Pymble Ladies' College
Mathematics Department

St Catherine's School

Year: 12
Subject: Extension I Mathematics
Time Allowed: 2 hours
(plus 5 mins reading time)
Date: August 2001

Exam number:______________

Directions to candidates:

- All questions are to be attempted.
- All questions are of equal value.
- All necessary working must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Each question attempted should be started on a new **booklet**.
- Approved calculators and geometrical instruments are required.
- This page is a cover sheet for **Section A**. Write a cover page for **Section B and C** and include your number.
- Hand in your work in **3 bundles**:
  Section A Questions 1, 2 and 3.
  Section B Questions. 4 and 5
  Section C Questions. 6 and 7.

TEACHER'S USE ONLY Total Marks
A
B
C
TOTAL

Question 1:

(a) Solve for x: $\dfrac{3}{x+5} \le 1$ (2 marks)

(b) A root of $e^x - x^2 = 0$ lies near $x = -0.5$. Use Newton's Method once to find a better approximation. (3 marks)

(c) Consider the function $f(x) = 2\sin^{-1}\dfrac{x}{2}$.

(i) Find the exact value of $f(\sqrt{2})$. (1 mark)

(ii) What is the domain and range of $f(x)$. (2 marks)

(iii) Sketch $f(x)$. (1 mark)

(iv) Find the equation of the tangent to the curve at $x = \sqrt{2}$ (3 marks)

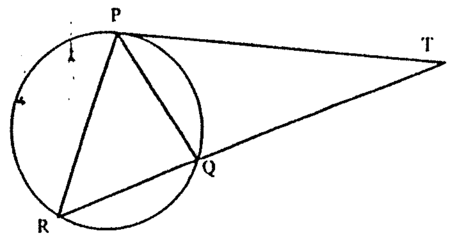
Question 2:

(a) Solve for x: $2|x-1| = 4x - 1$. (3 marks)

(b) If α, β, γ are the roots of $x^3 - 3x^2 - 2x + 1 = 0$, find:

(i) $\alpha + \beta + \gamma$ (1 mark) (ii) $\alpha\beta\gamma$ (1 mark)
(iii) $\alpha\beta + \alpha\gamma + \beta\gamma$ (1 mark) (iv) $\alpha^2 + \beta^2 + \gamma^2$ (2 marks)

(c) PT is a tangent to the circle PRQ. RQ is a secant intersecting the circle in Q and R. The line RQ intersects PT at T.



(3 marks)

Question 3:

(a) Find $\int x^2(x^3-5)^5\,dx$ using the substitution $u = x^3 - 5$ (3 marks)

(b) The angle between the lines $y = mx$ and $y = \frac{1}{2}x$ is 45° (3 marks)
Find two possible values of m.

(c) The rate at which a body cools in air is proportional to the difference between its temperature T and the temperature C of its surroundings. That is:

$\frac{dT}{dt} = -k(T - C)$ where t is the time in hours and k is a positive constant.

i) Show that $T = C + Ae^{-kt}$ is a solution to the differential equation above (where A is a real number). (2 marks)

A heated piece of metal is initially 90°C but cools to 70°C in one hour.
Given the surroundings are 25°C find:

ii) the constants A and C (3 marks)
iii) the temperature of the metal after three hours (to the nearest degree). (1 mark)

Question 4:

(a) Find $\int \cos^2 2x\,dx$ (2 marks)

(b) A team of 4 is to be chosen from 5 boys and 6 girls. How many teams are possible if:

(i) there are no restrictions (1 mark)

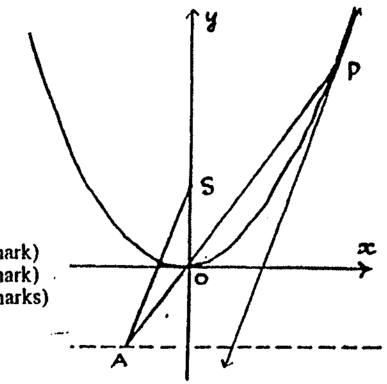
(ii) the shortest girl must be included (1 mark)

(c) Show by Mathematical Induction that the following statement: (4 marks)

$$(1\times2)+(2\times3)+(3\times4)+\ldots\ldots\ldots\ldots+(n-1)(n) = \frac{(n-1)n(n+1)}{3}$$

is true for all integers $n \geq 2$.

(d) The diagram shows the parabola with parametric coordinates $x = 2ap$ and $y = ap^2$
P is a point on the parabola. S is the focus and A is a point on the directrix.



The straight line drawn from point P $(2ap\ ap^2)$ on the parabola through the vertex at O(0,0) intersects the directrix at A.

(i) Find the equation of line PO (1 mark)
(ii) Show that the coordinates of A are $\left(\frac{-2a}{p}, -a\right)$ (1 mark)
(iii) Prove that AS is parallel to the tangent at P. (2 marks)

Question 5:

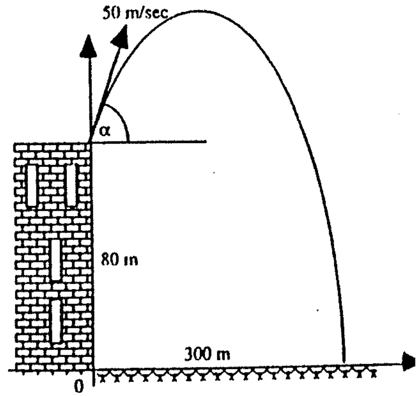
(a) Eight different coloured beads are arranged so that two particular colours are next to each other. In how many ways can they be arranged in:
(You may leave your answer in factorial notation)

(i) a line (1 mark)
(ii) a circle (1 mark)
(iii) a necklace (1 mark)

(b) Find the exact value of

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}}$$ (2 marks)

(c) Kelly throws a stone at an angle of elevation of α from the top of a tower 80 m high at an initial velocity of 50 m/sec, as in the diagram.



The acceleration due to gravity is assumed to be 10m/sec^2. Take the origin to be the base of the tower.

(i) Given that $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = -10$ show that

$x = 50t\cos\alpha$ and

$y = -5t^2 + 50t\sin\alpha + 80$,

where x and y are the horizontal and vertical displacements of the stone in metres from the origin at time t seconds after throwing. (3 marks)

(ii) Kelly wants the stone to land in the sea 300 metres from the base of the tower.

Question 6:

(a) Differentiate $f(x) = \ln(\tan^3 x)$ (2 marks)

(b) Consider the function $f(x) = \dfrac{x^2}{x^2 - 2}$

(i) Give the equations of any horizontal and vertical asymptotes. (2 marks)

(ii) Find the x and y intercepts if they exist. (1 mark)

(iii) Given that this curve has only one stationary point and it is a local maximum, find its coordinates. (2 marks)

(iv) Sketch the curve, indicating on your sketch all important features. (2 marks)

(c) Show that the equation $2x^3 - 5x^2 + 3x - 2 = 0$ has only one real root. (3 marks)

Question 7:

(a) The displacement x metres of a particle from the origin is in simple harmonic motion and is given by $x = 5\cos \pi t$, where the time t is in seconds.

(i) What is the period of the oscillation? (1 mark)

(ii) What is the speed v of the particle as it moves through the origin? (2 marks)

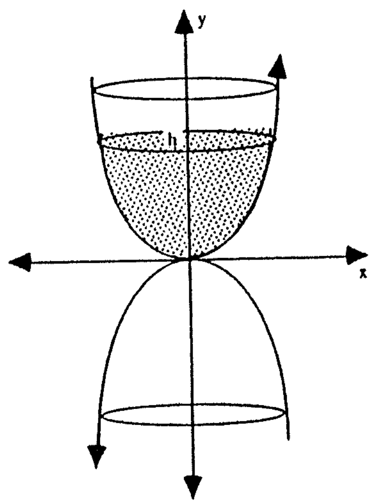
(b) Show that $(x-1)(x-2)$ is a factor of $P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + 2^n - 2^m$ where m, n are positive integers. (2 marks)

Question 7 (continued):

(c) An eggtimer has the same shape as the curve $y = x^3$ rotated about the y axis. The top half of the eggtimer is filled with sand to a depth of h units.

(i) Show that the volume V of sand needed is given by $V = \frac{3\pi}{5}\sqrt[3]{h^5}$. (3 marks)

The rate with which the sand falls into the bottom of the timer is found to be proportional to the height h of the sand in the top of the eggtimer.
(ie $\frac{dV}{dt} = kh$ where k is a real number)



(ii) Find the exact rate at which the height of the sand in the top of the eggtimer is falling when $h = \frac{27}{8}$ cm if the sand is flowing through the neck at 1 cm^3 / minute when $h = 5$ cm

(4 marks)