NSW GOVERNMENT

NSW Education Standards Authority

Sample **HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 8–16)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which expression is equal to $\int x^2 \sin x \, dx$?

A. $-x^2 \cos x - \int 2x \cos x \, dx$

B. $-2x \cos x + \int x^2 \cos x \, dx$

C. $-x^2 \cos x + \int 2x \cos x \, dx$

D. $-2x \cos x - \int x^2 \cos x \, dx$

2 Which of the following is a primitive of $\dfrac{\sin x}{\cos^3 x}$?

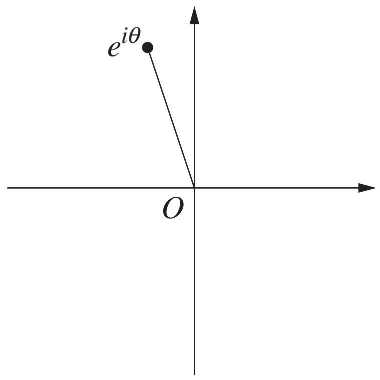
A. $\dfrac{1}{2}\sec^2 x$

B. $-\dfrac{1}{2}\sec^2 x$

C. $\dfrac{1}{4}\sec^4 x$

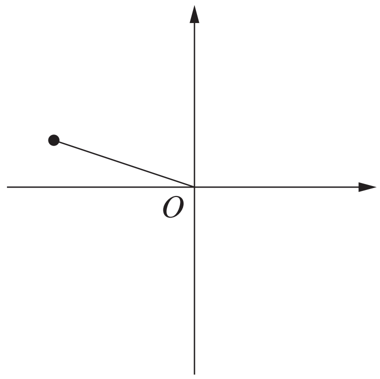
D. $-\dfrac{1}{4}\sec^4 x$

3 The Argand diagram shows the complex number $e^{i\theta}$.

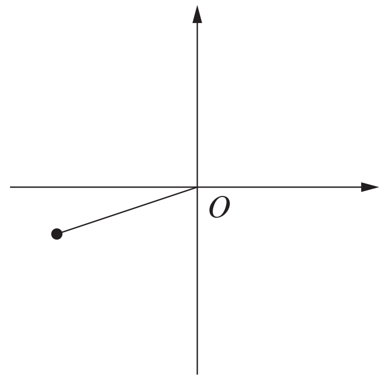


Which of the following diagrams best shows the complex number $-ie^{-i\theta}$?

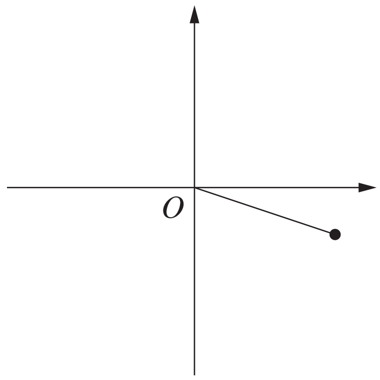
A.



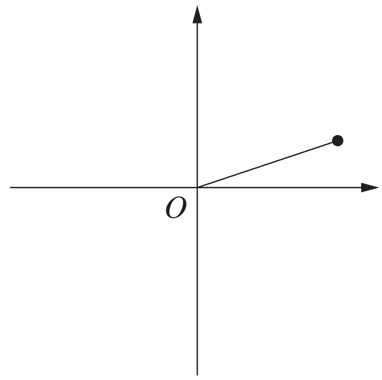
B.



C.



D.



4 Given that x and y are integers, which of the following is a true statement?

A. $\forall x \left(\exists y : y^2 = x \right)$

B. $\forall y \left(\exists x : y^2 = x \right)$

C. $\forall y \left(\forall x, y^2 = x \right)$

D. $\forall x \left(\forall y, y^2 = x \right)$

5 A particle is moving along the x-axis in simple harmonic motion. The displacement of the particle is x metres. The particle is at rest when $x = -2$ and when $x = 8$. It takes 6 seconds to travel from $x = -2$ to $x = 8$.

What is the maximum speed of the particle?

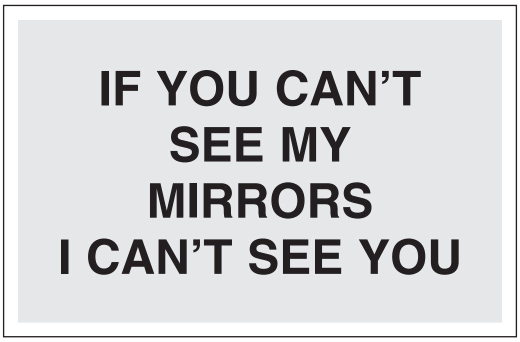
A. $\dfrac{5\pi}{6}\,\text{m s}^{-1}$

B. $\dfrac{4\pi}{3}\,\text{m s}^{-1}$

C. $\dfrac{5\pi}{3}\,\text{m s}^{-1}$

D. $\dfrac{10\pi}{3}\,\text{m s}^{-1}$

6 The sign shown appears on the rear of large vehicles.



Which of the following statements is logically equivalent to the statement on the sign?

A. If you can see my mirrors then I can see you.

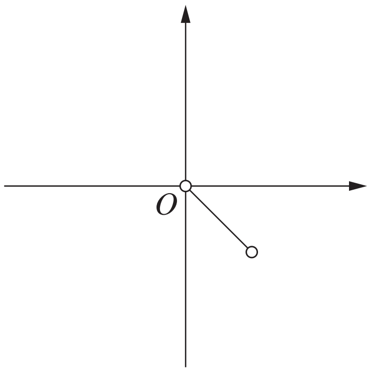
B. If I can see you then you can see my mirrors.

C. If I can't see you then you can't see my mirrors.

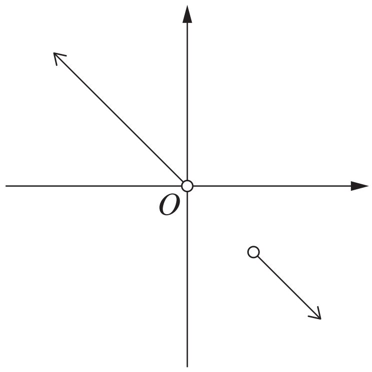
D. If I can't see your mirrors then you can't see me.

7 Which of the following diagrams best represents the solutions to the equation $\arg(z) = \arg(z + 1 - i)$?

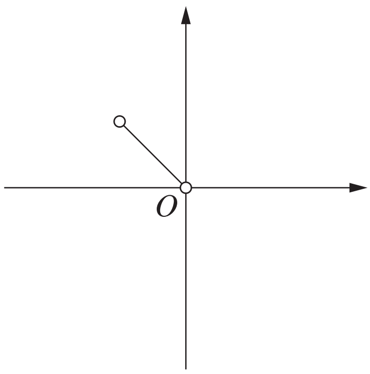
A.



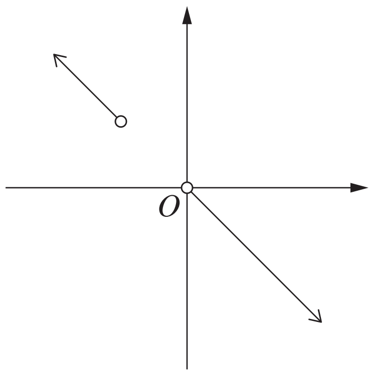
B.



C.



D.



8 A particle is moving along a straight line. Initially its displacement is at $x = 1$, its velocity is $v = 2$ and its acceleration is $a = 4$.

Which equation could describe the motion of the particle?

A. $v = 2\sin(x-1)+2$

B. $v = 2+4\log_e x$

C. $v^2 = 4\left(x^2-2\right)$

D. $v^2 = x^2+2x+4$

9 It is given that a, b are real and p, q are purely imaginary.

Which pair of inequalities must always be true?

A. $a^2p^2+b^2q^2 \le 2abpq, \quad a^2b^2+p^2q^2 \le 2abpq$

B. $a^2p^2+b^2q^2 \le 2abpq, \quad a^2b^2+p^2q^2 \ge 2abpq$

C. $a^2p^2+b^2q^2 \ge 2abpq, \quad a^2b^2+p^2q^2 \le 2abpq$

D. $a^2p^2+b^2q^2 \ge 2abpq, \quad a^2b^2+p^2q^2 \ge 2abpq$

10 Which pair of line segments intersects at exactly one point?

A. $$\begin{cases} \underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \end{pmatrix}, & 0 \le \lambda \le 1 \\ \underset{\sim}{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & 0 \le \lambda \le 1 \end{cases}$$

B. $$\begin{cases} \underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \end{pmatrix}, & 0 \le \lambda \le 1 \\ \underset{\sim}{v} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -9 \end{pmatrix}, & 0 \le \lambda \le 1 \end{cases}$$

C. $$\begin{cases} \underset{\sim}{u} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \end{pmatrix}, & 0 \le \lambda \le 1 \\ \underset{\sim}{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \end{pmatrix}, & 0 \le \lambda \le 1 \end{cases}$$

D. $$\begin{cases} \underset{\sim}{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}, & 0 \le \lambda \le 1 \\ \underset{\sim}{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \end{pmatrix}, & 0 \le \lambda \le 1 \end{cases}$$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Let $z = \sqrt{3} - i$.

(i) Express z in modulus–argument form. **2**

(ii) Show that z^6 is real. **1**

(iii) Find a positive integer n such that z^n is purely imaginary. **1**

(b) (i) Show that $(1 - 2i)^2 = -3 - 4i$. **1**

(ii) Hence solve the equation $z^2 - 5z + (7 + i) = 0$. **2**

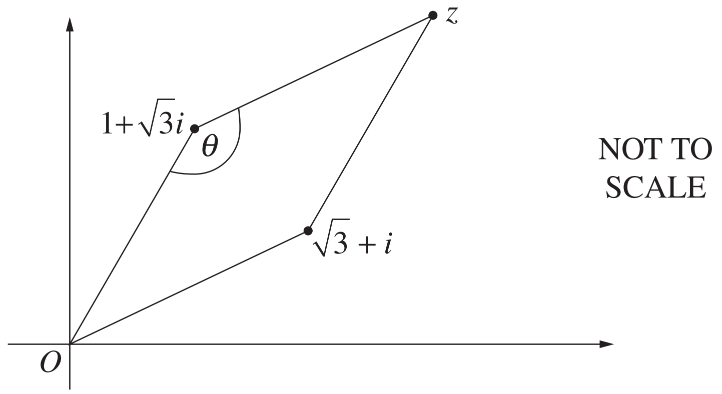
(c) (i) Find numbers A, B and C such that $\dfrac{x^2 + 8x + 11}{(x-3)\left(x^2+2\right)} = \dfrac{A}{x-3} + \dfrac{Bx + C}{x^2 + 2}$. **2**

(ii) Hence evaluate $\displaystyle\int \frac{x^2 + 8x + 11}{(x-3)\left(x^2+2\right)}\,dx$. **2**

(d) Using the substitution $u^2 = 4 - x^2$, or otherwise, evaluate $\displaystyle\int_0^2 x^3\sqrt{4 - x^2}\,dx$. **4**

Question 12 (14 marks) Use the Question 12 Writing Booklet.

(a) On the Argand diagram, the complex numbers $0, 1+\sqrt{3}i, \sqrt{3}+i$ and z form a rhombus.



(i) Find z in the form $a + bi$, where a and b are real numbers. **1**

(ii) An interior angle of the rhombus, θ, is marked on the diagram. **2**

Find the value of θ.

(b) (i) On the one diagram, draw neat sketches showing the subsets of the complex plane satisfying each of the relations **2**

$$|z-(3+2i)| = 2 \qquad \text{and} \qquad |z+3| = |z-5|.$$

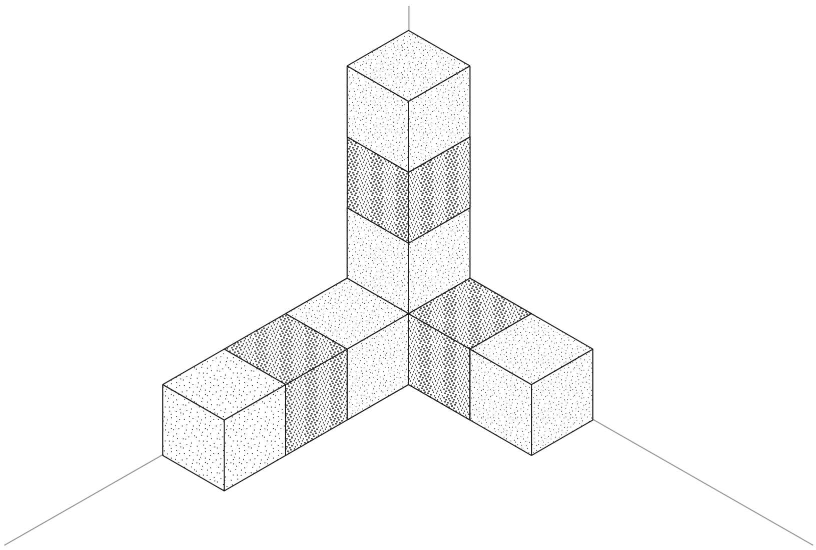
(ii) Hence write down all the values of z which simultaneously satisfy both relations. **1**

(iii) Use the diagram in part (i) to determine the values of k for which the simultaneous equations $|z-(3+2i)| = 2$ and $|z-2i| = k$ have exactly one solution for z. **2**

(c) Find all the solutions of $\left|e^{2i\theta}-1\right| = \sqrt{3}$ satisfying $-\pi < \theta \leq \pi$. **3**

Question 12 continues on page 10

(d) Two vertical walls and the floor meet at a corner of a room. One cube is placed in the corner. A solid shape is then formed by placing identical cubes to form horizontal rows on the floor against the walls or by stacking vertically against the two walls. An example is the solid shape shown in the diagram. This example is formed from nine cubes. **3**



Let n be the number of cubes used to make a solid shape in this way.

Use mathematical induction to show that the number of exposed faces of the cubes in this shape is $2n + 1$.

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) It is given that a and b are positive real numbers. **2**

Consider the statement $\forall a\left(\forall b, a^{\ln b} = b^{\ln a}\right)$.

Either prove that the statement is true or provide a counter-example.

(b) Consider the polynomial $P_n(x) = (x+1)^{2n+1} + x^{n+2}$. Let ω be a cube root of unity, $\omega \neq 1$. **4**

By considering $P_n(\omega)$ and $P_n\left(\omega^2\right)$ prove that $x^2 + x + 1$ is a factor of $P_n(x)$.

(c) A particle of mass 0.5 kilograms is dropped from the top of a tower of height h metres above the ground. The particle experiences a force due to air resistance of magnitude $\dfrac{v^2}{1000}$ newtons, where v is the speed of the particle in metres per second. Let the displacement, x metres, at time t seconds, be measured in a downwards direction.

(i) Show that the equation of motion of the particle is **1**

$$\ddot{x} = g - \frac{v^2}{500},$$

where g is the acceleration due to gravity.

Use $g = 9.8\ \mathrm{m\,s^{-2}}$ in the following parts.

(ii) Show, by integrating using partial fractions, that $v = 70\left(\dfrac{e^{0.28t} - 1}{e^{0.28t} + 1}\right)$. **5**

(iii) The particle takes 3 seconds to reach the ground. By finding a formula for x as a function of v, or otherwise, find h, the height of the tower. **3**

Question 14 (17 marks) Use the Question 14 Writing Booklet.

(a) Let $\alpha = \sqrt{4n-2}$ where n is a fixed positive integer.

(i) Prove that α is irrational. **3**

Let $\{\beta\}$ denote the 'fractional' part of β, so $\beta = k + \{\beta\}$ where k is an integer and $0 \le \{\beta\} < 1$.

(ii) Let N be a positive integer. Consider the numbers **3**

$$0, \{\alpha\}, \{2\alpha\}, \ldots \{(N-1)\alpha\}, \{N\alpha\}.$$

By dividing the interval $[0,1]$ into sections, or otherwise, prove that at least two of these numbers differ by less than $\frac{1}{N}$.

(iii) Hence prove that, for any positive integer N, there exist integers p and q such that $0 < |q\alpha - p| < \frac{1}{N}$. **1**

(b) A particle is thrown from the point O at the top of a very tall building. The equation of motion is

$$\ddot{\underset{\sim}{s}} = -r\dot{\underset{\sim}{s}} - g\underset{\sim}{j},$$

where $\underset{\sim}{s}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ is the position vector of the particle at time t, r is a constant and g is the magnitude of the contribution to the acceleration in the vertical direction due to gravity. The initial conditions are $\underset{\sim}{s}(0) = \underset{\sim}{0}$ and $\dot{\underset{\sim}{s}}(0) = u\cos(\alpha)\underset{\sim}{i} + u\sin(\alpha)\underset{\sim}{j}$.

(i) What is the physical interpretation of the constants u and α in the expression for the initial condition $\dot{\underset{\sim}{s}}(0)$? **2**

(ii) By considering the horizontal equation of motion, find an expression for $x(t)$. **3**

(iii) Verify that $\dot{y}(t) = u\sin(\alpha)e^{-rt} - \dfrac{g\left(1-e^{-rt}\right)}{r}$ satisfies the acceleration equation and initial conditions. **2**

Question 14 continues on page 13

Question 14 (continued)

(iv) Find the value of t when the particle reaches its maximum height. **2**

(v) Calculate how far you must be from the point at the base of the building directly below O to ensure that you cannot be hit by the particle. Give your answer in terms of the constants specified. **1**

End of Question 14

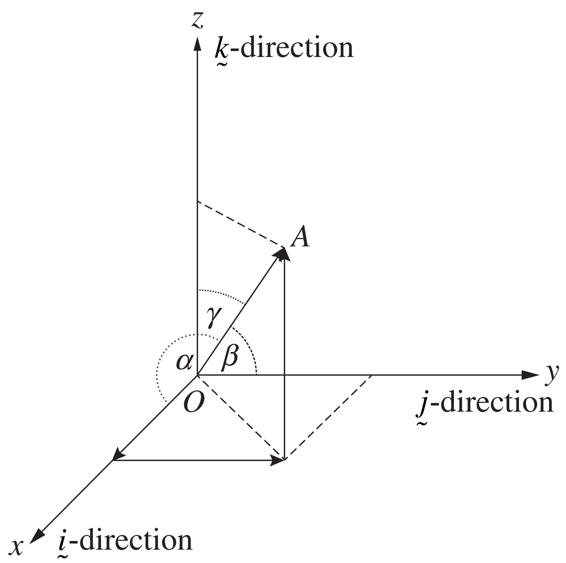
Question 15 (14 marks) Use the Question 15 Writing Booklet.

(a) A particle is moving in a straight line according to the equation $x = 5 + 6\cos 2t + 8\sin 2t$, where x is the displacement in metres and t is the time in seconds.

(i) Prove that the particle is moving in simple harmonic motion by showing that x satisfies an equation of the form $\ddot{x} = -n^2(x - c)$. **2**

(ii) When is the displacement of the particle zero for the first time? **3**

(b) The point A has (non-zero) position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and the vector $\overrightarrow{OA}$ makes angles α, β, γ with the x, y, z axes respectively. **4**



By taking a dot product with the three unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$, prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Question 15 continues on page 15

(c) (i) Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be three 3-dimensional vectors. **1**

Prove that $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$.

Let $\underset{\sim}{v}$ be the position vector of a point P on a sphere S with centre C and radius r, so that $\left| \underset{\sim}{v} - \underset{\sim}{c} \right| = r$, where $\underset{\sim}{c} = \overrightarrow{OC}$. (Do NOT prove this.)

(ii) The equation of the line ℓ through P in the direction of the vector $\underset{\sim}{m}$ is $\underset{\sim}{w} = \underset{\sim}{v} + \lambda \underset{\sim}{m}$. **2**

Find the values of λ that correspond to the intersection of the line ℓ and the sphere S. Give your answer in terms of $\underset{\sim}{v}$, $\underset{\sim}{c}$ and $\underset{\sim}{m}$.

(iii) Deduce that the line ℓ is tangent to the sphere S if and only if $\underset{\sim}{m} \cdot (\underset{\sim}{v} - \underset{\sim}{c}) = 0$. Interpret this result geometrically. **2**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) Recall that for positive real numbers x and y, $\sqrt{xy} \le \frac{x+y}{2}$. (Do NOT prove this.)

(i) Prove that $\sqrt{xy} \le \sqrt{\frac{x^2+y^2}{2}}$ for all positive real numbers x and y. **1**

(ii) Prove that $\sqrt[4]{abcd} \le \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$ for all positive real numbers a, b, c and d. **2**

(b) Let $I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$, for $n = 0, 1, 2, \ldots$

(i) Find the value of I_1. **1**

(ii) Using integration by parts, or otherwise, show that for $n > 2$ **3**

$$I_n = \left(\frac{n-1}{n+2}\right) I_{n-2}.$$

(iii) Find the value of I_5. **1**

(c) Suppose n is a positive integer.

(i) Show that $-x^{2n} \le \frac{1}{1+x^2} - \sum_{k=0}^{n-1} (-1)^k x^{2k} \le x^{2n}$. **3**

(ii) Use integration to deduce that **3**

$$-\frac{1}{2n+1} \le \frac{\pi}{4} - \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \le \frac{1}{2n+1}.$$

(iii) Hence deduce the value of $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. **1**

End of paper

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2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

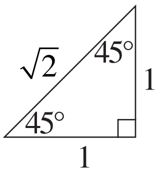
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

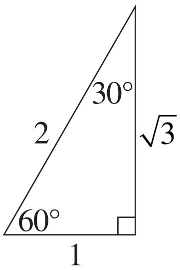


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

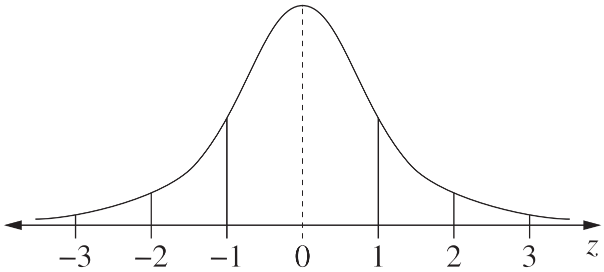
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x} p^x (1-p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1-p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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Mathematics Extension 2
Sample HSC Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	A
3	A
4	B
5	A
6	B
7	D
8	A
9	B
10	C

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Provides exact argument in radians, or equivalent merit	1

Sample answer:

$z = \sqrt{3} - i$

$|z| = 2, \qquad \text{Arg}(z) = -\dfrac{\pi}{6}$

$\text{So } z = 2\left(\cos\left(\dfrac{-\pi}{6}\right) + i\sin\left(\dfrac{-\pi}{6}\right)\right)$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\begin{aligned} z^6 &= 2^6(\cos(-\pi) + i\sin(-\pi)) \\ &= -2^6 \qquad \text{which is real} \end{aligned}$$

Question 11 (a) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Take $n = 3$

$$\begin{aligned} z^3 &= 2^3\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right) \\ &= -2^3 i \quad \text{which is purely imaginary} \end{aligned}$$

Question 11 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\begin{aligned}(1-2i)^2 &= 1-4i+4i^2\\ &= 1-4i-4\\ &= -3-4i\end{aligned}$$

Question 11 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Makes some progress towards correct solution such as using the quadratic formula	1

Sample answer:

$$\begin{aligned}z &= \frac{5 \pm \sqrt{25-4(7+i)}}{2}\\ &= \frac{5 \pm \sqrt{-3-4i}}{2}\\ &= \frac{5 \pm (1-2i)}{2} \qquad \text{from (i)}\\ &= 3-i \quad \text{or} \quad 2+i\end{aligned}$$

OR

$$\begin{aligned}z^2 - 5z + 6\frac{1}{4} &= -7 - i + 6\frac{1}{4}\\ \left(z - 2\frac{1}{2}\right)^2 &= \frac{-3-4i}{4}\\ z - 2\frac{1}{2} &= \pm\sqrt{\frac{-3-4i}{4}}\\ z - 2\frac{1}{2} &= \pm\frac{1}{2}(1-2i) \qquad \text{from (i)}\\ z &= 2\frac{1}{2} \pm \frac{1}{2}(1-2i)\\ &= 3-i \quad \text{or} \quad 2+i\end{aligned}$$

Question 11 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Finds the value of one of A, B or C	1

Sample answer:

$$\frac{x^2+8x+11}{(x-3)\left(x^2+2\right)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$$

$$= \frac{A\left(x^2+2\right) + (Bx+C)(x-3)}{(x-3)\left(x^2+2\right)}$$

We need: $x^2+8x+11 = A\left(x^2+2\right) + (Bx+C)(x-3)$

Make particular choices for x and substitute:

$x=3$: $\quad 44 = 11A$

$A = 4$

$x=0$: $\quad 11 = 2A - 3C$

$= 8 - 3C$

$C = -1$

$x=1$: $\quad 20 = 3A - 2B - 2C$

$20 = 12 - 2B + 2$

$B = -3$

Hence $A = 4, B = -3$ and $C = -1$

Question 11 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains one correct integral from the partial fraction	1

Sample answer:

$$\int \frac{x^2+8x+11}{(x-3)\left(x^2+2\right)}\,dx = \int \frac{4}{x-3}\,dx + \int \frac{-3x-1}{x^2+2}\,dx \qquad \text{(from part (i))}$$

$$= 4\ln(x-3) - \frac{3}{2}\int \frac{2x}{x^2+2}\,dx - \int \frac{dx}{x^2+2}$$

$$= 4\ln(x-3) - \frac{3}{2}\ln\left(x^2+2\right) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

Question 11 (d)

Criteria	Marks
• Provides correct solution	4
• Obtains appropriate definite integral, or equivalent merit	3
• Correctly rewrites integrand and limits, or equivalent merit	2
• Makes suggested substitution and rewrites integrand	1

Sample answer:

$$u^2 = 4 - x^2.$$

$$2u\,du = -2x\,dx.$$

$$\int_0^2 x^3\sqrt{4-x^2}\,dx = \int_{u=2}^{u=0} \left(4-u^2\right)\sqrt{u^2}\;\cdot\;(-u\,du)$$

$$= \int_0^2 u^2\left(4-u^2\right)du$$

$$= \int_0^2 4u^2 - u^4\,du$$

$$= \left[\frac{4u^3}{3} - \frac{u^5}{5}\right]_0^2$$

$$= \frac{32}{3} - \frac{32}{5} - 0$$

$$= \frac{64}{15}.$$

Question 12 (a) (i)

Criteria	Marks
• Provides correct answer	1

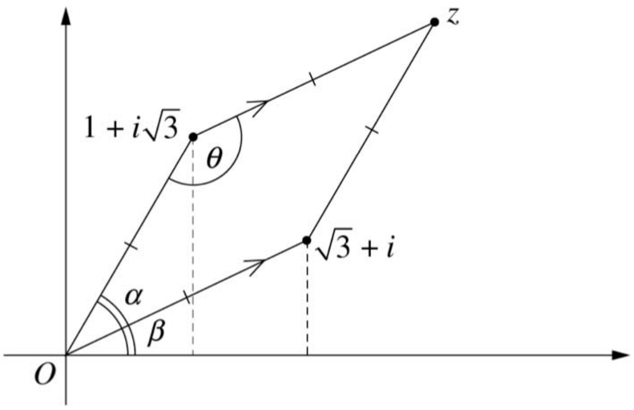
Sample answer:

$$\begin{aligned} z &= 1 + i\sqrt{3} + \sqrt{3} + i \\ &= \left(1+\sqrt{3}\right) + i\left(1+\sqrt{3}\right) \end{aligned}$$

Question 12 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Makes progress towards correct solution eg finds an angle in diagram	1

Sample answer:



$$\tan\beta = \frac{1}{\sqrt{3}} \qquad \tan\alpha = \frac{\sqrt{3}}{1}$$

$$\beta = \frac{\pi}{6} \qquad = \sqrt{3}$$

$$\therefore\ \alpha = \frac{\pi}{3}$$

$$\therefore\ \alpha - \beta = \frac{\pi}{6}$$

$$\therefore\ \theta = \pi - \frac{\pi}{6}$$

$$\therefore\ \theta = \frac{5\pi}{6}$$

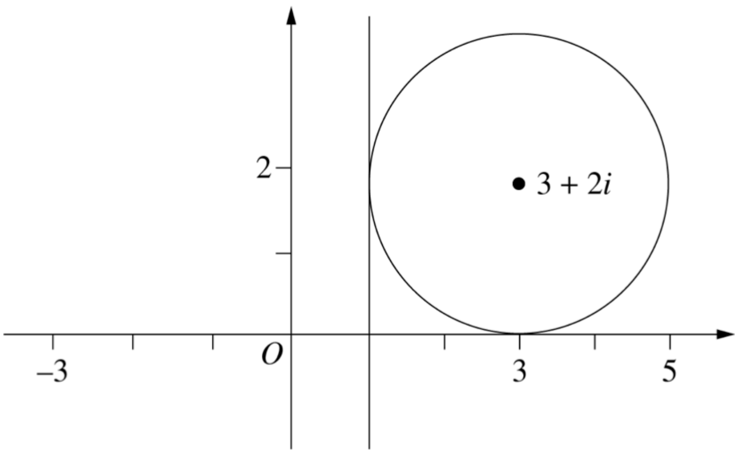
Question 12 (b) (i)

Criteria	Marks
• Provides correct sketches for the two relations	2
• Provides a correct sketch for one of the relations or equivalent merit	1

Sample answer:

$|z-(3+2i)|=2$ is a circle, centre $3+2i$, radius 2.

$|z+3|=|z-5|$ is the perpendicular bisector of the interval joining -3 and 5, that is, the line $x=1$.



Question 12 (b) (ii)

Criteria	Marks
• Provides correct answer	1

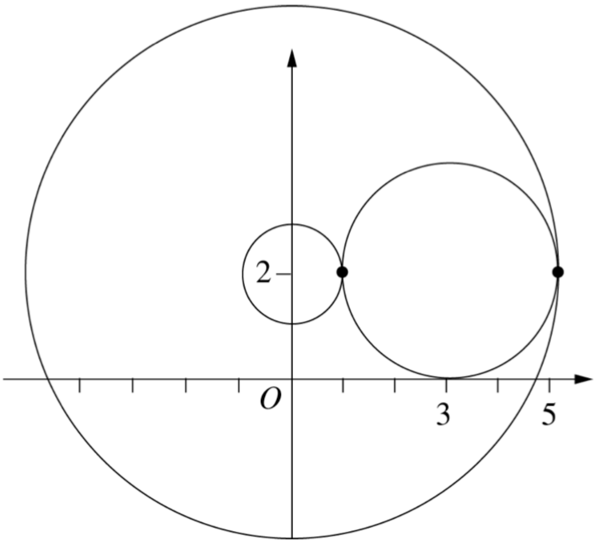
Sample answer:

The line is a tangent to the circle. Hence the only point of intersection is $z = 1 + 2i$.

Question 12 (b) (iii)

Criteria	Marks
• Provides correct solution	2
• Makes some progress towards correct solution	1

Sample answer:



$|z - 2i| = k$ represents a circle, centre $2i$, radius k.

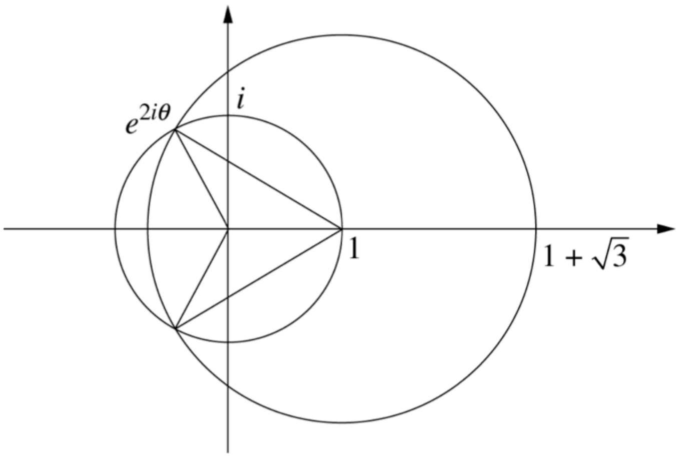
Hence the equations $|z - (3 + 2i)| = 2$ and $|z - 2i| = k$ have exactly one solution for z when $k = 1, 5$.

Question 12 (c)

Criteria	Marks
• Finds all four correct values for θ	3
• Determines that $\cos 2\theta = -\frac{1}{2}$	2
• Interprets some of the given information geometrically or writes $(\cos 2\theta - 1)^2 + \sin^2 2\theta = 3$	1

Sample answer:

$e^{2i\theta}$ lies on the intersection of the unit circle centred at 0 and the circle of radius $\sqrt{3}$ centred at 1.



Applying the cosine rule to the triangle with vertices at 0, 1 and $e^{2i\theta}$ gives

$\cos 2\theta = \frac{1+1-3}{2} = -\frac{1}{2}$, so $2\theta = \pm\frac{2\pi}{3}$ or $\pm\frac{4\pi}{3}$, yielding $\theta = \pm\frac{\pi}{3}$ or $\pm\frac{2\pi}{3}$.

Question 12 (d)

Criteria	Marks
• Provides correct solution	3
• Correctly states that when $n = 1$ there are 3 exposed faces, and attempts to deal with the change in the number of visible faces when another block is added	2
• Correctly states that when $n = 1$ there are 3 exposed faces, or equivalent merit	1

Sample answer:

When $n = 1$, there is one cube which is in the corner, so that 3 faces touch a surface (are obscured) and hence 3 faces are exposed. The number of exposed faces is $2 \times 1 + 1$ and so the statement is true for $n = 1$.

Suppose that the statement is true for $n = k$, that is, there are n cubes and $2n + 1$ exposed faces. Suppose that an extra cube is added to the existing layout. No matter whether the cube is attached to the vertical or a horizontal section the same number of exposed faces will be added.

Three faces of the new cube will be obscured so that 3 new faces will be added and one face of the existing cube will now be obscured. Thus the total number of exposed faces is $(2n + 1) + 3 - 1 = 2(n + 1) + 1$. Hence the statement is true for $n = k + 1$. The statement is true for all positive integers by induction.

Question 13 (a)

Criteria	Marks
• Provides correct proof	2
• Takes logs of both sides	1

Sample answer:

$a^{\ln b} = b^{\ln a} \quad \Leftrightarrow \quad \ln\left(a^{\ln b}\right)\ln\left(b^{\ln a}\right) \quad \Leftrightarrow \quad (\ln b)(\ln a) = (\ln a)(\ln b).$

Since the last statement is true for all positive real numbers a and b, so is the first.

Question 13 (b)

Criteria	Marks
• Provides correct solution	4
• Shows that $P_n\left(\omega^2\right) = 0$	3
• Shows that $P_n(\omega) = 0$	2
• Obtains $1 + \omega + \omega^2 = 0$, or equivalent merit	1

Sample answer:

Since ω is a cube root of unity, $\omega^3 = 1$ so $\omega^3 - 1 = \left(\omega - 1\right)\left(\omega^2 + \omega + 1\right) = 0$.

Hence $\omega^2 + \omega + 1 = 0$.

$$P_n(\omega) = (\omega + 1)^{2n+1} + \omega^{n+2} = \left(-\omega^2\right)^{2n+1} + \omega^{n+2} = \omega^{n+2}\left(1 - \omega^{3n}\right) = 0$$

Similarly, $P_n\left(\omega^2\right) = \left(\omega^2 + 1\right)^{2n+1} + \left(\omega^2\right)^{n+2} = (-\omega)^{2n+1} + \omega^{2n+4} = \omega^{2n+1}\left(-1 + \omega^3\right) = 0$.

Hence $\left(x - \omega\right)\left(x - \omega^2\right) = \left(x^2 + x + 1\right)$ is a factor of $P_n(x)$.

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

The forces acting on the particle are the force due to gravity in the same direction as the motion, and the resistive force in the opposite direction to the motion.

From Newton's 2nd Law, $m\ddot{x}$ = net force = $mg - \dfrac{1}{1000}v^2$, where $m = 0.5$.

$\therefore\ 0.5\ddot{x} = 0.5g - \dfrac{1}{1000}v^2$ and so $\ddot{x} = g - \dfrac{1}{500}v^2$.

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	5
• Obtains $t = \frac{25}{7}\ln\left(\frac{70+v}{70-v}\right)$, or equivalent merit	4
• Obtains correct integral, or equivalent merit	3
• Correctly uses partial fractions to rewrite integrand, or equivalent merit	2
• Obtains $\frac{dt}{dv} = \frac{500}{4900-v^2}$, or equivalent merit	1

Sample answer:

$$\frac{dv}{dt} = g - \frac{1}{500}v^2 = \frac{500 \times 9.8 - v^2}{500} = \frac{4900 - v^2}{500}$$

$$\frac{dt}{dv} = \frac{500}{4900 - v^2}$$

$$\therefore t = \int \frac{500}{4900 - v^2}\,dv.$$

Rewriting using partial fractions,

$$\frac{1}{4900 - v^2} = \frac{1}{(70-v)(70+v)} = \frac{a}{70-v} + \frac{b}{70+v}$$

$$1 = a(70+v) + b(70-v)$$

Substituting $v = 70$, $1 = 140a \quad \therefore a = \frac{1}{140}$

Substituting $v = -70$, $1 = 140b \quad \therefore b = \frac{1}{140}$

$$t = \int \frac{500}{4900 - v^2}\,dv = \frac{500}{140}\int \frac{1}{70-v} + \frac{1}{70+v}\,dv = \frac{25}{7}\left(-\ln(70-v) + \ln(70+v)\right) + C$$

$$= \frac{25}{7}\ln\left(\frac{70+v}{70-v}\right) + C$$

When $t = 0$, $v = 0$.

$$\therefore 0 = \frac{25}{7}\ln\left(\frac{70}{70}\right) + C$$

$$\therefore C = 0$$

Hence $t = \frac{25}{7}\ln\left(\frac{70+v}{70-v}\right)$

$$\frac{7t}{25} = \ln\left(\frac{70+v}{70-v}\right)$$

$$\therefore\ 0.28t = \ln\left(\frac{70+v}{70-v}\right)$$

$$\therefore\ e^{0.28t} = \frac{70+v}{70-v}$$

$$\therefore\ (70-v)e^{0.28t} = 70+v$$

$$\therefore\ 70e^{0.28t} - 70 = ve^{0.28t} + v$$

$$\therefore\quad v\left(e^{0.28t}+1\right) = 70\left(e^{0.28t}-1\right)$$

$$\therefore\ v = 70\frac{\left(e^{0.28t}-1\right)}{\left(e^{0.28t}+1\right)}.$$

Question 13 (c) (iii)

Criteria	Marks
• Provides correct solution	3
• Obtains $\therefore x = -250\ln\left(4900 - v^2\right) + 250\ln(4900)$	2
• Obtains $\dfrac{dx}{dv} = \dfrac{500v}{4900 - v^2}$	1

Sample answer:

$$v\frac{dv}{dx} = \frac{4900 - v^2}{500}$$

$$\therefore \frac{dx}{dv} = \frac{500v}{4900 - v^2}$$

$$\therefore \quad x = \int \frac{500v}{4900 - v^2}\,dv = \frac{500}{-2}\int \frac{-2v}{4900 - v^2}\,dv = -250\int \frac{-2v}{4900 - v^2}\,dv$$

$$= -250\ln\left(4900 - v^2\right) + k.$$

When $x = 0$, $v = 0$

$\therefore 0 = -250\ln(4900) + k$

$\therefore k = 250\ln(4900)$

$$\therefore x = -250\ln\left(4900 - v^2\right) + 250\ln(4900) = 250\ln\left(\frac{4900}{4900 - v^2}\right)$$

When $t = 3,\ v = 70\dfrac{\left(e^{0.84} - 1\right)}{\left(e^{0.84} + 1\right)} = 27.7851\ldots$

When $v = 27.7851\ldots,\ x = 250\ln\left(\dfrac{4900}{4900 - (27.7851)^2}\right) = 42.86$ metres.

ie $h = 42.86$ metres

Question 14 (a) (i)

Criteria	Marks
• Provides correct proof	3
• Deduces that if $\alpha = \dfrac{p}{q}$, then p is even	2
• Attempts proof by contradiction	1

Sample answer:

Suppose α was rational, so $\alpha = \dfrac{p}{q}$, where p and q are integers with no common factor.

Then $p^2 = 2(2n-1)q^2$. It follows that p is even, so $p = 2k$, from which it follows that $2k^2 = (2n-1)q^2$, from which it follows that q is also even, providing a contradiction. So α must be irrational.

Question 14 (a) (ii)

Criteria	Marks
• Provides correct proof	3
• Divides $[0,1)$ into N intervals AND attempts to apply the pigeon-hole principle	2
• Divides $[0,1)$ into N intervals OR attempts to apply the pigeon-hole principle	1

Sample answer:

The $N + 1$ numbers $0, \{\alpha\}, \{2\alpha\}, \ldots, \{(N-1)\alpha\}, \{N\alpha\}$ must each lie in one of the N intervals $\left[0, \dfrac{1}{N}\right), \left[\dfrac{1}{N}, \dfrac{2}{N}\right), \ldots, \left[1-\dfrac{1}{N}, 1\right)$, so by the pigeon-hole principle, at least one of these intervals contains two of the numbers. Those two numbers must differ by less than $1/N$.

Question 14 (a) (iii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

From part (ii) there are integers r and s with $0 \le r < s \le N$ with $\left|\{r\alpha\} - \{s\alpha\}\right| < 1/N$.
Since $r\alpha = R + \{r\alpha\}$ for some integer R and $s\alpha = S + \{s\alpha\}$ for some integer S, we have $|r\alpha - R + S - s\alpha| < 1/N$ or $|(r-s)\alpha - (R-S)| < 1/N$. Thus, if we take $q = s - r$ and $p = S - R$, we have $|q\alpha - p| < 1/N$ and $|q\alpha - p| \ne 0$ since $\alpha \ne \dfrac{p}{q}$.

Question 14 (b) (i)

Criteria	Marks
• Provides correct answer	2
• Provides correct interpretation for one constant	1

Sample answer:

u is the initial speed of the particle, and α is the angle between the direction it is thrown in and the horizontal.

Question 14 (b) (ii)

Criteria	Marks
• Provides correct answer	3
• Determines that $\dot{x}(t) = u\cos(\alpha)e^{-rt}$	2
• Determines $\ddot{x}(t) = -r\dot{x}$	1

Sample answer:

$\ddot{x}(t) = -r\dot{x}$, so $\dot{x}(t) = Ce^{-rt}$. Since $\dot{x}(0) = u\cos(\alpha)$, $\dot{x}(t) = u\cos(\alpha)e^{-rt}$.

So $x(t) = K - \dfrac{u}{r}\cos(\alpha)e^{-rt}$. Since $x(0) = 0$, we have

$x(t) = \dfrac{u\cos(\alpha)}{r}\left(1 - e^{-rt}\right)$

Question 14 (b) (iii)

Criteria	Marks
• Provides correct answer	2
• Verifies that the given expression satisfies the correct initial condition	1

Sample answer:

Observe that if $\dot{y}(t) = u\sin(\alpha)e^{-rt} - \dfrac{g\left(1-e^{-rt}\right)}{r}$, then

$$\begin{aligned}
\ddot{y}(t) &= -ur\sin(\alpha)e^{-rt} - \frac{g\left(0+re^{-rt}\right)}{r} \\
&= -r\left(u\sin(\alpha)e^{-rt} + \frac{g}{r}e^{-rt}\right) \\
&= -r\left(u\sin(\alpha)e^{-rt} - \frac{g\left(1-e^{-rt}\right)}{r}\right) - g \\
&= -r\dot{y} - g
\end{aligned}$$

and $\dot{y}(0) = u\sin(\alpha)e^{0} - \dfrac{g\left(1-e^{-0}\right)}{r} = u\sin(\alpha)$ as required.

Question 14 (b) (iv)

Criteria	Marks
• Provides correct value	2
• Identifies that the maximum occurs when $\dot{y} = 0$	1

Sample answer:

At the maximum, $\dot{y} = 0$ so $ur\sin(\alpha)e^{-rt} = g\left(1-e^{-rt}\right)$, which occurs when $\left(ur\sin(\alpha)+g\right)e^{-rt} = g$.

So $t = \dfrac{1}{r}\log_e\left(1+\dfrac{ur\sin(\alpha)}{g}\right)$

Question 14 (b) (v)

Criteria	Marks
• Provides correct answer	1

Sample answer:

From part (ii) we see that as $t \to \infty$, $x \to \dfrac{u\cos(\alpha)}{r}$. So you will be safe if you are more than $\dfrac{u\cos(\alpha)}{r}$ from the point at the base of the building directly below O.

Question 15 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Provides correct differentiation, or equivalent merit	1

Sample answer:

$$\dot{x} = -12\sin 2t + 16\cos 2t$$

$$\begin{aligned}\dot{x} &= -24\cos 2t - 32\sin 2t \\ &= -4(6\cos 2t + 8\sin 2t) \\ &= -4(5 + 6\cos 2t + 8\sin 2t - 5) \\ &= -4(x-5) \\ &= -2^2(x-5)\end{aligned}$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains equation $\sin(2t+\alpha)=-\frac{1}{2}$, or equivalent merit	2
• Attempts to write x in the form $5+A\sin(2t+\alpha)$, or equivalent merit	1

Sample answer:

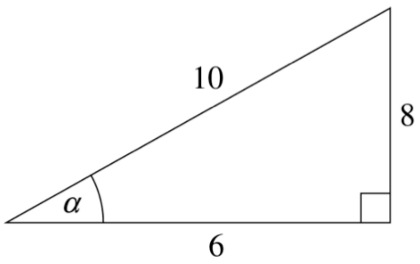
Displacement zero means $\quad x=0$

$$5+6\cos 2t+8\sin 2t=0$$
$$6\cos 2t+8\sin 2t=-5$$

Rewrite as $\quad 10\cos(2t-\alpha)=-5$

$$10(\cos 2t\alpha+\sin 2t\sin\alpha)=-5$$

Looking at the triangle



we see that

$$\tan x=\frac{8}{6}$$
$$x=\tan^{-1}\frac{8}{6}$$
$$x\approx 0.9273$$

Hence $\cos(2t-0.9273)\approx-\frac{5}{10}=-0.5$

$$2t-0.9273=\frac{2\pi}{3}\quad\left(\text{since }\cos\left(\frac{2\pi}{3}\right)=-0.5\right)$$
$$=2.0944$$
$$2t=3.02169$$
$$t=1.511$$

Question 15 (b)

Criteria	Marks
• Provides correct solution	4
• Solves at least one equation of the cosine, or equivalent merit	3
• Takes three correct dot products, or equivalent merit	2
• Attempts to use the dot product formula to find $\underset{\sim}{x} \cdot \underset{\sim}{i}$	1

Sample answer:

Let $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $\underset{\sim}{x} \cdot \underset{\sim}{i} = a$ and so $a = \sqrt{a^2 + b^2 + c^2} \cos\alpha$.

Similarly, then $\underset{\sim}{x} \cdot \underset{\sim}{i} = b$ and so $b = \sqrt{a^2 + b^2 + c^2} \cos\beta$ and $\underset{\sim}{x} \cdot \underset{\sim}{k} = c$ and so $c = \sqrt{a^2 + b^2 + c^2} \cos\gamma$.

Hence $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2} = 1$.

Question 15 (c) (i)

Criteria	Marks
• Provides correct proof	1

Sample answer:

Let $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$, $b = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$ and $\underset{\sim}{c} = c_1\underset{\sim}{i} + c_2\underset{\sim}{j} + c_3\underset{\sim}{k}$

Then

$$\begin{aligned}
\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) &= \left(a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}\right) \cdot \left((b_1 + c_1)\underset{\sim}{i} + (b_2 + c_2)\underset{\sim}{j} + (b_3 + c_3)\underset{\sim}{k}\right) \\
&= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\
&= \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}
\end{aligned}$$

Question 15 (c) (ii)

Criteria	Marks
• Provides correct answer	2
• Substitutes $\underset{\sim}{w} = \underset{\sim}{v} + \lambda \underset{\sim}{m}$ into $\left\|\underset{\sim}{w} - \underset{\sim}{c}\right\| = r$	1

Sample answer:

$\left|\underset{\sim}{v} + \lambda \underset{\sim}{m} - \underset{\sim}{c}\right| = r$ so $\left((\underset{\sim}{v} - \underset{\sim}{c}) + \lambda \underset{\sim}{m}\right) \cdot \left((\underset{\sim}{v} - \underset{\sim}{c}) + \lambda \underset{\sim}{m}\right) = r^2$.

Thus $\lambda^2 |\underset{\sim}{m}|^2 + 2\lambda \underset{\sim}{m} \cdot (\underset{\sim}{v} - \underset{\sim}{c}) = 0$, and so $\lambda = 0$ or $\dfrac{2\underset{\sim}{m} \cdot (\underset{\sim}{c} - \underset{\sim}{v})}{|\underset{\sim}{m}|^2}$.

Question 15 (c) (iii)

Criteria	Marks
• Provides correct answer	2
• Provides correct deduction or correct geometrical interpretation	1

Sample answer:

ℓ is a tangent if and only if there is only one point of intersection, and this occurs if and only if $\underset{\sim}{m} \cdot (\underset{\sim}{v} - \underset{\sim}{c}) = 0$.

Thus, a line through a point P on a sphere S with centre C is a tangent to the sphere if and only if its direction is perpendicular to the radius CP.

Question 16 (a) (i)

Criteria	Marks
• Provides correct proof	1

Sample answer:

Replacing x with x^2, y with y^2 in the given inequality we have

$$\sqrt{x^2 y^2} \le \frac{x^2 + y^2}{2}$$

ie $\quad xy \le \dfrac{x^2 + y^2}{2}$

so $\sqrt{xy} \le \sqrt{\dfrac{x^2 + y^2}{2}}$, since $x, y > 0$

Question 16 (a) (ii)

Criteria	Marks
• Provides correct proof	2
• Uses part (i) to obtain a valid expression in a^2, b^2, c^2 and d^2, or equivalent merit	1

Sample answer:

$$\begin{aligned}
\sqrt{(ab)(cd)} &= \sqrt{ab}\sqrt{cd} \\
&\le \sqrt{\frac{a^2+b^2}{2}}\sqrt{\frac{c^2+d^2}{2}} \qquad \text{by part (i)} \\
&\le \sqrt{\left(\frac{a^2+b^2}{2}\right)\left(\frac{c^2+d^2}{2}\right)} \\
&\le \frac{1}{2}\left(\frac{a^2+b^2+c^2+d^2}{2}\right) \qquad \text{by given inequality} \\
&= \frac{a^2+b^2+c^2+d^2}{4}
\end{aligned}$$

Taking positive square roots,

$$\sqrt[4]{(abcd)} \le \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$$

Question 16 (b) (i)

Criteria	Marks
• Provides the correct answer	1

Sample answer:

$$\begin{aligned}
I_1 &= \int_0^1 x^1\sqrt{1-x^2}\,dx \\
&= \frac{1}{2}\int_0^1 -2x\sqrt{1-x^2}\,dx \\
&= -\frac{1}{2}\left[\frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1 \\
&= -\frac{1}{3}(0-1) \\
&= \frac{1}{3}
\end{aligned}$$

Question 16 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly uses integration by parts, simplifying where possible, or equivalent merit	2
• Attempts to use integration by parts, or equivalent merit	1

Sample answer:

$$I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$$

Let $\quad u = x^{n-1} \qquad dv = x\sqrt{1-x^2}\,dx$

$$du = (n-1)x^{n-2}dx \qquad v = -\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}$$

$$\therefore\ I_n = \left[x^{n-1} \times \frac{-1}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1 + \frac{1}{3}\int_0^1 \left(1-x^2\right)^{\frac{3}{2}}(n-1)x^{n-2}\, dx$$

$$= \frac{(n-1)}{3}\int_0^1 x^{n-2}\left(1-x^2\right)\sqrt{1-x^2}\, dx$$

$$= \frac{(n-1)}{3}\int_0^1 x^{n-2}\sqrt{1-x^2}\, dx - \frac{(n-1)}{3}\int_0^1 x^n\sqrt{1-x^2}\, dx$$

$$\therefore\ I_n = \frac{n-1}{3} I_{n-2} - \frac{n-1}{3} I_n$$

$$I_n\left(1 + \frac{n-1}{3}\right) = \frac{n-1}{3} I_{n-2}$$

$$I_n\left(\frac{3+n-1}{3}\right) = \frac{n-1}{3} I_{n-2}$$

$$\therefore\ I_n = \frac{n-1}{n+2} I_{n-2}$$

Question 16 (b) (iii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$
\begin{aligned}
I_5 &= \frac{4}{7} \times I_3 \\
&= \frac{4}{7} \times \frac{2}{5} \times I_1 \\
&= \frac{4}{7} \times \frac{2}{5} \times \frac{1}{3} \qquad \text{from part (i)} \\
&= \frac{8}{105}
\end{aligned}
$$

Question 16 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Correctly sums and simplifies the middle term, or equivalent merit	2
• Applies formula for geometric series, or equivalent merit	1

Sample answer:

$$\frac{1}{1+x^2} - \left(1 - x^2 + x^4 - x^6 + \ldots + (-1)^{n-1} x^{2(n-1)}\right)$$

$$= \frac{1}{1+x^2} - \frac{1-\left(-x^2\right)^n}{1+x^2} = \frac{1-1+\left(-x^2\right)^n}{1+x^2}, \quad \text{using sum of geometric series.}$$

$$= \frac{\left(-x^2\right)^n}{1+x^2}$$

Since $1 + x^2 \geq 1$ for all $x, -x^{2n} \leq \dfrac{\left(-x^2\right)^n}{1+x^2} \leq x^{2n}$

We have

$$-x^{2n} \leq \frac{1}{1+x^2} - \left(1 - x^2 + x^4 \ldots + (-1)^{n-1} x^{2(n-1)}\right) \leq x^{2n}$$

Question 16 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Integrates terms in the equality in part (i) or equivalent merit	2
• Makes some progress towards the correct solution	1

Sample answer:

Inequalities are preserved by integration:

$$\pm\int_0^1 x^{2n}\,dx = \pm\left.\frac{x^{2n+1}}{2n+1}\right|_0^1 = \pm\frac{1}{2n+1}$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left.\tan^{-1}x\right|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\int_0^1 1 - x^2 + x^4 \ldots + (-1)^{n-1}x^{2n-2}\,dx$$

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1}\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^{n-1}\frac{1}{2n-1}$$

$$\text{so } -\frac{1}{2n+1} \le \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^{n-1}\frac{1}{2n-1}\right) \le \frac{1}{2n+1}$$

Question 16 (c) (iii)

Criteria	Marks
• Provides correct explanation	1

Sample answer:

$$\text{Since } \lim_{x\to\infty} -\frac{1}{2n+1} = \lim_{x\to\infty} -\frac{1}{2n+1} = 0,$$

then as $n \to \infty$

$$\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^{n-1}\frac{1}{2n-1}\right) \to 0$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\ldots$$

Mathematics Extension 2
Sample HSC Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes	Targeted performance bands
1	1	MEX-C1 Further Integration	MEX12-5	E2-E3
2	1	MEX-C1 Further Integration	MEX12-5	E2-E3
3	1	MEX-N1 Introduction to Complex Numbers	MEX12-4	E2-E3
4	1	MEX-P1 The Nature of Proof	MEX12-8	E2-E3
5	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6	E3-E4
6	1	MEX-P1 The Nature of Proof	MEX12-2	E2-E3
7	1	MEX-N1 Introduction to Complex Numbers	MEX12-4	E3-E4
8	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6	E3-E4
9	1	MEX-P1 The Nature of Proof	MEX12-2	E3-E4
10	1	MEX-V1 Further Work with Vectors	MEX12-3	E3-E4

Section II

Question	Marks	Content	Syllabus outcomes	Targeted performance bands
11 (a) (i)	2	MEX-N1 Introduction to Complex Numbers	MEX12-4	E2-E3
11 (a) (ii)	1	MEX-N1 Introduction to Complex Numbers	MEX12-4	E2-E4
11 (a) (iii)	1	MEX-N1 Introduction to Complex Numbers	MEX12-4	E2-E3
11 (b) (i)	1	MEX-N1 Introduction to Complex Numbers	MEX12-4	E2-E3
11 (b) (ii)	2	MEX-N2 Using Complex Numbers	MEX12-4	E2-E3
11 (c) (i)	2	MEX-C1 Further Integration	MEX12-5	E2-E3
11 (c) (ii)	2	MEX-C1 Further Integration	MEX12-5	E2-E3
11 (d)	4	MEX-C1 Further Integration	MEX12-5	E2-E4
12 (a) (i)	1	MEX-N2 Using Complex Numbers	MEX12-4	E2-E3
12 (a) (ii)	2	MEX-N2 Using Complex Numbers	MEX12-4	E3-E4
12 (b) (i)	2	MEX-N2 Using Complex Numbers	MEX12-4	E2-E3
12 (b) (ii)	1	MEX-N2 Using Complex Numbers	MEX12-4	E2-E3
12 (b) (iii)	2	MEX-N2 Using Complex Numbers	MEX12-4	E3-E4

Question	Marks	Content	Syllabus outcomes	Targeted performance bands
12 (c)	3	MEX-N2 Using Complex Numbers	MEX12-7, MEX12-8	E2-E4
12 (d)	3	MEX-P2 Further Proof by Mathematical Induction	MEX12-2	E2–E3
13 (a)	2	MEX-P1 The Nature of Proof	MEX12-2	E2-E3
13 (b)	4	MEX-N2 Using Complex Numbers	MEX12-4; MEX12-7; MEX12-8	E2-E4
13 (c) (i)	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6	E2-E3
13 (c) (ii)	5	MEX-C1 Further Integration	MEX12-5; MEX12-7	E2-E4
13(c) (iii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6; MEX12-7	E2-E4
14 (a) (i)	3	MEX-P1 The Nature of Proof	MEX12-2; MEX12-8	E2-E4
14 (a) (ii)	3	MEX-P1 The Nature of Proof	MEX12-2; MEX12-4; MEX12-8	E2-E4
14 (a) (iii)	1	MEX-P1 The Nature of Proof	MEX12-8	E3-E4
14 (b) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-8	E2-E3
14 (b) (ii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6; MEX12-7	E2-E4
14(b) (iii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6; MEX12-7	E2-E4
14 (b) (iv)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6; MEX12-7	E2-E4
14 (b) (v)	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6; MEX12-7; MEX12-8	E3-E4
15 (a) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6; MEX12-7	E2-E4
15 (a) (ii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6; MEX12-7; MEX12-8	E2-E4
15 (b)	4	MEX-V1 Further Work with Vectors	MEX12-3; MEX12-7	E2-E4
15 (c) (i)	1	MEX-V1 Further Work with Vectors	MEX12-3; MEX12-7	E2-E3
15 (c) (ii)	2	MEX-V1 Further Work with Vectors	MEX12-3; MEX12-7	E2-E4
15 (c) (iii)	2	MEX-V1 Further Work with Vectors	MEX12-3; MEX12-7	E2-E4
16 (a) (i)	1	MEX-P1 The Nature of Proof	MEX12-2	E2-E3
16 (a) (ii)	2	MEX-P1 The Nature of Proof	MEX12-2: MEX12-7	E3-E4
16 (b) (i)	1	MEX-C1 Further Integration	MEX12-5	E2-E3
16 (b) (ii)	3	MEX-C1 Further Integration	MEX12-5; MEX12-8	E2-E4
16 (b) (iii)	1	MEX-C1 Further Integration	MEX12-5	E3-E4

Question	Marks	Content	Syllabus outcomes	Targeted performance bands
16 (c) (i)	3	MEX-P1 The Nature of Proof	MEX12-2; MEX12-7	E3-E4
16 (c) (ii)	3	MEX-P1 The Nature of Proof MEX-C1 Further Integration	MEX12-2; MEX12-5; MEX12-8	E3-E4
16 (c) (iii)	1	MEX-P1 The Nature of Proof	MEX12-7; MEX12-8	E3-E4

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HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Writing Booklet

Question XX

SAMPLE

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Instructions

- Use this booklet for Question XX. → XX
- Write the number of this booklet and the total number of booklets that you have used for this question (eg: 1 of 3). → ☐ of ☐ (this booklet / number of booklets for this question)
- Write in black or blue pen (black is recommended).
- You may ask for an extra writing booklet if you need more space.
- If you have not attempted the question(s), you must still hand in the writing booklet, with 'NOT ATTEMPTED' written clearly on the front cover.
- You may NOT take any writing booklets, used or unused, from the examination room.

Start here for Question Number: XX

SAMPLE

Additional writing space on back page.

SAMPLE

SAMPLE

Tick this box if you have continued this answer in another writing booklet.