-1}{x-iy+i} = \frac{(x-1)+iy}{x-i(y-1)}$$

$$\frac{z-1}{\bar{z}+i} = \frac{[(x-1)+iy][x+i(y-1)]}{x^2+(y-1)^2}$$

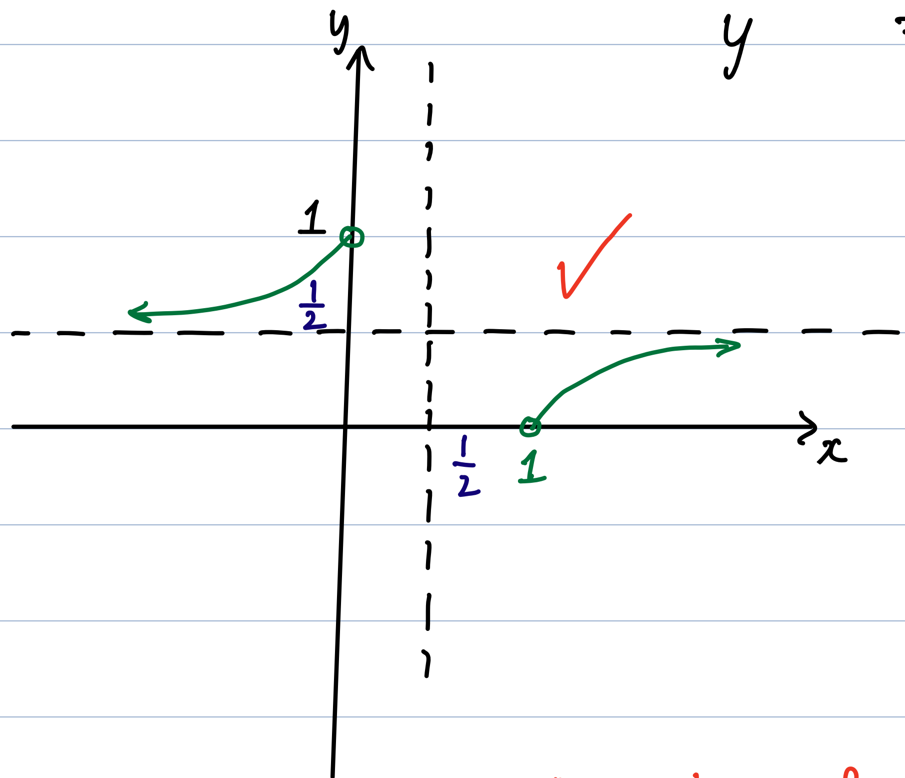
$$= \frac{x^2-x-y(y-1)+i[(x-1)(y-1)+xy]}{x^2+(y-1)^2}$$

$\text{Im} = 0$: $(x-1)(y-1) + xy = 0$

$$xy - x - y + 1 + xy = 0$$

$$y(2x-1) = x-1$$

$$y = \frac{x-1}{2x-1}$$ ✓



Arg(0) is undefined

$\therefore\ z-1 \neq 0,\ z \neq 1$

$\bar{z}+i \neq 0$

$\bar{z} \neq -i$

$z \neq i$

A number of students read $\text{Arg}(\bar{z}+i)$ as $\text{Arg}(z+i)$. It's important to recognise $z \neq 1$ and $z \neq i$

AW 1 for having $y = \frac{x-1}{2x-1}$.