ZETA
EXAMINATION
MATERIALS

Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 3 hours • Write using black pen • Calculators approved by NESA may be used • For questions in Section II, show relevant mathematical reasoning and/ or calculations
Total Marks: 100	Section I – 10 marks (pages 1–4) • Attempt Questions 1–10 • Allow about 15 minutes for this section Section II – 90 marks (pages 5–15) • Attempt Questions 11–16 • Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

1. Define the complex numbers $z = 1 + 3i$ and $w = 4 - 5i$. What is the projection of z onto w?

   A. $-\frac{44}{41} + \frac{55}{41}i$

   B. $-\frac{44}{\sqrt{41}} + \frac{55}{\sqrt{41}}i$

   C. $-\frac{55}{41} + \frac{44}{41}i$

   D. $-\frac{44}{\sqrt{41}} - \frac{55}{\sqrt{41}}i$

2. A sphere of unit radius is tangent to the xy-plane, the xz-plane and the yz-plane. Label A as the point on the sphere which is closest to the origin.

   What is the distance between A and the origin?

   A. 1

   B. $\sqrt{2}$

   C. $\sqrt{2} - 1$

   D. $\sqrt{3} - 1$

3. Which of the following is logically equivalent to the converse of the following statement?

   "If there are no teachers in the room, then there are no students in the room"

   A. If there is a teacher in the room, then there is a student in the room with them

   B. If there is a student in the room, then there is a teacher in the room with them

   C. If there are no teachers in the room, then there is a student in the room

   D. If there no students in the room, then there is a teacher in the room

4. Which of the following is a primitive of $\frac{x+1}{\sqrt{9-(x-5)^2}}$?

   A. $\sqrt{9-(x-5)^2}+12\sin^{-1}\left(\frac{x-5}{3}\right)$

   B. $\frac{1}{2}\sqrt{9-(x-5)^2}+6\sin^{-1}\left(\frac{x-5}{3}\right)$

   C. $x\sqrt{9-(x-5)^2}+4\tan^{-1}\left(\frac{x-5}{3}\right)$

   D. $\frac{1}{2}x\sqrt{9-(x-5)^2}+12\sin^{-1}\left(\frac{x-5}{3}\right)$

5. Two particles, A and B, initiate simple harmonic motion from the origin at the same time, moving in the same direction. Both particles oscillate about the origin. Particle A moves with period T, and Particle B moves with period $2T$. The particles collide after a time interval of $\frac{1}{4}T$. What is the ratio $a_B : a_A$ of their amplitudes?

   A. $1 : \sqrt{2}$

   B. $\sqrt{2} : 1$

   C. $\sqrt{3} : 2$

   D. $2 : \sqrt{3}$

6. Which of the following is a true statement?

   A. $\frac{n(n+1)}{2}$ is an odd number for $n \geq 1$

   B. $4n^2 - 1$ is divisible by 3 for $n \geq 1$

   C. $\sqrt{5n-4}$ is irrational for $n \geq 2$

   D. $n^3 - n$ is divisible by 6 for $n \geq 2$

7. The complex numbers z_1 and z_2 satisfy the following conditions:

$$z_1 + z_2 = 10 + 24i$$
$$|z_1 - z_2| = 2$$

What is the smallest possible value of $|z_1|$?

A. 9

B. 11

C. 24

D. 25

8. A particle is moving in 2-dimensional space, and is initially located at (4, 3). It is given that for the entirety of the particle's motion, $\boldsymbol{r} \bullet \boldsymbol{v} > 0$, where $\boldsymbol{r}$ and $\boldsymbol{v}$ are the particle's displacement and velocity vectors, respectively. Which of the following cannot be true?

A. The particle will eventually pass through the point (1,1)

B. The particle will approach, but not reach, a limiting displacement

C. The particle will eventually hit the y-axis

D. The particle's speed will eventually surpass its initial speed

9. A sphere S passes through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ and the origin. What is the centre of the sphere?

A. $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$

B. $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

C. $(a - b - c, b - a - c, c - a - b)$

D. $\left(\frac{a}{bc}, \frac{b}{ac}, \frac{c}{ab}\right)$

10. Define the complex polynomial $P(z) = z^n + az^{n-1} + 1$, where a is real. The complex number w is a non-real root of $P(z)$ with modulus 1. Which of the following is false?

A. $-2 < a < 2$

B. $1 + aw + w^n = 0$

C. w is an $(n-1)$-th root of unity

D. w is an $(n-2)$-th root of unity

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE Writing Book

a) Define the complex numbers $z = 3 + 4i$ and $w = 2 - 2i$. Find:

(i) $z + \overline{w}$ **1**

(ii) $\left|\frac{z}{w}\right|$ **2**

b) Find a line equation that passes through the point (3, -2, 1) and is parallel to the line with equation $\boldsymbol{r}(t) = \begin{pmatrix} -1 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ **1**

c) Write the complex number $2\sqrt{3} + 2i$ in exponential form **2**

d) The acceleration of a particle is given by $a = \frac{3}{2+v}$, where a represents the acceleration and v represents the velocity. **3**

Given that it is initially stationary and located at $x = 1$, find the displacement x of the particle when $v = 2$.

Question 11 continues on the next page

Question 11 (continued)

e) Label the points (2, 1, 5), (-1, 9, 4) and (6, -3, -2) as A, B and C respectively. It is given that $ABCD$ forms a parallelogram in 3D space. Find the point D. **2**

f) Evaluate $\int_0^{\frac{\pi}{4}} x \sin x \, dx$ **4**

End of Question 11

Question 12 (15 marks) Use a SEPARATE Writing Book

a) Consider the two lines l_1 and l_2, defined below:

$$l_1 \colon r_1(t) = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

$$l_2 \colon r_2(t) = \begin{pmatrix} -6 \\ 3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

(i) Find the intersection point of these two lines **2**

(ii) Find the acute angle formed between the two lines at this point. **2**

b) Consider the following partial fraction decomposition

$$\frac{6x^2 - 9x + 17}{(2x^2 + 3)(x + 5)} = \frac{Ax + B}{2x^2 + 3} + \frac{C}{x + 5}$$

(i) Identify the constants A, B and C in this partial fraction decomposition **2**

(ii) Hence evaluate $\int_0^2 \frac{6x^2-9x+17}{(2x^2+3)(x+5)}$ **2**

c) Define the complex polynomials $P_1(z) = z^2 + 6z + 10$ and $P_2(z) = z^3 + az^2 + bz + 20$, **2**

where a and b are real. Given that the polynomials share a common root, find a.

d) Let a and b be real positive numbers. You are given that $\frac{a+b}{2} \geq \sqrt{ab}$ (DO NOT PROVE THIS).

(i) Show that $\frac{a^2+b^2}{2} \geq ab$ **1**

(ii) Hence or otherwise show that, for positive c and d, **3**

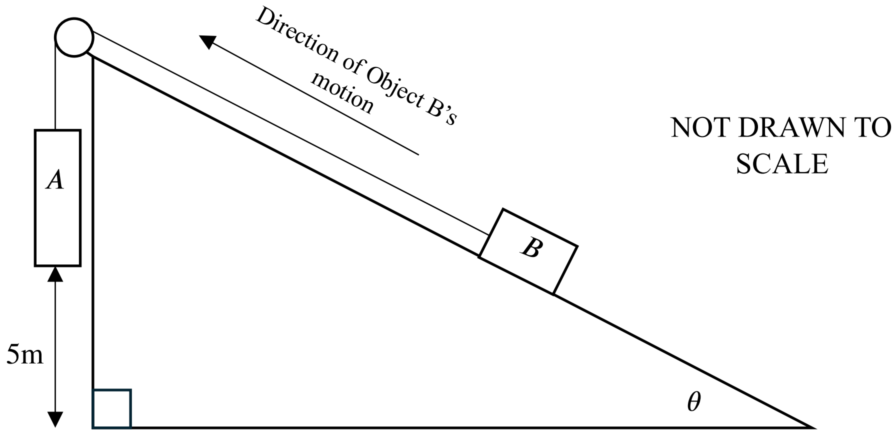
$$\frac{a^2 + b^2 + c^2 + d^2}{4} \geq \sqrt{abcd}$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE Writing Book

a) Prove that if $r^5 + 6r^3 + 2r > 3r^4 + 9r^2 + 6$, then r is positive. **2**

b) Objects A and B have masses 10kg and 6kg respectively, and are attached via a light inextensible string running over a pulley. The pulley is positioned at the top of an incline, angled at θ to the horizontal, upon which object B is situated. Object A hangs along the vertical side of the inclined block, 5m above the floor.



The incline's surface exerts a frictional force of $0.4g$ with object B as it slides on the ramp. Apart from tension, gravity and the normal force on Object B, no other forces act on either object. Initially, both objects are stationary, with the acceleration of the system such that Object A descends and Object B slides up the slope. Take $g = 10\ m/s^2$

(i) Show that the acceleration a of the system satisfies $a = \frac{3}{8}g(1.6 - \sin\theta)$ **2**

(ii) When Object A hits the floor, it is moving with a speed of $\sqrt{30}\ m/s$. Show that $\sin\theta = 0.8$ **2**

(iii) Once Object A hits the floor, it immediately stops moving. Object B's momentum causes it to slide slightly further up the ramp to a maximal position P, before sliding back down until the string is taut. **3**

Find the total distance travelled **up** the ramp by Object B.

Question 13 continues on the next page

Question 13 (continued)

c) Evaluate the following integral **3**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1+2\sin x}{3-\cos x}\,dx$$

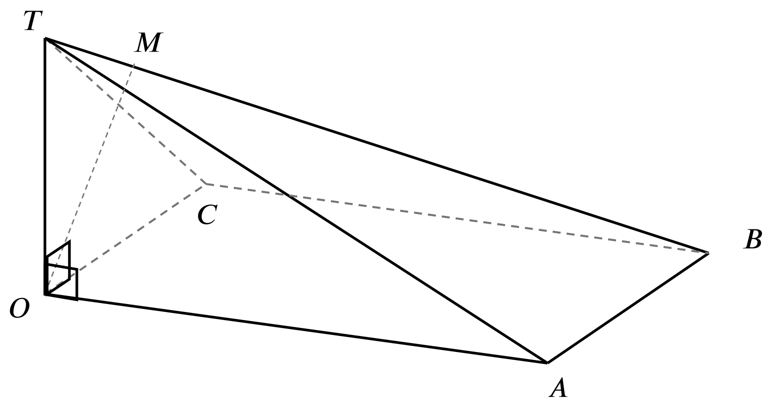
d) Use mathematical induction to prove that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} \geq \sqrt{n}$ for all $n \geq 1$ **3**

End of Question 13

Question 14 (14 marks) Use a SEPARATE Writing Book

a) The rectangular pyramid $OABCT$ is constructed such that the ratio $|OT|:|OA|:|AB|$ is 1:3:1.

The angles $\angle TOA$ and $\angle TOC$ are both right angles, and the point M is located on edge TB such that $OM \perp TB$.



(i) By considering a projection, show that $\overrightarrow{TM} = \frac{|\overrightarrow{TO}|^2}{|\overrightarrow{TB}|^2}\overrightarrow{TB}$ **3**

(ii) Hence find the ratio $TM: MB$ **1**

b) Consider the equation $z^5 = 1$.

(i) Write down the solutions to this equation in exponential form **1**

Let w be a non-real solution to this equation.

(ii) Show that $1 + w + w^2 + w^3 + w^4 = 0$ **1**

(iii) Show that $\frac{1}{2} + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = 0$ **2**

(iv) Hence show that $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$ **2**

Question 14 continues on the next page

Question 14 (continued)

c) A particle moves in simple harmonic motion according to the equation $\ddot{x} = -n^2(x - 2)$. It begins at the origin, moving toward the positive x-axis at a speed of u m/s, and it reaches the point $x = 5$ moving with speed $\frac{2}{3}u$.

(i) Find the amplitude of the particle's motion **2**

(ii) Given that the maximum acceleration experienced by the particle is 6m/s^2, find u **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE Writing Book

a) A particle of mass m is launched vertically upwards from the ground at a speed of U m/s. It experiences a downward gravitational force of mg, and air resistance of magnitude kv, where v is the particle's velocity and k is a positive constant.

The particle reaches its maximum height at $T = \frac{m}{k}\ln\left(1 + \frac{k}{mg}U\right)$

(i) Show that the particles maximum height is $H = \frac{m}{k}U\,\frac{m^2 g}{k^2}\ln\left(1 + \frac{k}{mg}U\right)$ **2**

A second particle of the same mass is projected upward from the same spot at time $\frac{T}{2}$, and collides with the first particle just as the first particle reaches its maximum height.

(ii) Show that the launch speed of the second particle is

$$V = \frac{mg}{k}\left(\frac{\sqrt{Q}}{\sqrt{Q}-1}\right)\left[Q - 1 - \ln\sqrt{Q}\right] - \frac{mg}{k}$$ **4**

Where $Q = 1 + \frac{k}{mg}U$

b) The complex number z lies on the circle with centre at $(1, 0)$ and radius 1 in the complex plane. Another complex number w also lies on this circle, with $\text{Im}(w) = \text{Im}(z)$. Let the points A and B represent the positions of w and $\overline{z}$ respectively. **3**

By writing the complex numbers z, w and $\overline{z}$ in the form $1 + e^{i\theta}$, or otherwise, show that $\angle AOB = \frac{\pi}{2}$

Question 15 continues on the next page

c) Define the recursive integral $I_n = \int_0^{\frac{1}{2}} \frac{x^n}{1-x} dx$

(i) Show that $I_n = -\frac{1}{n2^n} + I_{n-1}$, and hence show that **3**

$$I_n = \ln 2 - \sum_{k=1}^{n} \frac{1}{k2^k}$$

(ii) Explain why **1**

$$\int_0^{\frac{1}{2}} x^n dx < I_n < \int_0^{\frac{1}{2}} 2x^n dx$$

(iii) Deduce that **2**

$$\ln 2 = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k2^k}$$

End of Question 15

Question 16 (16 marks) Use a SEPARATE Writing Book

a) Let $w = \cos x + i \sin x$

(i) Show that **1**

$$w + w^2 + \cdots + w^n = \frac{e^{inx} - 1}{e^{ix} - 1} e^{ix}$$

(ii) Hence show that **3**

$$\sum_{k=1}^{n} \sin(kx) = \frac{\sin\left(\frac{nx}{2}\right) \sin\left(\frac{n+1}{2} x\right)}{\sin\left(\frac{x}{2}\right)}$$

b) Let $z_k = e^{\frac{2\pi k}{n} i}$ represent a generic n-th root of unity

(i) Show that $|z_k - 1| = 2\sin\left(\frac{\pi k}{n}\right)$ **2**

Define the points $A_0, A_1 \ldots A_{n-1}$ where A_0 is the point representing the origin on the complex plane, and

$$A_{k+1} = \sum_{j=0}^{k} z_j$$

The points A_k form the vertices of a regular n-sided polygon $\mathscr{P}$ with a side length of 1 (do NOT prove this)

(ii) Show that the distance between A_0 and A_k is $\frac{\sin\left(\frac{\pi k}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$ **2**

(iii) Using the result of part a) (ii), or otherwise, show that the sum of the lengths of the diagonals of the polygon $\mathscr{P}$ which have an endpoint at the origin is **3**

$$\frac{1}{2}\operatorname{cosec}^2\left(\frac{\pi}{2n}\right) - 2$$

(iv) Write down an expression for the sum of the lengths of all diagonals of $\mathscr{P}$ **1**

Question 16 continues on the next page

Question 16 (continued)

c) A sphere $\mathcal{S}_1$ is defined in the xyz co-ordinate space, with unit radius and centre at $(0, 5, 5)$. **4**

Another sphere $\mathcal{S}_2$ is centred at $(2, 0, 1)$ with radius r. A line l passing through the origin is tangent to $\mathcal{S}_1$.

Find the value of r for which the line l is also tangent to $\mathcal{S}_2$.

End of Exam

Mathematics Extension 2 Trial - Marking Guidelines

Section I

Multiple Choice Answer Key

Question	Answer
1	A
2	D
3	A
4	C
5	B
6	D
7	B
8	A
9	A
10	C

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$z + \overline{w} = 3 + 4i + 2 + 2i = 5 + 6i$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Calculates magnitudes correctly	1

Sample answer:

$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{\sqrt{3^2+4^2}}{\sqrt{2^2+2^2}} = \frac{5}{2\sqrt{2}}$

Question 11 (b)

Criteria	Marks
• Provides a correct solution	1

Sample answer:

A possible solution is the line l where

$$l = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

Question 11 (c)

Criteria	Marks
• Provides a correct solution	2
• Correctly finds either magnitude or argument	1

Sample answer:

$\left|2\sqrt{3}+2i\right| = \sqrt{12+4} = 4$

$\text{Arg}\left(2\sqrt{3}+2i\right) = \tan^{-1}\frac{2}{2\sqrt{3}} = \frac{\pi}{6}$

$\therefore 2\sqrt{3}+2i = 4e^{\frac{\pi}{6}i}$

Question 11 (d)

Criteria	Marks
• Provides a correct solution	3
• Finds the constant of integration	2
• Correctly integrates, without finding the constant of integration	1

Sample answer:

$a = v\frac{dv}{dx} = \frac{3}{2+v}$

$$\int (v^2+2v)dx = 3\int dx$$

$$\frac{1}{3}v^3 + v^2 = 3x + C$$

Initially, $v = 0$ and $x = 1 \Rightarrow 0 = 3 + C$

$\therefore C = -3$

$\frac{1}{3}v^3 + v^2 = 3x - 3$

Sub $v = 2$, $\frac{8}{3} + 4 = 3x - 3 \Rightarrow x = \frac{29}{9}$

Question 11 (e)

Criteria	Marks
• Provides a correct solution	2
• Correctly finds either $\overrightarrow{BC}$ or $\overrightarrow{BA}$	1

Sample answer:

$$\overrightarrow{BC} = \begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 9 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ -12 \\ -6 \end{pmatrix}$$

$\overrightarrow{AD} = \overrightarrow{BC}$ since ABCD is a parallelogram

$$\therefore \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 7 \\ -12 \\ -6 \end{pmatrix} = \begin{pmatrix} 9 \\ -11 \\ -1 \end{pmatrix}$$

Question 11 (f)

Criteria	Marks
• Provides a correct solution	4
• Correctly integrates, with an algebra mistake	3
• Obtains a correct expression for $uv - \int v du$	2
• Attempts to use integration by parts	1

Sample answer:

Let $u = x$ and $dv = \sin x\, dx$

$du = dx,\ v = -\cos x$

$$\int_0^{\frac{\pi}{4}} x \sin x\, dx = -[x \cos x]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos x\, dx$$

$$= -\left(\frac{\pi}{4} \times \frac{1}{\sqrt{2}} - 0\right) + [\sin x]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{\sqrt{2}} - 0\right) - \frac{\pi}{4\sqrt{2}} = \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)$$

Question 12 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to solve set of simultaneous equations	1

Sample answer:

$\boldsymbol{r}_1(\lambda) = \begin{pmatrix} 3+\lambda \\ 1-2\lambda \\ 3-\lambda \end{pmatrix}$ and $\boldsymbol{r}_2(\mu) = \begin{pmatrix} -6-5\mu \\ 3+2\mu \\ 4+\mu \end{pmatrix}$

Let $\boldsymbol{r}_1 = \boldsymbol{r}_2$. Then we have 3 simultaneous equations:

$3 + \lambda = -6 - 5\mu$ (1)

$1 - 2\lambda = 3 + 2\mu$ (2)

$3 - \lambda = 4 + \mu$ (3)

Add equations 1 and 3 $\Rightarrow 6 = -2 - 4\mu$,

$\therefore \mu = -2$

Substitute into $\boldsymbol{r}_2 \Rightarrow$ Intersection point is $(4, -1, 2)$

Question 12 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Equates two expressions for the dot product of the direction vectors of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$	1

Sample answer:

The acute angle between the two lines corresponds to the acute angle formed between the direction vectors at their point of intersection.

$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} = \left|\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}\right| \left|\begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}\right| \cos\theta$, where θ is the angle between the two lines

$-5 + (-4) + (-1) = \sqrt{6}\sqrt{30}\cos\theta$

$-10 = 6\sqrt{5}\cos\theta$

$\theta = \cos^{-1}\left(-\frac{5}{3\sqrt{5}}\right)$

This will give the value of the obtuse angle between the lines – the acute angle is found by subtracting this value from 180°

$\therefore$ The acute angle is $180° - \cos^{-1}\left(-\frac{5}{3\sqrt{5}}\right) \approx 42°$

Question 12 (b) (i)

Criteria	Marks
• Finds correct values of all constants	2
• Correctly finds the values of two constants	1

Sample answer:

$\frac{6x^2-9x+17}{(2x^2+3)(x+5)} = \frac{Ax+B}{2x^2+3} + \frac{C}{x+5}$

$6x^2 - 9x + 17 = (Ax + B)(x + 5) + C(2x^2 + 3)$

Let $x = -5 \quad \Rightarrow \quad 53C = 212$

$\therefore C = 4$

Let $x = 0 \quad \Rightarrow \quad 5B + 12 = 17$

$\therefore B = 1$

Let $x = 1 \quad \Rightarrow \quad 6(A + 1) + 20 = 14$

$\therefore A = -2$

Question 12 (b) (ii)

Criteria	Marks
• Provides a correct answer	2
• Finds a correct primitive expression	1

Sample answer:

$$\int_0^2 \frac{6x^2-9x+17}{(2x^2+3)(x+5)}\,dx = \int_0^2 \left(\frac{1}{2x^2+3} - \frac{2x}{2x^2+3} + \frac{4}{x+5}\right) dx$$

$$= \left[\frac{1}{2}\sqrt{\frac{2}{3}}\tan^{-1}\left(\sqrt{\frac{2}{3}}x\right) - \frac{1}{2}\ln\left|2x^2+3\right| + 4\ln|x+5|\right]_0^2$$

$$= \frac{1}{2}\sqrt{\frac{2}{3}}\tan^{-1}\left(2\sqrt{\frac{2}{3}}\right) - \frac{1}{2}\ln(11) + 4\ln(7) - \left(0 - \frac{1}{2}\ln(3) + 4\ln(5)\right)$$

$$= \frac{1}{2}\sqrt{\frac{2}{3}}\tan^{-1}\left(2\sqrt{\frac{2}{3}}\right) + \frac{1}{2}\ln\left(\frac{3}{11}\right) + 4\ln\left(\frac{7}{5}\right)$$

Question 12 (c)

Criteria	Marks
• Provides a correct answer	2
• Finds two common roots of the polynomials	1

Sample answer:

Using the quadratic formula, the roots of $P_1(z)$ are $3+i$ and $3-i$

Both polynomials have real co-efficients, which means that their complex roots come in conjugate pairs.

$\therefore$ If one of the roots of $P_1(z)$ is shared with $P_2(z)$, then so is the other root, since both roots are complex.

$\therefore\ P_2(3+i) = P_2(3-i) = 0$

Let the third root of $P_2(z)$ be labelled α.

Then the product of the roots of $P_2(z)$ is $(3+i)(3-i)\alpha = 10\alpha = -20$ by the formula for the product of the roots.

$\therefore\ \alpha = -2$

Using the formula for the sum of the roots, $a = -(3+i+3-i-2) = -4$

Question 12 (d) (i)

Criteria	Marks
• Provides a correct solution	1

Sample answer:

It is given that for positive a and b, $\frac{a+b}{2} \geq \sqrt{ab}$

Replace a with a^2 and b with b^2 $\Rightarrow$ $\frac{a^2+b^2}{2} \geq \sqrt{a^2b^2} = ab$

Question 12 (d) (ii)

Criteria	Marks
• Provides a correct solution	3
• Shows a progressed attempt to use the inequality $\frac{a+b}{2} \geq \sqrt{ab}$ with four variables	2
• Obtains $\frac{a^2+b^2+c^2+d^2}{2} \geq ab + cd$ or equivalent merit	1

Sample answer:

Using part (i), we have that $\frac{a^2+b^2}{2} \geq ab$ and $\frac{c^2+d^2}{2} \geq cd$

$$\therefore \frac{a^2+b^2+c^2+d^2}{2} \geq ab + cd$$

Using the identity given in part (i), we have that $\frac{ab+cd}{2} \geq \sqrt{abcd}$

$$\therefore \frac{a^2+b^2+c^2+d^2}{2} \geq 2\sqrt{abcd}$$

$$\frac{a^2+b^2+c^2+d^2}{4} \geq \sqrt{abcd}$$

Question 13 (a)

Criteria	Marks
• Provides correct solution	2
• Rewrites/Reframes the question using the contrapositive statement	1

Sample answer:

We are asked to show that $r^5 + 6r^3 + 2r > 3r^4 + 9r^2 + 6 \quad \Rightarrow \quad r > 0$

It suffices to prove the contrapositive of this statement, namely:

$$r < 0 \quad \Rightarrow \quad r^5 + 6r^3 + 2r < 3r^4 + 9r^2 + 6$$

Here we note that for $r < 0$, $r^n < 0$ for odd n and $r^n > 0$ for even n.

$\therefore r^5 + 6r^3 + 2r < 0 < 3r^4 + 9r^2 + 6$, and the proof is complete

Question 13 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Finds a correct net force expression for one of the objects	1

Sample answer:

The forces acting on Object A are gravity and the tension T in the string. Taking the direction of motion as positive, the net force on Object A is:

$$\sum F_A = m_A a = m_A g - T$$
$$10a = 10g - T$$

The forces acting on Object B in the direction of motion are tension, the normal force, friction and the component of gravity parallel to the plane.

Using trigonometry, the component of gravity parallel to the plane is $g \sin \theta$. The component perpendicular to the plane is opposed by the normal force, and so it can be disregarded. The net force on Object B is:

$$\therefore \sum F_B = m_B a = T - m_B g \sin \theta - 0.4g$$
$$6a = T - 6g \sin \theta - 0.4g$$

Adding both net force equations to eliminate T,

$$16a = 9.6g - 6g \sin \theta = 6g(1.6 - \sin \theta)$$
$$\therefore a = \frac{3}{8} g(1.6 - \sin \theta)$$

Question 13 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly finds an equation relating v and x	1

Sample answer:

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{3}{8}g(1.6 - \sin\theta) = \frac{30}{8}(1.6 - \sin\theta)$, since $g = 10$

$\frac{1}{2}v^2 = \frac{30x}{8}(1.6 - \sin\theta) + C$

We consider Object A's initial position as $x = 0$, where it is stationary

$\therefore$ When $x = 0$, $v = 0 \quad \Rightarrow \quad C = 0$

$\therefore \ \frac{1}{2}v^2 = \frac{30x}{8}(1.6 - \sin\theta)$

Substituting $v = \sqrt{30}$ and $x = 5$,

$15 = \frac{75}{4}(1.6 - \sin\theta)$

$$\sin\theta = 1.6 - \frac{4 \times 15}{75} = 0.8$$

Question 13 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly finds an equation relating v and x for the second leg of Object B's motion	2
• Correctly finds a new net force expression for Object B	1

Sample answer:

When Object A hits the ground, both objects have moved through a distance of 5m. When Object B continues to move up the ramp due to momentum, there is no tension force, since the string is not taut. Hence the only forces on Object B parallel to the plane of motion are friction and the component of gravity acting parallel to the plane.

$\therefore \sum F_B = m_B a = -0.4g - m_B g \sin\theta$

$6a = -4 - 6 \times 10 \times 0.8 = -52$

$\therefore a = -\frac{26}{3}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{26}{3}$

$\frac{1}{2}v^2 = -\frac{26}{3}x + C$

We take $x = 0$ to be the position of Object B when Object A lands on the ground, so that $v = \sqrt{30}$. Substituting these values:

$\frac{30}{2} = 0 + C \quad \Rightarrow \quad C = 15$

$v^2 = 30 - \frac{52}{3}x$

If we then let $v = 0$, we obtain the extra distance travelled up the ramp by Object B.

$0 = 30 - \frac{52}{3}x \quad \Rightarrow \quad x = \frac{45}{26}$

Therefore, the total distance travelled up the ramp by Object B is $5 + \frac{45}{26} = \frac{175}{26}$ metres

Question 13 (c)

Criteria	Marks
• Provides a correct answer	3
• Finds a correct primitive based on the new integral expression	2
• Makes a substitution to obtain a new integral equivalent to the first (including limits of integration)	1

Sample answer:

We use a t-substitution to find the integral. Letting $t = \tan\frac{x}{2}$, we have

$$1 + 2\sin x = 1 + 2\left(\frac{2t}{1+t^2}\right) = \frac{1 + t^2 + 4t}{1 + t^2}$$

$$3 - \cos x = 3 - \frac{1 - t^2}{1 + t^2} = \frac{3 + 3t^2 - 1 + t^2}{1 + t^2} = \frac{2 + 4t^2}{1 + t^2}$$

$$x = \frac{\pi}{3} \quad \rightarrow \quad t = \frac{1}{\sqrt{3}}$$

$$x = \frac{\pi}{2} \quad \rightarrow \quad t = 1$$

$$\int_{\pi/3}^{\pi/2} \frac{1 + 2\sin x}{3 - \cos x}\,dx = \frac{1}{2}\int_{1/\sqrt{3}}^{1} \frac{1 + 4t + t^2}{1 + 2t^2}\,dx$$

$$= \frac{1}{4}\int_{1/\sqrt{3}}^{1} \frac{2 + 2t^2}{1 + 2t^2}\,dx + \frac{1}{2}\int_{1/\sqrt{3}}^{1} \frac{4t}{1 + 2t^2}\,dx$$

$$= \frac{1}{4}\int_{1/\sqrt{3}}^{1} \left(1 + \frac{1}{1 + 2t^2}\right)dx + \frac{1}{2}\left[\ln\left|1 + 2t^2\right|\right]_{1/\sqrt{3}}^{1}$$

$$= \frac{1}{4}\left[t + \frac{\sqrt{2}}{2}\tan^{-1}\left(\sqrt{2}x\right)\right]_{1/\sqrt{3}}^{1} + \frac{1}{2}\left(\ln 3 - \ln\frac{5}{3}\right)$$

$$= \frac{1}{4}\left(1 + \frac{\sqrt{2}}{2}\tan^{-1}\sqrt{2} - \frac{1}{\sqrt{3}} - \frac{\sqrt{2}}{2}\tan^{-1}\frac{\sqrt{2}}{\sqrt{3}}\right) + \frac{1}{2}\ln\frac{9}{5}$$

$$= \frac{\sqrt{3} - 1}{4\sqrt{3}} + \frac{\sqrt{2}}{8}\left(\tan^{-1}\sqrt{2} - \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\right) + \frac{1}{2}\ln\frac{9}{5}$$

Question 13 (d)

Criteria	Marks
• Provides a sound, correctly structured proof	3
• Finds a correct primitive based on the new integral expression	2
• Proves the statement for the base case ($n = 1$)	1

Sample answer:

We first prove the statement for the base case, i.e. for $n = 1$

LHS $= 1 \geq \sqrt{1} =$ RHS

$\therefore$ The statement is true for $n = 1$

We now assume the truth of the statement for arbitrary $n = k$, i.e.

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} \geq \sqrt{k}$$

We now use this to prove the statement true for $n = k + 1$. We are hence required to show that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$$

LHS $= 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$\geq \sqrt{k} + \frac{1}{\sqrt{k+1}}$, by our inductive assumption

$= \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}}$

$\geq \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}}$

$= \frac{k+1}{\sqrt{k+1}}$

$= \sqrt{k+1}$, as required

Therefore, the statement is true by the principle of mathematical induction

Question 14 (a) (i)

Criteria	Marks
• Provides a correct answer	3
• Correctly projects $\overrightarrow{TO}$ onto $\overrightarrow{TB}$ without obtaining the required expression	2
• Attempts to project $\overrightarrow{TO}$ onto $\overrightarrow{TB}$	1

Sample answer:

We note that $\overrightarrow{TM}$ is the projection of $\overrightarrow{TO}$ onto $\overrightarrow{TB}$:

$$\therefore \overrightarrow{TM} = \frac{\overrightarrow{TO}\bullet\overrightarrow{TB}}{\overrightarrow{TB}\bullet\overrightarrow{TB}}\overrightarrow{TB}$$

$$= \frac{\overrightarrow{TO}\bullet(\overrightarrow{TO}+\overrightarrow{OB})}{|\overrightarrow{TB}|^2}\overrightarrow{TB}$$

$$= \frac{\overrightarrow{TO}\bullet\overrightarrow{TO}+\overrightarrow{TO}\bullet\overrightarrow{OB}}{|\overrightarrow{TB}|^2}\overrightarrow{TB}$$

$$= \frac{|\overrightarrow{TO}|^2}{|\overrightarrow{TB}|^2}\overrightarrow{TB}, \text{ since } TO \perp OB$$

Question 14 (a) (ii)

Criteria	Marks
• Provides a correct answer	1

Sample answer:

Using Pythagoras' Theorem, $|\overrightarrow{TB}|^2 = |\overrightarrow{TO}|^2 + |\overrightarrow{OB}|^2 = |\overrightarrow{TO}|^2 + |\overrightarrow{OA}|^2 + |\overrightarrow{AB}|^2 = 5|\overrightarrow{TO}|^2$

$\overrightarrow{TM} = \frac{|\overrightarrow{TO}|^2}{|\overrightarrow{TB}|^2}\overrightarrow{TB} = \frac{1}{5}\overrightarrow{TB} \Rightarrow$ The ratio $TM : MB$ is $1:4$

Question 14 (b) (i)

Criteria	Marks
• Provides a correct answer	1

Sample answer:

The solutions are $e^{\frac{2\pi k}{5}i}$, where $k = -2, -1, 0, 1, 2$

Question 14 (b) (ii)

Criteria	Marks
• Provides a correct answer	1

Sample answer:

w is a solution $\Rightarrow\ w^5 = 1\ \Rightarrow\ w^5 - 1 = 0\ \Rightarrow\ (w-1)\left(1 + w + w^2 + w^3 + w^4\right) = 0$

Since w is non-real, $w \neq 1$

$\therefore 1 + w + w^2 + w^3 + w^4 = 0$

Question 14 (b) (iii)

Criteria	Marks
• Provides a correct answer	2
• Uses the sum of roots of either $z^5 - 1 = 0$ or $1 + z + z^2 + z^3 + z^4 = 0$ to work towards a solution	1

Sample answer:

Using the sum of the roots of the polynomial equation $z^5 - 1 = 0$,

$1 + e^{\frac{2\pi}{5}i} + e^{-\frac{2\pi}{5}i} + e^{\frac{4\pi}{5}i} + e^{-\frac{4\pi}{5}i} = 0$

$1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$, since $e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta) = 2\cos\theta$

$\therefore \frac{1}{2} + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = 0$

Question 14 (b) (iv)

Criteria	Marks
• Provides a correct answer	2
• Forms a quadratic equation in $\cos\frac{\pi}{5}$	1

Sample answer:

$\frac{1}{2}+\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=\frac{1}{2}+\cos\frac{2\pi}{5}+\cos\left(\pi-\frac{\pi}{5}\right)=\frac{1}{2}+\cos\frac{2\pi}{5}-\cos\left(\frac{\pi}{5}\right)=0$

$\frac{1}{2}+\left(2\cos^2\frac{\pi}{5}-1\right)-\cos\left(\frac{\pi}{5}\right)=2\cos^2\frac{\pi}{5}-\cos\left(\frac{\pi}{5}\right)-\frac{1}{2}=0$

Using the quadratic equation, $\cos\frac{\pi}{5}=\frac{1\pm\sqrt{(-1)^2-4\times2\times\left(-\frac{1}{2}\right)}}{4}=\frac{1\pm\sqrt{5}}{4}$

Since $\frac{\pi}{5}$ is acute, $\cos\frac{\pi}{5}>0 \;\Rightarrow\; \cos\frac{\pi}{5}=\frac{1+\sqrt{5}}{4}$

Question 14 (c) (i)

Criteria	Marks
• Provides a correct answer	2
• Forms two equations with u, n and A, or equivalent merit	1

Sample answer:

$\ddot{x}=-n^2(x-2) \;\Rightarrow\;$ Centre of oscillation is $x=2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-n^2(x-2)$

$\frac{1}{2}v^2=-\frac{1}{2}n^2(x-2)^2+C$

Let the amplitude be A. When the particle reaches its positive extreme point, i.e. when $x=A+2$, its velocity is 0

$0=-\frac{1}{2}n^2A^2+C \;\Rightarrow\; C=\frac{1}{2}n^2A^2$

$\therefore v^2=-n^2(x-2)^2+n^2A^2=n^2(A^2-(x-2)^2)$

When $x=0$, $v=u \;\Rightarrow\; u^2=n^2(A^2-4)$

When $x=5$, $v=\frac{2}{3}u \;\Rightarrow\; \frac{4}{9}u^2=n^2(A^2-9)$

Dividing the first equation by the second yields $\frac{9}{4}=\frac{A^2-4}{A^2-9}$

$9A^2-81=4A^2-16$

$5A^2=65 \;\Rightarrow\; A=\sqrt{13}$

Question 14 (c) (i)

Criteria	Marks
• Provides a correct answer	2
• Finds n, or equivalent merit	1

Sample answer:

$\ddot{x} = -n^2(x-2)$

The particle's maximal positive value is $\sqrt{13}+2$, i.e. its centre of motion plus its amplitude. At this point, it experiences maximal negative acceleration, which is -6 m/s^2, as given in the question.

$\therefore -6 = -n^2\left(\sqrt{13}+2-2\right)$

$n = \sqrt{\frac{6}{\sqrt{13}}}$

From the previous part, $v^2 = n^2\left(A^2 - (x-2)^2\right) = \frac{6}{\sqrt{13}}(13-(x-2)^2)$

When $x = 0$, $v = u \Rightarrow u^2 = \frac{6}{\sqrt{13}}(13-4) = \frac{54}{\sqrt{13}}$

$$\therefore u = \sqrt{\frac{54}{\sqrt{13}}} = \frac{\sqrt{54}}{\sqrt[4]{13}}$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Integrates to find an expression relating v and x, with an algebra mistake	1

Sample answer:

$$m\ddot{x} = -mg - kv$$

$$mv\frac{dv}{dx} = -(mg + kv)$$

$$m\int \frac{v}{mg + kv}\,dv = -\int dx$$

$$\frac{m}{k}\int \left(1 - \frac{mg}{mg + kv}\right) dv = -\int dx$$

$$\frac{m}{k}\left(v - \frac{mg}{k}\ln|mg + kv|\right) = -x + C$$

When $x = 0, v = U \;\Rightarrow\; C = \frac{m}{k}\left(U - \frac{mg}{k}\ln|mg + kU|\right)$

$$\frac{m}{k}\left(v - \frac{mg}{k}\ln|mg + kv|\right) = -x + \frac{m}{k}\left(U - \frac{mg}{k}\ln|mg + kU|\right)$$

$$\therefore x = \frac{m}{k}\left((U - v) - \frac{mg}{k}\ln\left|\frac{mg + kU}{mg + kv}\right|\right)$$

The max height is reached when $v = 0$

$$\therefore H = \frac{m}{k}\left((U - 0) - \frac{mg}{k}\ln\left|\frac{mg + kU}{mg + 0}\right|\right) = \frac{m}{k}U - \frac{m^2 g}{k^2}\ln\left(1 + \frac{k}{mg}U\right)$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution	4
• Makes significant progress towards simplifying the expression for V	3
• Finds a correct expression relating x and t for particle 2	2
• Finds a correct expression relating v and t for particle 2	1

Sample answer:

Let V represent the launch speed of the second particle. This particle faces the same conditions of motion as the first, and so the net force remains the same

$$m\ddot{x} = -mg - kv$$

$$m\frac{dv}{dt} = -(mg + kv)$$

$$m\int \frac{1}{mg + kv}\,dv = -\int dt$$

$$\frac{m}{k}\ln|mg + kv| = -t + C$$

When $t = \frac{T}{2}$, $v = V \;\; \Rightarrow \;\; C = \frac{T}{2} + \frac{m}{k}\ln|mg + kV|$

$$\frac{m}{k}\ln\left|\frac{mg + kv}{mg + kV}\right| = -t + \frac{T}{2}$$

It is given that $T = \frac{m}{k}\ln\left(1 + \frac{k}{mg}U\right) = \frac{m}{k}\ln(Q)$

$$\ln\left|\frac{mg + kv}{mg + kV}\right| = -\frac{k}{m}t + \frac{1}{2}\ln Q$$

$$\frac{mg + kv}{mg + kV} = e^{\left(\frac{1}{2}\ln Q - \frac{k}{m}t\right)} = e^{\ln\sqrt{Q}}e^{-\frac{k}{m}t} = \sqrt{Q}e^{-kt/m}$$

$$v = \frac{(mg + kV)}{k}\sqrt{Q}e^{-\frac{kt}{m}} - \frac{mg}{k}$$

$$\frac{dx}{dt} = \left(\frac{mg}{k} + V\right)\sqrt{Q}e^{-\frac{kt}{m}} - \frac{mg}{k}$$

$$x = -\frac{m}{k}\left(\frac{mg}{k} + V\right)\sqrt{Q}e^{-\frac{kt}{m}} - \frac{mg}{k}t + C$$

When $t = \frac{T}{2}, x = 0 \;\; \Rightarrow \;\; C = \frac{m}{k}\left(\frac{mg}{k} + V\right)\sqrt{Q}e^{-\frac{kT}{2m}} + \frac{mg}{k}\frac{T}{2}$

$$x = -\frac{m}{k}\left(\frac{mg}{k} + V\right)\sqrt{Q}e^{-\frac{kt}{m}} - \frac{mg}{k}t + \frac{m}{k}\left(\frac{mg}{k} + V\right)\sqrt{Q}e^{-\frac{kT}{2m}} + \frac{mg}{2k}T$$

Again substituting $T = \frac{m}{k}\ln Q$,

$$x=-\frac{m}{k}\left(\frac{mg}{k}+V\right)\sqrt{Q}e^{-\frac{kt}{m}}-\frac{mg}{k}t+\frac{m}{k}\left(\frac{mg}{k}+V\right)\sqrt{Q}e^{-\frac{1}{2}\ln Q}+\frac{m^2g}{2k^2}\ln Q$$

$$x=\frac{m}{k}\left(\frac{mg}{k}+V\right)\sqrt{Q}e^{-\ln\sqrt{Q}}-\frac{m}{k}\left(\frac{mg}{k}+V\right)\sqrt{Q}e^{-\frac{kt}{m}}+\frac{m^2g}{2k^2}\ln Q-\frac{mg}{k}t$$

$$x=\frac{m}{k}\left(\frac{mg}{k}+V\right)\left(\sqrt{Q}\times\left(\sqrt{Q}\right)^{-1}-\sqrt{Q}e^{-\frac{kt}{m}}\right)+\frac{m^2g}{2k^2}\ln Q-\frac{mg}{k}t$$

$$x=\frac{m}{k}\left(\frac{mg}{k}+V\right)\left(1-\sqrt{Q}e^{-\frac{kt}{m}}\right)+\frac{mg}{2k}\left(\frac{m}{k}\ln Q-2t\right)$$

When $t=T, x=H$

$$H=\frac{m}{k}\left(\frac{mg}{k}+V\right)\left(1-\sqrt{Q}e^{-\frac{k}{m}T}\right)+\frac{mg}{2k}\left(\frac{m}{k}\ln Q-2T\right)$$

We now substitute $T=\frac{m}{k}\ln Q$ and $H=\frac{m}{k}U-\frac{m^2g}{k^2}\ln\left(1+\frac{k}{mg}U\right)=\frac{m}{k}\left(U-\frac{mg}{k}\ln Q\right)$, and solve for V

$$\frac{m}{k}\left(U-\frac{mg}{k}\ln Q\right)=\frac{m}{k}\left(\frac{mg}{k}+V\right)\left(1-\sqrt{Q}e^{-\frac{k\,m}{m\,k}\ln Q}\right)+\frac{mg}{2k}\left(\frac{m}{k}\ln Q-\frac{2m}{k}\ln Q\right)$$

$$\frac{m}{k}\left(U-\frac{mg}{k}\ln Q\right)=\frac{m}{k}\left(\frac{mg}{k}+V\right)\left(1-\sqrt{Q}e^{-\ln Q}\right)+\frac{mg}{2k}\left(-\frac{m}{k}\ln Q\right)$$

$$\frac{m}{k}\left(U-\frac{mg}{k}\ln Q\right)+\frac{m^2g}{2k^2}\ln Q=\frac{m}{k}\left(\frac{mg}{k}+V\right)\left(1-\sqrt{Q}\frac{1}{Q}\right)$$

$$\frac{m}{k}U-\frac{m^2g}{k^2}\ln Q+\frac{m^2g}{2k^2}\ln Q=\frac{m}{k}\frac{mg}{k}\left(1-\frac{1}{\sqrt{Q}}\right)+\frac{m}{k}\left(1-\frac{1}{\sqrt{Q}}\right)V$$

$$\frac{m}{k}\left(\frac{\sqrt{Q}-1}{\sqrt{Q}}\right)V=\frac{m}{k}U-\frac{m^2g}{2k^2}\ln Q-\frac{m}{k}\frac{mg}{k}\left(\frac{\sqrt{Q}-1}{\sqrt{Q}}\right)$$

$$V=\frac{k}{m}\left(\frac{\sqrt{Q}}{\sqrt{Q}-1}\right)\left[\frac{m}{k}U-\frac{m^2g}{2k^2}\ln Q\right]-\frac{mg}{k}$$

$$V=\left(\frac{\sqrt{Q}}{\sqrt{Q}-1}\right)\left[U-\frac{mg}{2k}\ln Q\right]-\frac{mg}{k}$$

Now by writing $U=\frac{mg}{k}(Q-1)$

$$V=\left(\frac{\sqrt{Q}}{\sqrt{Q}-1}\right)\left[\frac{mg}{k}(Q-1)-\frac{mg}{2k}\ln Q\right]-\frac{mg}{k}$$

$$V=\frac{mg}{k}\left(\frac{\sqrt{Q}}{\sqrt{Q}-1}\right)\left[Q-1-\frac{1}{2}\ln Q\right]-\frac{mg}{k}$$

$$V=\frac{mg}{k}\left(\frac{\sqrt{Q}}{\sqrt{Q}-1}\right)\left[Q-1-\ln\sqrt{Q}\right]-\frac{mg}{k}$$

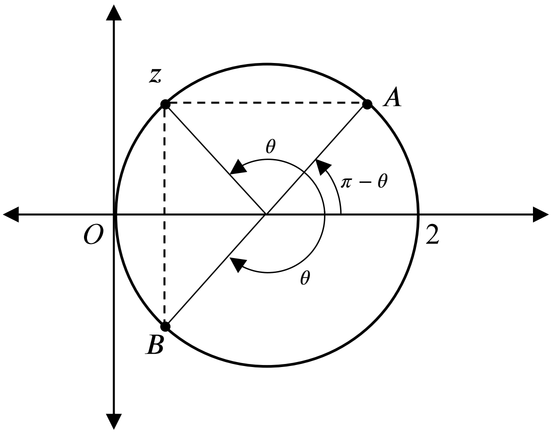
Question 15 (b)

Criteria	Marks
• Provides a correct solution	3
• Makes progress towards showing that $w = ki\overline{z}$	2
• Correctly expresses w, z and $\overline{z}$ in the form $1 + e^{i\phi}$	1

Sample answer:

Let $z = 1 + e^{i\theta}$

Then $w = 1 + e^{i(\pi-\theta)}$ and $\overline{z} = 1 + e^{-i\theta}$, as shown in the diagram below



If we are to show that $\angle AOB = \frac{\pi}{2}$, it would suffice to show that the vector $\overrightarrow{OA}$ can be obtained by rotating $\overrightarrow{OB}$ by $\frac{\pi}{2}$ and scaling it, i.e. that $w = ki\overline{z}$ where k is the scaling factor $\frac{|\overrightarrow{OA}|}{|\overrightarrow{OB}|}$

First consider $\frac{w}{\overline{z}}$:

$$\frac{w}{\overline{z}} = \frac{1 + e^{i(\pi-\theta)}}{1 + e^{-i\theta}} = \frac{e^{i\theta} + e^{i\pi}}{e^{i\theta} + 1} = \frac{e^{i\theta} - 1}{e^{i\theta} + 1} = \frac{e^{\frac{\theta}{2}i} - e^{-\frac{\theta}{2}i}}{e^{\frac{\theta}{2}i} + e^{-\frac{\theta}{2}i}} = \frac{2i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = i\tan\frac{\theta}{2} = ki$$

Hence $w = ki\overline{z}$ and the proof is complete.

NOTE (not required for full marks in this question): Even if $\tan\frac{\theta}{2}$ is negative (which will occur if z is in the fourth quadrant), the proof still holds true – in this case, the vector $\overrightarrow{OA}$ would be obtained by rotating $\overrightarrow{OB}$ clockwise by $\frac{\pi}{2}$, which would require $k < 0$.

Question 15 (c) (i)

Criteria	Marks
• Provides a correct answer	3
• Shows that $I_n = I_0 - \sum_{k=1}^{n} \frac{1}{k2^k}$	2
• Shows that $I_n = -\frac{1}{n2^n} + I_{n-1}$	1

Sample answer:

$$I_n = \int_0^{\frac{1}{2}} \frac{x^n}{1-x}\,dx = \int_0^{\frac{1}{2}} \frac{x^n - x^{n-1}}{1-x}\,dx + \int_0^{\frac{1}{2}} \frac{x^{n-1}}{1-x}\,dx$$

$$= -\int_0^{\frac{1}{2}} \frac{x^{n-1}(1-x)}{1-x}\,dx + I_{n-1}$$

$$= -\int_0^{\frac{1}{2}} x^{n-1}\,dx + I_{n-1}$$

$$= -\frac{1}{n}[x^n]_0^{1/2} + I_{n-1}$$

$$= -\frac{1}{n2^n} + I_{n-1}$$

$$\therefore I_n = -\frac{1}{n2^n} + I_{n-1} = -\frac{1}{n2^n} - \frac{1}{(n-1)2^{n-1}} + I_{n-2} = \cdots = I_0 - \sum_{k=1}^{n} \frac{1}{k2^k}$$

$$I_0 = \int_0^{\frac{1}{2}} \frac{1}{1-x}\,dx = -[\ln|1-x|]_0^{\frac{1}{2}} = -\ln\frac{1}{2} = \ln 2$$

$$\therefore I_n = I_0 - \sum_{k=1}^{n} \frac{1}{k2^k} = \ln 2 - \sum_{k=1}^{n} \frac{1}{k2^k}$$

Question 15 (c) (ii)

Criteria	Marks
• Provides a correct explanation	1

Sample answer:

The integral I_n is evaluated across the limits $0 \le x \le \frac{1}{2}$, for which $\frac{1}{2} \le 1 - x \le 1$ and so $1 \le \frac{1}{1-x} \le 2$.

Hence

$$\int_0^{\frac{1}{2}} x^n dx < \int_0^{\frac{1}{2}} \frac{x^n}{1-x}\,dx < \int_0^{\frac{1}{2}} 2x^n dx\text{, since equality doesn't always hold}$$

$$\therefore \int_0^{\frac{1}{2}} x^n dx < I_n < \int_0^{\frac{1}{2}} 2x^n dx$$

Question 15 (c) (i)

Criteria	Marks
• Provides a correct answer	2
• Evaluating $\int_0^{\frac{1}{2}} x^n dx$ in the inequality	1

Sample answer:

$$\int_0^{\frac{1}{2}} x^n dx = \frac{1}{n+1}\left[x^{n+1}\right]_0^{1/2} = \frac{1}{(n+1)2^{n+1}}$$

$$\therefore \frac{1}{(n+1)2^{n+1}} < I_n < \frac{1}{(n+1)2^n}$$

$$\frac{1}{(n+1)2^{n+1}} < \ln 2 - \sum_{k=1}^{n} \frac{1}{k2^k} < \frac{2}{(n+1)2^{n+1}}$$

As $n \to \infty$, $\frac{1}{(n+1)2^{n+1}} \to 0$, and so by a sandwiching argument,

$$\ln 2 - \sum_{k=1}^{\infty} \frac{1}{k2^k} = 0$$

$$\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k2^k}$$

Question 16 (a) (i)

Criteria	Marks
• Provides a correct answer	1

Sample answer:

$w = \cos x + i \sin x \quad \Rightarrow \quad w^n = \cos(nx) + i\sin(nx) = e^{inx}$ by De Moivre's Theorem

$$w + w^2 + \cdots + w^n = \frac{w(w^n - 1)}{w - 1} = \frac{e^{inx} - 1}{e^{ix} - 1} e^{ix}$$

Question 16 (a) (ii)

Criteria	Marks
• Provides a correct answer	3
• Rewrites the expression $\frac{e^{inx}-1}{e^{ix}-1} e^{ix}$ as $\frac{\sin\left(\frac{n}{2}x\right)}{\sin\left(\frac{x}{2}\right)} e^{\left(\frac{n+1}{2}\right)ix}$, or equivalent merit	2
• Obtains the LHS of the required expression by taking the imaginary part of $w + w^2 + \cdots + w^n$, or equivalent merit	1

Sample answer:

$w + w^2 + \cdots + w^n = \frac{w(w^n-1)}{w-1} = \frac{e^{inx}-1}{e^{ix}-1} e^{ix}$

$= \frac{e^{\frac{n}{2}ix}\left(e^{\frac{n}{2}ix} - e^{-\frac{n}{2}ix}\right)}{e^{\frac{x}{2}i}\left(e^{\frac{x}{2}i} - e^{-\frac{x}{2}i}\right)} e^{ix}$

$= \frac{2i\sin\left(\frac{n}{2}x\right)}{2i\sin\left(\frac{x}{2}\right)} e^{\left(\frac{n}{2}+1-\frac{1}{2}\right)ix}$

$= \frac{\sin\left(\frac{n}{2}x\right)}{\sin\left(\frac{x}{2}\right)} e^{\left(\frac{n+1}{2}\right)ix}$

Now consider $\text{Im}\left(w + w^2 + \cdots + w^n\right) = \text{Im}\left(\sum_{k=1}^{n} \cos(kx) + i\sum_{k=1}^{n} \sin(kx)\right) = \sum_{k=1}^{n} \sin(kx)$

This is equivalent to $\text{Im}\left(\frac{\sin\left(\frac{n}{2}x\right)}{\sin\left(\frac{x}{2}\right)} e^{\left(\frac{n+1}{2}\right)ix}\right) = \frac{\sin\left(\frac{n}{2}x\right)}{\sin\left(\frac{x}{2}\right)} \text{Im}\left(\cos\left(\frac{n+1}{2}x\right) + i\sin\left(\frac{n+1}{2}x\right)\right) = \frac{\sin\left(\frac{n}{2}x\right)}{\sin\left(\frac{x}{2}\right)} \sin\left(\frac{n+1}{2}x\right)$

$$\therefore \sum_{k=1}^{n} \sin(kx) = \frac{\sin\left(\frac{n}{2}x\right)}{\sin\left(\frac{x}{2}\right)} \sin\left(\frac{n+1}{2}x\right)$$

Question 16 (b) (i)

Criteria	Marks
• Provides a correct answer	2
• Obtains $\lvert z_k - 1\rvert^2 = 2 - 2\cos\frac{2\pi k}{n}$, or equivalent merit	1

Sample answer:

$z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$

$|z_k - 1| = \left|\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} - 1\right| = \left|\left(\cos\frac{2\pi k}{n} - 1\right) + i\sin\frac{2\pi k}{n}\right|$

$|z_k - 1|^2 = \left|\left(\cos\frac{2\pi k}{n} - 1\right) + i\sin\frac{2\pi k}{n}\right|^2$

$= \left(\cos\frac{2\pi k}{n} - 1\right)^2 + \sin^2\frac{2\pi k}{n}$

$= \cos^2\frac{2\pi k}{n} - 2\cos\frac{2\pi k}{n} + 1 + \sin^2\frac{2\pi k}{n}$

$= 2 - 2\cos\frac{2\pi k}{n}$

$= 2 - 2\left(1 - 2\sin^2\frac{\pi k}{n}\right)$

$= 4\sin^2\frac{\pi k}{n}$

$\therefore |z_k - 1| = 2\sin\frac{\pi k}{n}$

Question 16 (b) (ii)

Criteria	Marks
• Provides a correct answer	2
• Obtains $\lvert A_k - A_0\rvert = \frac{\left\lvert e^{\frac{2\pi k}{n}i} - 1\right\rvert}{\left\lvert e^{\frac{2\pi}{n}i} - 1\right\rvert}$, or equivalent merit	1

Sample answer:

$|A_k - A_0| = \left|\sum_{j=0}^{k-1} z_k - 0\right| = \left|1 + e^{\frac{2\pi}{n}i} + e^{\frac{4\pi}{n}i} + \cdots + e^{\frac{2\pi(k-1)}{n}i}\right| = \left|\frac{e^{\frac{2\pi k}{n}i} - 1}{e^{\frac{2\pi}{n}i} - 1}\right|$

$= \frac{\left|e^{\frac{2\pi k}{n}i} - 1\right|}{\left|e^{\frac{2\pi}{n}i} - 1\right|}$

$= \frac{|z_k - 1|}{|z_1 - 1|} = \frac{\sin\frac{\pi k}{n}}{\sin\frac{\pi}{n}}$ using the result from the previous part

Question 16 (b) (iii)

Criteria	Marks
• Provides a correct answer	3
• Obtains $\sum_{k=1}^{n-1}\lvert A_k - A_0\rvert = \frac{1}{2}\operatorname{cosec}^2\frac{\pi}{2n}$, without subtracting 2, or equivalent merit	2
• Uses part a) (ii) appropriately to aid in their computation, or equivalent merit	1

Sample answer:

Note that $A_0 = 0$, so the sum of the diagonals with A_0 as an endpoint is simply $\sum_{k=1}^{n-1}\lvert A_k - A_0\rvert - 2$. We subtract 2 to account for the distances $\lvert A_1 - A_0\rvert$ and $\lvert A_{n-1} - A_0\rvert$, which represent the sides of the polygon adjacent to A_0, both of which have a length of 1.

$$\begin{aligned}
\sum_{k=1}^{n-1}\lvert A_k - A_0\rvert &= \sum_{k=1}^{n-1}\frac{\sin\frac{\pi k}{n}}{\sin\frac{\pi}{n}} \\
&= \frac{1}{\sin\frac{\pi}{n}}\sum_{k=1}^{n-1}\sin\frac{\pi k}{n} \\
&= \frac{1}{\sin\frac{\pi}{n}} \times \frac{\sin\left(\frac{n-1}{2}\times\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{2n}\right)}\sin\left(\frac{n}{2}\times\frac{\pi}{n}\right) \\
&= \frac{1}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}} \times \frac{\sin\left(\frac{n-1}{n}\times\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2n}\right)}\sin\left(\frac{\pi}{2}\right) \\
&= \frac{1}{2\sin^2\frac{\pi}{2n}\cos\frac{\pi}{2n}} \times \cos\left(\frac{\pi}{2} - \frac{n-1}{n}\times\frac{\pi}{2}\right) \\
&= \frac{1}{2\sin^2\frac{\pi}{2n}\cos\frac{\pi}{2n}} \times \cos\left(\frac{\pi}{2n}\right) \\
&= \frac{1}{2}\operatorname{cosec}^2\frac{\pi}{2n}
\end{aligned}$$

$\therefore$ The sum of the diagonals with A_0 as an endpoint is $\frac{1}{2}\operatorname{cosec}^2\frac{\pi}{2n} - 2$

Question 16 (b) (iv)

Criteria	Marks
• Provides a correct answer (full explanation not required)	1

Sample answer:

The polygon $\mathscr{P}$ is regular and has n vertices, so the combined length of the diagonals 'sprouting' from any vertex is equivalent to the expression in part (iii). Hence we multiply this result by n. However, to avoid double-counting a diagonal (since each diagonal is adjacent to two of the n vertices), we also divide by 2.

$\therefore$ Combined sum of the lengths of all diagonals is $\frac{n}{2} \times \left(\frac{1}{2}\operatorname{cosec}^2 \frac{\pi}{2n} - 2\right) = \frac{n}{4}\operatorname{cosec}^2 \frac{\pi}{2n} - n$

Question 16 (c)

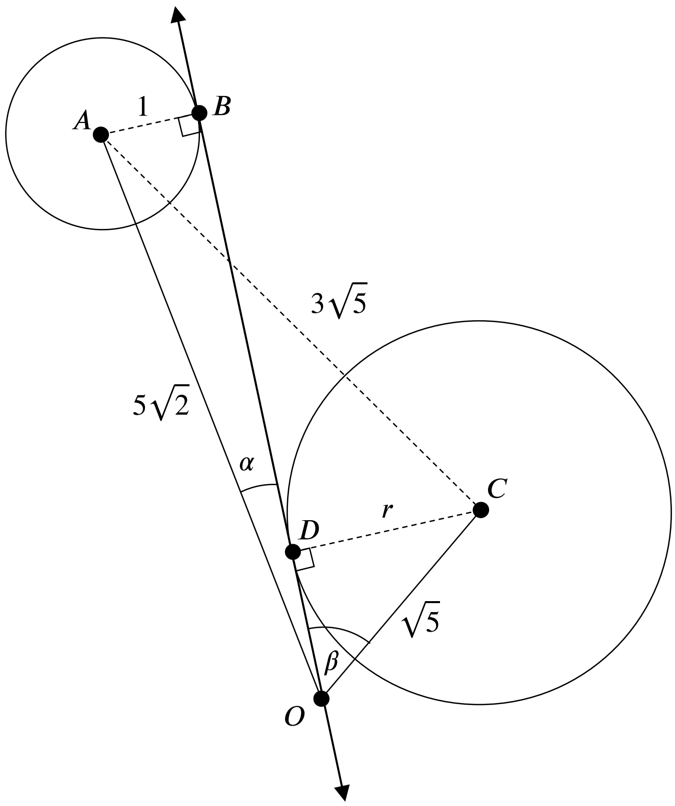
Criteria	Marks
• Provides a correct answer	4
• Forms an equation involving only r	3
• Uses the cosine rule to find $\cos(\angle AOB + \angle COD)$	2
• Identifies at least 4 distances in the problem setup	1

Sample answer:

Let A and B be the centre of $\mathcal{S}_1$ and its point of contact with l, respectively. Similarly, let C and D be the centre of $\mathcal{S}_2$ and its point of contact with l, respectively. Finally, let $\alpha = \angle AOB$ and $\beta = \angle COD$.

We note that:

- $AB = 1$ ($\mathcal{S}_1$ has unit radius)
- $OA = 5\sqrt{2}$ (distance of the point (0, 5, 5) from O)
- $OB = 7$ (using Pythagoras' theorem in ΔAOB)
- $OC = \sqrt{5}$ (distance of the point (2, 0, 1) from O)
- $AC = 3\sqrt{5}$ (distance between the points (0, 5, 5) and (2, 0, 1))



Now using the cosine rule in ΔAOC:

$AC^2 = OA^2 + OC^2 - 2 \times OA \times OC \times \cos(\alpha + \beta)$

$45 = 50 + 5 - 2 \times 5\sqrt{2} \times \sqrt{5} \times [\cos\alpha\cos\beta - \sin\alpha\sin\beta]$

$\cos\alpha\cos\beta - \sin\alpha\sin\beta = -\frac{10}{10\sqrt{10}} = -\frac{1}{\sqrt{10}}$

From ΔAOB, we have $\cos\alpha = \frac{7}{5\sqrt{2}}$ and $\sin\alpha = \frac{1}{5\sqrt{2}}$.

From ΔCOD, we have $\cos\beta = \frac{\sqrt{5-r^2}}{\sqrt{5}}$ and $\sin\beta = \frac{r}{\sqrt{5}}$

$\therefore \ \frac{7}{5\sqrt{2}}\frac{\sqrt{5-r^2}}{\sqrt{5}} - \frac{1}{5\sqrt{2}}\frac{r}{\sqrt{5}} = -\frac{1}{\sqrt{10}}$

$7\sqrt{5-r^2} - r = -5$

$7\sqrt{5-r^2} = r - 5$

$49\left(5 - r^2\right) = (r-5)^2$

$245 - 49r^2 = r^2 - 10r + 25$

$50r^2 - 10r - 220 = 0$

$5r^2 - r - 22 = 0$

$r = \frac{1 \pm \sqrt{1+4\times5\times22}}{10} = \frac{1\pm21}{10} = -2, \frac{11}{5}$

Since $r > 0$, we conclude that $r = \frac{11}{5}$