NSW GOVERNMENT

NSW Education Standards Authority

2021 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number on the Question 13 Writing Booklet attached

Total marks: 100

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–16)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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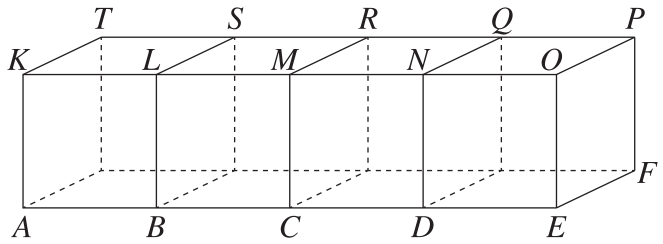
1 Four cubes are placed in a line as shown on the diagram.



Which of the following vectors is equal to $\overrightarrow{AB} + \overrightarrow{CQ}$?

A. $\overrightarrow{AQ}$

B. $\overrightarrow{CP}$

C. $\overrightarrow{PB}$

D. $\overrightarrow{RA}$

2 Which expression is equal to $\int x^5 e^{7x}\,dx$?

A. $\frac{1}{7}x^5e^{7x} - \frac{5}{7}\int x^4 e^{7x}\,dx$

B. $\frac{1}{7}x^5e^{7x} - \frac{5}{7}\int x^5 e^{7x}\,dx$

C. $\frac{5}{7}x^4e^{7x} - \frac{5}{7}\int x^4 e^{7x}\,dx$

D. $\frac{5}{7}x^4e^{7x} - \frac{5}{7}\int x^5 e^{7x}\,dx$

3 Which of the following is a vector equation of the line joining the points $A(4, 2, 5)$ and $B(-2, 2, 1)$?

A. $\underset{\sim}{r} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

B. $\underset{\sim}{r} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

C. $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$

D. $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$

4 Consider the statement:

'For all integers n, if n is a multiple of 6, then n is a multiple of 2'.

Which of the following is the contrapositive of the statement?

A. There exists an integer n such that n is a multiple of 6 and not a multiple of 2.

B. There exists an integer n such that n is a multiple of 2 and not a multiple of 6.

C. For all integers n, if n is not a multiple of 2, then n is not a multiple of 6.

D. For all integers n, if n is not a multiple of 6, then n is not a multiple of 2.

5 Which of the following statements is FALSE?

A. $\forall\, a, b \in \mathbb{R}, \quad a < b \Rightarrow a^3 < b^3$

B. $\forall\, a, b \in \mathbb{R}, \quad a < b \Rightarrow e^{-a} > e^{-b}$

C. $\forall\, a, b \in (0, +\infty), \quad a < b \Rightarrow \ln a < \ln b$

D. $\forall\, a, b \in \mathbb{R}$, with $a, b \neq 0, \quad a < b \Rightarrow \dfrac{1}{a} > \dfrac{1}{b}$

6 Which polynomial could have $2 + i$ as a zero, given that k is a real number?

A. $x^3 - 4x^2 + kx$

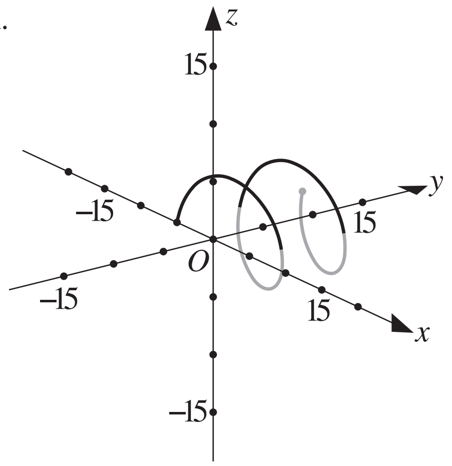
B. $x^3 - 4x^2 + kx + 5$

C. $x^3 - 5x^2 + kx$

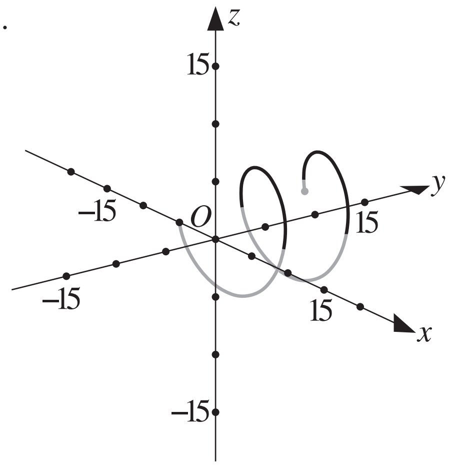
D. $x^3 - 5x^2 + kx + 5$

7 Which diagram best shows the curve described by the position vector
$\underset{\sim}{r}(t) = -5\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j} + t\underset{\sim}{k}$ for $0 \le t \le 4\pi$?

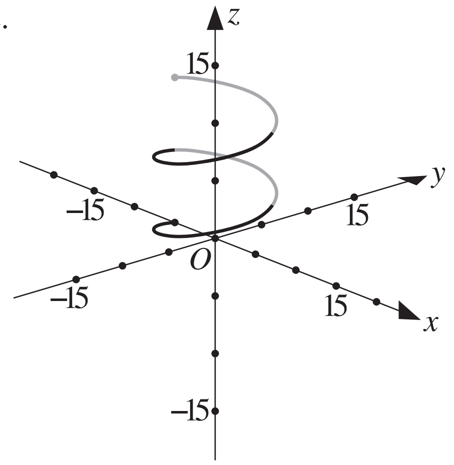
A.



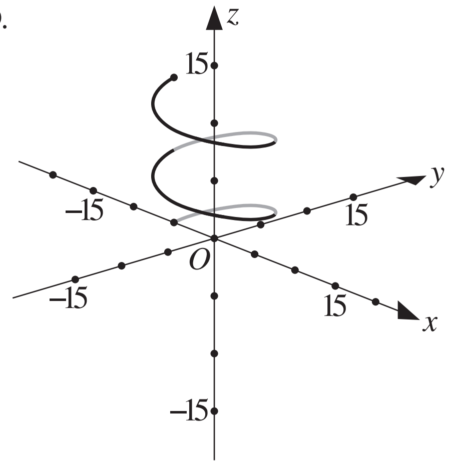
B.



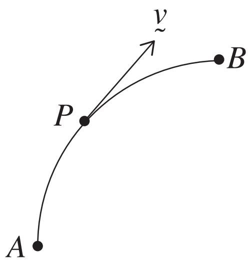
C.



D.



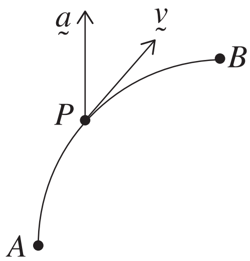
8 A particle is travelling from A on the curve joining A to B. At a particular time, the particle is at point P and has velocity $\underset{\sim}{v}$, as shown in the diagram.



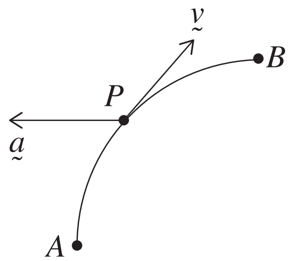
The speed of the particle is increasing.

Which of the following diagrams shows an acceleration, $\underset{\sim}{a}$, which would allow the particle to follow the curve to B?

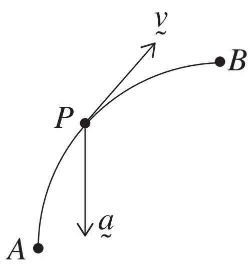
A.



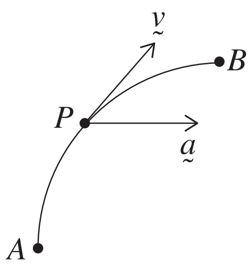
B.



C.

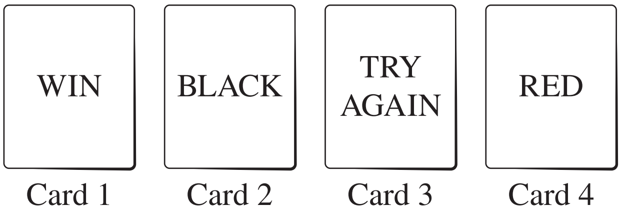


D.



9 Four cards have either RED or BLACK on one side and either WIN or TRY AGAIN on the other side.

Sam places the four cards on the table as shown below.



A statement is made: ‘If a card is RED, then it has WIN written on the other side’.

Sam wants to check if the statement is true by turning over the minimum number of cards.

Which cards should Sam turn over?

A. 1 and 4

B. 3 and 4

C. 1, 2 and 4

D. 1, 3 and 4

10 Consider the two non-zero complex numbers z and w as vectors.

Which of the following expressions is the projection of z onto w?

A. $\dfrac{\mathrm{Re}(zw)}{|w|}w$

B. $\left|\dfrac{z}{w}\right|w$

C. $\mathrm{Re}\left(\dfrac{z}{w}\right)w$

D. $\dfrac{\mathrm{Re}(z)}{|w|}w$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) The complex numbers $z = 2e^{i\frac{\pi}{2}}$ and $w = 6e^{i\frac{\pi}{6}}$ are given. **2**

Find the value of zw, giving the answer in the form $re^{i\theta}$.

(b) Find $\sum_{n=1}^{5} (i)^n$. **2**

(c) Find the angle between the vectors $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$, giving the angle in degrees correct to 1 decimal place. **3**

(d) (i) Find the two square roots of $-i$, giving the answers in the form $x + iy$, where x and y are real numbers. **2**

(ii) Hence, or otherwise, solve $z^2 + 2z + 1 + i = 0$ giving your solutions in the form $a + ib$ where a and b are real numbers. **2**

(e) The complex numbers $z = 5 + i$ and $w = 2 - 4i$ are given. **2**

Find $\frac{\overline{z}}{w}$, giving your answer in Cartesian form.

(f) Express $\dfrac{3x^2 - 5}{(x-2)(x^2 + x + 1)}$ as a sum of partial fractions over $\mathbb{R}$. **3**

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Find $\displaystyle\int \frac{2x+3}{x^2+2x+2}\,dx$. **3**

(b) Consider Statement A.

Statement A: 'If n^2 is even, then n is even.'

(i) What is the converse of Statement A? **1**

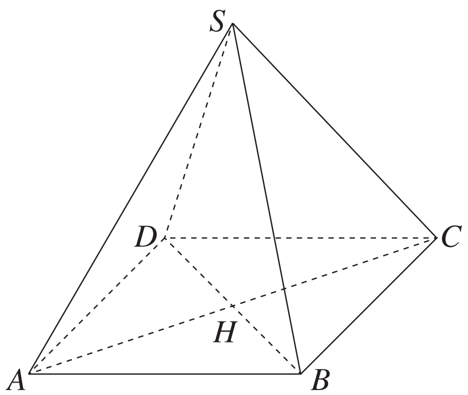
(ii) Show that the converse of Statement A is true. **1**

(c) Two lines are given by $\mathbf{r}_1 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 4 \\ -2 \\ q \end{pmatrix} + \mu \begin{pmatrix} p \\ 3 \\ -1 \end{pmatrix}$, where p and q are real numbers. These lines intersect and are perpendicular. **3**

Find the values of p and q.

(d) Prove by mathematical induction that $\sqrt{n!} > 2^n$, for integers $n \geq 9$. **3**

(e) The diagram shows the pyramid $ABCDS$ where $ABCD$ is a square. The diagonals of the square bisect each other at H.



(i) Show that $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD} = \underset{\sim}{0}$. **1**

Let G be the point such that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} + \overrightarrow{GS} = \underset{\sim}{0}$.

(ii) Using part (i), or otherwise, show that $4\overrightarrow{GH} + \overrightarrow{GS} = \underset{\sim}{0}$. **2**

(iii) Find the value of λ such that $\overrightarrow{HG} = \lambda\overrightarrow{HS}$. **1**

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) The location of the complex number $a + ib$ is shown on the diagram on page 1 of the Question 13 Writing Booklet. **2**

On the diagram provided in the writing booklet, indicate the locations of all of the fourth roots of the complex number $a + ib$.

(b) Use an appropriate substitution to evaluate $\int_{\sqrt{10}}^{\sqrt{13}} x^3 \sqrt{x^2 - 9}\, dx$. **3**

Question 13 continues on page 10

Question 13 (continued)

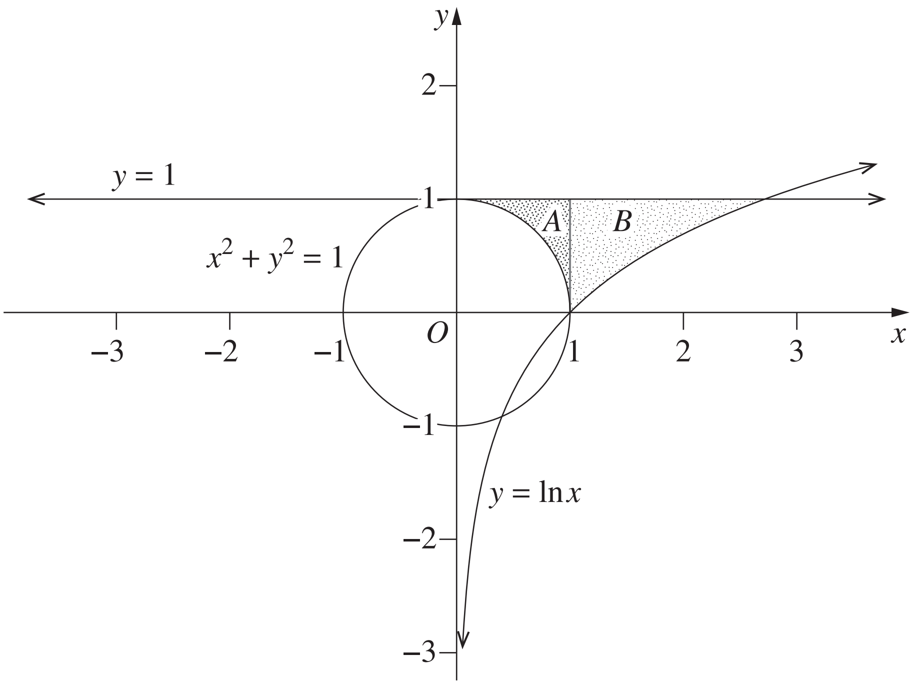
(c) (i) The integral I_n is defined by $I_n = \int_1^e (\ln x)^n \, dx$ for integers $n \geq 0$. **2**

Show that $I_n = e - nI_{n-1}$ for $n \geq 1$.

(ii) The diagram shows two regions. **4**

Region A is bounded by $y = 1$ and $x^2 + y^2 = 1$ between $x = 0$ and $x = 1$.

Region B is bounded by $y = 1$ and $y = \ln x$ between $x = 1$ and $x = e$.



The volume of the solid created when the region between the curve $y = f(x)$ and the x-axis, between $x = a$ and $x = b$, is rotated about the x-axis is given by $V = \pi \int_a^b [f(x)]^2 \, dx$.

The volume of the solid of revolution formed when region A is rotated about the x-axis is V_A.

The volume of the solid of revolution formed when region B is rotated about the x-axis is V_B.

Using part (i), or otherwise, show that the ratio $V_A : V_B$ is $1:3$.

Question 13 continues on page 11

Question 13 (continued)

(d) An object is moving in simple harmonic motion along the x-axis. The acceleration of the object is given by $\ddot{x} = -4(x-3)$ where x is its displacement from the origin, measured in metres, after t seconds.

Initially, the object is 5.5 metres to the right of the origin and moving towards the origin. The object has a speed of 8 m s^{-1} as it passes through the origin.

(i) Between which two values of x is the particle oscillating? **2**

(ii) Find the first value of t for which $x = 0$, giving the answer correct to 2 decimal places. **2**

End of Question 13

Please turn over

Question 14 (14 marks) Use the Question 14 Writing Booklet

(a) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\,dx$. **4**

(b) An object of mass 5 kg is on a slope that is inclined at an angle of 60° to the horizontal. The acceleration due to gravity is g m s^{-2} and the velocity of the object down the slope is v m s^{-1}.

As well as the force due to gravity, the object is acted on by two forces, one of magnitude $2v$ newtons and one of magnitude $2v^2$ newtons, both acting up the slope.

(i) Show that the resultant force down the slope is $\dfrac{5\sqrt{3}}{2}g - 2v - 2v^2$ newtons. **2**

(ii) There is one value of v such that the object will slide down the slope at a constant speed. **2**

Find this value of v in m s^{-1}, correct to 1 decimal place, given that $g = 10$.

(c) (i) Using de Moivre's theorem and the binomial expansion of $(\cos\theta + i\sin\theta)^5$, or otherwise, show that **3**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(ii) By using part (i), or otherwise, show that $\operatorname{Re}\left(e^{\frac{i\pi}{10}}\right) = \sqrt{\dfrac{5+\sqrt{5}}{8}}$. **3**

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) For all non-negative real numbers x and y, $\sqrt{xy} \le \dfrac{x+y}{2}$. (Do NOT prove this.)

(i) Using this fact, show that for all non-negative real numbers a, b and c, **2**

$$\sqrt{abc} \le \frac{a^2+b^2+2c}{4}.$$

(ii) Using part (i), or otherwise, show that for all non-negative real numbers a, b and c, **2**

$$\sqrt{abc} \le \frac{a^2+b^2+c^2+a+b+c}{6}.$$

(b) For integers $n \ge 1$, the triangular numbers t_n are defined by $t_n = \dfrac{n(n+1)}{2}$, giving $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, $t_4 = 10$ and so on.

For integers $n \ge 1$, the hexagonal numbers h_n are defined by $h_n = 2n^2 - n$, giving $h_1 = 1$, $h_2 = 6$, $h_3 = 15$, $h_4 = 28$ and so on.

(i) Show that the triangular numbers t_1, t_3, t_5, and so on, are also hexagonal numbers. **2**

(ii) Show that the triangular numbers t_2, t_4, t_6, and so on, are not hexagonal numbers. **1**

Question 15 continues on page 14

Question 15 (continued)

(c) An object of mass 1 kg is projected vertically upwards with an initial velocity of u m/s. It experiences air resistance of magnitude kv^2 newtons where v is the velocity of the object, in m/s, and k is a positive constant. The height of the object above its starting point is x metres. The time since projection is t seconds and acceleration due to gravity is g m/s^2.

(i) Show that the time for the object to reach its maximum height is $\frac{1}{\sqrt{gk}}\arctan\left(u\sqrt{\frac{k}{g}}\right)$ seconds. **3**

(ii) Find an expression for the maximum height reached by the object, in terms of k, g and u. **3**

(d) Prove that $2^n + 3^n \neq 5^n$ for all integers $n \geq 2$. **2**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

(a) (i) The point $P(x, y, z)$ lies on the sphere of radius 1 centred at the origin O. **2**

Using the position vector of P, $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, and the triangle inequality, or otherwise, show that $|x| + |y| + |z| \geq 1$.

(ii) Given the vectors $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, show that **3**

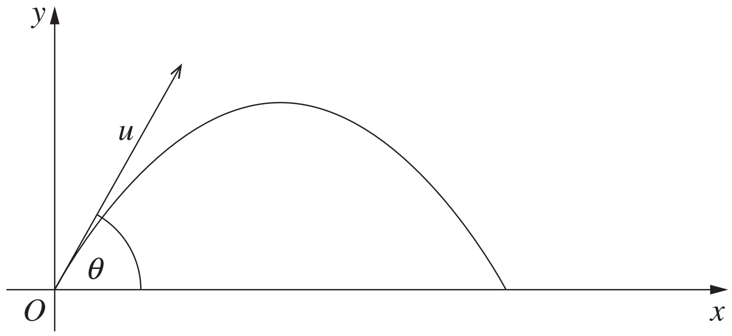
$$|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}.$$

(iii) As in part (i), the point $P(x, y, z)$ lies on the sphere of radius 1 centred at the origin O. **2**

Using part (ii), or otherwise, show that $|x| + |y| + |z| \leq \sqrt{3}$.

Question 16 continues on page 16

(b) A particle which is projected from the origin with initial speed $u\ \mathrm{m\,s^{-1}}$ at an angle of θ to the positive x-axis lands on the x-axis, as shown in the diagram. The particle is subject to an acceleration due to gravity of $g\ \mathrm{m\,s^{-2}}$. **5**



The position vector of the particle, $\underset{\sim}{r}(t)$, where t is the time in seconds after the particle is projected, is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ -\dfrac{gt^2}{2} + ut\sin\theta \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

For some value(s) of θ there will be two times during the time of flight when the particle's position vector is perpendicular to its velocity vector.

Find the value(s) of θ for which this occurs, justifying that both times occur during the time of flight.

(c) Sketch the region of the complex plane defined by $\mathrm{Re}(z) \geq \mathrm{Arg}(z)$ where $\mathrm{Arg}(z)$ is the principal argument of z. **3**

End of paper

1271 15260 5499310056

Centre Number

Student Number

NSW Government

NSW Education Standards Authority

2021 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Writing Booklet

Question 13

Instructions

- Use this Writing Booklet to answer Question 13.
- Write the number of this booklet and the total number of booklets that you have used for this question (eg: 1 of 3).
- Write your Centre Number and Student Number at the top of this page.
- Write using black pen.
- You may ask for an extra writing booklet if you need more space.
- If you have not attempted the question(s), you must still hand in the writing booklet, with 'NOT ATTEMPTED' written clearly on the front cover.
- You may NOT take any writing booklets, used or unused, from the examination room.

13

of

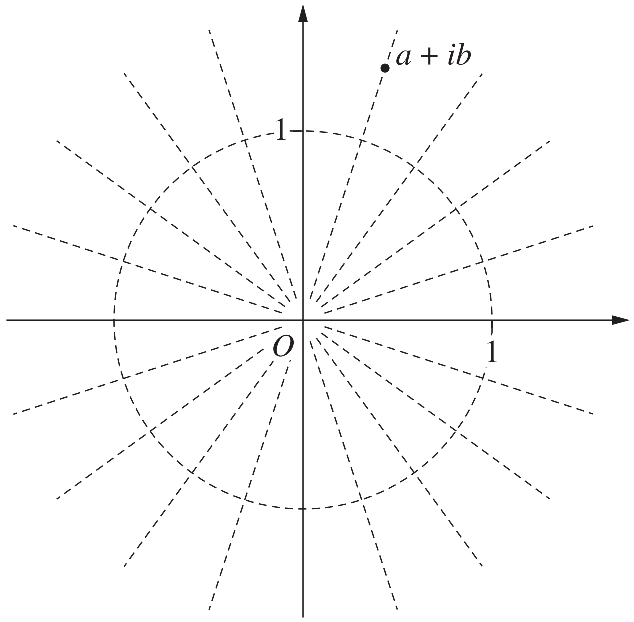
this booklet

number of booklets for this question

Office Use Only – Do NOT write anything, or make any marks below this line.

Start here for Question Number: **13**

(a) Indicate the locations of all of the fourth roots of the complex number $a + ib$.



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Tick this box if you have continued this answer in another writing booklet.

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2021 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

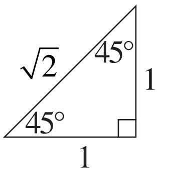
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

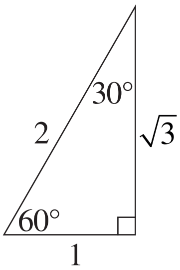
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

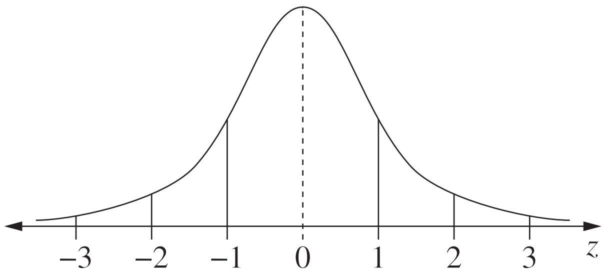
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)} dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)} dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx} dx = uv - \int v\frac{du}{dx} dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r}=\frac{n!}{(n-r)!}$$

$$\binom{n}{r}={}^{n}C_{r}=\frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n}=x^{n}+\binom{n}{1}x^{n-1}a+\cdots+\binom{n}{r}x^{n-r}a^{r}+\cdots+a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right|=\left|x\underset{\sim}{i}+y\underset{\sim}{j}\right|=\sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v}=\left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta=x_{1}x_{2}+y_{1}y_{2},$$

where $\underset{\sim}{u}=x_{1}\underset{\sim}{i}+y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v}=x_{2}\underset{\sim}{i}+y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned}z=a+ib&=r(\cos\theta+i\sin\theta)\\&=re^{i\theta}\end{aligned}$$

$$\begin{aligned}\left[r(\cos\theta+i\sin\theta)\right]^{n}&=r^{n}(\cos n\theta+i\sin n\theta)\\&=r^{n}e^{in\theta}\end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x=a\cos(nt+\alpha)+c$$

$$x=a\sin(nt+\alpha)+c$$

$$\ddot{x}=-n^{2}(x-c)$$

2021 HSC Mathematics Extension 2 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	B
2	A
3	B
4	C
5	D
6	A
7	D
8	D
9	B
10	C

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	2
• Finds the correct argument	1

Sample answer:

$$zw = (2 \times 6)e^{i\left(\frac{\pi}{2}+\frac{\pi}{6}\right)}$$

$$= 12e^{\frac{i2\pi}{3}}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Expands the notation and provides the correct expression with 5 terms OR • Correctly evaluates i^3 or i^4 or i^5	1

Sample answer:

$$\sum_{n=1}^{n}(i)^n = i + (i)^2 + (i)^3 + (i)^4 + (i)^5$$

$$= i - 1 - i + 1 + i$$

$$= i$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	3
• Correctly finds the dot product and both magnitudes OR • Uses the correct dot product or a correct magnitude in $a \cdot b = \lvert a \rvert \lvert b \rvert \cos\theta$	2
• Correctly finds the dot product of the two vectors OR • Correctly finds the magnitude of one vector	1

Sample answer:

Dot product $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -6 + 0 + 8$

$$= 2$$

Magnitudes:

$$\left| \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right| = \sqrt{2^2 + 0^2 + 4^2}$$

$$= \sqrt{20}$$

$$\left| \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \right| = \sqrt{(-3)^2 + 1^2 + 2^2}$$

$$= \sqrt{14}$$

$\therefore \sqrt{20}\sqrt{14}\cos\theta = 2$

$$\cos\theta = 0.1195\ldots$$

$$\theta = 83.135°\ldots$$

$$\approx 83.1° \qquad \text{(1 decimal place)}$$

Question 11 (d) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the square of $(x + iy)$, or equivalent merit OR • Attempts to use the modulus/argument form of $-i$ OR • Plots $-i$ on an Argand diagram and attempts to locate the square roots	1

Sample answer:

Let $(x+iy)^2 = -i$ where $x, y \in \mathbb{R}$

$\therefore x^2 - y^2 = 0$ and $2xy = -1$

$$\therefore y = \frac{-1}{2x}$$

$$\therefore x^2 - \left(\frac{-1}{2x}\right)^2 = 0$$

$$4x^4 - 1 = 0$$

$$x^4 = \frac{1}{4}$$

$$x^2 = \frac{1}{2} \quad (x \in \mathbb{R})$$

$$\therefore \quad x = \pm\frac{1}{\sqrt{2}}$$

$\therefore$ square roots are $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$ and $\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

Question 11 (d) (ii)

Criteria	Marks
• Provides correct answers	2
• Uses the quadratic formula to obtain $z = -1 \pm \sqrt{-i}$, or equivalent merit	1

Sample answer:

$$z^2 + 2z + 1 + i = 0$$

$$\begin{aligned} \therefore \quad z &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(1+i)}}{2} \\ &= \frac{-2 \pm 2\sqrt{1-1-i}}{2} \\ &= -1 \pm \sqrt{-i} \end{aligned}$$

From part (i):

$$z = -1 + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \quad \text{or} \quad z = -1 - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

that is,

$$z = \frac{\left(1-\sqrt{2}\right)}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \quad \text{or} \quad z = \frac{-\left(1+\sqrt{2}\right)}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	2
• Identifies $\overline{z}$, or equivalent merit	1

Sample answer:

$z = 5 + i \qquad w = 2 - 4i$

$\therefore \overline{z} = 5 - i$

$$\frac{\overline{z}}{w} = \frac{(5-i)}{(2-4i)} \times \frac{(2+4i)}{(2+4i)}$$

$$= \frac{10 + 20i - 2i + 4}{4 + 16}$$

$$= \frac{14 + 18i}{20}$$

$$= \frac{14}{20} + \frac{18}{20}i$$

$$= \frac{7}{10} + \frac{9}{10}i$$

Question 11 (f)

Criteria	Marks
• Provides correct solution	3
• Evaluates one of the three coefficients	2
• Provides a correct expression for the sum of fractions with unknown coefficients, or equivalent merit	1

Sample answer:

Let $\dfrac{3x^2-5}{(x-2)(x^2+x+1)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+x+1}$

$$= \frac{A(x^2+x+1)+(Bx+C)(x-2)}{(x-2)(x^2+x+1)}$$

$\therefore 3x^2-5 = A\left(x^2+x+1\right)+(Bx+C)(x-2)$

when $x=2$

$$3(2)^2-5 = A(2^2+2+1)$$
$$7 = 7A$$
$$A = 1$$

Equating coefficients of x^2

$3 = A+B$

$\therefore B = 2$

Equating constants:

$-5 = A - 2C$

$2C = 6$

$C = 3$

$$\therefore \frac{3x^2-5}{(x-2)(x^2+x+1)} = \frac{1}{x-2} + \frac{2x+3}{x^2+x+1}$$

Question 12 (a)

Criteria	Marks
• Provides correct solution	3
• Integrates one of the correct fractions OR • Writes the integrand as the correct sum of fractions and completes the square in a denominator, or equivalent merit	2
• Attempts to write the integrand as a sum of two suitable fractions, or equivalent merit	1

Sample answer:

$$\int \frac{2x+3}{x^2+2x+2}\,dx = \int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

$$= \ln\left(x^2+2x+2\right) + \int \frac{1}{1+(x+1)^2}\,dx + C$$

$$= \ln\left(x^2+2x+2\right) + \tan^{-1}(x+1) + k$$

Question 12 (b) (i)

Criteria	Marks
• States the converse	1

Sample answer:

The converse is ‘If n is even, then n^2 is even’.

Question 12 (b) (ii)

Criteria	Marks
• Provides correct proof	1

Sample answer:

If n is even then $n = 2k$ for some integer k, then $n^2 = (2k)^2 = 2(2k^2)$ which is also even.

Question 12 (c)

Criteria	Marks
• Provides correct solution	3
• Finds the value of p or μ	2
• Evaluates the dot product of the direction vectors OR • Recognises that $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$ OR • Equates component(s) of $\mathbf{r}_1$ and $\mathbf{r}_2$	1

Sample answer:

$$\mathbf{r}_1 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \mathbf{r}_2 = \begin{pmatrix} 4 \\ -2 \\ q \end{pmatrix} + \mu \begin{pmatrix} p \\ 3 \\ -1 \end{pmatrix}$$

They are perpendicular $\therefore\ \mathbf{r}_1 \cdot \mathbf{r}_2 = 0$

$$1 \times p + 0 \times 3 + 2 \times -1 = 0$$
$$p - 2 = 0$$
$$p = 2$$

They intersect $\therefore$ components are equal

① $-2 + \lambda = 4 + p\mu$

② $1 = -2 + 3\mu \qquad \Rightarrow 3 = 3\mu$ so $\mu = 1$

③ $3 + 2\lambda = q - \mu$

Substitute $\mu = 1$ and $p = 2$ into ①

$$-2 + \lambda = 4 + 2 \times 1$$
$$\lambda = 8$$

Substitute $\lambda = 8$ and $\mu = 1$ into ③

$$3 + 2 \times 8 = q - 1$$
$$19 = q - 1$$
$$q = 20$$

$\therefore\ p = 2$ and $q = 20$.

Question 12 (d)

Criteria	Marks
• Provides correct solution	3
• Shows that $p(k) \Rightarrow p(k+1)$ is a true statement, or equivalent merit	2
• Establishes initial case, or equivalent merit	1

Sample answer:

$\sqrt{n!} > 2^n, \quad n \geq 9$

Prove it's true for $n = 9$

$$\sqrt{9!} = 602.39\ldots$$
$$2^9 = 512$$
$$\therefore \sqrt{9!} > 2^9$$

$\therefore$ It's true for $n = 9$

Assume it's true for $n = k$: Assume $\sqrt{k!} > 2^k$

Prove it's true for $n = k + 1$: Required to prove $\sqrt{(k+1)!} > 2^{k+1}$

$$\begin{aligned}
\text{LHS} &= \sqrt{(k+1)!} \\
&= \sqrt{k!} \times \sqrt{(k+1)} \\
&> 2^k \times \sqrt{(k+1)} && \text{using the assumption} \\
&> 2^k \times 2 && \text{because } \sqrt{k+1} \text{ is greater than 2 for } k \geq 9 \\
&= 2^{k+1} \\
&= \text{RHS}
\end{aligned}$$

$\therefore \sqrt{(k+1)!} > 2^{k+1}$ as required

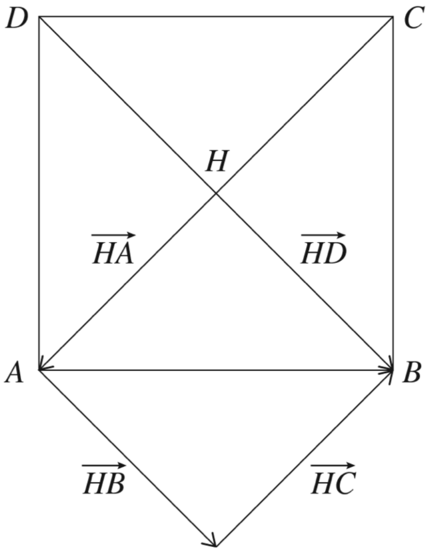
$\therefore$ By principle of mathematical induction, the inequality is true for $n \geq 9$.

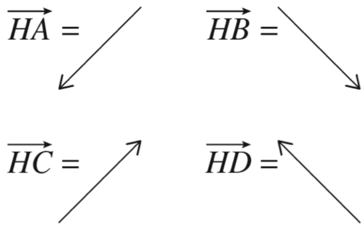
Question 12 (e) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Consider the square base





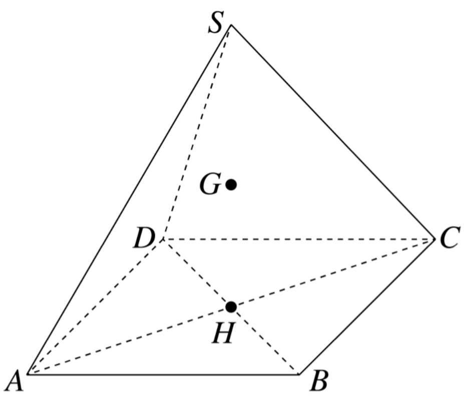
As H is the midpoint of AC and BD, when you add them head to tail, you end up back where you started.

$\therefore\ \overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD} = \underset{\sim}{0}$

Question 12 (e) (ii)

Criteria	Marks
• Provides correct solution	2
• Writes $\overrightarrow{GA} = \overrightarrow{GH} + \overrightarrow{HA}$, or equivalent merit	1

Sample answer:



$$\overrightarrow{GA} = \overrightarrow{GH} + \overrightarrow{HA}$$
$$\overrightarrow{GB} = \overrightarrow{GH} + \overrightarrow{HB}$$
$$\overrightarrow{GC} = \overrightarrow{GH} + \overrightarrow{HC}$$
$$\overrightarrow{GD} = \overrightarrow{GH} + \overrightarrow{HD}$$

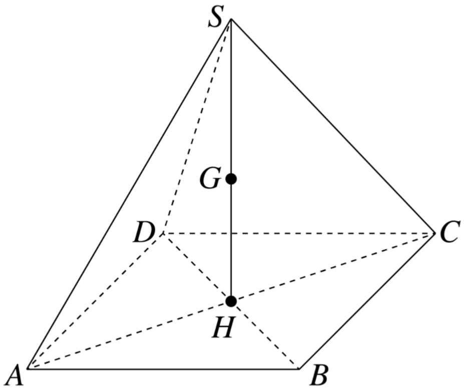
Adding these

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} = 4\overrightarrow{GH} + \overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD}$$
$$-\overrightarrow{GS} = 4\overrightarrow{GH} + \underset{\sim}{0}$$
$$\therefore\ 4\overrightarrow{GH} + \overrightarrow{GS} = \underset{\sim}{0}$$

Question 12 (e) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:



From part (ii)

$$4\overrightarrow{GH} + \overrightarrow{GS} = \underset{\sim}{0}$$

$$4\overrightarrow{GH} + \overrightarrow{GH} + \overrightarrow{HS} = \underset{\sim}{0}$$

$$5\overrightarrow{GH} + \overrightarrow{HS} = \underset{\sim}{0}$$

$$\overrightarrow{HS} = -5\overrightarrow{GH}$$

$$= 5\overrightarrow{HG}$$

$$\overrightarrow{HG} = \frac{1}{5}\overrightarrow{HS}$$

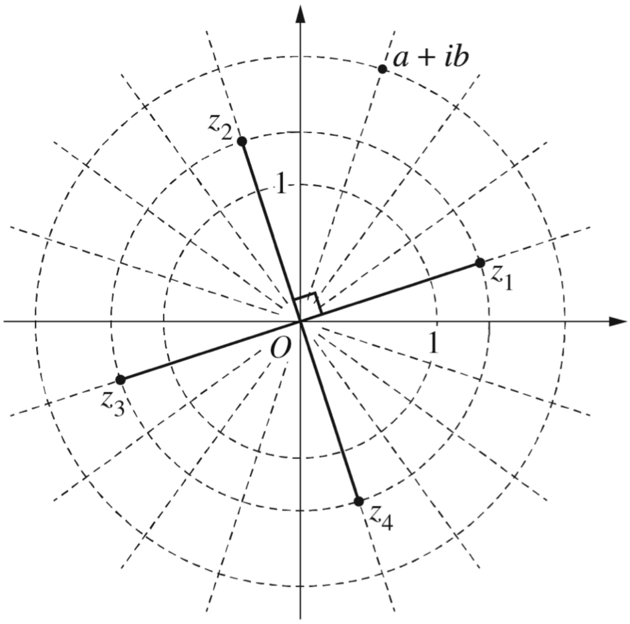
$$\therefore \lambda = \frac{1}{5}$$

Question 13 (a)

Criteria	Marks
• Provides appropriate sketch	2
• Indicates one correct argument OR • Indicates correct modulus	1

Sample answer:

z_1, z_2, z_3 and z_4 represent the fourth roots.



Question 13 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains correct primitive OR correct definite integral in terms of u, or equivalent merit	2
• Attempts to use a suitable substitution	1

Sample answer:

$$I = \int_{\sqrt{10}}^{\sqrt{13}} x^3\sqrt{x^2-9}\,dx$$

$$= \int_{\sqrt{10}}^{\sqrt{13}} x^2\left(x^2-9\right)\frac{x}{\sqrt{x^2-9}}\,dx$$

$$= \int_1^2 \left(u^2+9\right)u^2\,du$$

$$= \int_1^2 u^4+9u^2\,du$$

$$= \left[\frac{u^5}{5}+3u^3\right]_1^2$$

$$= \frac{2^5}{5}+3\times 2^3-\left(\frac{1}{5}+3\right)$$

$$= \frac{32}{5}-\frac{1}{5}+24-3$$

$$= \frac{31}{5}+21$$

$$= \frac{136}{5}$$

Let $u = \sqrt{x^2-9}$

$du = \dfrac{x}{\sqrt{x^2-9}}dx$

and $u^2 = x^2-9$

When $x=\sqrt{10}$, $u=\sqrt{10-9}=1$

When $x=\sqrt{13}$, $u=\sqrt{13-9}=2$

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Uses integration by parts, or equivalent	1

Sample answer:

$$I_n = \int_1^e (\ln x)^n \, dx, \qquad n \geq 0$$

Let $\quad u = (\ln x)^n \qquad$ and $\quad v' = 1$

$$\therefore \quad u' = \frac{n(\ln x)^{n-1}}{x} \qquad\qquad v = x$$

$$\begin{aligned}\therefore I_n &= \Big[x(\ln x)^n \Big]_1^e - \int_1^e n(\ln x)^{n-1} \, dx \\ &= e(\ln e)^n - (\ln 1)^n - nI_{n-1} \\ &= e - 0 - nI_{n-1} \\ &= e - nI_{n-1}\end{aligned}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	4
• Shows $V_B = \pi$, or equivalent	3
• Obtains volume V_A • Writes volume V_A or V_B as a difference of volumes involving an integral expression	2
• States volume of a relevant sphere OR hemisphere OR cylinder OR • Provides correct integral expression for a relevant volume	1

Sample answer:

$$V_A = \pi \int_0^1 (1)^2 - \left(1 - x^2\right) dx$$

$$= \pi \int_0^1 x^2\, dx$$

$$= \pi \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{\pi}{3}$$

$$V_B = \pi \int_1^e (1)^2 - (\ln x)^2\, dx$$

$$= \pi \Big[x \Big]_1^e - \pi \int_1^e (\ln x)^2\, dx$$

$$= \pi (e-1) - \pi I_2 \qquad \text{from part (i)}$$

Now $I_2 = e - 2I_1$

$$I_1 = e - I_0$$

$$I_0 = \int_1^e 1\, dx$$

$$= \Big[x \Big]_1^e$$

$$= e - 1$$

$$\therefore I_2 = e - 2\big[e - (e-1)\big] = e - 2$$

$$\therefore V_B = \pi(e-1) - \pi(e-2)$$

$$= -\pi + 2\pi$$

$$= \pi$$

$$\therefore V_A : V_B = \frac{\pi}{3} : \pi$$

$$= 1 : 3$$

Question 13 (d) (i)

Criteria	Marks
• Provides correct solution	2
• States the amplitude of the motion, or equivalent merit	1

Sample answer:

$\ddot{x} = -4(x-3)$

$\therefore n^2 = 4, \quad \text{centre is } x = 3$

$n = 2$

$\text{period} = \dfrac{2\pi}{2} = \pi$

$v^2 = n^2\left(a^2 - (x-c)^2\right)$

when $\quad v = 8, x = 0$

$64 = 4\left(a^2 - (0-3)^2\right)$

$16 = a^2 - 9$

$a^2 = 25$

$a = 5$

$\therefore$ The particle oscillates between

$x = 3+5 \quad \text{and} \quad x = 3-5$

$= 8 \qquad\qquad\qquad = -2$

Question 13 (d) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds the displacement function, or equivalent merit	1

Sample answer:

Consider $x = a\cos(nt+\alpha)+c$, where

$a=5, \quad n=2, \quad c=3$

$\therefore\ x = 5\cos(2t+\alpha)+3$

When $\quad t=0, \quad x=5.5$

$$5.5 = 5\cos\alpha + 3$$

$$\frac{2.5}{5} = \cos\alpha$$

$$\therefore \quad \alpha = \cos^{-1}\left(\frac{2.5}{5}\right)$$

$$= 1.04719\ldots\text{radians}$$

$$0 = 5\cos(2t+1.04719\ldots)+3$$

$$\cos(2t+1.04719\ldots) = \frac{-3}{5}$$

$$2t+1.04719\ldots = 2.214\ldots\text{radians}$$

$$2t = 1.167\ldots$$

$$t = 0.583\ldots$$

$\therefore$ First value of t when $x=0$ is $t=0.58$ seconds (2 decimal places).

Question 14 (a)

Criteria	Marks
• Provides correct solution	4
• Applies partial fraction to the correct integral, or equivalent merit	3
• Completes t-substitution, including the limits of integration, or equivalent merit	2
• Attempts to use a t-substitution, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\,dx \qquad \text{let } t = \tan\frac{x}{2} \quad \therefore \cos x = \frac{1-t^2}{1+t^2}$$

when $x = 0, \quad t = 0$

$x = \dfrac{\pi}{2}, \quad t = 1$

$$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left(1+t^2\right)dx$$

$$\therefore dx = \frac{2}{1+t^2}\,dt$$

$$\therefore \text{Integral} = \int_0^1 \frac{1}{3+\dfrac{5\left(1-t^2\right)}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{3+3t^2+5-5t^2}\,dt$$

$$= \int_0^1 \frac{2}{8-2t^2}\,dt$$

$$= \int_0^1 \frac{1}{4-t^2}\,dt$$

$$= \int_0^1 \frac{1}{(2-t)(2+t)}\,dt$$

$$= \frac{1}{4}\int_0^1 \frac{1}{2-t} + \frac{1}{2+t}\,dt$$

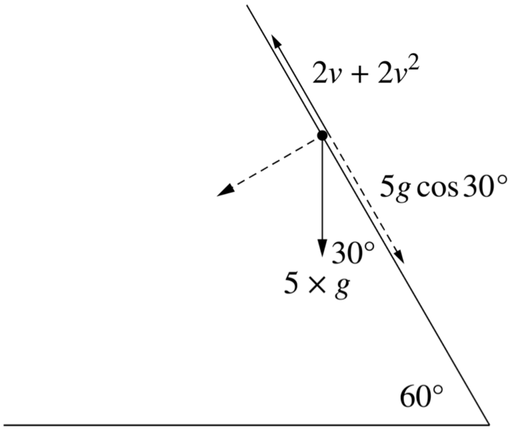
$$= \frac{1}{4}\left[\ln\left|\frac{2+t}{2-t}\right|\right]_0^1$$

$$= \frac{1}{4}(\ln 3 - \ln 1) \;= \frac{1}{4}\ln 3$$

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Finds appropriate component of force due to gravity OR • Provides appropriate diagram, or equivalent merit	1

Sample answer:



Resultant force down slope $= 5g\cos 30° - 2v - 2v^2$

$$= \frac{5\sqrt{3}}{2}g - 2v - 2v^2$$

Question 14 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• States that the resultant force is $\underset{\sim}{0}$, or equivalent merit	1

Sample answer:

Constant speed $\Rightarrow$ Resultant force = 0

$$\therefore \frac{5\sqrt{3}}{2}g = 2v + 2v^2$$

$$g = 10 \quad \therefore 25\sqrt{3} = 2v + 2v^2$$

$$2v^2 + 2v - 25\sqrt{3} = 0$$

$$\therefore v = \frac{-2 \pm \sqrt{4 + 4(2)\left(25\sqrt{3}\right)}}{4}$$

$$= 4.1798\ldots \text{ or } -5.1798\ldots$$

$\therefore$ speed $= 4.1798\ldots$ as moving down the slope

$\simeq 4.2\ \mathrm{m\,s^{-1}}$ (1 decimal place)

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Expands using binomial theorem AND • Expands using De Moivre's theorem	2
• Expands using binomial theorem OR • Expands using De Moivre's theorem	1

Sample answer:

Using De Moivre:

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

Using binomial expansion

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 &= (\cos\theta)^5 + 5(\cos\theta)^4(i\sin\theta)\\ &\quad +10(\cos\theta)^3(i\sin\theta)^2\\ &\quad +10(\cos\theta)^2(i\sin\theta)^3\\ &\quad +5(\cos\theta)(i\sin\theta)^4\\ &\quad +(i\sin\theta)^5\\ &= \cos^5\theta + 5\cos^4\theta\sin\theta i\\ &\quad -10\cos^3\theta\sin^2\theta\\ &\quad -10\cos^2\theta\sin^3\theta i\\ &\quad +5\cos\theta\sin^4\theta\\ &\quad +\sin^5\theta i\end{aligned}$$

Equating real parts:

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\ &= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-\cos^2\theta\right)^2\\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta\left(1-2\cos^2\theta+\cos^4\theta\right)\\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Finds the solutions to the degree 5 polynomial, or equivalent merit	2
• Uses $\theta = \frac{\pi}{10}$ in the equation from part (i), or equivalent merit	1

Sample answer:

$$\mathrm{Re}\left(e^{i\frac{\pi}{10}}\right) = \mathrm{Re}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$$

$$= \cos\frac{\pi}{10}$$

When $\theta = \frac{\pi}{10}$, $\cos 5\theta = \cos\frac{\pi}{2} = 0$

$\therefore \cos\frac{\pi}{10}$ is a root of $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$$

$$\cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right) = 0$$

$$\cos\frac{\pi}{10} \neq 0$$

$\therefore \cos\frac{\pi}{10}$ is a root of $16\cos^4\theta - 20\cos^2\theta + 5 = 0$

$$\therefore \cos^2\theta = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}$$

$$= \frac{20 \pm 4\sqrt{5}}{32}$$

$$= \frac{5 \pm \sqrt{5}}{8}$$

$$\therefore \cos\frac{\pi}{10} = \pm\sqrt{\frac{5 \pm \sqrt{5}}{8}}$$

But roots of $\cos 5\theta = 0$ are:

$$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$$

$$\therefore \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$$

so roots of $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$ are $\cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{\pi}{2} = 0, \cos\frac{7\pi}{10}$ and $\cos\frac{9\pi}{10}$

From the graph of $y = \cos x$, we see that $\cos\frac{\pi}{10}$ is the largest of these.

$$\therefore \cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$$

$$\therefore \operatorname{Re}\left(e^{i\frac{\pi}{10}}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to use appropriate expressions for x and y	1

Sample answer:

$$\sqrt{abc} = \sqrt{(ab)c} \le \frac{ab+c}{2} \qquad \text{using the result provided}$$

$$= \frac{\sqrt{a^2b^2}+c}{2}$$

$$\le \frac{\frac{a^2+b^2}{2}+c}{2} \qquad \text{using the same result a second time}$$

$$= \frac{a^2+b^2+2c}{4}$$

Answers could include:

Let $x = \dfrac{a^2+b^2}{2}$ and $y = c$

$$\therefore \sqrt{\frac{a^2+b^2}{2}\cdot c} \le \frac{\frac{a^2+b^2}{2}+c}{2}$$

$$\sqrt{\frac{a^2c+b^2c}{2}} \le \frac{a^2+b^2+2c}{4}$$

Let $m = a^2c$ and $n = b^2c$

$$\therefore \sqrt{mn} = \sqrt{a^2c\cdot b^2c} = \sqrt{a^2b^2c^2} = abc$$

$$\sqrt{\frac{m+n}{2}} = \frac{a^2c+b^2c}{2}$$

$$\therefore \quad abc \le \frac{a^2c+b^2c}{2}$$

$$\therefore \sqrt{abc} \le \sqrt{\frac{a^2c+b^2c}{2}} \le \frac{a^2+b^2+2c}{4}$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Permutes the result in (i), or equivalent merit	1

Sample answer:

$$\sqrt{abc} \le \frac{a^2 + b^2 + 2c}{4}$$

$$\sqrt{acb} \le \frac{a^2 + c^2 + 2b}{4}$$

$$\sqrt{bca} \le \frac{b^2 + c^2 + 2a}{4}$$

Adding

$$3\sqrt{abc} \le \frac{2a^2 + 2b^2 + 2c^2 + 2a + 2b + 2c}{4}$$

$$3\sqrt{abc} \le \frac{a^2 + b^2 + c^2 + a + b + c}{2}$$

$$\therefore \sqrt{abc} \le \frac{a^2 + b^2 + c^2 + a + b + c}{6} \quad \text{as required}$$

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to use an expression for an odd number in the formula for t_n, or equivalent merit	1

Sample answer:

$$t_n = \frac{n(n+1)}{2} \qquad h_n = 2n^2 - n$$

The odd numbers 1, 3, 5, … can be expressed as $2m - 1$ where m is an integer.
$\therefore$ the odd triangular numbers are t_{2m-1}

$$\begin{aligned} t_{2m-1} &= \frac{(2m-1)(2m-1+1)}{2} \\ &= \frac{(2m-1)2m}{2} \\ &= m(2m-1) \\ &= 2m^2 - m \\ &= h_m \end{aligned}$$

$\therefore$ The odd triangular numbers are hexagonal.

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

The even numbers 2, 4, 6 … can be expressed as $2m$ where m is an integer.

$$\begin{aligned} t_{2m} &= \frac{2m(2m+1)}{2} \\ &= m(2m+1) \\ &= 2m^2 + m \end{aligned}$$

Use proof by contradiction to show it is not hexagonal. Assume it is hexagonal.

$$\begin{aligned} 2m^2 + m &= 2k^2 - k \qquad \text{where } k \text{ is an integer} \\ m + k &= 2k^2 - 2m^2 \\ m + k &= 2\left(k^2 - m^2\right) \\ m + k &= 2(k-m)(k+m) \\ \therefore\ 1 &= 2(k-m) \qquad \text{since } k + m \neq 0 \end{aligned}$$

So, 1 is even, which is not true $\therefore$ the original statement is false.

Question 15 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Integrates to obtain an expression for t in terms of v, or equivalent merit	2
• Provides an expression for the resultant force, or equivalent merit	1

Sample answer:

$$\text{Force} = -mg - kv^2$$

$$ma = -mg - kv^2$$

$$a = -g - \frac{kv^2}{m}$$

$$a = -g - kv^2 \qquad \text{given } m = 1$$

$$\frac{dv}{dt} = -g - kv^2 \qquad \text{need } t \text{ in terms of } v\text{, so use } a = \frac{dv}{dt}$$

$$\frac{dt}{dv} = -\frac{1}{g + kv^2}$$

$$t = -\frac{1}{\sqrt{gk}}\tan^{-1}\left(v\sqrt{\frac{k}{g}}\right) + c$$

When $t = 0$, $v = u$

$$0 = -\frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) + c$$

$$c = \frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

$$\therefore \quad t = \frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \frac{1}{\sqrt{gk}}\tan^{-1}\left(v\sqrt{\frac{k}{g}}\right)$$

Max height, $v = 0 \ \therefore \ t = \frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$ as required.

(Either notation, arctan or $\tan^{-1}$, is acceptable.)

Question 15 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Integrates to obtain an expression for x in terms of v, or equivalent merit	2
• Provides an integral expression for x in terms of v, or equivalent merit	1

Sample answer:

$$v\frac{dv}{dx} = -g - kv^2$$

$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$x = -\frac{1}{2k}\ln\left(g + kv^2\right) + c$$

When $x = 0,\ v = u$

$$0 = -\frac{1}{2k}\ln\left(g + ku^2\right) + c$$

$$c = \frac{1}{2k}\ln\left(g + ku^2\right)$$

$$\therefore\ x = \frac{1}{2k}\left(\ln\left(g + ku^2\right) - \ln\left(g + kv^2\right)\right) = \frac{1}{2k}\ln\frac{g + ku^2}{g + kv^2}$$

Maximum height, sub in $v = 0$

$$\therefore \text{ maximum height } x = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right)$$

Question 15 (d)

Criteria	Marks
• Provides correct solution	2
• Writes $5^n = (2+3)^n$ or equivalent merit	1

Sample answer:

$$5 = 2 + 3$$

so $\quad 5^n = (2+3)^n$

$$= 2^n + 3^n + \binom{n}{1}2 \times 3^{n-1} + \text{other terms}$$

$$> 2^n + 3^n$$

so $\quad 2^n + 3^n \neq 5^n \qquad$ if $n \geq 2$

Question 16 (a) (i)

Criteria	Marks
• Provides correct solution	2
• States that $\overrightarrow{OP}$ is a unit vector, or equivalent merit.	1

Sample answer:

Point P is on the unit sphere so

$$\begin{aligned}
1 &= \left|\overrightarrow{OP}\right| \\
&= \left|x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}\right| \\
&\le \left|x\underset{\sim}{i}\right| + \left|y\underset{\sim}{j} + z\underset{\sim}{k}\right| && \text{(triangular inequality)} \\
&\le \left|x\underset{\sim}{i}\right| + \left|y\underset{\sim}{j}\right| + \left|z\underset{\sim}{k}\right| && \text{(triangular inequality)} \\
&= |x| + |y| + |z| && \underset{\sim}{i},\ \underset{\sim}{j},\ \underset{\sim}{k}\ \text{unit vectors}
\end{aligned}$$

so $|x| + |y| + |z| \ge 1$

Answers could include:

P is on the unit sphere $\therefore \sqrt{x^2 + y^2 + z^2} = 1$

$\therefore |x|^2 + |y|^2 + |z|^2 = 1$

But $|x| \le 1$, $|y| \le 1$ and $|z| \le 1$

$\therefore |x|^2 \le |x|$, $|y|^2 \le |y|$ and $|z|^2 \le |z|$

$$\begin{aligned}
\therefore 1 &= |x|^2 + |y|^2 + |z|^2 \\
&\le |x| + |y| + |z|
\end{aligned}$$

Question 16 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains $\left\lvert a_1b_1 + a_2b_2 + a_3b_3 \right\rvert = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}\sqrt{b_1^{\,2} + b_2^{\,2} + b_3^{\,2}}\cos\theta \ldots$	2
• Attempts to apply the dot product, or equivalent merit	1

Sample answer:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left\lvert \underset{\sim}{a} \right\rvert \left\lvert \underset{\sim}{b} \right\rvert \cos\theta$$

$$a_1b_1 + a_2b_2 + a_3b_3 = \left\lvert \underset{\sim}{a} \right\rvert \left\lvert \underset{\sim}{b} \right\rvert \cos\theta$$

$$-1 \le \cos\theta \le 1$$

$$-\left\lvert \underset{\sim}{a} \right\rvert \left\lvert \underset{\sim}{b} \right\rvert \le a_1b_1 + a_2b_2 + a_3b_3 \le \left\lvert \underset{\sim}{a} \right\rvert \left\lvert \underset{\sim}{b} \right\rvert$$

so
$$\left\lvert a_1b_1 + a_2b_2 + a_3b_3 \right\rvert \le \left\lvert \underset{\sim}{a} \right\rvert \left\lvert \underset{\sim}{b} \right\rvert$$

$$= \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}\sqrt{b_1^{\,2} + b_2^{\,2} + b_3^{\,2}}$$

Question 16 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Chooses one suitable vector to use with the result from part (ii), or equivalent merit	1

Sample answer:

If $P(x,y,z)$ is on the unit sphere $x^2+y^2+z^2=1$, $|x|^2+|y|^2+|z|^2=1$.
Hence $Q(|x|,|y|,|z|)$ is on the unit sphere.

In part (ii) let $\underset{\sim}{a}=\begin{pmatrix}|x|\\|y|\\|z|\end{pmatrix}$

And $\underset{\sim}{b}=\begin{pmatrix}1\\1\\1\end{pmatrix}$

Then

$$\big||x|+|y|+|z|\big|\le\sqrt{|x|^2+|y|^2+|z|^2}\sqrt{1^2+1^2+1^2}$$
$$=\sqrt{3}$$

$|x|,|y|$ and $|z|$ are not negative, so
$|x|+|y|+|z|=\big||x|+|y|+|z|\big|\le\sqrt{3}$

Answers could include:

$P(x,y,z)$ on unit sphere so $x^2+y^2+z^2=1$.
Choose b_1 to be 1 or –1 so that $xb_1\ge 0$ so $xb_1=|x|$, and similarly choose b_2 and b_3

so $yb_2\ge 0$ and $zb_3\ge 0$.

Let $\underset{\sim}{a}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

By part (ii)

$$\left|xb_1+yb_2+zb_3\right|\le\sqrt{x^2+y^2+z^2}\sqrt{b_1{}^2+b_2{}^2+b_3{}^2}$$

$$=1\times\sqrt{1+1+1}$$

$$=\sqrt{3}$$

$\big||x|+|y|+|z|\big|\le\sqrt{3}$

$|x|,|y|,|z|$ are not negative so

$|x|+|y|+|z|\le\sqrt{3}$

Question 16 (b)

Criteria	Marks
• Provides correct solution, with justification	5
• Shows that the relevant times are positive OR • Shows that the relevant times occur before the particle lands, or equivalent merit	4
• Obtains a quadratic that will identify all possible angles, or equivalent merit	3
• Uses the fact that the dot product of the two vectors is 0, or equivalent merit	2
• Obtains velocity vector, or equivalent merit	1

Sample answer:

Position $\quad \underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ -\dfrac{gt^2}{2} + ut\sin\theta \end{pmatrix}$

Velocity $\quad \dot{\underset{\sim}{r}}(t) = \begin{pmatrix} u\cos\theta \\ -gt + u\sin\theta \end{pmatrix}$

If position vector is perpendicular to velocity vector then

$$\begin{aligned}
0 &= \underset{\sim}{r}(t) \cdot \dot{\underset{\sim}{r}}(t) \\
&= \begin{pmatrix} ut\cos\theta \\ -\dfrac{gt^2}{2} + ut\sin\theta \end{pmatrix} \cdot \begin{pmatrix} u\cos\theta \\ -gt + u\sin\theta \end{pmatrix} \\
&= u^2 t\cos^2\theta + \left(-\frac{gt^2}{2} + ut\sin\theta\right)(-gt + u\sin\theta) \\
&= u^2 t\cos^2\theta + \frac{g^2t^3}{2} - \frac{gt^2u}{2}\sin\theta - gt^2u\sin\theta + u^2t\sin^2\theta \\
&= u^2 t - \frac{3ugt^2\sin\theta}{2} + \frac{g^2t^3}{2} \\
&= \frac{t}{2}\left(g^2t^2 - 3ugt\sin\theta + 2u^2\right)
\end{aligned}$$

During time of flight $t > 0$ so above can only be zero when $g^2t^2 - 3ugt\sin\theta + 2u^2 = 0$.

$y = g^2t^2 - 3ugt\sin\theta + 2u^2$ is the graph of a concave up parabola where y is a function of t.

Want two zeros so $\triangle = b^2 - 4ac > 0$ and we also know $0 < \theta < \frac{\pi}{2}$

$$9u^2g^2\sin^2\theta - 4 \times g^2 \times 2u^2 > 0$$

$$u^2g^2\left(9\sin^2\theta - 8\right) > 0$$

$$9\sin^2\theta - 8 > 0 \qquad \text{as } u^2g^2 > 0$$

$$\sin^2\theta > \frac{8}{9}$$

$$\sin\theta > \frac{\sqrt{8}}{3} \qquad \left(0 < \theta < \frac{\pi}{2} \text{ so } \sin\theta > 0\right)$$

$$\theta > 1.23 \qquad \text{(2 decimal places)}$$

(70.52°)

This shows that we may have two points during the time of flight if $\frac{\pi}{2} > \theta > 1.23$

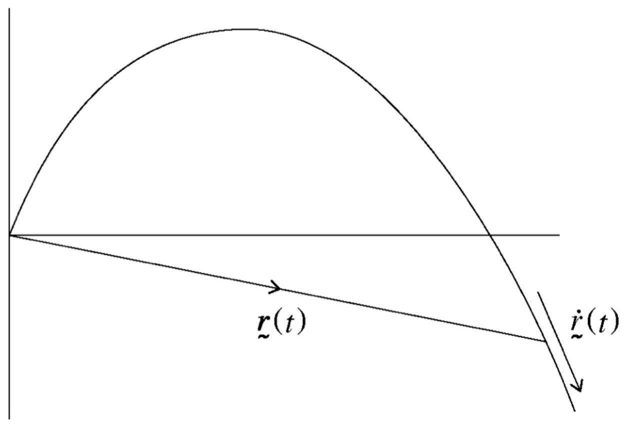
$$t = \frac{3ug\sin\theta \pm ug\sqrt{9\sin^2\theta - 8}}{2g^2} = \frac{u}{2g}\left(3\sin\theta \pm \sqrt{9\sin^2\theta - 8}\right)$$

$9\sin^2\theta - 8 < 9\sin^2\theta = (3\sin\theta)^2$

so $t > 0$

so both points occur after projection.

If we ignore the ground, and consider points on the trajectory below the point of projection.



Both $\underset{\sim}{r}(t)$ and $\dot{\underset{\sim}{r}}(t)$ point into the 4th quadrant so the angle between them is less than $\frac{\pi}{2}$.

Thus the two points must occur after projection but before the projectile lands.

Question 16 (c)

Criteria	Marks
• Provides correct sketch	3
• Considers inequalities involving $x\tan(x)$ AND has included or excluded at least one section of the Argand plane • Obtains the region below <diag>	2
• States that a nominated section of the Argand plane is included in the required region OR • States that a nominated section of the Argand plane is excluded from the required region	1

Sample answer:

If $z = x + iy$ and $Arg(z) = \theta$

then $-\pi < \theta \le \pi$

$\tan\theta = \dfrac{y}{x}$ when $x \ne 0$

$\text{Re}(z) = x$

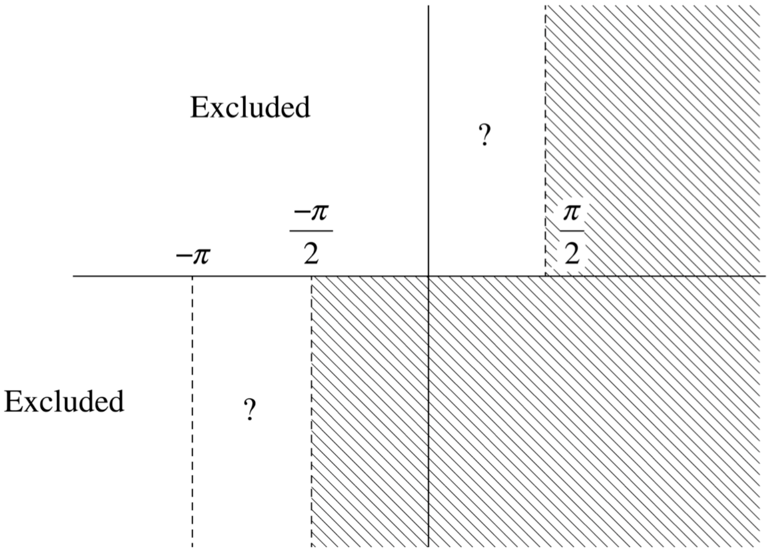
In 2nd quadrant $\text{Re}(z) = x < 0 < \dfrac{\pi}{2} < Arg(z)$ so 2nd quadrant not in region

In 4th quadrant $\text{Re}(z) > 0 > Arg(z)$ so 4th quadrant included in region

In 1st quadrant $Arg(z) < \dfrac{\pi}{2}$ so if $x \ge \dfrac{\pi}{2}$ then z is included in region

In 3rd quadrant $-\pi < Arg(z) < -\dfrac{\pi}{2}$ so $x < -\pi$ excluded and $x > -\dfrac{\pi}{2}$ included.

So far



If $0 \le x < \dfrac{\pi}{2}$ then $\tan x$ is an increasing function

$$\operatorname{Re}(z) \ge Arg(z)$$

$$x \ge \theta$$

$$\tan x \ge \tan\theta = \frac{y}{x} \quad \left(\text{tan increases on}\left[0, \frac{\pi}{2}\right]\right)$$

$$x\tan x \ge y \qquad (x > 0)$$

If $-\pi < x < \dfrac{-\pi}{2}$ tan also increasing

$$Arg(z) = \arctan\left(\frac{y}{x}\right) - \pi$$

$$x \ge \arctan\left(\frac{y}{x}\right) - \pi$$

$$x + \pi \ge \arctan\left(\frac{y}{x}\right)$$

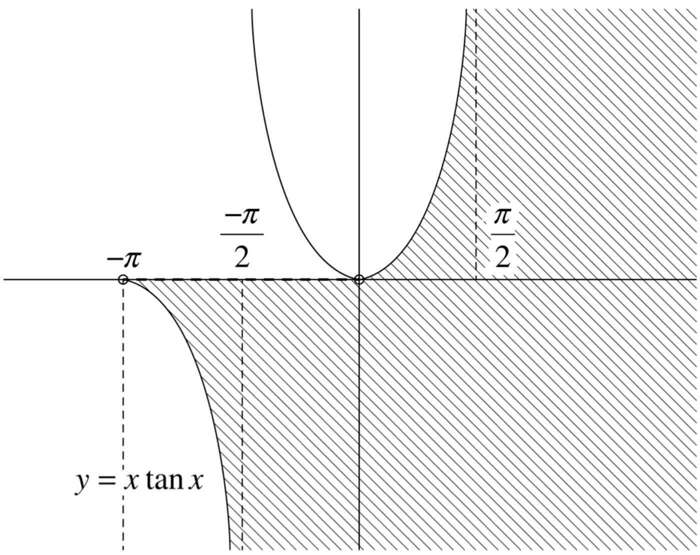
$$\tan(x+\pi) \ge \frac{y}{x}$$

$$x\tan(x) \le y$$

as $\tan(x+\pi) = \tan x$ and $x < 0$

Also, if $z = 0$ and if $y = 0, x < 0, Arg(z) = \pi > x$

$\therefore$ not included.



2021 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	MEX V1 Further work with vectors	MEX 12 3
2	1	MEX C1 Further integration	MEX 12–5
3	1	MEX V1 Further work with vectors	MEX 12–3
4	1	MEX P1 The nature of proof	MEX 12–2
5	1	MEX P1 The nature of proof	MEX 12–1
6	1	MEX N2 Using complex numbers	MEX 12–4
7	1	MEX V1 Further work with vectors	MEX 12–3
8	1	MEX M1 Applications of calculus to mechanics	MEX 12–6
9	1	MEX P1 The nature of proof	MEX 12–2
10	1	MEX N2 Using complex numbers	MEX 12–4

Section II

Question	Marks	Content	Syllabus outcomes
11(a)	2	MEX N1 Introduction to complex numbers	MEX 12–4
11 (b)	2	MEX P2 Further proof by mathematical induction MEX N1 Introduction to complex numbers	MEX 12–1
11 (c)	3	MEX V1 Further work with vectors	MEX 12–3
11 (d) (i)	2	MEX N2 Using complex numbers	MEX 12–4
11 (d) (ii)	2	MEX N2 Using complex numbers	MEX 12–4
11 (e)	2	MEX N1 Introduction to complex numbers	MEX 12–4
11 (f)	3	MEX C1 Further integration	MEX 12–1
12 (a)	3	MEX C1 Further integration	MEX 12–5
12 (b) (i)	1	MEX P1 The nature of proof	MEX 12–2
12 (b) (ii)	1	MEX P1 The nature of proof	MEX 12–2
12 (c)	3	MEX V1 Further work with vectors	MEX 12–3
12 (d)	3	MEX P2 Further proof by mathematical induction	MEX 12–2
12 (e) (i)	1	MEX V1 Further work with vectors	MEX 12–3
12 (e) (ii)	2	MEX V1 Further work with vectors	MEX 12–3

Question	Marks	Content	Syllabus outcomes
12 (e) (iii)	1	MEX V1 Further work with vectors	MEX 12–3
13 (a)	2	MEX N2 Using complex numbers	MEX 12–4
13 (b)	3	MEX C1 Further integration	MEX 12–5
13 (c) (i)	2	MEX C1 Further integration	MEX 12–5
13 (c) (ii)	4	MEX C1 Further integration	MEX 12–5
13 (d) (i)	2	MEX M1 Application of calculus to mechanics	MEX 12–6
13 (d) (ii)	2	MEX M1 Application of calculus to mechanics	MEX 12–6, MEX 12–7
14 (a)	4	MEX C1 Further integration	MEX 12–5
14 (b) (i)	2	MEX M1 Application of calculus to mechanics	MEX 12–6
14 (b) (ii)	2	MEX M1 Application of calculus to mechanics	MEX 12–6, MEX 12–7
14 (c) (i)	3	MEX N2 Using complex numbers	MEX 12–4
14 (c) (ii)	3	MEX N2 Using complex numbers	MEX 12–4
15 (a) (i)	2	MEX P1 The nature of proof	MEX 12–2
15 (a) (ii)	2	MEX P1 The nature of proof	MEX 12–2
15 (b) (i)	2	MEX P1 The nature of proof	MEX 12–1, MEX 12–2
15 (b) (ii)	1	MEX P1 The nature of proof	MEX 12–1, MEX 12–2
15 (c) (i)	3	MEX M1 Application of calculus to mechanics	MEX 12–6
15 (c) (ii)	3	MEX M1 Application of calculus to mechanics	MEX 12–6
15 (d)	2	MEX P1 The nature of proof	MEX 12–2
16 (a) (i)	2	MEX P1 The nature of proof	MEX 12–2
16 (a) (ii)	3	MEX V1 Further work with vectors	MEX 12–3
16 (a) (iii)	2	MEX V1 Further work with vectors	MEX 12–3
16 (b)	5	MEX M1 Application of calculus to mechanics	MEX 12–6, MEX 12–7
16 (c)	3	MEX N1 Introduction to complex numbers	MEX 12–4

Mathematics Extension 2

HSC Marking Feedback 2021

Question 11

Part (a)

Students should:

- apply index laws to multiply complex numbers using Euler form
- add the arguments when multiplying complex numbers.

In better responses, students were able to:

- immediately identify the complex form required and add the arguments when multiplying complex numbers.

Areas for students to improve include:

- using efficient techniques for questions with small mark values in an examination.

Part (b)

Students should:

- understand the use of sigma notation in the Mathematics Extension 2 course.

In better responses, students were able to:

- immediately write down the integer powers of i.

Areas for students to improve include:

- writing $i = \sqrt{-1}$ and manipulating it with powers.

Part (c)

Students should:

- know the syllabus formulae well, in particular the formulae around vectors
- use the Reference Sheet.

In better responses, students were able to:

- substitute the correct values into the required formula $\cos\theta = \frac{a \cdot b}{|a||b|}$ and move quickly to the correct solution.

Areas for students to improve include:

- working with simple arithmetic

- rounding off where instructed.

Part (d) (i)

Students should:

- apply a range of efficient strategies when finding square roots of complex numbers.

In better responses, students were able to:

- concisely find the two square roots of $-i$ using geometry or algebraic methods
- manipulate complex equations like $z^2 = -i$ in an efficient and effective manner.

Areas for students to improve include:

- using the geometry of complex numbers to quickly assess responses
- having versatility across different approaches instead of applying a rehearsed approach to all questions.

Part (d) (ii)

Students should:

- apply a range of efficient strategies when solving quadratic equations.

In better responses, students were able to:

- complete the square quickly and accurately to make z the subject before using part (i).

Areas for students to improve include:

- connecting parts (i) and (ii) of the questions.

Part (e)

Students should:

- perform division of complex numbers in Cartesian form.

In better responses, students were able to:

- rationalise the denominator correctly and collect like terms.

Areas for students to improve include:

- avoiding simple arithmetic errors
- using $i^2 = -1$.

Part (f)

Students should:

- manipulate partial fractions across a range of forms.

In better responses, students were able to:

- use a combination of techniques involving identities and in particular, substitute appropriate values to determine the unknown coefficients.

Areas for students to improve include:

- solving simultaneous equations with three variables.

Question 12

Part (a)

Students should:

- decompose an integral with quadratic denominator into components.

In better responses, students were able to:

- complete the square and separate the integrand into integrals that lead to arctan and logarithmic functions
- recognise the degree of the numerator was one less than the degree of the denominator.

Areas for students to improve include:

- recognising the integrals of the form $\int \frac{P(x)}{Q(x)}\, dx$ that can be decomposed
- completing the square accurately to facilitate integration.

Part (b) (i)

Students should:

- know the definition of a converse.

In better responses, students were able to:

- write the converse: 'If n is even, then n^2 is even'.

Areas for students to improve include:

- know the definitions of converse, contrapositive, counter-example and contradiction.

Part (b) (ii)

Students should:

- know how to prove simple numerical statements that involve odd and even numbers
- use algebraic approaches to prove number results.

In better responses, students were able to:

- succinctly set up and show that 'If n was even, then n^2 was even'
- establish an even number, $n = 2k$, and show $n^2 = 2(2k^2)$.

Areas for students to improve include:

- improving clarity of arguments when proving results
- avoiding careless algebraic errors, such as $4k^2 = 2(2k)$
- avoiding simple arithmetic and algebraic errors.

Part (c)

Students should:

- know the implications of the statement that vectors are perpendicular and intersect.

In better responses, students were able to:

- write the direction vectors and then find their scalar product, which provided $p = 2$

- equate the two vectors and solve simultaneously for μ and λ, which then allowed them to find q using p as well.

Areas for students to improve include:

- practising the skill of determining direction vectors
- using the scalar product effectively
- solving simultaneous equations without error.

Part (d)

Students should:

- use mathematical induction to prove inequalities.

In better responses, students were able to:

- clearly set up the induction process and express their logic
- clearly show where they used $P(k)$: $\sqrt{k!} > 2^k$
- clearly justify why $\sqrt{k+1} > 2$, for $k \geq 9$.

Areas for students to improve include:

- setting out a proof by mathematical induction with a simple logical process
- showing why inequalities are true for specified values of k.

Part (e) (i)

Students should:

- use vectors to show geometric results.

In better responses, students were able to:

- use the information provided to show the result required
- explain their use of vectors to arrive at the desired result.

Areas for students to improve include:

- defining the vectors used if they are not defined in the question. For example, Let $\overrightarrow{AB} = \underset{\sim}{a}$
- using vector notation correctly.

Part (e) (ii)

Students should:

- show the result given using the suggested expressions.

In better responses, students were able to:

- define the vectors used in showing the result given
- show each step of the process and explain where the given results were used.

Areas for students to improve include:

- showing the result using vectors, not describing the situation and jumping to given conclusions.

Part (e) (iii)

Students should:

- manipulate vector expressions
- determine the scalar quantity that one vector is of another vector, knowing the difference between positive and negative vectors.

In better responses, students were able to:

- use the given results in the previous part to successfully determine the value of λ.

Areas for students to improve include:

- utilising the continuity between parts of a question
- understanding that $\overrightarrow{HS} = -\overrightarrow{SH}$ and other such relationships may be used to show given results.

Question 13

Part (a)

Students should:

- apply De Moivre's theorem to find the arguments and approximate moduli for the fourth roots of a complex number and correctly place these on an Argand diagram
- know the geometrical relationship between the n^{th} roots of a complex number. That is, the roots have the same length, and the roots are equally spaced about the Argand diagram.

In better responses, students were able to:

- generate the correct arguments for the four roots
- determine that the correct modulus of each root lay in the interval $1 \le \left|z^{\frac{1}{4}}\right| \le |a+ib|$
- calculate the argument of the first fourth root and use this to determine the argument of the following roots since the roots are equally spaced about the Argand diagram
- calculate the argument of each root using De Moivre's theorem.

Areas for students to improve include:

- understanding a complex number in polar form and applying De Moivre's theorem to find the arguments and moduli of the roots of complex numbers.
- using the Reference Sheet to assist with clarity and precision
- finding the n^{th} root of a complex number algebraically
- transferring their knowledge from solving equations with numerical values to this generalised form of $a + ib$
- representing the n^{th}roots of a complex number geometrically
- improving knowledge of De Moivre's theorem with the understanding of both transformations to the argument and modulus separately. That is, if $z = r\operatorname{cis}\theta$, $\arg\left(z^{\frac{1}{4}}\right) = \frac{1}{4}\theta$ and $\left|z^{\frac{1}{4}}\right| = r^{\frac{1}{4}}$.

Part (b)

Students should:

- provide an appropriate substitution, apply this substitution with correct limits and then integrate the new expression correctly to determine the definite integral
- select an appropriate substitution
- have thorough knowledge of all steps in integration by substitution, from changing the limits to calculating $\frac{dx}{du}$
- manipulate algebra and substitute correctly.

In better responses, students were able to:

- use $u = x^2 - 9$ or $u^2 = x^2 - 9$ as their substitution and correctly determined the definite integral
- use $x = 3\sec\theta$ successfully, although this method proved longer with some difficult limits
- use other approaches such as integration by parts, algebraic manipulation or use $u = x^2$ as the substitution, although this proved to be the most challenging.

Areas for students to improve include:

- avoiding simple errors when converting the integral in terms of x to an integral in term of u
- improving the setting out of solutions and showing all working. Splitting the x^3 term into x^2 and x would have avoided errors.
- thinking of a suitable substitution before persisting with inappropriate substitutions. For example, some students tried to use $u = 3\sin\theta$ or $u = x^3$
- recognising which trigonometric substitutions are suited to which algebraic expressions
- finding the correct limits for u. This was particularly an issue when using the substitution $x = 3\sec\theta$
- avoiding rounding their limits instead of providing exact values
- avoiding unnecessary steps such as converting the expression in terms of u back into an expression in terms of x
- avoiding simple errors such as dropping the $\frac{1}{2}$ in front of the integral sign
- avoiding simple errors when integrating, particularly when fractional powers are involved.

Part (c) (i)

Students should:

- recognise the need to use integration by parts to solve the recurrence relation
- explicitly state all four parts involved in integration by parts.

In better responses, students were able to:

- use the recurrence relation to clearly write u, du, v and dv and apply this correctly with $u = (\ln x)^n$ and $dv = 1$
- successfully use the substitution method of $u = \ln x$ and $x = e^u$.

Areas for students to improve include:

- using the correct choices for u and dv when performing integration by parts and clearly show the proof step by step
- clearly setting out solutions when using integration by parts
- remembering the exponent on $u = (\ln x)^n$ when manipulating algebra
- remembering to multiply by the derivative of $\ln x$ when differentiating this expression.

Part (c) (ii)

Students should:

- apply their knowledge of volumes of solids of revolution from Mathematics Extension 1 to recognise that a difference of volumes is required for this question
- use part (i) and the recurrence formula to help solve difficult integrals efficiently
- follow the prompt within the question
- recognise the different types of shapes that are generated when the specified regions are rotated around the x-axis and find their volumes using integration skills and the given volume formula where required.

In better responses, students were able to:

- write correct expressions for V_A and V_B
- clearly state the volumes required and express these as a difference of volumes
- use volumes of cylinders and spheres to efficiently answer the question
- notice the connection between this question and part (i) and use the recurrence relationship correctly
- generate an expression for I_0, or find I_1 and then use integration by parts
- use the 'or otherwise' approach by integrating $(\ln x)^2$ by parts.

Areas for students to improve include:

- setting out recurrence formula clearly
- setting work out clearly and in a logical order and looking to use part (i) when prompted
- generating an expression for I_0 , rather than finishing at I_1 which simplifies more easily
- gaining a deeper understanding of volumes of solids of revolution, by visualising the volumes of the cylinders and hemispheres that are generated and the limits that are used for these regions
- understanding that the formula $V = \pi \int_a^b y_1{}^2 - y_2{}^2 dy \neq \pi \int_a^b (y_1 - y_2)^2 dy$
- understanding that the formula $V = \pi \int_a^b y_1{}^2 - y_2{}^2 dy = \pi \int_a^b y_1{}^2 dy - \pi \int_a^b y_2{}^2 dy$
- identifying the correct limits of integration
- visualising and drawing solids of revolution and connecting this to the formulae above.

Question 14

Part (a)

Students should:

- identify the style of definite integral question that is appropriate for a t-substitution.

In better responses, students were able to:

- fully simplify fractions before proceeding to partial fractions
- clearly show the substitution of dx, change the limits of integration and develop and simplify the expression involving $\cos x$ in terms of t
- take algebraic steps to allow their path to the solution to involve integers.

Areas for students to improve include:

- simplifying algebraic fractions before proceeding, keeping their next steps as straightforward as possible
- factorising using the difference of two squares accurately
- knowing the expression for $\frac{dx}{dt}$ when using the t results and have a deeper understanding of the process involved with the t-substitution approach.

Part (b) (i)

Students should:

- decompose mg parallel to the slope and then add the resistive force to find resultant force
- find the components of vectors.

In better responses, students were able to:

- construct an accurate diagram representing the problem
- know where and how to construct the triangle to decompose mg into appropriate components.

Areas for students to improve include:

- translating a word problem into a force diagram
- knowing how to decompose mg parallel to the plane
- being familiar with mechanical models of systems involving gravity forces.

Part (b) (ii)

Students should:

- set acceleration to zero due to constant velocity
- be able to solve a quadratic equation
- identify which direction is positive in the question
- correctly round to one decimal place.

In better responses, students were able to:

- keep the quadratic equation as simple as possible
- use the quadratic formula correctly to solve a quadratic equation

- correctly apply the quadratic equation, identifying which of the two solutions were correct
- correctly answer the question to one decimal place as requested.

Areas for students to improve include:

- reading the question to respond to the appropriate number of decimal places
- solving a quadratic equation using an appropriate method
- understanding which direction is positive and which is negative and making an appropriate selection
- being familiar with mechanical models of systems involving gravity forces.

Part (c) (i)

Students should:

- use De Moivre's theorem correctly
- complete a binomial expansion
- equate real parts in an expansion and express their result in terms of $\cos\theta$ only.

In better responses, students were able to:

- state the binomial expansion and De Moivre's theorem separately
- write the most generic form of the binomial expansion and not work with it algebraically until equating real parts, thereby avoiding introducing algebraic errors into the expansion
- clearly write the binomial expansion with its numerous powers and operators correctly
- connect the real components of both expressions to successfully finalise the question.

Areas for students to improve include:

- being accurate with writing down the powers correctly
- taking care with basic number and algebraic skills when expanding a complex binomial expansion when it involves powers and negatives and powers of i.

Part (c) (ii)

Students should:

- correctly identify the link form part (i) to establish the connection between $\mathrm{Re}\left(e^{\frac{i\pi}{10}}\right)$ and $\cos\left(\frac{\pi}{10}\right)$.
- solve the resulting polynomial equation
- make decisions about the values of $\sqrt{\frac{5\pm\sqrt{5}}{8}}$ and choose the appropriate one, using trigonometric theory.

In better responses, students were able to:

- apply part (i) to form a quadratic equation that could be solved via factorising and the quadratic equation after using a substitution
- form a polynomial by solving $\cos 5\theta = 0$ and determining its distinct roots
- identify that $\cos\left(\frac{\pi}{10}\right)$ is not equal to zero and therefore a root of the quartic

- decide that the result is either $\cos\left(\frac{\pi}{10}\right)$ or $\cos\left(\frac{3\pi}{10}\right)$
- compare the value of $\cos\left(\frac{\pi}{10}\right)$ to $\cos\left(\frac{3\pi}{10}\right)$ using either a graph and/or stating it is a decreasing function in the appropriate domain and decide that it is the greater of the two.

Areas for students to improve include:

- finding the roots of the polynomial from part (i) that results in 0 using $\cos 5\theta = 0$
- comparing $\cos\left(\frac{3\pi}{10}\right)$ to a root and not an arbitrary value
- correctly justifying that $\cos\left(\frac{\pi}{10}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$ through the use of a correct comparison or demonstrating the cosine function is decreasing over the domain of $\left(0, \frac{\pi}{2}\right)$.

Question 15

Part (a) (i)

Students should:

- use the relationship between the arithmetic mean and geometric mean for two terms more than once to achieve a desired result
- prove the statement given for all non-negative values of a, b and c.

In better responses, students were able to:

- know what substitutions for x and y would be most beneficial, for example $x = \frac{a^2+b^2}{2}$ and $y = c$ or similar.

Areas for students to improve include:

- showing all steps in the proof, using the given inequality as required
- using the relationship between the arithmetic mean and geometric mean in its many forms.

Part (a) (ii)

Students should:

- use the suggested result, or otherwise
- know the extended versions of the relationship between the arithmetic mean and geometric mean.

In better responses, students were able to:

- use the extended version; $\sqrt[6]{abcdef} \leq \frac{a+b+c+d+e+f}{6}$
- use the result from part (i) recognising that $\sqrt{abc} \leq \frac{a^2+c^2+2b}{4}$ and $\sqrt{abc} \leq \frac{b^2+c^2+2a}{4}$ are also true.

Areas for students to improve include:

- using part (i) to assist in understanding and solving this part
- knowing the different forms of the relationship between the arithmetic mean and geometric mean.

Part (b) (i)

Students should:

- use a general expression for an odd number, such as $n = 2k - 1$ in this case
- prove the statement for all odd values of n.

In better responses, students were able to:

- substitute $n = 2k - 1$ or $n = 2k + 1$ into t_n, students using $n = 2k - 1$ were more successful as it simplified easily. Those substituting $n = 2k + 1$ had an extra step to show that $t_{2k+1} = 2k^2 + 3k + 1 = 2(k + 1)^2 - (k + 1) = h_{k+1}$.

Areas *for students to improve include:*

- understanding that proving for all odd values of n does not mean simply trying a few odd numbers, like the ones given in the question
- reducing simple algebraic errors by writing every step of the proof and not skipping steps.

Part (b) (ii)

Students should:

- prove that the expression for t_n where n is even cannot equal the expression for a hexagonal number
- prove that there cannot be any value of t_n for even n which is a hexagonal number.

In better responses, students were able to:

- prove that $t_{2k} = h_n$ was only possible if one solution was negative and the other was a positive fraction
- show from part (i) that the odd indexed triangular numbers covered all hexagonal numbers and so there are no hexagonal numbers that map to the even indexed triangular numbers.

Areas for students to improve include:

- understanding how to disprove statements and that providing counter-examples does not disprove a statement for all even n
- showing every step to avoid careless and costly arithmetic and algebraic errors.

Part (c) (i)

Students should:

- create a correct equation of motion making $\frac{dv}{dt}$ the subject and integrating until the time when $v = 0$.

In better responses, students were able to:

- use a definite integral approach, such as $\int_0^T dt = -\int_u^0 \frac{dv}{g+kv^2} = \int_0^u \frac{dv}{1+\frac{k}{g}v^2}$, which was generally quicker and more successful
- reverse the limits of integration to eliminate the negative from the equation
- use the limit method or substituted $v = 0$ to find the given equation for t.

Areas for students to improve include:

- drawing a force diagram and determining a correct equation of motion
- knowing what is required as a result, t or x, which then determines the use of $\frac{dv}{dt}$ or $v\frac{dv}{dx}$.

Part (c) (ii)

Students should:

- create a correct equation of motion making $v\frac{dv}{dx}$ the subject and integrating until the maximum height when $v = 0$, is reached.

In better responses, students were able to:

- integrate using $v\frac{dv}{dx}$ using definite or indefinite integrals with definite integrals providing the solution quicker and easier
- draw a force diagram and indicate the positive direction to be used.

Areas for students to improve include:

- accurately applying the arithmetic of logarithms
- taking care with the algebra and not losing constants
- ensuring that the solution is logical, clear and legible.

Part (d)

Students should:

- prove that there is no value of $n \geq 2$ for which $2^n + 3^n = 5^n$.

In better responses, students were able to:

- state that $5^n = (2+3)^n$ then use a binomial expansion and noting the middle terms in the expansion are all positive, so $2^n + 3^n \neq 5^n$
- successfully show that $2^n + 3^n < 5^n$ using mathematical induction, and so $2^n + 3^n \neq 5^n$
- use contradiction, showing that $5^n - 3^n = (5-3)(5^{n-1} + 5^{n-2}3 + 5^{n-3}3^2 + \cdots + 3^{n-1})$ which gives a statement like $2^{n-1} = 5^{n-1} + 5^{n-2}3^1 + 5^{n-3}3^2 + \cdots + 3^{n-1}$, for $n \geq 2$.

Areas for students to improve include:

- understanding that mathematical induction cannot be used to disprove a statement, but only prove that a statement is true for all values
- using binomial expansions and remembering that there are coefficients $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}$ within the expansion.

Question 16

Part (a) (i)

Students should:

- operate with the magnitude of a vector, using appropriate notation
- utilise appropriate identities from the Vectors topic.

In better responses, students were able to:

- state and apply the triangle inequality as an identity and use it show the lower bound to the expression $|x| + |y| + |z|$ is 1.

Areas for students to improve include:

- understanding the difference between $\left|x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}\right|$ and $|x + y + z|$.

Part (a) (ii)

Students should:

- be clear with their intent in show questions that involve inequalities.

In better responses, students were able to:

- identify a correct relationship initially, typically $\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|} = \cos\theta$, leading to a correct inequality involving $|\underset{\sim}{a} \cdot \underset{\sim}{b}|$ and $|\underset{\sim}{a}|\,|\underset{\sim}{b}|$.

Areas for students to improve include:

- using absolute value or modulus symbols across vectors and scalars.

Part (a) (iii)

Students should:

- seek to find the intent behind a multi-part question.

In better responses, students were able to:

- carefully choose vectors from part (ii) to apply in this part, leading to the required upper bound for $|x| + |y| + |z|$ of $\sqrt{3}$.

Areas for students to improve include:

- understanding inequalities using vectors.

Part (b)

Students should:

- be able to answer questions involving links between multiple topics.

In better responses, students were able to:

- link the ideas of projectile motion, dot products, discriminant theory and trigonometry to find a range of angles
- justify their answers.

Areas for students to improve include:

- manipulating algebraic expansions
- applying discriminant theory to quadratic equations.

Part (c)

Students should:

- make geometric and algebraic connections across complex numbers.

In better responses, students were able to:

- realise a functional relationship, $y = x \tan x$, and use it to guide them to a correct region satisfying the required relationship in z.

Areas for students to improve include:

- using the Cartesian form when dealing with graphs involving complex numbers.