2023
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your student name and/or number at the top of every page

Total marks – 100

Section I – 10 marks (pages 3 - 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 - 11)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

This paper MUST NOT be removed from the examination room.

STUDENT NAME/NUMBER..

STUDENT NAME/NUMBER..

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Select the alternative A, B, C, D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

STUDENT NAME/NUMBER..

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

---

1. Which of the following statements is FALSE?

(A) $\forall a, b, x \in (1, \infty),\quad a < b \Rightarrow a^x < b^x$

(B) $\forall a, b, x \in (1, \infty),\quad a < b \Rightarrow x^a < x^b$

(C) $\forall a, b, x \in (1, \infty),\quad a < b \Rightarrow \log_a x < \log_b x$

(D) $\forall a, b, x \in (1, \infty),\quad a < b \Rightarrow \log_x a < \log_x b$

2. In an Argand diagram the points $A(-3, 2)$ and $B(5, -4)$ lie at opposite ends of a diameter of a circle. What is the equation of the circle?

(A) $|z - 1 + i| = 10$

(B) $|z - 1 + i| = 5$

(C) $|z + 1 - i| = 5$

(D) $|z + 1 - i| = 10$

3. What is the size of the acute angle θ between the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{k}$?

(A) $\theta = \frac{\pi}{6}$

(B) $\theta = \frac{\pi}{5}$

(C) $\theta = \frac{\pi}{4}$

(D) $\theta = \frac{\pi}{3}$

4. Which of the following is an expression for $\int \frac{1}{x^2 - \sqrt{3}x + 1}\,dx$?

(A) $\tan^{-1}\left(x - \sqrt{3}\right) + c$

(B) $2\tan^{-1}\left(x - \sqrt{3}\right) + c$

(C) $\tan^{-1}\left(2x - \sqrt{3}\right) + c$

(D) $2\tan^{-1}\left(2x - \sqrt{3}\right) + c$

5. A particle is moving in simple harmonic motion along the x axis. At time t seconds it has displacement $x = 4\cos\left(t + \frac{\pi}{4}\right)$ metres from the origin O. How many times does the particle pass through O in the first minute of its motion?

(A) 17

(B) 18

(C) 19

(D) 20

6. Considering the statement $P(x)\ odd \Rightarrow P'(x)\ even$, where $P(x)$ is a non-zero polynomial, which of the following is correct?

(A) The contrapositive statement is false and the converse statement is false.

(B) The contrapositive statement is true and the converse statement is false.

(C) The contrapositive statement is false and the converse statement is true.

(D) The contrapositive statement is true and the converse statement is true.

7. A sequence of complex numbers $z_1, z_2, z_3, z_4, \ldots$ is given by the rule $z_1 = Z$ and $z_{n+1} = c\,\overline{z}_n + c - 1$ for $n = 1, 2, 3, 4, \ldots$ where c is a complex number with modulus 1. What is the value of z_3 ?

(A) $z_3 = Z$

(B) $z_3 = -2 + Z$

(C) $z_3 = 2c + Z$

(D) $z_3 = -2 + 2c + Z$

8. A particle has initial velocity $2\underset{\sim}{i}$ ms^{-1}. At time t seconds it has acceleration $(2t-3)\underset{\sim}{i} + \frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)\underset{\sim}{j}$ ms^{-2}. When is the particle at rest?

(A) Never

(B) At time $t = 1$ second only

(C) At time $t = 2$ seconds only

(D) At times $t = 1$ and $t = 2$ seconds

9. What is the value of $\int_{-k}^{k}\{f(x)-f(-x)\}dx$?

(A) 0

(B) $\int_{0}^{k} f(x)dx$

(C) $2\int_{0}^{k} f(x)dx$

(D) $4\int_{0}^{k} f(x)dx$

10. A body of mass m kg moves in a horizontal straight line with initial speed U ms^{-1} subject to a resistance of magnitude $m(1+v^2)$ Newtons where v ms^{-1} is its speed. What is the distance travelled by the body in coming to rest?

(A) $\dfrac{1}{1+U}$ metres

(B) $\tan^{-1}U$ metres

(C) e^{1+U^2} metres

(D) $\ln\sqrt{1+U^2}$ metres

STUDENT NAME/NUMBER.......................................

Marks

Section II

90 Marks
Attempt Questions 11-16.
Allow about 2 hours and 45 minutes for this section.

Answer the questions in writing booklets provided. Use a separate writing booklet for each question.
In Questions 11-16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

(a) Express $\dfrac{1+2i}{2+i}$ in the form $a+ib$ where a and b are real. **2**

(b) Use the substitution $t=\tan\frac{x}{2}$ to find $\displaystyle\int \frac{1}{1+\sin x}\,dx$. **2**

(c) The complex numbers $z_1=-1+i$ and z_2 are such that $|z_1z_2|=\sqrt{6}$ and $\arg(z_1z_2)=\frac{7\pi}{12}$.

(i) Express z_1 in the form $re^{i\theta}$ where $r>0$ and $-\pi<\theta\le\pi$. **1**

(ii) Find z_2 in the form $a+ib$ where a and b are real. **2**

(d) A particle is moving along the x axis. At time t seconds it has displacement x metres from the origin O, velocity v ms^{-1} and acceleration a ms^{-2} where $a=12-4x$. Initially the particle is at rest 5 metres to the right of O.

(i) Use integration to show that $v^2=-4x^2+24x-20$. **2**

(ii) Find the range of possible values of x. **2**

(e)(i) On an Argand diagram shade the region containing all points representing complex numbers z that satisfy both $|\,z-2i\,|\le 2$ and $\frac{\pi}{6}\le\arg z\le\frac{\pi}{4}$. **2**

(ii) Find in simplest exact form the area of the shaded region. **2**

STUDENT NAME/NUMBER.......................................

Marks

Question 12 (15 marks) **Use a separate writing booklet.**

(a)(i) Show that $\left(2^p-1\right)\left(1+2^p+2^{2p}+\ldots+2^{(q-1)p}\right)=2^{pq}-1$ for positive integers p and q. **1**

(ii) Prove the statement *If* 2^n-1 *is prime then n is prime* by proving the contrapositive statement. **2**

(b) Use the substitution $x=\tan^2\theta,\ 0\le\theta<\frac{\pi}{2}$ to evaluate in simplest exact form **4**

$$\int_0^1 \frac{\sqrt{x}}{(1+x)^3}\,dx\ .$$

(c) With respect to a fixed origin O, the point P has position vector $2\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}$ and the line L that passes through P has vector equation $\underset{\sim}{r}=\left(2\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}\right)+\lambda\left(\underset{\sim}{i}-2\underset{\sim}{j}-\underset{\sim}{k}\right)$ for some scalar parameter λ. The line L is perpendicular to the plane $x-2y-z=3$.

(i) Show that the point P does not lie in the plane $x-2y-z=3$ and find the position vector of the point Q where the line L meets the plane. **3**

(ii) Hence find in simplest exact form the shortest distance from the point P to the plane $x-2y-z=3$. **1**

(d) A particle is moving in simple harmonic motion along the x axis with amplitude $a=3$ metres. At time t seconds it has displacement x metres from the origin O and velocity v ms^{-1} given by $v^2=n^2\left\{a^2-(x-c)^2\right\}$ for some constants $n>0,\ c>0$. **4**

The particle has speed $2\sqrt{5}$ ms^{-1} when it is at O and speed 6 ms^{-1} when it is 2 metres to the right of O. Find the centre and the period of the motion.

STUDENT NAME/NUMBER.......................................

Marks

Question 13 (15 marks) **Use a separate writing booklet.**

(a) Find the projection of the vector $\underset{\sim}{a} = 2\underset{\sim}{i} + 4\underset{\sim}{j}$ on the vector $\underset{\sim}{b} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$. **2**

(b) The roots $\alpha, \beta, \gamma, \delta$ of the equation $z^4 + az^3 + bz^2 + cz + d = 0$, where *a*, *b*, *c*, *d* are real, are represented by the vertices of square *PQRS* in an Argand diagram. Each of the four quadrants contains exactly one vertex of the square, and one of the roots of the equation is $1+2i$. Find the values of *a* and *d*. **3**

(c) A particle of mass *m* kg is projected vertically upwards with speed $2g$ ms^{-1} under gravity in a medium in which the resistance to motion has magnitude $\dfrac{mv^2}{g}$ Newtons where the speed of the particle is v ms^{-1} and the acceleration due to gravity is g ms^{-2}. The particle reaches a maximum height of $\frac{1}{2}g\ln 5$ metres before falling vertically downwards back to its starting point. During its descent, at time *t* seconds the particle has fallen *x* metres, has velocity v ms^{-1} and acceleration a ms^{-2} given by $a = \dfrac{g^2 - v^2}{g}$.

(i) Show that during its descent $x = \frac{1}{2}g\ln\left(\dfrac{g^2}{g^2 - v^2}\right)$. **3**

(ii) Find in simplest exact form the speed with which the particle returns to its starting point. **1**

(d) With respect to a fixed origin *O*, the lines L_1 and L_2 have vector equations $\underset{\sim}{r}_1 = (-9+2\lambda)\underset{\sim}{i} + \lambda\underset{\sim}{j} + (10-\lambda)\underset{\sim}{k}$ and $\underset{\sim}{r}_2 = (3+3\mu)\underset{\sim}{i} + (1-\mu)\underset{\sim}{j} + (17+5\mu)\underset{\sim}{k}$ respectively where λ and μ are scalar parameters. The point *A* with position vector $5\underset{\sim}{i} + 7\underset{\sim}{j} + 3\underset{\sim}{k}$ lies on L_1. The point *B* is the reflection of the point *A* in the line L_2.

(i) Find the position vector of the point of intersection *P* of the lines L_1 and L_2. **2**

(ii) Show that L_1 and L_2 are perpendicular to each other. **2**

(iii) Find the position vector of the point *B*. **2**

Marks

Question 14 (15 marks) **Use a separate writing booklet.**

(a) The numbers $a>0,\ b>0,\ c>0$ are such that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are consecutive terms in an arithmetic sequence. Show that $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are also consecutive terms in an arithmetic sequence. **3**

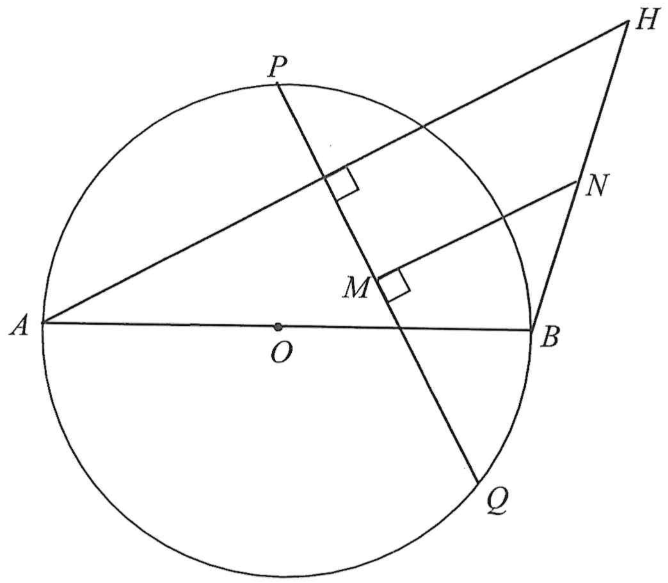
(b)(i) Show that $r\,{}^{p}C_{r} = p\,{}^{p-1}C_{r-1}$ for integers p and r such that $0<r<p$. **1**

(ii) Deduce that if p is prime and $0<r<p$ then p is a factor of ${}^{p}C_{r}$. **1**

(iii) Show that $(n+1)^{p}-(n+1)=(n^{p}-n)+\sum_{r=1}^{p-1}{}^{p}C_{r}\,n^{r}$ for positive integers n and $p\geq 2$. **1**

(iv) Prove by Mathematical Induction that if p is prime then $n^{p}-n$ is divisible by p for all positive integers $n\geq 1$. **3**

(c)



In the diagram, AB is a diameter of a circle with centre O and PQ is a chord of the circle that is not perpendicular to AB. The perpendicular from A to PQ is produced to the point H outside the circle. M is the midpoint of PQ and N is the point on BH such that $MN \perp PQ$. Let $\overrightarrow{OA}=\underset{\sim}{a}$, $\overrightarrow{OP}=\underset{\sim}{p}$, $\overrightarrow{OQ}=\underset{\sim}{q}$ and $\overrightarrow{AH}=\underset{\sim}{h}$.

(i) By writing $\overrightarrow{OM}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$, show that the points O, M and N are collinear. **2**

(ii) If $\overrightarrow{BN}=\lambda\overrightarrow{BH}$ for some scalar λ, express $\overrightarrow{ON}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{h}$ and λ. **1**

(iii) Hence show that N is the midpoint of BH. **3**

STUDENT NAME/NUMBER.......................................

Marks

Question 15 (15 marks) **Use a separate writing booklet.**

(a) Use proof by contradiction to show that there exists no positive integer n such that n^2+1 is divisible by 3. **3**

(b) A missile is fired from point O with speed V ms^{-1} at an angle α above the horizontal where $\frac{\pi}{4}<\alpha<\frac{\pi}{2}$. The missile moves in a vertical plane under gravity where the acceleration due to gravity is g ms^{-2}. At time t seconds the position vector of the missile relative to O is $\underset{\sim}{r}(t)=(Vt\cos\alpha)\underset{\sim}{i}+\left(Vt\sin\alpha-\frac{1}{2}gt^2\right)\underset{\sim}{j}$.

(i) Show that the greatest height reached by the missile is $H=\dfrac{V^2\sin^2\alpha}{2g}$ metres and that when the missile is at its greatest height its angle of elevation from O is $\tan^{-1}\left(\frac{1}{2}\tan\alpha\right)$. **3**

(ii) When the missile reaches a height of h metres its angle of inclination to the horizontal falls to $\frac{\pi}{4}$. Show that $\dfrac{h}{H}=2-\text{cosec}^2\alpha$. **3**

(c) Let $I_n=\displaystyle\int_0^1 \frac{1}{\left(4-x^2\right)^n}\,dx$ for $n=1,2,3,\ldots$.

(i) Find in simplest exact form the value of I_1. **3**

(ii) Show that $I_{n+1}=\dfrac{1}{8n3^n}+\dfrac{2n-1}{8n}I_n$ for $n=1,2,3,\ldots$. **3**

STUDENT NAME/NUMBER......................................

Marks

Question 16 (15 marks) **Use a separate writing booklet.**

(a) Show that $\log_e(1+x) > \dfrac{2x}{2+x}$ for $x > 0$. **3**

(b)(i) Show that the limiting sum S of the series $1+\frac{1}{2}e^{i\theta}+\frac{1}{4}e^{2i\theta}+\frac{1}{8}e^{3i\theta}+\ldots$ exists and is given by $S=\dfrac{2}{2-e^{i\theta}}$. **3**

(ii) Hence show that $\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+\frac{1}{16}\sin 4\theta+\ldots=\dfrac{2\sin\theta}{5-4\cos\theta}$. **2**

(iii) Show that there exists no real value of θ such that S is purely imaginary. **1**

(c)(i) Show that $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$. **1**

(ii) Hence show that $\int_0^{\pi}\sin 2x\log_e\left(e^x+e^{\frac{\pi}{2}}\right)dx=\int_0^{\pi}\sin 2x\left\{\left(x-\frac{\pi}{2}\right)-\log_e\left(e^x+e^{\frac{\pi}{2}}\right)\right\}dx$. **2**

(iii) Hence evaluate $\int_0^{\pi}\sin 2x\log_e\left(e^x+e^{\frac{\pi}{2}}\right)dx$. **3**

END OF PAPER

Section 1 Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1	C	For $1<a<b$, $\log_a x$ is an increasing function so that $\log_a b>\log_a a=1$ Then $\log_b x=\dfrac{\log_a x}{\log_a b}\Rightarrow \log_a x=\log_a b\log_b x>\log_b x$. C is false. A, B, D are true.	MEX12-2
2	B	Circle has centre $(1,-1)$ and radius $\frac{1}{2}AB=\frac{1}{2}\sqrt{8^2+6^2}=5$ Equation is $\lvert z-(1-i)\rvert=5$, giving $\lvert z-1+i\rvert=5$	MEX12-4
3	A	$\cos\theta=\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert}=\dfrac{4+0+2}{\sqrt{6}\sqrt{8}}=\dfrac{3}{2\sqrt{3}}=\dfrac{\sqrt{3}}{2}\qquad\therefore\theta=\frac{\pi}{6}$	MEX12-3
4	D	$\displaystyle\int\frac{1}{x^2-\sqrt{3}x+1}dx=\int\frac{4}{\left(2x-\sqrt{3}\right)^2+1}dx=2\tan^{-1}\left(2x-\sqrt{3}\right)+c$	MEX12-5
5	C	$x=0$ for $t+\frac{\pi}{4}=(2m-1)\frac{\pi}{2}$, $t=(4m-3)\frac{\pi}{4}$, $m=1,2,3,\ldots$ $(4m-3)\frac{\pi}{4}\le 60\Rightarrow m\le\frac{1}{4}\left(60\div\frac{\pi}{4}+3\right)\approx 19\cdot 8\qquad\therefore$ At O 19 times	MEX12-6
6	B	If $P(x)$ is odd, terms of $P(x)$ all have the form ax^n, $n=1,3,5,\ldots$ so that terms of $P'(x)$ all have the form bx^m, $m=0,2,4,\ldots$ and $P'(x)$ is even. Hence the given statement and its contrapositive are both true. Consider the polynomial $P(x)=x+1$ with $P'(x)=1$. $P'(x)$ is even, but $P(x)$ is not odd. Hence the converse of the given statement is false	MEX12-2
7	A	$z_2=c\overline{Z}+c-1\quad\therefore\overline{z}_2=\overline{c}\,Z+\overline{c}-1$ $z_3=c\left(\overline{c}\,Z+\overline{c}-1\right)+c-1$ $=c\overline{c}\,Z+c\overline{c}-c+c-1$ But $c\overline{c}=1$ $\therefore z_3=Z$	MEX12-4
8	C	$\underset{\sim}{v}=\left(t^2-3t+c\right)\underset{\sim}{i}+\left\{\sin\left(\frac{\pi}{2}t\right)+d\right\}\underset{\sim}{j}$, c,d constant $\underset{\sim}{v}=2\underset{\sim}{i}$ for $t=0\Rightarrow c=2$ and $d=0$ $\therefore\underset{\sim}{v}=\left(t^2-3t+2\right)\underset{\sim}{i}+\left\{\sin\left(\frac{\pi}{2}t\right)\right\}\underset{\sim}{j}$ $\therefore\underset{\sim}{v}=0\Rightarrow t=0,2,4,\ldots$ and $(t-2)(t-1)=0$. $\therefore$ At rest only for $t=2$.	MEX12-3
9	A	$u=-x\qquad x=-k\Rightarrow u=k$ $du=-dx\qquad x=k\Rightarrow u=-k$ $\displaystyle\therefore\int_{-k}^{k}f(-x)dx=-\int_{k}^{-k}f(u)du=\int_{-k}^{k}f(u)du=\int_{-k}^{k}f(x)dx$ $\displaystyle\int_{-k}^{k}\{f(x)-f(-x)\}dx=\int_{-k}^{k}f(x)dx-\int_{-k}^{k}f(-x)dx=0$	MEX12-5
10	D	If body travels D metres in coming to rest, $v\dfrac{dv}{dx}=-\left(1+v^2\right)$ gives $\displaystyle\int_U^0\frac{-v}{1+v^2}dv=\int_0^D dx\qquad\therefore D=-\frac{1}{2}\left[\ln\left(1+v^2\right)\right]_U^0=\frac{1}{2}\ln\left(1+U^2\right)=\ln\sqrt{1+U^2}$	MEX12-6

Section II

Question 11

a. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Simplifies expression into required form	2
Substantial progress eg. correct procedure with one error	1

Answer

$$\frac{1+2i}{2+i}=\frac{(1+2i)(2-i)}{2^2+1^2}=\frac{4+3i}{5}=\tfrac{4}{5}+\tfrac{3}{5}i$$

b. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Makes given substitution to find the integral in terms of x	2
Substantial progress eg. correct procedure with one error or omission	1

Answer

$$t=\tan\tfrac{x}{2}$$
$$dt=\tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$
$$dx=\frac{2}{1+t^2}dt$$

$$1+\sin x=\frac{1+t^2+2t}{1+t^2}$$
$$=\frac{(1+t)^2}{1+t^2}$$

$$\int\frac{1}{1+\sin x}dx=\int\frac{1+t^2}{(1+t)^2}\,\frac{2}{1+t^2}dt$$
$$=\frac{-2}{1+t}+c$$
$$=\frac{-2}{1+\tan\frac{x}{2}}+c$$

c.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Writes z_1 in the required form	1

Answer

$$z_1=\sqrt{2}\left(-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)=\sqrt{2}\left(\cos\tfrac{3\pi}{4}+i\sin\tfrac{3\pi}{4}\right)=\sqrt{2}e^{i\frac{3\pi}{4}}$$

c.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Finds z_2 in required form	2
Substantial progress eg. finds modulus and argument of z_2	1

Answer

$|z_1|=\sqrt{2}$ and $|z_1z_2|=\sqrt{6}\Rightarrow|z_2|=\sqrt{3}$

$\arg(z_1)=\tfrac{3\pi}{4}$ and $\arg(z_1z_1)=\tfrac{7\pi}{12}\Rightarrow\arg(z_2)=\tfrac{7\pi}{12}-\tfrac{3\pi}{4}=-\tfrac{\pi}{6}$

$\therefore z_2=\sqrt{3}\left\{\cos\left(-\tfrac{\pi}{6}\right)+i\sin\left(-\tfrac{\pi}{6}\right)\right\}$

$z_2=\tfrac{3}{2}-\tfrac{\sqrt{3}}{2}i$

Q11 (cont)

d.i. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Obtains required result by integration	2
Substantial progress eg. correct procedure with one error	1

Answer

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right)=12-4x \qquad \left.\begin{array}{l} t=0 \\ x=5 \\ v=0 \end{array}\right] \Rightarrow \begin{array}{l} 0=60-50+c \\ c=-10 \end{array} \qquad \therefore v^2=-4x^2+24x-20$$

$$\tfrac{1}{2}v^2=12x-2x^2+c$$

d.ii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Writes and solves an appropriate quadratic inequality for x	2
Substantial progress eg. writes a quadratic inequality for x in factored form	1

Answer

$$v^2=-4\left(x^2-6x+5\right)=-4(x-1)(x-5)$$

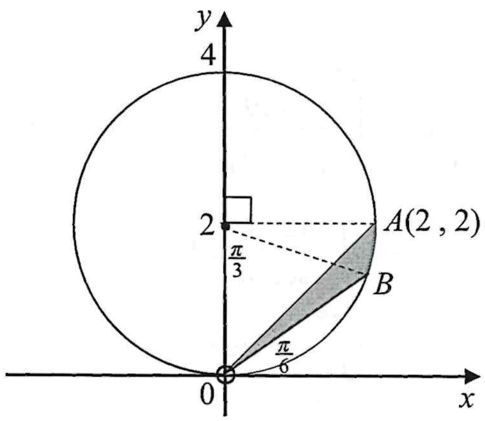
$$v^2\ge 0 \Rightarrow (x-1)(x-5)\le 0 \qquad \therefore 1\le x\le 5$$

e.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Shades required region in an Argand diagram showing sufficient detail	2
Substantial progress eg. region mostly correct but one error or omission	1

Answer



e.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Finds area in simplest exact form	2
Substantial progress eg. finds area of one of the two segments cut off by the chords OA, OB	1

Answer

Area is $\tfrac{1}{4}\pi\times 2^2-\tfrac{1}{2}\times 2\times 2-\left\{\tfrac{1}{2}\times 2^2\times\tfrac{\pi}{3}-\tfrac{1}{2}\times 2^2\times\sin\tfrac{\pi}{3}\right\}=\sqrt{3}-2+\tfrac{\pi}{3}$ sq. units

Question 12

a.i. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses the sum of a geometric series to establish result	1

Answer

$1+2^p+2^{2p}+\ldots+2^{(q-1)p}$ is q terms of a geometric series with $a=1,\ r=2^p$.

$$\therefore \left(2^p-1\right)\left(1+2^p+2^{2p}+\ldots+2^{(q-1)p}\right)=\left(2^p-1\right)\left\{\frac{\left(2^p\right)^q-1}{2^p-1}\right\}=2^{pq}-1$$

a.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves truth of contrapositive statement	2
Substantial progress eg. correct procedure but explanation lacks clarity	1

Answer

If n is not prime, then $\exists$ integers $p>1,\ q>1$ such that $n=pq$ and then from (i), 2^n-1 has a factor $\left(2^p-1\right)>1$. Hence $n\ is\ not\ prime \Rightarrow 2^n-1\ is\ not\ prime.\ \therefore\ 2^n-1\ is\ prime \Rightarrow n\ is\ prime$

b. Outcomes assessed: 12MEX12-5

Marking Guidelines

Criteria	Marks
Makes substitution then evaluates transformed integral in simplest exact form	4
Substantial progress eg. correct procedure with one error in execution	3
Moderate progress eg. substitutes then simplifies new integrand using appropriate trig. identities	2
Some progress eg. carries out given substitution	1

Answer

$x=\tan^2\theta,\ 0\le\theta<\frac{\pi}{2}$

$dx=2\tan\theta\sec^2\theta\, d\theta$

$x=0\Rightarrow\theta=0$

$x=1\Rightarrow\theta=\frac{\pi}{4}$

$1+x=1+\tan^2\theta$

$\quad=\sec^2\theta$

$$\begin{aligned}\int_0^1\frac{\sqrt{x}}{(1+x)^3}\,dx&=\int_0^{\frac{\pi}{4}}\frac{\tan\theta}{\left(\sec^2\theta\right)^3}\,2\tan\theta\sec^2\theta\, d\theta\\&=2\int_0^{\frac{\pi}{4}}\frac{\tan^2\theta}{\sec^4\theta}\,d\theta\\&=2\int_0^{\frac{\pi}{4}}\sin^2\theta\cos^2\theta\, d\theta\\&=\tfrac{1}{2}\int_0^{\frac{\pi}{4}}\sin^2 2\theta\, d\theta\\&=\tfrac{1}{4}\int_0^{\frac{\pi}{4}}\left(1-\cos4\theta\right)d\theta\\&=\tfrac{1}{4}\left[\theta-\tfrac{1}{4}\sin4\theta\right]_0^{\frac{\pi}{4}}\\&=\tfrac{\pi}{16}\end{aligned}$$

Q12(cont)

c.i. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Shows P does not lie in the plane and finds the coordinates of Q	3
Substantial progress eg. correct procedure with one error or omission	2
Some progress eg, shows P does not lie in the plane	1

Answer

At $P(2,1,3)$, $x-2y-z=2-2-3=-3\neq 3$. Hence P does not lie in the plane.

At $Q(2+\lambda,1-2\lambda,3-\lambda)$, $(2+\lambda)-2(1-2\lambda)-(3-\lambda)=3$. $\quad \therefore \lambda=1,\ Q(3,-1,2)$

$6\lambda-3=3 \qquad \overrightarrow{OQ}=3\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}$

c.ii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Finds the required distance	1

Answer

$PQ^2=1^2+2^2+1^2$ Hence required distance is $\sqrt{6}$ units.

d. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Writes and solves simultaneous equations for c and n to find the centre and period	4
Substantial progress eg. correct procedure with one error or omission	3
Moderate progress eg. writes simultaneous equations and eliminates n	2
Some progress eg. writes simultaneous equations	1

Answer

$x=0\Rightarrow v^2=20 \quad \therefore \quad n^2\{9-c^2\}=20 \quad (1)$

$x=2\Rightarrow v^2=36 \quad \therefore \quad n^2\{9-(2-c)^2\}=36 \quad (2)$

$$\frac{(2)}{(1)} \Rightarrow \frac{9-c^2+4c-4}{9-c^2}=\frac{9}{5}$$

$$\frac{4(c-1)}{9-c^2}=\frac{4}{5}$$

$5(c-1)=9-c^2$

$c^2+5c-14=0$

$(c+7)(c-2)=0$

$c>0 \Rightarrow c\neq -7 \qquad \therefore c=2$

Then $n>0,\ 5n^2=20\Rightarrow n=2$

Centre is 2 m to right of O. Period is π seconds.

Question 13

a. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Finds required projection	2
Substantial progress eg. correct procedure with one error	1

Answer

$$proj_{\underset{\sim}{b}}\underset{\sim}{a}=\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\underset{\sim}{b}=\frac{8+12}{4^2+3^2}\left(4\underset{\sim}{i}+3\underset{\sim}{j}\right)=\tfrac{16}{5}\underset{\sim}{i}+\tfrac{12}{5}\underset{\sim}{j}$$

b. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Finds the values of all four roots and deduces the values of a and d	3
Substantial progress eg. finds the value of all four roots and the value of one of a, d.	2
Some progress eg. finds the coordinates of all four vertices or the values of all 4 roots	1

Answer

a, b, c, d *real* $\Rightarrow 1-2i$ is a second root of the equation. If the vertices represented by $1+2i,\ 1-2i$ are endpoints of a diagonal of the square, then since the diagonals are equal and bisect at right angles, the other vertices would be the real numbers -1 and 3 which contradicts the information that each quadrant contains exactly one vertex. Hence the join of $1+2i,\ 1-2i$ is a side of the square. The opposite side has vertices in quadrants 2 and 3, and has endpoints $-3+2i,\ -3-2i$ since the sides of *PQRS* are equal and adjacent sides are perpendicular. $\therefore a=-(\textit{sum of roots})=4$ and $d=\left(1^2+2^2\right)\left(3^2+2^2\right)=65$.

c.i. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Uses integration to establish required result	3
Substantial progress eg. correct procedure with one error	2
Some progress eg. finds the anti-derivative	1

Answer

$$v\frac{dv}{dx}=\frac{\left(g^2-v^2\right)}{g}$$

$$\int\frac{-2v}{g^2-v^2}\,dv=-\frac{2}{g}\int dx$$

$$\ln\left(A\left(g^2-v^2\right)\right)=-\frac{2x}{g},\quad A\ const.$$

$$x=0, v=0\ \Rightarrow\ A=\frac{1}{g^2}$$

$$x=-\tfrac{1}{2}g\ln\left(\frac{g^2-v^2}{g^2}\right)$$

$$\therefore\ x=\tfrac{1}{2}g\ln\left(\frac{g^2}{g^2-v^2}\right)$$

Q13(cont)

c.ii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Finds speed in simplest exact form	1

Answer $x=\frac{1}{2}g\ln 5 \Rightarrow \dfrac{g^2}{g^2-v^2}=5 \quad \therefore v^2=\frac{4}{5}g^2$ Returns to start with speed $\frac{2}{\sqrt{5}}g$ ms^{-1}.

d.i. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Finds the coordinates of P and hence the position vector of P	2
Substantial progress eg. correct procedure except consistency of the three equations not verified	1

Answer

At intersection point P,

$-9+2\lambda = 3+3\mu \quad (1)$ $\qquad (1)+3\times(2) \Rightarrow -9+5\lambda=6 \qquad \lambda=3,\ \mu=-2$

$\lambda = 1-\mu \quad (2)$ $\qquad$ *Testing* (3): $LHS=10-3=7,\ RHS=17-10=7$

$10-\lambda=17+5\mu \quad (3)$ $\qquad$ Hence L_1, L_2 intersect at $(-9+6,3,10-3)$ $\quad \therefore$ at $P(-3,3,7)$

$\overrightarrow{OP}=-3\underset{\sim}{i}+3\underset{\sim}{j}+7\underset{\sim}{k}$

d.ii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Uses direction vectors to show lines are perpendicular	2
Substantial progress eg. correct procedure with one error	1

Answer

L_1, L_2 have direction vectors $\underset{\sim}{u}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$, $\underset{\sim}{v}=\begin{pmatrix}3\\-1\\5\end{pmatrix}$ respectively. $\quad \underset{\sim}{u}\cdot\underset{\sim}{v}=6-1-5=0 \quad \therefore L_1 \perp L_2$

d.iii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Finds the position vector of B	2
Substantial progress eg. finds the vector from P to A.	1

Answer

B lies on L_1 such that $\overrightarrow{PB}=-\overrightarrow{PA}$. Then $\overrightarrow{OB}=\overrightarrow{OP}+\overrightarrow{PB}=\overrightarrow{OP}-\overrightarrow{PA}=\left(-3\underset{\sim}{i}+3\underset{\sim}{j}+7\underset{\sim}{k}\right)-\left(8\underset{\sim}{i}+4\underset{\sim}{j}-4\underset{\sim}{k}\right)$

$\therefore \overrightarrow{OB}=-11\underset{\sim}{i}-\underset{\sim}{j}+11\underset{\sim}{k}$

Question 14

a. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves the required result	3
Substantial progress eg. adopts a suitable method leading to the result but proof incomplete	2
Some progress eg. applies the condition for the given sequence to be arithmetic	1

Answer

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b} \qquad \therefore \frac{b+c+a}{b}-\frac{a+b+c}{a}=\frac{c+a+b}{c}-\frac{b+c+a}{b}$$

$$1+\frac{c+a}{b}-\left(1+\frac{b+c}{a}\right)=1+\frac{a+b}{c}-\left(1+\frac{c+a}{b}\right)$$

$$\frac{c+a}{b}-\frac{b+c}{a}=\frac{a+b}{c}-\frac{c+a}{b}$$

Hence $\dfrac{b+c}{a},\ \dfrac{c+a}{b},\ \dfrac{a+b}{c}$ are consecutive terms in an arithmetic sequence.

b.i. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Establishes required result	1

Answer

For $0<r<p$, $\quad {}^pC_r=\dfrac{p!}{r!(p-r)!}=\dfrac{p(p-1)!}{r(r-1)!(p-r)!}=\dfrac{p}{r}\,{}^{p-1}C_{r-1}\ . \quad \therefore\ r\,{}^pC_r=p\,{}^{p-1}C_{r-1}$

b.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Deduces the required result	1

Answer

For $0<r<p$ and p prime, the only common factor of r and p is 1. Hence $p\,|\left(r\,{}^pC_r\right)\ \Rightarrow\ p\,|\,{}^pC_r$

b.iii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Establishes required result	1

Answer

For integers $n\geq 1,\ p\geq 2$

$$(n+1)^p=\sum_{r=0}^{p}{}^pC_r n^r=1+n^p+\sum_{r=1}^{p-1}{}^pC_r n^r$$

$$\therefore (n+1)^p-(n+1)=\left(n^p-n\right)+\sum_{r=1}^{p-1}{}^pC_r n^r$$

Q14(cont)

b.iv. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses Mathematical Induction to prove true members of an appropriate sequence of propositions	3
Substantial progress eg. correct procedure but explanation lacks clarity	2
Soms progress eg. defines an appropriate sequence of propositions and shows 1st is true	1

Answer

For p prime, let P_n, $n = 1, 2, 3, \ldots$ be the sequence of propositions $n^p - n = p\,I$ *for some integer* I.

Consider P_1 : $1^p - 1 = 0$ Hence P_1 is true.

If P_k is true : $k^p - k = p\,I$ for some integer I *

$$\text{Consider } P_{k+1}:\ (k+1)^p - (k+1) = (k^p - k) + \sum_{r=1}^{p-1} {}^pC_r\, k^r \quad (\text{since } k \ge 1 \text{ and } p \text{ prime} \Rightarrow p \ge 2\ )$$

$$= pI + \sum_{r=1}^{p-1} {}^pC_r\, k^r \qquad I \text{ an integer, if } P_k \text{ is true, using } *$$

$$= p\,I + \sum_{r=1}^{p-1} p\,I_r k^r \qquad \text{where } I_r \text{ is an integer, } r = 1,2,3,\ldots \text{ using (ii)}$$

$$= p\left(I + \sum_{r=1}^{p-1} I_r k^r\right) \quad \text{where } \left(I + \sum_{r=1}^{p-1} I_r k^r\right) \text{ is integral.}$$

Hence if P_k is true then P_{k+1} is true. But P_1 is true. Therefore P_n is true for all positive integers n by Mathematical Induction. Hence if p is prime then $n^p - n$ is divisibile by p for all integers $n \ge 1$.

c.i. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Expresses $\overrightarrow{OM}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$ then deduces required collinearity	2
Substantial progress eg. expresses $\overrightarrow{OM}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$	1

Answer

$$\overrightarrow{OM} = \tfrac{1}{2}\left(\underset{\sim}{p} + \underset{\sim}{q}\right) \qquad \text{Then } \overrightarrow{OM} \cdot \overrightarrow{PQ} = \tfrac{1}{2}\left(\underset{\sim}{p} + \underset{\sim}{q}\right) \cdot \left(\underset{\sim}{q} - \underset{\sim}{p}\right) = \tfrac{1}{2}\left(\underset{\sim}{q} \cdot \underset{\sim}{q} - \underset{\sim}{p} \cdot \underset{\sim}{p}\right) = 0 \text{ since } OP = OQ \text{ (radii)}$$

Then $\angle OMN = \angle OMQ + \angle QMN = 90° + 90° = 180°$. Hence O, M, N are collinear.

c.ii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Expresses $\overrightarrow{ON}$ in required form	1

Answer

$$\overrightarrow{ON} = \overrightarrow{OB} + \overrightarrow{BN} = -\underset{\sim}{a} + \lambda\, \overrightarrow{BH} = -\underset{\sim}{a} + \lambda\left(\overrightarrow{BA} + \overrightarrow{AH}\right) = -\underset{\sim}{a} + \lambda\left(2\underset{\sim}{a} + \underset{\sim}{h}\right)$$

$$\therefore\ \overrightarrow{ON} = (2\lambda - 1)\underset{\sim}{a} + \lambda \underset{\sim}{h}$$

Q14(cont)
c.iii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Uses *ON*, *AH* both perpendicular to *PQ* to show $\lambda=\frac{1}{2}$	3
Substantial progress eg. correct procedure but fails to explain why $2\lambda-1=0$	2
Some progress eg. uses the perpendicularity to write one appropriate dot product equal to 0	1

Answer

$AH \perp PQ \Rightarrow \underset{\sim}{h}\cdot\left(\underset{\sim}{q}-\underset{\sim}{p}\right)=0$ and O, M, N collinear $\Rightarrow \overrightarrow{ON}\cdot\overrightarrow{PQ}=0$ since $MN \perp PQ$.

$$\therefore \left\{\left(2\lambda-1\right)\underset{\sim}{a}+\lambda\underset{\sim}{h}\right\}\cdot\left(\underset{\sim}{q}-\underset{\sim}{p}\right)=0$$

$$\therefore \left(2\lambda-1\right)\underset{\sim}{a}\cdot\left(\underset{\sim}{q}-\underset{\sim}{p}\right)=0$$

But *AB*, *PQ* are not perpendicular $\Rightarrow$ $\underset{\sim}{a}\cdot\left(\underset{\sim}{q}-\underset{\sim}{p}\right)\neq 0$. Hence $2\lambda-1=0$. $\therefore \lambda=\frac{1}{2}$ and $BN=\frac{1}{2}BH$.

Hence *N* is the midpoint of *BH*.

Question 15

a. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses proof by contradiction to prove the required result	3
Substantial progress eg. correct procedure but lack of clarity or insufficient detail	2
Some progress eg. shows that the square of a positive integer has one of the forms $3k$, $3k+1$	1

Answer

A positive integer has one of the forms $3m,\ 3m+1,\ 3m+2$ for some integer $m\geq 0$.

But $\left(3m\right)^2=3\left(3m^2\right)$, $\left(3m+1\right)^2=3\left(3m^2+2m\right)+1$, $\left(3m+2\right)^2=3\left(3m^2+4m+1\right)+1$

so that the square of a positive integer has one of the forms $3k$, $3k+1$ for some integer $k\geq 0$.

If $n^2+1=3h$ for positive integers *h* and *n*, then $n^2=3h-1=3\left(h-1\right)+2$ which is a contradiction.

Hence for all positive integers *n*, there is no positive integer *h* such that $n^2+1=3h$.

Hence there is no positive integer *n* such that n^2+1 is divisible by 3.

b.i. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Establishes both required results	3
Substantial progress eg. finds *t* at greatest height then *H*	2
Some progress eg. finds *t* at greatest height	1

Answer

$$\underset{\sim}{r}\left(t\right)=\left(Vt\cos\alpha\right)\underset{\sim}{i}+\left(Vt\sin\alpha-\tfrac{1}{2}gt^2\right)\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}}\left(t\right)=\left(V\cos\alpha\right)\underset{\sim}{i}+\left(V\sin\alpha-gt\right)\underset{\sim}{j}$$

$$\dot{y}=0\Rightarrow t=\frac{V\sin\alpha}{g}$$

$$\therefore H=V\left\{\frac{V\sin\alpha}{g}\right\}\sin\alpha-\tfrac{1}{2}g\left\{\frac{V\sin\alpha}{g}\right\}^2=\frac{V^2\sin^2\alpha}{2g}$$

Let θ be angle of elevation from *O*. $\quad \dot{y}=0\Rightarrow x=\dfrac{V^2\sin\alpha\cos\alpha}{g}$ $\quad \therefore \tan\theta=\dfrac{V^2\sin^2\alpha}{2g}\times\dfrac{g}{V^2\sin\alpha\cos\alpha}$

$$\therefore \tan\theta=\tfrac{1}{2}\tan\alpha \qquad \therefore \theta=\tan^{-1}\left(\tfrac{1}{2}\tan\alpha\right)$$

Q15(cont)

b.ii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Writes and solves simultaneous equations to establish required result	3
Substantial progress eg. writes and solves an equation for t then writes an expression for h	2
Some progress eg. writes an equation for t	1

Answer

Let ϕ be angle of inclination to horizontal at time t. Then $\tan\phi = \dfrac{\dot{y}}{\dot{x}} = \dfrac{V\sin\alpha - gt}{V\cos\alpha}$.

$$\left.\begin{array}{l} y = h \\ \phi = \frac{\pi}{4} \end{array}\right] \Rightarrow \begin{array}{ll} Vt\sin\alpha - \frac{1}{2}gt^2 = h & (1) \\ V\sin\alpha - gt = V\cos\alpha & (2) \end{array}$$

$$(2) \Rightarrow \quad t = \tfrac{1}{g}V(\sin\alpha - \cos\alpha)$$

$$(1) - \tfrac{1}{2}t \times (2) \Rightarrow \tfrac{1}{2}Vt\sin\alpha = h - \tfrac{1}{2}Vt\cos\alpha$$

$$h = \tfrac{1}{2}Vt(\sin\alpha + \cos\alpha)$$

$$= \frac{V^2}{2g}(\sin^2\alpha - \cos^2\alpha)$$

$$\therefore \frac{h}{H} = \frac{\sin^2\alpha - \cos^2\alpha}{\sin^2\alpha} = \frac{2\sin^2\alpha - 1}{\sin^2\alpha} = 2 - \operatorname{cosec}^2\alpha$$

c.i. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Evaluates the required definite integral in simplest exact form	3
Substantial progress eg. correct procedure with one error or omission	2
Some progress eg. expresses the integrand as sum of partial fractions	1

Answer

$$\int_0^1 \frac{1}{4-x^2}\,dx = \tfrac{1}{4}\int_0^1 \frac{1}{2-x} + \frac{1}{2+x}\,dx = \tfrac{1}{4}\left[\ln\left(\frac{2+x}{2-x}\right)\right]_0^1 = \tfrac{1}{4}\ln 3 \qquad \therefore I_1 = \tfrac{1}{4}\ln 3$$

c.ii. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Uses integration by parts to establish required result	3
Substantial progress eg. correct procedure with one error	2
Some progress eg. applies integration by parts to transform integral	1

Answer

$$I_n = \int_0^1 \frac{1}{(4-x^2)^n}\,dx$$

$$= \left[\frac{x}{(4-x^2)^n}\right]_0^1 - \int_0^1 x\frac{2nx}{(4-x^2)^{n+1}}\,dx$$

$$= \frac{1}{3^n} + 2n\int_0^1 \frac{4-x^2-4}{(4-x^2)^{n+1}}\,dx$$

$$\therefore \quad I_n = \frac{1}{3^n} + 2n\{I_n - 4I_{n+1}\}$$

$$(1-2n)I_n = \frac{1}{3^n} - 8nI_{n+1}$$

$$8nI_{n+1} = \frac{1}{3^n} + (2n-1)I_n$$

$$I_{n+1} = \frac{1}{8n3^n} + \frac{(2n-1)}{8n}I_n$$

Question 16

a. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves required result	3
Substantial progress eg. correct procedure but some lack of clarity in explanation	2
Some progress eg. considers the function $f(x)$ and finds its derivative	1

Answer

Let $f(x)=\ln(1+x)-\dfrac{2x}{2+x}$. Then $f(x)=\ln(1+x)-2+\dfrac{4}{2+x}$

$$f'(x)=\frac{1}{1+x}-\frac{4}{(2+x)^2}$$

$$=\frac{(2+x)^2-4(1+x)}{(1+x)(2+x)^2}$$

$$=\frac{x^2}{(1+x)(2+x)^2}$$

Hence $f'(0)=0$ and $f'(x)>0$ for $x>0$, so that $f(x)$ is stationary at $x=0$ and increasing for $x>0$.

But $f(0)=0$. Hence $f(x)>0$ for $x>0$. $\therefore \ln(1+x)>\dfrac{2x}{2+x}$ for $x>0$.

b.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Proves the required result after considering a geometric series of non-real terms	3
Substantial progress eg. adapts results for geometric series of real terms, attempting justification	2
Some progress eg. quotes result for geometric sequence of real terms, ignoring fact r is non-real	1

Answer

Let $S_n=1+\frac{1}{2}e^{i\theta}+\frac{1}{4}e^{2i\theta}+\ldots+\frac{1}{2^{n-1}}e^{(n-1)i\theta}$, $n=1,2,3,\ldots$ Then using results for a geometric sequence with

first term 1 and common ratio $\frac{1}{2}e^{i\theta}\neq 1$, $S_n=\dfrac{1-\left(\frac{1}{2}e^{i\theta}\right)^n}{1-\frac{1}{2}e^{i\theta}}=\dfrac{2}{2-e^{i\theta}}-\left(\frac{1}{2}\right)^{n-1}\left(\dfrac{e^{ni\theta}}{2-e^{i\theta}}\right)$.

$\left|\dfrac{e^{ni\theta}}{2-e^{i\theta}}\right|=\dfrac{1}{\left|2-e^{i\theta}\right|}$, where $1\le\left|2-e^{i\theta}\right|\le 3$ (considering the distance in the Argand diagram from the point $(2,0)$ to the circle $z=e^{i\theta}$ centre O, radius 1)

and hence $\frac{1}{3}\le\dfrac{1}{\left|2-e^{i\theta}\right|}\le 1$.

$\therefore\left|\left(\frac{1}{2}\right)^{n-1}\left(\dfrac{e^{ni\theta}}{2-e^{i\theta}}\right)\right|=\left|\dfrac{1}{2-e^{i\theta}}\right|\left(\frac{1}{2}\right)^{n-1}\le\left(\frac{1}{2}\right)^{n-1}\to 0$ as $n\to\infty$. Considering S_n as the sum of vectors in an

Argand diagram, $\lim\limits_{n\to 0}S_n$ exists and is given by $S=\dfrac{2}{2-e^{i\theta}}$.

Q16(cont)

b.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Establishes required result	2
Substantial progress eg. rationalises the denominator of the expression for S and simplifies	1

Answer

$$S=\frac{2}{\left(2-\cos\theta\right)-i\sin\theta}=\frac{2\left(2-\cos\theta\right)+2i\sin\theta}{\left(2-\cos\theta\right)^2+\sin^2\theta}$$

$$\therefore \operatorname{Im}\left(S\right)=\frac{2\sin\theta}{4-4\cos\theta+\cos^2\theta+\sin^2\theta}=\frac{2\sin\theta}{5-4\cos\theta}$$

$$\therefore \tfrac{1}{2}\sin\theta+\tfrac{1}{4}\sin 2\theta+\tfrac{1}{8}\sin 3\theta+\tfrac{1}{16}\sin 4\theta+\ldots=\frac{2\sin\theta}{5-4\cos\theta}\qquad \textit{(equating imaginary parts)}$$

b.iii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Deduces result by considering the real part of S	1

Answer

$\operatorname{Re}\left(S\right)=\dfrac{2\left(2-\cos\theta\right)}{5-4\cos\theta}\neq 0$ for any real value of θ since $\cos\theta\neq 2$.

Hence there is no real value of θ for which S is purely imaginary.

c.i. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Applies an appropriate substitution to produce required result	1

Answer

$u=a-x$

$du=-dx$

$x=0\Rightarrow u=a$

$x=a\Rightarrow u=0$

$$\begin{aligned}\int_0^a f\left(x\right)dx&=\int_a^0 -f\left(a-u\right)du\\&=\int_0^a f\left(a-u\right)du\\&=\int_0^a f\left(a-x\right)dx\end{aligned}$$

Q16(cont)
c.ii. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Applies (i) then uses appropriate trig. identity and log laws to establish required result	2
Substantiol progress eg. applies (i) and appropriate trig. identity to simplify integrand	1

Answer

$$\int_0^{\pi} \sin 2x \log_e\left(e^x + e^{\frac{\pi}{2}}\right) dx = \int_0^{\pi} \sin 2(\pi - x) \log_e\left(e^{\pi - x} + e^{\frac{\pi}{2}}\right) dx$$

$$= \int_0^{\pi} -\sin 2x \log_e\left\{e^{\frac{\pi}{2}-x}\left(e^{\frac{\pi}{2}} + e^x\right)\right\} dx$$

$$= \int_0^{\pi} -\sin 2x \left\{\left(\tfrac{\pi}{2} - x\right) + \log_e\left(e^{\frac{\pi}{2}} + e^x\right)\right\} dx$$

$$= \int_0^{\pi} \sin 2x \left\{\left(x - \tfrac{\pi}{2}\right) - \log_e\left(e^x + e^{\frac{\pi}{2}}\right)\right\} dx$$

c.iii. Outcomes assessed:

Marking Guidelines

Criteria	Marks
Uses (ii) and applies integration by parts to evaluate the required definite integral	3
Substantial progress eg. correct procedure with one error in evaluation	2
Some progress eg. applies (ii) to find simpler integral for twice the required definite integral	1

Answer

$$\int_0^{\pi} \sin 2x \log_e\left(e^x + e^{\frac{\pi}{2}}\right) dx = \int_0^{\pi} \sin 2x \left\{\left(x - \tfrac{\pi}{2}\right) - \log_e\left(e^x + e^{\frac{\pi}{2}}\right)\right\} dx$$

$$= \int_0^{\pi} \left(x - \tfrac{\pi}{2}\right) \sin 2x \, dx - \int_0^{\pi} \sin 2x \log_e\left(e^x + e^{\frac{\pi}{2}}\right) dx$$

$$2\int_0^{\pi} \sin 2x \log_e\left(e^x + e^{\frac{\pi}{2}}\right) dx = \int_0^{\pi} \left(x - \tfrac{\pi}{2}\right) \sin 2x \, dx$$

$$= -\tfrac{1}{2}\left[\left(x - \tfrac{\pi}{2}\right)\cos 2x\right]_0^{\pi} + \tfrac{1}{2}\int_0^{\pi} \cos 2x \, dx$$

$$= -\tfrac{1}{2}\left\{\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right)\right\} + \tfrac{1}{4}\left[\sin 2x\right]_0^{\pi}$$

$$= -\tfrac{\pi}{2}$$

$$\int_0^{\pi} \sin 2x \log_e\left(e^x + e^{\frac{\pi}{2}}\right) dx = -\tfrac{\pi}{4}$$

Independent HSC Trial Examinations 2023 Mathematics Extension 2 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1	1	The nature of proof	MEX12-2	E2-E3
2	1	Introduction to complex numbers	MEX12-4	E2-E3
3	1	Further work with vectors	MEX12-3	E2-E3
4	1	Further integration	MEX12-5	E2-E3
5	1	Application of calculus to mechanics	MEX12-6	E2-E3
6	1	The nature of proof	MEX12-2	E3-E4
7	1	Introduction to complex numbers	MEX12-4	E3-E4
8	1	Further work with vectors	MEX12-3	E3-E4
9	1	Further integration	MEX12-5	E3-E4
10	1	Application of calculus to mechanics	MEX12-6	E3-E4
11 a	2	Introduction to complex numbers	MEX12-4	E2-E3
b	2	Further integration	MEX12-5	E2-E3
c i	1	Introduction to complex numbers	MEX12-4	E2-E3
ii	2	Introduction to complex numbers	MEX12-4	E2-E3
d i	2	Application of calculus to mechanics	MEX12-6	E2-E3
ii	2	Application of calculus to mechanics	MEX12-6	E2-E3
e i	2	Introduction to complex numbers	MEX12-4	E2-E3
e ii	2	Introduction to complex numbers	MEX12-4	E2-E3
12 a i	1	The nature of proof	MEX12-2	E2-E3
ii	2	The nature of proof	MEX12-2	E2-E3
b	4	Further integration	MEX12-5	E3-E4
c i	3	Further work with vectors	MEX12-3	E2-E3
ii	1	Further work with vectors	MEX12-3	E2-E3
d	4	Application of calculus to mechanics	MEX12-6	E3-E4
13 a	2	Further work with vectors	MEX12-3	E2-E3
b	3	Introduction to complex numbers	MEX12-4	E2-E3
c i	3	Application of calculus to mechanics	MEX12-6	E3-E4
ii	1	Application of calculus to mechanics	MEX12-6	E2-E3
d i	2	Further work with vectors	MEX12-3	E2-E3
ii	2	Further work with vectors	MEX12-3	E2-E3
iii	2	Further work with vectors	MEX12-3	E3-E4
14 a	3	The nature of proof	MEX12-2	E2-E3
b i	1	The nature of proof	MEX12-2	E2-E3
ii	1	The nature of proof	MEX12-2	E2-E3
iii	1	The nature of proof	MEX12-2	E2-E3
iv	3	Further proof by Mathematical Induction	MEX12-2	E3-E4
c i	2	Further work with vectors	MEX12-3	E3-E4
ii	1	Further work with vectors	MEX12-3	E2-E3
iii	3	Further work with vectors	MEX12-3	E3-E4
15 a	3	The nature of proof	MEX12-2	E2-E3
b i	3	Application of calculus to mechanics	MEX12-6	E3-E4
ii	3	Application of calculus to mechanics	MEX12-6	E3-E4
c i	3	Further integration	MEX12-5	E3-E4
ii	3	Further integration	MEX12-5	E3-E4

16 a	3	The nature of proof	MEX12-2	E2-E3
b i	3	Introduction to complex numbers	MEX12-4	E3-E4
ii	2	Using complex numbers	MEX12-4	E3-E4
iii	1	Using complex numbers	MEX12-4	E2-E3
c i	1	Further integration	MEX12-5	E2-E3
ii	2	Further integration	MEX12-5	E3-E4
iii	3	Further integration	MEX12-5	E3-E4