CSSA

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW

Centre Number

Student Number

DO NOT REMOVE PAPER FROM EXAM ROOM

2020
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Thursday, 20 August 2020

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A Reference Sheet is provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 100

Section I Pages 2 - 5

10 marks

- Attempt Questions 1 - 10
- Allow 15 minutes for this section

Section II Pages 6 - 11

90 marks

- Attempt Questions 11 - 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1–10.

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1 Let $z = 1 + \sqrt{3}i$. What is z in exponential form?

(A) $e^{\frac{i\pi}{3}}$

(B) $e^{\frac{i\pi}{6}}$

(C) $2e^{\frac{i\pi}{3}}$

(D) $2e^{\frac{i\pi}{6}}$

2 What is the distance of the point $(2, 3, 7)$ from the x-z plane?

(A) 2 units

(B) 3 units

(C) 7 units

(D) 9 units

3 Consider the conditional statement for $n > 2$: If n is a prime number, then n is odd.

What is the contrapositive of the conditional statement?

(A) If n is prime, then n is not odd.

(B) If n is not prime, then n is not odd.

(C) If n is odd, then n is not prime.

(D) If n is not odd, then n is not prime.

4 Which of the following expressions is equal to $\int \frac{1}{x(\log_e x)^2}$?

(A) $\frac{1}{\log_e x}+c$

(B) $\frac{1}{(\log_e x)^3}+c$

(C) $\log_e\left(\frac{1}{x}\right)+c$

(D) $\frac{-1}{\log_e x}+c$

5 Let $\alpha = 1-i$.

Which of the following is true about the value of α^{10}?

(A) It is purely real

(B) It is purely imaginary

(C) 0

(D) $32\left(\cos\frac{5\pi}{2}+i\sin\frac{5\pi}{2}\right)$

6 Which of the following statements does not have a counter-example?

(A) If a person does not own a pet, then they do not own a cat.

(B) A quadrilateral is formed by joining any four points in a plane.

(C) All primes are odd.

(D) If x is even, then x^2 is odd.

7 A particle of unit mass travels horizontally through a medium. When time $t = 0$, the particle is at point O with initial speed U. The resistance on the particle due to the medium is kv^2, where v is the velocity of the particle at time t and k is a positive constant.

Which expression gives the correct velocity of the particle?

(A) $\dfrac{1}{v} = kt + \dfrac{1}{U}$

(B) $v = kt + \dfrac{1}{U}$

(C) $\dfrac{1}{v} = kt$

(D) $v = kt$

8 Which expression is equal to $\displaystyle\int \frac{dx}{\sqrt{8 - 2x - x^2}}$?

(A) $\sin^{-1}\left(\dfrac{1-x}{2\sqrt{2}}\right) + c$

(B) $\sin^{-1}\left(\dfrac{1-x}{3}\right) + c$

(C) $\sin^{-1}\left(\dfrac{1+x}{2\sqrt{2}}\right) + c$

(D) $\sin^{-1}\left(\dfrac{1+x}{3}\right) + c$

9 A particle is moving in simple harmonic motion about the origin according to the equation $x = 3\cos nt$, where x metres is its displacement after t seconds.

Given that the particle passes through the origin with a speed of $\sqrt{3}$ ms^{-1}, what is the period of the motion?

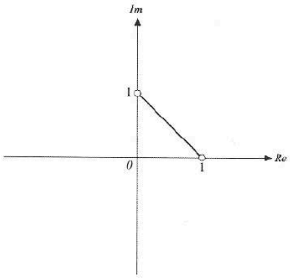
(A) $\dfrac{2\sqrt{3\pi}}{3}$ seconds

(B) $\dfrac{2\sqrt{3}}{3\pi}$ seconds

(C) $\dfrac{6\pi}{\sqrt{3}}$ seconds

(D) $\dfrac{6}{\sqrt{3\pi}}$ seconds

10 The locus of z is displayed on the Argand diagram below.



Which of the following is the equation of the locus of z?

(A) $\arg\left(\dfrac{z-i}{z-1}\right)=0$

(B) $\arg\left(\dfrac{z-i}{z-1}\right)=\pm\pi$

(C) $\arg\left(\dfrac{z+i}{z+1}\right)=0$

(D) $\arg\left(\dfrac{z+i}{z+1}\right)=\pm\pi$

End of Section I

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z=1+i$ and $w=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$.

(i) Express $\frac{w}{z}$ in polar form. **2**

(ii) Hence or otherwise, express $\left(w\bar{z}\right)^8$ in the form $a+ib$, where a and b are real. **2**

(b) Let $\underset{\sim}{a}=\begin{pmatrix}2\\1\\3\end{pmatrix}$ and $\underset{\sim}{b}=\begin{pmatrix}-1\\1\\2\end{pmatrix}$. **2**

What is the angle between the vectors to the nearest degree?

(c) Let $P(z)=z^4-4z^3-3z^2+50z-52$. **3**

Solve $P(z)=0$ if $z=3-2i$ is a root of the polynomial.

(d) A line passes through the points $A(1,3,-2)$ and $B(2,-1,5)$.

(i) Show that the vector equation of the line AB is given by: **2**

$$\underset{\sim}{r}=\left(\underset{\sim}{i}+3\underset{\sim}{j}-2\underset{\sim}{k}\right)+\lambda_1\left(\underset{\sim}{i}-4\underset{\sim}{j}+7\underset{\sim}{k}\right)\ ,\ \lambda_1\in\mathbb{R}$$

(ii) Determine if the point $C(3,4,9)$ lies on the line. **1**

(iii) Consider a line with parametric equations $x=1-\lambda_2$, $y=2+3\lambda_2$, $z=-1+\lambda_2$. **3**
Assuming this line is neither parallel nor perpendicular to AB, determine whether the lines intersect or are skew.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the equation $z^2-2(1+2i)z+(1+i)=0$.

(i) Show that $\left(z-(1+2i)\right)^2=-4+3i$. **1**

(ii) Hence solve $z^2-2(1+2i)z+(1+i)=0$. **3**

(b) Sketch the intersection of the regions defined by **2**

$$|z-2i|\le 1 \text{ and } 0<Arg(z-2i)\le\frac{3\pi}{4}.$$

(c) Use the substitution $t=\tan\frac{x}{2}$ to show that $\int_0^{\frac{\pi}{3}}\frac{1}{8\sin x+6\cos x-10}\,dx=\frac{1}{2}\left(\frac{1}{2-\sqrt{3}}\right)$. **4**

(d) A function is defined by $f(x)=\frac{4x}{5-x}$. **3**

Solve the inequality $f(|x|)\le 2$.

(e) It is given that p and q are real numbers. **2**

Consider the statement $\forall p\left(\forall q,\frac{1}{p^2}<\frac{1}{q^2}\right)$.

Either prove that the statement is true or provide a counter-example.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) (i) Express $\dfrac{-x^2+2x+5}{(x^2+2)(1-x)}$ in the form $\dfrac{ax+b}{x^2+2}+\dfrac{c}{1-x}$. **2**

(ii) Hence find $\displaystyle\int \frac{-x^2+2x+5}{(x^2+2)(1-x)}\,dx$. **2**

(b) A particle of mass M kilograms is projected vertically upward with a velocity of 120 ms^{-1}. The air resistance acting on the particle is $3Mv$ newtons, where v is the velocity of the particle.

(i) Show that if the acceleration due to gravity is 10 ms^{-2}, the equation of motion is given by $\ddot{x}=-(10+3v)$. **1**

(ii) Find the maximum height reached by the particle, correct to the nearest metre. **3**

(iii) Find the time at which the particle reaches its maximum height, correct to one decimal place. **2**

(c) A sphere S_1 with centre $\underset{\sim}{c}=2\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$ passes through $\underset{\sim}{a}=4\underset{\sim}{i}+4\underset{\sim}{j}+4\underset{\sim}{k}$.

(i) Find the Cartesian equation of S_1. **2**

(ii) A second sphere, S_2, has equation $(x-2)^2+(y-2)^2+(z-5)^2=1$. Find the equation of the circle in which S_1 and S_2 intersect and state the centre and radius of this circle. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Prove that $\frac{1+\sqrt{5}}{2}$ is irrational. **3**

(b) The price, p, of fuel rises and falls in simple harmonic motion according to the equation $p = 1.5 + \frac{1}{2}\sin\frac{\pi}{7}t + \frac{1}{2}\cos\frac{\pi}{7}t$, where the price is measured in dollars and t is the number of days after midnight Sunday.

(i) What is the amplitude and period of the fuel price? **3**

(ii) What is the price of fuel, to the nearest cent, at 12 midday on Monday? **1**

(iii) At what time and day, correct to the nearest hour, will the fuel price first be at a minimum? **3**

(c) A particle is moving such that $\ddot{x} = 2x^3 + 6x^2 + 4x$.
Initially $x = 1$ and $v = -3$.

(i) Show that $v = -x(x+2)$. **2**

(ii) Find an expression for x in terms of t. **2**

(iii) Hence, find the limiting position of the particle. **1**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int e^{-x}\sin(-x)\,dx$. 3

(b) A projectile is fired from ground level with an initial velocity u ms^{-1} at an angle of θ to the horizontal. The air resistance is directly proportional to the velocity, with k the constant of proportionality.

Assume that the equations of motion are:

$$x = \frac{u\cos\theta}{k}\left(1-e^{-kt}\right)$$

$$y = \frac{10+ku\sin\theta}{k^2}\left(1-e^{-kt}\right)-\frac{10t}{k}$$

where $(x,\ y)$ are the coordinates of the projectile at time t seconds. Do NOT prove these equations.

The projectile is fired at an angle of $60°$, with initial velocity $10\sqrt{3}$ ms^{-1} and $k = 0.4$.

(i) Find the time when the projectile reaches its greatest height. 2

(ii) The projectile hits the ground when $t \approx 2.6$ seconds. Find the magnitude and direction of the velocity of the projectile when it hits the ground. 3

(c) Consider the equation $z^5 = 1$.

(i) Write down, in polar form, the five roots of $z^5 = 1$. 2

(ii) Show that, for $z \neq 1$, 2

$$\frac{z^5-1}{z-1} = \left(z^2-2z\cos\left(\frac{2\pi}{5}\right)+1\right)\left(z^2-2z\cos\left(\frac{4\pi}{5}\right)+1\right).$$

(iii) Deduce that $\cos\left(\frac{2\pi}{5}\right)$ and $\cos\left(\frac{4\pi}{5}\right)$ are roots of the equation $4x^2+2x-1=0$. 3

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) A sequence is defined recursively as $u_1 = 0$, $u_2 = 25$ and $u_n = 10u_{n-1} - 25u_{n-2}$, for $n \geq 3$. **3**

Using mathematical induction, prove that $u_n = (n-1)5^n$ for $n \geq 1$.

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x \, dx$.

(i) Show that $I_n = \left(\frac{n-1}{n+2}\right) I_{n-2}$ for $n \geq 2$. **4**

(ii) Hence show that $I_2 = \frac{\pi}{16}$. **1**

(c) (i) Show that $0 < \frac{1}{x\sqrt{x}} < \frac{1}{x} < \frac{1}{\sqrt{x}}$ for $x > 1$. **1**

(ii) Show that $0 < \frac{1}{\sqrt{x}} < \frac{1}{x} < \frac{1}{x\sqrt{x}}$ for $0 < x < 1$. **1**

(iii) Show that, for $t \geq 1$, $\frac{2\left(\sqrt{t}-1\right)}{\sqrt{t}} \leq \ln t \leq 2\left(\sqrt{t}-1\right)$. **2**

(iv) Find $\lim_{t \to 0}(t \ln t)$ and $\lim_{t \to \infty}\left(\frac{\ln t}{t}\right)$ **3**

End of paper