Neap

Trial Examination 2021

HSC Year 12 Mathematics Extension 1

General Instructions	• Reading time – 10 minutes • Working time – 2 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • For questions in Section II, show relevant rmathematical easoning and/or calculations
Total marks: 70	**Section I – 10 marks (pages 2–6)** • Attempt Questions 1– 10 • Allow about 15 minutes for this section **Section II – 60 marks (pages 7–14)** • Attempt Questions 11– 14 • Allow about 1 hour and 45 minutes for this section

SECTION I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is NOT a factor of the polynomial $P(x)=x^3-x^2-8x+12$?

A. $(x+3)$

B. $(x+3)^2$

C. $(x-2)$

D. $(x-2)^2$

2 What is the derivative of $\sin x . \cos^{-1} x$?

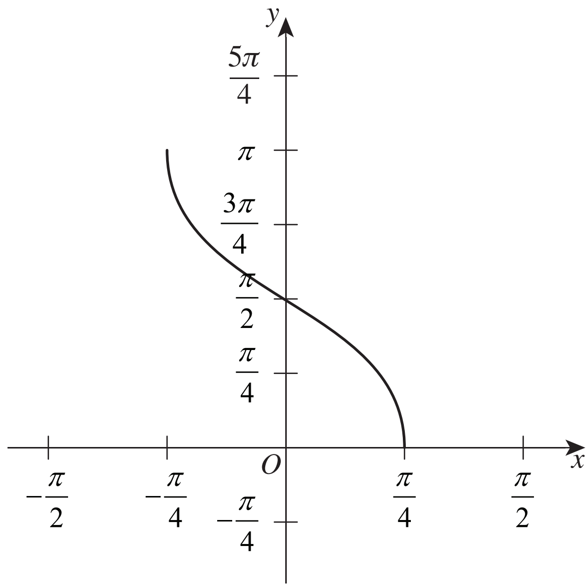
A. $+\dfrac{\cos x}{\sqrt{1-x^2}}$

B. $-\dfrac{\cos x}{\sqrt{1-x^2}}$

C. $\cos x . \cos^{-1} x+\dfrac{\sin x}{\sqrt{1-x^2}}$

D. $\cos x . \cos^{-1} x-\dfrac{\sin x}{\sqrt{1-x^2}}$

3 The graph of an inverse trigonometric function is shown.



What is the equation of this function?

A. $y = \frac{4}{\pi}\cos^{-1} x$

B. $y = \frac{\pi}{4}\cos^{-1} x$

C. $y = \cos^{-1}\left(\frac{4x}{\pi}\right)$

D. $y = \cos^{-1}\left(\frac{\pi x}{4}\right)$

4 A mathematician wants to put a total of eight different books on her shelf. She can choose from eight different algebra books and eight different calculus books.

Given that the book ‘Calculus I’ must be on the shelf, how many ways can she select the remaining books to go on the shelf if she wants at least four algebra books and at least two other calculus books?

A. 3626

B. 3822

C. 8036

D. 8820

5 $ABCD$ is a rectangle. $\overrightarrow{AB} = \underset{\sim}{b}$ and $\overrightarrow{AC} = \underset{\sim}{c}$.

Which of the following statements is FALSE?

A. $\underset{\sim}{c} \cdot \left(2\underset{\sim}{b} - \underset{\sim}{c}\right) = 0$

B. $\underset{\sim}{b} \cdot \left(\underset{\sim}{c} - \underset{\sim}{b}\right) = 0$

C. $\left|\underset{\sim}{c}\right| = \left|2\underset{\sim}{b} - \underset{\sim}{c}\right|$

D. $\overrightarrow{AD} = \underset{\sim}{c} - \underset{\sim}{b}$

6 What is the primitive of $\dfrac{\ln 2}{\sqrt{\pi - x^2}}$?

A. $\ln 2.x.\sin^{-1}\left(\dfrac{x}{\pi}\right) + C$

B. $\ln 2.\sin^{-1}\left(\dfrac{x}{\sqrt{\pi}}\right) + C$

C. $\dfrac{\ln 2}{\sqrt{\pi}}.\sin^{-1}\left(\dfrac{x}{\sqrt{\pi}}\right) + C$

D. $\ln 2.\sin^{-1}\left(\dfrac{x}{\pi}\right) + C$

7 A bag contains 2 red, 5 blue, 6 white, 11 green and 14 yellow marbles.
What is the minimum number of marbles that need to be chosen randomly from the bag to ensure that 6 marbles of the same colour have been chosen?

A. 16

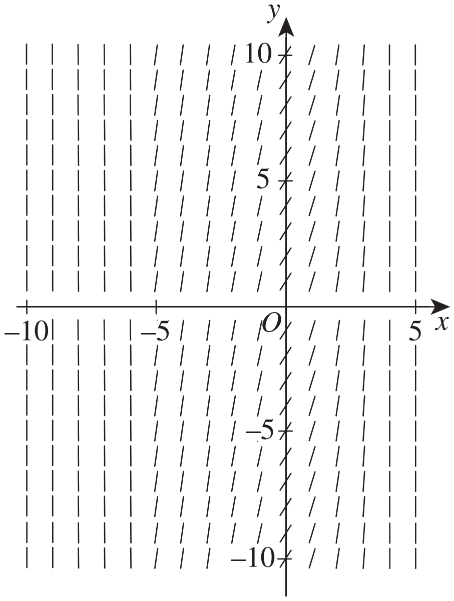
B. 17

C. 23

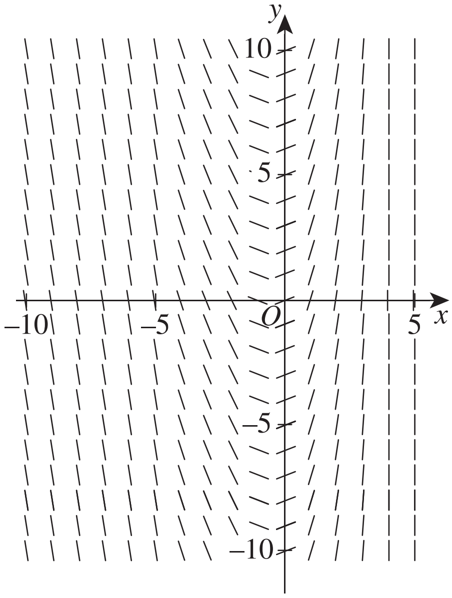
D. 34

8 Which of the following diagrams best represents the direction field for the differential equation $\dfrac{dy}{dx} = x + e^x$?

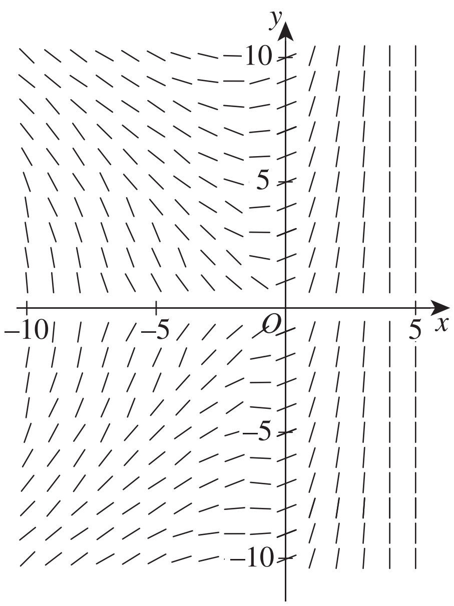
A.



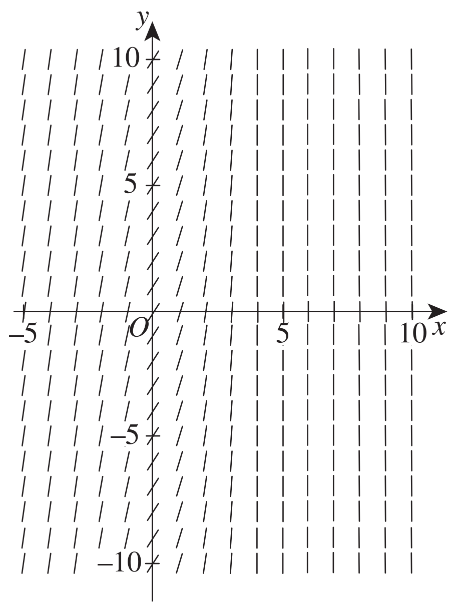
B.



C.



D.



9 Given that $y = f(x)$, what is the solution to the differential equation $\dfrac{dy}{dx} = 5 - y$, where $f(3) = 4$?

A. $y = 5 - e^{x-3}$

B. $y = 5 - e^{3-x}$

C. $x = 5y - \dfrac{y^2}{2} - \dfrac{25}{2}$

D. $x = 5y - \dfrac{y^2}{2} - 9$

10 The function $f(x) = x^2 - 4x + 7$ has an inverse function $f^{-1}(x)$ in the domain $x \geq 2$.
What is the value of $f^{-1}(f(m))$, where m is a real number NOT in the domain $x \geq 2$?

A. m

B. $2 - m$

C. $2 + m$

D. $4 - m$

SECTION II

60 marks

Attempt Questions 11–14

Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Write the expression $\cos 7x \sin 5x$ as a sum/difference of trigonometric ratios. **1**

(ii) Hence, solve the equation $2\cos 7x \sin 5x = \sin 12x - \dfrac{\sqrt{3}}{2}$ for $0 \le x \le 2\pi$. **3**

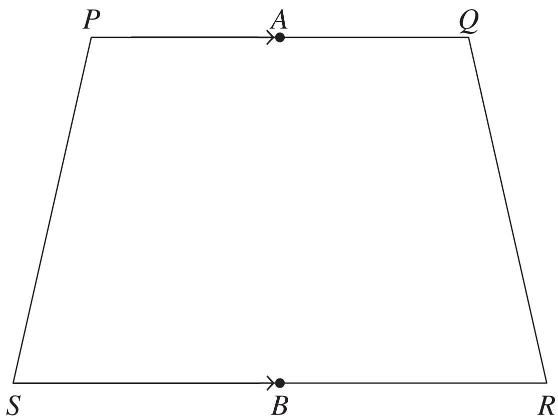
(b) Prove by mathematical induction that $2^{2n} + 6n - 1$ is divisible by 3 for all integers $n \ge 1$. **3**

(c) Consider the polynomial $P(x) = x^3 - 2px + q$, where p and q are constants and $p \ne 0$.
It is given that α, β and $\alpha + \beta$ are the roots of the equation $P(x) = 0$.

(i) Prove that $\alpha = -\beta$. **1**

(ii) Hence, or otherwise, find all the roots of $P(x)$ in terms of p. **3**

(d) *PQRS* is a trapezium with *A* and *B* being the midpoints of *PQ* and *RS* respectively.



Let $\overrightarrow{PA} = \underset{\sim}{a}$ and $\overrightarrow{SB} = \underset{\sim}{b}$.

(i) Express $\overrightarrow{QR}$ in terms of $\underset{\sim}{a}, \underset{\sim}{b}$ and $\overrightarrow{AB}$. **2**

(ii) Hence, or otherwise, show that $\overrightarrow{AB} = \dfrac{1}{2}\left(\overrightarrow{PS} + \overrightarrow{QR}\right)$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

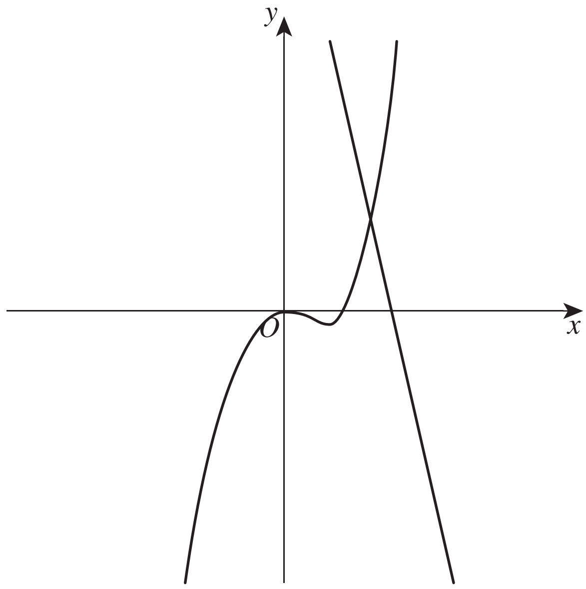
(a) (i) Write the expression $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$. **2**

(ii) Hence, by drawing the graphs on the number plane and showing clearly important points, determine the number of solutions that the equation $\sqrt{3}\sin x + \cos x = \ln x$ has in the domain $(0, \infty)$. **3**

(b) Use the identity $(1+x)^m(1+x)^n = (1+x)^{m+n}$ to show that **2**

$$\binom{m+n}{4} = \binom{n}{4} + \binom{n}{3}\binom{m}{1} + \binom{n}{2}\binom{m}{2} + \binom{n}{1}\binom{m}{3} + \binom{m}{4}.$$

(c) The graphs of the functions $g(x) = mx + b$ and $f(x) = 3x^3 - 2x^2$ are shown. **3**



The solution to the equation $mx + b < 3|x|^3 - 2|x|^2$ is $x \in (-\infty, -2) \cup (1, \infty)$.

By sketching the graph of $y = 3|x|^3 - 2|x|^2$, find the values of *m* and *b* to complete the function $g(x)$.

Question 12 continues on page 9

Question 12 (continued)

(d) An online furniture store claims that 90% of all orders are shipped within 54 hours of a customer placing an order through the store's website. Ty ordered a total of 150 various pieces of furniture from the store for his company.

(i) According to the store's claim, 135 pieces of furniture should be delivered within 54 hours of Ty's order being placed through the store's website. **1**

What is the probability that this delivery will occur? Give your answer to three significant figures.

(ii) Show that the expected value and the standard deviation of the sampling proportion are 0.9 and 0.0245 respectively. **2**

(iii) Part of a table of $P(Z \leq z)$ values, where Z is a standard normal variable, is shown. **2**

Z	*0*	*0.01*	*0.02*	*0.03*	*0.04*	*0.05*	*0.06*	*0.07*	*0.08*	*0.09*
1	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147

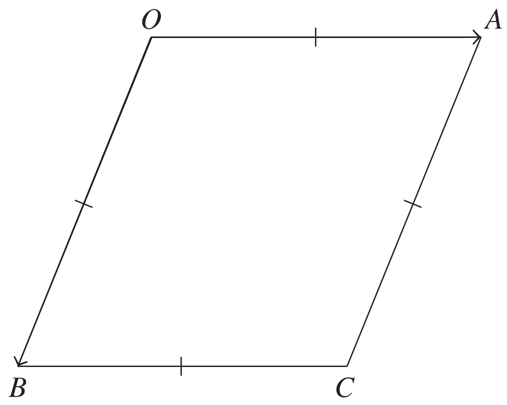
Ty expected that at least a certain number of pieces would be delivered within 54 hours. He used a computer program and calculated the probability of this happening to be 90.15%.

Using the table, find the minimum number of furniture pieces that would be delivered within 54 hours.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) $OACB$ is a rhombus with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. **2**

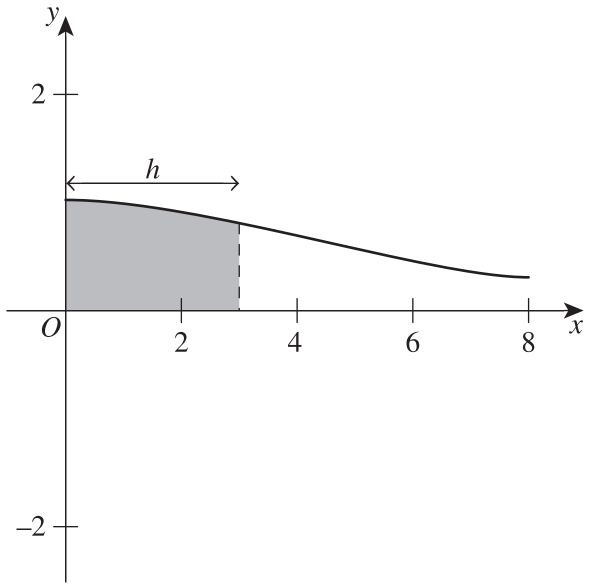


Using vector methods, prove that the diagonal OC bisects $\angle AOB$.

Question 13 continues on page 11

Question 13 (continued)

(b) The graph of the function $y = \dfrac{3}{\sqrt{9+x^2}}$ for $x > 0$ is shown.

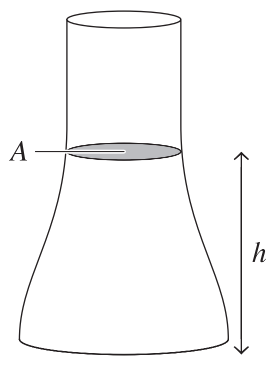


(i) The area bounded by the curve $y = \dfrac{3}{\sqrt{9+x^2}}$, the axes and the lines $x = 0$ and $x = h$ **2**

is rotated about the x-axis to create a solid of revolution.

Show that the volume of this solid is given by $V = 3\pi \tan^{-1}\left(\dfrac{h}{3}\right)$ cubic units.

(ii) The solid of revolution from the graph has the shape of a vase lying down on its side. The diagram shows a vase of the same shape standing upright. **2**



As water is being poured into the vase, the height h of the water is increasing at 3 cm/s.

Find the exact rate at which the volume V of the water is increasing when $h = 6$ cm.

(iii) The surface of the water forms a circular shape with the area A. **2**

By using the original graph, or otherwise, find the exact rate at which the area A is decreasing when $h = 6$ cm.

Question 13 continues on page 12

Question 13 (continued)

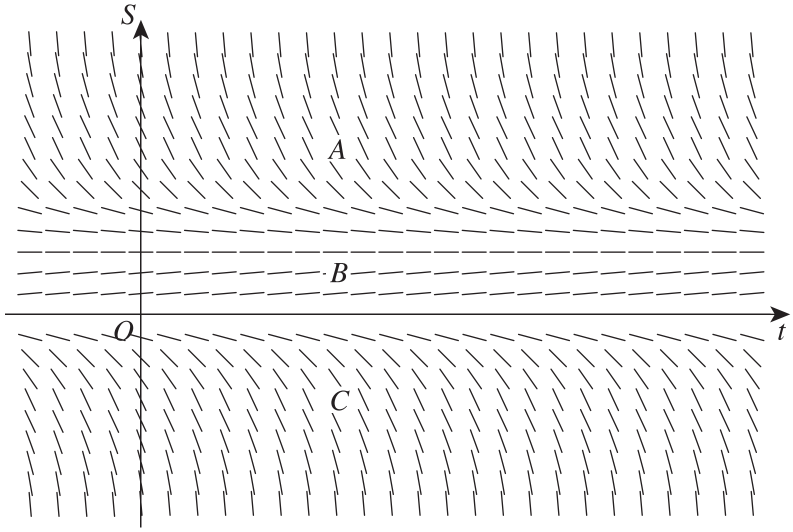
(c) The spread of a flu through a student population is modelled by the equation

$$S = \frac{2000}{1 + 199e^{-0.4t}}$$

where S is the total number of students infected after t days.

(i) Show that the given equation for S satisfies the differential equation $\frac{dS}{dt} = \frac{S}{5}\left(2 - \frac{S}{1000}\right)$. **3**

(ii) The slope field of $\frac{dS}{dt} = \frac{S}{5}\left(2 - \frac{S}{1000}\right)$ is shown. **2**



There are three regions labelled A, B and C in which a solution curve can be found.

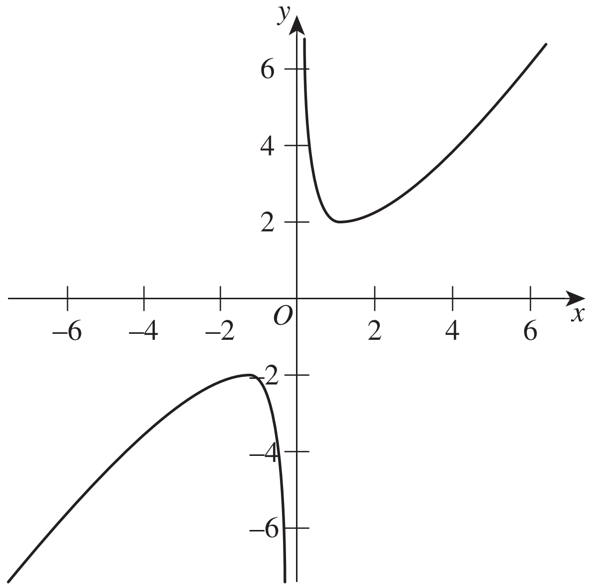
In which one of these three regions can the solution curve S exist? Justify your answers with reference to constant solutions and initial value.

(iii) According to this model, after how many days does the rate of increase reach maximum? Give your answer to the nearest number of days. **2**

End of Question 13

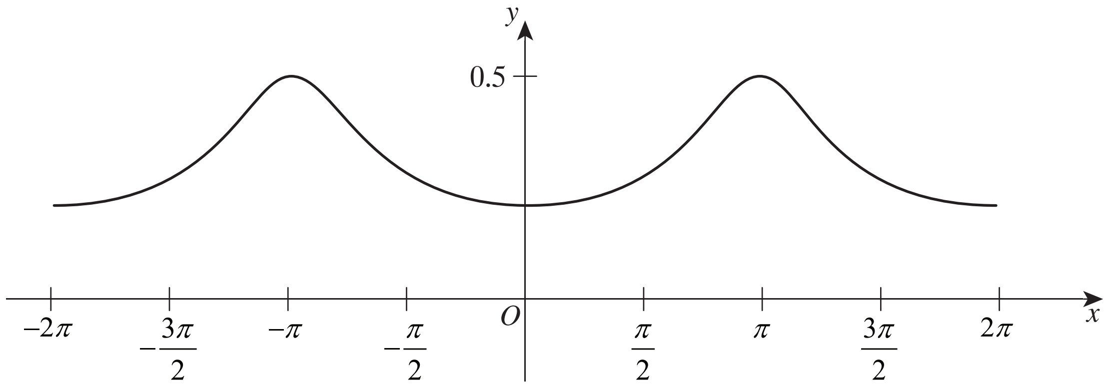
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of the function $f(x)=\dfrac{x^2+1}{x}$ is shown. **3**



Sketch the graph of $y=\dfrac{1}{\sqrt{f(x)}}$, showing all important features including turning point(s), intercept(s) and asymptote(s).

(b) The graph of the function $y=\dfrac{1}{5+3\cos x}$ for $-2\pi \le x \le 2\pi$ is shown.



(i) By using the substitution $t=\tan\dfrac{x}{2}$ and t-formula, show that **3**

$\displaystyle\int \frac{1}{5+3\cos x}\,dx=\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)+C$, where C is a constant.

(ii) Hence, show that the area bounded by the curve $y=\dfrac{1}{5+3\cos x}$, the lines $x=0$ **2**

and $x=\pi$ and the x-axis is $\dfrac{\pi}{4}$.

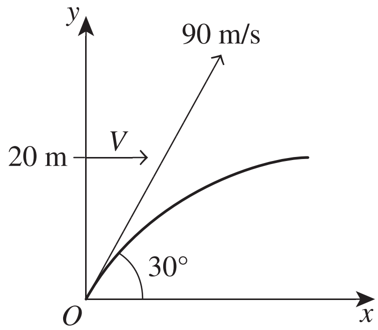
Question 14 continues on page 14

Question 14 (continued)

(c) (i) Initially, a golf ball was hit from the ground at a velocity of 90 m s^{-1} at an angle of 30° to the horizontal and $g = 10\text{m s}^{-2}$. **2**

Show that the position vector of the golf ball after t seconds is given by
$\underset{\sim}{s}(t) = \left(45\sqrt{3}t\right)\underset{\sim}{i} + \left(45t - 5t^2\right)\underset{\sim}{j}$.

(ii) Five seconds after the golf ball was hit, a small stone was fired at a velocity V horizontally from a point 20 m above the ground. **3**



By finding the position vector of the stone, or otherwise, show that the two objects collided after the golf ball has travelled for 21 seconds.

(iii) What is the stone's speed at collision? Give your answer to three significant figures. **2**

End of paper

MATHEMATICS ADVANCED
MATHEMATICS EXTENSION 1
MATHEMATICS EXTENSION 2
REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \ r \neq 1$$

$$S = \frac{a}{1-r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

$A = \frac{1}{2}ab\sin C$

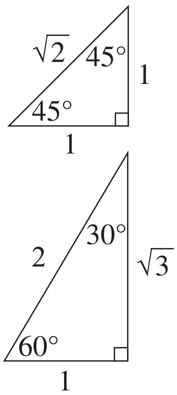
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \frac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$\cos A = \frac{1 - t^2}{1 + t^2}$

$\tan A = \frac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

$\sin A\sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

$\sin A\cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$

$\cos A\sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

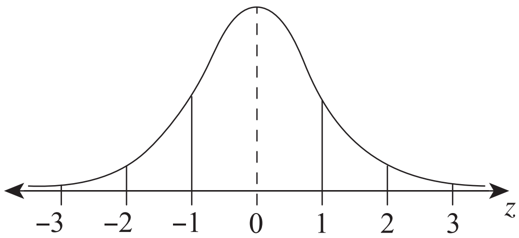
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score less than $Q_1 - 1.5 \times IQR$ or more than $Q_3 - 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq r) = \int_a^r f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$

$X \sim \text{Bin}(n, p)$

$\Rightarrow P(X = x)$

$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\text{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1+\left[f(x)\right]^2}}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$
where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 - \left[f(x)\right]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$
$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \ldots + f\left(x_{n-1}\right)\right]\right\}$$
where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \ldots + \binom{n}{r}x^{n-r}a^{r} + \ldots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + x\underset{\sim}{j}\right| = \sqrt{x^{2} + y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1 x_2 + y_1 y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^{2}(x - c)$$

Neap

Trial Examination 2021

HSC Year 12 Mathematics Extension 1

Section II Writing Booklet

Student Name/Number: ____________________

Instructions

Use a separate writing booklet for each question in Section II.

Write the number of this booklet and the total number of booklets that you have used for this question (e.g. $\boxed{1}$ of $\boxed{3}$) $\Rightarrow$ $\boxed{}$ of $\boxed{}$

this booklet | number of booklets for this question

Write in black or blue pen (black is recommended).

You may ask for an extra writing booklet if you need more space.

If you have not attempted the question(s), you must still hand in a writing booklet, with 'NOT ATTEMPTED' written clearly on the front cover.

You may NOT take any writing booklets, used or unused, from the examination room.

Tick this box if you have continued this answer in another writing booklet. ☐

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Trial Examination 2021

HSC Year 12 Mathematics Extension 1

DIRECTIONS:

Write your name in the space provided.

Write your student number in the boxes provided below. Then, in the columns of digits below each box, fill in the oval which has the same number as you have written in the box. Fill in **one** oval only in each column.

Read each question and its suggested answers. Select the alternative A, B, C, or D that best answers the question. Fill in the response oval completely, using blue or black pen. Mark only **one oval** per question.

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and draw an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

STUDENT NAME: ______________________________

STUDENT NUMBER:

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9
0	0	0	0	0	0	0	0	0

SECTION I

MULTIPLE-CHOICE ANSWER SHEET

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

STUDENTS SHOULD NOW CONTINUE WITH SECTION II

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Trial Examination 2021

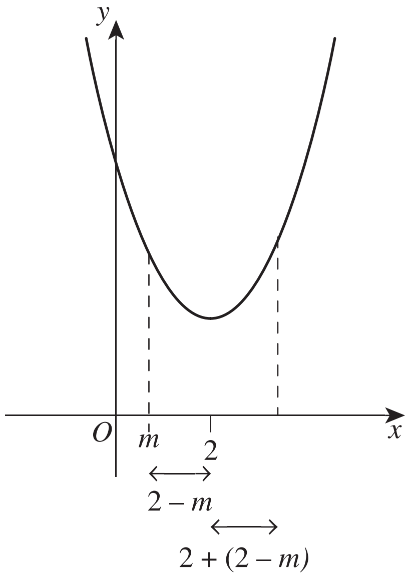
HSC Year 12 Mathematics Extension 1

Solutions and marking guidelines

SECTION I

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 1 B $P(-3)=0$ $P'(-3)\neq 0$	ME–F2 Polynomials ME11–1 Bands E2–E3
Question 2 D $\dfrac{dy}{dx}=\cos x.\cos^{-1}x-\dfrac{\sin x}{\sqrt{1-x^2}}$	ME– C2 Further Calculus Skills ME12–1 Bands E2–E3
Question 3 C Domain: $-1\leq\dfrac{4x}{\pi}\leq 1$ $-\pi\leq 4x\leq\pi$ $-\dfrac{\pi}{4}\leq x\leq\dfrac{\pi}{4}$ Range: $0\leq y\leq\pi$	ME–T1 Inverse Trigonometric Functions ME11–3 Bands E2–E3
Question 4 A Scenario 1: 4 algebra and 3 other calculus books ${}^8C_4\times{}^7C_3=2450$ Scenario 2: 5 algebra and 2 other calculus books ${}^8C_5\times{}^7C_2=1176$ Therefore, the total is $2450+1176=3626$.	ME–A1 Working with Combinatorics ME11–5 Bands E2–E3
Question 5 A $\overrightarrow{AC}$ is NOT perpendicular to $\overrightarrow{BC}$. $\therefore$ **A** is correct. $\overrightarrow{AB}\perp\overrightarrow{BC}$ This statement is true. $\therefore$ **B** is incorrect. $AC=DB$ This statement is true. $\therefore$ **C** is incorrect. $\overrightarrow{AD}=\overrightarrow{BC}=\underset{\sim}{c}-\underset{\sim}{b}$ This statement is true. $\therefore$ **D** is incorrect.	ME–V1 Introduction to Vectors ME12–2 Bands E2–E3

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 6 B $\int \frac{\ln 2}{\sqrt{\pi - x^2}}dx = \ln 2 \times \int \frac{1}{\sqrt{\pi - x^2}}dx$ Now: $a^2 = \pi$ $a = \sqrt{\pi}$ Also: $\int \frac{1}{\sqrt{\pi - x^2}}dx = \sin^{-1}\left(\frac{x}{\sqrt{\pi}}\right) + C$ $\ln 2 \times \int \frac{1}{\sqrt{\pi - x^2}}dx = \ln 2 \times \sin^{-1}\left(\frac{x}{\sqrt{\pi}}\right) + C$	ME– C2 Further Calculus Skills ME12–1 Bands E2–E3
Question 7 C $23 = 2 + 5 + 5 \times 3 + 1$	ME–A1 Working with Combinatorics ME11–5 Bands E2–E3
Question 8 B **B** is correct. It shows $y' = x + e^x$. **A** is incorrect. It shows $y' = e^{-x} + e^x$. **C** is incorrect. It shows $y' = \frac{x}{y} + e^x$. **D** is incorrect. It shows $y' = e^{-x} + x$.	ME– C3 Applications of Calculus ME12–1, 12– 4 Bands E3–E4
Question 9 B $\frac{dy}{dx} = 5 - y$ $\frac{1}{5-y}dy = 1dx$ $-\ln(5+y) = x + C$ $5 - y = Ae^{-x}$, where $A = e^{-C}$ $y = 5 - Ae^{-x}$ Substituting $y = 4$, $x = 3$ results in $A = e^3$. $\therefore y = 5 - e^{3-x}$	ME– C3 Applications of Calculus ME12–1, 12– 4 Bands E3–E4

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 10 D $f(x)$ maps m outside the domain to $2+2-m=4-m$, which is inside the domain. Now $f^{-1}(f(m))=f^{-1}\left(f(4-m)\right)=4-m$. 	ME–T1 Inverse Trigonometric Functions ME11–1 Bands E3–E4

SECTION II

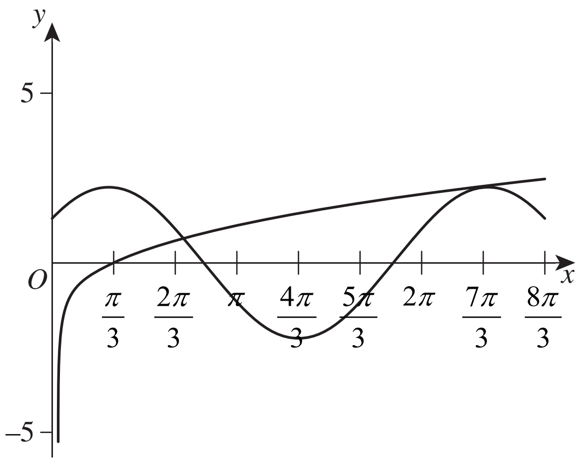
<table>
<tr><th>Sample answer</th><th>Syllabus content, outcomes, targeted performance bands and marking guide</th></tr>
<tr><td>Question 11</td><td></td></tr>
<tr><td>(a) (i) $\cos 7x \sin 5x = \frac{1}{2}(\sin 12x - \sin 2x)$</td><td>ME–T2 Further Trigonometric Identities
ME11–3 Band E2
• Gives the correct solution ....1</td></tr>
<tr><td>(ii) $2\cos 7x \sin 5x = \sin 12x - \frac{\sqrt{3}}{2}$

Consider the LHS:
$2\cos 7x \sin 5x = 2 \times \frac{1}{2}(\sin 12x - \sin 2x)$
(using the part (a) (i) result)
$= \sin 12x - \sin 2x$
$\sin 12x - \sin 2x = \sin 12x - \frac{\sqrt{3}}{2}$
$-\sin 2x = -\frac{\sqrt{3}}{2}$
$\sin 2x = \frac{\sqrt{3}}{2}$
$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$
$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$</td><td>ME–T3 Trigonometric Equations
ME12–3 Bands E2–E3
• Gives the correct solutions 3
<hr>• Correctly simplifies the equation to $\sin 2x = \frac{\sqrt{3}}{2}$.
AND
• Solves for at least ONE value of x 2
<hr>• Correctly simplifies the equation to $\sin 2x = \frac{\sqrt{3}}{2}$ ....1</td></tr>
</table>

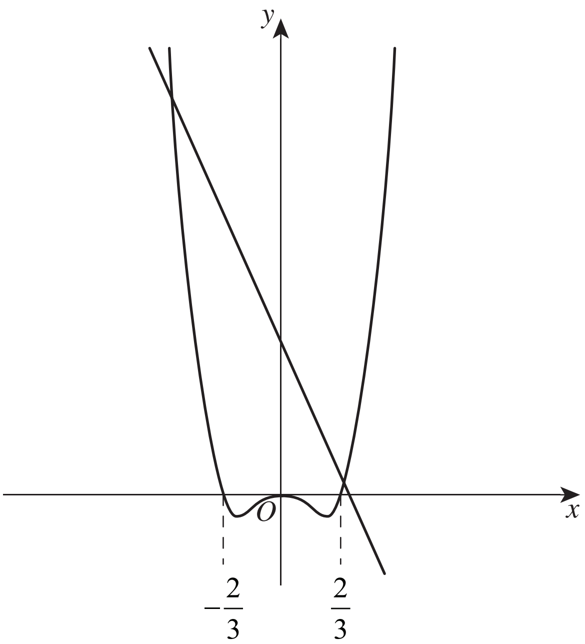
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) Let $S(n)$ be the statement $2^{2n}+6n-1$ is divisible by 3 for all integers $n \geq 1$. Step 1: Prove that $S(1)$ is true. $2^{2\times1}+6\times1-1=9$, which is divisible by 3. Therefore, $S(1)$ is true. Step 2: Assume that $S(k)$ is true. $2^{2k}+6k-1=3M$, where M is an integer. $2^{2k}=3M-6k+1$ Step 3: Prove that $S(k+1)$ is true. $2^{2(k+1)}+6(k+1)-1=3N$, where N is an integer. $2^{2(k+1)}+6(k+1)-1=2^{2k+2}+6k+6-1$ $=4\times2^{2k}+6k+5$ $=4\times(3M-6k+1)+6k+5$ $=12M-24k+4+6k+5$ $=12M-18k+9$ $=3(4M-6k+3)$ $=3N$, where N is an integer Therefore, $S(k+1)$ is true if $S(k)$ is true. By mathematical induction, $S(n)$ is true for all integers $n \geq 1$.	ME–P1 Proof by Mathematical Induction ME12–1 Bands E2–E3 • Gives the correct proof for all steps. 3 • Gives the correct proof for step 1. AND • Makes some progress in using the assumption for step 2 . . . . 2 • Gives the correct proof for step 1 1
(c) (i) $a=1, b=0, c=-2p, d=q$ Taking sum of the roots, one at a time: $\alpha+\beta+(\alpha+\beta)=-\frac{b}{a}$ $2(\alpha+\beta)=0$ $\alpha+\beta=0$ $\therefore \alpha+\beta=0$ $\alpha=-\beta$	ME–F2 Polynomials ME11–1, 11–2 Bands E2–E3 • Gives the correct proof 1

	Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii)	Taking sum of the roots, three at a time OR taking product of the roots: $\alpha\beta(\alpha+\beta)=-q$ $q=0$ as $\alpha+\beta=0$ $P(x)=x^3-2px+q$ $=x^3-2px$ $=x\left(x^2-2p\right)$ $x\left(x^2-2p\right)=0$ $x=0$ or $x^2-2p=0$ $x^2-2p=0$ $x^2=2p$ $x=\pm\sqrt{2p}$ Therefore, the roots are $x=0,\pm\sqrt{2p}$.	ME–F2 Polynomials ME11–1, 11–2 Bands E2–E3 • Gives the correct solutions . . . . 3 • Correctly finds the value of q AND ONE root. . . . . 2 • Correctly finds the value of q. 1
(d) (i)	If $\overrightarrow{PA}=\underset{\sim}{a}$ and $\overrightarrow{SB}=\underset{\sim}{b}$, then $\overrightarrow{AQ}=\underset{\sim}{a}$ and $\overrightarrow{BR}=\underset{\sim}{b}$, as A and B are the midpoints of PQ and RS respectively. $\overrightarrow{QR}=\overrightarrow{QA}+\overrightarrow{AB}+\overrightarrow{BR}$ $=-\underset{\sim}{a}+\overrightarrow{AB}+\underset{\sim}{b}$ $=\underset{\sim}{b}-\underset{\sim}{a}+\overrightarrow{AB}$	ME–V1 Introduction to Vectors ME12–2 Bands E2–E3 • Gives the correct solution . . . . 2 • Demonstrates that $\overrightarrow{AQ}=\underset{\sim}{a}$ and $\overrightarrow{BR}=\underset{\sim}{b}$ OR equivalent merit. 1
(ii)	$\overrightarrow{PS}=\overrightarrow{PA}+\overrightarrow{AB}+\overrightarrow{BS}$ $=\underset{\sim}{a}+\overrightarrow{AB}-\underset{\sim}{b}$ $=\underset{\sim}{a}-\underset{\sim}{b}+\overrightarrow{AB}$ Adding $\overrightarrow{QR}$ to $\overrightarrow{PS}$: $\overrightarrow{QR}+\overrightarrow{PS}=\underset{\sim}{b}-\underset{\sim}{a}+\overrightarrow{AB}+\underset{\sim}{a}-\underset{\sim}{b}+\overrightarrow{AB}$ (using the part (d) (i) result) $=2\overrightarrow{AB}$ $\therefore \overrightarrow{AB}=\frac{1}{2}\left(\overrightarrow{PS}+\overrightarrow{QR}\right)$	ME–V1 Introduction to Vectors ME12–2 Bands E2–E3 • Gives the correct proof . . . . 2 • Correctly finds $\overrightarrow{PS}$ in terms of $\underset{\sim}{a},\underset{\sim}{b}$ and $\overrightarrow{AB}$ 1

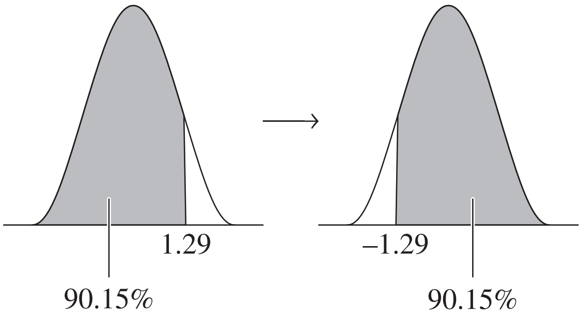
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 12	
(a) (i) $\sqrt{3}\sin x + \cos x \equiv R\sin(x+\alpha)$ $= R\sin x\cos\alpha + R\cos x\sin\alpha$ Therefore: $R\cos\alpha = \sqrt{3}$ $R\sin\alpha = 1$ $R^2\left((\cos x)^2 + (\sin x)^2\right) = 4$ $R = 2$ In addition: $\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{1}{\sqrt{3}}$ $\tan\alpha = \dfrac{1}{\sqrt{3}}$ $\alpha = \dfrac{\pi}{6}$ $\therefore \sqrt{3}\sin x + \cos x = 2\sin\left(x + \dfrac{\pi}{6}\right)$	ME–T3 Trigonometric Equations ME12–3 Bands E2–E3 • Gives the correct solution . . . . 2 ——— • Finds the correct value for R or α 1

Sample answer	**Syllabus content, outcomes, targeted performance bands and marking guide**
(ii) The graphs of $y = 2\sin\left(x + \frac{\pi}{6}\right)$ and $y = \ln x$:  Inspect number of points of interest near $x = \frac{7\pi}{3}$: $\sqrt{3}\sin x + \cos x = 2\sin\left(x + \frac{\pi}{6}\right)$ (from part (a) (i) result) $= 2\sin\left(\frac{7\pi}{3} + \frac{\pi}{6}\right)$ $= 2\sin\left(\frac{5\pi}{2}\right)$ $= 2$ $\ln x = 1.99$ Therefore, there exist two POI at $x = \frac{7\pi}{3}$. Hence, there are three solutions to the equation $\sqrt{3}\sin x + \cos x = \ln x$.	ME–T3 Trigonometric Equations ME12–3, 12–7 Bands E2–E4 • Gives the correct solution AND clearly justifies all POI at $x = \frac{7\pi}{3}$ . . . . 3 • Correctly sketches both graphs . . . . 2 • Correctly sketches the graph of $y = 2\sin\left(x + \frac{\pi}{6}\right)$. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) Consider the identity $(1+x)^m(1+x)^n=(1+x)^{m+n}$. The term x^4 of $(1+x)^{m+n}$ is: $\binom{m+n}{4}x^4 \quad (1)$ The term x^4 of $(1+x)^m(1+x)^n$ is: $\binom{m}{4}x^4\times 1+\binom{m}{3}x^3\times\binom{n}{1}x+\binom{m}{2}x^2\times\binom{n}{2}x^2+\binom{m}{1}x\times\binom{n}{3}x^3+1\times\binom{n}{4}x^4=\left[\binom{m}{4}+\binom{m}{3}\binom{n}{1}+\binom{m}{2}\binom{n}{2}+\binom{m}{1}\binom{n}{3}+\binom{n}{4}\right]x^4 \quad (2)$ Compare the coefficients of x^4 from (1) and (2): $\binom{m+n}{4}=\binom{m}{4}+\binom{m}{3}\binom{n}{1}+\binom{m}{2}\binom{n}{2}+\binom{m}{1}\binom{n}{3}+\binom{n}{4}$	ME–A1 Working with Combinatorics ME11–5 Bands E2–E3 • Gives the correct proof . . . . 2 • Finds the correct coefficient of x^4 of $(1+x)^{m+n}$ AND demonstrates some progress in achieving the coefficient of $(1+x)^m(1+x)^n$ 1

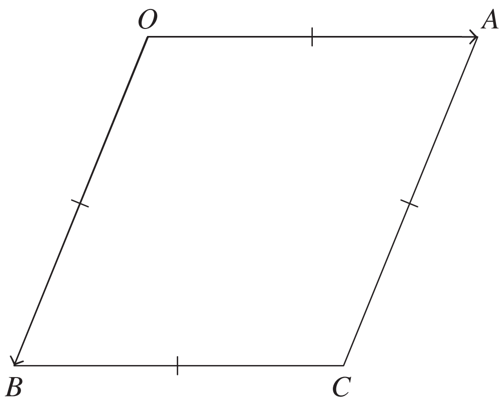
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) The graph of $y = 3\|x\|^3 - 2\|x\|^2$ is $y = f\|x\|$, which can be drawn by taking the graph for $x \geq 0$ and reflecting it over the y-axis. $f(x) = 3x^2 - 2x^2$ $= x^2(3x - 2)$ The function $f(x) = x^2(3x-2)$ has two x-intercepts at 0 and $\frac{2}{3}$. Therefore, when reflected, the graph of $y = 3\|x\|^3 - 2\|x\|^2$ should have three x-intercepts at 0, $\frac{2}{3}$ and $-\frac{2}{3}$. There are now two POI between the straight line and $y = 3\|x\|^3 - 2\|x\|^2$.  The solutions $x \in (-\infty, -2) \cup (1, \infty)$ give the x-coordinates of these POI to be –2 and 1. Substituting these into $y = 3\|x\|^3 - 2\|x\|^2$ gives: $y = 3\|-2\|^3 - 2\|-2\|^2$ $= 16$ $y = 3\|1\|^3 - 2\|1\|^2$ $= 1$ (continues on page 12)	ME–F1 Further Work with Functions ME11–1, 11–2, 11–7 Bands E2–E3 • Gives the correct solution . . . . 3 ——— • Sketches the correct graph of $y = 3\|x\|^3 - 2\|x\|^2$. AND • Finds the POI between $f(x)$ and $g(x)$ . . . . 2 ——— • Sketches the correct graph of $y = 3\|x\|^3 - 2\|x\|^2$. OR • Finds the POI between $f(x)$ and $g(x)$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (continued) Therefore, the straight line passes through the points (–2, 16) and (1, 1). $m = \frac{16-1}{-2-1}$ $= -5$ $y = -5x + b$ Substituting (1, 1): $1 = -5 \times 1 + b$ $b = 6$ $\therefore g(x) = -5x + 6.$	
(d) (i) $^{150}C_{135}(0.9)^{135}(0.1)^{15} = 0.107970$ $= 0.108$	ME–S1 The Binomial Distribution ME12–5 Bands E2–E3 • Gives the correct solution 1
(ii) $E(\hat{p}) = E\left(\frac{X}{n}\right)$ $= \frac{E(X)}{n}$ $= \frac{np}{n}$ $= p$ $= 0.9$ $\sigma(\hat{p}) = \sigma\left(\frac{X}{n}\right)$ $= \frac{\sigma(X)}{n}$ $= \frac{\sqrt{npq}}{n}$ $= \frac{\sqrt{150 \times 0.9 \times 0.1}}{150}$ $= 0.0245$	ME–S1 The Binomial Distribution ME12–5 Bands E2–E3 • Gives the correct proofs. . . . . 2 • Correctly shows the expected value OR standard deviation 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) From the table, $0.90147 = 90.15\%$ probability. This corresponds to a z-score of 1.29. However, for the least number of furniture pieces expected to be delivered, we need to use the symmetry of the normal distribution curve for a z-score of –1.29.  We now need to find x (the sample proportion) that corresponds to this z-score. $-1.29 = \dfrac{x - 0.9}{0.0245}$ (using part (d) (ii) result) $x = -1.29 \times 0.0245 + 0.9$ $= 0.868395$ Therefore, the minimum number of furniture expected to be delivered within 54 hours is $0.868395 \times 150 = 130.25925$, which would round to 130 pieces of furniture.	ME–S1 The Binomial Distribution ME12–5 Bands E3–E4 • Gives the correct solution .... 2 • Correctly finds the z-score of –1.29........................ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 13	

(a)



If $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$ then $\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b}$.

$$\begin{aligned}\cos\angle AOC &= \frac{\underset{\sim}{a}\cdot\left(\underset{\sim}{a}+\underset{\sim}{b}\right)}{\left|\underset{\sim}{a}\right|\cdot\left|\underset{\sim}{a}+\underset{\sim}{b}\right|}\\ &= \frac{\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{a}\cdot\underset{\sim}{b}}{\left|\underset{\sim}{a}\right|\cdot\left|\underset{\sim}{a}+\underset{\sim}{b}\right|}\\ &= \frac{\left|\underset{\sim}{a}\right|^2+\underset{\sim}{a}\cdot\underset{\sim}{b}}{\left|\underset{\sim}{a}\right|\cdot\left|\underset{\sim}{a}+\underset{\sim}{b}\right|} \quad (1)\end{aligned}$$

$$\begin{aligned}\cos\angle BOC &= \frac{\underset{\sim}{b}\cdot\left(\underset{\sim}{a}+\underset{\sim}{b}\right)}{\left|\underset{\sim}{b}\right|\cdot\left|\underset{\sim}{a}+\underset{\sim}{b}\right|}\\ &= \frac{\underset{\sim}{b}\cdot\underset{\sim}{b}+\underset{\sim}{a}\cdot\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|\cdot\left|\underset{\sim}{a}+\underset{\sim}{b}\right|}\\ &= \frac{\left|\underset{\sim}{b}\right|^2+\underset{\sim}{a}\cdot\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|\cdot\left|\underset{\sim}{a}+\underset{\sim}{b}\right|} \quad (2)\end{aligned}$$

But $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right|$ ($OA = OB$, adjacent sides are equal in a rhombus).

From (1) and (2), $\cos\angle AOC$ or $\cos\angle BOC$ and, as none of these angles is a reflex angle, $\angle AOC = \angle BOC$.

Therefore, the diagonal OC bisects $\angle AOB$.

ME–V1 Introduction to Vectors
ME12–2 Bands E2–E4

- Gives the correct proof 2

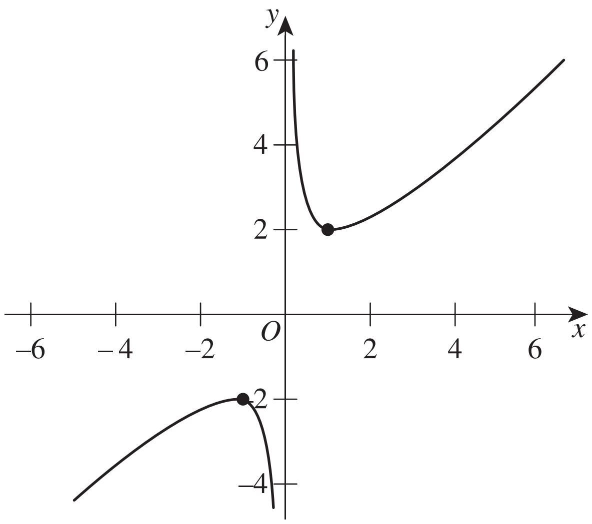
- Finds $\cos\angle AOC$ or $\cos\angle BOC$ . . 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) (i) $V = \pi\int_0^h y^2 dx$ $= \pi\int_0^h \left(\frac{3}{\sqrt{9+x^2}}\right)^2 dx$ $= \pi\int_0^h \frac{9}{9+x^2} dx$ $= 9\pi\int_0^h \frac{1}{9+x^2} dx$ $= \frac{9\pi}{3}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_0^h$ $= 3\pi\left[\tan^{-1}\left(\frac{h}{3}\right) - 0\right]$ $= 3\pi\tan^{-1}\left(\frac{h}{3}\right)$ cubic units	ME–C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives the correct proofs. . . . . 2 • Correctly integrates $\left(\frac{3}{\sqrt{9+x^2}}\right)^2$ 1
(ii) $\frac{dh}{dt} = 3$ cm/s $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ (1) Now: $V = 3\pi\tan^{-1}\left(\frac{h}{3}\right)$ $\frac{dV}{dh} = 3\pi \times \frac{3}{9+h^2}$ $= \frac{9\pi}{9+h^2}$ Substitute $\frac{dh}{dt} = 3$ cm/s, $\frac{dV}{dh} = \frac{9\pi}{9+h^2}$ and $h = 6$ cm into (1): $\frac{dV}{dt} = \frac{9\pi}{9+6^2} \times 3$ $= \frac{3\pi}{5}$ cm^3/s	ME–C1 Rates of Changes ME11–4 Bands E2–E3 • Gives the correct solution . . . . 2 • Correctly finds $\frac{dV}{dh}$ 1

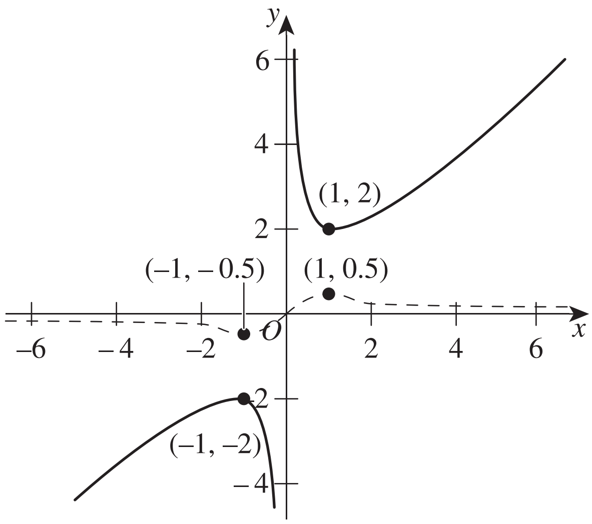
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) Refer to the original graph; as $x = h, y = \dfrac{3}{\sqrt{9+x^2}}$, this y-value is the radius of the water surface area A. Therefore: $A = \pi r^2$ $= \pi\left(\dfrac{3}{\sqrt{9+x^2}}\right)^2$ $= \dfrac{9\pi}{9+h^2}$ $= 9\pi\left(9+h^2\right)^{-1}$ $\dfrac{dA}{dh} = -9\pi\left(9+h^2\right)^{-2} \times 2h$ $= \dfrac{-18\pi h}{\left(9+h^2\right)^2}$ $\dfrac{dA}{dt} = \dfrac{dA}{dh} \times \dfrac{dh}{dt}$ Substitute and $\dfrac{dh}{dt} = 3$ cm/s, $\dfrac{dA}{dh} = \dfrac{-18\pi h}{\left(9+h^2\right)^2}$ $h = 6$ cm into $\dfrac{dA}{dt} = \dfrac{dA}{dh} \times \dfrac{dh}{dt}$: $\dfrac{dA}{dt} = \dfrac{-18\pi \times 6}{\left(9+6^2\right)^2} \times 3$ $= \dfrac{-4\pi}{25}$ Therefore the area is decreasing at $\dfrac{4\pi}{25}$ cm^2/s.	ME–C1 Rates of Changes ME11–4 Bands E3–E4 • Gives the correct solution . . . . 2 • Correctly finds $\dfrac{dA}{dh}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) $S=\dfrac{2000}{1+199e^{-0.4t}}$ $=2000\left(1+199e^{-0.4t}\right)^{-1}$ $\dfrac{dS}{dt}=-2000\left(1+199e^{-0.4t}\right)^{-2}\times-0.4\times199e^{-0.4t}$ $=\dfrac{-2000\times-0.4\times199e^{-0.4t}}{\left(1+199e^{-0.4t}\right)^2}$ $=\dfrac{159\,200e^{-0.4t}}{\left(1+199e^{-0.4t}\right)^2}$ (1) $\dfrac{S}{5}\left(2-\dfrac{S}{1000}\right)=\dfrac{2000}{5\times\left(1+199e^{-0.4t}\right)}\times\left(2-\dfrac{2000}{1000\left(1+199e^{-0.4t}\right)}\right)$ $=\dfrac{400}{1+199e^{-0.4t}}\left(2-\dfrac{2}{1+199e^{-0.4t}}\right)$ $=\dfrac{800}{1+199e^{-0.4t}}\left(1-\dfrac{1}{1+199e^{-0.4t}}\right)$ $=\dfrac{800}{1+199e^{-0.4t}}\left(\dfrac{1+199e^{-0.4t}-1}{1+199e^{-0.4t}}\right)$ $=\dfrac{800}{1+199e^{-0.4t}}\left(\dfrac{199e^{-0.4t}}{1+199e^{-0.4t}}\right)$ $=\dfrac{159\,200e^{-0.4t}}{\left(1+199e^{-0.4t}\right)^2}$ (2) From (1) and (2): $\dfrac{dS}{dt}=\dfrac{S}{5}\left(2-\dfrac{S}{1000}\right)$	ME– C3 Applications of Calculus ME12– 4 Bands E3–E4 • Gives the correct proof .... 3 • Finds $\dfrac{dS}{dt}$ AND demonstrates some progress .... 2 • Finds $\dfrac{dS}{dt}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Region B From the differential equation proven in part (c) (i), the solution curve S must be bounded by two constant solutions, 0 and 2000. These constant solutions are shown by horizontal slope line segments. Substituting $t = 0$: $S = \frac{2000}{1+199e^0}$ $= 10$ This initial S value is in the interval [0, 2000]. All these characteristics of the solution curve S occur in region B.	ME– C3 Applications of Calculus ME12– 4, 12–7 Bands E3–E4 • Gives the correct solution . . . . 2 • Correctly finds constant solutions 1
(iii) The graph $\frac{dS}{dt}$ versus S is a parabola with two intercepts with the t-axis: 0 and 2000. As the parabola concaves down, the maximum is the vertex that occurs at $S = \frac{1}{2}(0+2000) = 1000$. Substituting $S = 1000$ into $S = \frac{2000}{1+199e^{-0.4t}}$ gives: $1000 = \frac{2000}{1+199e^{-0.4t}}$ $1+199e^{-0.4t} = 2$ $199e^{-0.4t} = 1$ $e^{-0.4t} = \frac{1}{199}$ $-0.4t = \ln\frac{1}{199}$ $t = \frac{\ln\frac{1}{199}}{-0.4}$ $= 13.233$ ≈ 13 days	ME– C3 Applications of Calculus ME12– 4 Bands E3–E4 • Gives the correct solution . . . . 2 • Correctly finds the maximum point with correct S values. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 14	
(a) Consider $f(x)=\dfrac{x^2+1}{x}=x+x^{-1}$. At stationary points, $f'(x)=1-x^{-2}=0$. $1-\dfrac{1}{x^2}=0$ $x^2=1$ $x=\pm 1$ Substituting these values into $f(x)=\dfrac{x^2+1}{x}$ gives the stationary points (1, 2) and (–1, –2). By inspection, the graph of $f(x)$ can be completed as shown.  Consider $g(x)=\dfrac{1}{f(x)}$. The minimum point on $f(x)$ becomes the maximum point on $\dfrac{1}{f(x)}$. $\therefore y=\dfrac{1}{f(x)}$ has a maximum point at $\left(1,\dfrac{1}{f(x)}\right)=\left(1,\dfrac{1}{2}\right)$. (continues on page 20)	ME–F1 Further Work with Functions ME11–1, 11–2, 11–7 Bands E2–E4 • Gives the correct solution . . . . 3 • Correctly draws the graph of $g(x)=\dfrac{1}{\sqrt{f(x)}}$ without turning points. OR • Correctly draws the graph of $g(x)=\dfrac{1}{f(x)}$ with turning points. . . . . 2 • Correctly draws the graph of $g(x)=\dfrac{1}{f(x)}$ with turning points. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(a) (continued) As $x \to \infty, f(x) \to \infty, \dfrac{1}{f(x)} \to 0^+$. At $x = 0, \dfrac{1}{f(x)} = 0$. Both $y = f(x)$ and $y = \dfrac{1}{f(x)}$ are odd functions. Hence, the graphs of $y = f(x)$ (full) and $y = g(x) = \dfrac{1}{f(x)}$ (dashed) are as shown.	



Consider:

$$h(x) = \frac{1}{\sqrt{f(x)}}$$
$$= \sqrt{\frac{1}{f(x)}}$$
$$= \sqrt{g(x)}$$

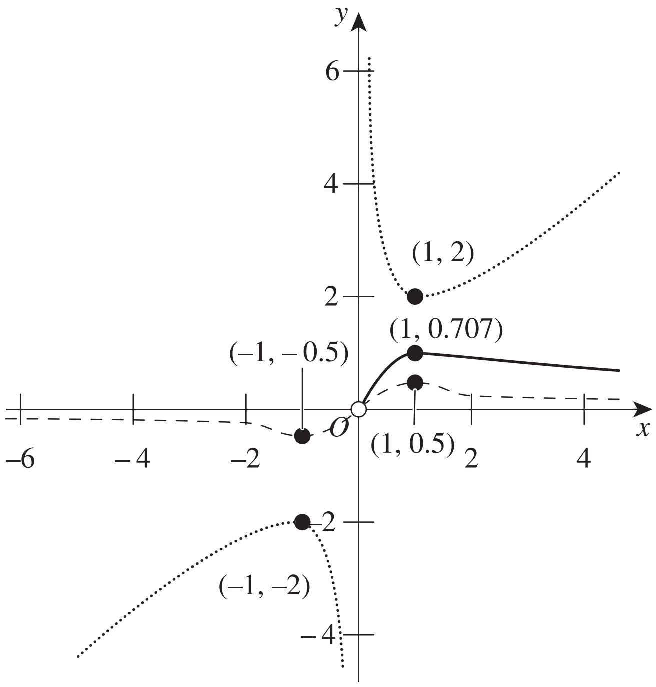
x only exists when $g(x) \geq 0$.

$g(0) = 0; h(0) = \sqrt{g(0)} = 0$

The maximum point at $x = 1$ remains a maximum point on $h(x)$.

$g(1) = 2; h(1) = \sqrt{g(2)} = \sqrt{0.5} \approx 0.7071$

(continues on page 21)

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(a) (continued) As $g(x) < 1$ for $x > 0$, the square root function $\sqrt{g(x)}$ is above $g(x)$ (the square root of a number less than 1 is more than the original number). Hence, the graph of $y = \frac{1}{f(x)}$ is shown (full). 	

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) (i) For $t=\tan\frac{x}{2}$, $\frac{dt}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)$ $=\frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)$ $=\frac{1}{2}\left(1+t^2\right)$ $\int\frac{1}{5+3\cos x}dx=\int\frac{1}{5+3\left(\frac{1-t^2}{1+t^2}\right)}\times\frac{2}{1+t^2}dt$ since $\frac{dx}{dt}=\frac{2}{1+t^2}, dx=\frac{2}{1+t^2}dt.$ $=\int\frac{1}{\frac{5\left(1+t^2\right)+3\left(1-t^2\right)}{1+t^2}}\times\frac{2}{1+t^2}dt$ $=\int\frac{1}{\frac{5\left(1+t^2\right)+3\left(1-t^2\right)}{1+t^2}}\times\frac{2}{1+t^2}dt$ $=\int\frac{2}{5\left(1+t^2\right)+3\left(1-t^2\right)}dt$ $=\int\frac{2}{8+2t^2}dt$ $=\int\frac{1}{4+t^2}dt$ $=\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+C$ $=\frac{1}{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right)+C$ $=\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)+C$	ME–T2 Further Trigonometric Identities, ME– C2 Further Calculus Skills ME11–3, 12–3 Bands E3–E4 • Gives the correct proof . . . . 3 • Achieves the complete integrand in terms of t . . . . 2 • Correctly finds $\frac{dx}{dt}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $A = \int_0^{\pi} \frac{1}{5+3\cos x}\,dx$ $= \left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)\right]_0^{\pi}$ However, $\tan\frac{\pi}{2}$ is undefined. Therefore, due to the symmetry of the curve about $x = \pi$: $A = \int_0^{\pi} \frac{1}{5+3\cos x}\,dx$ Let $t = \tan\frac{x}{2}$. When $x = 0, t = \tan 0 = 0$. When $x = \pi, t = \tan\frac{\pi}{2} = \infty$. $\int_0^{\infty} \frac{1}{4+t^2}\,dt = \left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{\infty}$ $= \frac{1}{2}\tan^{-1}(\infty) - \frac{1}{2}\tan^{-1}(0)$ $= \frac{1}{2}\cdot\frac{\pi}{2} - 0$ $= \frac{\pi}{4}$	ME– C3 Applications of Calculus ME12–1, 12–7 Bands E3–E4 • Gives the correct proof . . . . 2 • Achieves the result that $A = \frac{1}{2}\times\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)\right]_0^{2\pi}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) At $t = 0, u = 90$ m/s, $\theta = 30°$, $g = 10$ m/s^2 and $\underset{\sim}{a} = -10\underset{\sim}{j}$. $\underset{\sim}{v} = \int \underset{\sim}{a}\,dt$ $= \int -10\underset{\sim}{j}\,dt$ $= -10t\underset{\sim}{j} + C_1$ At $t = 0$, $\underset{\sim}{v} = 90\cos 30°\underset{\sim}{i} + 90\sin 30°\underset{\sim}{j}$ $= 45\sqrt{3}\underset{\sim}{i} + 45\underset{\sim}{j}$ $\therefore C_1 = 45\sqrt{3}\underset{\sim}{i} + 45\underset{\sim}{j}$ $\therefore \underset{\sim}{v} = 45\sqrt{3}\underset{\sim}{i} + (45 - 10t)\underset{\sim}{j}$ $\underset{\sim}{s} = \int \underset{\sim}{v}\,dt$ $= \int 45\sqrt{3}\underset{\sim}{i} + (45 - 10t)\underset{\sim}{j}\,dt$ $= 45\sqrt{3}t\underset{\sim}{i} + (45t - 5t^2)\underset{\sim}{j} + C_2$ At $t = 0, \underset{\sim}{s} = 0$. $\therefore C_2 = 0$ $\therefore \underset{\sim}{s} = 45\sqrt{3}t\underset{\sim}{i} + (45t - 5t^2)\underset{\sim}{j}$	ME–V1 Introduction to Vectors ME12–2 Bands E2–E4 • Gives the correct proof . . . . 2 • Correctly derives the velocity vector 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Let t, in seconds, be the time travelled by the golf ball. The time travelled by the stone is then $t^* = (t-5)$ seconds. At $t^* = 0, u = V$ m/s, $\theta = 0°$, $g = 10$ m/s^2 and $\underset{\sim}{a} = -10\underset{\sim}{j}$. $\underset{\sim}{v} = \int \underset{\sim}{a}\,dt^*$ $= \int -10\underset{\sim}{j}\,dt^*$ $= -10t^*\underset{\sim}{j} + C_1$ At $t^* = 0, v = V\cos 0\underset{\sim}{i} + V\sin 0\underset{\sim}{j} = V\underset{\sim}{i}$. $\therefore C_1 = V\underset{\sim}{i}$ $\therefore \underset{\sim}{v} = V\underset{\sim}{i} - 10t^*\underset{\sim}{j}$ $\underset{\sim}{s} = \int \underset{\sim}{v}\,dt$ $= \int V\underset{\sim}{i} - 10t^*\underset{\sim}{j}\,dt^*$ $= Vt^*\underset{\sim}{i} - 5t^{*2}\underset{\sim}{j} + C_2$ At $t^* = 0, \underset{\sim}{s} = 20\underset{\sim}{j}$. $\therefore C_2 = 20\underset{\sim}{j}$ $\therefore \underset{\sim}{s}_2 = Vt^*\underset{\sim}{i} + \left(20 - 5t^{*2}\right)\underset{\sim}{j}$ Substitute in $t^* = (t-5)$ since $\frac{dt^*}{dt} = 1$: *Note: This substitution does not affect any of the integration processes above.* $\therefore \underset{\sim}{s}_2 = V(t-5)\underset{\sim}{i} + \left(20 - 5(t-5)^2\right)\underset{\sim}{j}$, where $\underset{\sim}{s}_2$ is the position vector of the stone, and $\underset{\sim}{s}_1 = 45\sqrt{3}t\underset{\sim}{i} + \left(45t - 5t^2\right)\underset{\sim}{j}$, where $\underset{\sim}{s}_1$ is the position vector of the golf ball. At the time of collision, the two position vectors are equal: $45\sqrt{3}t\underset{\sim}{i} + \left(45t - 5t^2\right)\underset{\sim}{j} = V(t-5)\underset{\sim}{i} + \left(20 - 5(t-5)^2\right)\underset{\sim}{j}$ (continues on page 26)	ME–V1 Introduction to Vectors ME12–2 Bands E3–E4 • Gives the correct proof .... 3 • Finds the displacement/position vector for the stone .... 2 • Finds the velocity vector of the stone 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) (continued) $\therefore 45\sqrt{3}t = V(t-5)$ (1) and $(45t-5t^2) = \left(20-5(t-5)^2\right)$ (2) Using (2): $9t-t^2 = 4-(t-5)^2$ $9t-t^2 = 4-t^2+10t-25$ $9t = 10t-21$ $t = 21$ Therefore, the two objects collided after the golf ball has travelled for 21 seconds.	
(iii) Substituting $t = 21$ into $45\sqrt{3}t = V(t-5)$ (from part (c) (ii)) gives: $45\sqrt{3}\times 21 = V(21-5)$ $V = \dfrac{45\sqrt{3}\times 21}{21-5}$ $= \dfrac{945\sqrt{3}}{16}$ m/s Using the velocity vector $\underset{\sim}{v} = V\underset{\sim}{i} - 10t^*\underset{\sim}{j}$, substitute $V = \dfrac{945\sqrt{3}}{16}$ and $t^* = 21-5 = 16$: $\underset{\sim}{v} = \dfrac{945\sqrt{3}}{16}\underset{\sim}{i} - 10\times 16\underset{\sim}{j}$ Therefore, the speed of the stone at the time of collision is: $\lvert\underset{\sim}{v}\rvert = \sqrt{\left(\dfrac{945\sqrt{3}}{16}\right)^2 + (160)^2}$ $= 189.908$ ≈ 190 m/s	ME–V1 Introduction to Vectors ME12–2 Bands E3–E4 • Gives the correct solution . . . . 2 • Finds the initial speed of the stone. 1