CRANBROOK SCHOOL

Student Number: ____________________

2023 **YEAR 12 TASK 4**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 180 Minutes
- Write using blue or black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 20 minutes for this section

Section II – 90 marks

- Attempt Questions 11 – 14
- Allow about 160 minutes for this section

Section I

10 marks
Attempt Questions 1–10

Use the multiple-choice answer sheet for Questions 1–10.

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1 In an Argand diagram the points $A(-3,2)$ and $B(5,-4)$ lie at opposite ends of a diameter of a circle. What is the equation of the circle?

A. $|z-1+i|=5$

B. $|z+1-i|=5$

C. $|z-1+i|=10$

D. $|z+1-i|=10$

2 What is the size of the acute angle θ between the vectors $\underset{\sim}{a}=2\underset{\sim}{i}-\underset{\sim}{j}-\underset{\sim}{k}$ and $\underset{\sim}{b}=2\underset{\sim}{i}-2\underset{\sim}{k}$?

A. $\theta=\dfrac{\pi}{6}$

B. $\theta=\dfrac{\pi}{5}$

C. $\theta=\dfrac{\pi}{4}$

D. $\theta=\dfrac{\pi}{3}$

3 Which of the following is an expression for $\displaystyle\int \frac{1}{x^2 - \sqrt{3}x + 1}\, dx$?

A. $\tan^{-1}\left(x - \sqrt{3}\right) + c$

B. $2\tan^{-1}\left(x - \sqrt{3}\right) + c$

C. $\tan^{-1}\left(2x - \sqrt{3}\right) + c$

D. $2\tan^{-1}\left(2x - \sqrt{3}\right) + c$

4 The amount of apples, bananas and oranges sold by a fruit seller over a year is shown in the table below

Fruit	Amount Sold (tonnes)	Profit ($/tonne)
Apples	25	530
Bananas	55	380
Oranges	10	410

Let $\underset{\sim}{a} = \begin{bmatrix} 25 \\ 55 \\ 10 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 530 \\ 380 \\ 410 \end{bmatrix}$ and $\underset{\sim}{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Which of the following expressions calculates the average profit in dollars per tonne of fruit sold over the year?

A. $\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{c}}{\underset{\sim}{b} \cdot \underset{\sim}{c}}$

B. $\dfrac{\underset{\sim}{b} \cdot \underset{\sim}{c}}{\underset{\sim}{a} \cdot \underset{\sim}{c}}$

C. $\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{c}}$

D. $\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{a} \cdot \underset{\sim}{c}}$

5 Which of the following expressions is equivalent to $\displaystyle\int \ln\left(x^2+1\right)\,dx$?

A. $x\ln\left(x^2+1\right)-2x+2\tan^{-1}x+c$

B. $x\ln\left(x^2+1\right)-2\ln\left(x^2+1\right)+c$

C. $\ln\left(x^2+1\right)-2x+2\tan^{-1}x+c$

D. $\ln\left(x^2+1\right)-x\ln\left(x^2+1\right)+c$

6 Consider the complex, non-real cube roots of unity ω and ω^2.

What is the value of $\left(1-\omega+\omega^2\right)\left(1+\omega-\omega^2\right)$?

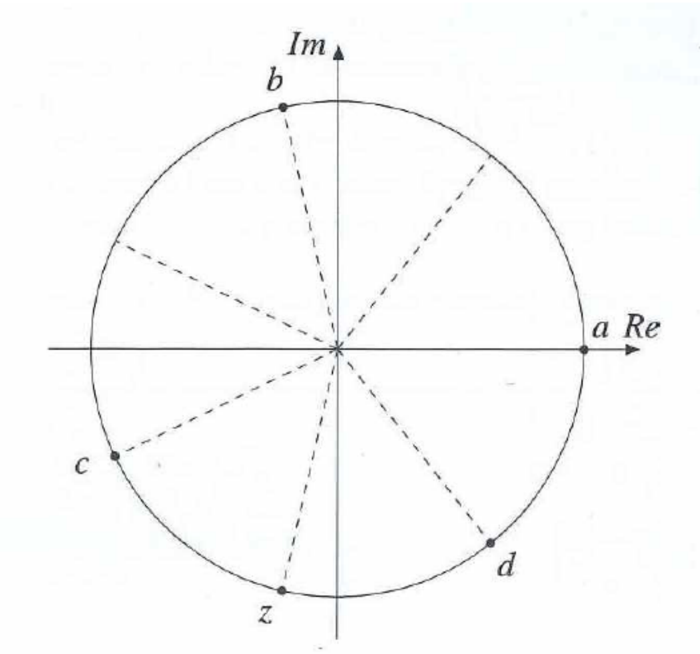
A. 0

B. 1

C. 2

D. 4

7 The complex numbers a, b, c, d and z are solutions to $z^7 = 1$ as shown in the Argand diagram below.



Which of the following is a cube root of z?

A. a

B. b

C. c

D. d

8 A sequence of complex numbers $z_1, z_2, z_3, z_4, \ldots$ is given by the rule $z_1 = Z$ and $z_{n+1} = c\bar{z}_n + c - 1$ for $n \in \mathbb{Z}^+$ where c is a complex number with modulus 1. What is the value of z_3?

A. $z_3 = Z$

B. $z_3 = -2 + Z$

C. $z_3 = 2c + Z$

D. $z_3 = -2 + 2c + Z$

9 Consider a line that passes through the point $(5, 2, 1)$ and is parallel to the $x - y$ plane and the $x - z$ plane.

Which of the following is the vector equation of the line?

A. $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

10 The function $F(x)$ is the primitive of $f(x)$, that is $F'(x) = f(x)$. Which of the following is true?

A. $$\int \left(\frac{d}{dx} \int_a^b f(x)\,dx \right) dx = f(b) - f(a)$$

B. $$\int_a^b \left(\frac{d}{dx} \int f(x)\,dx \right) dx = f(b) - f(a)$$

C. $$\frac{d}{dx} \int \left(\int_a^b f(x)\,dx \right) dx = F(b) - F(a)$$

D. $$\frac{d}{dx} \int_a^b \left(\int f(x)\,dx \right) dx = F(b) - F(a)$$

Section II

90 marks
Attempt Questions 11 – 16

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Please begin a new Writing Booklet.

(a) Express $2\sqrt{2}e^{-\frac{3\pi}{4}i}$ in the form $x + iy$. **2**

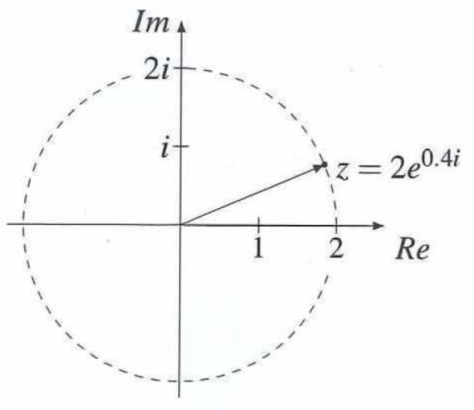
(b) Consider the two points $A(2,2,2)$ and $B(2,-2,2)$.

(i) Find $\overrightarrow{AB}$ **1**

(ii) Find $|\overrightarrow{AB}|$ **1**

(iii) Find $\angle AOB$. Give your answer to the nearest degree. **2**

(c) Consider the complex number $z = 2e^{0.4i}$, as sketched below. **3**



Copy and clearly label this Argan diagram, and sketch the four points represented by z, $\bar{z}$, $-\bar{z}$ and $z - \bar{z}$.

(d) Use the substitution $t = \tan\dfrac{x}{2}$ to find $\displaystyle\int \frac{1}{1+\sin x}\,dx$. **3**

(e) Show for any complex numbers z and w, that $\overline{zw} = \overline{z} \times \overline{w}$. **3**

End of Question 11

Question 12 (15 marks) Please begin a new Writing Booklet.

(a) Consider the two lines

$$L_1 : \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} \text{ and } L_2 : \begin{bmatrix} -1 \\ 2 \\ -6 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}.$$

(i) Find the values of k given $B(9,k,24)$ lies on L_2. **2**

(ii) Find the point of intersection of L_1 and L_2. **3**

(b) Find $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\sqrt{1-x^2}}\,dx$ **4**

(c) Solve $z^2 + (7-i)z + 16 + 4i = 0$ **3**

(d) Find $\displaystyle\int \frac{x^3-2}{x^3-x}\,dx.$ **3**

End of Question 12

Question 13 (15 marks) Please begin a new Writing Booklet.

(a) On an Argand diagram shade the region containing all points representing the complex numbers z that satisfy both $|z-2i| \leq 2$ and $\dfrac{\pi}{6} \leq \arg z \leq \dfrac{\pi}{4}$. **2**

(b) The graph of a polynomial function $f(x)=(x+3)(x-2)(x^2+bx+c)$ has a y intercept of -6 and passes through $(1,-4)$.

(i) Find the two complex roots of the equation $f(x)=0$. **2**

(ii) Express these two complex roots in the form $r(\cos\theta + i\sin\theta)$. **2**

(iii) Plot all four solutions to $f(x)=0$ on an Argand diagram. **2**

(iv) Write down the name of the quadrilateral formed by these four points. **1**

(c) Consider the sphere with vector equation $\left| \underset{\sim}{r} - \begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix} \right| = 3$.

(i) Show that the point $(5,-10,3)$ lies on the sphere. **1**

(ii) Find the point on the sphere farthest from the origin. **2**

(d) Prove by Mathematical induction that $3n^2 - 3n \leq 2^n - 1$ for $n \geq 7$. **3**

End of Question 13

Question 14 (15 marks) Please begin a new Writing Booklet.

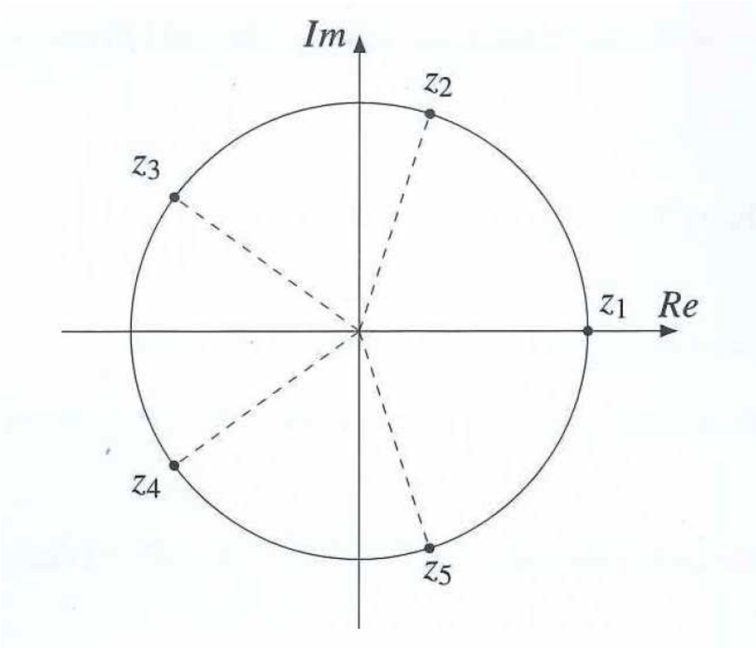
(a) The number z is a fifth root of unity where $z \neq 1$, that is $z^5 = 1$

(i) Show that $\left(z+z^{-1}\right)^2 + \left(z+z^{-1}\right) - 1 = 0$. **2**

(ii) If $z = e^{i\theta}$, show $\cos\theta = \dfrac{z+z^{-1}}{2}$. **1**

(iii) Hence show $\cos\dfrac{2\pi}{5} = \dfrac{\sqrt{5}-1}{4}$ **2**

(iv) Consider the five fifth roots of unity z_1, z_2, z_3, z_4 and z_5 as shown in the diagram below. **3**



Show that $\left|\dfrac{z_3 - z_1}{z_2 - z_1}\right| = \dfrac{1+\sqrt{5}}{2}$.

Question 14 continues on page 13

(b) The position of an object afer t seconds is given by the vector equation

$$\underset{\sim}{r} = \cos\frac{\pi t}{4}\underset{\sim}{i} + \left(\cos\frac{\pi t}{4} + \sin\frac{\pi t}{4}\right)\underset{\sim}{j} + \sin\frac{\pi t}{4}\underset{\sim}{k}.$$

(i) What is the position of the object after 3 seconds? **1**

(ii) Find the vector equation of the tangent to the path taken by the object after 3 seconds. **3**

(c) Find $\displaystyle\int_0^9 \frac{1}{\sqrt{1+\sqrt{x}}}\,dx$ **3**

End of Question 14

Question 15 (15 marks) Please begin a new Writing Booklet.

(a) Consider the function $f(x) = e^{-x}\cos x$, with domain $x \in \left[0, \dfrac{3\pi}{2}\right]$. Show that the ratio of the area above the x-axis to the area below the x-axis is **4**

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}$$

(b) A series is defined by $T_n = 6T_{n-1} - 9T_{n-2}$ for $n \geq 2$ where $T_0 = 1$ and $T_1 = 6$. **4**

Use Mathematical induction to prove $T_n = 3^n + n \times 3^n$.

(c) Let $I_n = \displaystyle\int \frac{\cos nx}{\sin x}\, dx$.

(i) Show that $\cos((n-2)x) - \cos nx = 2\sin((n-1)x)\sin x$. **2**

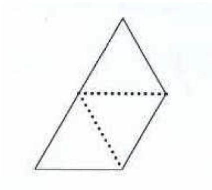
(ii) Show that $I_n - I_{n-2} = \dfrac{2\cos((n-1)x)}{n-1} + c$ **2**

(iii) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\cos 2x - \cos 6x}{\sin x}\, dx$ **3**

End of Question 15

Question 16 (15 marks) Please begin a new Writing Booklet.

(a) Consider the tile below consisting of three equilateral triangles of side length 1 unit. **3**



Prove the following result for positive integers n, using Mathematical induction:

An equilateral triangle of side length 2^n units may be covered by the tiles shown above (in any orientation) such that a single equilateral triangle of side length 1 unit is left over at one of the vertices. The tiles may not overlap.

(b) The limiting sum of series is be given by $S = \dfrac{a}{1-r}$ where a is the first term of the series and r is the common ratio between consecutive terms. It only exists when $|r| \leq 1$.

(i) Show that the limiting sum S of the series $1 + \dfrac{1}{2}e^{i\theta} + \dfrac{1}{4}e^{2i\theta} + \dfrac{1}{8}e^{3i\theta} + \ldots$ is given by $S = \dfrac{2}{2-e^{i\theta}}$. **2**

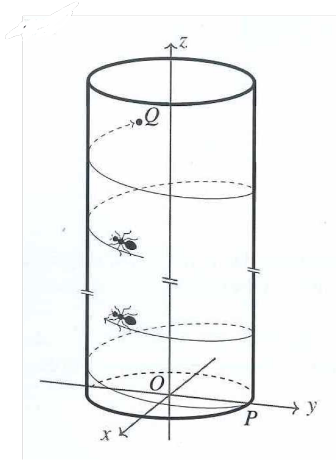
(ii) Hence show that $\dfrac{1}{2}\sin\theta + \dfrac{1}{4}\sin 2\theta + \dfrac{1}{8}\sin 3\theta + \ldots = \dfrac{2\sin\theta}{5-4\cos\theta}$. **2**

(iii) Show that there exists no real values of θ such that S is purely imaginary. **1**

Question 16 continues on page 16

Question 16 (continued)

(c) An ant follows a spiral path up a cylindrical column from $P(0,7,0)$ to the point Q as shown in the diagram below.



The ant's position $\underset{\sim}{r}$ in centimetres after t seconds is given by the vector equation below.

$$\underset{\sim}{r}(t) = 7\sin\frac{\pi t}{32}\underset{\sim}{i} + 7\cos\frac{\pi t}{32}\underset{\sim}{j} + \frac{t}{15}\underset{\sim}{k}$$

(i) Find the coordinates of the point Q if $\left|\overrightarrow{OQ}\right| = 25$. **3**

(ii) How many times has the ant crossed the line $x = 7$ on its journey to Q? **2**

(iii) The ant crawls back from Q to P along the shortest path possible. How far did it crawl on this leg of its journey? Give you answer correct to one decimal place. **2**

End of Exam

NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

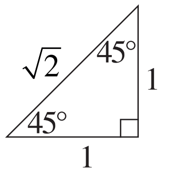
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

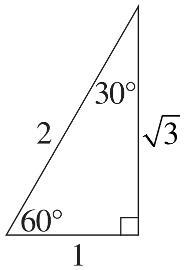


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

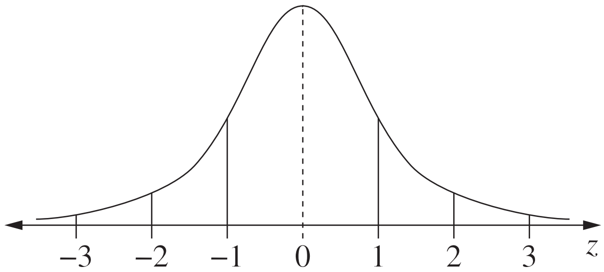
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \text{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)} dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)} dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx} dx = uv - \int v\frac{du}{dx} dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^{2}(x-c)$$

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