Western Mathematics Exams

2024	TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen.
- Calculators approved by NESA may be used.
- A reference sheet is provided at the back of this paper.
- In Questions in Section II, show relevant mathematical reasoning and/or calculations.
- Write your Name and Student Number on the Question 11 Writing Booklet attached.

Total Marks: 100

Section I – 10 marks (pages 2 – 4)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II – 90 marks (pages 5 – 11)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. What are the centre and radius of the sphere with equation :

   $x^2 + y^2 + z^2 - 2x + 6y + 4z - 2 = 0$?

   A. Centre (1, –3, –2), Radius = 4

   B. Centre (–1, 3, 2), Radius = 4

   C. Centre (1, –3, –2), Radius = 16

   D. Centre (–1, 3, –2), Radius = 16

2. Consider the following statement.

   "If water turns to ice, then the temperature is below 0° C."

   Which of the following is the contrapositive of this statement?

   A. If the temperature is below 0° C, then water turns to ice.

   B. If water does not turn to ice, then the temperature is not below 0°.

   C. If water turns to ice, then the temperature is not below 0° C.

   D. If the temperature is not below 0° C, then water does not turn to ice.

3. A particle is moving in Simple Harmonic Motion in a straight line.

   In one minute of its motion, it completes exactly 15 oscillations and travels 120 metres.

   What is the amplitude of the motion?

   A. 2 metres

   B. 4 metres

   C. 8 metres

   D. 16 metres

4. If $\omega = \frac{1+2i}{3+4i}$, then what is the modulus of ω?

A. $\frac{\sqrt{13}}{\sqrt{7}}$

B. $\frac{\sqrt{13}}{5}$

C. $\frac{\sqrt{5}}{5}$

D. $\frac{5}{\sqrt{5}}$

5. Find $\int \frac{x^2+2x-3}{x+1} dx$.

A. $-4\ln(x+1)+c$

B. $x+1-4\ln(x+1)+c$

C. $2x+x-4\ln(x+1)+c$

D. $\frac{x^2}{2}+x-4\ln(x+1)+c$

6. A particle is performing Simple Harmonic Motion in a straight line.

At time t seconds it has displacement x metres from a fixed-point O on the line where x is given by $x = 4sin^2t - 1$.

Where is the centre of motion?

A. $x = -1$

B. $x = 0$

C. $x = 1$

D. $x = 2$

7. If $\frac{1}{x(x+1)} \equiv \frac{a}{x} + \frac{b}{x+1}$, what are the values of a and b?

A. $a = -1, b = -1$

B. $a = 1, b = -1$

C. $a = -1, b = 1$

D. $a = 1, b = 1$

8. Consider the vector $\underset{\sim}{a} = 2\underset{\sim}{i} - b\underset{\sim}{j} + 6\underset{\sim}{k}$.

If $\underset{\sim}{a}$ has a magnitude of $4\sqrt{3}$ units, what is the exact value of b ?

A. $2\sqrt{2}$

B. $4\sqrt{2}$

C. 4

D. 8

9. Which of the following is a square root of $16 + 30i$.

A. $4 + \sqrt{30}i$

B. $5 + 3i$

C. $5 - 3i$

D. $-5 + 3i$

10. Evaluate : $\displaystyle\int_1^e \frac{(\ln x)^3}{x}\, dx$.

A. $\frac{1}{5}$

B. $\frac{1}{4}$

C. $\frac{1}{3}$

D. $\frac{1}{2}$

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 writing booklet.

(a) On the Argand diagram provided in the Question 11 Writing Booklet, sketch the region represented by the intersection of the inequalities: **3**

$$1 \leq |z+1| < 2 \text{ and } \frac{\pi}{4} < \arg(z+1) < \frac{2\pi}{3}.$$

(b) Consider the complex numbers $z = 4\sqrt{3} + 4i$ and $\omega = 2(cos\frac{\pi}{4} + i\,sin\,\frac{\pi}{4})$.

(i) Find the value of $z\omega$, giving your answer in the form $re^{i\theta}$. **2**

(ii) Hence, or otherwise, find $\sqrt[4]{z\,\omega}$. **1**

(c) If $(1 - i)$ is a root of the equation $z^2 - (2 + i)z + c = 0$.

(i) Show that the other root is $(1 + 2i)$. **1**

(ii) Find the value of c. **1**

Question 11 continues on page 6

Question 11 (continued)

(d) Find the equation of the vector which begins at the point $(3, 4, 5)$ and passes through the point $(-1, 1, -2)$. **2**

(e) Draw a neat sketch of the locus represented by $3|z - (2 + 2i)| = |z - (6 + 6i)|$. **3**

(f) Express $\frac{3 - 6i}{4 - i}$ in the form $a + ib$, where a and b are real. **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 writing booklet.

(a) (i) Find $\int x^2 e^{-x}\, dx$. **3**

(ii) Find $\int \frac{dx}{\sqrt{6-5x-x^2}}$. **3**

(iii) Evaluate $\int_0^{\frac{\pi}{6}} \sin 2x \cos^3 2x\, dx$. **3**

(b) Find the size of the angle between the vectors $\hat{P} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\hat{Q} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, correct to the nearest degree. **2**

(c) A line *l* has vector equation $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$. **2**

A point $B(x, 0, 5)$ lies on the line *l*.

Find the value of *x*.

(d) Prove that for all real numbers *x* and *y*, where $x \neq y$, **2**

$x^4 + y^4 + z^4 > x^2y^2 + x^2z^2 + z^2y^2$.

End of Question 12

Question 13 (15 marks) Use the Question 13 writing booklet.

(a) Solve the equation $x^5 + 2x^4 + 3x^3 + 6x^2 - 4x - 8 = 0$ given that $x = 2i$ is a root. **3**

(b) If $a, b > 0$, prove $\left(\frac{a+b}{2}\right)^3 \leq \frac{a^3 + b^3}{2}$ with equality when a = b. **3**

(c) Let n be a positive integer and let $I_n = \int_1^2 (\log_e x)^n \; dx$.

(i) Prove that $I_n = 2(\log_e 2)^n - nI_{n-1}$. **2**

(ii) Hence evaluate $\int_1^2 (\log_e x)^3 \; dx$ as a polynomial in $\log_e 2$. **3**

(d) (i) Given $z = \cos\theta + i\sin\theta$, show that $z^n - z^{-n} = 2\cos n\theta$. **1**

(ii) Hence, express $\cos^6\theta$ in terms of $\cos n\theta$. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 writing booklet.

(a) (i) If $\frac{x^2+1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$, find the values of A, B and C. **3**

(ii) Hence find $\int \frac{x^2+1}{x^3+2x^2+x}\,dx$. **2**

(b) (i) Using the process of Mathematical Induction, prove that: **3**

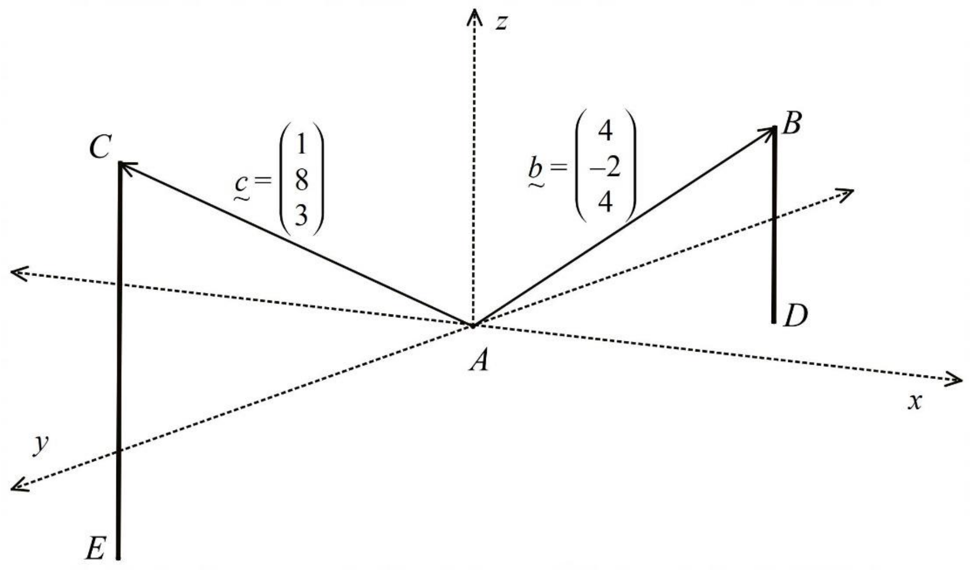
$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots + \frac{1}{n^2} \le 2 - \frac{1}{n}$ for $n \ge 1$.

(ii) Hence show that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots + \frac{1}{99^2} \le 1 \cdot 99$. **2**

(c) Find the intersections of the line $\underset{\sim}{r} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} + t(2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k})$ with the sphere $(x-3)^2 + (y-2)^2 + (z+1)^2 = 36$. **3**

(d) Two vertical posts CE and BD stand on a horizontal plane ADE. **2**

Using A as the origin, the vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ represent the locations of the top of each post, B and C.



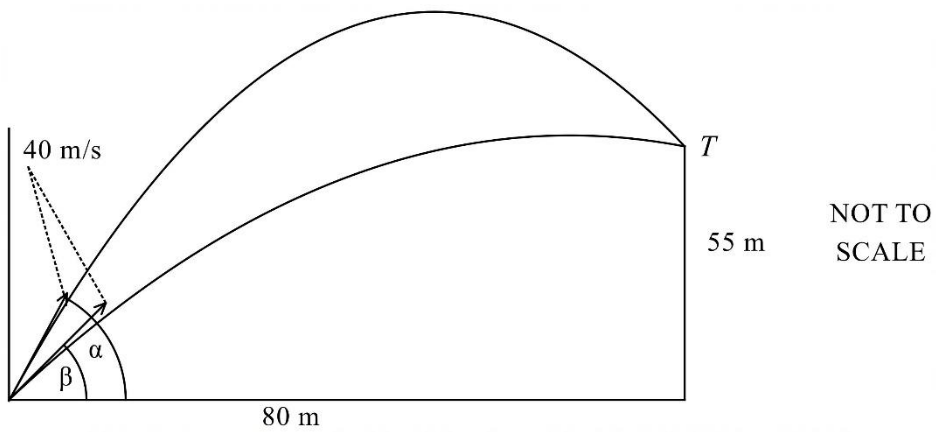
Show that $\Delta\, ABC$ is right angled at A.

End of Question 14

Question 15 (15 marks) Use the Question 15 writing booklet.

(a) A missile is fired into the air at a velocity of 40 m s^{-1} at an angle of α to the horizontal.

Shortly afterward a second missile is fired from the same point with the same velocity, but at a different angle, β, as shown in the diagram.



The missiles both simultaneously hit their target, *T*, which is at a horizontal distance of 80 m from the point of firing and at a vertical height of 55 m.

(i) Show that the equation of the path of the first missile is given by: **2**

$y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2 x$.

(ii) Show that $\tan\alpha = \frac{5}{2}$ and $\tan\beta = \frac{3}{2}$. **2**

Use g = 10 m s^{-2} as an approximation for the acceleration due to gravity.

(iii) Determine the time elapsed between the firing of the two missiles. **3**

(b) Prove by contradiction that for a ≥2; $\sqrt{a} + \sqrt{a+2} > \sqrt{a+8}$. **3**

(c) (i) If $x \geq 0,$ show that $\frac{x}{x^2+9} \leq \frac{1}{6}$. **2**

(ii) By integrating both sides of the inequality above with respect to *x* between the limits $x = 0$ and $x = \alpha$, show that: **3**

$e^{\frac{1}{3}\alpha} \geq \frac{1}{9}\alpha^2 + 1, \alpha \geq 0.$

End of Question 15

Question 16 (15 marks) Use the Question 16 writing booklet.

(a) A body of mass one kilogram is projected vertically upwards at a speed of 40 metres per second. The particle is under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{20}v^2$, where v is the magnitude of the particle's velocity at that time.

Assuming the acceleration due to gravity is 10 metres per second per second.

(i) Show that while the body is travelling upwards the equation of the motion is: **1**

$$\ddot{x} = -\left(10 + \frac{1}{20}v^2\right).$$

(ii) Taking $\ddot{x} = v\frac{dv}{dx}$, calculate the greatest height reached by the particle. **2**

(iii) Taking $\ddot{x} = \frac{dv}{dt}$, calculate the time taken to reach the greatest height. **2**

(b) Having reached its greatest height, the particle in part (a) falls back to its starting point. The particle is still under the effects of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{20}v^2$.

(i) Write down the equation of motion of the particle as it falls. **1**

(ii) Find the speed of the particle when it returns to its starting point. **3**

(c) It is known that the arithmetic mean of three numbers is greater than the geometric mean of those numbers. That is:

$$\frac{a+b+c}{3} \geq \sqrt[3]{(abc)}$$

(i) Given $a + b + c = t$ and $(a, b, c > 0)$, prove that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{t}$. **3**

(ii) Hence, given $a, b, c > 0$ and $a + b + c = 1$, prove $\left\{\frac{1}{a} - 1\right\} \cdot \left\{\frac{1}{b} - 1\right\} \cdot \left\{\frac{1}{c} - 1\right\} \geq 8$. **3**

End of Paper

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Mathematics Advanced

Mathematics Extension 1

Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

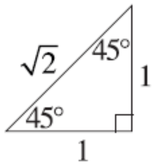
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

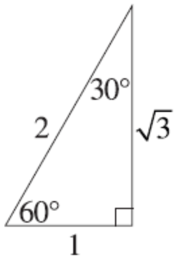
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

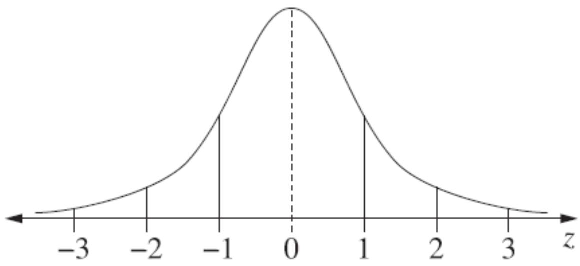
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x) = \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2\,,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

2024 Trial HSC Examination
Mathematics Extension 2 Course

Name ________________________________ Teacher ________________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	$2 + 4 =$	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A ⊗ B ⊗ correct C ○ D ○

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

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Student Name | Student Number

Western Mathematics Exams

2024

TRIAL HIGHER SCHOOL CERTIFICATE

EXAMINATION

Mathematics Extension 2

Writing Booklet

Question 11

Instructions

- Use this booklet for Question 11 ⇨ $\boxed{11}$

- Write the number of this booklet and the number of booklets that you have used for this question (e.g. $\boxed{1}$ of $\boxed{3}$) ⇨ $\boxed{}$ of $\boxed{}$
  this booklet | number of booklets for this question

- Write your name and Student Number at the top of this page.

- Write in a black pen

- You may ask for an extra writing booklet if you need more space

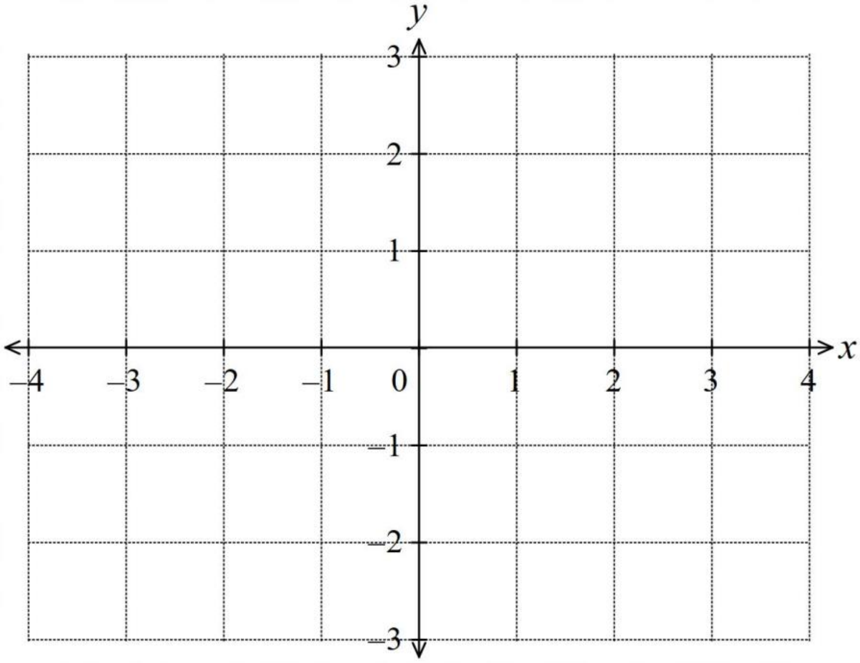
- If you have not attempted the question(s), you must still hand in the writing booklet with “NOT ATTEMPTED” written clearly on the front cover.

- You may not take any writing booklets, used or unused, from the examination room.

Start here for

Question Number 11

(a)



Tick this box if you have continued this answer in another writing booklet.

Western Mathematics Exams

2024
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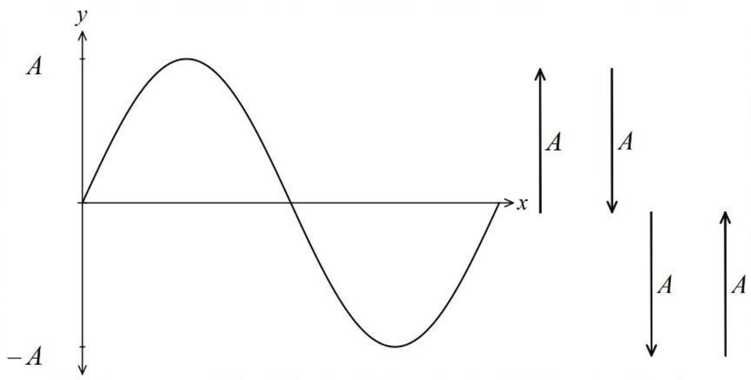
EXAMINATION

Mathematics
Extension 2

SOLUTIONS

WME 2024 Ext 2 Multiple Choice Worked Solutions

No	Working	Answer
1	$x^2 + y^2 + z^2 - 2x + 6y + 4z - 2 = 0$ $x^2 - 2x + y^2 + 6y + z^2 + 4z = 2$ $x^2 - 2x + 1 + y^2 + 6y + 9 + z^2 + 4z + 4 = 2 + 1 + 9 + 4$ $(x-1)^2 + (y+3)^2 + (z+2)^2 = 16$ Centre (1, –3, –2) Radius = 4	**A**
2	For the statement *If P then Q*, the contrapositive is *If not Q then not P*. So, for "If water turns to ice, then the temperature is below 0° C," the contrapositive is: "If the temperature is not below 0° C, then water does not turn to ice." NB "If the temperature is below 0° C, then water turns to ice," is the converse.	**D**
3	Let the amplitude be A metres. In completing one oscillation, the particle will travel four times the amplitude, i.e. $4A$ metres.  The distance travelled in 15 oscillations = 15 × 4A = 120 $\therefore A = 2$metres	**A**
4	$\omega = \frac{1+2i}{3+4i} \times \frac{3-4i}{3-4i} \quad = \frac{3-4i+6i+8}{9+16}$ $= \dfrac{11+2i}{25}$ $\lvert\omega\rvert = \left\lvert\dfrac{11+2i}{25}\right\rvert$ $= \sqrt{\left(\dfrac{11}{25}\right)^2 + \left(\dfrac{2}{25}\right)^2}$ $= \sqrt{\dfrac{121+4}{625}}$ $= \sqrt{\dfrac{125}{625}}$ $= \dfrac{5\sqrt{5}}{25}$ $= \dfrac{\sqrt{5}}{5}$ Alternatively, $\lvert\omega\rvert = \dfrac{\lvert 1+2i\rvert}{\lvert 3+4i\rvert} = \dfrac{\sqrt{5}}{5}$	**C**

5	$\int \frac{x^2+2x-3}{x+1}\,dx. = \int \frac{x^2\ +\ 2x+1-\ 4}{x\ +\ 1}dx$ $= \int \frac{(x+1)^2-\ 4}{x\ +\ 1}dx$ $= \int \left(x+1-\frac{4}{x+1}\right)\,dx$ $= \frac{x^2}{2}+x-4\ln(x+1)+c$	**D**
6	$x = 4\sin^2 t-1$ $= 2(1-\cos 2t)-1$ $= 1-2\cos 2t$ $\therefore$ centre is at $x = 1$.	**C**
7	$\frac{1}{x(x+1)} \equiv \frac{a}{x}+\frac{b}{x+1}$ $1 \equiv a(x+1)+bx$ $Let\ \ x = 0 \qquad 1 = a$ Let $x = -1 \quad 1 = -b, \qquad \therefore b = -1$	**B**
8	$\lvert \underset{\sim}{a} \rvert = \sqrt{2^2+\ (-b)^2+\ 6^2} = 4\sqrt{3}$ $4+\ b^2+36 = 48$ $b^2+40 = 48$ $b^2 = 8$ $b = \sqrt{8} = 2\sqrt{2}$	**A**
9	Let $(x+iy)^2 = 16+30i$ Then $x^2+2xyi-\ y^2 = 16+30i$ $\therefore x^2-\ y^2 = 16\ \ -\ -\ -\ \ (1)$ And $2xy =\ 30$ $(x^2+\ y^2)^2 =\ (x^2-\ y^2)^2+4x^2y^2$ $=\ 16^2+\ (30)^2$ $= 1\,156$ $\therefore\ x^2+\ y^2 = 34 \qquad -\ -\ -\ (2)$ (1) + (2) $\ 2x^2 = 50$ $x^2 = 25$ $x =\ \pm 5$ (2) – (1) $\ 2y^2 = 18$ $y^2 = 9$ $y =\ \pm 3$ Since $2xy =\ 30$ Square roots are $\pm(5+3i)$, so $5+3i$ is a root. Can also be done by squaring the possible solutions and comparing, For example $A.\ \ \left(4+\sqrt{30i}\right)^2 = 16+8\sqrt{30}\,i\ -\ 30$ $= -14+8\sqrt{30}\,i$ So $4+\sqrt{30i}$ **is not** a root of $16+30i$ $B.\ \ (5+3i)^2 = 25+30i-9$ $= 16+30i$ So $5+3i$ **is** a root of $16+30i$	**B**

10	Let $u = \log_e x$ $\frac{du}{dx} = \frac{1}{x}, \therefore du = \frac{1}{x}\, dx$ when $x = e, u = \log_e e = 1$ when $x = 1, u = \log_e 1 = 0$ $$\int_1^e \frac{(\log_e x)^3}{x}\, dx = \int_0^1 u^3\, du$$ $$= \left[\frac{1}{4}u^4\right]_0^1$$ $$= \frac{1}{4}[1^4 - 0^4]$$ $$= \frac{1}{4}$$	**B**

Trial HSC Examination
2024

Mathematics Extension 2

Name ______________________________ Teacher ______________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	2 + 4 =	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

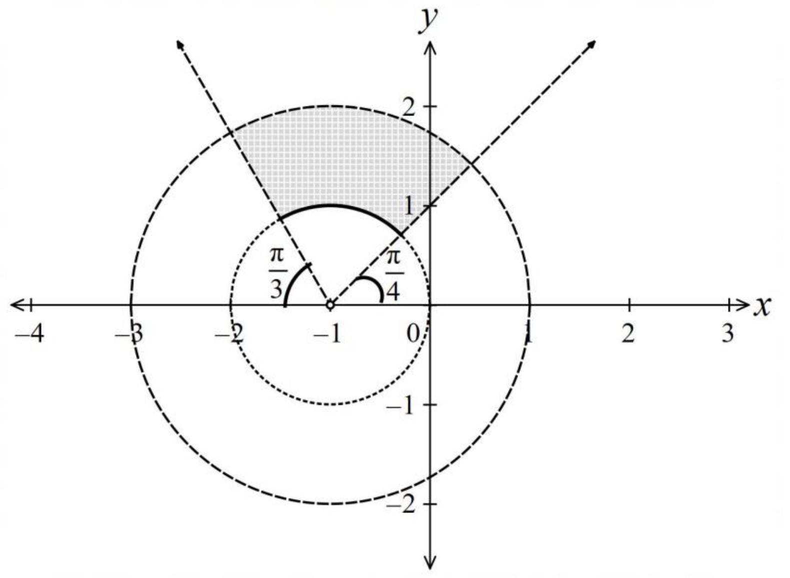
If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ●	B ~~●~~	C ○	D ○

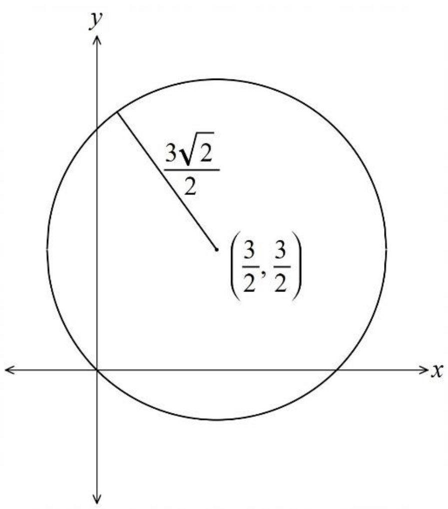
If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A ~~●~~	B ~~●~~ correct	C ○	D ○

1.	A ●	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ●
3.	A ●	B ○	C ○	D ○
4.	A ○	B ○	C ●	D ○
5.	A ○	B ○	C ○	D ●
6.	A ○	B ○	C ●	D ○
7.	A ○	B ●	C ○	D ○
8.	A ●	B ○	C ○	D ○
9.	A ○	B ●	C ○	D ○
10.	A ○	B ●	C ○	D ○

11		WME Ext 2 HSC 2024 **Question 11 Worked Solutions**	Marks	**Allocation and Comments**
(a)		$1 \leq \|z+1\| < 2$ and $\frac{\pi}{4} < \arg(z+1) < \frac{2\pi}{3}$.  	**3**	3 marks for correct graph 2 marks for graph with minor error such as incorrect line marking or shading 1 mark for one correct graph
(b)	(i)	$z = 4\sqrt{3} + 4i \quad r = \sqrt{(4\sqrt{3})^2 + (4)^2} = \sqrt{64} = 8$ $\theta = \tan^{-1}\left(\frac{4}{4\sqrt{3}}\right) = \frac{\pi}{6}$ $so \;\; z = 8e^{i\frac{\pi}{6}}$. $\omega = 2cis\,\frac{\pi}{4}, so \;\; \omega = 2e^{i\frac{\pi}{4}}$ $\therefore z\omega = \left(8e^{i\frac{\pi}{6}}\right)\left(2e^{i\frac{\pi}{4}}\right) = 16e^{i\frac{5\pi}{12}}$	**2**	2 marks for correct answer 1 mark for correct value of r or θ
	(ii)	$\sqrt[4]{z\omega} = \sqrt[4]{16e^{i\frac{5\pi}{12}}} = 2e^{i\frac{5\pi}{48}}$	**1**	1 mark for correct answer

11		WME Ext 2 HSC 2024 Question 11 Worked Solutions	Marks	Allocation and Comments
(c)	(i)	Let the other root be $(x+iy)$. Sum of roots $= -\frac{b}{a}$ $\therefore (1-i)+(x+iy) = \dfrac{2+i}{1}$ $x+1 = 2,\ y-1=1$ $x = 1,\ y=2$ $\therefore$ other root is $(1+2i)$	**1**	1 mark for correct answer
	(ii)	Product of the roots is $\frac{c}{a}$ $\therefore\ (1-i)(1+2i) = c$ $c = 1+2i-i-2i^2 = 1+2i-i+2$ $c = 3+i$	**1**	1 mark for correct answer
(d)		Let $\hat{a} = \begin{pmatrix}3\\4\\5\end{pmatrix}$ and $\hat{b} = \begin{pmatrix}-1\\1\\-2\end{pmatrix}$ The vector joining A to B is $\overrightarrow{AB} = \hat{b}\cdot\hat{a} = \begin{pmatrix}-1-3\\1-4\\-2-5\end{pmatrix} = \begin{pmatrix}-4\\-3\\-7\end{pmatrix}$ The general vector beginning at A and passing through B is $r = \begin{pmatrix}3\\4\\5\end{pmatrix} + \lambda\begin{pmatrix}-4\\-3\\-7\end{pmatrix}$	**2**	2 marks for correct answer 1 mark for vector $\overrightarrow{AB}$

11		WME Ext 2 HSC 2024 Question 11 Worked Solutions	Marks	Allocation and Comments
(e)		$3\|z-(2+2i)\| = \|z-(6+6i)\|$ $3\|(x-2)+i(y-2)\| = \|(x-6)+i(y-6)\|$ $9(x-2)^2+9(y-2)^2 = (x-6)^2+(y-6)^2$ $9x^2-36x+36+9y^2-36y+36 = x^2-12x+36+y^2-12y+36$ $8x^2+8y^2-24x-24y=0$ $x^2+y^2-3x-3y=0$ $x^2-3x+\left(\frac{-3}{2}\right)^2+y^2-3y+\left(\frac{-3}{2}\right)^2 = \left(\frac{-3}{2}\right)^2+\left(\frac{-3}{2}\right)^2$ $(x-\frac{3}{2})^2+(y-\frac{3}{2})^2 = \frac{18}{4}$ Which is a circle, centre $(\frac{3}{2},\frac{3}{2})$ with radius $\sqrt{\frac{18}{4}} = \frac{3\sqrt{2}}{2}$. 	3	3 marks for correct graph 2 marks for correct equation or equivalent merit 1 mark for some correct algebraic reasoning toward equation or equivalent merit
(f)		$\frac{3-6i}{4-i} \times \frac{4+i}{4+i} = \frac{12+3i-24i+6}{16+1}$ $= \frac{18-21i}{17}$, which is in the form $a+ib$ where $a = \frac{18}{17}$ and $b = -\frac{21}{17}$	2	2 marks for correct answer 1 mark for multiplication by conjugate or equivalent merit

12		**WME Ext 2 HSC 2024** **Question 12 Worked Solutions**	Marks	**Allocation and Comments**
(a)	(i)	Using integration by parts. $\int uv' = uv - \int vu'$ Let $u = x^2 \quad v' = e^{-x}$ $u' = 2x \quad v = -e^{-x}$ $\int x^2 e^{-x} dx = uv - \int vu'$ $= -x^2 e^{-x} - \int 2x\,(-e^{-x}) dx$ $= -x^2 e^{-x} + \int 2x\,(e^{-x}) dx$ Using Integration by parts again Let $u = 2x \quad v' = e^{-x}$ $u' = 2 \quad v = -e^{-x}$ $\int x^2 e^{-x} dx = -x^2 e^{-x} - 2xe^{-x} - \int -2e^{-x} dx$ $= -x^2 e^{-x} - 2xe^{-x} + -2e^{-x} + c$ $= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$ $= -e^{-x}(x^2 + 2x + 2) + c$	**3**	3 marks for correct answer (either of the last two lines) 2 marks for two uses of Integration by parts with errors or equivalent merit 1 mark for one use of integration by parts or equivalent merit
	(ii)	$6 - 5x - x^2 = 6 - (x^2 + 5x)$ $= 6 - \left(x^2 + 5x + \frac{25}{4}\right) + \frac{25}{4}$ $= \frac{49}{4} - \left(x + \frac{5}{2}\right)^2$ $\int \frac{dx}{\sqrt{6 - 5x - x^2}} = \int \frac{dx}{\sqrt{\frac{49}{4} - \left(x + \frac{5}{2}\right)^2}}$ $= \sin^{-1} \frac{\left(x + \frac{5}{2}\right)}{\frac{7}{2}} + c$ $= \sin^{-1} \left[\frac{2\left(x + \frac{5}{2}\right)}{7}\right] + c$ $= \sin^{-1} \left(\frac{2x + 5}{7}\right) + c$	**3**	3 marks for correct answer 2 marks for completing the square and forming the integral or equivalent merit 1 mark for completing the square and attempt to form integral or equivalent merit

<table>
<tr><th colspan="2">12</th><th>WME Ext 2 HSC 2024
Question 12 Worked Solutions</th><th>Marks</th><th>Allocation and Comments</th></tr>
<tr><td></td><td>(iii)</td><td>$$\int_0^{\frac{\pi}{6}} \sin 2x \cos^3 2x \, dx$$
Let $u = \cos 2x$
$du = -2\sin 2x$
When $x = 0, u = 1$
$x = \frac{\pi}{6}, u = \frac{1}{2}$
$$\begin{aligned}\int_0^{\frac{\pi}{6}} \sin 2x \cos^3 2x \, dx &= -\frac{1}{2}\int_0^{\frac{\pi}{6}} -2\sin 2x \cos^3 2x \, dx \\ &= -\frac{1}{2}\int_1^{\frac{1}{2}} u^3 du \\ &= \frac{1}{2}\int_{\frac{1}{2}}^{1} u^3 du \\ &= \frac{1}{2}\left[\frac{u^4}{4}\right]_{\frac{1}{2}}^{1} \\ &= \frac{1}{8}\left[(1)^4 - \left(\frac{1}{2}\right)^4\right] \\ &= \frac{1}{8}\left(1 - \frac{1}{16}\right) \\ &= \frac{1}{8} \times \frac{15}{16} \\ &= \frac{15}{128}\end{aligned}$$</td><td>3</td><td>3 marks for correct answer

2 marks for changing limits and correct substitution or equivalent merit

1 marks for changing limits and subsequent errors or equivalent merit</td></tr>
<tr><td>(b)</td><td></td><td>$|P| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$
$|Q| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$
$P \cdot Q = 2(2) + 1(-3) + 3(1) = 4$
Now: $P \cdot Q = |P||Q| \, cos\theta$
So, $\cos\theta = \frac{P \cdot Q}{|P||Q|} = \frac{4}{\sqrt{14}\sqrt{14}} = \frac{2}{7}$
$\theta = cos^{-1}\left(\frac{2}{7}\right)$
$= 73°$ (nearest degree)</td><td>2</td><td>2 marks for correct answer

1 mark for correct magnitudes of $|P|$ and $|Q|$ and substituting into formula or equivalent merit.</td></tr>
<tr><td>(c)</td><td></td><td>$$\begin{pmatrix} x \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$
Form Parametric equations.
$x = -3t + 2,$
$2t + 4 = 0$
$-t + 3 = 5$
Here, $t = -2$
$\therefore x = -3(-2) + 2$
$x = 8$</td><td>2</td><td>2 marks for correct answer

1 mark for forming equations and finding correct value of t or equivalent merit</td></tr>
</table>

12		WME Ext 2 HSC 2024 **Question 12 Worked Solutions**	Marks	**Allocation and Comments**
(d)		$(x^2 - y^2)^2 > 0$ $x^4 - 2x^2y^2 + y^4 > 0$ $x^4 + y^4 > 2x^2y^2$ $x^4 + z^4 > 2x^2z^2$ $z^4 + y^4 > 2z^2y^2$ $2x^4 + 2y^4 + 2z^4 > 2x^2y^2 + 2x^2z^2 + 2z^2y^2$ $x^4 + y^4 + z^4 > x^2y^2 + x^2z^2 + z^2y^2$	**2**	2 marks for any complete and valid proof 1 mark for any partially complete proof or proof with a minor logical or algebraic error

<table>
<tr><td colspan="2">13</td><td>WME Ext 2 HSC 2024
Question 13 Worked Solutions</td><td>Marks</td><td>Allocation and Comments</td></tr>
<tr><td>(a)</td><td></td><td>$$x^5 + 2x^4 + 3x^3 + 6x^2 - 4x - 8 = 0$$ Given $x = 2i$ is a root, then $x = -2i$ is also a root.
Therefore, the equation is divisible by:
$(x - 2i)(x + 2i) = x^2 + 4.$
By division,
$$x^5 + 2x^4 + 3x^3 + 6x^2 - 4x - 8$$ $$= (x^2 + 4)(x^3 + 2x^2 - x - 2)$$ Using the factor theorem and further division
$$x^5 + 2x^4 + 3x^3 + 6x^2 - 4x - 8$$ $$= (x^2 + 4)(x^2 - 1)(x + 2)$$ Therefore, roots are: $x = \pm 2i, \pm 1$ and $x = -2$.</td><td>3</td><td>3 marks for correct roots

2 marks for correct division and initial factorisation leading to incorrect roots or equivalent merit

1 mark for division or equivalent merit</td></tr>
<tr><td>(b)</td><td></td><td>$$\begin{aligned}\left(\frac{a+b}{2}\right)^3 - \frac{a^3 + b^3}{2} &= \frac{1}{8}(a^3 + 3a^2b + 3ab^2 + b^3) - \frac{a^3 + b^3}{2}\\ &= \frac{1}{8}(a^3 + 3a^2b + 3ab^2 + b^3) - \frac{4a^3 + 4b^3}{8}\\ &= \frac{1}{8}(a^3 + 3a^2b + 3ab^2 + b^3 - 4a^3 - 4b^3)\\ &= \frac{1}{8}(-3a^3 + 3a^2b + 3ab^2 - 3b^3)\\ &= -\frac{3}{8}(a^3 - a^2b - ab^2 + b^3)\\ &= -\frac{3}{8}[a^2(a - b)] - b^2(a - b)]\\ &= -\frac{3}{8}(a - b)(a^2 - b^2)\\ &= -\frac{3}{8}(a - b)(a - b)(a + b)\\ &= -\frac{3}{8}(a + b)(a - b)^2\\ &\le 0 \text{ with equality when } a = \text{b}.\end{aligned}$$ $\therefore \left(\frac{a+b}{2}\right)^3 \le \frac{a^3 + b^3}{2}$ with equality when $a = b$.</td><td>3</td><td>3 marks for correct derivation

2 marks for expanding and finding common denominator and manipulation toward result or equivalent merit

1 mark for some correct steps toward proof or equivalent merit</td></tr>
<tr><td>(c)</td><td>(i)</td><td>$$I_n = \int_1^2 (\log_e x)^n dx$$ $$Let\ u = (\log_e x)^n, \qquad v' = 1$$ $$u' = n\left(\frac{1}{x}\right)(\log_e x)^{n-1} \qquad v = x$$ $$uv - \int vu'$$ $$[x(\log_e x)^n]_1^2 - \int_1^2 x.n(\log_e x)^{n-1}\left(\frac{1}{x}\right)dx$$ $$\{2(\log_e 2)^n\} - n\int_1^2 (\log_e x)^{n-1}\,dx$$ $$= 2(\log_e 2)^n - nI_{n-1}$$</td><td>2</td><td>2 marks for correct derivation of result

1 mark for using integration by parts and manipulating toward recurrence formula or equivalent merit</td></tr>
</table>

13		WME Ext 2 HSC 2024 Question 13 Worked Solutions	Marks	Allocation and Comments
	(ii)	$\int_1^2 (\log_e x)^3\,dx = I_3 = \{2(\log_e 2)^3\} - 3I_2$ $= \{2(\log_e 2)^3\} - 3[2(\log_e 2)^2 - 2I_1]$ $= \{2(\log_e 2)^3\} - 6(\log_e 2)^2 + 6I_1$ $= \{2(\log_e 2)^3\} - 6(\log_e 2)^2 + 6(2\log_e 2 - 1)$ $= \{2(\log_e 2)^3\} - 6(\log_e 2)^2 + 12\log_e 2 - 6$ $= 2(\log_e 2)^3 - 3\{2(\log_e 2)^2 - 4\log_e 2 + 2\}$	**3**	3 marks for correct result 2 marks for use of recurrence formulas with an error or equivalent merit 1 mark for some progress toward result or equivalent merit
(d)	(i)	$z^n = \cos n\theta + i\sin n\theta$ $z^{-n} = \cos n\theta - i\sin n\theta$ $z^n + z^{-n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$ $= 2\cos n\theta$	**1**	1 mark for correct answer
	(ii)	$\left(z + \frac{1}{z}\right)^6 = (2\cos\theta)^6$ $= 64\cos^6\theta$ Also, $\left(z+\frac{1}{z}\right)^6 = z^6 + 6z^5\frac{1}{z} + 15z^4\frac{1}{z^2} + 20z^3\frac{1}{z^3} + 15z^2\frac{1}{z^4} + 6z\frac{1}{z^5} + \frac{1}{z^6}$ $64\cos^6\theta = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$ $= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$	**3**	3 marks for correct expression 2 marks for equating and grouping with a subsequent error or equivalent merit 1 mark for some progress toward expression or equivalent merit

14		WME Ext 2 HSC 2024 Question 14 Worked Solutions	Marks	Allocation and Comments
(a)	(i)	$\frac{x^2+1}{x^3+2x^2+x} = \frac{x^2+1}{x(x^2+2x+1)}$ $= \frac{x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ Multiplying by $x(x+1)^2$ $x^2 + 1 = A(x+1)^2 + Bx(x+1) + Cx$ Try $x = 0$: $0^2 + 1 = A(0+1)^2 \quad \therefore A = 1$ Try $x = -1$: $-1^2 + 1 = (-1)C \quad \therefore C = -2$ Equate coefficients of x^2: $1 = A + B$ $1 = 1 + B \quad \therefore B = 0$	**3**	3 marks for correct values of A B C 2 marks for correct partial fractions to achieve values for *A*, *B* and *C* with an error or equivalent merit. 1 mark for some progress including use of partial fractions or equivalent merit
	(ii)	$\int \frac{x^2+1}{x^3+2x^2+x}\,dx = \int \left(\frac{1}{x} + \frac{0}{x+1} - \frac{2}{(x+1)^2}\right) dx$ $= \int \left(\frac{1}{x} - \frac{2}{(x+1)^2}\right) dx$ $= \int \left[\frac{1}{x} - 2(x+1)^{-2}\right] dx$ $= \ln x + 2(x+1)^{-1} + c$ $= \ln x + \frac{2}{x+1} + c$	**2**	2 marks for correct result 1 mark for substitution and manipulation toward result or equivalent merit

14		**WME Ext 2 HSC 2024** **Question 14 Worked Solutions**	Marks	**Allocation and Comments**
(b)	(i)	Test n = 1. LHS = 1 RHS = 2 – 1 = 1 so LHS $\leq RHS$ $\therefore$ True for n = 1. Assume true for some integer $n = k$ i.e. that: $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\ldots\ldots+\frac{1}{k^2} \leq 2-\frac{1}{k}$ Wish to show true for n = k + 1, i.e. that: $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\ldots\ldots+\frac{1}{(k+1)^2} \leq 2-\frac{1}{(k+1)}$ Now adding $\frac{1}{(k+1)^2}$ to both sides of assumption above: $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{k^2}+\frac{1}{(k+1)^2} \leq 2-\frac{1}{k}+\frac{1}{(k+1)^2}$ $\leq 2-\left[\frac{1}{k}-\frac{1}{(k+1)^2}\right]$ $\leq 2-\left[\frac{(k+1)^2-k}{k(k+1)^2}\right]$ $\leq 2-\left[\frac{k^2+2k+1-k}{k(k+1)^2}\right]$ $\leq 2-\left[\frac{k^2+k+1}{k(k+1)^2}\right]$ $\leq 2-\left[\frac{k^2+k}{k(k+1)^2}+\frac{1}{k(k+1)^2}\right]$ $\leq 2-\left[\frac{k(k+1)}{k(k+1)^2}+\frac{1}{k(k+1)^2}\right]$ $\leq 2-\frac{1}{(k+1)}-\frac{1}{k(k+1)^2}$ $\leq 2-\frac{1}{(k+1)}$ since $\frac{1}{k(k+1)^2}>0$ $\therefore$ If true for $n = k$, then also true for $n = k+1$ But since true for n = 1, by induction also true for all integers $n > 1$. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\ldots\ldots+\frac{1}{n^2} \leq 2-\frac{1}{n}$ for $n \geq 1$	**3**	3 marks for complete and valid proof 2 marks for proof missing one item such as proof for n = 1 or missing conclusion or equivalent merit 1 mark for error in manipulation and missing conclusion or equivalent merit

<table>
<tr><td colspan="2">14</td><td>WME Ext 2 HSC 2024
Question 14 Worked Solutions</td><td>Marks</td><td>Allocation and Comments</td></tr>
<tr><td></td><td>(ii)</td><td>$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots+\frac{1}{99^2} \le 2-\frac{1}{99}$ From (a)
$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots+\frac{1}{99^2} \le \frac{197}{99}$
Now $\frac{197}{99} = \frac{19700}{9900} \le \frac{19701}{9900} = \frac{199}{100}$
so $\frac{197}{99} < \frac{199}{100}$
$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots+\frac{1}{99^2} \le 1.99$
Or using a calculator (Part marks)
$1+\frac{1}{2^2}$
$+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots+\frac{1}{99^2} \approx 1.98989899 < 1.99$
$\therefore\ 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots+\frac{1}{99^2} < 1.99$</td><td>2</td><td>2 marks for correctly showing result.

1 mark for substitution and some correct manipulation, or for using a calculator to add terms to show that the sum is less than 1.99 or equivalent merit</td></tr>
<tr><td>(c)</td><td></td><td>Parametric Equations are:
$x = 3+2t$
$y = 2-t$
$z = -1-2t$

$\therefore\ (3+2t-3)^2+(2-t-2)^2+(-1-2t+1)^2 = 36$
$(2t)^2+(-t)^2+(-2t)^2 = 36$
$4t^2+t^2+4t^2 = 36$
$9t^2 = 36$
$t^2 = 4$
$t = \pm 2$
$t = -2;\ x = 3+2(-2) = -1$ $\quad t = 2;\ x = 3+2(2) = 7$
$t = -2;\ y = 2-(-2) = 4$ $\quad t = 2;\ y = 2-(2) = 0$
$t = -2;\ z = -1-2(-2) = 3$
$t = 2;\ z = -1-2(2) = -5$

$\therefore$ Points of intersection are $(-1,4,3)$ and $(7,0,-5)$</td><td>3</td><td>3 marks for correct points

2 marks for solution with a minor error or equivalent merit

1 mark for obtaining parametric equations or equivalent merit</td></tr>
<tr><td>(d)</td><td></td><td>$\angle ABC$ can be found using the dot product of $\underset{\sim}{b}$ and $\underset{\sim}{c}$
$\underset{\sim}{b}\cdot\underset{\sim}{c} = \lvert\underset{\sim}{b}\rvert.\lvert\underset{\sim}{c}\rvert\cos\theta$
If $\cos 90° = 0$, then $\underset{\sim}{b}\cdot\underset{\sim}{c} = 0$ and $\angle ABC = 90°$
$\underset{\sim}{b}\cdot\underset{\sim}{c} = 1\times 4+8\times(-2)+3\times 4$
$= 4-16+12$
$= 0$
$\therefore\ \angle ABC = 90°$ and $\Delta\, ABC$ is a right triangle.</td><td>2</td><td>2 marks for correctly using dot product to show sides at right angles

1 mark for finding the dot product or equivalent merit</td></tr>
</table>

15		**WME Ext 2 HSC 2024** **Question 15 Worked Solutions**	Marks	**Allocation and Comments**
(a)	(i)	H: $\ddot{x} = 0$ $\quad$ $V: \ddot{y} = -g$ $\dot{x} = V\cos\alpha$ $\quad$ $\dot{y} = V\sin\alpha - gt$ $x = Vt\cos\alpha \;\; (1)$ $\quad$ $y = Vt\sin\alpha - \frac{1}{2}gt^2 \quad (2)$ From (1) $t = \frac{x}{V\cos\alpha}$ sub into (2) $y = V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2$ $y = x\frac{\sin\alpha}{\cos\alpha} - \frac{1}{2}g\left(\frac{x^2}{V^2\cos^2\alpha}\right)$ $= x\tan\alpha - \frac{gx^2}{2V^2}\sec^2 x$	**2**	2 marks for correct result 1 mark for displacement equations and substitution to combine equations or equivalent merit
	(ii)	By substituting $x = 80$, $y = 55$, $V = 40$ and $g = 10$: $55 = 80\tan\alpha - \frac{10(80)^2}{2(40)^2}(1 + \tan^2\alpha)$ $55 = 80\tan\alpha - 20(1 + \tan^2\alpha)$ $55 = 80\tan\alpha - 20 - 20\tan^2\alpha$ $\therefore\ 20\tan^2\alpha - 80\tan\alpha + 75 = 0$ $4\tan^2\alpha - 16\tan\alpha + 15 = 0$ $(2\tan\alpha - 5)(2\tan\alpha - 3) = 0$ $\therefore \tan\alpha = \frac{5}{2}$ or $\tan\alpha = \frac{3}{2}$ $\frac{3}{2}$ refers to the second missile. $\therefore\ tan\,\beta = \frac{3}{2}$.	**2**	2 marks for correct result 1 mark for forming quadratic equation and factorising or equivalent merit

15		**WME Ext 2 HSC 2024** **Question 15 Worked Solutions**	Marks	**Allocation and Comments**
	(iii)	Using $x = V_1 t_1 \cos\alpha$ and $x = V_2 t_2 \cos\beta$ $\therefore 80 = 40t_1 \cos\alpha$ and $80 = 40t_2 \cos\beta$ $\therefore t_1 = \frac{2}{\cos\alpha}, \quad t_2 = \frac{2}{\cos\beta}$ $t_1 - t_2 = \frac{2}{\cos\alpha} - \frac{2}{\cos\beta}$ $= 2\left(\frac{1}{\cos\alpha} - \frac{1}{\cos\beta}\right)$ $= 2(\sec\alpha - \sec\beta)$ But $\sec\alpha = \sqrt{1+\tan^2\alpha}$ and $\sec\beta = \sqrt{1+\tan^2\beta}$ $= \sqrt{1^2 + \left(\frac{5}{2}\right)^2} \qquad = \sqrt{1^2 + \left(\frac{3}{2}\right)^2}$ $= \sqrt{1 + \frac{25}{4}} \qquad = \sqrt{1 + \frac{9}{4}}$ $= \sqrt{\frac{29}{4}} \qquad = \sqrt{\frac{13}{4}}$ $= \frac{\sqrt{29}}{2} \qquad = \frac{\sqrt{13}}{2}$ $\therefore t_1 - t_2 = 2\left(\frac{\sqrt{29}}{2} - \frac{\sqrt{13}}{2}\right)$ $= \sqrt{29} - \sqrt{13}$ $= 1 \cdot 8$ seconds (1dp)	**3**	3 marks for correct result 2 marks for expression for $t_1 - t_2$ and working to find values for $sec\,\alpha$ and $sec\,\beta$ 1 mark for some correct and relevant working
(b)		Assume that the converse is true $\sqrt{a} + \sqrt{a+2} \leq \sqrt{a+8}$ $\left(\sqrt{a} + \sqrt{a+2}\right)^2 \leq a + 8$ $a + 2\sqrt{a}\sqrt{a+2} + a + 2 \leq a + 8$ $2a + 2\sqrt{a^2 + 2a} + 2 \leq a + 8$ $a + 2\sqrt{a^2 + 2a} \leq 6$ Since proof is for $a \geq 2$, consider when $a = 2$ $2 + 2\sqrt{2^2 + 2 \times 2} = 7.65685425$ which is not < 6 And as all terms in the expression are positive $2 + 2\sqrt{2^2 + 2 \times 2} < 3 + 2\sqrt{3^2 + 2 \times 3}$ and all successive terms will be larger so also not less than 6 So the assumption is false for $a \geq 2$ So by contradiction $\sqrt{a} + \sqrt{a+2} > \sqrt{a+8}$ for all $a \geq 2$	**3**	3 marks for correct result 2 marks for squaring both sides and obtaining expression and evaluating and showing the contradiction with a minor error 1 mark for some correct and relevant working

<table>
<tr><td colspan="2">15</td><td>WME Ext 2 HSC 2024
Question 15 Worked Solutions</td><td>Marks</td><td>Allocation and Comments</td></tr>
<tr><td>(c)</td><td>(i)</td><td>$$(x-3)^2 \geq 0$$ $$x^2-6x+9 \geq 0$$ $$x^2+9 \geq 6x$$ $$\frac{6x}{x^2+9} \leq 1$$ $$\therefore \frac{x}{x^2+9} \leq \frac{1}{6}$$</td><td>2</td><td>2 marks for correctly showing result.

1 mark for substitution and some correct manipulation or equivalent merit</td></tr>
<tr><td></td><td>(ii)</td><td>$$\int_0^\alpha \frac{x}{x^2+9}\,dx \leq \int_0^\alpha \frac{1}{6}\,dx$$ $$\frac{1}{2}\left[\log_e(x^2+9)\right]_0^\alpha \leq \frac{1}{6}[x]_o^\alpha$$ $$\frac{1}{2}\left[\log_e(\alpha^2+9)-\log_e 9\right] \leq \frac{1}{6}\alpha$$ $$\log_e\left(\frac{\alpha^2+9}{9}\right) \leq \frac{1}{3}\alpha$$ $$\frac{\alpha^2+9}{9} \leq e^{\frac{1}{3}\alpha}$$ $$\therefore e^{\frac{1}{3}\alpha} \geq \frac{1}{9}\alpha^2+1, \alpha \geq 0$$</td><td>3</td><td>3 marks for correctly showing result.

2 marks for correct use of integral with a minor error

1 mark for some correct and relevant working or equivalent merit</td></tr>
</table>

16		**WME Ext 2 HSC 2024** **Question 16 Worked Solutions**	Marks	**Allocation and Comments**
(a)	(i)	$m = 1$ $\quad \ddot{x} = -g - \frac{1}{20}v^2$ $g = 10$ $\quad = -10 - \frac{1}{20}v^2$ $= -\left(10 + \frac{1}{20}v^2\right)$	**1**	1 mark for correct working
	(ii)	$v\frac{dv}{dx} = -\frac{1}{20}(v^2 + 200)$ At ground; $x = 0$, $v = 40$ $\int_{40}^{0} \frac{v}{v^2 + 200}\,dv = -\frac{1}{20}\int_0^H dx$ At max height, $v = 0$, $x = H$. $\therefore \frac{1}{2}[ln(v^2 + 200)]_{40}^{0} = -\frac{1}{20}[x]_0^H$ $\therefore \frac{1}{2}[ln\,200 - ln\,1800] = -\frac{1}{20}[H - 0]$ $\therefore -10\,ln\left(\frac{200}{1800}\right) = H$ $\therefore H = -10\,ln\frac{1}{9}$ $= -10\,(ln\,1 - ln\,9)$ $= 10\,ln\,9$ Maximum height is $10\,ln\,9$ metres	**2**	2 marks for correctly showing result. 1 mark for integration and substitution with a minor error in working or equivalent merit
	(iii)	$\ddot{x} = -\frac{1}{20}(v^2 + 200)$ $\frac{dv}{dt} = -\frac{1}{20}(v^2 + 200)$ At Ground $t = 0$, $v = 40$ At Max Height $t = T$, $v = 0$ $\int_{40}^{0} \frac{dv}{v^2 + 200} = -\frac{1}{20}\int_0^T dt$ $\frac{1}{\sqrt{200}}\left[\tan^{-1}\left(\frac{v}{\sqrt{200}}\right)\right]_{40}^{0} = -\frac{1}{20}[t]_0^T$ $-\frac{20}{10\sqrt{2}}\left[\tan^{-1}0 - \tan^{-1}\left(\frac{40}{\sqrt{200}}\right)\right] = T$ $\therefore T = -\sqrt{2}[0 - \tan^{-1}(2\sqrt{2})]$ $T \approx 1.74$ sec	**2**	2 marks for correctly showing result. 1 mark for integration and substitution with a minor error in working or equivalent merit

16		WME Ext 2 HSC 2024 **Question 16 Worked Solutions**	Marks	**Allocation and Comments**
(b)	(i)	Particle Falls: $\ddot{x} = g - \frac{1}{20}v^2$ $(g = 10)$ $v\frac{dv}{dx} = 10 - \frac{1}{20}v^2$	**1**	1 mark for correct equation
(b)	(ii)	The particle falls from rest through a height, H, to reach a final velocity of V. $\therefore \int_0^V \frac{vdv}{200 - v^2} = \frac{1}{20}\int_0^H dx$ $-\frac{1}{2}[ln(200 - v^2)]_0^V = \frac{1}{20}[x]_0^H$ Now $H = 10\,ln\,9\ from\,part\,(a)$(i) $\therefore -10\ln\left(\frac{200 - v^2}{200}\right) = 10\,ln\,9$ $ln\left(\frac{200 - v^2}{200}\right) = -ln\,9$ $= ln\left(\frac{1}{9}\right)$ $\left(1 - \frac{v^2}{200}\right) = \frac{1}{9}$ $\frac{v^2}{200} = \frac{8}{9}$ $9v^2 = 1600$ $v^2 = \frac{1600}{9}$ $v = \frac{40}{3}$ Speed when particle hits the ground is $13\frac{1}{3}$ m/s.	**3**	3 marks for correct answer. 2 marks for correct use of integral and prior result with a minor error or equivalent merit 1 mark for some correct and relevant working or equivalent merit

<table>
<tr><td colspan="2">16</td><td>WME Ext 2 HSC 2024
Question 16 Worked Solutions</td><td>Marks</td><td>Allocation and Comments</td></tr>
<tr><td>(c)</td><td>(i)</td><td>$\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}}$ (by AM ≥ GM)
$a+b+c \geq 3(abc)^{\frac{1}{3}}$

$t \geq 3(abc)^{\frac{1}{3}}$
$\frac{t}{3} \geq (abc)^{\frac{1}{3}}$
$\frac{3}{t} \leq \frac{1}{(abc)^{\frac{1}{3}}}$
$\frac{1}{(abc)^{\frac{1}{3}}} \geq \frac{3}{t}$

Now,
$\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{3} \geq \left\{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\right\}^{\frac{1}{3}}$ (by AM ≥ GM)
$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3\left\{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\right\}^{\frac{1}{3}}$
$\geq 3\left\{\frac{1}{abc}\right\}^{\frac{1}{3}}$
$\geq 3\frac{1}{(abc)^{\frac{1}{3}}}$

$\geq 3\cdot\frac{3}{t}$
$\geq \frac{9}{t}$</td><td>3</td><td>3 marks for correct derivation

2 marks for manipulation toward result with a minor error or equivalent merit

1 mark for some correct steps toward proof or equivalent merit</td></tr>
<tr><td>(c)</td><td>(ii)</td><td>$\left\{\frac{1}{a}-1\right\}\cdot\left\{\frac{1}{b}-1\right\}\cdot\left\{\frac{1}{c}-1\right\} = \frac{(1-a)(1-b)(1-c)}{abc}$
$= \frac{1-(a+b+c)+(ab+ac+bc)-(abc)}{abc}$
Since $a+b+c=1$
$= \frac{ab+ac+bc-abc}{abc}$
$= \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1$
But $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$ from (i)
$\therefore \left\{\frac{1}{a}-1\right\}\cdot\left\{\frac{1}{b}-1\right\}\cdot\left\{\frac{1}{c}-1\right\} \geq 8.$</td><td>3</td><td>3 marks for correct derivation

2 marks for manipulation toward result with a minor error or equivalent merit

1 mark for some correct steps toward proof or equivalent merit</td></tr>
</table>