2021
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your student name and/or number at the top of every page

Total marks – 100

Section I – 10 marks (pages 3 - 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 - 11)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

This paper MUST NOT be removed from the examination room.

STUDENT NAME/NUMBER...

STUDENT NAME/NUMBER.......................................

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Select the alternative A, B, C, D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

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Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

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1. Given that x and y are real numbers, which of the following is a true statement ?

(A) $\forall y \left(\exists x : x^2 - y^2 = x\right)$

(B) $\forall y \left(\exists x : x^2 - y^2 = y\right)$

(C) $\forall y \left(\exists x : x^2 + y^2 = x\right)$

(D) $\forall y \left(\exists x : x^2 + y^2 = y\right)$

2. What is the radius of the circle $\left((x+2)\, \underset{\sim}{i} + (y-3)\, \underset{\sim}{j}\right) . \left((x-6)\, \underset{\sim}{i} + (y+1)\, \underset{\sim}{j}\right) = 0$?

(A) $\frac{1}{2}\sqrt{15}$

(B) $\sqrt{5}$

(C) $\sqrt{15}$

(D) $2\sqrt{5}$

3. Given that $z = 1 + 2i$ is a root of the equation $z^2 - (3+i)z + k = 0$, what is the value of k ?

(A) $k = 3i$

(B) $k = 1 - 2i$

(C) $k = 2 - i$

(D) $k = 4 + 3i$

4. What is the value of $\int_{-1}^{1} \left(\sin^{-1} x + \cos^{-1} x\right) dx$

(A) 0

(B) $\frac{\pi}{2}$

(C) π

(D) 2π

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5. Given the statement $In\ \Delta ABC,\ \sin B = 0\cdot 5 \Rightarrow B = 30°$, which of the following is correct?

(A) The contrapositive statement is false and the converse statement is false.

(B) The contrapositive statement is false and the converse statement is true.

(C) The contrapositive statement is true and the converse statement is false.

(D) The contrapositive statement is true and the converse statement is true.

6. What is an expression for $\int \frac{1}{\sin x}\,dx$?

(A) $\frac{1}{4}\ln\left|\tan\frac{x}{2}\right| + c$

(B) $\frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + c$

(C) $\ln\left|\tan\frac{x}{2}\right| + c$

(D) $2\ln\left|\tan\frac{x}{2}\right| + c$

7. A particle is moving on a line with Simple Harmonic Motion. At time t seconds it has displacement x metres from a fixed point on the line and velocity $v\ \text{ms}^{-1}$ given by $v^2 = -0\cdot 5\,x^2 + 2x + 2\cdot 5$. What is the period of the motion?

(A) π seconds

(B) $\pi\sqrt{2}$ seconds

(C) 2π seconds

(D) $2\pi\sqrt{2}$ seconds

8. The lines $\underset{\sim}{r}_1 = \begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ a \end{bmatrix}$ and $\underset{\sim}{r}_2 = \mu \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, where λ and μ are scalar parameters and a is constant, are skew. Which of the following is NOT correct ?

(A) a can equal -2

(B) a can equal -1

(C) a can equal 1

(D) a can equal 2

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9. A body of mass m kg moves in a straight line with initial speed U ms^{-1} subject to a resistance force of magnitude $m(1+v)$ Newtons when its speed is v ms^{-1}. What is the time taken by the body in coming to rest ?

(A) $\frac{1}{1+U}$ seconds

(B) $\sqrt{1+U}$ seconds

(C) $\ln(1+U)$ seconds

(D) e^{1+U} seconds

10. Given $z_1 = \cos A + i\sin A$ and $z_2 = \cos B + i\sin B$, what is the value of $\arg\left(\frac{z_1 z_2}{z_1+z_2}\right)$?

(A) $\frac{1}{2}AB$

(B) $\frac{1}{2}(A+B)$

(C) AB

(D) $A+B$

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Marks

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer the questions in writing booklets provided. Use a separate writing booklet for each question. In Questions 11-16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

(a)(i) In an Argand diagram draw the locus of a point representing the complex number z such that $\left| z-(3+3i) \right| = \sqrt{3}$. **2**

(ii) If the point P on the locus that is closest to the origin represents the complex number z_1, find the modulus and principal argument of z_1. **2**

(b)(i) Find numbers A, B and C such that $\dfrac{1}{x^2(1+x)} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$. **2**

(ii) Hence evaluate in simplest exact form $\displaystyle\int_1^3 \frac{1}{x^2(x+1)}\,dx$. **3**

(c)(i) Express $z = -1+\sqrt{3}i$ in the form $re^{i\theta}$ where $r>0$ and $-\pi<\theta\le\pi$. **2**

(ii) Express $\overline{z}$, z^2 and $\dfrac{1}{z}$ in the form $re^{i\theta}$ where $r>0$ and $-\pi<\theta\le\pi$. **3**

(iii) Show the points Q, R and S representing $\overline{z}$, z^2 and $\dfrac{1}{z}$ respectively in an Argand diagram. **1**

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Marks

Question 12 (15 marks) **Use a separate writing booklet.**

(a)(i) Use the substitution $x = 4\sin^2\theta,\ 0 \le \theta \le \frac{\pi}{2}$ to show that **2**

$$\int_2^3 \frac{\sqrt{x}}{(4-x)^{\frac{3}{2}}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\tan^2\theta\,d\theta\,.$$

(ii) Hence evaluate in simplest exact form $\displaystyle\int_2^3 \frac{\sqrt{x}}{(4-x)^{\frac{3}{2}}}\,dx$. **2**

(b)(i) Express $1+\cos\theta+i\sin\theta$ in modulus / argument form. **2**

(ii) Hence show that $1+4\cos\theta+6\cos 2\theta+4\cos 3\theta+\cos 4\theta = 16\cos^4\frac{\theta}{2}\cos 2\theta$. **3**

(c) With respect to a fixed origin O, the lines L_1 and L_2 have equations

$\underset{\sim}{r}_1 = \begin{bmatrix} 11 \\ 2 \\ 17 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}$ and $\underset{\sim}{r}_2 = \begin{bmatrix} -5 \\ 11 \\ p \end{bmatrix} + \mu \begin{bmatrix} q \\ 2 \\ 2 \end{bmatrix}$ respectively, where λ and μ

are scalar parameters and p and q are constants.

(i) If L_1 and L_2 intersect at right angles, show $q=-3$ and find the value of p. **4**

(ii) Find the coordinates of the point of intersection. **2**

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Marks

Question 13 (15 marks) **Use a separate writing booklet.**

(a) Given the vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ satisfy $\underset{\sim}{u}+\underset{\sim}{v}=17\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}$ and $\underset{\sim}{u}-\underset{\sim}{v}=\underset{\sim}{i}+9\underset{\sim}{j}-4\underset{\sim}{k}$, find the acute angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **3**

(b) Prove that for all positive integers n and p, where p is prime, there exists no positive integer m such that $(3m+2)^2 = n^2 + p$. **4**

(c) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} and acceleration a ms^{-2} where $a = 6x^2$. Initially the particle is 1 m to the right of O moving towards O with speed 2 ms^{-1}.

(i) Find expressions for v as a function of x and for x as a function of t. **3**

(ii) Describe the limiting motion of the particle. **1**

(d) A particle is performing Simple Harmonic Motion as it moves in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, where x is given by $x = 1 + 2\cos\left(2t - \frac{\pi}{6}\right)$.

(i) Find the time taken for the particle to first reach maximum speed. **2**

(ii) Find in simplest exact form the distance travelled by the particle in first reaching O. **2**

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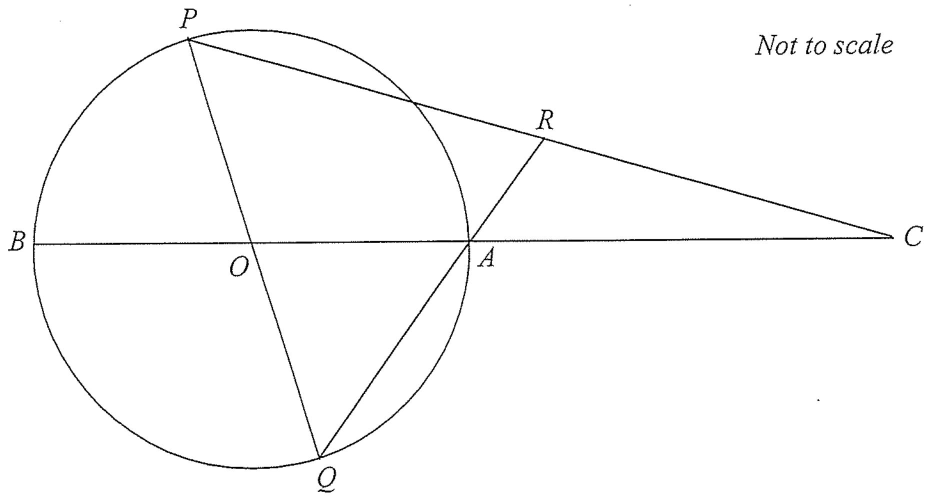
Marks

Question 14 (15 marks) **Use a separate writing booklet.**

(a)



In the diagram, PQ and AB are diameters of a circle with centre O. BA is produced to C so that $BA = AC$. QA produced meets PC in R. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OP} = \underset{\sim}{p}$. Given $\overrightarrow{PR} = \lambda \overrightarrow{PC}$ and $\overrightarrow{QR} = \mu \overrightarrow{QA}$ for some scalars λ and μ, show that R is the midpoint of PC. **4**

(b) It is given that $a^2 + b^2 \geq 2ab$ for any real numbers $a > 0$ and $b > 0$.
DO NOT PROVE THIS RESULT.

(i) If $a > 0,\ b > 0,\ c > 0$ and $d > 0$ are real numbers, show that $a^4 + b^4 + c^4 + d^4 \geq 4abcd$. **2**

(ii) If additionally $a^4 + b^4 + c^4 + d^4 \leq 4$, show that $a^{-4} + b^{-4} + c^{-4} + d^{-4} \geq 4$. **3**

(c) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x \, dx$ for $n = 0, 1, 2, 3, \ldots$

(i) Use one application of integration by parts to show that **4**

$$I_n = \frac{n-1}{n+2} I_{n-2} \quad \text{for} \quad n = 2, 3, 4, 5, \ldots$$

(ii) Hence evaluate I_5 in simplest exact form. **2**

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Question 15 (15 marks) **Use a separate writing booklet.**

(a)(i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

(ii) Hence, or otherwise, find the value of $\int_0^{\pi} \frac{\sin x}{e^{\frac{\pi}{2}} + e^x}\,dx$. **4**

(b)(i) Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $a_0, a_1, \ldots, a_{n-1}, a_n$ are integers such that $a_n \neq 0$ and $a_0 \neq 0$. Show that if p and q are integers with no common factor and $\alpha = \frac{p}{q}$ is a rational root of $P(x) = 0$, then p is a divisor of a_0 and q is a divisor of a_n. **3**

(ii) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and hence show $x = \cos\frac{\pi}{9}$ is a root of $8x^3 - 6x - 1 = 0$. **3**

(iii) Hence show that $\cos\frac{\pi}{9}$ is irrational. **3**

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Marks

Question 16 (15 marks) **Use a separate writing booklet.**

(a)(i) By considering the graph of $y = \dfrac{1}{x\sqrt{x}}$, or otherwise, show that for all positive integers $k \geq 1$, $\dfrac{1}{(k+1)\sqrt{k+1}} < \dfrac{2}{\sqrt{k}} - \dfrac{2}{\sqrt{k+1}}$. **2**

(ii) Hence use Mathematical Induction to show that for all positive integers $n \geq 2$, **4**

$$\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \ldots + \frac{1}{n\sqrt{n}} < 3 - \frac{2}{\sqrt{n}}.$$

(b) A particle of mass m kg falls vertically from rest under gravity in a medium where the resistance to motion has magnitude $\frac{1}{g}mv^2$ Newtons when the speed of the particle is v ms^{-1}, the acceleration due to gravity being g ms^{-2}. At time t seconds the particle has fallen x metres and has velocity v ms^{-1}.

(i) Show that $\ddot{x} = \dfrac{1}{g}\left(g^2 - v^2\right)$. **1**

(ii) Show that $v = g\left(\dfrac{e^{2t}-1}{e^{2t}+1}\right)$ and $v^2 = g^2\left(1 - e^{-\frac{2x}{g}}\right)$. **5**

(iii) Find in simplest exact form the time taken and the distance fallen by the particle in reaching half of its terminal velocity. **3**

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

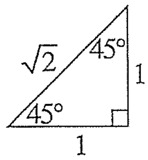
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

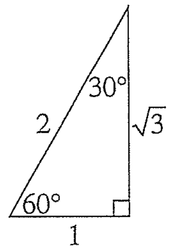
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \; \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \; \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \; \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A\cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A\sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

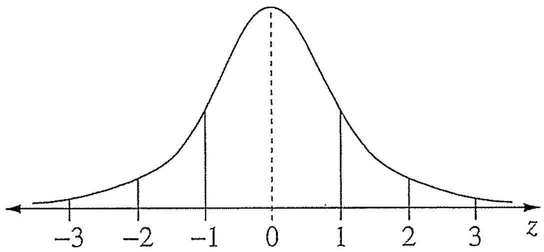
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \; P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$$

$$X \sim \text{Bin}(n, p)$$

$$\Rightarrow \; P(X = x)$$

$$= \binom{n}{x} p^x (1-p)^{n-x}, \; x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\text{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

Independent Trial HSC 2021 — Mathematics Extension 2 — Marking Guidelines

Section 1 — Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1	A	$\forall$ real y, $x^2-y^2=x$ is a quadratic equation in x with real solutions given by $(x-\frac{1}{2})^2=y^2+\frac{1}{4}$. Hence A is true. $y=-\frac{1}{2}$ provides a counter example for B since $\exists$ no real x : $x^2=\frac{1}{4}-\frac{1}{2}$ $y=1$ provides a counter example for C since the quadratic equation $x^2+1=x$ can be written $(x-\frac{1}{2})^2=-1+\frac{1}{4}$ and has no real solutions $y=2$ provides a counter example for D since $\exists$ no real x : $x^2+4=2$	MEX12-2
2	D	Equation is $(x+2)(x-6)+(y-3)(y+1)=0$, giving $x^2-4x+y^2-2y=15$ Hence circle has equation $(x-2)^2+(y-1)^2=20$. Radius is $\sqrt{20}=2\sqrt{5}$	MEX12-3
3	D	Sum of the roots is $3+i$. $\therefore$ roots are $1+2i$, $2-i$ with product $4+3i=k$	MEX12-4
4	C	$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$. Hence definite integral has value $\frac{\pi}{2}\left(1-(-1)\right)=\pi$	MEX12-5
5	B	The contrapositive is $B\neq 30°\Rightarrow \sin B\neq 0\cdot 5$ False, as $\sin 150°=0\cdot 5$ The converse is $B=30°\Rightarrow \sin B=0\cdot 5$ True	MEX12-2
6	C	$t=\tan\frac{x}{2}$ $dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$ $\frac{2dt}{1+t^2}=dx$ $\frac{1}{\sin x}=\frac{1+t^2}{2t}$ $\int\frac{1}{\sin x}dx=\int\frac{1}{t}dt=\ln\lvert t\rvert+c=\ln\left\lvert\tan\frac{x}{2}\right\rvert+c$	MEX12-5
7	D	$v^2=\frac{1}{2}\left(-x^2+4x+5\right)=\left(\frac{1}{\sqrt{2}}\right)^2\left\{9-\left(x-2\right)^2\right\}$ $\therefore n=\frac{1}{\sqrt{2}}$ and $T=\frac{2\pi}{n}=2\sqrt{2}\pi$	MEX12-6
8	A	$\begin{aligned}1+\lambda&=\mu\\-6+2\lambda&=-2\mu\end{aligned}$ has solution $\begin{aligned}\lambda&=1\\\mu&=2\end{aligned}$. Hence for lines to be skew $4+a\lambda\neq\mu$ for $\lambda=1,\ \mu=2$ $4+a\neq 2$ $\therefore a\neq -2$	MEX12-3
9	C	$\frac{dv}{dt}=-(1+v)$ $\int\frac{1}{1+v}dv=-\int dt$ Let $t=T$ when $v=0$ $\int_U^0\frac{1}{1+v}dv=-\int_0^T dt$ $\therefore T=\ln(1+U)$ $\left[\ln(1+v)\right]_U^0=-T$	MEX12-6
10	B	$z_1+z_2=\left(\cos A+\cos B\right)+i\left(\sin A+\sin B\right)$ $=2\left\{\cos\left(\frac{1}{2}(A+B)\right)\cos\left(\frac{1}{2}(A-B)\right)+i\sin\left(\frac{1}{2}(A+B)\right)\cos\left(\frac{1}{2}(A-B)\right)\right\}$ $=2\cos\left(\frac{1}{2}(A-B)\right)\left\{\cos\left(\frac{1}{2}(A+B)\right)+i\sin\left(\frac{1}{2}(A+B)\right)\right\}$ $\therefore \arg\left(\frac{z_1z_2}{z_1+z_2}\right)=\arg\left(z_1z_2\right)-\arg\left(z_1+z_2\right)=A+B-\frac{1}{2}\left(A+B\right)=\frac{1}{2}\left(A+B\right)$	MEX12-4

Section II

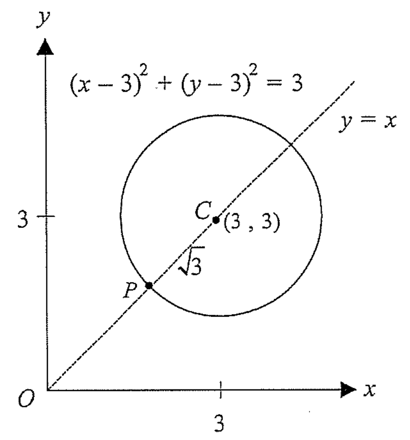
Question 11

a.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Sketches circle with correct centre and radius	2
Substantial progress eg. circle with correct centre	1

Answer



a.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Finds modulus and principal argument of z_1	2
Substantial progress eg. finds one of the modulus and argument	1

Answer

OPC is a line with equation $y=x$ $\quad\therefore \text{Arg } z_1=\frac{\pi}{4}$

$OP=OC-CP=3\sqrt{2}-\sqrt{3}$ $\quad\therefore |z_1|=3\sqrt{2}-\sqrt{3}$

b.i. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Uses properties of an identity to find the values of A, B, C	2
Substantial progress eg. finds one of A, B, C	1

Answer

$$1\equiv Ax(1+x)+B(1+x)+Cx^2$$

put $x=-1$: $\quad 1=C$

put $x=0$: $\quad 1=B$

equate coeffs of x^2: $0=A+C \quad \therefore A=-1$

$$\therefore \frac{1}{x^2(1+x)}\equiv\frac{-1}{x}+\frac{1}{x^2}+\frac{1}{x+1}$$

Q11b (cont)

b.ii. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Finds the anti-derivative and evaluates in simplest exact form	3
Substantial progress eg. finds the anti-derivative but one error in simplest exact evaluation	2
Some progress eg. finds anti-derivative	1

Answer

$$\int_1^3 \frac{1}{x^2(x+1)}\,dx = \int_1^3 \left(-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx$$

$$= \left[-\frac{1}{x}+\ln\left(\frac{x+1}{x}\right)\right]_1^3$$

$$= -\left(\tfrac{1}{3}-1\right)+\ln\tfrac{4}{3}-\ln 2$$

$$= \tfrac{2}{3}-\ln\tfrac{3}{2}$$

c.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Writes z in required form	2
Substantial progress eg. finds one of r and θ	1

Answer

$$z = 2\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) = 2e^{i\frac{2\pi}{3}}$$

c.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Writes all three functions of z in required form	3
Substantial progress eg. writes two of the expressions correctly	2
Some progress eg. writes one of the expressions correctly	1

Answer

$\overline{z} = 2e^{-i\frac{2\pi}{3}}$, $z^2 = 4e^{-i\frac{2\pi}{3}}$ and $\frac{1}{z} = \frac{1}{2}e^{-i\frac{2\pi}{3}}$

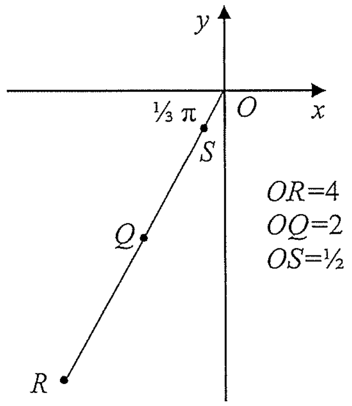
Q11c (cont)

c.iii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Shows points in correct relative positions in an Argand diagram.	1

Answer



Question 12

a.i. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Performs substitution to give required result	2
Substantial progress eg. writes integrand in terms of θ, or converts dx and limits	1

Answer

$$x = 4\sin^2\theta,\quad 0 \le \theta \le \tfrac{\pi}{2}$$
$$dx = 8\sin\theta\cos\theta\, d\theta$$
$$x = 2 \Rightarrow \theta = \tfrac{\pi}{4}$$
$$x = 3 \Rightarrow \theta = \tfrac{\pi}{3}$$

$$\frac{\sqrt{x}}{(4-x)^{\frac{3}{2}}} = \frac{2\sin\theta}{8\cos^3\theta} = \frac{2\tan^2\theta}{8\sin\theta\cos\theta}$$

$$\therefore \int_2^3 \frac{\sqrt{x}}{(4-x)^{\frac{3}{2}}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\tan^2\theta\, d\theta$$

a.ii. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Uses an appropriate trig identity to find anti-derivative and evaluates	2
Substantial progress eg. finds anti-derivative	1

Answer

$$\begin{aligned}
\int_2^3 \frac{\sqrt{x}}{(4-x)^{\frac{3}{2}}}\,dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\tan^2\theta\, d\theta \\
&= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\sec^2\theta - 1\right) d\theta \\
&= 2\left[\tan\theta - \theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
&= 2\left\{\left(\sqrt{3}-1\right) - \left(\tfrac{\pi}{3} - \tfrac{\pi}{4}\right)\right\} \\
&= 2\sqrt{3} - 2 - \tfrac{\pi}{6}
\end{aligned}$$

Q12 (cont)

b.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Uses appropriate trig identities to express the complex number in the required form	2
Substantial progress eg. uses double angle identities	1

Answer

$$\begin{aligned}1+\cos\theta+i\sin\theta &= 2\cos^2\tfrac{\theta}{2}+2i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\\ &= 2\cos\tfrac{\theta}{2}(\cos\tfrac{\theta}{2}+i\sin\tfrac{\theta}{2})\end{aligned}$$

b.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Equates real parts of two equivalent expressions for $(1+\cos\theta+i\sin\theta)^4$ to obtain result	3
Substantial progress eg. expands $(1+\cos\theta+i\sin\theta)^4$ using Binomial and de Moivre's theorems	2
Some progress eg. recognises RHS as $\left\{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right\}^4$	1

Answer

Using the Binomial theorem and de Moivre's theorem,

$$\left\{1+\left(\cos\theta+i\sin\theta\right)\right\}^4=1+4\left(\cos\theta+i\sin\theta\right)+6\left(\cos2\theta+i\sin2\theta\right)+4\left(\cos3\theta+i\sin3\theta\right)+\left(\cos4\theta+i\sin4\theta\right)$$

$$\left\{2\cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}+i\sin\tfrac{\theta}{2}\right)\right\}^4=16\cos^4\tfrac{\theta}{2}\left(\cos2\theta+i\sin2\theta\right)$$

Equating real parts gives $1+4\cos\theta+6\cos2\theta+4\cos3\theta+\cos4\theta=16\cos^4\frac{\theta}{2}\cos2\theta$

c.i. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Uses zero dot product of direction vectors to find q then consistent system of equations to find p	4
Substantial progress eg. finds q and writes system of equations but makes one error in solution	3
Moderate progress eg. finds q and writes system of equations	2
Some progress eg. finds q	1

Answer

L_1, L_2 perpendicular $\quad \therefore \begin{bmatrix}-2\\1\\-4\end{bmatrix}\cdot\begin{bmatrix}q\\2\\2\end{bmatrix}=0 \qquad \therefore\ -2q+2-8=0$

$$q=-3$$

L_1, L_2 intersect. Hence the system of equations

$$\begin{aligned}11-2\lambda&=-5-3\mu \quad (1)\\ 2+\lambda&=11+2\mu \quad (2)\\ 17-4\lambda&=p+2\mu \quad (3)\end{aligned}$$

is consistent.

$(1)+2\times(2) \Rightarrow 15=17+\mu \quad \therefore \mu=-2$

Then from (2) $\quad \lambda=5 \qquad$ Then from (3) $\quad 17-20=p-4 \quad \therefore p=1$

Q12c (cont)

c.ii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Finds all three coordinates	2
Substantial progress eg. finds one of the coordinates	1

Answer

At intersection point, $\begin{aligned} x &= 11-2\lambda = 1 \\ y &= 2+\lambda = 7 \\ z &= 17-4\lambda = -3 \end{aligned}$. Hence lines intersect at $(1,7,-3)$

Question 13

a. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Finds vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ and then the acute angle between them.	3
Substantial progress eg. correct process but makes one error	2
Some progress eg. finds $\underset{\sim}{u}$ and $\underset{\sim}{v}$.	1

Answer

$$\underset{\sim}{u}+\underset{\sim}{v} = 17\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k} \quad (1)$$

$$\underset{\sim}{u}-\underset{\sim}{v} = \underset{\sim}{i}+9\underset{\sim}{j}-4\underset{\sim}{k} \quad (2)$$

$$(1)+(2) \Rightarrow 2\underset{\sim}{u} = 18\underset{\sim}{i}+8\underset{\sim}{j}-2\underset{\sim}{k}$$

$$(1)-(2) \Rightarrow 2\underset{\sim}{v} = 16\underset{\sim}{i}-10\underset{\sim}{j}+6\underset{\sim}{k}$$

$\therefore \underset{\sim}{u} = \begin{bmatrix} 9 \\ 4 \\ -1 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} 8 \\ -5 \\ 3 \end{bmatrix}$. If the angle between them is θ,

$$\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \frac{72-20-3}{\sqrt{9^2+4^2+1^2}\times\sqrt{8^2+5^2+3^2}} = \frac{1}{2}$$

Hence the acute angle between the vectors is $\frac{\pi}{3}$.

b. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Constructs a proof with full explanation using properties of positive integers and prime numbers	4
Substantial progress eg. correct process but some explanatory detail unclear	3
Moderate progress eg. factorises p and realises these factors must be 1 and p	2
Some progress eg. factorises expression for p	1

Answer

Let m, n and p be positive integers where p is prime and $(3m+2)^2 = n^2+p$.

Then $\begin{aligned} p &= (3m+2)^2 - n^2 \\ &= (3m+2-n)(3m+2+n) \end{aligned}$.

Now m, n and p are positive integers $\Rightarrow (3m+2-n)$ and $(3m+2+n)$ are positive integers such that

$$0 < (3m+2-n) < (3m+2+n)$$

But p is prime. Hence $3m+2-n=1$ and $3m+2+n=p$.

Then $\begin{aligned} p+1 &= 6m+4 \\ p &= 3(2m+1) \end{aligned}$. Now p prime $\Rightarrow 2m+1=1$ and hence $m=0$.

Hence by contradiction, $\forall$ positive integers n and p where p is prime, $\exists$ no positive integer m such that $(3m+2)^2 = n^2+p$.

Q13 (cont)

c.i. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Uses integration to solve a pair of D.E.'s to find v as a function of x and x as a function of t	3
Substantial progress eg. correct procedure but one error made	2
Some progress eg. finds v^2 in terms of x	1

Answer

$$v\frac{dv}{dx}=6x^2$$
$$\int 2v\,dv=\int 12x^2\,dx$$
$$v^2=4x^3+c$$
$$x=1,\ v=-2\Rightarrow c=0$$
$$\therefore\quad v^2=4x^3$$
$$\therefore\quad v=-2x\sqrt{x}$$

$$\frac{dx}{dt}=-2x^{\frac{3}{2}}$$
$$\int -\tfrac{1}{2}x^{-\frac{3}{2}}\,dx=\int dt$$
$$x^{-\frac{1}{2}}=t+d$$
$$t=0,\ x=1\Rightarrow d=1$$
$$\therefore\quad x^{-\frac{1}{2}}=t+1$$
$$x=\frac{1}{(t+1)^2}$$

c.ii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Describes the limiting behavior of the particle	1

Answer

Particle continues moving towards O with speed approaching zero, but the particle never reaches O which is its limiting position.

d.i. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Solves trig.equation to find first time when particle is at centre of its motion.	2
Substantial progress eg writes trig. equation for t	1

Answer

Maximum speed occurs at the centre of the motion where $x=1$ and $\cos(2t-\frac{\pi}{6})=0$.

First reaches max speed when $2t-\frac{\pi}{6}=\frac{\pi}{2}$, that is at time $\frac{\pi}{3}$ seconds

d.ii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Determines path travelled and hence the required distance	2
Substantial progress eg. finds its initial position and direction of travel	1

Answer

$x=1+2\cos(2t-\frac{\pi}{6})$ $\qquad$ $\dot{x}=-4\sin(2t-\frac{\pi}{6})$ $\qquad$ $t=0\Rightarrow x=1+\sqrt{3}$ and $\dot{x}=2$

Initially particle is at $x=1+\sqrt{3}$ and moving away from the centre $x=1$ towards the extreme at $x=3$, then travelling back to $x=0$. The total distance travelled is $5-\sqrt{3}$ metres.

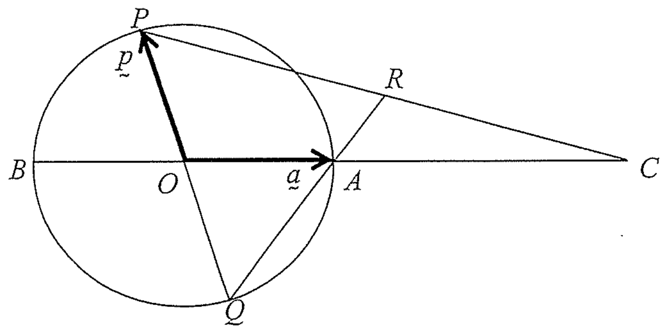
Question 14

a. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Writes vectors $\overrightarrow{QP}$, $\overrightarrow{PR}$ and $\overrightarrow{QR}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{p}$, λ and μ to find value of λ to deduce result	4
Substantial progress eg. obtains a relationship between $\underset{\sim}{a}$, $\underset{\sim}{p}$, λ and μ	3
Moderate progress eg. writes $\overrightarrow{QP}$, $\overrightarrow{PR}$ and $\overrightarrow{QR}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{p}$, λ and μ	2
Some progress eg. writes $\overrightarrow{OQ}$ and $\overrightarrow{OC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{p}$	1

Answer



$\overrightarrow{OQ} = -\underset{\sim}{p}$ and $\overrightarrow{OC} = 3\underset{\sim}{a}$. $\quad \therefore \overrightarrow{PR} = \lambda \overrightarrow{PC} = \lambda(3\underset{\sim}{a} - \underset{\sim}{p})$ and $\overrightarrow{QR} = \mu \overrightarrow{QA} = \mu(\underset{\sim}{a} + \underset{\sim}{p})$

$$\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PR}$$

$$\mu(\underset{\sim}{a} + \underset{\sim}{p}) = 2\underset{\sim}{p} + \lambda(3\underset{\sim}{a} - \underset{\sim}{p})$$

$$(\mu - 3\lambda)\underset{\sim}{a} = (2 - \mu - \lambda)\underset{\sim}{p}$$

But $\underset{\sim}{a}$ and $\underset{\sim}{p}$ are not parallel. Hence $\mu - 3\lambda = 0 \quad (1)$

and $\mu + \lambda = 2 \quad (2)$

$(2)-(1)$ gives $\quad 4\lambda = 2 \qquad \therefore \lambda = \frac{1}{2}$ and $\overrightarrow{PR} = \frac{1}{2}\overrightarrow{PC}$

Hence R is the midpoint of PC

b.i. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses the given result with appropriate replacements for a, b to obtain required inequality	2
Substantial progress eg. replaces a, b by a^2, b^2 and by c^2, d^2 in given inequality	1

Answer

$a^4 + b^4 \geq 2a^2b^2$

$c^4 + d^4 \geq 2c^2d^2$

$$\therefore a^4 + b^4 + c^4 + d^4 \geq 2(a^2b^2 + c^2d^2) = 2\left\{(ab)^2 + (cd)^2\right\} \geq 4abcd$$

Q14b (cont)

b.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Replaces a, b, c, d in (i) by reciprocals then completes deduction using (i) and new condition	3
Substantial progress eg. uses appropriate replacements in (i); writes inequality for $\frac{1}{abcd}$ from (i)	2
Some progress eg. uses appropriate replacements in (i)	1

Answer

$a \to a^{-1},\ b \to b^{-1}, c \to c^{-1}, d \to d^{-1}$ gives $a^{-4}+b^{-4}+c^{-4}+d^{-4} \geq \frac{4}{abcd}$

But from result in (i), $\frac{a^4+b^4+c^4+d^4}{4} \geq abcd$

Hence if $a^4+b^4+c^4+d^4 \leq 4$, then $\frac{1}{abcd} \geq \frac{4}{a^4+b^4+c^4+d^4} \geq 1$ $\quad \therefore a^{-4}+b^{-4}+c^{-4}+d^{-4} \geq 4$

c.i. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Carries out an appropriate integration by parts with suitable rearrangement to produce result	4
Substantial progress eg. correct process but one error	3
Moderate progress eg. carries out an appropriate integration by parts with evaluation of first term	2
Some progress eg. attempts integration by parts with some success	1

Answer

For $n \geq 2$

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x\, dx \\
&= \int_0^{\frac{\pi}{2}} \cos^{n-1} x\, (\cos x \sin^2 x)\, dx \\
&= \tfrac{1}{3}\left[\sin^3 x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)\cos^{n-2} x\,(-\sin x)\,\tfrac{1}{3}\sin^3 x\, dx \\
&= 0 + \frac{n-1}{3}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\,(1-\cos^2 x)\sin^2 x\, dx \\
3I_n &= (n-1)\{I_{n-2} - I_n\} \\
(n+2)I_n &= (n-1)I_{n-2} \\
\therefore I_n &= \frac{n-1}{n+2} I_{n-2} \quad \text{for } n = 2, 3, 4, 5, \ldots
\end{aligned}$$

c.ii. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Uses the recurrence relation and evaluates I_1 to evaluate I_5	2
Substantial progress eg. uses the recurrence relation	1

Answer

$I_5 = \frac{4}{7} I_3 = \frac{4}{7} \times \frac{2}{5} I_1$ where $I_1 = \int_0^{\frac{\pi}{2}} \cos x \sin^2 x\, dx = \frac{1}{3}\left[\sin^3 x\right]_0^{\frac{\pi}{2}} = \frac{1}{3}$ $\quad \therefore I_5 = \frac{8}{105}$

Question 15

a.i. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Uses appropriate substitution to prove required result by applying property of definite integral	2
Substantial progress eg. correct procedure but with one error or incomplete explanation	1

Answer

$$u = a - x$$
$$du = -dx$$

$$x = 0 \Rightarrow u = a$$
$$x = a \Rightarrow u = 0$$

$$\begin{aligned}\int_0^a f(x)\,dx &= -\int_a^0 f(a-u)\,du \\ &= \int_0^a f(a-u)\,du \\ &= \int_0^a f(a-x)\,dx\end{aligned}$$

a.ii. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Applies result from (i) or uses substitution to transform integral then completes evaluation	4
Substantial progress eg. correct procedure with one error or incomplete explanation	3
Moderate progress eg. result from (i), using trig identity, removing factor $e^{\frac{\pi}{2}-x}$ from denominator	2
Some progress eg. applies result from (i)	1

Answer

Let $I = \displaystyle\int_0^{\pi} \frac{\sin x}{e^{\frac{\pi}{2}} + e^x}\,dx$.

Then from (i) $I = \displaystyle\int_0^{\pi} \frac{\sin(\pi - x)}{e^{\frac{\pi}{2}} + e^{\pi - x}}\,dx$

$$= \int_0^{\pi} \frac{e^{-\frac{\pi}{2}} e^x \sin x}{e^x + e^{\frac{\pi}{2}}}\,dx$$

$$\therefore\ e^{\frac{\pi}{2}} I = \int_0^{\pi} \frac{e^x \sin x}{e^{\frac{\pi}{2}} + e^x}\,dx$$

$$\begin{aligned}\therefore\ 2e^{\frac{\pi}{2}} I &= \int_0^{\pi} \frac{e^{\frac{\pi}{2}} \sin x}{e^{\frac{\pi}{2}} + e^x}\,dx + \int_0^{\pi} \frac{e^x \sin x}{e^{\frac{\pi}{2}} + e^x}\,dx \\ &= \int_0^{\pi} \frac{\left(e^{\frac{\pi}{2}} + e^x\right)\sin x}{e^{\frac{\pi}{2}} + e^x}\,dx \\ &= \int_0^{\pi} \sin x\,dx \\ &= -\left[\cos x\right]_0^{\pi} \\ &= 2\end{aligned}$$

$$\therefore\ I = e^{-\frac{\pi}{2}}$$

Q15 (cont)

b.i. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Rearranges $P(\frac{p}{q})=0$ in two different ways and uses number properties to deduce results	3
Substantial progress eg. uses one rearrangement to deduce one of the required results	2
Some progress eg. states $(qx-p)$ is a factor of $P(x)$ but further argument is unconvincing	1

Answer

$$P\left(\tfrac{p}{q}\right)=a_n\left(\tfrac{p}{q}\right)^n+a_{n-1}\left(\tfrac{p}{q}\right)^{n-1}+a_{n-2}\left(\tfrac{p}{q}\right)^{n-2}+\ldots+a_1\left(\tfrac{p}{q}\right)+a_0$$

$$P\left(\tfrac{p}{q}\right)=0 \Rightarrow -q^n a_0=p\left(a_n p^{n-1}+a_{n-1}qp^{n-2}+\ldots+a_1 q^{n-1}\right)$$

But the bracketed expression is an integer and p and q have no common factors. Hence p is a factor of a_0 .

Also $P\left(\tfrac{p}{q}\right)=0 \Rightarrow -p^n a_n=q\left(a_{n-1}p^{n-1}+a_{n-2}qp^{n-2}+\ldots+a_1 q^{n-2}p+a_0 q^{n-1}\right)$

Again the bracketed expression is an integer and p and q have no common factors. Hence q is a factor of a_n

b.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves identity then shows equation equivalent to $x=\cos\theta,\ \cos 3\theta=\frac{1}{2}$ to deduce $\cos\frac{\pi}{9}$ is a root	3
Substantial progress eg. proves identity and uses $x=\cos\theta$ to transform equation	2
Some progress eg. proves identity	1

Answer

$$\begin{aligned}\cos 3\theta&=\cos\theta\cos 2\theta-\sin\theta\sin 2\theta\\&=\cos\theta\left(2\cos^2\theta-1\right)-2\cos\theta\sin^2\theta\\&=\cos\theta\left\{\left(2\cos^2\theta-1\right)-2\left(1-\cos^2\theta\right)\right\}\\&=\cos\theta\left(4\cos^2\theta-3\right)\\&=4\cos^3\theta-3\cos\theta\end{aligned}$$

$\cos 3\theta=\frac{1}{2} \Leftrightarrow 8\cos^3-6\cos\theta-1=0$

$\therefore$ roots of $8x^3-6x-1=0$ have the form

$x=\cos\theta$ where $\cos 3\theta=\frac{1}{2}$

$$3\theta=\pm\tfrac{\pi}{3}+2m\pi,\ m=0,\pm1,\pm2,\ldots$$

$$\theta=\tfrac{\pi}{9}(6m\pm1)$$

$\therefore x=\cos\frac{\pi}{9}$ is a root

b.iii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses (i) to make a list of possible rational values of $\cos\frac{\pi}{9}$ and shows none satisfy the identity	3
Substantial progress eg. lists possible positive rational values and eliminates two of them	2
Some progress eg. lists possible rational values	1

Answer

If $\cos\frac{\pi}{9}$ is rational, then $\exists$ integers p, q with no common factor such that $\cos\frac{\pi}{9}=\frac{p}{q}$.

Then using (i), p is a divisor of -1 and q is a divisor of 8.

Since $\cos\frac{\pi}{9}>0$, the only possible rational values of $\cos\frac{\pi}{9}$ are $1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8}$

Clearly $\cos\frac{\pi}{9}\neq 1$ and $\cos\frac{\pi}{9}\neq\frac{1}{2}$ since $y=\cos x$ is strictly decreasing for $0\le x\le\frac{\pi}{2}$ and $\cos 0=1,\ \cos\frac{\pi}{3}=\frac{1}{2}$.

$\cos\frac{\pi}{9}=\frac{1}{4}\Rightarrow 4\cos^3\frac{\pi}{9}-3\cos\frac{\pi}{9}=4\times\left(\frac{1}{4}\right)^3-3\left(\frac{1}{4}\right)=-\frac{11}{16}\neq\frac{1}{2}=\cos\frac{3\pi}{9} \quad \therefore \cos\frac{\pi}{9}\neq\frac{1}{4}$

$\cos\frac{\pi}{9}=\frac{1}{8}\Rightarrow 4\cos^3\frac{\pi}{9}-3\cos\frac{\pi}{9}=4\times\left(\frac{1}{8}\right)^3-3\left(\frac{1}{8}\right)=-\frac{47}{128}\neq\frac{1}{2}=\cos\frac{3\pi}{9} \quad \therefore \cos\frac{\pi}{9}\neq\frac{1}{8}$

Hence $\cos\frac{\pi}{9}$ is irrational.

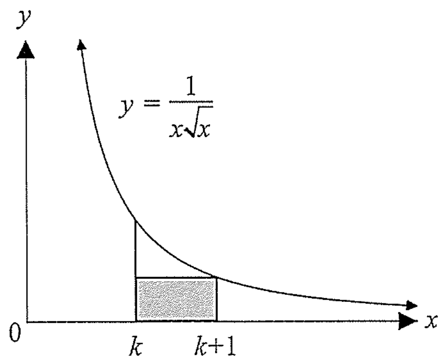
Question 16

a.i. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Considers area under curve between $x=k$ and $x=k+1$ to obtain required inequality	2
Substantial progress eg. correct process but error in evaluating definite integral	1

Answer



Area of shaded rectangle is $\frac{1}{(k+1)\sqrt{k+1}}$

$$\therefore \frac{1}{(k+1)\sqrt{k+1}} < \int_k^{k+1} \frac{1}{x\sqrt{x}}\,dx$$
$$= -\left[\frac{2}{\sqrt{x}}\right]_k^{k+1}$$
$$= \frac{2}{\sqrt{k}} - \frac{2}{\sqrt{k+1}}$$

a.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Applies the process of Mathematical Induction correctly to prove the required result	4
Substantial progress eg. process applied correctly but some aspect of explanation unclear	3
Moderate progress eg. establishes truth of P_2 and incorporates P_k in LHS of P_{k+1}	2
Some progress eg. establishes truth of P_2	1

Answer

Let P_n, $n=2,3,4,\ldots$ be the sequence of propositions $\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\ldots+\frac{1}{n\sqrt{n}}<3-\frac{2}{\sqrt{n}}$.

Consider P_2 : $\text{LHS}=\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}<1+\left(\frac{2}{\sqrt{1}}-\frac{2}{\sqrt{2}}\right)=3-\frac{2}{\sqrt{2}}=\text{RHS}$ (using (i) with $k=1$) $\therefore$ P_2 is true.

If P_k is true : $\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\ldots+\frac{1}{k\sqrt{k}}<3-\frac{2}{\sqrt{k}}$ *

Consider P_{k+1} : $\text{LHS}=\left(\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\ldots+\frac{1}{k\sqrt{k}}\right)+\frac{1}{(k+1)\sqrt{k+1}}$

$$<3-\frac{2}{\sqrt{k}}+\frac{1}{(k+1)\sqrt{k+1}} \quad \text{if } P_k \text{ is true, using *}$$
$$<3-\frac{2}{\sqrt{k}}+\frac{2}{\sqrt{k}}-\frac{2}{\sqrt{k+1}} \quad \text{using result from (i)}$$
$$=3-\frac{2}{\sqrt{k+1}}$$
$$=\text{RHS}$$

Hence if P_k is true then P_{k+1} is true. But P_2 is true. Hence by Mathematical Induction, P_n is true for all positive integers $n \geq 2$.

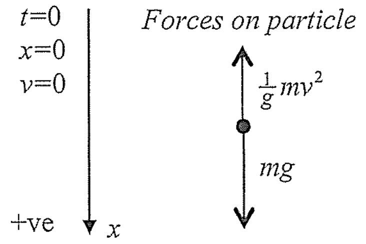
Q16 (cont)

b.i. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Considers the forces on the particle to deduce the result	1

Answer



By Newton's 2nd Law

$$m\ddot{x} = mg - \tfrac{1}{g}mv^2$$

$$\therefore \quad \ddot{x} = g - \tfrac{1}{g}v^2$$

$$\ddot{x} = \tfrac{1}{g}\left(g^2 - v^2\right)$$

b.ii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Applies integration after selecting appropriate representation for $\ddot{x}$ to obtain required expressions	5
Extensive progress eg. correct procedures for both integrations, but one error	4
Substantial progress eg obtains one of the required results by integration	3
Moderate progress eg. obtains anti-derivative and evaluates constant for one of the D.E.'s	2
Some progress eg. chooses appropriate expression for $\ddot{x}$ and finds anti-derivative in one case	1

Answer

$$\frac{dv}{dt} = \frac{1}{g}\left(g^2 - v^2\right)$$

$$\int \frac{2g}{g^2 - v^2}\,dv = 2\int dt$$

$$\int \left(\frac{1}{g+v} + \frac{1}{g-v}\right)dv = 2t$$

$$\ln A\left(\frac{g+v}{g-v}\right) = 2t, \quad A > 0 \text{ constant}$$

$$t = 0,\ v = 0 \Rightarrow A = 1$$

$$\therefore \ln\left(\frac{g+v}{g-v}\right) = 2t$$

$$\frac{g+v}{g-v} = e^{2t}$$

$$g + v = (g - v)e^{2t}$$

$$v(e^{2t} + 1) = g(e^{2t} - 1)$$

$$v = g\left(\frac{e^{2t} - 1}{e^{2t} + 1}\right)$$

$$\tfrac{1}{2}\frac{dv^2}{dx} = \frac{1}{g}\left(g^2 - v^2\right)$$

$$\int \frac{-1}{g^2 - \left(v^2\right)}\,d\left(v^2\right) = \frac{-2}{g}\int dx$$

$$\ln B\left(g^2 - v^2\right) = -\tfrac{2}{g}x\ , \quad B > 0 \text{ constant}$$

$$x = 0,\ v = 0 \Rightarrow B = \tfrac{1}{g^2}$$

$$\therefore \ln\left(\frac{g^2 - v^2}{g^2}\right) = \frac{-2x}{g}$$

$$\frac{g^2 - v^2}{g^2} = e^{\frac{-2x}{g}}$$

$$g^2 - v^2 = g^2 e^{\frac{-2x}{g}}$$

$$v^2 = g^2\left(1 - e^{\frac{-2x}{g}}\right)$$

Q16b (cont)

b.iii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Finds terminal velocity then time and distance fallen when half this velocity attained	3
Substantial progress eg. finds the terminal velocity and one of the time or distance required	2
Some progress eg. finds the terminal velocity	1

Answer

$\ddot{x} \to 0$ as $v \to g$. Hence terminal velocity is g ms^{-1} .

$$2t = \ln\left(\frac{g+v}{g-v}\right) \qquad\qquad \frac{-2x}{g} = \ln\left(\frac{g^2-v^2}{g^2}\right)$$

$$v = \tfrac{1}{2}g \Rightarrow t = \tfrac{1}{2}\ln\left(\frac{\tfrac{3}{2}g}{\tfrac{1}{2}g}\right) = \tfrac{1}{2}\ln 3 \qquad\qquad v = \tfrac{1}{2}g \Rightarrow x = \frac{-g}{2}\ln\left(\frac{g^2-\tfrac{1}{4}g^2}{g^2}\right) = \tfrac{1}{2}g\ln\tfrac{4}{3}$$

Hence particle attains half its terminal velocity $\tfrac{1}{2}\ln 3$ seconds after it begins to fall, and when it has fallen a distance $\tfrac{1}{2}g\ln\tfrac{4}{3}$ metres.