CSSA
CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NSW

Centre Number

Student Number

2014
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Thursday 31 July 2014

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a separate sheet
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 100

Section I Pages 2–5

10 marks
- Attempt Questions 1–10
- Allow 15 minutes for this section

Section II Pages 6–15

90 marks
- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Write $\dfrac{40}{1-3i}$ in the form $a+ib$, where a and b are real.

(A) $4-12i$

(B) $4+12i$

(C) $-5-15i$

(D) $-5+15i$

2 What is the eccentricity of the hyperbola $16x^2-25y^2=400$?

(A) $\dfrac{3}{5}$

(B) $\dfrac{3}{4}$

(C) $\dfrac{\sqrt{41}}{5}$

(D) $\dfrac{\sqrt{41}}{4}$

3 The equation $y^3-xy+x^3=7$ implicitly defines y in terms of x.

Which of the following is an expression for $\dfrac{dy}{dx}$?

(A) $\dfrac{-3x^2}{3y^2-1}$

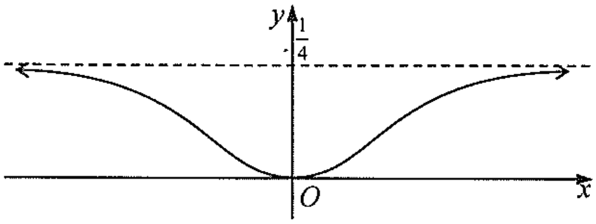
(B) $\dfrac{y-3x^2}{3y^2-x}$

(C) $\dfrac{y-3x^2+7}{3y^2-x}$

(D) $\dfrac{3y^2-y+3x^2}{x}$

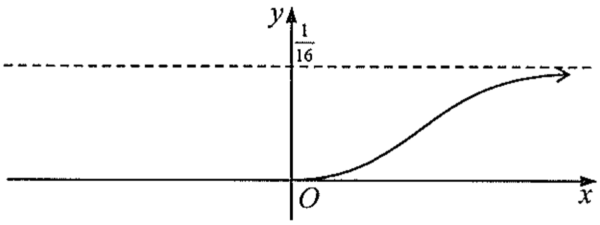
4 The diagram shows the graph of $y = f(x)$.

DIAGRAMS NOT TO SCALE.

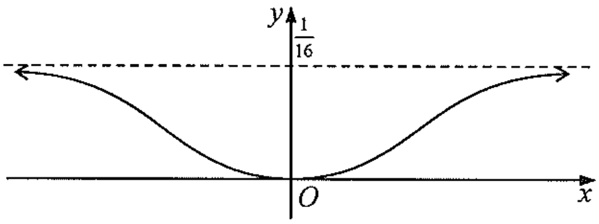


Which of the following best represents the graph of $y = \sqrt{f(x)}$?

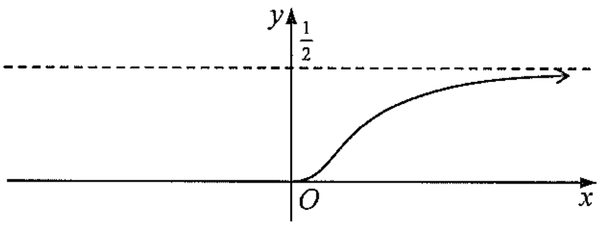
(A)



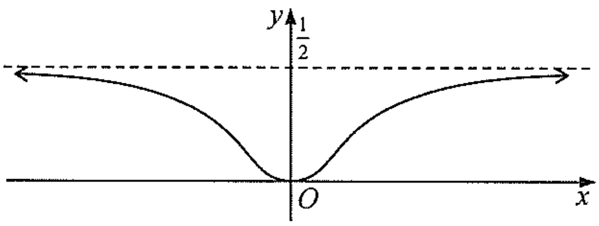
(B)



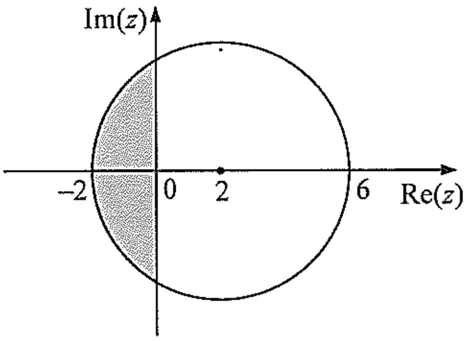
(C)



(D)



5 A circle with centre $(2, 0)$ and radius 4 units is shown on an Argand diagram below.



Which of the following inequalities represents the shaded region?

(A) $\text{Re}(z) \le 0$ and $|z-2| \le 4$

(B) $\text{Re}(z) \le 0$ and $|z-2| \le 16$

(C) $\text{Im}(z) \le 0$ and $|z-2| \le 4$

(D) $\text{Im}(z) \le 0$ and $|z-2| \le 16$

6 A particle moves in a circle of radius 40 cm with a constant angular speed of 15 revolutions per minute. What is the speed of the particle?

(A) $\frac{\pi}{5}$ ms^{-1}

(B) 6 ms^{-1}

(C) 12π ms^{-1}

(D) 20π ms^{-1}

7 The cube roots of unity are 1, ω and ω^2. Simplify $(1-\omega+\omega^2)(1+\omega-\omega^2)$.

(A) 0

(B) 1

(C) 2

(D) 4

8 Which integral is obtained when the substitution $t=\tan\frac{x}{2}$ is applied to $\int\frac{dx}{5+4\cos x}$?

(A) $\int\frac{2}{9-4t^2}dt$

(B) $\int\frac{2}{9+t^2}dt$

(C) $\int\frac{1+t^2}{9+t^2}dt$

(D) $\int\frac{2(1-t^2)}{(1+t^2)(9-t^2)}dt$

9 Given α, β and γ are the roots of the equation $x^3-4x+7=0$, find the cubic equation with roots α^2, β^2 and γ^2.

(A) $x^3-4\sqrt{x}+7=0$

(B) $x^3+16x+49=0$

(C) $x^3-4x^2+16x-49=0$

(D) $x^3-8x^2+16x-49=0$

10 Given z and w are non-zero complex numbers, $z\neq\pm w$, such that $z\bar{z}=w\bar{w}$, which of the following statements is true?

(A) $\arg\left(\frac{z+w}{z-w}\right)=0$

(B) $\arg\left(\frac{z+w}{z-w}\right)=\pi$

(C) $\arg\left(\frac{z+w}{z-w}\right)=\pm\frac{\pi}{2}$

(D) $\arg\left(\frac{z+w}{z-w}\right)$ cannot be determined

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11– 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $\omega = \sqrt{3} + i$.

(i) Express ω in modulus-argument form. **1**

(ii) Hence, or otherwise, express $z^3\omega$ in modulus-argument form. **2**

(b) By completing the square, find $\displaystyle\int \frac{9}{x^2 + 4x + 13}\, dx$. **2**

(c) Evaluate $\displaystyle\int_0^1 xe^{4x}\, dx$. **3**

(d) (i) Find real numbers a and b such that **2**

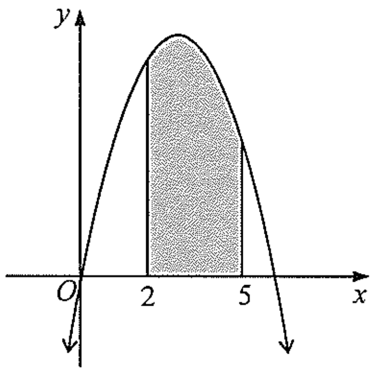
$$\frac{3x}{(x-2)^2(x-3)} = \frac{a}{(x-2)^2} + \frac{b}{x-2} + \frac{9}{x-3}$$

(ii) Hence, or otherwise, find $\displaystyle\int \frac{3x}{(x-2)^2(x-3)}\, dx$. **2**

Question 11 continues on page 7

Question 11 (continued)

(e) The region enclosed between $y = 6x - x^2$, the x-axis and the lines $x = 2$ and $x = 5$ is shaded in the diagram below. **3**



The shaded region is rotated about the y-axis.

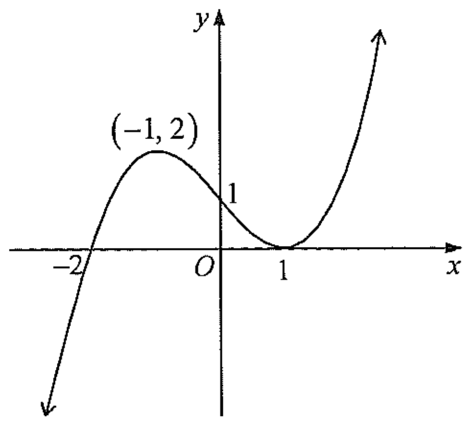
Using the method of cylindrical shells, find the volume of the solid generated.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below is a sketch of the function $y = f(x)$,

where $f(x) = \frac{1}{2}(x+2)(x-1)^2$.



Draw separate one-third page diagrams of the graphs of each of the following.

(i) $y = |f(x)|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

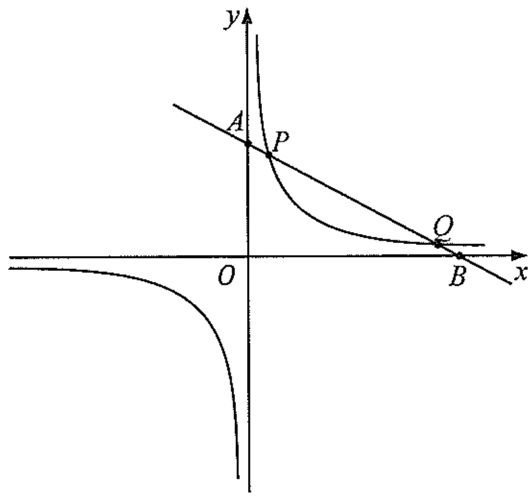
(b) It is given that $1+i$ is a root of $p(x) = x^4 - 2x^3 - 7x^2 + 18x - 18$. **3**

Express $p(x)$ as the product of quadratic and linear factors with real coefficients.

Question 12 continues on page 9

Question 12 (continued)

(c) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the same branch of the rectangular hyperbola $xy = c^2$. The line PQ intersects the asymptotes at A and B as shown in the diagram.

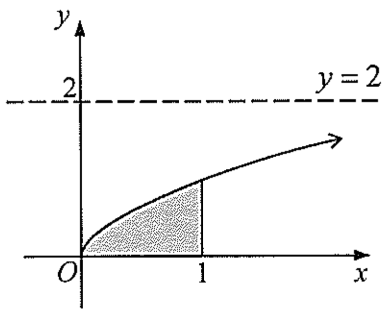


(i) Show that the equation of PQ is given by $x + pqy = c(p+q)$. **2**

(ii) The midpoint M of PQ is $\left(\frac{c(p+q)}{2}, \frac{c(p+q)}{2pq}\right)$. (Do NOT prove this.) **2**

Using this given information, or otherwise, show that $AP = BQ$.

(d) The area under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$ is rotated about the line $y = 2$. **3**

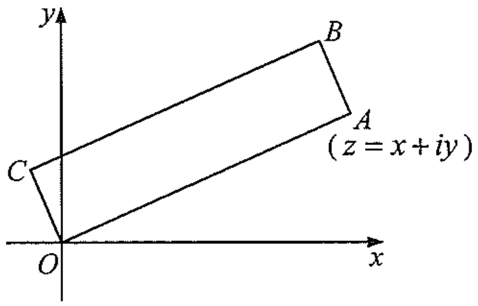


By taking slices perpendicular to the line $y = 2$, find the volume of the solid generated.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) In the Argand diagram below, $OABC$ is a rectangle. O is the origin and the distance OA is four times the distance AB. The vertex A is represented by the complex number $z = x + iy$. **2**

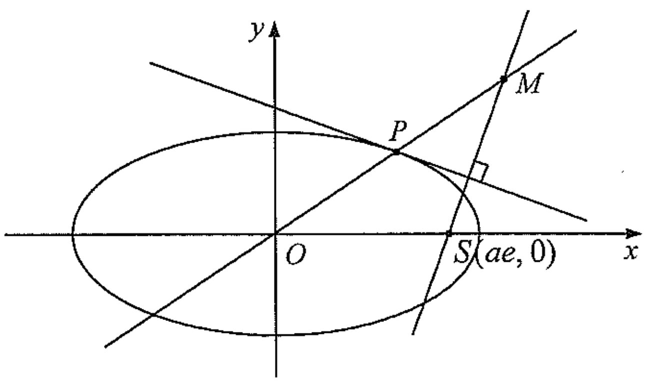


Find an expression for the complex number that represents the vertex B. Leave your answer in the form $a + ib$.

(b) (i) Show that if α is a zero of multiplicity 2 of a polynomial $f(x)$, then $f(\alpha) = f'(\alpha) = 0$. **2**

(ii) The polynomial $g(x) = px^3 - 3qx + r$ has a zero of multiplicity 2. Show that $4q^3 = pr^2$. **3**

(c) The diagram shows the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with focus $S(ae, 0)$ and origin O. $P(a\cos\theta, b\sin\theta)$ is any point on the ellipse. The line through S perpendicular to the tangent at P and the line OP produced meet at M.



(i) Show that the gradient of the tangent to the ellipse at P is given by $-\dfrac{b\cos\theta}{a\sin\theta}$. **1**

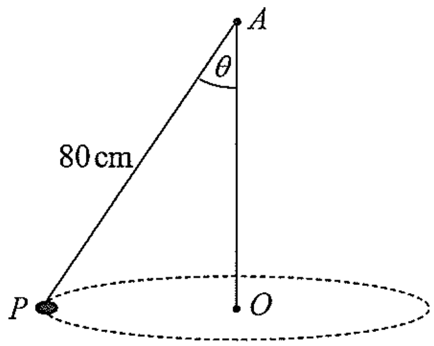
(ii) Show that M lies on the corresponding directrix to S. **4**

Question 13 continues on page 11

Question 13 (continued)

(d) A particle P of mass 3 kg is attached by a string of length 80 cm to a point A. The particle moves with constant angular velocity ω in a horizontal circle with centre O which lies directly below A. The angle the string makes with OA is θ. **3**

The forces acting on the particle are the tension, T, in the string and the force due to gravity. The greatest tension that can safely be allowed in the string is 200 Newtons.



By considering the forces acting on the particle in the horizontal direction, find the maximum angular velocity ω of the particle. Give your answer correct to 1 decimal place.

End of Question 13

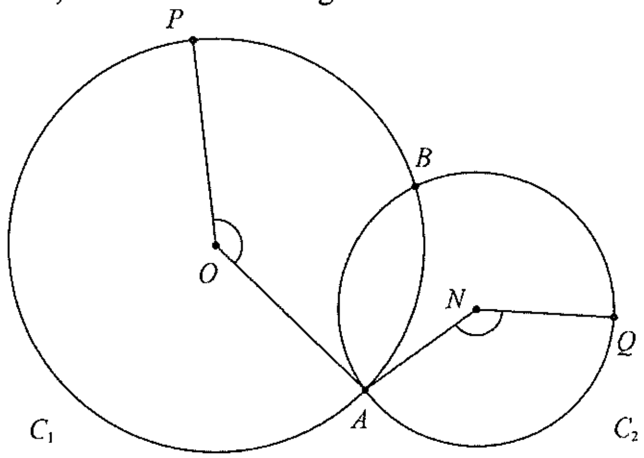
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int_1^e (\ln x)^n \, dx$, where $n \geq 0$.

(i) Show that $I_n = e - nI_{n-1}$ for $n \geq 1$. **2**

(ii) Hence evaluate $\int_1^e (\ln x)^3 \, dx$. **2**

(b) Two circles C_1 and C_2 with centres O and N respectively intersect at A and B. P lies on C_1 and Q lies on C_2 such that $\angle AOP = \angle ANQ$ and $\angle AOP > \angle AOB$, as shown in the diagram below. **3**



Prove that the points P, B and Q are collinear.

(c) (i) Given $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$, plot the roots of $z^6 + z^3 + 1 = 0$ on an Argand diagram. **2**

(ii) Show that **2**

$$z^6 + z^3 + 1 = \left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$

(iii) Show that $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} + \cos\frac{2\pi}{9}\cos\frac{8\pi}{9} + \cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -\frac{3}{4}$ **1**

(d) The inequality $x > \ln(1 + x)$ holds for all real $x > 0$. (Do NOT prove this.) **3**

Use this result and the method of mathematical induction to prove that for all positive integers n,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n + 1).$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

(ii) Hence, or otherwise, find $\int_0^{\frac{\pi}{2}} x\sin 2x\,dx$. **3**

(b) An object of mass 70 kg, initially at rest, is pulled along a horizontal surface by a constant force of 140 N. It experiences a resistance proportional to its speed.
When the speed is $10\ \text{ms}^{-1}$, the acceleration is $1\ \text{ms}^{-2}$.
Let x represent the displacement in metres from the initial position of the object.

(i) Show that the equation of motion is $\ddot{x} = 2 - \frac{1}{10}v$. **2**

(ii) Find an expression for x as a function of v. **3**

(iii) Show that the object's speed cannot exceed $20\ \text{ms}^{-1}$. **1**

(c) A nine letter arrangement consists of 3 A's, 3 B's and 3 C's such that there are:

- no A's in the first three letters
- no B's in the next three letters
- no C's in the last three letters

(i) Find the number of nine letter arrangements if the first three letters are 2 B's and 1 C in some order. **2**

(ii) Find the total number of nine letter arrangements. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

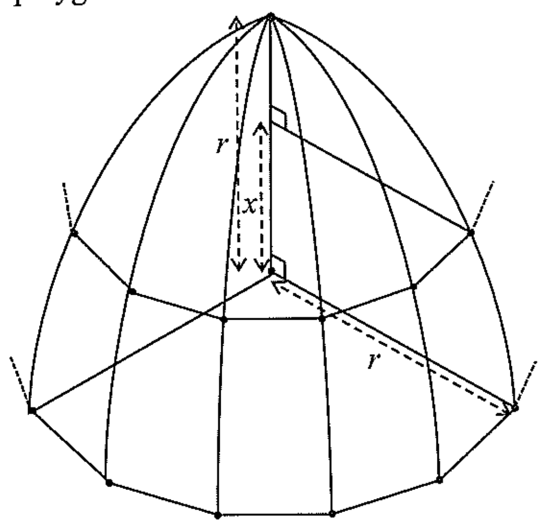
(a) (i) Prove that $x^2+y^2+z^2 \geq xy+yz+xz$ for x, y and z positive real numbers. **2**

(ii) The inequality $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z} \geq \dfrac{9}{x+y+z}$ holds for x, y and z positive real numbers. (Do NOT prove this). **2**

Given x, y and z are positive real numbers with $x^2+y^2+z^2=9$, prove that

$$\frac{1}{1+xy}+\frac{1}{1+yz}+\frac{1}{1+xz} \geq \frac{3}{4}.$$

(b) The diagram below shows part of a polygonal dome. Each cross-section is a regular n-sided polygon.



The vertex of the dome is r units directly above the centre of the polygonal base, which is r units from each vertex. A circular arc joins the top of the dome to each vertex of the base.

(i) Show that the area of the horizontal cross-section x units from the base is given by $\dfrac{n}{2}\sin\left(\dfrac{2\pi}{n}\right)\times\left(r^2-x^2\right)$. **2**

(ii) Hence show that the volume of the dome is given by **2**

$$\frac{nr^3}{3}\sin\left(\frac{2\pi}{n}\right).$$

(iii) Show that as $n \to \infty$, the volume of the dome approaches that of a hemisphere. **1**

Question 16 continues on page 15

Question 16 (continued)

(c) (i) Show that $\dfrac{x^{2^n}}{1-x^{2^{n+1}}} = \dfrac{1}{1-x^{2^n}} - \dfrac{1}{1-x^{2^{n+1}}}$. **2**

(Note that $x^{2^n} = x^{(2^n)}$).

(ii) Using the result in part (i), **1**

show that $\displaystyle\sum_{n=0}^{N} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{1}{1-x} - \frac{1}{1-x^{2^{N+1}}}$.

(iii) Let x be a real number with $-1 < x < 1$. **1**

Given $\displaystyle\lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}}$,

show that $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{x}{1-x}$.

(iv) Hence find $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2014^{2^n} - 2014^{-2^n}}$. **2**

End of Paper

EXAMINERS

Gerry Sozio (Convenor)	Catholic Education Office, Wollongong Diocese
Robert Muscatello	Mount Carmel Catholic High School, Varroville
Sebastian Puntillo	St Scholastica's College, Glebe
Frank Reid	University of New South Wales, Kensington
Thanom Shaw	University of New South Wales, Kensington