CSSA

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NSW

Centre Number

Student Number

DO NOT REMOVE PAPER FROM EXAM ROOM

2020

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Afternoon Session
Friday, 28 August 2020

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A Reference Sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 70

Section I Pages 2 – 6

10 marks

- Attempt Questions 1 – 10
- Allow 15 minutes for this section

Section II Pages 7 – 13

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 In a group of 110 students, what is the minimum number of students that have a birthday in at least one of the months of the year?

(A) 9

(B) 10

(C) 11

(D) 12

2 A polynomial $P(x)$ has a triple root at $x = -2$.

Which of the following statements is true?

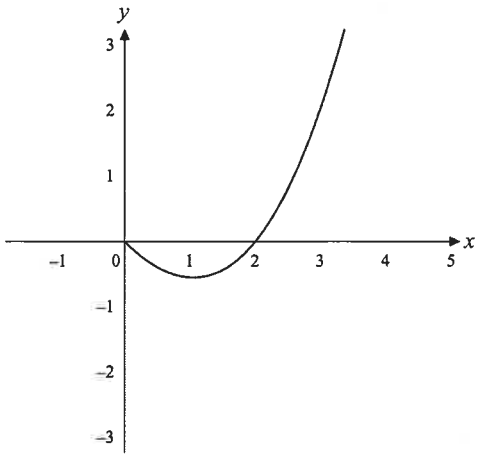
(A) $(x+2)^2$ is a factor of $P'(x)$.

(B) $(x-2)^2$ is a factor of $P'(x)$.

(C) $(x+2)^3$ is a factor of $P'(x)$.

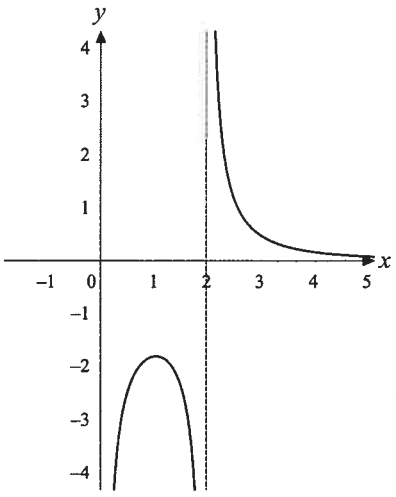
(D) $(x-2)^3$ is a factor of $P'(x)$.

3 The diagram shows the graph of the function $y = f(x)$.

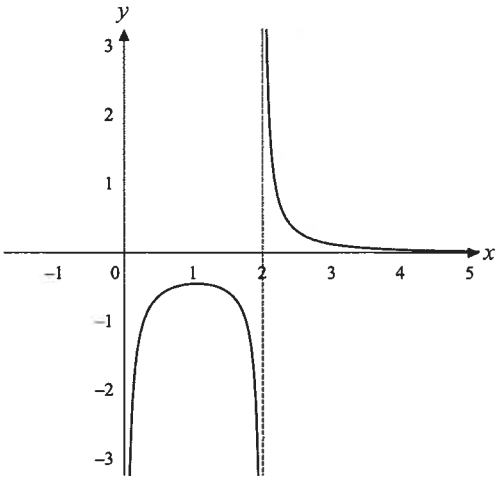


If the local minimum is $\left(1, -\frac{1}{2}\right)$, which of the following is the graph of $y = \frac{1}{f(x)}$?

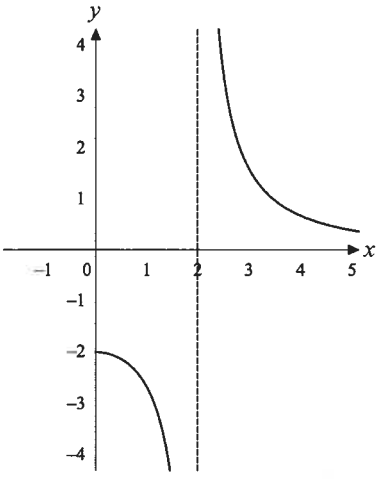
(A)



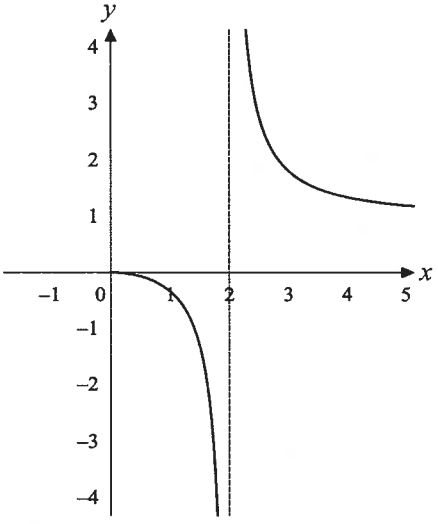
(B)



(C)



(D)



4 Using the substitution $t = \tan\frac{\theta}{2}$, simplify the expression $\frac{\cos\theta - 1}{2\sin\theta}$.

(A) t

(B) $-t$

(C) $\frac{t}{2}$

(D) $-\frac{t}{2}$

5 In how many ways can the letters of the word TROOPS be arranged if the two Os are to be separated?

(A) 60

(B) 240

(C) 300

(D) 600

6 What is the variance of the following Bernoulli distribution?

X	0	1
$P(X = x)$	0.2	0.8

(A) 0.16

(B) 0.20

(C) 0.40

(D) 0.64

7 Which expression is equal to $\int\left(\sin^2 x+x^2\right)dx$?

(A) $\frac{1}{2}x-\frac{1}{4}\sin 2x+2x+c$

(B) $x-\frac{1}{2}\sin 2x+\frac{x^3}{3}+c$

(C) $\frac{1}{2}x-\frac{1}{4}\sin 2x+\frac{x^3}{3}+c$

(D) $x-\frac{1}{2}\sin 2x+2x+c$

8 Given the vectors $\underset{\sim}{a}=2\underset{\sim}{i}-5\underset{\sim}{j}$ and $\underset{\sim}{b}=3\underset{\sim}{i}+4\underset{\sim}{j}$, what is $\text{proj}_{\underset{\sim}{a}}\underset{\sim}{b}$?

(A) $\frac{14}{29}\left(2\underset{\sim}{i}-5\underset{\sim}{j}\right)$

(B) $\frac{14}{\sqrt{29}}\left(2\underset{\sim}{i}-5\underset{\sim}{j}\right)$

(C) $\frac{-14}{29}\left(2\underset{\sim}{i}-5\underset{\sim}{j}\right)$

(D) $\frac{-14}{\sqrt{29}}\left(2\underset{\sim}{i}-5\underset{\sim}{j}\right)$

9 The graph of the function $y=\cos^{-1}x+1$ is transformed by being dilated vertically by a scale factor of 2 and then translated up by 3.

What is the equation of the transformed graph?

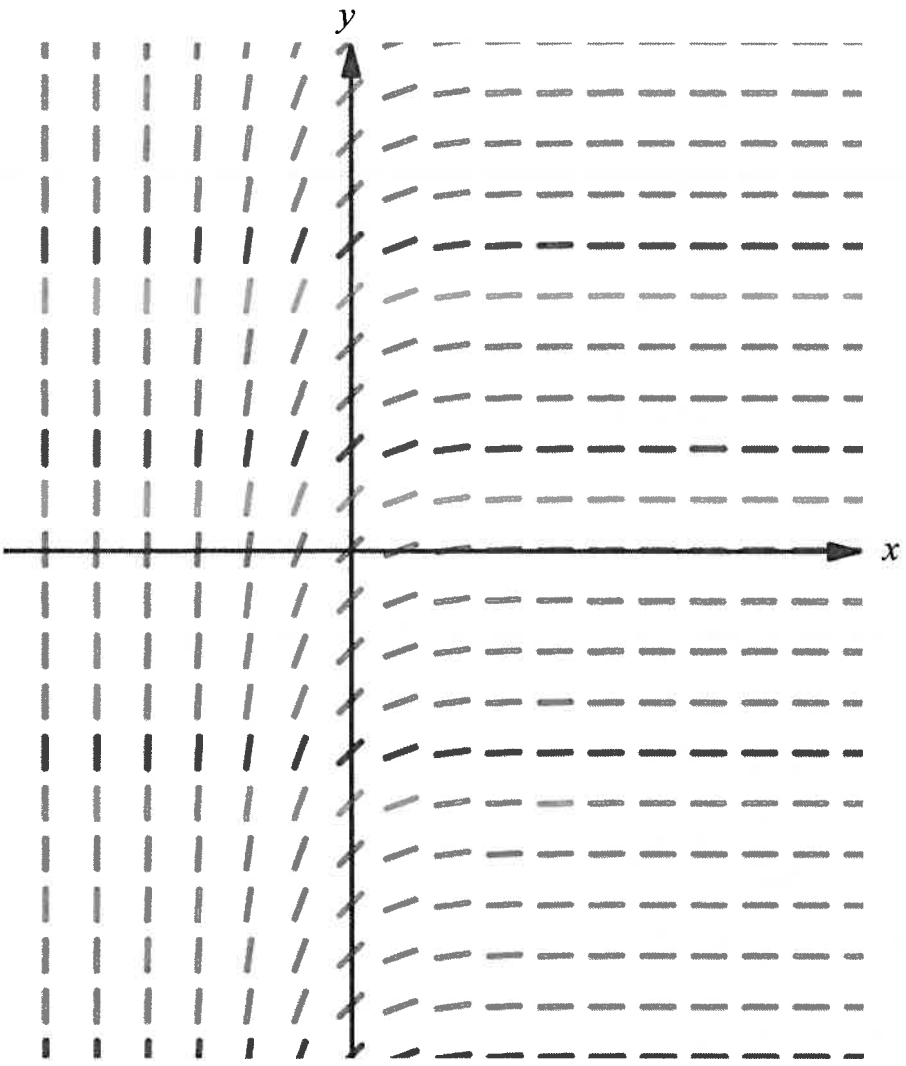
(A) $y=2\cos^{-1}x+4$

(B) $y=2\cos^{-1}x+5$

(C) $y=2\cos^{-1}x+6$

(D) $y=2\cos^{-1}x+8$

10 Which of the following could be the differential equation represented by the slope field below?



(A) $\frac{dy}{dx} = \ln x$

(B) $\frac{dy}{dx} = -\ln x$

(C) $\frac{dy}{dx} = e^{x}$

(D) $\frac{dy}{dx} = e^{-x}$

End of Section I

Section II

60 marks
Attempt Questions 11 - 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $\frac{2x}{x+3} \leq 1$. **3**

(b) The polynomial $P(x) = 2x^3 + kx^2 - 1$ is divided by $x+2$ and the remainder is 7. **2**

Find the value of k.

(c) A luxury bed and breakfast has 6 standard and 5 executive rooms. On any day the probability that a standard room is unoccupied is 6.4% and the probability that an executive room is unoccupied is 13.1%. **3**

On any random day, find the probability that 2 standard rooms and 2 executive rooms are unoccupied.

(d) A committee of five is to be chosen from four men and six women.

(i) How many committees are possible if there are no restrictions? **1**

(ii) How many committees are possible if the majority of members are to be women? **2**

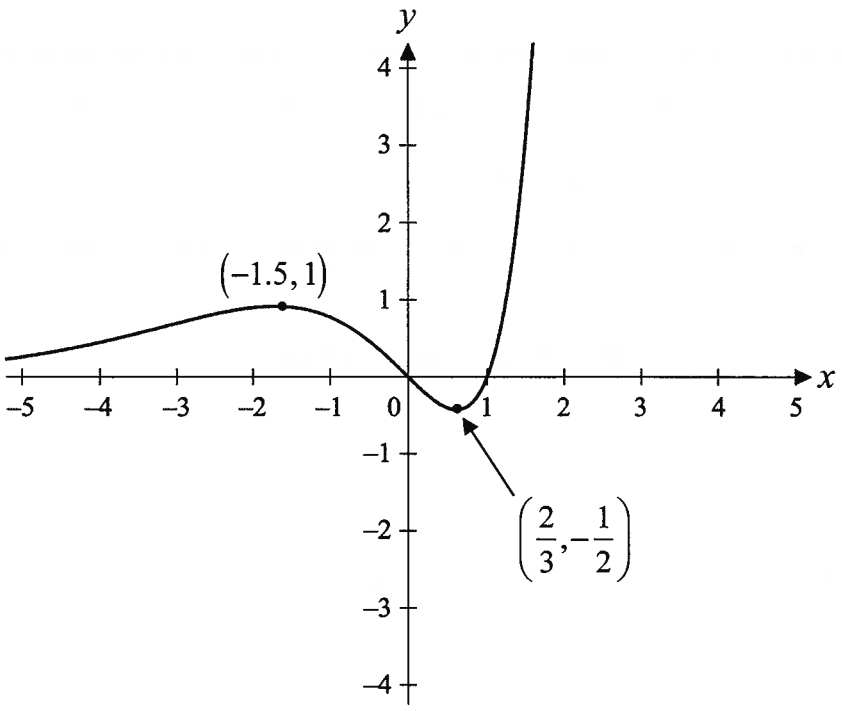
(e) Find the exact magnitude and direction (to the nearest degree) of vector $\overrightarrow{AB}$ where **2**
$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

(f) Find the value of x if the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + x\underset{\sim}{j}$ are perpendicular. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $y = f(x)$.



Draw a separate half-page diagram for each of the following, showing all asymptotes and intercepts.

(i) $y^2 = f(x)$ **2**

(ii) $y = x f(x)$ **2**

(b) Solve $\sin x + \sqrt{3}\cos x = 1$ for $x \in [0, 2\pi]$. **3**

(c) Find $\int_0^{\frac{4}{3}} \frac{dx}{16+9x^2}$. **3**

Question 12 continues on page 9

Question 12 (continued)

(d) Use the substitution $u = x^3 - 3$ to find the exact volume of the solid formed when the curve $y = x\left(x^3 - 3\right)^3$ is rotated about the x–axis between $x = 0$ and $x = 1$. **3**

(e) The relationship between the volume, V, measured in cm^3, of a balloon and the internal air pressure, P, measured in grams/cm^2, is given by $PV = 45\ 000$. **2**

The volume of a balloon is increasing at a rate of 100 cm^3s^{-1}. Find the rate of change of the air pressure inside the balloon when the volume is 4 000 cm^3.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Find the Cartesian equation for the curve with parametric equations 2

$$x = 1 + 2\cos 2t \quad \text{and} \quad y = 2 + 2\sin 2t.$$

(b) Jerry stands at the top of a building 30 m tall and throws a ball with a velocity of 15 m s^{-1} at an angle of $45°$ to the horizontal. The ball eventually reaches the ground.

(i) Derive the vector displacement of the ball in the form $\underset{\sim}{s} = x\underset{\sim}{i} + y\underset{\sim}{j}$. You may assume that $g = 10 \text{ m s}^{-2}$. 3

(ii) Find the time taken for the ball to reach the ground. 1

(c) (i) The functions f and g are defined by $f(x) = \sqrt{2x-1}$ and $g(x) = x + 2$. Find the inverse of the composite function $f \circ g$. 2

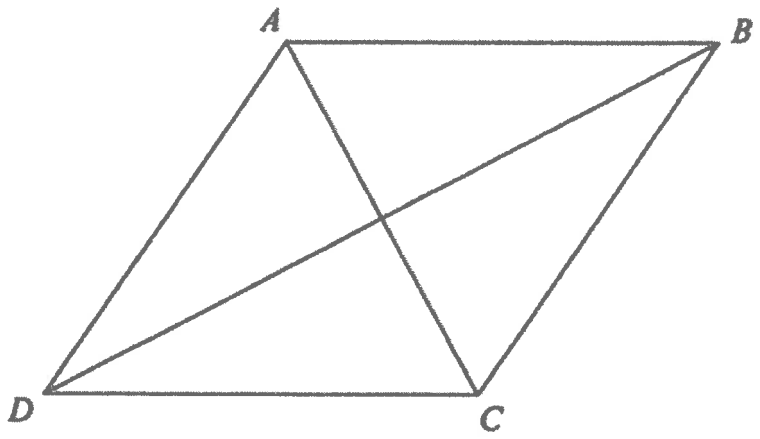
(ii) Sketch the graph of $y = f^{-1}(g(x))$. 2

(d) Prove using mathematical induction that $7^n - 3^n$ is divisible by 4 for $n \geq 1$. 3

Question 13 continues on page 11

Question 13 (continued)

(e) In the diagram $ABCD$ is a parallelogram. If $AB = BC$, prove that the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular. **2**



End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove that $\frac{d}{dx}\left(x\cos^{-1}x\right)=\cos^{-1}x-\frac{x}{\sqrt{1-x^2}}$. **1**

(ii) Hence use the substitution $x=\sin\theta$ to show that **3**

$$\int_0^{\frac{1}{2}}\cos^{-1}x\,dx=\frac{\pi}{6}+\left(1-\frac{\sqrt{3}}{2}\right).$$

(b) Solve the equation $\sec x\,\frac{dy}{dx}=\frac{e^{\sin x}}{y}$, given $x=0$ when $y=0$. **3**

(c) The population, P, of kangaroos in a reservation is modelled by the logistic equation $\frac{dP}{dt}=2P\left(5-\frac{P}{10000}\right)$ where t is measured in days. It is estimated that initially the population of the kangaroos is 15% of the carrying capacity of the reservation.

(i) What is the initial population of the kangaroos? **2**

(ii) What is the population of the kangaroos when the rate of increase is a maximum? **1**

Question 14 continues on page 13

Question 14 (continued)

(d) A standard quality check found that out of 200 light globes, 6 had a defect.

(i) If the sample proportion is approximately normally distributed, show that the mean and sample standard deviation are 0.03 and 0.01206 respectively. **2**

(ii) Find the z–score if 4 light globes in this sample are defective. **1**

(iii) Use the section of the table showing the values of $P(Z < z)$ to find the probability that the number of defective light globes in this sample was between 4 and 5. **2**

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7258	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7518	0.7549
0.7	0.7580	0.7612	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7996	0.8023	0.8051	0.8079	0.8106	0.8133

End of Paper

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EXAMINERS

John Abi-Elias (Convenor)	MLC School, Burwood
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John Wheatley	St Patrick's College, Strathfield
Vito Zurlo	St Scholastica's College, Glebe Point

Additional Disclaimer

Users of CSSA Trial HSC Exams are advised that due to changing NESA examination policies it cannot be assumed that CSSA Exams and NESA Exams will from year to year always fully align with respect to either or both the format and content of examination questions. Candidates for HSC exams and their teacher should anticipate a dynamic assessment environment.

Student Number: Teacher's Name:

Mathematics Extension 1 Replacement Questions for Trial HSC Examination

Replacement Question 1 on page 2

12 people are running for election as school captain. There are 109 people voting.

What is the least number of votes needed for someone to win the election?

(A) 9

(B) 10

(C) 11

(D) 12

Replacement Question 13(c)(i) on Page 10

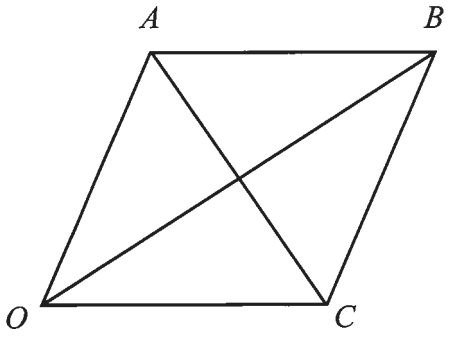
(c) (i) The functions f and g are defined by $f(x)=\sqrt{2x-1}$ and $g(x)=x+2$. Find the inverse of the composite function $f(g(x))$. **2**

(ii) Sketch the graph of $y=f^{-1}(g(x))$. **2**

Replacement Question 13(e) on Page 11

(e) In the diagram $OABC$ is a parallelogram. Given $OA = OC$. Let $\overrightarrow{OA}=\underset{\sim}{a}$ and $\overrightarrow{OC}=\underset{\sim}{b}$. **2**

Prove that the vectors, $\overrightarrow{OB}$ and $\overrightarrow{AC}$ are perpendicular.



Replacement Question 14(a)(ii) on Page 12

(a) (i) Prove that $\frac{d}{dx}\left(x\cos^{-1}x\right)=\cos^{-1}x-\frac{x}{\sqrt{1-x^2}}$. **1**

(ii) Hence use the substitution $x=\sin\theta$, or otherwise, to show that **3**

$$\int_0^{\frac{1}{2}}\cos^{-1}x\,dx=\frac{\pi}{6}+\left(1-\frac{\sqrt{3}}{2}\right).$$

CSSA

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2020 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION
MATHEMATICS EXTENSION 1 - MARKING GUIDELINES

Section I
10 Marks
Multiple-choice Answer Key

Question	Answer
1	B
2	A
3	A
4	D
5	B
6	A
7	C
8	C
9	B
10	D

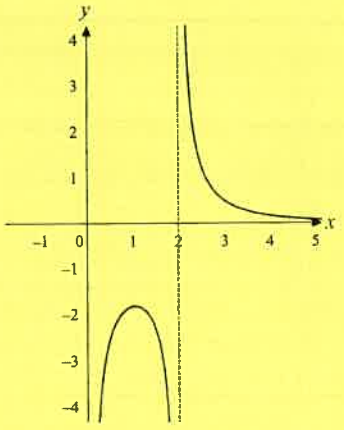
Question 1 (1 mark)
Outcomes Assessed: ME11-5
Targeted Performance Bands: E2

Solution	Mark
Using the pigeonhole principle. $\frac{110}{12} = 9\frac{1}{6}$ $\therefore$ if there are 110 students, the minimum number of students that have a birthday in at least one month is 10. Hence (B).	1

Question 2 (1 mark)
Outcomes Assessed: ME11-2
Targeted Performance Bands: E2

Solution	Mark
If a polynomial $P(x)$ has a triple root at $x=-2$ it can be written as: $P(x)=(x+2)^3 Q(x)$ $P'(x)=3(x+2)^2 Q(x)+(x+2)^3 Q'(x)$ $=(x+2)^2\left[3Q(x)+(x+2)Q'(x)\right]$ $\therefore (x+2)^2$ is a factor of $P'(x)$ Hence (A).	1

Question 3 (1 mark)
Outcomes Assessed: ME11-1
Targeted Performance Bands: E2-E3

Solution	Mark
at $x=0^+$, $\dfrac{1}{f(x)}\to-\infty$ at $x=2^-$, $\dfrac{1}{f(x)}\to-\infty$ at $x=2^+$, $\dfrac{1}{f(x)}\to\infty$ at $x=1$, $\dfrac{1}{f(x)}=-2$ Hence (A) 	1

Question 4 (1 mark)
Outcomes Assessed: ME11-3
Targeted Performance Bands: E2-E3

Solution	Mark
let $t = \tan\frac{\theta}{2}$ $\therefore \cos\theta = \frac{1-t^2}{1+t^2}$ and $\sin\theta = \frac{2t}{1+t^2}$ $\frac{\cos\theta - 1}{2\sin\theta} = \frac{\frac{1-t^2}{1+t^2} - 1}{2 \times \frac{2t}{1+t^2}}$ $= \frac{1-t^2-\left(1+t^2\right)}{4t}$ $= \frac{-2t^2}{4t}$ $= \frac{-t}{2}$ Hence (D)	1

Question 5 (1 mark)
Outcomes Assessed: ME11-5
Targeted Performance Bands: E2-E3

Solution	Mark
$n(\text{two Os separated}) = n(\text{unrestricted}) - n(\text{two Os together})$ $= \frac{6!}{2!} - 5!$ $= 240$ ways Hence (B)	1

Question 6 (1 mark)
Outcomes Assessed: ME12-5
Targeted Performance Bands: E3

Solution	Mark
The variance of a Bernoulli distribution is given by: $\sigma^2 = p(1-p)$ $= 0.8(1-0.8)$ $= 0.16$ Hence (A)	1

Question 7 (1 mark)
Outcomes Assessed: ME12-4
Targeted Performance Bands: E3

Solution	Mark
$\cos^2 x - \sin^2 x = \cos 2x$ $\int \sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x)\, dx$ $= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + c$ $= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$ $\therefore \int \left(\sin^2 x + x^2\right) dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + \frac{x^3}{3} + c$ Hence (C)	1

Question 8 (1 mark)
Outcomes Assessed: ME12-2
Targeted Performance Bands: E3

Solution	Mark
$\underset{\sim}{a} = 2\underset{\sim}{i} - 5\underset{\sim}{j},\ \underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ $\text{proj}_{\underset{\sim}{a}}\underset{\sim}{b} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{a}\rvert^2}\underset{\sim}{a}$ $= \frac{2\times 3 + (-5\times 4)}{\left(\sqrt{2^2+5^2}\right)^2}\left[2\underset{\sim}{i} - 5\underset{\sim}{j}\right]$ $= \frac{-14}{29}\left[2\underset{\sim}{i} - 5\underset{\sim}{j}\right]$ Hence (C)	1

Question 9 (1 mark)
Outcomes Assessed: ME11-3
Targeted Performance Bands: E3-E4

Solution	Mark
The graph of $y = \cos^{-1} x + 1$ is dilated vertically by a scale factor of 2 $y = 2\left(\cos^{-1} x + 1\right)$ $= 2\cos^{-1} x + 2$ The graph of $y = 2\cos^{-1} x + 2$ is then shifted up 3 units $y = 2\cos^{-1} x + 2 + 3$ $= 2\cos^{-1} x + 5$ Hence (B)	1

Question 10 (1 mark)
Outcomes Assessed: ME12-4
Targeted Performance Bands: E3-E4

Solution	Mark
The slope field follows the following pattern: as $x \to -\infty, \frac{dy}{dx} \to \infty$ for $x \in (-\infty, \infty), \frac{dy}{dx} > 0$ and decreasing as $x \to \infty, \frac{dy}{dx} \to 0$ $\therefore \frac{dy}{dx} = e^{-x}$ Hence (D)	1

Section II
60 marks

Question 11 (15 marks)

11 (a) (3 marks)
Outcomes assessed: ME11-2
Targeted Performance Bands: E2

Criteria	Marks
• Correct solution	3
• Multiplies both sides of the inequality by $(x+3)^2$ or equivalent merit	2
• Recognises that $x=-3$ cannot be part of solution	1

Sample Answer:

$\dfrac{2x}{x+3} \le 1$ Note $x \neq -3$

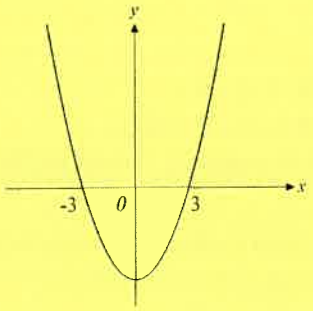
$2x(x+3) \le (x+3)^2$

$2x(x+3)-(x+3)^2 \le 0$

$(x+3)\big(2x-(x+3)\big) \le 0$

$(x+3)(x-3) \le 0$

From graph, solution is $-3 < x \le 3$.



11 (b) (2 marks)
Outcomes Assessed: ME11-2
Targeted Performance Bands: E2

Criteria	Marks
• Correct solution	2
• Attempts to use the remainder theorem	1

Sample Answer:

$P(x) = 2x^3 + kx^2 - 1$

Given $P(-2) = 7$

$7 = 2(-2)^3 + k(-2)^2 - 1$

$7 = -17 + 4k$

$\therefore k = 6$

11 (c) (3 marks)

Outcomes Assessed: ME12-5

Targeted Performance Bands: E2-E3

Criteria	Marks
• Correct solution	3
• Two correct binomial probabilities	2
• Attempted use of binomial probability	1

Sample Answer:

Standard Room: $X \sim B(6, 0.064)$

$$P(X=2) = {}^{6}C_{2}(0.064)^{2}(0.936)^{4}$$
$$= 0.0471 \quad (4 \text{ dp})$$

Executive Room: $X \sim B(5, 0.131)$

$$P(X=2) = {}^{5}C_{2}(0.131)^{2}(0.869)^{3}$$
$$= 0.1126 \quad (4 \text{ dp})$$

$$P(2 \text{ Standard and } 2 \text{ Executive unoccupied}) = 0.0471 \times 0.1126$$
$$= 0.0053 \quad (4 \text{ dp})$$

11 (d) (i) (1 mark)

Outcomes Assessed: ME11-5

Targeted Performance Bands: E2-E3

Criteria	Mark
• Correct solution	1

Sample Answer:

${}^{10}C_{5} = 252$

$\therefore$ there are 252 possible committees that can be formed with no restrictions.

11 (d)(ii) (2 marks)

Outcomes Assessed: ME11-5

Targeted Performance Bands: E2-E3

Criteria	Marks
• Correct solution	2
• Makes some progress towards solution	1

Sample Answer:

Majority of women can be formed with 5 women, 4 women or 3 women.

$${}^6C_4 \times {}^4C_1 = 60$$

$${}^6C_3 \times {}^4C_2 = 120$$

$${}^6C_5 \times {}^4C_0 = 6$$

$\therefore$ there are 186 possible committees that can be formed if the majority of members are women.

11 (e) (2 marks)

Outcomes Assessed: ME12-2

Targeted Performance Bands: E2-E3

Criteria	Marks
• Correct solution	2
• Correct magnitude or direction	1

Sample Answer:

$$\left|\overrightarrow{AB}\right| = \sqrt{(3-2)^2 + (-1-1)^2}$$

$$= \sqrt{5}$$

$$\tan\theta = \frac{-1-1}{3-1}$$

$$\theta = \tan^{-1}(-2)$$

$$= 297° \text{ (nearest degree)}$$

11 (f) (2 marks)

Outcomes Assessed: ME12-2

Targeted Performance Bands: E2-E3

Criteria	Marks
• Correct solution	2
• Attempts to use the dot product to calculate the value of x	1

Sample Answer:

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular then $\underset{\sim}{a}.\underset{\sim}{b}=0$

$2\times(-3)+(3)\times x=0$

$\therefore x=2$

Question 12 (15 marks)

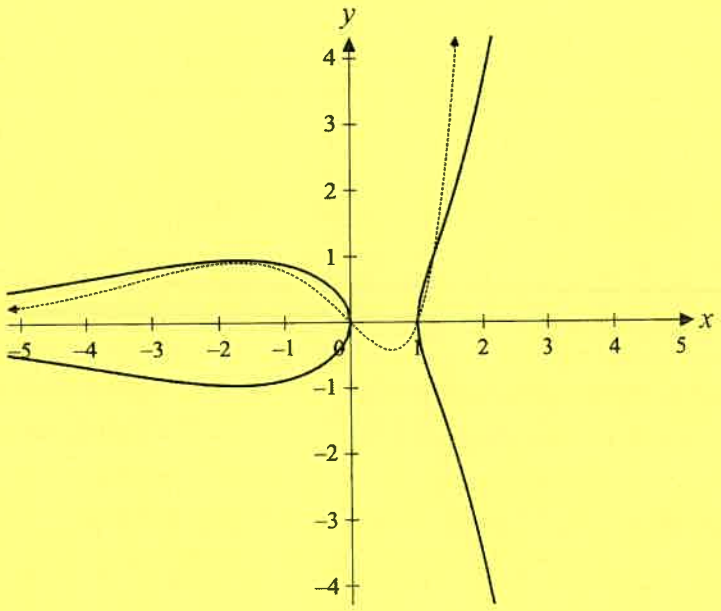
12 (a)(i) (2 marks)

Outcomes assessed: ME11-2

Targeted Performance Bands: E2-E3

Criteria	Marks
• Correct solution	2
• Shows some features of the graph of $y^2 = f(x)$ or the correct graph of $y = \sqrt{f(x)}$	1

Sample Answer:



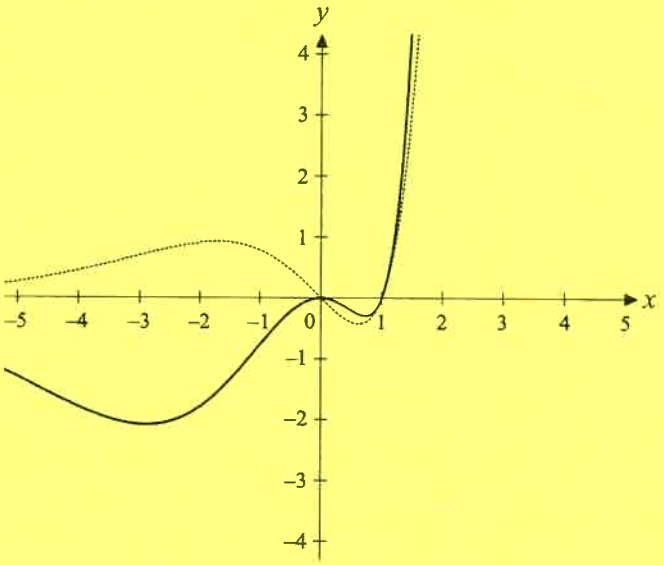
12 (a)(ii) (2 marks)

Outcomes assessed: ME11-2

Targeted Performance Bands: E2-E3

Criteria	Marks
• Correct solution	2
• Shows some features of the graph of $y = x\,f(x)$	1

Sample Answer:



12 (b) (3 marks)

Outcomes Assessed: ME12-3

Targeted Performance Bands: E3

Criteria	Marks
• Correct solution	3
• Finds correct values of x outside of the specified domain	2
• Forms a correct equation using the auxiliary method	1

Sample Answer:

$$\sin x + \sqrt{3}\cos x = 2\sin\left(x + \frac{\pi}{3}\right)$$

$\therefore \sin x + \sqrt{3}\cos x = 1$ becomes

$$2\sin\left(x + \frac{\pi}{3}\right) = 1$$

$$\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$

$$\therefore x = \frac{\pi}{2}, \frac{11\pi}{6} \text{ for } x \in [0, 2\pi]$$

12 (c) (3 marks)

Outcomes assessed: ME12-4

Targeted Performance Bands: E3

Criteria	Marks
• Correct solution	3
• Correct integration	2
• Attempts to form an integral involving $\tan^{-1} x$	1

Sample Answer:

$$\int_0^{\frac{4}{3}} \frac{dx}{16+9x^2}$$

$$= \frac{1}{3}\int_0^{\frac{4}{3}} \frac{3dx}{16+(3x)^2}$$

$$= \frac{1}{4}\times\frac{1}{3}\left[\tan^{-1}\frac{3x}{4}\right]_0^{\frac{4}{3}}$$

$$= \frac{1}{12}\left[\tan^{-1}\left(\frac{3}{4}\times\frac{4}{3}\right)-\tan^{-1}(0)\right]$$

$$= \frac{1}{12}\tan^{-1}1$$

$$= \frac{\pi}{48}$$

12 (d) (3 marks)

Outcomes Assessed: ME12-4

Targeted Performance Bands: E3

Criteria	Marks
• Correct solution	3
• Correct substitution and change of variable or equivalent merit	2
• Forms correct integral	1

Sample Answer:

$$V = \pi \int_0^1 y^2 \, dx$$

$$= \pi \int_0^1 x^2 \left(x^3 - 3\right)^6 dx$$

$$= \pi \int_0^1 x^2 \left(x^3 - 3\right)^6 dx$$

$$= \pi \int_{-3}^{-2} u^6 . \frac{1}{3} du$$

$$= \frac{\pi}{3}\left[\frac{u^7}{7}\right]_{-3}^{-2}$$

$$= \frac{\pi}{3}\left[\frac{2059}{7}\right]$$

$$= \frac{2059\pi}{21} \text{ units}^3$$

$u = x^3 - 3$

$\frac{du}{dx} = 3x^2$

$\frac{1}{3} du = x^2 dx$

$x = 1, u = -2$

$x = 0, u = -3$

12 (e) (2 marks)

Outcomes Assessed: ME11-4

Targeted Performance Bands: E3

Criteria	Marks
• Correct solution	2
• Obtains expression for $\frac{dP}{dt}$	1

Sample Answer:

$$\frac{dV}{dt} = 100$$

$$\frac{dP}{dt} = \frac{dV}{dt} \times \frac{dP}{dV}$$

$$PV = 45000 \rightarrow P = \frac{45000}{V}$$

$$\frac{dP}{dV} = \frac{-45000}{V^2}$$

$$\therefore \frac{dP}{dt} = 100 \times \frac{-45000}{V^2}$$

When $V = 4000$, $\frac{dP}{dt} = 100 \times \frac{-45000}{4000^2}$

$= -0.28$

$\therefore$ the air pressure is decreasing at a rate of 0.28 g/cm^2/s.

Question 13 (15 marks)

13 (a) (2 marks)
Outcomes assessed: ME11-2
Targeted Performance Bands: E3

Criteria	Marks
• Correct solution	2
• Attempts to eliminate parameter by squaring both equations	1

Sample Answer:

$x = 1 + 2\cos 2t$ and $y = 2 + 2\sin 2t$

$x - 1 = 2\cos 2t \qquad (1)$

$y - 2 = 2\sin 2t \qquad (2)$

$(1)^2 + (2)^2$ gives:

$$(x-1)^2 + (y-2)^2 = 4\cos^2 2t + 4\sin^2 2t$$
$$= 4\left(\cos^2 2t + \sin^2 2t\right)$$
$$= 4$$

$\therefore$ the Cartesian equation is $(x-1)^2 + (y-2)^2 = 4$

13 (b)(i) (3 marks)

Outcomes Assessed: ME12-2

Targeted Performance Bands: E4

Criteria	Marks
• Correct solution	3
• Correct integration to displacement	2
• Correct components of velocity	1

Sample Answer:

Find the components of displacement

$\ddot{y} = -g$

$\dot{y} = -gt + c_1$

when $t = 0,\ \dot{y} = \dfrac{15\sqrt{2}}{2} \quad \therefore c_1 = \dfrac{15\sqrt{2}}{2}$

$\therefore \dot{y} = -gt + \dfrac{15\sqrt{2}}{2}$

$y = -\dfrac{gt^2}{2} + \dfrac{15\sqrt{2}}{2}t + c_2$

when $t = 0,\ y = 30 \ \therefore c_2 = 30$

$\therefore y = -\dfrac{gt^2}{2} + \dfrac{15\sqrt{2}}{2}t + 30$

$\therefore y = -5t^2 + \dfrac{15\sqrt{2}}{2}t + 30 \quad$ using $g = 10$

$\therefore \underset{\sim}{s} = \left(\dfrac{15\sqrt{2}}{2}t\right)\underset{\sim}{i} + \left(-5t^2 + \dfrac{15\sqrt{2}}{2}t + 30\right)\underset{\sim}{j}$

$\ddot{x} = 0$

$\dot{x} = c_3$

when $t = 0,\ \dot{x} = \dfrac{15\sqrt{2}}{2} \quad \therefore c_3 = \dfrac{15\sqrt{2}}{2}$

$\therefore \dot{x} = \dfrac{15\sqrt{2}}{2}$

$x = \dfrac{15\sqrt{2}}{2}t + c_4$

when $t = 0,\ x = 0 \ \therefore c_4 = 0$

$\therefore x = \dfrac{15\sqrt{2}}{2}t$

13 (b)(ii) (1 mark)

Outcomes Assessed: ME12-2

Targeted Performance Bands: E3

Criteria	Mark
• Correct solution	1

Sample Answer:

For the ball to reach the ground the component of $\underset{\sim}{j}$ must equal zero.

$$-5t^2 + \frac{15\sqrt{2}}{2}t + 30 = 0$$

$$t = \frac{-\frac{15\sqrt{2}}{2} \pm \sqrt{\left(-\frac{15\sqrt{2}}{2}\right)^2 - 4(-5)(30)}}{2(-5)}$$

$$= \frac{-\frac{15\sqrt{2}}{2} \pm \sqrt{712.5}}{-10}$$

$= 3.7$ seconds (1 dp) taking $t > 0$

13 (c)(i) (2 marks)

Outcomes Assessed: ME11-1

Targeted Performance Bands: E3

Criteria	Marks
• Correct solution	2
• Correct equation of f^{-1} without specifying domain	1

Sample Answer:

$$f \circ g = \sqrt{2(x+2)-1}$$

$$= \sqrt{2x+3}$$

$(f \circ g)^{-1}$ is given by rearranging:

$$x = \sqrt{2y+3}$$

$$\therefore y = \frac{x^2-3}{2}, \text{ for } x \geq 0$$

13 (c)(ii) (2 marks)

Outcomes Assessed: ME11-1

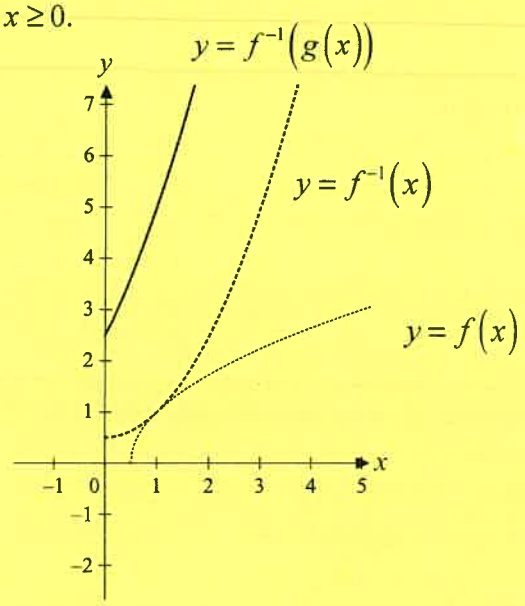
Targeted Performance Bands: E3

Criteria	Marks
• Correct solution	2
• Correct graph without correct domain	1

Sample Answer:

The domain of $f^{-1}(x)$ is $x \geq 0$

$\therefore$ the domain of $f^{-1}(g(x))$ is also $x \geq 0$.



13 (d) (3 marks)

Outcomes Assessed: ME12-1

Targeted Performance Bands: E3-E4

Criteria	Marks
• Correct solution	3
• Correctly sets up the statement that needs to be proved	2
• Shows that $P(n)$ is true for $n=1$	1

Sample Answer:

Let $P(n)$ be the proposition that 7^n-3^n is divisible by 4 for $n\geq 1$.

When $n=1$, LHS $=7^1-3^1=4$

$=4\times 1$

$\therefore P(n)$ is true when $n=1$.

Assume that $P(n)$ is true when $n=k$.

i.e. assume that $7^k-3^k=4M$, where $M\in\mathbb{Z}^+$

Required to prove that $P(n)$ is true when $n=k+1$.

i.e. required to prove that $7^{k+1}-3^{k+1}=4P$, where $P\in\mathbb{Z}^+$

$$\begin{aligned} LHS &= 7^{k+1}-3^{k+1} \\ &= 7.7^k-3.3^k \\ &= 7\left(4M+3^k\right)-3.3^k \quad \text{by assumption} \\ &= 7\times 4M+7.3^k-3.3^k \\ &= 7\times 4M+4.3^k \\ &= 4\left(7M+3^k\right) \\ &= 4P, \quad \text{where } P\in\mathbb{Z}^+ \end{aligned}$$

$\therefore$ if $P(k)$ is true then $P(k+1)$ is true.

$\therefore$ by the process of mathematical induction $P(n)$ is true for $n\geq 1$.

13 (e) (2 marks)

Outcomes Assessed: ME12-2

Targeted Performance Bands: E3-E4

Criteria	Marks
• Correct solution	2
• Progress towards solution using the dot product	1

Sample Answer:

let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$

since $ABCD$ is a parallelogram

$\overrightarrow{CD} = -\underset{\sim}{a}$ and $\overrightarrow{DA} = -\underset{\sim}{b}$

$\therefore \overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$ and $\overrightarrow{DB} = \underset{\sim}{a} - \underset{\sim}{b}$

to prove $\overrightarrow{AC}$ and $\overrightarrow{DB}$ are perpendicular we show $\overrightarrow{AC} \cdot \overrightarrow{DB} = 0$

$$\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{a} - \underset{\sim}{b}\right) = \left|\underset{\sim}{a}\right|^2 - \left|\underset{\sim}{b}\right|^2$$

$$= 0 \qquad \text{since } AB = BC$$

$\therefore \overrightarrow{AC}$ and $\overrightarrow{DB}$ are perpendicular

Question 14 (15 marks)

14 (a)(i) (1 mark)
Outcomes assessed: ME12-4
Targeted Performance Bands: E3

Criteria	Mark
• Correct solution	1

Sample Answer:

using the product rule:

$$\frac{d}{dx}\left(x\cos^{-1}x\right)=x\times\frac{-1}{\sqrt{1-x^2}}+1\times\cos^{-1}x$$

$$=\cos^{-1}x-\frac{x}{\sqrt{1-x^2}}$$

14 (a)(ii) (3 marks)

Outcomes assessed: ME12-4

Targeted Performance Bands: E3-E4

Criteria	Marks
• Correct solution	3
• Correctly evaluates one integral	2
• Integration of all terms between $x = 0$ and $x = \frac{1}{2}$	1

Sample Answer:

integrating both sides of $\frac{d}{dx}\left(x\cos^{-1}x\right) = \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$

$$\int_0^{\frac{1}{2}} \frac{d}{dx}\left(x\cos^{-1}x\right)dx = \int_0^{\frac{1}{2}} \cos^{-1}x\,dx - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$$

$$\left[x\cos^{-1}x\right]_0^{\frac{1}{2}} = \int_0^{\frac{1}{2}} \cos^{-1}x\,dx - \int_0^{\frac{\pi}{6}} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta \qquad \text{using } x = \sin\theta$$

$$\left[\frac{1}{2}\cos^{-1}\frac{1}{2} - 0\cos^{-1}0\right] = \int_0^{\frac{1}{2}} \cos^{-1}x\,dx - \int_0^{\frac{\pi}{6}} \frac{\sin\theta}{|\cos\theta|}\cos\theta\,d\theta$$

$$\frac{\pi}{6} = \int_0^{\frac{1}{2}} \cos^{-1}x\,dx - \int_0^{\frac{\pi}{6}} \sin\theta\,d\theta \qquad \sqrt{\cos^2\theta} > 0 \text{ since } 0 \le \theta \le \frac{\pi}{2}$$

$$\frac{\pi}{6} = \int_0^{\frac{1}{2}} \cos^{-1}x\,dx - \left(-\frac{\sqrt{3}}{2} + 1\right)$$

$$\therefore \int_0^{\frac{1}{2}} \cos^{-1}x\,dx = \frac{\pi}{6} + \left(1 - \frac{\sqrt{3}}{2}\right)$$

14 (b) (3 marks)

Outcomes assessed: ME12-4

Targeted Performance Bands: E3-E4

Criteria	Marks
• Correct solution	3
• Correct integration	2
• Correctly separates the differential equation	1

Sample Answer:

$$\sec x \frac{dy}{dx} = \frac{e^{\sin x}}{y}$$

$$y\,dy = \cos x e^{\sin x}\,dx$$

integrating both sides:

$$\int y\,dy = \int \cos x e^{\sin x}\,dx$$

$$\frac{y^2}{2} = e^{\sin x} + c \qquad \text{since } \frac{d}{dx}\left(e^{\sin x}\right) = \cos x e^{\sin x}$$

given $x = 0,\ y = 0$

$$\frac{0^2}{2} = e^{\sin 0} + c$$

$$c = -1$$

$$\therefore y^2 = 2e^{\sin x} - 2$$

14 (c)(i) (2 marks)

Outcomes assessed: ME12-4

Targeted Performance Bands: E3-E4

Criteria	Marks
• Correct solution	2
• Attempts to let $\frac{dP}{dt}=0$	1

Sample Answer:

The carrying capacity is found by letting $\frac{dP}{dt}=0$

i.e. $P=0$ or $5-\frac{P}{10000}=0$

$\therefore P=50000$ is the carrying capacity.

The initial population of kangaroos is 15% of the carrying capacity.

$\therefore P_0=0.15\times 50000$

$=7500$

14 (c)(ii) (1 mark)

Outcomes assessed: ME11-4

Targeted Performance Bands: E3-E4

Criteria	Mark
• Correct solution	1

Sample Answer:

The rate of increase is a maximum when the population is half the carrying capacity.

$\therefore \frac{dP}{dt}$ is a maximum when $P=25000$ kangaroos.

14 (d)(i) (2 marks)

Outcomes assessed: ME12-5

Targeted Performance Bands: E3-E4

Criteria	Marks
• Correct solution	2
• Finds the correct value of μ or σ.	1

Sample Answer:

$$\hat{p} = \frac{6}{200}$$
$$= 0.03$$

$$\mu_{proportion} = \hat{p}$$
$$= 0.03$$

$$\sigma_{proportion} = \frac{\sigma}{n}$$
$$= \frac{\sqrt{np(1-p)}}{n}$$
$$= \frac{\sqrt{200 \times 0.03 \times 0.97}}{200}$$
$$= 0.01206 \text{ (5 decimal places)}$$

14 (d)(ii) (1 mark)

Outcomes assessed: ME12-5

Targeted Performance Bands: E3

Criteria	Mark
• Correct solution	1

Sample Answer:

For 4 defective globes $\hat{p} = \dfrac{4}{200}$

$= 0.02$

$$z = \frac{0.02 - 0.03}{0.01206}$$

$= -0.829187...$

$= -0.83$ (2 decimal places)

14 (d)(iii) (2 marks)

Outcomes assessed: ME12-5

Targeted Performance Bands: E3-E4

Criteria	Marks
• Correct solution	2
• Attempts use of the table of $P(Z < z)$ values	1

Sample Answer:

To find $P(4 < X < 5)$ we need to calculate the $z-$score of 5 defective globes

$$x = \frac{5}{200} = 0.025$$

$$z = \frac{0.025 - 0.03}{0.01206}$$

$= -0.41459...$

$= -0.41$ (2 decimal places)

$$\therefore P(4 < X < 5) = P(-0.83 < z < -0.41)$$

$$= P(z < -0.41) - P(z < -0.83)$$

$$= \left(1 - P(z < 0.41)\right) - \left(1 - P(z < 0.83)\right)$$

$$= (1 - 0.6591) - (1 - 0.7967)$$

$$= 0.1376$$