| Student No: | |
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| | |

# Mathematics Extension 2

## Year 12, Assessment task 3, Term 2 2024

**General Instruction:**

- Reading time – 10 minutes
- Working time – 180 minutes
- Write using black/blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

**Total marks:**

**Section I – 10 marks (pages 1–3)**

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II – 90 marks (pages 4–9)**

- Attempt Questions 11–16
- Allow about 2 hour and 35 minutes for this section

**Class Teacher (please tick one name)**

- ◯ Mr Berry
- ◯ Ms Lee
- ◯ Mr Umakanthan
- ◯ Dr Vranešević

| Question No. | 1-10 | 11 | 12 | 13 | 14 | 15 | 16 | Total |
|---|---|---|---|---|---|---|---|---|
| Marks | /10 | /15 | /15 | /15 | /15 | /15 | /15 | /100 |

**Section I**

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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1. Which of the following is **NOT** equivalent to 'If $n$ is even, then $n^2$ is a multiple of 4'?

   A) $n$ is even $\therefore n^2$ is a multiple of 4;

   B) $n^2$ is a multiple of 4 if $n$ is even;

   C) $n^2$ is a multiple of 4 $\therefore n$ is even;

   D) $n$ is even implies that $n^2$ is a multiple of 4.

2. Let $I_n = \displaystyle\int x^n e^{ax}\, dx$. Which of the following is the correct expression for $I_n$?

   A) $I_n = \dfrac{x^n e^{ax}}{a} - nI_{n-1}$;

   B) $I_n = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}I_{n-1}$;

   C) $I_n = \dfrac{x^n e^{ax}}{a} + nI_{n-1}$;

   D) $I_n = \dfrac{x^n e^{ax}}{a} + \dfrac{n}{a}I_{n-1}$.

3. The acceleration of a particle moving in a straight line with velocity $v$ is given by $\ddot{x} = v^2$. Initially $v = 1$. What is $v$ as a function of $t$?

   A) $v = 1 - t$;

   B) $v = \ln|1 - t|$;

   C) $v = \dfrac{t}{1-t}$;

   D) $v = \dfrac{1}{1-t}$.

4. The negation of the statement $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $2x + 3y = 12$ is:

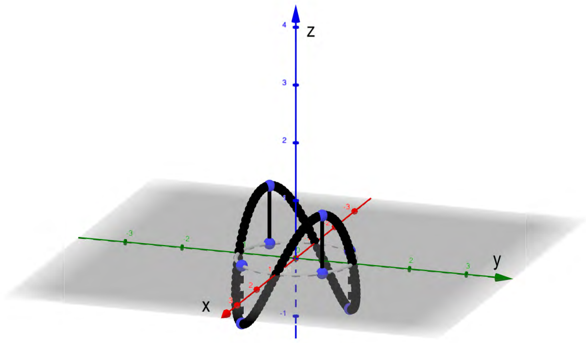
   A) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $2x + 3y \neq 12$;

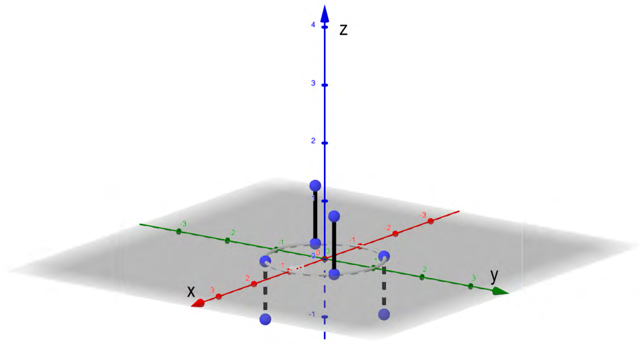
   B) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}\ 2x + 3y \neq 12$;

   C) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}\ 2x + 3y \neq 12$;

   D) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $2x + 3y \neq 12$.

5. The sketch of the curve in the shape of saddle is drawn on the left. The curve has two peaks at $t = \frac{\pi}{4}, \frac{5\pi}{4}$ and two troughs at $t = \frac{3\pi}{4}, \frac{7\pi}{4}$, shown on the right. Which of following parametric equations are represented by the given curve:




   A) $x = \sin t, y = \cos 2t, z = \sin t$;
   B) $x = \sin t, y = \cos t, z = \sin 2t,$;
   C) $x = \sin 2t, y = \cos t, z = \sin t$;
   D) $x = \sin 2t, y = \cos t, z = \sin 2t$.

6. Which of the following complex numbers equals $(\sqrt{3}+i)^4$

   A) $-2 + \frac{2}{\sqrt{3}}i$;
   B) $-8 + \frac{8}{\sqrt{3}}i$;
   C) $-2 + 2\sqrt{3}i$;
   D) $-8 + 8\sqrt{3}i$.

7. The line $l_1$ has vector equation $\mathbf{r_1} = \mathbf{i} + \lambda(\mathbf{j} - \mathbf{k})$ and the line $l_2$ has vector equation $\mathbf{r}_2 = (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \mu(2\mathbf{i} + \mathbf{k})$. Which of the following statements is correct?

   A) $l_1$ and $l_2$ are parallel;
   B) $l_1$ and $l_2$ are perpendicular;
   C) $l_1$ and $l_2$ intersect at a point;
   D) $l_1$ and $l_2$ are skew.

8. Which of the following is an expression for $\int \frac{4x^2 - 5x - 1}{(x-3)(x^2+1)}\,dx$

   A) $\ln[(x-3)(x^2+1)] + C$;
   B) $\ln[(x-3)^2(x^2+1)] + C$;
   C) $\ln[(x-3)(x^2+1)] + \tan^{-1} x + C$;
   D) $\ln[(x-3)^2(x^2+1)] + \tan^{-1} x + C$.

9. What is the solution to the equation $z^2 = i\bar{z}$

   A) $(0,0)$ and $(0,1)$;

   B) $(0,0)$ and $(0,-1)$;

   C) $(0,0)$, $(0,-1)$, $(-\frac{\sqrt{3}}{2},\frac{1}{2})$ and $(\frac{\sqrt{3}}{2},\frac{1}{2})$;

   D) $(0,0)$, $(0,1)$, $(-\frac{\sqrt{3}}{2},\frac{1}{2})$ and $(\frac{\sqrt{3}}{2},\frac{1}{2})$;

10. A particle of unit mass $m$ is projected vertically upwards with an initial velocity of $u$ ms$^{-1}$ in a medium in which the resistance to the motion is proportional to the square of the velocity $v$ ms$^{-1}$ of the particle or $mkv^2$. Let $x$ be the displacement in metres of the particle above the point of projection, O, so that the equation of motion is $\ddot{x} = -(g+kv^2)$ where $g$ ms$^{-2}$ is the acceleration due to gravity and $k$ is constant proportional to $g$, given as $k = 0.01g$.
    Which of the following gives the correct expression for the time taken?

    A) $t = \frac{100}{g}\big(\tan^{-1}\frac{u}{10} - \tan^{-1}\frac{v}{10}\big)$;

    B) $t = \frac{10}{g}\big(\tan^{-1}u - \tan^{-1}v\big)$;

    C) $t = \frac{10}{g}\big(\tan^{-1}\frac{u}{10} - \tan^{-1}\frac{v}{10}\big)$;

    D) $t = \frac{100}{g}\big(\tan^{-1}u - \tan^{-1}v\big)$.

**End of Section I**

**Section II**

**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hour and 35 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11**. (15 marks) Use the Question 11 Writing Booklet

(a) State whether each statement is true or false, justifying your answer.

(i) $\forall n \in \mathbb{N}, n^2 > n$ **1**

(ii) $\exists x \in \mathbb{R}$ such as $x^2 + 1 = 0$ **1**

(iii) $\forall x \in \mathbb{R}, \exists y \in \mathbb{N}$ such as $x + y = 10$ **1**

(b) It is given that $f(z) = z^4 + 2\sqrt{2}z^3 + z^2 + 8\sqrt{2}z - 12$ . One of the roots of the equation $f(z) = 0$ is given by $2i$. By factorising $f(z)$ as a product of two quadratic factors, obtain the other roots of the equation. **3**

(c) Find the integral: **2**

$$\int \frac{e^{\arctan x} + x\ln(1+x^2) + 1}{1+x^2} dx$$

(d) The displacement of a particle moving in a straight line, $s$ m, at time $x$ seconds is given by $s = \sin 4x + 2\sin 2x + 2$.

(i) Show that the velocity of the particle at time $x$ seconds is given by $v = \dfrac{8}{(1+t^2)^2}(1-3t^2)\ \text{ms}^{-1}$, where $t = \tan x$ **2**

(ii) Hence find the value of $x$ where $0 \le x \le \pi$ for which the displacement is maximised. **2**

(e) Find the solutions of the integral equation with respect to $a$, for $2 \le a \le 3$, of **3**

$$\int_0^a \cos(x + a^2)dx = \sin a$$

**Question 12.** (15 marks) Use the Question 12 Writing Booklet

(a) A complex number $z_1$ is given by $z_1 = a + (a-3)i$, where $a$ is a positive real constant. It is given that $\arg z_1 = \theta$, where $-\dfrac{\pi}{2} < \theta < 0$.

(i) Find, leaving your answer in terms of $\theta$,

($\alpha$) $\arg(-2z_1)$, **1**

($\beta$) $\arg(z_1 - 2a)$. **2**

(ii) Another complex number $z_2$ is given by $z_2 = 1 + 3i$. Without using a calculator, find the range of values of $a$ such that

$$\frac{|z_1 z_2|^2}{\operatorname{Im}(z_1 z_2)} \leq 10$$

**4**

(b) Prove that a four-digit number is divisible by 11 if and only if the difference between the sum of its even digits and the sum of its odd digits is a multiple of 11 (including zero).
**Hint**: Divide 1001 by 11. **3**

(c) (i) Differentiate $\cot^3 x$ **1**

(ii) Differentiate $\cot^5 x$ **1**

(iii) Using derivatives from (i) and (ii) find the integral of

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^6 x \, dx$$

**3**

**Question 13.** (15 marks) Use the Question 13 Writing Booklet

(a) (i) if $k$ is an integer where $k \geq 3$ and $(k-1)(k+1) < k^2$, show that

$$\frac{1}{(k-1)k(k+1)} > \frac{1}{k^3}$$ **1**

(ii) Given that $S_n = \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots + \frac{1}{n^3} = \sum_{3}^{n} \frac{1}{k^3}$, use partial fractions from previous part or otherwise to prove that $S_n < \frac{1}{12}$. **3**

(b) Let $p$ and $q$ be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that

(i)

$$\frac{1}{3} \leq \frac{1}{p(p+1)} + \frac{1}{q(q+1)} \leq \frac{1}{2}$$ **2**

(ii)

$$\frac{1}{p(p-1)} + \frac{1}{q(q-1)} \geq 1$$ **2**

(c) An object is moving on a horizontal plane and at position $x$ metres from the origin it has acceleration ($\ddot{x}$ m s$^{-2}$) given by $\ddot{x} = 0.01\left(x + \frac{64}{x^3}\right)$. Initially the object is 4 metres to the left of the origin and moving to the left at a speed of $\frac{\sqrt{3}}{5}$ m s$^{-1}$.

(i) If the velocity of the object is $v$ m s$^{-1}$, show that

$$v^2 = 0.01\left(\frac{x^4 - 64}{x^2}\right).$$ **2**

(ii) Explain why the velocity of the object at a position $x$ metres from the origin is given by

$$v = \frac{\sqrt{x^4 - 64}}{10x}.$$ **1**

(iii) Find, correct to the nearest second, the time to reach a position 50 metres to the left of the origin. **4**

**Question 14.** (15 marks) Use the Question 14 Writing Booklet

(a) Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$(a_1^2 + a_1)(a_2^2 + a_2)\ldots(a_n^2 + a_n) \geq 2^n.$$

**3**

(b) (i) A parachutist of mass 90 kg jumps out of an aeroplane at a height of 100 m. The acceleration due to gravity is 10 m/s$^2$.
Derive the vertical velocity functions $v$ m/s$^2$ in terms of displacement $x$ for the parachutist, if the initial velocity is $u$ m/s$^2$ and there is:

($\alpha$) **no** resistance force during free fall; **2**

($\beta$) a resistance force of $0.27v^2$ Newtons when parachute is open. **3**

(ii) A parachutist jumps from rest, and opens his parachute 60 m from the ground.

($\alpha$) Find the vertical velocity when parachute is open. **1**

($\beta$) Find the vertical velocity (1 decimal place) on landing. **2**

($\gamma$) What percentage (to the nearest whole percent) is the landing velocity compared to the terminal velocity? **2**

(c) Let $a, b, c$ be positive integers. Prove that it is impossible for all three numbers $a^2 + b + c$, $b^2 + a + c$, $c^2 + a + b$ to be perfect squares.

**2**

**Question 15.** (15 marks) Use the Question 15 Writing Booklet

(a) Consider the following vector equation $\mathbf{r} = (2+\lambda)\,\mathbf{i} + (3-\lambda)\,\mathbf{j} + (4-2\lambda)\,\mathbf{k}$.

(i) Find the coordinates of the points where the line $\mathbf{r}$ cuts the coordinate planes, where point $A$ is on $xy$ plane, point $B$ is on $xz$ plane, and point $C$ is on $yz$ plane. **3**

(ii) Find the vector projection of $\overrightarrow{OC}$ onto $\overrightarrow{AB}$ **1**

(iii) Find the area of $\Delta OAC$ correct to 3 significant figures. **2**

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \left(\sin x\right)^n \, dx$, where $n$ is an integer $n \geq 0$.

(i) Using integration by parts, show that for $n \geq 2$, **3**

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

(ii) Deduce that **3**

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \ldots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{and} \quad I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \ldots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

(iii) Explain why **1**

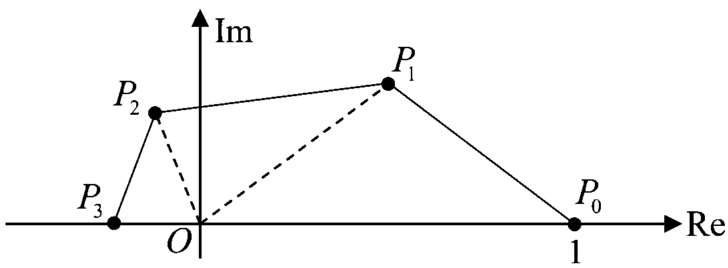
$$I_k > I_{k+1}$$

(iv) Hence, using the fact that $I_{2n-1} > I_{2n}$ and $I_{2n} > I_{2n+1}$, show that **2**

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2 \cdot 4^2 \cdot \ldots (2n)^2}{1 \cdot 3^2 \cdot 5^2 \cdot \ldots (2n-1)^2 \cdot (2n+1)} < \frac{\pi}{2}$$

**Question 16.** (15 marks) Use the Question 16 Writing Booklet

(a) A complex number $z$ is given by $z = re^{i\frac{\pi}{n}}$, where $0 < r < 1$ and $n$ is a positive integer with $n \geq 3$. The numbers $1,\ z,\ z^2,\ \dots,\ z^n$ can be represented by the points $P_0,\ P_1,\ P_2,\ \dots,\ P_n$ respectively in an Argand diagram. The $(n+1)$-sided polygon formed by using $P_0,\ P_1,\ P_2,\ \dots,\ P_n$ is called the $(n+1)$-*polygon generated by* $z$. An example of a $(3+1)$-polygon generated by $z$ is shown in the following Argand diagram (not drawn to scale).




Let $z = \dfrac{1}{4}(1 + \sqrt{3}i)$ for parts (i) to (iii).

(i) Express $z$ in the form $re^{i\theta}$ where $r > 0$ and $0 < \theta \leq \pi$. **2**

(ii) Hence write down $z^2$ and $z^3$ in similar form. **2**

(iii) Find the exact area of triangle $OP_0P_1$, where $O$ is the origin. Hence find the exact area of the $(3+1)$-polygon generated by $z$. **4**

Let $z = \dfrac{1}{2}re^{i\frac{\pi}{n}}$ for parts (iv)

(iv) Find the area of an $(n+1)$- polygon generated by $z$ in terms of $n$, leaving your answer in the form $a(1 - b^n)\sin\dfrac{\pi}{n}$, where $a$ and $b$ are real numbers to be determined. **3**

(b) (i) Consider triangle $ABC$, where $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$. Find a vector (in terms of $\mathbf{b}$ and $\mathbf{c}$) which bisects the angle $\angle CAB$. **1**

(ii) On the same triangle $ABC$, $D$ and $E$ are points on the sides $AC$ and $AB$ of the triangle, respectively. Also, $DE$ is not parallel to $CB$. Suppose $F$ and $G$ are points of $BC$ and $ED$, respectively, such that $BF : FC = EG : GD = BE : CD$. Show that $GF$ is parallel to the angle bisector of $\angle CAB$. **3**

**End of Examination**

Q1 A) $n$ is even $\Rightarrow$ $n^2$ is multiple of 4

B) $n^2$ is a multiple of 4 if $n$ is even

C) $n^2$ is a multiple of 4 $\Rightarrow$ $n$ is even

D) $n$ is even implies $n^2$ is a multiple of 4

$\therefore$ C

Q2 $I_n = \int x^n e^{ax}\,dx$

(by parts)

$u = x^n \qquad v' = e^{ax}$

$u' = n x^{n-1} \qquad v = \frac{1}{a} e^{ax}$

$$= \frac{x^n e^{ax}}{a} - \int n x^{n-1} \times \frac{1}{a} e^{ax}\,dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx$$

$$\therefore I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

B

Q3 $\ddot{x} = v^2$, $t=0, v=1$

$v = f(t) = ?$

(hor. equation of motion)

$$\ddot{x} = \frac{dv}{dt} = v^2$$

$dt = \frac{dv}{v^2}$ Q3 continues

$$\int_0^t dt = \int_1^v v^{-2}\,dv$$

$$t = (-1)\, v^{-1} \Big|_1^v$$

$$= (-1)\left(\frac{1}{v} - \frac{1}{1}\right)$$

$$= (-1)\,\frac{1-v}{v}$$

$$t = \frac{v-1}{v}$$

$$tv = v-1$$

$$v(t-1) = -1$$

$$v = \frac{1}{1-t}$$

D

Q4. $\neg(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $2x+3y=12)$

$\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$ $2x+3y \neq 12$

$\therefore$ B (quantifiers)

Q6. $(\sqrt{3}+i)^4 = \left[2\left(\frac{\sqrt{3}}{2}+i\,\frac{1}{2}\right)\right]^4$

$= 2^4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^4$

$= 2^4\left(\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6}\right)$ (CN-powers De-M)

$= 16\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$= 16\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$= -8+8\sqrt{3}i$

D

Q7 $\underset{\sim}{r_1} = \hat{\underset{\sim}{i}}+\lambda(\hat{\underset{\sim}{j}}-\hat{\underset{\sim}{k}}) = \begin{pmatrix}1\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\-1\end{pmatrix}$

$\underset{\sim}{r_2} = \begin{pmatrix}3\\2\\-1\end{pmatrix}+\mu\begin{pmatrix}2\\0\\1\end{pmatrix}$ (vector equation of straight line)

$\begin{pmatrix}1\\\lambda\\-\lambda\end{pmatrix} = \begin{pmatrix}3+2\mu\\2\\-1+\mu\end{pmatrix} \Rightarrow \lambda = 2$

$-2 = -1+\mu \Rightarrow \mu = -1$

$1 = 3+2(-1) = 1$

$\therefore$ $\underset{\sim}{r_1}$ & $\underset{\sim}{r_2}$ has a point of intersection

C

Q5 for $t = \frac{\pi}{4}$

A) $x = \frac{\sqrt{2}}{2}, \; y = 1, \; z = \frac{\sqrt{2}}{2}$ ×

B) $x = \frac{\sqrt{2}}{2}, \; y = \frac{\sqrt{2}}{2}, \; z = 1$

$t = \frac{5\pi}{4}$ : $x = -\frac{\sqrt{2}}{2}, \; y = -\frac{\sqrt{2}}{2}, \; z = 1$ } for peaks

$t = \frac{3\pi}{4}$ : $x = -\frac{\sqrt{2}}{2}, \; y = \frac{\sqrt{2}}{2}, \; -1$

$t = \frac{7\pi}{4}$ : $x = \frac{\sqrt{2}}{2}, \; y = -\frac{\sqrt{2}}{2}, \; -1$ } for throughts

$\therefore$ B

Q8 $\displaystyle\int \frac{4x^2 - 5x - 1}{(x-3)(x^2+1)} \, dx$ (partial fractions)

$$\frac{4x^2 - 5x - 1}{(x-3)(x^2+1)} = \frac{a}{x-3} + \frac{bx+1}{x^2+1}$$

$$4x^2 - 5x - 1 = a(x^2+1) + (bx+1)(x-3)$$

$$= (a+b)x^2 + (1-3b)x + a - 3$$

$a + b = 4 \Rightarrow$

$-5 = 1 - 3b \Rightarrow -6 = -3b \Rightarrow b = 2$

$-1 = a - 3 \Rightarrow a = 2$

$\therefore \int \left( \frac{2}{x-3} + \frac{2x+1}{x^2+1} \right) dx =$

$= \int \frac{2}{x-3}\, dx + \int \frac{2x\, dx}{x^2+1} + \int \frac{dx}{1+x^2}$

$= 2 \ln|x-3| + \ln|x^2+1| + \tan^{-1} x + C$

$= \ln\left[ (x^2+1)(x-3)^2 \right] + \tan^{-1} x + C$

$\therefore$ D

Q9. $z^2 = i\bar{z}$ $\quad z = x+iy \quad \bar{z} = x - iy$

$z^2 = i\bar{z}$

$(x+iy)^2 = i(x-iy)$

(CN arithmetics equations)

$x^2 - y^2 + 2xyi = +y + xi$

$x^2 - y^2 = y \quad (1)$

$2xy = x \quad (2) \Rightarrow x(2y-1) = 0 \Rightarrow x=0$ or $y = \frac{1}{2}$

subs into (1)

for $x=0$ : $-y^2=y$

$y(1+y)=0 \Rightarrow y=0$ or $y=-1$

for $y=\frac{1}{2}$ : $x^2-\frac{1}{4}=\frac{1}{2}$

$x^2=\frac{3}{4} \Rightarrow x=\pm\frac{\sqrt{3}}{2}$

$\therefore (0,0),(0,-1),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

$\therefore$ C

Q10 $m=1$, $v_0=4$, $R=mkv^2$, $g$, $k=0.01g$

$m\ddot{x}=-mg-mkv^2$

$\ddot{x}=-(g+0.01gv^2)$

$$\frac{dv}{dt}=-g(1+0.01v^2)$$

$$\frac{dt}{dv}=\frac{-1}{g}\frac{1}{1+0.01v^2}, \quad (0.1)^2=0.01$$

$$\int_0^t dt=-\frac{1}{g}\int\frac{dv}{1+(0.1v)^2}$$

$$t = \frac{-1}{g}\frac{1}{0.1}\int_u^v \frac{0.1\,dv}{1+(0.1v)^2}$$

$$= \frac{-10}{g}\left[\tan^{-1} 0.1v\right]_u^v = \frac{-10}{g}\left[\tan^{-1}\frac{v}{10}\right]_u^v$$

$$= \frac{-10}{g}\left[\tan^{-1}\frac{v}{10} - \tan^{-1}\frac{u}{10}\right]$$

$$= \frac{10}{g}\left(\tan^{-1}\frac{u}{10} - \tan^{-1}\frac{v}{10}\right)$$

$\therefore$ (C)

| | |
|---|---|
| Q1 | C |
| Q2 | B |
| Q3 | D |
| Q4 | B |
| Q5 | B |
| Q6 | D |
| Q7 | C |
| Q8 | D |
| Q9 | C |
| Q10 | C |

Q11 a)

i) $\forall n \in N, n^2 > n$ False as for $n=1$, $1>1$ not true (1)

ii) $\exists x \in R$, such as $x^2+1=0$ False as (1)
$x^2=-1 \Rightarrow x=\sqrt{-1}=\pm i \notin R$

iii) $\forall x \in R, \exists y \in N$ such as $x+y=10$ False (1)
as for $x=\frac{1}{2}$, $y= 9\frac{1}{2} \notin N$

b) $f(z)=z^4+2\sqrt{2}z^3+z^2+8\sqrt{2}z-12$
$f(z)=0$ for $z=2i$ find other roots

since all the coef. of $f(z)$ are real: then
$f(2i)=0$, then $f(-2i)=0$ $\therefore$ $-2i$ is another root (1)

Thus product of quadratic factors =
$[z-(2i)][z-(-2i)]$
$= z^2+4$

$\therefore f(z)=(z^2+4)(z^2+Az-3)$
$=z^4+4z^2+Az^3+4Az-3z^2-12$ (1)
$=z^4+Az^3+z^2+4Az-12$ $\therefore$ $A=2\sqrt{2}$ or $4A=8\sqrt{2}$ $\Rightarrow A=2\sqrt{2}$

$\Rightarrow A = 2\sqrt{2}$

$\therefore f(z) = (z^2+4)(z^2+2\sqrt{2}z-3)$

$$z^2+2\sqrt{2}z-3=0$$

$$z_{1,2} = \frac{-2\sqrt{2} \pm \sqrt{8-4\times1\times(-3)}}{2}$$

$$= \frac{-2\sqrt{2} \pm \sqrt{8+12}}{2}$$

$$= \frac{-2\sqrt{2} \pm 2\sqrt{5}}{2}$$

$$= -\sqrt{2} \pm \sqrt{5}$$

1

$\therefore$ roots are $2i, -2i, -\sqrt{2}+\sqrt{5}, -\sqrt{2}-\sqrt{5}$

c) $$\int \frac{e^{\arctan x} + x\ln(1+x^2) + 1}{1+x^2}\,dx =$$

$$= \int \frac{e^{\arctan x}}{1+x^2}\,dx + \int \frac{x\ln(1+x^2)}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

sub. $\arctan x = t$ & $\ln(1+x^2) = u$

$\frac{dx}{1+x^2} = dt$ $\qquad \frac{2x\,dx}{1+x^2} = du$ (1)

$$= \int e^t\,dt + \frac{1}{2}\int u\,du + \arctan x + C$$

$$= e^t + \frac{1}{2}\,\frac{u^2}{2} + \arctan x + C$$ (1)

$$= e^{\arctan x} + \frac{1}{4}\left[\ln(1+x^2)\right]^2 + \arctan x + C$$

d) $s(x) = \sin 4x + 2\sin 2x + 2$

$x$ - time [sec]

i) $v = \frac{8}{(1+t^2)^2}(1-3t^2)\ \frac{m}{s}$ , $t = \tan x$

$$\frac{ds}{dx} = 4\cos 4x + 4\cos 2x$$

$t = \tan x$

$$\cos 2x = \frac{1-t^2}{1+t^2} \qquad \sin 2x = \frac{2t}{1+t^2}$$

$$\cos 2 \times 2x = \cos^2 2x - \sin^2 2x$$

1

$$\therefore \frac{ds}{dx} = v = 4\times\left(\left(\frac{1-t^2}{1+t^2}\right)^2 - \left(\frac{2t}{1+t^2}\right)^2\right)$$

$$+ 4\times\frac{1-t^2}{1+t^2}$$

$$= 4\ \frac{1-2t^2+t^4 - 4t^2}{(1+t^2)^2} + \frac{4-4t^2}{1+t^2}$$

$$= \frac{4-24t^2+4t^4+4+\cancel{4t^2}-\cancel{4t^2}-\cancel{4t^4}}{(1+t^2)^2}$$

$$= \frac{8 - 24t^2}{(1+t^2)^2} = \frac{8(1-3t^2)}{(1+t^2)^2} \ \frac{m}{s}$$ $\square$ 1

ii) $0 \le x \le \pi$ $\qquad S_{max} = ?$

for $S_{max}$, $v = 0$

$\therefore \ 1 - 3t^2 = 0 \Rightarrow t^2 = \frac{1}{3} \Rightarrow t = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$

$\pm\frac{\sqrt{3}}{3} = \tan x \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$ 1

$\frac{d^2s}{dx^2} = -16\sin 4x - 8\sin 2x$

$\frac{d^2s}{dx^2}\left(x = \frac{\pi}{6}\right) = -16\frac{\sqrt{3}}{2} - 8\frac{\sqrt{3}}{2} = -24\frac{\sqrt{3}}{2} = -12\sqrt{3} < 0$

$\cap$ ✓

$\frac{d^2s}{dx^2}\left(x = \frac{5\pi}{6}\right) = -16\left(-\frac{\sqrt{3}}{2}\right) - 8\left(-\frac{\sqrt{3}}{2}\right) = 12\sqrt{3} > 0$

$\cup$

$\therefore$ max of $s(x)$ is for $x = \frac{\pi}{6}$ 1

e) $2 \leq a \leq 3$

$$\int_0^a \cos(x + a^2)\,dx = \sin a$$

$$\sin(x+a^2)\Big|_0^a = \sin a$$

$$\sin(a+a^2) - \sin a^2 = \sin a \quad (1)$$

using Product to sum trig. id.

$$2\cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$\left.\begin{aligned} A+B &= a+a^2 \\ A-B &= a^2 \end{aligned}\right\} \Rightarrow B = A - a^2$$

$$2A = 2a^2 + a$$

$$A = \frac{2a^2+a}{2}$$

$$= \frac{2a^2+a-2a^2}{2} = \frac{a}{2}$$

$\therefore$ (1) becomes

$$2\cos\frac{2a^2+a}{2}\sin\frac{a}{2}=\sin a$$

then

using double angle on right

$$2\cos\frac{2a^2+a}{2}\sin\frac{a}{2}=2\sin\frac{a}{2}\cos\frac{a}{2}$$

$$2\sin\frac{a}{2}\left[\cos\frac{2a^2+a}{2}-\cos\frac{a}{2}\right]=0 \quad (1)$$

$$2\sin\frac{a}{2}$$

$$2\sin A\sin B=\cos(A-B)-\cos(A+B)$$

$$\left.\begin{aligned}A-B&=\frac{2a^2+a}{2}\\ A+B&=\frac{a}{2}\end{aligned}\right\} \quad B=\frac{a^2+a}{2}-\frac{2a^2+a}{2}=\frac{-a^2}{2}$$

$$2A=\frac{2a^2+2a}{2}=a^2+a$$

$$A=\frac{a^2+a}{2}$$

$$\therefore 2\sin\frac{a}{2}\times 2\sin\frac{a^2+a}{2}\sin\left(\frac{-a^2}{2}\right)=0$$

$$-4\sin\frac{a}{2}\sin\frac{a^2+a}{2}\sin\frac{a^2}{2}=0$$

$$\sin\frac{a}{2}\ \sin\frac{a^2+a}{2}\ \sin\frac{a^2}{2}=0$$

$$\therefore\ \frac{a}{2}=k\pi \text{ or } \frac{a^2+a}{2}=n\pi \text{ or } \frac{a^2}{2}=m\pi$$

$$a=2k\pi \text{ or } a=\pm\sqrt{2m\pi} \text{ or } a=\frac{-1\pm\sqrt{1+8n\pi}}{2}$$

(1)

where $k, m, n \in \mathbb{Z}$

as $2\le a\le 3$ all values should be positive

$$a^2+a-2n\pi=0$$

$$a=\frac{-1\pm\sqrt{1-4\times1\times(-2n\pi)}}{2}$$

$$a=\frac{-1\pm\sqrt{1+8n\pi}}{2}$$

now: $2\le a\le 3$ ;

$$\therefore\ 2\le 2k\pi\le 3$$

$$\frac{1}{\pi}\le k\le\frac{3}{2\pi}$$

$$0.318\le k\le 0.477$$

$$2\le\sqrt{2m\pi}\le 3$$

$$\frac{4}{2\pi}\le m\le\frac{9}{2\pi}$$

$$0.64\le m\le 1.432$$

$$\therefore\ m=1$$

$$2\le\frac{-1+\sqrt{1+8n\pi}}{2}\le 3$$

$$5\le\sqrt{1+8n\pi}\le 7$$

$$24\le 8n\pi\le 48$$

$$\frac{24}{8\pi}\le n\le\frac{48}{8\pi}$$

$$\frac{3}{\pi} \le n \le \frac{6}{\pi}$$

$$0.95 \le n \le 1.91$$

$$\therefore \; \underline{\underline{n=1}}$$

Final solution for $a$ is

$$a = \sqrt{2\pi} \text{ or } \frac{-1+\sqrt{1+8\pi}}{2}$$

1

or in set notation

$$a \in \left\{ \sqrt{2\pi}, \frac{-1+\sqrt{1+8\pi}}{2} \right\}$$

# Q12

a) $z_1 = a + (a-3)i$ , $a \in \mathbb{R}^+$

i) $\arg z_1 = \theta$ , $-\frac{\pi}{2} < \theta < 0$

$\arg(-2z_1) = \arg(-2) + \arg(z_1)$

$= \pi + \theta$ ①

b) $\arg(z_1 - 2a) = ?$

$z_1 - 2a = a + (a-3)i - 2a = -a + (a-3)i$

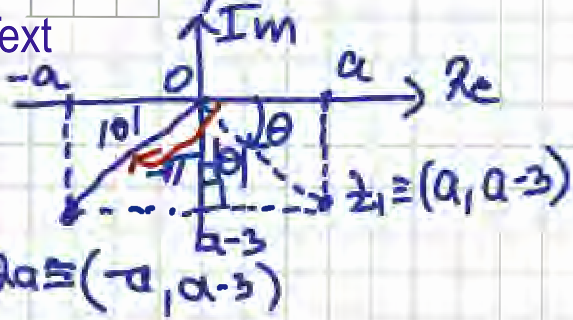
$\arg(z_1 - 2a) = \arg(-a + (a-3)i)$ ①

$= -\pi + |\theta|$

$= -\pi + (-\theta)$

$= -\pi + \theta$ ①

Text




ii) $z_2 = 1+3i$ , find $a$ such that

$$\frac{|z_1 z_2|^2}{\operatorname{Im}(z_1 z_2)} \leq 10$$

$z_1 z_2 = (a + (a-3)i)(1+3i)$

$= (a - 3a + 9) + (3a + a - 3)i$

$= (-2a + 9) + (4a - 3)i$

$\therefore \operatorname{Im}(z_1 z_2) = 4a-3$ 1

$$|z_1 z_2|^2 = |z_1|^2 |z_2|^2 = (a^2 + (a-3)^2)(1^2+3^2)$$
$$= 10(2a^2 - 6a + 9)$$ 1

$$\therefore \frac{|z_1 z_2|^2}{\operatorname{Im}(z_1 z_2)} \leq 10$$

$$\frac{10(2a^2-6a+9)}{4a-3} \leq 10$$

$$\frac{2a^2-6a+9}{4a-3} \leq 1$$

$$\frac{2a^2-6a+9 - (4a-3)}{4a-3} \leq 0$$

$$\frac{2a^2-6a+9-4a+3}{4a-3} \leq 0$$

$$\frac{2a^2-10a+12}{4a-3} \leq 0$$

$$\frac{2(a-2)(a-3)}{4a-3} \leq 0$$ 1


$\therefore a < \frac{3}{4}$ or $2 \leq a < 3$ 1

since $-\frac{\pi}{2} < \arg z_1 < 0$ $\therefore \operatorname{Im}(z_1) = a-3 < 0$ $2 \leq a < 3$

also it is given $a > 0$ $\therefore a < 3$ $\therefore 0 < a < \frac{3}{4}$ or

b) Let the digit of the number, in order, be $a, b, c, d$

$$N = 1000a + 100b + 10c + d \quad ①$$

If $N$ divisible by 11, then $1000a + 100b + 10c + d = 11k$
$k \in \mathbb{Z}$

$$1001 \div 11 = 91$$

$\therefore$ $1001a - a + 99b + b + 11c - c + d = 11k$

$$11 \times 91a - a + 11 \times 9b + b + 11c - c + d = 11k$$

$$11 \times 91a + 11 \times 9b + 11c - 11k = a - b + c - d \quad ①$$

$$11(91a + 9b + c - k) = (a+b) - (b+d)$$

$\therefore$ if $N$ divisible by 11, the difference of sum of even digits and the sum of odd digits is a multiple of 11. ①

Q12c) $\cot^3 x$ , $\cot^5 x$

$$\int_{\pi/4}^{\pi/3} \cot^6 x \, dx = ?$$

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \operatorname{cosec}^2 x$

let $y = \cot^5 x$

$y' = -5\cot^4 x \operatorname{cosec}^2 x$

$= -5\cot^4 x \,(1 + \cot^2 x)$

$= -5\cot^4 x - 5\cot^6 x$ ①

$5\cot^6 x = -5\cot^4 x - y'$

$\cot^6 x = -\frac{1}{5} y' - \cot^4 x$

$$\therefore \int_{\pi/4}^{\pi/2} \cot^6 x \, dx = -\left[\frac{1}{5}\int_{\pi/4}^{\pi/2} y' + \int_{\pi/4}^{\pi/2} \cot^4 x \, dx\right]$$

$$= -\frac{1}{5} y \Big|_{\frac{\pi}{4}}^{\pi/2} - \int_{\pi/4}^{\pi/2} \cot^4 x \, dx$$

$$= -\frac{1}{5} \cot^5 x \Big|_{\frac{\pi}{4}}^{\pi/2} - \int_{\pi/4}^{\pi/2} \cot^4 x \, dx$$

$= -\frac{1}{5}(0-1) - I$ 1

$I = \int_{\frac{\pi}{4}}^{\pi/2} \cot^4 x\,dx$

$y = \cot^3 x$

$y' = -3\cot^2 x \operatorname{cosec}^2 x = 3\cot^2 x(1+\cot^2 x)$ 1

$= -3\cot^2 x - 3\cot^4 x$

$\therefore \cot^4 x = -\frac{1}{3}y' - \cot^2 x$

$\therefore \int_{\pi/4}^{\pi/2} \cot^4 x\,dx = -\frac{1}{3}y\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\pi/4}^{\frac{\pi}{2}} \cot^2 x\,dx$ 1

$= -\frac{1}{3}\cot^3 x\Big|_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2}(\operatorname{cosec}^2 x - 1)\,dx$

$= -\frac{1}{3}(0-1) - (\cot x - x)\Big|_{\pi/4}^{\pi/2}$

$= +\frac{1}{3} - \left[0 - 1 - \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$

$= \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{\pi}{4} + \frac{4}{3} = I$ 1

$\therefore \int_{\pi/4}^{\pi/2} \cot^6 dx =$ $= \frac{13}{15} - \frac{\pi}{4}$

Q13

a) i) $k \geq 3$ $\quad (k-1)(k+1) < k^2$

show $\dfrac{1}{(k-1)(k+1)} > \dfrac{1}{k^3}$

Since $(k-1)(k+1) < k^2$, $k \geq 3$

$\therefore (k-1)k(k+1) < k^3$

$\therefore \dfrac{1}{(k-1)k(k+1)} > \dfrac{1}{k^3}$ ①

ii) $S_n = \dfrac{1}{3^3} + \dfrac{1}{4^3} + \dfrac{1}{5^3} + \cdots + \dfrac{1}{n^3} = \sum_{3}^{n} \dfrac{1}{k^3}$

to prove $S_n < \dfrac{1}{12}$

Let $\dfrac{1}{(k-1)k(k+1)} = \dfrac{A}{(k-1)k} + \dfrac{B}{k(k+1)} = \dfrac{1/2}{(k-1)k} - \dfrac{\frac{1}{2}}{k(k+1)}$ (✱) ①

Now $S_n = \dfrac{1}{3^3} + \dfrac{1}{4^3} + \dfrac{1}{5^3} + \cdots + \dfrac{1}{n^3}$

using (i) inequal

$\therefore S_n < \dfrac{1}{2\times3\times4} + \dfrac{1}{3\times4\times5} + \dfrac{1}{4\times5\times6} + \cdots + \dfrac{1}{(n-1)n(n+1)}$

$= \sum_{k=3}^{n} \dfrac{1}{(k-1)k(k+1)}$

$\therefore S_n < \dfrac{1}{2} \sum_{k=3}^{n} \left[\dfrac{1}{(k-1)k} - \dfrac{1}{k(k+1)}\right]$ from (✱) ①

$\therefore 2S_n < \left[\dfrac{1}{2\times3} - \cancel{\dfrac{1}{3\times4}} + \cancel{\dfrac{1}{3\times4}} - \cancel{\dfrac{1}{4\times5}} + \cancel{\dfrac{1}{4\times5}} - \cdots\right.$

$$\cdots + \cancel{\frac{1}{(n-1)n}} - \frac{1}{n(n+1)} \Big] = \frac{1}{2\times 3} - \frac{1}{n(n+1)}$$

$$\therefore \quad 2S_n < \frac{1}{6} - \frac{1}{n(n+1)}$$

$$\therefore 2S_n < \frac{1}{6} \quad \text{for } n \geq 3$$

$$\therefore \quad S_n < \frac{1}{12} \qquad 1$$

Q13 b) i.) if $\frac{1}{p} + \frac{1}{q} = 1$

$\therefore p+q = pq = S$

$(p+q)^2 = p^2+q^2+2pq \quad (1)$

$(p+q)^2 = (pq)^2 = S^2 \quad (2)$ ①

$(1) \;\&\; (2) \Rightarrow (p+q)^2 \geq 4pq \Rightarrow S \geq 4$

$$\frac{1}{p(p+1)} + \frac{1}{q(q+1)} = \frac{1}{p} - \frac{1}{p+1} + \frac{1}{q} - \frac{1}{q+1} =$$

$$= 1 - \frac{p+q+2}{(p+1)(q+1)}$$

$$= 1 - \frac{S+2}{2S+1} = \frac{S-1}{2S+1}$$

$\therefore$ we have to prove

$$\frac{1}{3} \leq \frac{S-1}{2S+1} \leq \frac{1}{2}$$

but $2S+1 \leq 3S-3 \Leftrightarrow 4 \leq S$ and ①

$2S-2 \leq 2S+1 \Leftrightarrow -2 \leq 1 \quad \square$

b) ii)

$$\frac{1}{p(p-1)} + \frac{1}{q(q-1)} = \frac{1}{p-1} - \frac{1}{p} + \frac{1}{q-1} - \frac{1}{q} =$$

$$= \frac{p+q-2}{(p-1)(q-1)} - 1 \quad 1$$

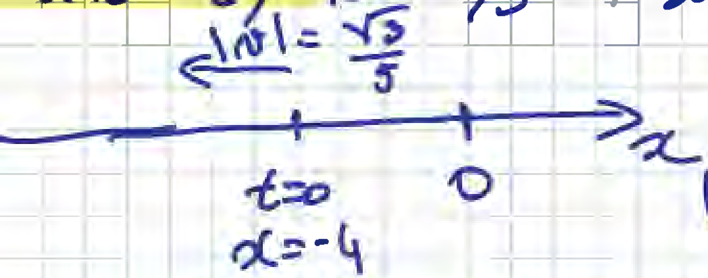
$$= \frac{s-2}{s-s+1} - 1$$

$= s-3$ and as $s \geq 4$ from (i)

$\therefore \quad s-3 \geq 1 \quad 1$

$$\therefore \frac{1}{p(p-1)} + \frac{1}{q(q-1)} \geq 1$$

Q13 c) $v$ m/s   show $v^2 = 0.01\frac{x^4-64}{x^2}$




$t=0,\ x=-4$ m

$|v| = \frac{\sqrt{3}}{5}$ m/s

$\therefore v = -\frac{\sqrt{3}}{5}$

$$\ddot{x} = 0.01\left(x + \frac{64}{x^3}\right) = f(x)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 0.01\left(x + \frac{64}{x^3}\right) \quad (1)$$

$$\frac{1}{2}v^2 = 0.01\left(\frac{x^2}{2} - \frac{64}{2x^2}\right) + c$$

or

$$\frac{1}{2}\int_{-\frac{\sqrt{3}}{5}}^{v} v^2\,dv = 0.01\int_{-4}^{x}\left(x + \frac{64}{x^3}\right)dx$$

$$v^2 = 0.01\left(x^2 - \frac{64}{x^2}\right) + C$$

$x=-4,\ v = -\frac{\sqrt{3}}{5}$

$$\frac{3}{25} = \frac{1}{100}\left(16 - \frac{64}{16}\right) + C$$

$$12 = 12 + C \Rightarrow C = 0 \quad (1)$$

$$\therefore v^2 = 0.01\left(\frac{x^4-64}{x^2}\right)$$

ii) why $v = +\frac{\sqrt{x^4-64}}{10x}$

Object will be stationary when $v=0$

$\therefore x^4 - 64 = 0$ ①

$\therefore x = \pm 2\sqrt{2}$

Since object starts at $x=-4$ and moves left it will never come to rest ($x=-2\sqrt{2}$)

$\therefore$ velocity is always negative, so when we take the $\sqrt{\ }$ we need to choose an expression that is always negative and since $x<0$ $\therefore v = \frac{\sqrt{x^4-64}}{10x}$

or For $x<0$, $x$ and $\frac{1}{x^3}$ are both negative

$\therefore$ the acceleration is always negative for $x<0$ and since the initial direction of motion is to the left from position $x=-4$ then the object will always move towards the left, $\therefore$ the velocity will be always negative. Therefore choose the positive $\sqrt{x^2-64}$ & $10x<0$

iii) $t = ?$ $x = -50m$

$$v = \frac{\sqrt{x^4 - 64}}{10x}$$

$$\frac{dx}{dt} = \frac{\sqrt{x^4-64}}{10x}$$ ①

$$\frac{dt}{dx} = \frac{10x}{\sqrt{x^4-64}}$$

$$\int_0^t dt = \int_{-4}^{-50} \frac{10x}{\sqrt{x^4-64}}\,dx$$

let $u = x^2$ $\quad x=-4, \; u = 16$ ①
$\quad x = 50, \; u = 2500$

$du = 2x\,dx$

$$t = \int_{16}^{2500} \frac{5\,du}{\sqrt{u^2 - 64}}$$

RCR for

$$\int \frac{f'(a)}{\sqrt{f(x)^2 - a^2}} = \ln\left|f(x) + \sqrt{f(x)^2 - a^2}\right| + C$$

$$= 5\left[\ln\left|u + \sqrt{u^2 - 64}\right|\right]_{16}^{2500}$$ ①

$$= 5\left[\ln\left(2500 + \sqrt{2500^2 - 64^2}\right) - \ln\left(16 + \sqrt{16^2 - 64}\right)\right]$$

$\approx 25.604$ sec

time, correct to nearest second, to reach a position 50m to the left of the origin is 26 seconds. 1

13 (a)

(i) $(k-1)(k+1) < k^2$

Multiplying both sides by $k$ $(k \geq 3$ given$)$

$(k-1)k(k+1) < k^3$

$$\Rightarrow \frac{1}{(k-1)k(k+1)} > \frac{1}{k^3}$$

or equivalent working

(ii) Let $f(k) = -\frac{1}{2k(k+1)}$

$$\Rightarrow f(k) - f(k-1) = \frac{1}{2(k-1)(k)} - \frac{1}{2k(k+1)}$$

$$= \frac{(k+1)-(k-1)}{2(k-1)k(k+1)}$$

$$= \frac{1}{(k-1)k(k+1)}$$

$$\therefore \frac{1}{k^3} < \frac{1}{k(k-1)(k+1)} \Rightarrow \frac{1}{k^3} < f(k) - f(k-1)$$

$$\Rightarrow \frac{1}{3^3} < \cancel{f(3)} - f(2)$$

$$\frac{1}{4^3} < \cancel{f(4)} - \cancel{f(3)}$$

$$\frac{1}{5^3} < \cancel{f(5)} - \cancel{f(4)}$$

$$\vdots \qquad \vdots \quad \cancel{f(n-3)}$$

$$\frac{1}{(n-2)^3} < \cancel{f(n-2)} - \cancel{f(n-3)}$$

$$\frac{1}{(n-1)^3} < \cancel{f(n-1)} - \cancel{f(n-2)}$$

$$\frac{1}{n^3} < f(n) - \cancel{f(n-1)}$$

falling sum

$$\Rightarrow \text{Adding all} \quad S_n < f(n) - f(2)$$

$$= -\frac{1}{2n(n+1)} - \frac{1}{2.2.(3)}$$

$$= \frac{1}{12} - \frac{1}{2n(n+1)}$$

$$< \frac{1}{12}$$

13(b) $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow \boxed{p+q=pq}$ and since $\frac{1}{p}+\frac{1}{q} \geq \frac{2}{\sqrt{pq}}$ (AM≥GM)

$\Rightarrow 1 \geq \frac{2}{\sqrt{pq}}$

$\Rightarrow \boxed{pq \geq 4}$

let $t = \frac{1}{p(p+1)} + \frac{1}{q(q+1)}$

$= \frac{1}{p} - \frac{1}{p+1} + \frac{1}{q} - \frac{1}{q+1}$

$= 1 - \left(\frac{1}{p+1} + \frac{1}{q+1}\right)$ $\left(\because \frac{1}{p}+\frac{1}{q}=1\right)$

$= 1 - \frac{2+pq}{1+2pq}$ $(\because p+q=pq)$

$= \frac{1}{2} - \frac{3}{2(1+2pq)}$

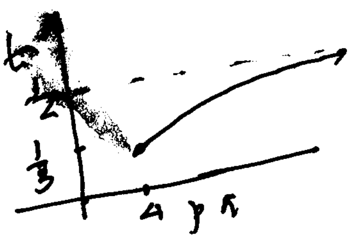
$t$ is continuous $\forall\, pq$

$\frac{dt}{d(pq)} = \frac{3}{(1+2pq)^2}$

$\Rightarrow t$ is monotonic increasing w.r.t $pq$

when $pq=4$ $\quad t_1 = \frac{1}{2} - \frac{3}{2(1+8)} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

as $pq \to \infty$ $\quad t \to \frac{1}{2}$




$\Rightarrow \frac{1}{3} \leq t < \frac{1}{2}$

i.e. $\frac{1}{3} \leq \frac{1}{p(p+1)} + \frac{1}{q(q+1)} < \frac{1}{2}$

13 (b) (ii) $\frac{1}{p(p-1)} + \frac{1}{q(q-1)}$

$= \frac{1}{p-1} - \frac{1}{p} + \frac{1}{q-1} - \frac{1}{q}$

$= \frac{1}{p-1} + \frac{1}{q-1} - 1$ ($\because \frac{1}{p}+\frac{1}{q}=1$)

$= \frac{pq-2}{pq-p-q+1} - 1$

$= \frac{pq-2}{1} - 1$

$= pq-3$

$\geq 1$ (since $pq \geq 4$ from (i))

13 (c) (i) $\ddot{x} = 0.01\left(x + \frac{64}{x^3}\right)$

$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 0.01\left(x + \frac{64}{x^3}\right)$

$\int d\left(\tfrac{1}{2}v^2\right) = \int 0.01\left(x + \frac{64}{x^3}\right)dx$

$\frac{v^2}{2} = 0.01\left(\frac{x^2}{2} - \frac{32}{x^2}\right) + C$

Initially $x = -4$, $v = -\frac{\sqrt{3}}{5}$

$\Rightarrow C = \frac{3}{50} - 0.01(8-2) = 0$

$\Rightarrow v^2 = \frac{0.01}{x^2}(x^4 - 64)$

$v = \pm \cancel{\phantom{xx}} \pm \sqrt{\frac{x^4-64}{10x}}$

Initially $v < 0$, $x < 0 \Rightarrow v = \sqrt{\frac{x^4-64}{10x}}$

$v = -\sqrt{\frac{x^4-64}{10x}}$ is not acceptable as it contradicts the initial condition

13 (c) (iii)

$$v = \frac{\sqrt{x^4 - 64}}{10x}$$

$$\frac{dx}{dt} = \frac{\sqrt{x^4 - 64}}{10x}$$

$$\Rightarrow \int_{-4}^{-50} \frac{10x}{\sqrt{x^4 - 64}}\,dx = \int_0^t dt$$

$$\Rightarrow \left[5 \ln\left[x^2 + \sqrt{x^4 - 64}\right]\right]_{-4}^{-50} = t$$

$$\Rightarrow t = 26.220\ldots$$

$$\simeq \underline{26 \text{ min}}$$

Q14 a) $a_1, a_2, a_3, \dots, a_n \in \mathbb{R}^+$ &

$a_1 \times a_2 \times a_3 \times \dots \times a_n = 1$

Prove $(a_1^2+a_1)(a_2^2+a_2)\dots(a_n^2+a_n) \geq 2^n$

---

$a_1 \geq \sqrt{a_1}$

$(a_1-\sqrt{a_1})^2 \geq 0$

$a_1^2 - 2a_1\sqrt{a_1} + a_1 \geq 0$

$a_1^2 + a_1 \geq 2a_1\sqrt{a_1}$ (1)

Similarly for $a_2$ to $a_n$ ①

$a_2^2 + a_2 \geq 2a_2\sqrt{a_2}$ (2)

$\vdots$

$a_n^2 + a_n \geq 2a_n\sqrt{a_n}$ (n) ①

(1) × (2) × ... × (n) :

$\therefore (a_1^2+a_1)(a_2^2+a_2)(a_3^2+a_3)\dots(a_n^2+a_n) \geq$

$2 \times a_1\sqrt{a_1} \times 2a_2\sqrt{a_2} \times \dots \times 2a_n\sqrt{a_n}$

$\geq 2^n(a_1 a_2 \dots a_n)\sqrt{a_1 a_2 \times \dots a_n}$

①

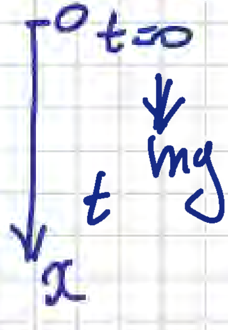
$\geq 2^n$

Q14 b)

i) $m = 90\,kg$ $\quad h = 100\,m$ $\quad g = 10\,m/s^2$

$v = f(x) = ?$ $\quad v_0 = u$

α) no resistance




$$m\ddot{x} = mg$$

$$\ddot{x} = g$$

$$v\frac{dv}{dx} = g$$

$$\int_u^v v\,dv = \int_0^x 10\,dx \qquad (1)$$

$$\frac{v^2}{2}\Big|_u^v = 10x\Big|_0^x$$

$$\frac{v^2 - u^2}{2} = 10x$$

$$\frac{v^2}{2} = 10x + \frac{u^2}{2}$$

$$v^2 = 20x + u^2 \Rightarrow v = \sqrt{u^2 + 20x}, \quad v > 0 \qquad (1)$$

(a) $m\ddot{x} = mg - 0.27v^2$

$$\ddot{x} = g - \frac{0.27}{90}v^2$$

$$\ddot{x} = 10 - 0.003v^2 \quad (1) \quad (*)$$

$$v\frac{dv}{dx} = 10 - 0.003v^2$$

$$\int_0^x dx = \int_u^v \frac{v\,dv}{10-0.003v^2}$$

(1)

$$x = -\frac{1}{0.003}\,\frac{1}{2}\Big[\ln|10-0.003v^2|\Big]_u^v$$

$$= -\frac{1}{0.006}\ln\frac{10-0.003v^2}{10-0.003u^2}$$

$$\ln\frac{10-0.003v^2}{10-0.003u^2} = -0.006x$$

$$\frac{10-0.003v^2}{10-0.003u^2} = e^{-0.006x}$$

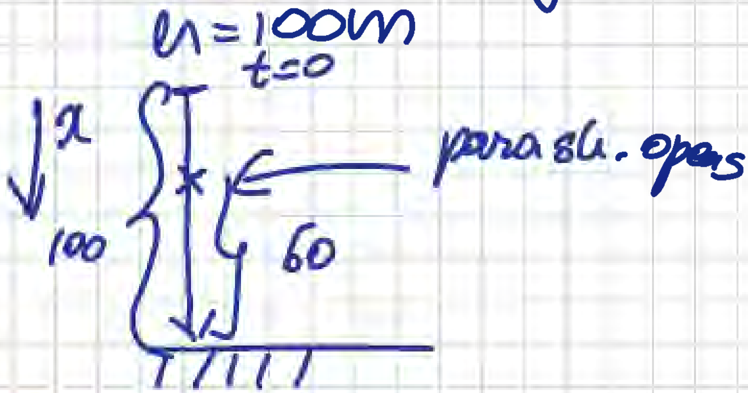
$$(10-0.003v^2) = (10-0.003u^2)e^{-0.006x}$$

(1)

$$v = \sqrt{\frac{10-(10-0.003u^2)e^{-0.006x}}{0.003}}, \quad v>0$$

5) ii) $t=0$, $v=0$, 60 m from the ground

(a) $v = \dot{x} = ?$ after 40m




$x = 40\,m$

from i)(a):

$v = \sqrt{u^2 + 20x}$

$v = \sqrt{0^2 + 20 \times 40}$

$= 20\sqrt{2}\ m/s$

$\simeq 28.28\ m/s$

(1)

b) $v = \dot{x} = ?$ on landing, $x = 60\,m$, $u = 20\sqrt{2}$

from i)(b) (now with resistance) (1)

$$v = \sqrt{\frac{10 - (10 - 0.003 \times (20\sqrt{2})^2)\, e^{-0.006 \times 60}}{0.003}}$$

$$v = \sqrt{\frac{10 - (10 - 0.003 \times 800)\, e^{-0.036}}{0.003}}$$

$$v = \sqrt{\frac{10 - 5.3023}{0.003}} = \sqrt{1565.8866}$$

$v = 39.571285 \simeq 39.6\ m/s$ (1)

5) $\gamma$) $v_L = 39.6$ (from 6)(b))

$$\frac{v_c}{v_T} = ?$$

$v_T = ?$ when $\ddot{x} = 0$

from (*) (i)(b)) $\ddot{x} = 10 - 0.003v^2 \Rightarrow$

(1)

$$v_T = \sqrt{\frac{10}{0.003}}$$

$$v_T = 57.73502 \ \frac{m}{s}$$
$$= 57.7 \ m/s$$

$$\frac{v_c}{v_T} = \frac{39.6}{57.7} \ 100\% = 68.539\%$$

$$\approx 69\%$$ (1)

14 c)

$a, b, c \in \mathbb{Z}^+$

$a^2 + b + c$ , $b^2 + a + c$ , $c^2 + a + b$ can't be perfect squares

Let assume $a \leq b \leq c$

$\therefore c^2 < c^2 + a + b \leq c^2 + 2c < (c+1)^2$ 1

$\therefore c^2 + a + b < (c+1)^2$

$\therefore c^2 + a + b$ cannot be perfect square

Similarly for the other two. 1

Q15

a) $r = (2+\lambda)\underline{i} + (3-\lambda)\underline{j} + (4-2\lambda)\underline{k}$

i) in $x$-$y$ plane: $z=0$, $4-2\lambda=0 \Rightarrow \lambda=2$

point A $(4, 1, 0)$ ①

in $xz$ plane: $y=0$, $3-\lambda=0 \Rightarrow \lambda=3$

point B $(5, 0, -2)$ ①

in $yz$ plane: $x=0$, $2+\lambda=0 \Rightarrow \lambda=-2$

point C $(0, 5, 8)$ ①

iii) $\text{proj}_{\vec{AB}} \vec{OC} = \dfrac{\vec{OC}\cdot\vec{AB}}{|\vec{AB}|^2}\vec{AB}$

$\vec{AB} = \begin{pmatrix}1\\-1\\-2\end{pmatrix}$

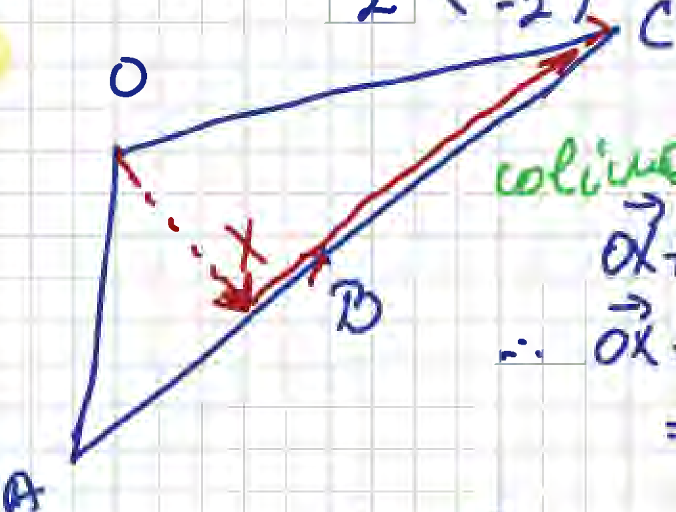
$\vec{OC} = \begin{pmatrix}0\\5\\8\end{pmatrix}$

$$= \frac{(0-5-16)}{\left(\sqrt{1+1+4}\right)^2}\begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

$$= \frac{-21}{6}\begin{pmatrix}1\\-1\\-2\end{pmatrix} \quad 1$$

$$= -\frac{7}{2}\begin{pmatrix}1\\-1\\-2\end{pmatrix} = \vec{XC}$$

iv)




if not used collinearity of A, B, C no marks for iv)

$\vec{OX}+\vec{XC}=\vec{OC}$

$\therefore \vec{OX} = \vec{OC}-\vec{XC}$

$$= \begin{pmatrix}0\\5\\8\end{pmatrix} + \frac{7}{2}\begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

Area $\Delta OAC = \frac{1}{2}|\vec{AC}| \times |\vec{OX}|$ 1

$\vec{AC} = \begin{pmatrix}-4\\4\\8\end{pmatrix}$ $\quad \vec{OX} = \begin{pmatrix}7/2\\5-\frac{7}{2}\\8-7\end{pmatrix} = \begin{pmatrix}7/2\\\frac{3}{2}\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}7\\3\\2\end{pmatrix}$

$$\text{Area } \Delta_{OAC} = \frac{1}{2}\sqrt{16+16+64} \times \sqrt{\frac{1}{4}(49+9+4)}$$

$$= \frac{1}{2}\sqrt{98}\sqrt{\frac{1}{4}62}$$

$$= \frac{1}{4}4\sqrt{6}\sqrt{62}$$

$$= \sqrt{3\times2\times2\times31}$$

$$= 2\sqrt{93}$$

$$= 19.287301|52$$

$$\approx 19.3 \ (3 \text{ s.f.})$$ 1

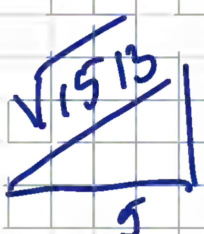
point to use part iii)

or $A = \frac{1}{2}|\vec{OA}||\vec{OC}|\sin(\angle COA)$

$$\vec{OC}\cdot\vec{OA} = 5 = |\vec{OA}||\vec{OC}|\cos(\angle COA)$$

$$= \sqrt{17}\sqrt{89}\cos(\angle COA)$$

$$\cos(\angle COA) = \frac{5}{\sqrt{1513}} \Rightarrow \sin(\angle COA) = \frac{\sqrt{1488}}{\sqrt{1513}}$$




$$\therefore A = \frac{1}{2}\sqrt{1513}\,\frac{\sqrt{1488}}{\sqrt{1513}} = 2\sqrt{93} \approx 19.3$$

15b) $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n \, dx \quad , n \geq 0$

i) $n \geq 2 \quad , I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

$I_n = \int_0^{\pi/2} (\sin x)^{n-1} \frac{d}{dx}(-\cos x) \, dx$

use only trig. fun. when int. by parts

$= \left[(\sin x)^{n-1} \cdot (-\cos x)\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} (-\cos x) \frac{d}{dx}(\sin x)^{n-1} dx$ 1

$= [0 - 0] + (n-1) \int_0^{\pi/2} (\sin x)^{n-2} \cos x \cos x \, dx$

$= (n-1) \int_0^{\pi/2} (\sin x)^{n-2} \cos^2 x \, dx$

$= (n-1) \int_0^{\pi/2} (\sin x)^{n-2} (1 - \sin^2 x) \, dx$ 1

$= (n-1) \int_0^{\pi/2} (\sin x)^{n-2} dx - (n-1) \int_0^{\pi/2} \sin^n x \, dx$

$\therefore I_n = (n-1) I_{n-2} - (n-1) I_n$

$\therefore I_n (1 + (n-1)) = (n-1) I_{n-2}$ 1

$\therefore I_n = \frac{n-1}{n} I_{n-2} \quad , n \geq 2 \quad \square$

ii) from i) $I_{2n} = \left(\frac{2n-1}{2n}\right) I_{2n-2}$

$$= \frac{2n-1}{2n}\left(\frac{2n-3}{2n-2} I_{2n-4}\right)$$ re-using the i) results

$$= \frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\,\frac{2n-5}{2n-4} I_{2n-6}$$

$$= \frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\,\frac{2n-5}{2n-4}\cdots\frac{(2n-2n)+1}{(2n-2n)+1} I_{2n-2n}$$

$$= \frac{(2n-1)(2n-3)(2n-5)\cdots 1}{(2n)(2n-2)(2n-4)\cdots 2} I_0$$ 1

$$I_0 = \int_0^{\pi/2} (\sin x)^0\,dx = x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore I_{2n} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$ 1

Also, from (i) $I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$

$$= \frac{2n}{2n+1}\left(\frac{2n-2}{2n-1} I_{2n-3}\right)$$ using (i) again

$$= \frac{2n}{2n+1}\,\frac{2n-2}{2n-1}\,\frac{2n-4}{2n-3} I_{2n-5}$$ again (i)

$$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \ldots \cdot \frac{2}{3} \cdot I_1$$

$$I_1 = \int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2}$$

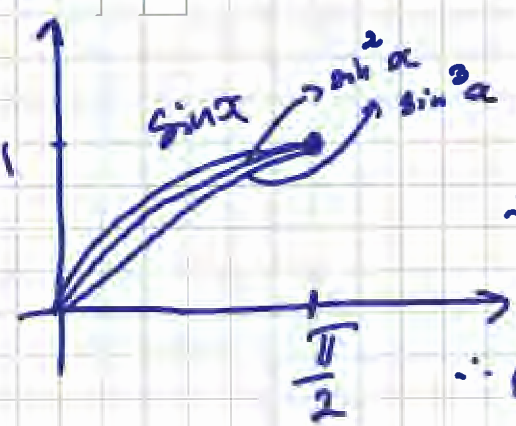
$$= -(0-1) = 1$$

1

$$\therefore I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \ldots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

iii) $\left.\begin{aligned} I_k &= \int_0^{\pi/2} (\sin x)^k \, dx \\ I_{k+1} &= \int_0^{\pi/2} (\sin x)^{k+1} \, dx \end{aligned}\right\} k \geq 0$

$I_k$ & $I_{k+1}$ represent the area under the curves $y_k = (\sin x)^k$ & $y_{k+1} = (\sin x)^{k+1}$ for $0 \leq x \leq \frac{\pi}{2}$




for $0 \leq x \leq \frac{\pi}{2}$

$0 \leq \sin x \leq 1$

$\therefore (\sin x)^{k+1} \leq (\sin x)^k$

$\therefore$ area under the

curve $y_k$ is greather than the area under the curve $y_{k+1}$ for $0 \le x \le \frac{\pi}{2}$

$\therefore I_k > I_{k+1}$ 1

iv) using the results in (ii) for $I_{2n+1}$

$$I_{2n-1} = \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \frac{2n-6}{2n-5} \cdots \frac{4}{5}\,\frac{2}{3} \cdot 1$$

from (iii), $I_k > I_{k+1}$ and $I_{2n-1} > I_{2n}$ and $I_{2n-1} > I_{2n+1}$

since $I_{2n-1} > I_{2n}$, then

$$\frac{2n-2}{2n-1}\;\frac{2n-4}{2n-3} \cdots \frac{4}{5}\;\frac{2}{3} \cdot \frac{1}{1} > \frac{2n-1}{2n}\;\frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Thus:

$$\frac{(2n)(2n-2)^2(2n-4)^2 \cdots 4^2 \cdot 2^2}{(2n-1)^2(2n-3)^2 \cdots 5^2\, 3^2\, 1} > \frac{\pi}{2}$$ 1

hence

$$\frac{2n}{2n+1}\;\frac{2n(2n-2)^2(2n-4)^2 \cdots 4^2\, 2^2}{(2n-1)^2(2n-3)^2 \cdots 5^2 3^2 \cdot 1} > \frac{2n}{2n+1}\;\frac{\pi}{2}$$

i.e. $$\frac{2^2 \cdot 4^2 \cdot \;\cdots\; \cdot (2n)^2}{1 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2(2n+1)} > \frac{2n}{2n+1}\,\frac{\pi}{2} \quad (1)$$

Since $I_{2n} > I_{2n+1}$, then

$$\underbrace{\frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\,\frac{2n-5}{2n-4}\cdots\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}}_{(*)} > \frac{2n}{2n+1}\,\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{4}{5}\,\frac{2}{3}\cdot 1$$

$$(*)\ \frac{\pi}{2} > \frac{(2n)^2(2n-2)^2\cdots 4^2\cdot 2^2}{(2n+1)\,(2n-1)^2(2n-3)^2\cdots 3^2\cdot 1}$$

$$(*)\ \frac{\pi}{2} > \frac{2^2\cdot 4^2\cdots(2n)^2}{1\cdot 3^2\cdot 5^2\cdots(2n-1)^2(2n+1)} \quad (2)$$

Now from (1) and (2)

1

$$\therefore\ \frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2\cdot 4^2\cdots(2n)^2}{1\cdot 3^2\,5^2\cdots(2n-1)^2(2n+1)} < \frac{\pi}{2}$$

$\square$

or to look at

$$I_{2n+1}\cdot I_{2n} < I_{2n+1}\,I_{2n-1} < I_{2n}\,I_{2n-1}$$

Q16 a)

$z = \frac{1}{4}(1+\sqrt{3}i)$

i) $|1+\sqrt{3}| = \sqrt{1+3} = \sqrt{4} = 2$ (1)

$\arg(1+\sqrt{3}i) = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$

$\therefore \quad z = \frac{1}{4}\left(2e^{i\pi/3}\right) = \frac{1}{2}e^{i\pi/3}$ (1)

ii) $z^2 = \left(\frac{1}{2}e^{i\pi/3}\right)^2 = \frac{1}{4}e^{i\frac{2\pi}{3}}$ (1)

$z^3 = \left(\frac{1}{2}e^{i\pi/3}\right)^3 = \frac{1}{8}e^{i\pi}$ (1)

iii) $Ar\,\Delta OP_0P_1 = \frac{1}{2}\times(1)\times\left(\frac{1}{2}\right)\sin\frac{\pi}{3}$ (1)

$= \frac{\sqrt{3}}{8}$ units$^2$

$Ar\,OP_1P_2 = \frac{1}{2}\times\left(\frac{1}{2}\right)\times\left(\frac{1}{4}\right)\sin\frac{\pi}{3} = \frac{\sqrt{3}}{32}$ units$^2$ (1)

$Ar\,\Delta OP_2P_3 = \frac{1}{2}\times\left(\frac{1}{4}\right)\times\left(\frac{1}{8}\right)\times\sin\frac{\pi}{3} = \frac{\sqrt{3}}{128}$ units$^2$ (1)

$\therefore$ Area of (3+1)-polygon

$= \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{32} + \frac{\sqrt{3}}{128} = \frac{21\sqrt{3}}{128}$ units$^2$ (1)

iv) $\frac{1}{2}(1)\left(\frac{1}{2}\right)\sin\frac{\pi}{n} + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\sin\frac{\pi}{n} +$

$+ \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)\sin\frac{\pi}{n} \cdots + \frac{1}{2}\left(\frac{1}{2^{n-1}}\right)\left(\frac{1}{2^n}\right)\sin\frac{\pi}{n}$ 1

$= \frac{1}{2}\left(1\times\frac{1}{2} + \frac{1}{2}\frac{1}{4} + \frac{1}{4}\frac{1}{8} + \cdots + \frac{1}{2^{n-1}}\frac{1}{2^n}\right) \times \sin\frac{\pi}{n}$

$= \frac{1}{2}\left(\underbrace{\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2n-1}}}_{n \text{ terms}}\right)\sin\frac{\pi}{n}$ 1

Area of $(n+1)$-polygon :

$S_n = a\frac{1-r^n}{1-r}$

$\therefore = \frac{1}{2}\left(\frac{\frac{1}{2}\left(1-\left(\frac{1}{2^2}\right)^n\right)}{1-\frac{1}{2^2}}\right)\sin\frac{\pi}{n}$

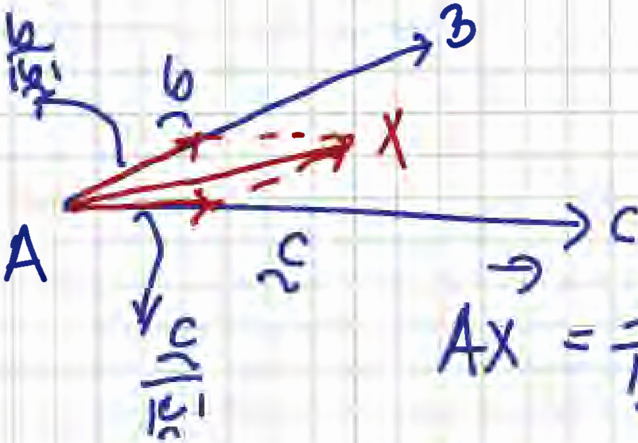
$= \frac{1}{4}\frac{4}{3}\left(1-\left(\frac{1}{4}\right)^n\right)\sin\frac{\pi}{n}$

$= \frac{1}{3}\left(1-\left(\frac{1}{4}\right)^n\right)\sin\frac{\pi}{n}$ 1

$\therefore a = \frac{1}{3}, b = \frac{1}{4}$

1(G5)

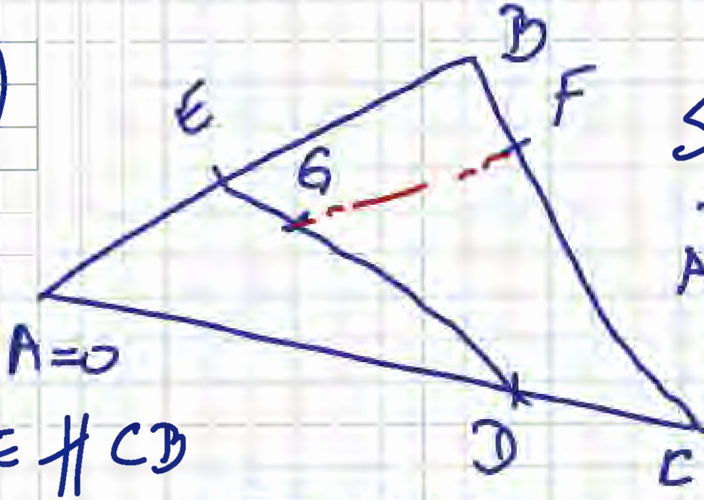
i)




$$\vec{AX} = \frac{b}{|b|} + \frac{c}{|c|}$$

(1)

ii)




DE // CB

BF : FC = EG : GD = BE : CD

Show GF // AX

Set origin at A:

$\vec{AB} = \vec{OB} = b$

$\vec{AC} = \vec{OC} = c$

$\vec{AD} = \vec{OD} = d$

$\vec{AE} = \vec{OE} = e$

$\vec{AG} = \vec{OG} = g$

$\vec{AF} = \vec{OF} = f$

---

$$\left.\begin{array}{l} e = p\,b \\ d = q\,c \end{array}\right\} \begin{array}{l}(1)\\ p, q \in (0,1) \\ (2)\end{array}$$

let $\frac{\vec{BF}}{FC} = t$

$$\vec{BF} + \underset{\sim}{b} = \underset{\sim}{f} \Rightarrow t\vec{FC} + \underset{\sim}{b} = \underset{\sim}{f}$$
$$\vec{FC} + \underset{\sim}{f} = \underset{\sim}{c}$$
$$\Rightarrow t(\underset{\sim}{c} - \underset{\sim}{f}) + \underset{\sim}{b} = \underset{\sim}{f}$$

$$\therefore t\underset{\sim}{c} - t\underset{\sim}{f} + \underset{\sim}{b} = \underset{\sim}{f} \Rightarrow \underset{\sim}{f} = \frac{t\underset{\sim}{c} + \underset{\sim}{b}}{1+t} \quad (3)$$

1

also $\dfrac{\vec{EG}}{\vec{GD}} = t \Rightarrow \vec{EG} = t\vec{GD}$

$$\underset{\sim}{e} + \vec{EG} = \underset{\sim}{g} \Rightarrow \underset{\sim}{e} + t\vec{GD} = \underset{\sim}{g}$$
$$\underset{\sim}{g} + \vec{GD} = \underset{\sim}{d}$$
$$\Rightarrow \underset{\sim}{e} + t(\underset{\sim}{d} - \underset{\sim}{g}) = \underset{\sim}{g}$$

$$\therefore \underset{\sim}{e} + t\underset{\sim}{d} - t\underset{\sim}{g} = \underset{\sim}{g} \Rightarrow \underset{\sim}{g} = \frac{\underset{\sim}{e} + t\underset{\sim}{d}}{1+t}$$ then from

(1) & (2)
$$\underset{\sim}{g} = \frac{p\underset{\sim}{b} + tq\underset{\sim}{c}}{1+t} \quad (4)$$

1

also $\dfrac{\vec{BE}}{\vec{CD}} = t \Rightarrow |\vec{BE}| = t|\vec{CD}|$

$$|\vec{BE}| = |\underset{\sim}{b} - \underset{\sim}{e}| = |\underset{\sim}{b} - p\underset{\sim}{b}| = (1-p)|\underset{\sim}{b}|$$

$$|\vec{CD}| = |\underset{\sim}{c} - \underset{\sim}{d}| = |\underset{\sim}{c} - g\underset{\sim}{c}| = (1-g)|\underset{\sim}{c}|$$

$$\therefore (1-p)|\underset{\sim}{b}| = t(1-g)|\underset{\sim}{c}| \quad (5)$$

then using (3) and (4)

$$\vec{GF} = \underset{\sim}{f} - \underset{\sim}{g} = \frac{t\underset{\sim}{c}+b}{1+t} - \frac{p\underset{\sim}{b} + tg\underset{\sim}{c}}{1+t}$$

$$\underset{\sim}{f} - \underset{\sim}{g} = \frac{t - tg}{1+t}\underset{\sim}{c} + \frac{1-p}{1+t}\underset{\sim}{b}$$

$$= \frac{t(1-g)}{1+t}\underset{\sim}{c} + \frac{1-p}{1+t}\underset{\sim}{b}$$

and (5) $\Rightarrow$ $t(1-g) = \dfrac{(1-p)|\underset{\sim}{b}|}{|\underset{\sim}{c}|}$

$$\therefore \underset{\sim}{f} - \underset{\sim}{g} = \frac{(1-p)}{1+t}\frac{|\underset{\sim}{b}|}{|\underset{\sim}{c}|}\underset{\sim}{c} + \frac{1-p}{1+t}\underset{\sim}{b}$$

$\therefore$

$$\vec{GF} = \frac{1-p}{1+t}|\underset{\sim}{b}|\left(\frac{\underset{\sim}{c}}{|\underset{\sim}{c}|} + \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}\right) \quad 1$$

$\therefore \vec{GF} = \text{const} \cdot \vec{AX} \quad \therefore \vec{GF} \parallel \vec{AX}$ $\square$