NSW Education Standards Authority

2022 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–15)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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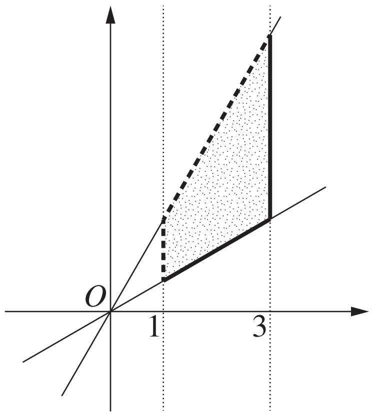
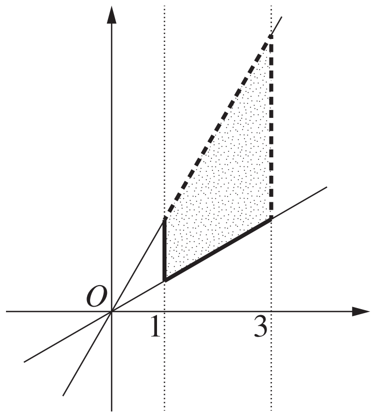
1 Let R be the region in the complex plane defined by $1 < \text{Re}(z) \le 3$ and $\dfrac{\pi}{6} \le \text{Arg}(z) < \dfrac{\pi}{3}$.

Which diagram best represents the region R?

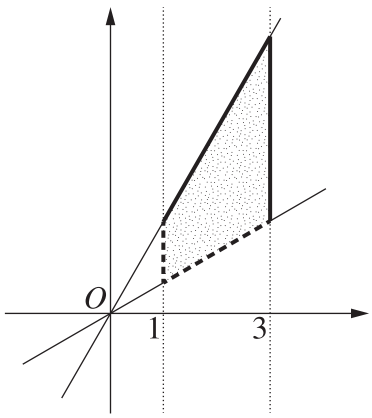
A.



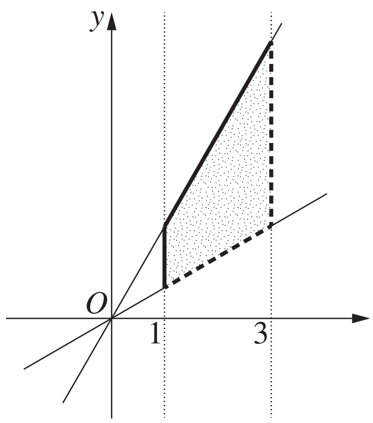
B.



C.



D.



2 The following proof aims to establish that $-4 = 0$.

$$
\begin{array}{llr}
\text{Let} & a = -4 & \\
\Rightarrow & a^2 = 16 \text{ and } 4a + 4 = -12 & \text{Line 1} \\
\Rightarrow & a^2 + 4a + 4 = 4 & \text{Line 2} \\
\Rightarrow & (a+2)^2 = 2^2 & \text{Line 3} \\
\Rightarrow & a + 2 = 2 & \text{Line 4} \\
\Rightarrow & a = 0 &
\end{array}
$$

At which line is the implication incorrect?

A. Line 1

B. Line 2

C. Line 3

D. Line 4

3 Let A, B, P be three points in three-dimensional space with $A \neq B$.

Consider the following statement.

If P is on the line AB, then there exists a real number λ such that $\overrightarrow{AP} = \lambda \overrightarrow{AB}$.

Which of the following is the contrapositive of this statement?

A. If for all real numbers λ, $\overrightarrow{AP} = \lambda \overrightarrow{AB}$, then P is on the line AB.

B. If for all real numbers λ, $\overrightarrow{AP} \neq \lambda \overrightarrow{AB}$, then P is not on the line AB.

C. If there exists a real number λ such that $\overrightarrow{AP} = \lambda \overrightarrow{AB}$, then P is on the line AB.

D. If there exists a real number λ such that $\overrightarrow{AP} \neq \lambda \overrightarrow{AB}$, then P is not on the line AB.

4 Of the following expressions, which one need NOT contain a term involving a logarithm in its anti-derivative?

A. $\dfrac{x+2}{x^2+4x+5}$

B. $\dfrac{x+2}{x^2-4x-5}$

C. $\dfrac{x-1}{x^3-x^2+x-1}$

D. $\dfrac{x+1}{x^3-x^2+x-1}$

5 If $\int_a^x f(t)\,dt = g(x)$, which of the following is a primitive of $f(x)g(x)$?

A. $\frac{1}{2}[f(x)]^2$

B. $\frac{1}{2}[f'(x)]^2$

C. $\frac{1}{2}[g(x)]^2$

D. $\frac{1}{2}[g'(x)]^2$

6 It is known that a particular complex number z is NOT a real number.

Which of the following could be true for this number z?

A. $\overline{z} = iz$

B. $\overline{z} = |z^2|$

C. $\mathrm{Re}(iz) = \mathrm{Im}(z)$

D. $\mathrm{Arg}\left(z^3\right) = \mathrm{Arg}(z)$

7 Consider the statement P.

P: For all integers $n \geq 1$, if n is a prime number then $\dfrac{n(n+1)}{2}$ is a prime number.

Which of the following is true about this statement and its converse?

A. The statement P and its converse are both true.

B. The statement P and its converse are both false.

C. The statement P is true and its converse is false.

D. The statement P is false and its converse is true.

8 As a projectile of mass m kilograms travels through air, it experiences a frictional force. The magnitude of this force is proportional to the square of the speed v of the projectile. The constant of proportionality is the positive number k. The position of the particle at time t is denoted by $\begin{pmatrix} x \\ y \end{pmatrix}$. The acceleration due to gravity is $g \text{ m s}^{-2}$.

Based on Newton's laws of motion, which equation models the motion of this projectile?

A. $\begin{pmatrix} 0 \\ -mg \end{pmatrix} + kv\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$

B. $\begin{pmatrix} 0 \\ -mg \end{pmatrix} - kv\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$

C. $\begin{pmatrix} 0 \\ -mg \end{pmatrix} + kv^2\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$

D. $\begin{pmatrix} 0 \\ -mg \end{pmatrix} - kv^2\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$

9 Let A and B be two distinct points in three-dimensional space. Let M be the midpoint of AB.

Let S_1 be the set of all points P such that $\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$.

Let S_2 be the set of all points N such that $\left|\overrightarrow{AN}\right| = \left|\overrightarrow{MN}\right|$.

The intersection of S_1 and S_2 is the circle S.

What is the radius of the circle S?

A. $\dfrac{\left|\overrightarrow{AB}\right|}{2}$

B. $\dfrac{\left|\overrightarrow{AB}\right|}{4}$

C. $\dfrac{\sqrt{3}\left|\overrightarrow{AB}\right|}{2}$

D. $\dfrac{\sqrt{3}\left|\overrightarrow{AB}\right|}{4}$

10 A particle is moving vertically in a resistive medium under the influence of gravity. The resistive force is proportional to the velocity of the particle.

The initial speed of the particle is NOT zero.

Which of the following statements about the motion of the particle is always true?

A. If the particle is initially moving downwards, then its speed will increase.

B. If the particle is initially moving downwards, then its speed will decrease.

C. If the particle is initially moving upwards, then its speed will eventually approach a terminal speed.

D. If the particle is initially moving upwards, then its speed will not eventually approach a terminal speed.

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) Express $\dfrac{3-i}{2+i}$ in the form $x + iy$, where x and y are real numbers. **2**

(b) Evaluate $\displaystyle\int \sin^3 2x \cos 2x \, dx$. **2**

(c) (i) Write the complex number $-\sqrt{3} + i$ in exponential form. **2**

(ii) Hence, find the exact value of $\left(-\sqrt{3} + i\right)^{10}$ giving your answer in the form $x + iy$. **2**

(d) A triangle is formed in three-dimensional space with vertices $A(1, -1, 2)$, $B(0, 2, -1)$ and $C(2, 1, 1)$. **3**

Find the size of $\angle ABC$, giving your answer to the nearest degree.

(e) Let ℓ_1 be the line with equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$. **2**

The line ℓ_2 passes through the point $A(-6, 5)$ and is parallel to ℓ_1.

Find the equation of the line ℓ_2 in the form $y = mx + c$.

(f) Using the substitution $t = \tan \dfrac{x}{2}$, find **3**

$$\int \frac{dx}{1 + \cos x - \sin x}.$$

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) For real numbers a, $b \geq 0$ prove that $\dfrac{a+b}{2} \geq \sqrt{ab}$. **2**

(b) A particle is moving in a straight line with acceleration $a = 12 - 6t$. The particle starts from rest at the origin. **3**

What is the position of the particle when it reaches its maximum velocity?

(c) A particle with mass 1 kg is moving along the x-axis. Initially, the particle is at the origin and has speed u m s^{-1} to the right. The particle experiences a resistive force of magnitude $v + 3v^2$ newtons, where v m s^{-1} is the speed of the particle after t seconds. The particle is never at rest.

(i) Show that $\dfrac{dv}{dx} = -(1 + 3v)$. **1**

(ii) Hence, or otherwise, find x as a function of v. **2**

(d) Using partial fractions, evaluate $\displaystyle\int_2^n \frac{4+x}{(1-x)(4+x^2)}\,dx$, giving your answer in the form $\dfrac{1}{2}\ln\left(\dfrac{f(n)}{8(n-1)^2}\right)$, where $f(n)$ is a function of n. **4**

(e) Given the complex number $z = e^{i\theta}$, show that $w = \dfrac{z^2-1}{z^2+1}$ is purely imaginary. **3**

Question 13 (14 marks) Use the Question 13 Writing Booklet

(a) Prove that for all integers n with $n \geq 3$, if $2^n - 1$ is prime, then n cannot be even. **3**

(b) The numbers a_n, for integers $n \geq 1$, are defined as **4**

$$a_1 = \sqrt{2}$$
$$a_2 = \sqrt{2 + \sqrt{2}}$$
$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \text{ and so on.}$$

These numbers satisfy the relation $a_{n+1}^2 = 2 + a_n$, for $n \geq 1$. (Do NOT prove this.)

Use mathematical induction to prove that $a_n = 2\cos\dfrac{\pi}{2^{n+1}}$, for all integers $n \geq 1$.

(c) Consider the equation $z^5 + 1 = 0$, where z is a complex number.

(i) Solve the equation $z^5 + 1 = 0$ by finding the 5th roots of -1. **2**

(ii) Show that if z is a solution of $z^5 + 1 = 0$ and $z \neq -1$, then $u = z + \dfrac{1}{z}$ is a solution of $u^2 - u - 1 = 0$. **2**

(iii) Hence find the exact value of $\cos\dfrac{3\pi}{5}$. **3**

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) (i) The two non-parallel vectors $\vec{u}$ and $\vec{v}$ satisfy $\lambda\vec{u} + \mu\vec{v} = \vec{0}$ for some real numbers λ and μ. **2**

Show that $\lambda = \mu = 0$.

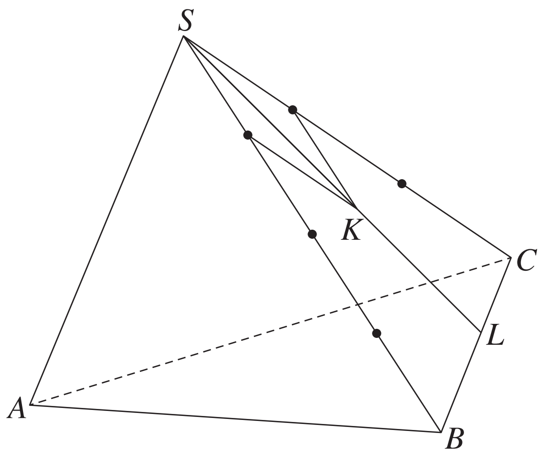
(ii) The two non-parallel vectors $\vec{u}$ and $\vec{v}$ satisfy $\lambda_1\vec{u} + \mu_1\vec{v} = \lambda_2\vec{u} + \mu_2\vec{v}$ for some real numbers $\lambda_1, \lambda_2, \mu_1$ and μ_2. **1**

Using part (i), or otherwise, show that $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$.

The diagram below shows the tetrahedron with vertices A, B, C and S.

The point K is defined by $\overrightarrow{SK} = \frac{1}{4}\overrightarrow{SB} + \frac{1}{3}\overrightarrow{SC}$, as shown in the diagram.

The point L is the point of intersection of the straight lines SK and BC.



(iii) Using part (ii), or otherwise, determine the position of L by showing that $\overrightarrow{BL} = \frac{4}{7}\overrightarrow{BC}$. **2**

(iv) The point P is defined by $\overrightarrow{AP} = -6\overrightarrow{AB} - 8\overrightarrow{AC}$. **2**

Does P lie on the line AL? Justify your answer.

Question 14 continues on page 11

(b) Let $J_n = \int_0^1 x^n e^{-x}\,dx$, where n is a non-negative integer.

(i) Show that $J_0 = 1 - \dfrac{1}{e}$. **1**

(ii) Show that $J_n \le \dfrac{1}{n+1}$. **2**

(iii) Show that $J_n = nJ_{n-1} - \dfrac{1}{e}$, for $n \ge 1$. **2**

(iv) Using parts (i) and (iii), show by mathematical induction, or otherwise, that for all $n \ge 0$, **2**

$$J_n = n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}.$$

(v) Using parts (ii) and (iv) prove that $e = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}$. **1**

End of Question 14

Please turn over

Question 15 (15 marks) Use the Question 15 Writing Booklet

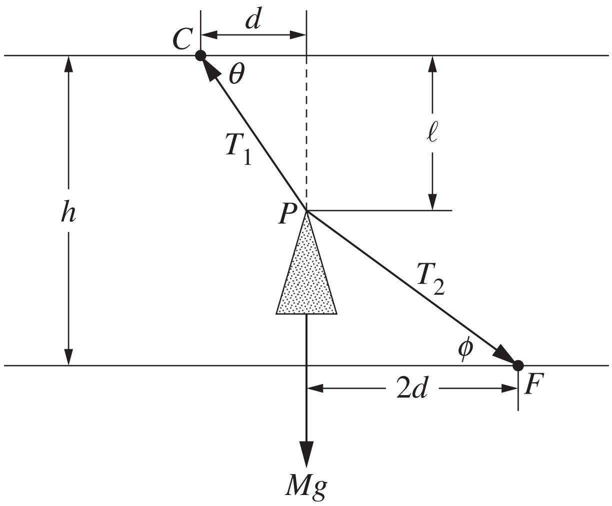
(a) A machine is lifted from the floor of a room using two ropes. The two ropes ensure that the horizontal components of the forces are balanced at all times. It is assumed that at all times the machine moves vertically upwards at a constant velocity.

The machine is located in a room with height h metres.

One of the ropes is attached to the point P on the machine and to the fixed point C on the ceiling of the room. The point C is a distance d metres to the left of P. Let the vertical distance from P to the ceiling be ℓ metres and let θ be the angle this rope makes with the horizontal.

The other rope is attached to the point P and to the fixed point F on the floor of the room. The point F is a distance $2d$ metres to the right of P. Let ϕ be the angle this rope makes with the horizontal.

Let the tension in the first rope be T_1 newtons, the tension in the second rope be T_2 newtons, the mass of the machine be M kilograms and the acceleration due to gravity be g m s^{-2}.



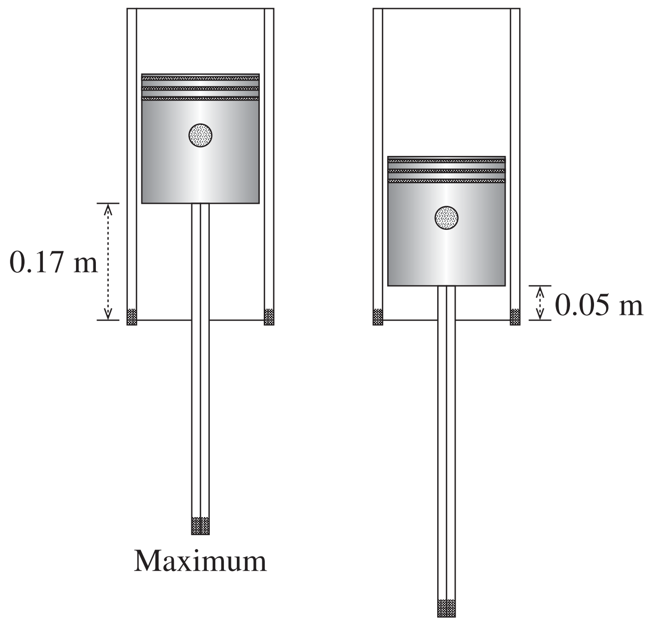
(i) By considering horizontal and vertical components of the forces at P, show that **3**

$$\tan\theta = \tan\phi + \frac{Mg}{T_2\cos\phi}.$$

(ii) Hence, or otherwise, show that the point P cannot be lifted to a position $\frac{2h}{3}$ metres above the floor. **2**

Question 15 continues on page 13

(b) The diagrams show two positions of a single piston in the cylinder chamber of a motorcycle. The piston moves vertically, in simple harmonic motion, between a maximum height of 0.17 metres and a minimum height of 0.05 metres. **3**



The mass of the piston is 0.8 kg. The piston completes 40 cycles per second.

What is the resultant force on the piston, in newtons, that produces the maximum acceleration of the piston? Give your answer correct to the nearest newton.

(c) Using the substitution $x = \tan^2\theta$, evaluate **4**

$$\int_0^1 \sin^{-1}\sqrt{\frac{x}{1+x}}\, dx.$$

(d) The complex number z satisfies $\left| z - \frac{4}{z} \right| = 2$. **3**

Using the triangle inequality, or otherwise, show that $|z| \le \sqrt{5} + 1$.

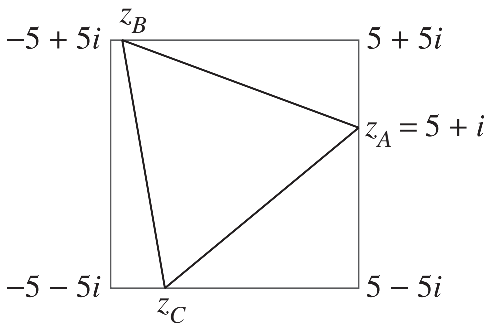
End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

(a) A square in the Argand plane has vertices **4**

$$5 + 5i, \qquad 5 - 5i, \qquad -5 - 5i \qquad \text{and} \qquad -5 + 5i.$$

The complex numbers $z_A = 5 + i$, z_B and z_C lie on the square and form the vertices of an equilateral triangle, as shown in the diagram.



NOT TO SCALE

Find the exact value of the complex number z_B.

(b) A projectile of mass M kg is launched vertically upwards from a horizontal plane with initial speed v_0 m s^{-1} which is less than 100 m s^{-1}. The projectile experiences a resistive force which has magnitude $0.1Mv$ newtons, where v m s^{-1} is the speed of the projectile. The acceleration due to gravity is 10 m s^{-2}. **4**

The projectile lands on the horizontal plane 7 seconds after launch.

Find the value of v_0, correct to 1 decimal place.

Question 16 continues on page 15

(c) It is given that for positive numbers $x_1, x_2, x_3, \ldots, x_n$ with arithmetic mean A,

$$\frac{x_1 \times x_2 \times x_3 \times \cdots \times x_n}{A^n} \le 1. \qquad \text{(Do NOT prove this.)}$$

Suppose a rectangular prism has dimensions a, b, c and surface area S.

(i) Show that $abc \le \left(\dfrac{S}{6}\right)^{\frac{3}{2}}$. **2**

(ii) Using part (i), show that when the rectangular prism with surface area S is a cube, it has maximum volume. **2**

(d) Find all the complex numbers z_1, z_2, z_3 that satisfy the following three conditions simultaneously. **3**

$$\begin{cases} |z_1| = |z_2| = |z_3| \\ z_1 + z_2 + z_3 = 1 \\ z_1 z_2 z_3 = 1 \end{cases}$$

End of paper

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NSW GOVERNMENT

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2022 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}\left(a + b\right)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$\left(x - h\right)^2 + \left(y - k\right)^2 = r^2$$

Financial Mathematics

$$A = P\left(1 + r\right)^n$$

Sequences and series

$$T_n = a + \left(n - 1\right)d$$

$$S_n = \frac{n}{2}\left[2a + \left(n - 1\right)d\right] = \frac{n}{2}\left(a + l\right)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r} = \frac{a\left(r^n - 1\right)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

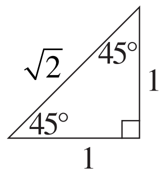
$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

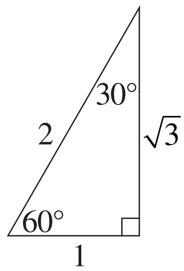
Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$



$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

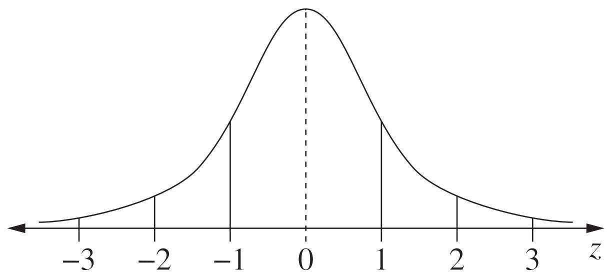
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2} + y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^{2}(x - c)$$

2022 HSC Mathematics Extension 2 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	A
2	D
3	B
4	C
5	C
6	A
7	D
8	B
9	D
10	C

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	2
• Multiplies top and bottom by the conjugate of the denominator, or equivalent merit	1

Sample answer:

$$\frac{3-i}{2+i} = \frac{3-i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{6-2i-3i-1}{4+1}$$

$$= \frac{5-5i}{5}$$

$$= 1-i$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Chooses a suitable substitution, or equivalent merit	1

Sample answer:

$$\int \sin^3 2x \cos 2x \, dx$$

$$= \frac{1}{4}\int 4\sin^3 2x \cos 2x \, dx$$

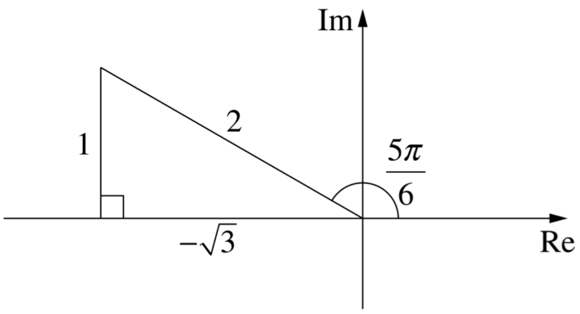
$$= \frac{1}{4}\left(\frac{\sin^4(2x)}{2}\right) + C$$

$$= \frac{1}{8}\sin^4(2x) + C$$

Question 11 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains correct modulus OR • Obtains correct argument	1

Sample answer:



$$-\sqrt{3}+i=2e^{i\frac{5\pi}{6}}$$

Question 11 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Attempts to apply de Moivre's theorem, or equivalent merit	1

Sample answer:

$$\left(-\sqrt{3}+i\right)^{10}=\left(2e^{i\frac{5\pi}{6}}\right)^{10} \qquad \text{from part (i)}$$

$$=2^{10}e^{i\frac{50\pi}{6}}$$

$$=2^{10}e^{i\frac{\pi}{3}}$$

$$=2^{10}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$=2^9+2^9\sqrt{3}i$$

Question 11 (d)

Criteria	Marks
• Provides correct solution	3
• Obtains correct dot product OR attempts to use the cosine rule using 3 correct lengths	2
• Obtains one relevant vector OR obtains one relevant length	1

Sample answer:

$A(1, -1, 2) \qquad B(0, 2, -1) \qquad C(2, 1, 1)$

$$\underset{\sim}{a} = \overrightarrow{BA} = \begin{pmatrix} 1 - 0 \\ -1 - 2 \\ 2 - (-1) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$$

$$\underset{\sim}{b} = \overrightarrow{BC} = \begin{pmatrix} 2 - 0 \\ 1 - 2 \\ 1 - (-1) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \cos\angle ABC &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}| \cdot |\underset{\sim}{b}|} = \frac{2+3+6}{\sqrt{1+9+9} \cdot \sqrt{4+1+4}} \\ &= \frac{11}{3\sqrt{19}} \end{aligned}$$

$$\begin{aligned} \therefore\ \angle ABC &= \cos^{-1}\left(\frac{11}{3\sqrt{19}}\right) \\ &\approx 33^\circ \end{aligned}$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	2
• Finds equation of ℓ_2 in any form OR finds the correct slope of ℓ_2 OR equivalent merit	1

Sample answer:

$$\ell_1: \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\ell_2 \parallel \ell_1$ passes through (–6, 5)

$$\ell_2: \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$x = -6 + 3\mu$$
$$y = 5 + 2\mu$$

$$2x = -12 + 6\mu$$
$$3y = 15 + 6\mu$$

Subtract

$$3y - 2x = 27$$
$$3y = 2x + 27$$
$$y = \frac{2}{3}x + 9$$

Question 11 (f)

Criteria	Marks
• Provides correct solution	3
• Obtains integrand in terms of t in simplest form, or equivalent merit	2
• Correctly replaces dx in terms of t and dt, or equivalent merit	1

Sample answer:

$$t = \tan\frac{x}{2} \qquad \Rightarrow \qquad dx = \frac{2dt}{1+t^2}$$

$$\int \frac{dx}{1+\cos x - \sin x}$$

$$= \int \frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}}$$

$$= \int \frac{2dt}{2-2t}$$

$$= \int \frac{dt}{1-t}$$

$$= -\ln|1-t| + k$$

$$= -\ln\left|1-\tan\frac{x}{2}\right| + k$$

Question 12 (a)

Criteria	Marks
• Provides correct solution	2
• Chooses a suitable strategy	1

Sample answer:

$a,\ b \geq 0$ so $\sqrt{a}$, $\sqrt{b}$ are real

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$

$$a - 2\sqrt{a}\sqrt{b} + b \geq 0$$

$$a + b \geq 2\sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Question 12 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains the displacement of the particle at time t, or equivalent merit	2
• Obtains the position of the particle at time t OR • Finds the time at which the velocity of the particle is a maximum	1

Sample answer:

$\ddot{x} = 12 - 6t$

$t = 0, \quad v = 0, \quad x = 0$

$$\frac{dv}{dt} = 12 - 6t$$

$$v = \int 12 - 6t \, dt$$

$$= 12t - 3t^2 + c$$

$t = 0, \quad v = 0, \quad \Rightarrow \quad c = 0$

$\therefore \quad v = 12t - 3t^2$

$$\frac{dx}{dt} = 12t - 3t^2$$

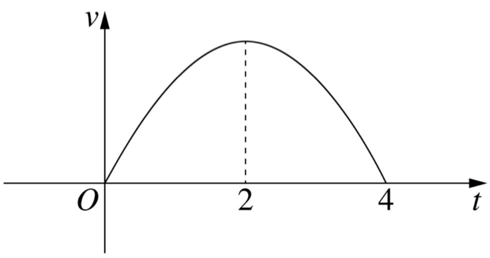
$\therefore \quad x = 6t^2 - t^3 + c$

$t = 0, \quad v = 0, \quad \Rightarrow \quad c = 0$

$\therefore \quad x = 6t^2 - t^3$

<u>Maximum velocity</u>

$v = 12t - 3t^2$



v is maximum when $t = 2$.

$\therefore$ Position at maximum velocity is

$x = 6 \times 2^2 - 2^3$

$= 16$ units to right of origin

Question 12 (c) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$F = ma = 1 \times a = a$$

$$a = -\left(v + 3v^2\right)$$

$$v\frac{dv}{dx} = -v(1+3v) \qquad \text{and } v \neq 0$$

$$\frac{dv}{dx} = -(1+3v)$$

Question 12 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Separates the variables from the DE in part (i), or equivalent merit	1

Sample answer:

$$\frac{dv}{dx} = -(1+3v)$$

$$\int \frac{dv}{1+3v} = -\int dx$$

$$\frac{1}{3}\ln|1+3v| = -x + c$$

When $x = 0$, $v = u$

So $\frac{1}{3}\ln|1+u| = c$

$$\therefore \quad x = \frac{1}{3}\ln|1+3u| - \frac{1}{3}\ln|1+3v|$$

$$= \frac{1}{3}\ln\left(\frac{1+3u}{1+3v}\right)$$

Since $1 + 3v > 0$ and $1 + 3u > 0$

Question 12 (d)

Criteria	Marks
• Provides correct solution	4
• Obtains correct primitive	3
• Obtains the correct partial fraction decomposition, or equivalent merit	2
• Writes the correct general form of the partial fraction decomposition, or equivalent merit	1

Sample answer:

$$\frac{4+x}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{4+x^2}$$

$$4+x = A(4+x^2) + (Bx+C)(1-x)$$

$x=1 \qquad 5=5A, \qquad A=1$

$$\frac{4+x}{(1-x)(4+x^2)} - \frac{1}{1-x} = \frac{4+x-(4+x^2)}{(1-x)(4+x^2)}$$

$$= \frac{x-x^2}{(1-x)(4+x^2)}$$

$$= \frac{x}{4+x^2}$$

$$\int_2^n \frac{4+x}{(1-x)(4+x^2)}\,dx = \int_2^n \frac{1}{1-x} + \frac{x}{4+x^2}\,dx$$

$$= \int_2^n \frac{1}{1-x} + \frac{1}{2}\left(\frac{2x}{4+x^2}\right)dx$$

$$= \left[-\ln|1-x| + \frac{1}{2}\ln|4+x^2|\right]_2^n$$

$$= -\ln|1-n| + \frac{1}{2}\ln|4+n^2| - \frac{1}{2}\ln 8$$

$$= \ln\frac{1}{|n-1|} + \frac{1}{2}\ln|4+n^2| - \frac{1}{2}\ln 8$$

$$= \frac{1}{2}\left(2\ln\frac{1}{|n-1|} + \ln|4+n^2| - \ln 8\right)$$

$$= \frac{1}{2}\ln\left(\frac{4+n^2}{8|n-1|^2}\right)$$

Question 12 (e)

Criteria	Marks
• Provides correct solution	3
• Uses the properties of complex numbers with modulus 1 to simplify their correct fraction, or equivalent merit	2
• Divides the numerator and denominator by a suitable power of z, or equivalent merit	1

Sample answer:

If $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta}$

$$w = \frac{z^2 - 1}{z^2 + 1}$$

$$= \frac{z\left(z - \frac{1}{z}\right)}{z\left(z + \frac{1}{z}\right)}$$

$$= \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$

$$= \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}$$

$$= \frac{2i\sin\theta}{2\cos\theta}$$

$= i\tan\theta$ which is purely imaginary.

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Uses the assumption that $n = 2k$ to factor $2^n - 1$, or equivalent merit	2
• Attempts to use the contrapositive, or equivalent merit	1

Sample answer:

It is equivalent to prove the contrapositive: 'if n is even, then $2^n - 1$ is <u>not</u> prime'.

Assume n is even. With $n \geq 3$, that means $n = 2k$ where k is an integer with $k \geq 2$.

$$2^n - 1 = 2^{2k} - 1 = \left(2^k\right)^2 - 1 = \left(2^k + 1\right)\left(2^k - 1\right)$$

With $k \geq 2$, $2^k + 1 \geq 5$ and $2^k - 1 \geq 3$, so $2^n - 1$ has two proper factors so it is not prime.

Question 13 (b)

Criteria	Marks
• Provides correct proof	4
• Proves the inductive step, or equivalent merit	3
• Establishes the base case and makes some progress with the inductive step, or equivalent merit	2
• Establishes base case, or equivalent merit	1

Sample answer:

$a_1 = \sqrt{2} \qquad a_2 = \sqrt{2+\sqrt{2}} \qquad a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} \qquad \ldots$

$a_{n+1}^2 = 2 + a_n, \qquad n \geq 1$

Consider $n = 1$ case

RHS $= 2\cos\frac{\pi}{4} = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2} = a_1 =$ LHS, so the result is true for $n = 1$

Assume the result is true for $n = k$.

Thus, $a_k = 2\cos\frac{\pi}{2^{k+1}}, \qquad k \geq 1$

Prove the result is true for $n = k + 1$

$$\begin{aligned} a_{k+1}^2 &= 2 + a_k \\ &= 2 + 2\cos\frac{\pi}{2^{k+1}} \\ &= 2\left[1 + \cos\frac{\pi}{2^{k+1}}\right] \qquad\qquad 1+\cos\theta = 2\cos^2\frac{\theta}{2} \\ &= 2 \times 2\cos^2\frac{\pi}{2^{k+2}} \\ &= 4\cos^2\frac{\pi}{2^{k+2}} \end{aligned}$$

$$\therefore\ a_{k+1} = 2\cos\frac{\pi}{2^{k+2}} \qquad\qquad (a_{k+1} > 0)$$

Thus if the result is true for k, it is also true for $k + 1$.

Hence, using the principle of mathematical induction, the result is true for all $k \geq 1$.

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Writes -1 in general exponential form, or equivalent merit	1

Sample answer:

$$z^5 + 1 = 0$$

$$z^5 = -1 = e^{i\pi}$$

$$\therefore \quad z = e^{i\left(\frac{2k\pi + \pi}{5}\right)}, \qquad k = 0, \ \pm 1, \ \pm 2$$

$$\therefore \quad z = e^{i\frac{\pi}{5}}, \quad z = e^{i\frac{3\pi}{5}}, \quad z = e^{-\frac{\pi}{5}i}, \quad z = e^{i\frac{\pi}{5}}, \quad z = e^{i\pi} = -1$$

$$z = -1, \quad e^{\pm i\frac{\pi}{5}}, \quad e^{\pm i\frac{3\pi}{5}}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Uses the given substitution and attempts to simplify, or equivalent merit	1

Sample answer:

Let z be a solution to $z^5 + 1 = 0$ with $z \neq -1$.

Let $u = z + \frac{1}{z}$

$$
\begin{aligned}
u^2 - u - 1 &= \left(z + \frac{1}{z}\right)^2 - \left(z + \frac{1}{z}\right) - 1 \\
&= z^2 + 2 + \frac{1}{z^2} - z - \frac{1}{z} - 1 \\
&= z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} \\
&= \frac{z^4 - z^3 + z^2 - z + 1}{z^2} \\
&= \frac{1}{z^2}\left(\frac{1-(-z)^5}{1-(-z)}\right) \qquad \text{using Geometric series, as } z \neq -1 \\
&= \frac{1+z^5}{z^2(1+z)} \\
&= 0 \qquad \text{since } z^5 + 1 = 0
\end{aligned}
$$

$\therefore\ u = z + \frac{1}{z}$ is a solution of $u^2 - u - 1 = 0$

Question 13 (c) (iii)

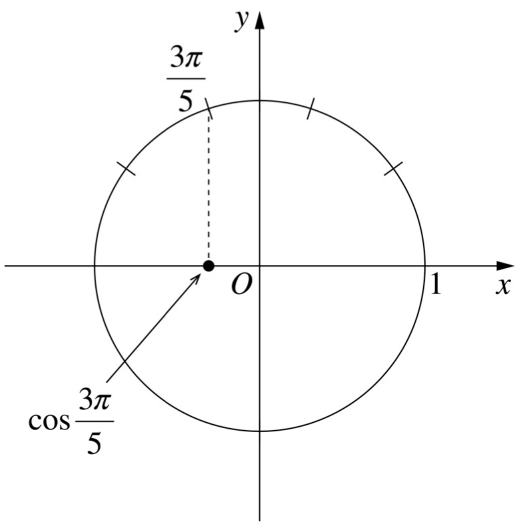
Criteria	Marks
• Provides correct solution	3
• Uses the connection between one of the solutions in part (i) and $\cos\frac{3\pi}{5}$	2
• Finds the values of u, or equivalent merit	1

Sample answer:

By part (i), $z = e^{\frac{3i\pi}{5}}$ is a solution of $z^5 + 1 = 0$.

Since $e^{\frac{3i\pi}{5}} \neq -1$, by part (ii), $u = e^{\frac{3i\pi}{5}} + \dfrac{1}{e^{\frac{3i\pi}{5}}} = 2\cos\dfrac{3\pi}{5}$ is a solution of $u^2 - u - 1 = 0$.

This equation has two solutions, $\dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$.



$\cos\dfrac{3\pi}{5} < 0$ whereas $\dfrac{1+\sqrt{5}}{2} > 0$

So $\quad 2\cos\dfrac{3\pi}{5} = \dfrac{1-\sqrt{5}}{2}$

$\therefore \quad \cos\dfrac{3\pi}{5} = \dfrac{1-\sqrt{5}}{4}$

Question 14 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to use a proof by contradiction or contrapositive, or equivalent merit	1

Sample answer:

Assume $\lambda\vec{u} + \mu\vec{v} = \vec{0}$

If λ and μ are both non-zero, then $\vec{u} = -\dfrac{\mu}{\lambda}\vec{v}$ and $\vec{v}$ are parallel.

But $\vec{u}$, $\vec{v}$ are not parallel.

Hence, one of λ or μ is zero.

Assume the other one is not zero.

Without loss of generality, assume $\lambda = 0$ but $\mu \neq 0$ then $\mu\vec{v} = \vec{0}$.

But $\vec{v} \neq \vec{0}$ so $\mu = 0$, a contradiction.

Hence $\lambda = \mu = 0$

Question 14 (a) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

If $\quad \lambda_1\vec{u} + \mu_1\vec{v} = \lambda_2\vec{u} + \mu_2\vec{v}$

Then $\left(\lambda_1 - \lambda_2\right)\vec{u} + \left(\mu_1 - \mu_2\right)\vec{v} = \vec{0}$

So $\lambda_1 - \lambda_2 = 0$ and $\mu_1 - \mu_2 = 0$ From part (i)

And hence $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$

Question 14 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Writes $\overrightarrow{SL}$ or $\overrightarrow{BL}$ as a linear combination of $\overrightarrow{SB}$ and $\overrightarrow{SC}$, or equivalent merit	1

Sample answer:

L is on the line SK so $\overrightarrow{SL} = \lambda\overrightarrow{SK}$ for some real number λ.

$$\overrightarrow{SL} = \frac{\lambda}{4}\overrightarrow{SB} + \frac{\lambda}{3}\overrightarrow{SC}$$

Similarly, there exists a real number μ such that

$$\overrightarrow{BL} = \mu\overrightarrow{BC} = -\mu\overrightarrow{SB} + \mu\overrightarrow{SC}$$

$$\begin{aligned}\overrightarrow{SL} &= \overrightarrow{SB} + \overrightarrow{BL}\\ &= (1-\mu)\overrightarrow{SB} + \mu\overrightarrow{SC}\end{aligned}$$

Now, $\frac{\lambda}{4}\overrightarrow{SB} + \frac{\lambda}{3}\overrightarrow{SC} = (1-\mu)\overrightarrow{SB} + \mu\overrightarrow{SC}$

$\overrightarrow{SB}$ and $\overrightarrow{SL}$ are not parallel, so from part (ii),

$$\frac{\lambda}{4} = 1-\mu \quad \text{and} \quad \frac{\lambda}{3} = \mu$$

$$\begin{aligned}\lambda &= 3\mu\\ \lambda &= 4-4\mu\\ 3\mu &= 4-4\mu\\ \mu &= \frac{4}{7}\end{aligned}$$

Hence $\overrightarrow{BL} = \mu\overrightarrow{BC} = \frac{4}{7}\overrightarrow{BC}$

Question 14 (a) (iv)

Criteria	Marks
• Provides correct solution	2
• Finds the vector $\overrightarrow{AL}$, or equivalent merit	1

Sample answer:

Write both $\overrightarrow{AP}$ and $\overrightarrow{AL}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\begin{aligned}\overrightarrow{AL} &= \overrightarrow{AB} + \overrightarrow{BL} \\ &= \overrightarrow{AB} + \frac{4}{7}\overrightarrow{BC} \\ &= \overrightarrow{AB} + \frac{4}{7}\left(\overrightarrow{BA} + \overrightarrow{AC}\right) \\ &= \frac{3}{7}\overrightarrow{AB} + \frac{4}{7}\overrightarrow{AC}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AP} &= -6\overrightarrow{AB} - 8\overrightarrow{AC} \\ &= -14\left(\frac{3}{7}\overrightarrow{AB} + \frac{4}{7}\overrightarrow{AC}\right) \\ &= -14\overrightarrow{AL}\end{aligned}$$

$\overrightarrow{AP} = -14\overrightarrow{AL}$, so $\overrightarrow{AP}$ and $\overrightarrow{AL}$ are parallel, and A is common to those vectors, so P lies on the line AL.

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$J_n = \int_0^1 x^n e^{-x}\,dx \qquad n \ge 0$$

$$n = 0 \qquad J_0 = \int_0^1 e^{-x}\,dx$$

$$= -\left[e^{-x}\right]_0^1$$

$$= -\left(\frac{1}{e} - 1\right)$$

$$= 1 - \frac{1}{e}$$

Question 14 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• States that $e^{-x} \le 1$ for $0 \le x \le 1$, or equivalent merit	1

Sample answer:

$$J_n = \int_0^1 x^n e^{-x}\,dx \le \int_0^1 x^n\,dx \qquad \text{since } x^n e^{-x} \le x^n \text{ for } [0, 1]$$

$$= \left[\frac{x^{n+1}}{n+1}\right]_0^1$$

$$= \frac{1}{n+1}$$

$$\therefore \; J_n \le \frac{1}{n+1}$$

Question 14 (b) (iii)

Criteria	Marks
• Provides correct solution	2
• Attempts to apply integration by parts, correctly dealing with the limits of integration, or equivalent merit	1

Sample answer:

$$J_n = \int_0^1 x^n e^{-x}\,dx$$

$$= \left[x^n\left(-e^{-x}\right)\right]_0^1 - \int_0^1 nx^{n-1}\left(-e^{-x}\right)dx$$

$$= -\frac{1}{e} + nJ_{n-1}$$

Question 14 (b) (iv)

Criteria	Marks
• Provides correct solution	2
• Uses part (iii) to write J_{k+1} in terms of J_k, or equivalent merit	1

Sample answer:

$$J_n = -\frac{1}{e} + n \times J_{n-1} \qquad n \geq 0$$

$n = 0$

$$\text{RHS} = 0! - \frac{0!}{e}\sum_{r=0}^{0}\frac{1}{1!}$$

$$= 1 - \frac{1}{e} = J_0 \qquad \text{by part (i)}$$

$$= \text{LHS}$$

$\therefore$ The property is true for $n = 0$.

Assume the property holds for k, so $J_k = k! - \dfrac{k!}{e}\displaystyle\sum_{r=0}^{k}\frac{1}{r!}$

By part (iii)

$$J_{k+1} = (k+1)J_k - \frac{1}{e}$$

$$= (k+1)\left(k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right) - \frac{1}{e}$$

$$= (k+1)! - \left(\frac{(k+1)!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right) - \frac{1}{e}\frac{(k+1)!}{(k+1)!}$$

$$= (k+1)! - \frac{(k+1)!}{e}\left(\left(\sum_{r=0}^{k}\frac{1}{r!}\right) + \frac{1}{(k+1)!}\right)$$

$$= (k+1)! - \frac{(k+1)!}{e}\left(\sum_{r=0}^{k+1}\frac{1}{r!}\right)$$

Thus, the result is true for $n = k + 1$, if true for $n = k$.

Thus, using the principle of mathematical induction, the result is proven.

Question 14 (b) (v)

Criteria	Marks
• Provides correct solution	1

Sample answer:

From part (ii)

$$0 \le J_n \le \frac{1}{n+1}$$

$$\text{But } \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$$\text{So } \lim_{n\to\infty} J_n = 0$$

From part (iv)

$$\frac{J_n}{n!} = 1 - \frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!}$$

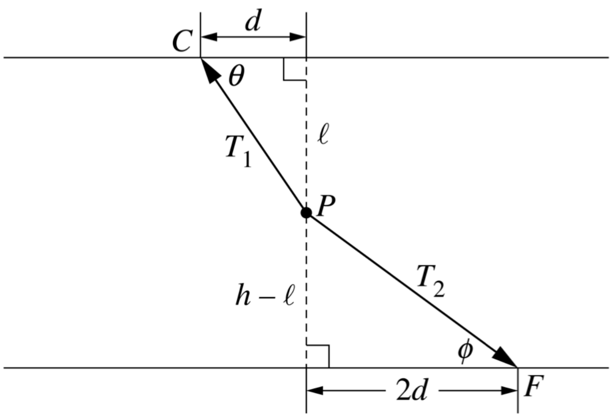
$$\text{So } \sum_{r=0}^{n}\frac{1}{r!} = e - \frac{e\,J_n}{n!}$$

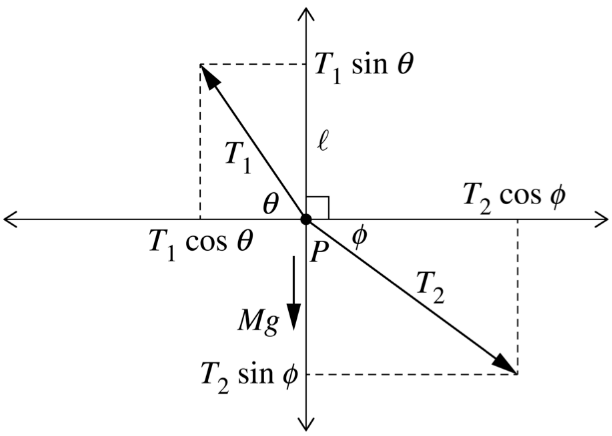
$$\begin{aligned}\lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{r!}\right) &= \lim_{n\to\infty}\left(e - \frac{e\,J_n}{n!}\right)\\ &= e - 0\\ &= e\end{aligned}$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct solution	3
• Obtains equations for the forces in both the horizontal and vertical directions	2
• Obtains an equation for the forces in either the horizontal or vertical direction	1

Sample answer:





Resolving the forces horizontally,

$$T_1 \cos\theta = T_2 \cos\phi \qquad \text{(I)}$$

Resolving the forces vertically,

$$T_1 \sin\theta = Mg + T_2 \sin\phi \qquad \text{(II)}$$

$\dfrac{\text{(II)}}{\text{(I)}}$ gives, $\tan\theta = \dfrac{Mg + T_2 \sin\phi}{\cos\phi}$

$$\therefore \quad \tan\theta = \frac{Mg}{T_2 \cos\phi} + \tan\phi$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Writes the equation from part (i) in terms of ℓ, d and h OR shows that $\theta > \phi$ OR considers the case where P is $\frac{2h}{3}$ OR equivalent merit	1

Sample answer:

$$\tan\theta = \frac{Mg}{T_2\cos\phi} + \tan\phi$$

From the diagram

$$\tan\theta = \frac{\ell}{d} \quad \text{and} \quad \tan\phi = \frac{h-\ell}{2d}$$

$$\text{Therefore} \quad \frac{\ell}{d} = \frac{Mg}{T_2\cos\phi} + \frac{h-\ell}{2d}$$

$$\frac{Mg}{T_2\cos\phi} > 0$$

Thus

$$\frac{\ell}{d} > \frac{h-\ell}{2d}$$

$$\ell > \frac{h-\ell}{2}$$

$$\frac{3\ell}{2} > \frac{h}{2}$$

$$\ell > \frac{h}{3}$$

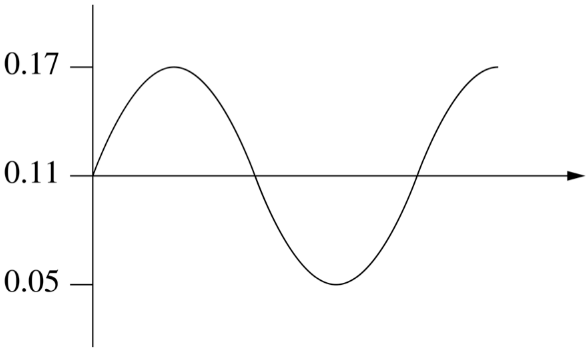
So $h - \ell < \frac{2h}{3}$

$\therefore$ P cannot be lifted to a position $\frac{2h}{3}$ metres above the floor.

Question 15 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains an equation of motion of the piston, or equivalent merit	2
• Finds the period of the motion, or equivalent merit	1

Sample answer:



Amplitude $a = \dfrac{0.17 - 0.05}{2} = 0.06$ m

Period of motion = $\dfrac{1}{40} = 0.025$ s

Equation of motion simple harmonic motion (SHM) is

$\ddot{x} = -n^2(x - b)$

$n = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.025} = 80\pi$ and

Centre of motion $b = 0.11$

$\therefore \quad \ddot{x} = -(80\pi)^2(x - 0.11)$

Acceleration is maximum at an extremity,

$$\begin{aligned}\ddot{x}_{\text{max}} &= -(80\pi)^2(0.05 - 0.11)\\ &= 384\pi^2 \text{ m s}^{-2}\end{aligned}$$

The maximum force on the piston $f = m\ddot{x}$

$= 0.8 \times 384\pi^2$ newtons

≈ 3032 newtons

Question 15 (c)

Criteria	Marks
• Provides correct solution	4
• Attempts to use integration by parts with the correct integral, or equivalent merit	3
• Obtains integral in terms of θ, or equivalent merit	2
• Attempts to use the given substitution, or equivalent merit	1

Sample answer:

Let $x = \tan^2\theta$

at $x = 0 \quad \theta = 0$

at $x = 1 \quad \theta = \dfrac{\pi}{4}$

$$dx = 2\tan\theta\sec^2\theta\, d\theta$$

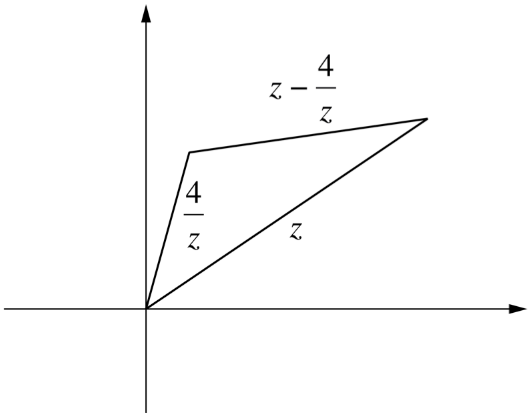
$$\begin{aligned}\sin^{-1}\sqrt{\frac{x}{1+x}} &= \sin^{-1}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}}\\ &= \sin^{-1}\sqrt{\sin^2\theta}\\ &= \theta \qquad \text{for } 0 \le \theta \le \frac{\pi}{4}\text{, as } \sin\theta \ge 0\end{aligned}$$

$$\begin{aligned}\int_0^1 \sin^{-1}\sqrt{\frac{x}{1+x}}\,dx &= \int_0^{\frac{\pi}{4}} 2\theta\tan\theta\sec^2\theta\, d\theta \qquad \text{Integration by part}\\ &= \left[\theta\tan^2\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\tan^2\theta\, d\theta\\ &= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} -1 + \sec^2\theta\, d\theta\\ &= \frac{\pi}{4} - \left[-\theta + \tan\theta\right]_0^{\frac{\pi}{4}}\\ &= \frac{\pi}{4} + \frac{\pi}{4} - 1\\ &= \frac{\pi}{2} - 1\end{aligned}$$

Question 15 (d)

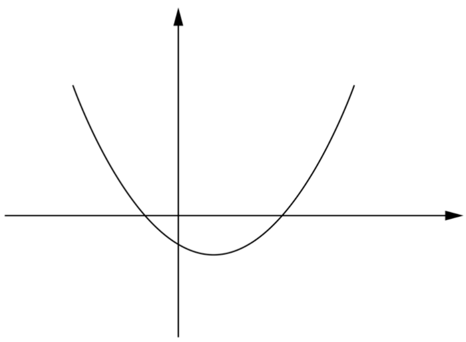
Criteria	Marks
• Provides correct solution	3
• Uses the triangle inequality to obtain a relevant inequality involving only $\lvert z \rvert$, or equivalent merit	2
• Correctly uses the triangle inequality, or equivalent merit	1

Sample answer:



By triangle inequality

$$\begin{aligned}
\lvert z\rvert &\le \left\lvert z-\frac{4}{z}\right\rvert + \left\lvert \frac{4}{z}\right\rvert \\
\lvert z\rvert &\le 2+\frac{4}{\lvert z\rvert} \\
\lvert z\rvert^2 &\le 2\lvert z\rvert + 4 \\
\lvert z\rvert^2 - 2\lvert z\rvert - 4 &\le 0 \\
\lvert z\rvert &\le \frac{2+\sqrt{4+4\times 4}}{2} \\
&= \frac{2+\sqrt{20}}{2} \\
&= 1+\sqrt{5}
\end{aligned}$$



Hence $\lvert z\rvert \le 1+\sqrt{5}$

Question 16 (a)

Criteria	Marks
• Provides correct solution	4
• Equates real or imaginary parts from $z_C - z_A = e^{i\frac{\pi}{3}}(z_B - z_A)$ OR Obtains a quadratic in x from $\lvert z_B - z_A \rvert = \lvert z_C - z_A \rvert = \lvert z_B - z_C \rvert$	3
• Attempts to use either $z_C - z_A = e^{i\frac{\pi}{3}}(z_B - z_A)$, or equivalent merit OR $\lvert z_B - z_A \rvert = \lvert z_C - z_A \rvert = \lvert z_B - z_C \rvert$	2
• Observes that $z_B = x + 5i$, for some real number x, or equivalent merit	1

Sample answer:

Let $z_B = b + 5i$ and $z_C = c - 5i$ where b and c are real numbers.

$$e^{i\frac{\pi}{3}}(z_B - z_A) = z_C - z_A$$

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left((b-5) + 4i\right) = (c-5) - 6i$$

Equating the imaginary parts, we get

$$\begin{aligned} 2 + \frac{\sqrt{3}}{2}(b-5) &= -6 \\ \frac{\sqrt{3}}{2}(b-5) &= -8 \\ b &= 5 - \frac{16}{\sqrt{3}} \end{aligned}$$

$$\therefore\ z_B = \left(5 - \frac{16}{\sqrt{3}}\right) + 5i$$

Question 16 (b)

Criteria	Marks
• Provides correct solution	4
• Obtains the velocity, or equivalent merit	3
• Attempts to solve differential equation, or equivalent merit	2
• Attempts to obtain the force equation	1

Sample answer:

$$M\ddot{y} = -Mg - 0.1Mv_y$$

$$\frac{dv_y}{dt} = -\left(g + 0.1v_y\right)$$

$$\int \frac{dv_y}{g + 0.1v_y} = -\int dt$$

$$10\left[\ln\left|g + 0.1v_y\right|\right] = -t + k$$

When $t = 0$, $v_y = v_0$

$$10\ln\left|g + 0.1v_0\right| = k$$

$$\ln\left|g + 0.1v_y\right| = -\frac{t}{10} + \ln\left|g + 0.1v_0\right|$$

$$\ln\left|\frac{g + 0.1v_y}{g + 0.1v_0}\right| = -\frac{t}{10}$$

$$\ln\frac{g + 0.1v_y}{g + 0.1v_0} = -\frac{t}{10}$$ as $g + 0.1v > 0$ because all speeds are less than 100 m s^{-1}

$$g + 0.1v_y = \left(g + 0.1v_0\right)e^{-\frac{t}{10}}$$

$$0.1v_y = -g + \left(g + 0.1v_0\right)e^{-\frac{t}{10}}$$

$$\therefore \quad v_y = 10\left[-g + \left(g + 0.1v_0\right)e^{-\frac{t}{10}}\right]$$

Integrating with respect to t,

$$y = 10\left[-gt + \left(g + 0.1v_0\right)\frac{e^{-\frac{t}{10}}}{-\frac{1}{10}}\right] + c$$

$t = 0, \qquad y = 0$

$$0 = 10\left[-10\left(g + 0.1v_0\right)\right] + c$$

$$\therefore \quad c = 100\left(g + 0.1v_0\right)$$

$$\therefore \quad y = 10\left[-gt - 10\left(g + 0.1v_0\right)e^{-\frac{t}{10}}\right] + 100\left(g + 0.1v_0\right)$$

$t = 7, \qquad y = 0, \qquad g = 10$

$$\begin{aligned} 0 &= 10\left[-70 - 10\left(10 + 0.1v_0\right)e^{-0.7}\right] + 100\left(10 + 0.1v_0\right) \\ &= -700 - 100\left(10 + 0.1v_0\right)e^{-0.7} + 100\left(10 + 0.1v_0\right) \\ 700 &= 100\left[10 + 0.1v_0\right]\left[1 - e^{-0.7}\right] \\ 10 + 0.1v_0 &= \frac{7}{1 - e^{-0.7}} \\ \therefore \quad v_0 &= \left(\frac{7}{1 - e^{-0.7}} - 10\right) \times 10 \\ &= 39.05\ldots \text{ m s}^{-1} \end{aligned}$$

v_0 is 39.1 m s^{-1} correct to 1 decimal place.

Question 16 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to apply given information with three numbers, or equivalent merit	1

Sample answer:

Consider ab, ac, bc

Then $S = 2(ab + ac + bc)$

From the given result

$$ab \times ac \times bc \le \left(\frac{ab + ac + bc}{3}\right)^3$$

$$(abc)^2 \le \left(\frac{S}{6}\right)^3$$

$$abc \le \left(\frac{S}{6}\right)^{\frac{3}{2}}$$

Question 16 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Applies part (i) to ab, ac and bc, or equivalent merit	1

Sample answer:

Given a cube, we have $a = b = c$

So $V = abc = a^3$

And $S = 2(ab + ac + bc) = 6a^2$

$$\left(\frac{S}{6}\right)^{\frac{3}{2}} = \left(\frac{6a^2}{6}\right)^{\frac{3}{2}} = a^3$$

So $V = \left(\frac{S}{6}\right)^{\frac{3}{2}}$ From part (i) we know that $V \le \left(\frac{S}{6}\right)^{\frac{3}{2}}$ for all rectangular prisms

Hence a cube has maximum possible volume for a fixed surface area.

Question 16 (d)

Criteria	Marks
• Provides correct solution	3
• Shows, $z_1z_2 + z_1z_3 + z_2z_3 = 1$	2
• Treats z_1, z_2 and z_3 as zeros of a polynomial OR Shows $\lvert z_1 \rvert = 1$	1

Sample answer:

As $\quad z_1z_2z_3 = 1$

We have $\quad |z_1||z_1||z_3| = 1$

But $\quad |z_1| = |z_2| = |z_3|$

So $\quad |z_1|^3 = 1$

$$|z_1| = 1$$

Hence $\quad |z_1| = |z_2| = |z_3| = 1$

$$z_1 = e^{i\theta}, \quad \text{for some } \theta$$

$$\frac{1}{z_1} = e^{-i\theta} = \overline{e^{i\theta}} = \overline{z_1}$$

Similarly $\quad \dfrac{1}{z_2} = \overline{z_2}$

$$\frac{1}{z_3} = \overline{z_3}$$

$$z_1z_2z_3 = 1$$

So $\quad z_1z_2 = \dfrac{1}{z_3} = \overline{z_3}$

Similarly $\quad z_1z_3 = \overline{z_2}$ and $z_2z_3 = \overline{z_1}$

Hence $\quad z_1z_2 + z_1z_3 + z_2z_3 = \overline{z_3} + \overline{z_2} + \overline{z_1}$

$$= \overline{z_1 + z_2 + z_3}$$

$$= 1$$

Therefore z_1, z_2, z_3 are the zeros of the polynomial

$$z^3 - z^2 + z - 1 = 0$$
$$(z-1)(z^2+1) = 0$$
$$z = 1, \quad i, \quad -i$$

In some order z_1, z_2, z_3 are 1, i, $-i$.

2022 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	MEX-N2 Using Complex Numbers	MEX12-4
2	1	MEX-P1 The Nature of Proof	MEX12-2
3	1	MEX-P1 The Nature of Proof	MEX12-8
4	1	MEX-C1 Further Integration	MEX12-5
5	1	MEX-C1 Further Integration	MEX12-5
6	1	MEX-N2 Using Complex Numbers	MEX12-4
7	1	MEX-P1 The Nature of Proof	MEX12-8
8	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
9	1	MEX-V1 Further Work with Vectors	MEX12-3
10	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	2	MEX-N1 Introduction to Complex Numbers	MEX12-1
11 (b)	2	MEX-C1 Further Integration	MEX12-5
11 (c) (i)	2	MEX-N1 Introduction to Complex Numbers	MEX12-1
11 (c) (ii)	2	MEX-N1 Introduction to Complex Numbers	MEX12-1
11 (d)	3	MEX-V1 Further Work with Vectors	MEX12-3
11 (e)	2	MEX-V1 Further Work with Vectors	MEX12-3
11 (f)	3	MEX-C1 Further Integration	MEX12-5
12 (a)	2	MEX-P1 The Nature of Proof	MEX12-2
12 (b)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
12 (c) (i)	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
12 (c) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
12 (d)	4	MEX-C1 Further Integration	MEX12-5, MEX12-7
12 (e)	3	MEX-N1 Introduction to Complex Numbers	MEX12-4
13 (a)	3	MEX-P1 The Nature of Proof	MEX12-2, MEX12-8
13 (b)	4	MEX-P2 Further Proof by Mathematical Induction	MEX12-2

Question	Marks	Content	Syllabus outcomes
13 (c) (i)	2	MEX-N2 Using Complex Numbers	MEX12-4
13 (c) (ii)	2	MEX-N1 Introduction to Complex Numbers	MEX12-4
13 (c) (iii)	3	MEX-N2 Using Complex Numbers	MEX12-4
14 (a) (i)	2	MEX-V1 Further Work with Vectors	MEX12-3
14 (a) (ii)	1	MEX-V1 Further Work with Vectors	MEX12-3
14 (a) (iii)	2	MEX-V1 Further Work with Vectors	MEX12-3
14 (a) (iv)	2	MEX-V1 Further Work with Vectors	MEX12-3
14 (b) (i)	1	MEX-C1 Further Integration	MEX12-5
14 (b) (ii)	2	MEX-C1 Further Integration	MEX12-5
14 (b) (iii)	2	MEX-C1 Further Integration	MEX12-5
14 (b) (iv)	2	MEX-P2 Further Proof by Mathematical Induction	MEX12-2
14 (b) (v)	1	MEX-P1 The Nature of Proof	MEX12-7
15 (a) (i)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
15 (a) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
15 (b)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
15 (c)	4	MEX-C1 Further Integration	MEX12-5
15 (d)	3	MEX-P1 The Nature of Proof MEX-N1 Introduction to Complex Numbers	MEX12-2, MEX12-8
16 (a)	4	MEX-N1 Introduction to Complex Numbers MEX-N2 Using Complex Numbers	MEX12-4, MEX12-7
16 (b)	4	MEX-M1 Applications of Calculus to Mechanics	MEX12-6, MEX12-7
16 (c) (i)	2	MEX-P1 The Nature of Proof	MEX12-7
16 (c) (ii)	2	MEX-P1 The Nature of Proof	MEX12-7
16 (d)	3	MEX-N2 Using Complex Numbers	MEX12-4

Mathematics Extension 2

HSC Marking Feedback 2022

Question 11

Part (a)

Students should:

- know to multiply the numerator and denominator by the conjugate of the denominator
- show all calculations.

In better responses, students were able to:

- expand the product of the numerators correctly showing all four terms
- show the denominator as the sum of the squares of the moduli
- simplify the complex number.

Areas for students to improve include:

- multiplying two negative multiples of i
- showing relevant mathematical reasoning and/or calculations.

Part (b)

Students should:

- recognise the integrand as the product of a compound trigonometric function and a multiple of the derivative of its inner function
- rearrange in reverse chain rule form or let u equal the inner function of the compound function or use the double angle results to find the primitive function in terms of x.

In better responses, students were able to:

- rewrite the integrand in reverse chain rule form
- find the primitive of the integrand efficiently.

Areas for students to improve include:

- understanding and using the reverse chain rule
- choosing the simplest suitable u substitution
- using the correct constant factor with the double angle results
- looking for the most efficient method to find a primitive function.

Part (c) (i)

Students should:

- find the modulus and argument of $-\sqrt{3}+i$ showing all calculations and/or a sketch of its position on the complex plane
- write the complex number in exponential form.

In better responses, students were able to:

- sketch the position of the complex number on the complex plane with an exact ratio triangle in the correct position
- use the exact ratio triangle or formula, with suitable working, to find the modulus and argument
- check that the argument matched the quadrant in which the complex number lies.

Areas for students to improve include:

- identifying the correct quadrant in which a complex number lies in order to evaluate the correct argument
- showing relevant mathematical reasoning and/or calculations.

Part (c) (ii)

Students should:

- use their answer from part (i)
- use De Moivre's theorem to find the modulus and argument of a complex number
- convert the complex number from exponential form to Cartesian form showing all calculations.

In better responses, students were able to:

- use De Moivre's theorem to find the modulus and argument of the complex number
- find the principal argument
- convert $e^{i\theta}$ to Cartesian form using exact triangles
- simplify the complex number in Cartesian form.

Areas for students to improve include:

- showing relevant mathematical reasoning and/or calculations.

Part (d)

Students should:

- find the 2 vectors whose tails are at B, find their dot product and moduli, and use the angle between 2 vectors formula to find the angle to the nearest degree
- find the 3 vectors forming the triangle, find their lengths and use the cosine rule to find the correct angle to the nearest degree.

In better responses, students were able to:

- use vector formulae correctly.

Areas for students to improve include:

- recognising the correct vectors to use to find the required angle
- using vector formulae

- understanding that the angle between two vectors can be acute or obtuse.

Part (e)

Students should:

- find the gradient of the line then use the point-gradient formula or find the equation of the line in vector form then solve simultaneous equations, to find the equation of the line in gradient-intercept form.

In better responses, students were able to:

- recognise the gradient of the line from the direction vector, then use the point-gradient formula.

Areas for students to improve include:

- understanding the equation of a two-dimensional line in both vector and Cartesian form
- choosing the most efficient method when using lines in vector form
- writing their answer in the form required.

Part (f)

Students should:

- find the primitive function of an integrand involving trigonometric functions using t-results.

In better responses, students were able to:

- substitute the appropriate t-results and determine dx
- simplify the integrand correctly
- find the primitive function correctly using absolute value signs and rewrite their answer in terms of x.

Areas for students to improve include:

- using t-results and simplifying the resulting integrands
- writing the final answer using the same variable as the original integrand
- Understanding when absolute value signs are required.

Question 12

Part (a)

Students should:

- prove the arithmetic mean is greater than or equal to the geometric mean for non-negative values of a and b.

In better responses, students were able to:

- use the result $(\sqrt{a}-\sqrt{b})^2 \geq 0$ or $(a-b)^2 \geq 0$ and replace a with $\sqrt{a}$ and b with $\sqrt{b}$
- use the result $(a-b)^2 \geq 0$ and complete the square
- compare the difference between the LHS and RHS to show $\frac{(\sqrt{a}-\sqrt{b})(\sqrt{a}-\sqrt{b})^2}{2} \geq 0.$

Areas for students to improve include:

- clarity of justifications when proving results

- learning a standard method to prove this result.

Part (b)

Students should:

- integrate the acceleration function twice with respect to time, to obtain functions for velocity and displacement respectively
- know that maximum velocity occurs when acceleration is zero.

In better responses, students were able to:

- obtain a correct equation for x in terms of t, recognise maximum velocity occurred when $t = 2$, and substitute $t = 2$ into their equation to find the value of x.

Areas for students to improve include:

- clarity of their solutions
- using the velocity equation to find the displacement equation
- calculating constants of integration using given information
- setting acceleration equal to zero to find when the maximum velocity occurs.

Part (c) (i)

Students should:

- use Newtons second law $F = ma$ and the resistive equation given in the question
- use the equation $a = v\frac{dv}{dx}$ to determine an expression for $\frac{dv}{dx}$.

In better responses, students were able to:

- recognise the particle was moving horizontally
- recognise the resistive force was negative and given in terms of velocity
- simplify $F = ma$ by substituting the mass as 1 kg and then use the equation $a = v\frac{dv}{dx}$.

Areas for students to improve include:

- identifying the initial conditions, particularly the direction of travel
- using the equation $a = v\frac{dv}{dx}$ when acceleration is given in terms of velocity.

Part (c) (ii)

Students should:

- integrate $\frac{dv}{dx}$ to find the displacement in terms of velocity, using the given function from part (i).

In better responses, students were able to:

- integrate the expression and find the constants of integration correctly, keeping the equation in factored form
- use the laws of logarithms to simplify the answer.

Areas for students to improve include:

- applying integration rules and procedures

- using initial conditions to find the constant of integration.

Part (d)

Students should:

- decompose an integral with a linear and quadratic denominator into components
- integrate an expression accurately, using logarithmic laws to simplify the expression into the required form.

In better responses, students were able to:

- decompose the expression into partial fractions
- use integration techniques correctly and simplify the expression using logarithmic laws to express the final answer in the form $\frac{1}{2} ln\left(\frac{4+n^2}{8(n-1)^2}\right)$.

Areas for students to improve include:

- recognising the need for a remainder of $Bx + C$ when given a denominator of $4 + x^2$
- clear setting out of partial fractions to accurately find the required integral
- using a variety of logarithmic laws.

Part (e)

Students should:

- apply a variety of complex number properties to prove an expression is purely imaginary.

In better responses, students were able to:

- work in exponential form and factorise using a form of $e^{i\theta}$ as a common factor
- factorise by taking out a common factor of z
- apply De Moivre's theorem and double angle formulae to simplify the expression
- multiply the numerator and denominator by the conjugate of the denominator.

Areas for students to improve include:

- understanding and working with complex numbers in exponential form
- considering both the real and imaginary parts, before proving that $Re(z) = 0$
- applying De Moivre's theorem and working accurately in polar form
- applying appropriate double angle results accurately.

Question 13

Part (a)

Students should:

- know how to work with even (and odd) numbers in proof
- not equate prime with odd
- state either the contrapositive or contradiction correctly.

In better responses, students were able to:

- use the contrapositive
- factorise $(2^k)^2 - 1$ correctly
- recognise that the expression had 2 factors other than 1 and itself
- show the expression was divisible by 3 when n was even, either by factorising using the binomial expansion or with an inductive proof.

Areas for students to improve include:

- writing a proof statement, in this case the contrapositive or contradiction
- recognising when an inductive proof is best
- setting out solutions to proofs
- writing a concluding statement using the language of proof.

Part (b)

Students should:

- know the process of proof by mathematical induction
- know how to use an assumption in an inductive proof
- test for the base case.

In better responses, students were able to:

- be clear and logical in their presentation of the inductive proof
- apply the double angle results or use a similar approach
- establish that only the positive result was valid when taking the square root.

Areas for students to improve include:

- understanding that when taking a square root there are two possible results, and recognising which of the 2 cases to include
- using the double angle or sum to product results efficiently
- setting out a proof by mathematical induction.

Part (c) (i)

Students should:

- understand that there are 5 evenly spaced solutions around the unit circle
- establish that -1 is a solution which is positioned at $\theta = \pi$ around the circle.

In better responses, students were able to:

- draw a diagram
- use the exponential form $\left(z = re^{i\theta}\right)$ or $z = r[\cos\theta + i \sin\theta]$ to find the roots
- clearly state their solutions in the domain $[-\pi, \pi][-\pi, \pi]$
- indicate that the solutions were $\frac{2\pi}{5}$ apart.

Areas for students to improve include:

- ensuring complete solutions are provided, not just the arguments of the complex numbers

- using different notations when working with roots of complex numbers.

Part (c) (ii)

Students should:

- substitute into, and expand, algebraic expressions involving quadratics.

In better responses, students were able to:

- factorise $z^5 + 1 = 0$ and use the result to simplify the expression
- manipulate the algebraic expression obtained to show that it equalled zero.

Areas for students to improve include:

- understanding the difference between the solution to a polynomial and the argument for complex equations.

Part (c) (iii)

Students should:

- know how to factorise a quadratic
- understand and be able to show that $z + \frac{1}{z} = 2\cos\theta$.

In better responses, students were able to:

- use the quadratic formula to factorise $u^2 - u - 1 = 0$
- determine that $z + \frac{1}{z} = 2\cos\frac{3\pi}{5}$ and that this is a second quadrant angle
- equate $2\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{2}$ and simplify to give the required result
- justify which of the quadratic solutions is correct.

Areas for students to improve include:

- connecting parts (i) and (ii) of the question
- determining which quadrant and consequently which solution is required.

Question 14

Part (a) (i)

Students should:

- understand the conditions for vectors to be parallel, and hence non-parallel
- use correct mathematical language, for example equating components (not coefficients).

In better responses, students were able to:

- use logic, via contradiction or otherwise, to explain why $\lambda = \mu = 0$
- assume the vectors were non-parallel, then use $\mu \neq \lambda \neq 0$ to show why this leads to a contradiction
- explain that if 2 vectors are not parallel, then it is not possible to express one as a scalar multiple of the other

- focus on the geometric definition of parallel vectors and use $\lambda\vec{u} = -\mu\vec{v}$ to explain that the only way the vectors lead to the origin is if $\lambda = \mu = 0$
- use mathematical language in their solutions, including *scalar multiple, linear combination, equating components.*

Areas for students to improve include:

- understanding that non-parallel lines have different direction vectors
- distinguishing between direct and indirect approaches of proof (like proof by contradiction) when applying it to 'if – then' statements.

Part (a) (ii)

Students should:

- connect the information in part (i) to part (ii)
- manipulate the equations to show the parameters are equal.

In better responses, students were able to:

- group the components so the equation resembled part (i)
- use the result of part (i) to explain why $\lambda_1 - \lambda_2 = 0$ and $\mu_1 - \mu_2 = 0$.

Areas for students to improve include:

- identifying connections between parts of a question
- not drawing incorrect assumptions from previous parts, for example the vector sum of zero in part (i) does not mean the vector sum is zero in part (ii).

Part (a) (iii)

Students should:

- understand vector notation and geometry
- know a tetrahedron can be a regular or irregular shape
- use their knowledge of vectors to show the result.

In better responses, students were able to:

- identify and use the connections between parts (i), (ii) and (iii)
- work algebraically to simplify expressions accurately and equate components
- solve simultaneous equations to find the parameter $\left(\lambda = \frac{4}{7}\right)$
- develop a relationship between linear combination of vectors and use part (ii) to show the required result
- use $\overrightarrow{SK} = \frac{1}{4}\overrightarrow{SB} + \frac{1}{3}\overrightarrow{SC}$ to create relationships involving $\overrightarrow{SL}$ or $\overrightarrow{BL}$ in the presence of $\overrightarrow{SC}$, $\overrightarrow{SB}$ and $\overrightarrow{BC}$ and apply part (ii) to derive $\mu = \frac{4}{7}$.

Areas for students to improve include:

- drawing or copying diagrams and marking with the information given in the question
- identifying key vector relationships in a geometric setting and develop the necessary relationships
- understanding the necessity for a parameter to establish the required relationship

- representing and distinguishing between vectors in different forms such as vector form and equation form
- using vector notation, particularly writing the direction of vectors without error.

Part (a) (iv)

Students should:

- show $\overrightarrow{AP}$ lies on $\overrightarrow{AL}$ by using the information in the question and the result from part (iii).

In better responses, students were able to:

- use the result in part (iii) to make a connection with the point required
- recognise the relevance of the negative in the obtained parameter as meaning point P was not located between A and L but was still on the line
- manipulate the vector $\overrightarrow{BC}$ into $\overrightarrow{AP} = -6\overrightarrow{AB} - 8\overrightarrow{AC}$ by using $\overrightarrow{BL} = \frac{4}{7}\overrightarrow{BC}$ and realising $\overrightarrow{AL} = \overrightarrow{AB} + \overrightarrow{BL}$ and $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ by using substitution and the use of factorising.

Areas for students to improve include:

- drawing diagrams if referring to their own vectors or variables
- understanding the difference between an interval and a line/vector
- applying the information found in previous parts
- understanding the necessity for the use of a parameter to map out all the points on the line AL
- being mindful of negative signs when simplifying expressions.

Part (b) (i)

Students should:

- show all steps in their solution, especially for a 'show' question where the result is given.

In better responses, students were able to:

- identify that when $n = 0$, $J_0 = \int_0^1 e^{-x} dx$
- substitute and integrate accurately.

Areas for students to improve include:

- interpreting a general formula
- showing detail in solutions including substitution of limits
- taking care when substituting limits to avoid errors
- understanding and using index notation.

Part (b) (ii)

Students should:

- consider the effect on the integral of increasing the value of n
- recognise how to create an inequality, in this case consider what relationships can be compared
- integrate to show that the inequality holds.

In better responses, students were able to:

- recognise that $e^{-x} \leq 1$ and so $e^{-x}x^n \leq x^n$ in the interval [0,1]
- draw a graph to show their understanding of the question
- see the connection between the definite integral and area.

Areas for students to improve include:

- practising the use of arguments such as $\int_0^1 e^{-x}x^n dx \leq \int_0^1 x^n dx$ rather than relying on specific cases such as when $n = 0$ or when $n = 1$
- approaching the problem geometrically by drawing an appropriate graph
- understanding how inequalities in integration can be verified using area graphs.

Part (b) (iii)

Students should:

- recognise the need to use integration by parts
- write the formula for integration by parts before substituting
- show full working including substitution of limits.

In better responses, students were able to:

- identify their use of integration by parts, integrate correctly and show the substitution of limits leading to the result.

Areas for students to improve include:

- setting out solutions when using integration by parts
- operating with negative signs and simplifying with care
- converting expressions back into recursive form $\left(J_n = nJ_{n-1} - \frac{1}{e}\right)$ from integral form
- using subscripts.

Part (b) (iv)

Students should:

- understand the process of mathematical induction
- work efficiently and effectively with sigma notation and subscripts
- apply mathematical induction or use repeated substitution.

In better responses, students were able to:

- use sigma notation and factorise in order to arrive at the result
- develop a pattern
- write down accurate expressions for J_k and J_{k+1}
- understand how to manipulate the sigma notation to obtain the required expression
- correctly manipulate the summation operator to show the statement holds for $n = k + 1$ by using the condition in part (iii).

Areas for students to improve include:

- taking care when dealing with series expressions and ensuring that all terms and signs are accounted for when simplifying expressions
- taking care with subscripts and the use of brackets with sigma notation
- understanding the changes to the value on the top of the sigma notation, in this case k changing to $k+1$ when a new term is added.

Part (b) (v)

Students should:

- understand how limits work
- know that sigma notation limits are important
- use part (ii) to place a bound on J_n and evaluate the limit after suitable algebraic manipulations.

In better responses, students were able to:

- use the limit and sigma notations clearly and efficiently
- show full working
- manipulate the expression, then take the limit as $n \to \infty$ on both sides to arrive at the result
- rearrange the given expression to make the required subject e
- apply the information from part (ii)
- recognise $n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!} = 0$.

Areas for students to improve include:

- identifying connections between parts of a question
- understanding limits and using limit notation correctly
- manipulating algebraic expressions involving factorial notation
- operating with inequalities.

Question 15

Part (a) (i)

Students should:

- decompose forces into horizontal and vertical components
- eliminate tension forces using simultaneous equations
- obtain the expression for $\tan\theta$ by solving the vertical forces and horizontal forces simultaneously.

In better responses, students were able to:

- draw diagrams which clearly showed the horizontal and vertical components of the forces
- identify the horizontal component as $T_1\cos\theta = T_2\cos\emptyset$ and the vertical component as $T_1\sin\theta = Mg + T_2\sin\emptyset$
- use a division strategy to solve the simultaneous equations

- clearly show all steps in solving the simultaneous equations to obtain $\tan\theta = \tan\emptyset + \frac{Mg}{T_2\cos\emptyset}$.

Areas for students to improve include:

- understanding that equilibrium means that opposite forces are equal
- understanding that constant velocity means acceleration is zero, so the force is zero
- equating horizontal and vertical forces
- using correct symbols and being careful with similar symbols such as θ and ϕ
- applying trigonometry correctly to the described forces
- using the correct notation and variables for the forces.

Part (a) (ii)

Students should:

- use the result given in part (i)
- equate $\frac{Mg}{T_2\cos\phi}$ as always greater than zero
- represent $\tan\theta$ and $\tan\phi$ as ratios of distances
- identify that $\tan\theta$ and $\tan\phi$ are equal at the given value
- use $M > 0$ and $g > 0$ to construct an inequality or to identify a contradiction.

In better responses, students were able to:

- equate $\tan\theta$ and $\tan\phi$ clearly from part (i)
- demonstrate clear and logical steps to show $P < \frac{2h}{3}$ by using contradiction or establishing an inequality.

Areas for students to improve include:

- identifying when a problem can be solved using a contradiction.

Part (b)

Students should:

- draw a diagram to describe the simple harmonic motion
- determine the amplitude, the period and the centre of motion
- find the maximum force using $F = ma$.

In better responses, students were able to:

- find the amplitude, period, centre of motion and $n = \frac{2\pi}{T} = 80\pi$
- find $\ddot{x} = -(80\pi)^2(x - 0.11)$, substitute an extremity to determine the maximum acceleration and then multiply by the mass 0.8 kg to obtain the maximum force.

Areas for students to improve include:

- accuracy in calculations
- deriving the second derivative of a trigonometric function with a coefficient being a multiple of π
- understanding what the unit of measurement, Newtons, represents
- understanding the difference between frequency and period.

Part (c)

Students should:

- recognise the need to use integration by parts
- explicitly state all four parts involved in integration by parts.

In better responses, students were able to:

- correctly change the limits of the integration
- simplify the integral before applying by parts
- separate the question into clear logical steps
- select optimal integration by parts components.

Areas for students to improve include:

- simplifying inverse trigonometric function expressions
- expanding brackets correctly
- integration by parts, particularly using the correct choices for u and dv when performing integration by parts
- showing all steps in working.

Part (d)

Students should:

- use the triangle inequality
- interpret the inequality $|z| > 0$
- use the hint to find $|z| \leq \left|z - \frac{4}{z}\right| + \left|\frac{4}{z}\right|$.

In better responses, students were able to:

- draw a clear diagram to display given information
- demonstrate their knowledge of the triangle inequality and select the appropriate one to use
- use the $|z|$ definition to restrict the solution to the quadratic inequality
- equate $|z| \leq \left|z - \frac{4}{z}\right| + \left|\frac{4}{z}\right|$ to $|z|^2 - 2|z| - 4 \leq 0$ and correctly solve the quadratic inequation.

Areas for students to improve include:

- understanding the triangle inequality
- knowing how to interact algebraically with moduli.

Question 16

Part (a)

Students should:

- determine the relationships between the three vertices using complex vectors or two-dimensional geometry
- solve vector equations or multiple equations arising from two-dimensional geometry
- find the exact value of z_B.

In better responses, students were able to:

- define z_B in Cartesian form as $b + 5i$
- find vectors representing two of the sides of the equilateral triangle and create an equation linking them using exponential form
- equate the imaginary parts of the equation and solve to find the real part of z_B.

Areas for students to improve include:

- determining efficient techniques to approach questions involving complex numbers which are not scaffolded
- manipulating vector equations
- understanding complex vectors and using proper vector notation.

Part (b)

Students should:

- find a single force equation representing upward and downward flight and integrate with respect to time to find v, then integrate with respect to time to find the initial velocity
- find separate force equations for the upward and downward flights, find expressions for the time and maximum height on the upward flight, and time and maximum height on the downward flight, and solve for the initial velocity.

In better responses, students were able to:

- use a single force equation for the whole flight and efficiently choose the integration methods needed to find the initial velocity.

Areas for students to improve include:

- recognising that a single force equation can be used for upward and downward flight with resisted motion
- choosing integration methods efficiently in questions which are not scaffolded, in this case definite integrals were generally more efficient than indefinite integrals
- working with the algebra to simplify integrands or work with integrals.

Part (c) (i)

Students should:

- start with a result known to be true and proceed to the result to be proved
- use the given result, or the arithmetic mean-geometric mean (AM-GM) result, together with an expression for the surface area of a rectangular prism to proceed to the result to be proved.

In better responses, students were able to:

- select ab, bc and ca as the terms that needed to be substituted into either the given result or the AM-GM result with three terms
- efficiently manipulate indices and roots to achieve the result to be proved.

Areas for students to improve include:

- knowing how to find the surface area and volume of a rectangular prism
- understanding what the arithmetic mean of a set of numbers represents
- recognising which terms to substitute into the AM-GM result or the given result

- manipulating inequalities
- starting all proofs with a result known to be true rather than the result to be proved.

Part (c) (ii)

Students should:

- clearly link the proof to the result from part (i), recognising that the left-hand side of the inequality represents the volume of the rectangular prism
- state the sides of a cube are equal, and provide its volume and surface area
- substitute the expressions for volume and surface area into the inequality for part (i) to show that the left- and right-hand sides are equal
- clearly explain why this proves that the volume of a cube is the maximum possible volume for any rectangular prism with a fixed surface area.

In better responses, students were able to:

- set out their justification in a clear and concise manner
- clearly state each step of logic that was needed to prove the result
- clearly show that their proof answered the question.

Areas for students to improve include:

- knowing how to find the volume and surface area of a cube
- understanding the logical steps that are needed to prove a result
- setting out working in a clear and logical order
- explaining why their solution has proved the result.

Part (d)

Students should:

- recognise that the given information can refer to the complex roots of a polynomial, or to complex vectors
- show that the moduli of each complex number equals 1, then use conjugate rules to find the sum of the roots two at a time, then factorise the cubic equation to find that the only three solutions are $1, \pm i$
- show that the moduli of each complex number equals 1, draw a clear diagram showing that three unit vectors starting at 0 and ending at 1 form a rhombus, show that one of the vectors is 1 and the other two are negatives of each other, then use the given results to show that the other two roots are $\pm i$.

In better responses, students were able to:

- recognise that the question involved complex numbers as the roots of a polynomial
- manipulate equations involving complex numbers leaving the variable as z, rather than converting to exponential or Cartesian form
- recognise the need for and use conjugate rules
- use the sum and products of roots to find the cubic, then factorise to find the solutions
- prove that $1, \pm i$ are the only three possible solutions.

Areas for students to improve include:

- understanding and manipulating equations involving complex numbers
- applying conjugate rules in different contexts.