2024
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your student name and/or number at the top of every page

Total marks – 100

Section I – 10 marks (pages 3 - 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 - 11)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

This paper MUST NOT be removed from the examination room.

STUDENT NAME/NUMBER..

STUDENT NAME/NUMBER...

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Select the alternative A, B, C, D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

STUDENT NAME/NUMBER..

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

---

1. Which of the following statements is FALSE ?

(A) $\forall a, b \in (-\infty, \infty),\ a<b \Rightarrow \dfrac{a}{1+|a|} < \dfrac{b}{1+|b|}$

(B) $\forall a, b \in (-\infty, \infty),\ a<b \Rightarrow \dfrac{a}{1+|b|} < \dfrac{b}{1+|a|}$

(C) $\forall a, b \in (-\infty, \infty),\ a<b \Rightarrow \dfrac{a}{1+a^2} < \dfrac{b}{1+b^2}$

(D) $\forall a, b \in (-\infty, \infty),\ a<b \Rightarrow \dfrac{a}{1+b^2} < \dfrac{b}{1+a^2}$

2. ABC is a triangle. Which of the following is an expression for $\dfrac{e^{iB} \times e^{iC}}{e^{iA}}$?

(A) $-\cos 2A + i\sin 2A$
(B) $-\cos 2A - i\sin 2A$
(C) $\cos 2A - i\sin 2A$
(D) $\cos 2A + i\sin 2A$

3. With respect to a fixed origin O, the points P and Q have position vectors $\overrightarrow{OP} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\overrightarrow{OQ} = 8\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}$ respectively. What is the length of PQ ?

(A) $PQ = 5$
(B) $PQ = 6$
(C) $PQ = 7$
(D) $PQ = 8$

4. Which of the following is an expression for $\int \cos^3 x\, dx$?

(A) $\sin x \cos^2 x + \dfrac{1}{3}\sin^3 x + c$

(B) $\sin x \cos^2 x - \dfrac{1}{3}\sin^3 x + c$

(C) $\sin x + \dfrac{1}{3}\sin^3 x + c$

(D) $\sin x - \dfrac{1}{3}\sin^3 x + c$

5. A particle is moving in simple harmonic motion along the x axis. At time t seconds it has displacement x metres from the origin O and velocity v ms^{-1} given by $v^2 = 6 - 4x - 2x^2$. Where is the centre of motion of the particle?

(A) 2 metres to the left of O
(B) 1 metre to the left of O
(C) 1 metre to the right of O
(D) 2 metres to the right of O

6. $f(x)$ and $g(x)$ are both functions with domain $(-\infty, \infty)$. Consider the statement:

$$f(x) \textit{ is odd and } g(x) \textit{ is odd} \Rightarrow f(g(x)) \textit{ is odd}$$

Which of the following is correct?

(A) the contrapositive statement is false and the converse statement is false
(B) the contrapositive statement is true and the converse statement is true
(C) the contrapositive statement is false and the converse statement is true
(D) the contrapositive statement is true and the converse statement is false

7. Consider the complex numbers z of the form $z = \cos^2\theta + i\sin^2\theta$ where $0 \le \theta \le \frac{\pi}{2}$. What is the range of the possible values of $|z|$?

(A) $0 \le |z| \le 1$

(B) $\frac{1}{\sqrt{2}} \le |z| \le 1$

(C) $\frac{1}{2} \le |z| \le 1$

(D) $\frac{1}{4} \le |z| \le 1$

8. Consider the plane $2x + y + z = 2$ and the line $\underset{\sim}{r} = (1+\lambda)\underset{\sim}{i} + (1-\lambda)\underset{\sim}{j} + (1-\lambda)\underset{\sim}{k}$, where λ is a scalar parameter. Which of the following is correct ?

(A) the line lies completely outside the plane
(B) the line intersects the plane in exactly one point and is perpendicular to it
(C) the line intersects the plane in exactly one point and is not perpendicular to it
(D) the line lies completely in the plane

9. Which of the following is equal to $\int_0^a \{f(a-x)+f(2a-x)\}dx$?

(A) 0

(B) $\int_0^{2a} f(x)dx$

(C) $2\int_0^a f(x)dx$

(D) $2\int_0^{2a} f(x)dx$

10. A stone is thrown horizontally with speed $V\,\text{ms}^{-1}$ from the top point O of a vertical building of height H metres that stands on horizontal ground. It moves in a vertical plane under gravity in a medium where the acceleration due to gravity is $g\,\text{ms}^{-2}$. A time t seconds the stone has position vector $Vt\underset{\sim}{i} - \frac{1}{2}gt^2\underset{\sim}{j}$ relative to O. The stone hits the ground at time T seconds at a distance X metres from the foot of the building. At what angle to the horizontal does the stone hit the ground ?

(A) $\tan^{-1}\left(\frac{H}{2X}\right)$

(B) $\tan^{-1}\left(\frac{H}{X}\right)$

(C) $\tan^{-1}\left(\frac{2H}{X}\right)$

(D) $\tan^{-1}\left(\frac{4H}{X}\right)$

STUDENT NAME/NUMBER.......................................

Marks

Section II

90 Marks
Attempt Questions 11-16.
Allow about 2 hours and 45 minutes for this section.

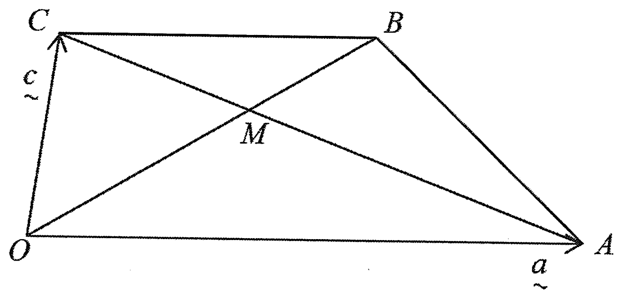
Answer the questions in writing booklets provided. Use a separate writing booklet for each question.
In Questions 11-16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

(a) Use the substitution $t=\tan\frac{x}{2}$ to find $\int \frac{1-\tan\frac{x}{2}}{\cos x}dx$. **2**

(b) Find in simplest exact form the value of $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$. **2**

(c)



In the diagram $OABC$ is a trapezium in which $\overrightarrow{CB}=k\overrightarrow{OA}$ for some constant $k>0$.
The diagonals OB and AC intersect at M. Let $\overrightarrow{OA}=\underset{\sim}{a}$ and $\overrightarrow{OC}=\underset{\sim}{c}$.
Let $\overrightarrow{OM}=\lambda\overrightarrow{OB}$ and $\overrightarrow{AM}=\mu\overrightarrow{AC}$ for some constants $\lambda>0$ and $\mu>0$.

(i) Use vector methods to show that $\overrightarrow{OM} = \frac{1}{k+1}\overrightarrow{OB}$. **2**

(ii) If $\overrightarrow{OM}=\frac{1}{2}\overrightarrow{OB}$ explain what type of quadrilateral $OABC$ is. **1**

(d) The equation $z^3-12z^2+cz+d=0$, where c and d are real, has a root $5+2i$.

(i) Find the other two roots of the equation in the form $a+ib$, where a and b are real. **2**

(ii) Hence find the values of c and d. **2**

(e) A particle is moving along the x axis. At time t seconds it has displacement x metres from the origin O, velocity v ms^{-1} and acceleration a ms^{-2} given by $a=x-10$. Initially the particle is 1 metre to the right of O and moving away from O with speed 9 ms^{-1}.

(i) Find an expression for v in terms of x. **2**

(ii) Find an expression for x in terms of t. **2**

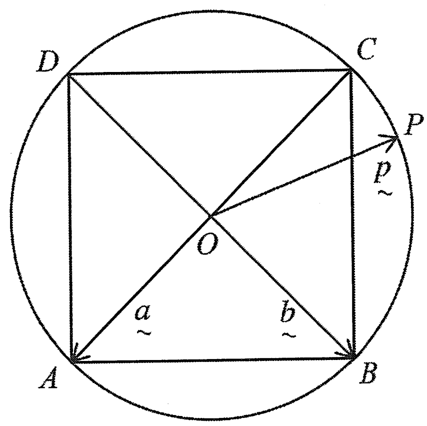
End of Question 11

STUDENT NAME/NUMBER......................................

Marks

Question 12 (15 marks) **Use a separate writing booklet.**

(a)



In the diagram, $ABCD$ is a square with diagonals intersecting at O. The circle with centre O and radius r passes through the vertices of the square. P is any point on the circle. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OP} = \underset{\sim}{p}$. Use vector methods to show that **3**

$|\overrightarrow{PA}|^2 + |\overrightarrow{PB}|^2 + |\overrightarrow{PC}|^2 + |\overrightarrow{PD}|^2 = 8r^2$.

(b) Consider the following statement about all positive integers $p > 3$:

If p is prime then $p = 6k \pm 1$ for some positive integer $k = 1, 2, 3, \ldots$

(i) State the contrapositive statement. **1**

(ii) Prove that the contrapositive statement is true. **3**

(c)(i) On an Argand diagram sketch the point P representing the complex number $w = 8e^{-\frac{\pi}{6}i}$ and the locus of all complex numbers that satisfy $\arg(z + 10i) = \frac{\pi}{3}$. **2**

(ii) Hence find the shortest distance between P and this locus. **2**

(d) A particle is moving in simple harmonic motion along the x axis. At time t seconds it has displacement x metres from the origin O given by $x = b + a\cos(nt + \pi)$ for some constants b, a and n, where $a > 0$ and $n > 0$. Initially the particle is at O. After 1 second it is 6 metres to the right of O, and at time 2 seconds it is 12 metres to the right of O. The maximum speed of the particle is $3\pi\,\text{ms}^{-1}$. Find the values of b, a and n. **4**

End of Question 12

STUDENT NAME/NUMBER......................................

Marks

Question 13 (15 marks) **Use a separate writing booklet.**

(a) Use Mathematical Induction to prove that ${}^{2n}C_n < 2^{2n-2}$ for all positive integers $n \geq 5$. **3**

(b) Use integration by parts to find in simplest exact form $\int_1^e 2^{\log_e x}\,dx$. **3**

(c) The complex numbers w and z are such that $|w| = |z| = 1$. Show that the complex number $v = \dfrac{w+z}{1+wz}$ is real. **3**

(d) With respect to a fixed origin O, the points A, B and C have position vectors $\overrightarrow{OA} = 3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\overrightarrow{OB} = 7\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\overrightarrow{OC} = 13\underset{\sim}{i} + 6\underset{\sim}{j} + 7\underset{\sim}{k}$. The line L has vector equation $\underset{\sim}{r} = 6\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k} + \lambda\left(3\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}\right)$, where λ is a scalar parameter.

(i) Show that the point A lies on the line L. **1**

(ii) Show the points A, B and C are collinear. **2**

(iii) Find the acute angle between the line L and the line through points A, B and C. **2**

(iv) Hence find the acute angle between the line through the points A, B and C, and the plane through the point A that is perpendicular to L. **1**

End of Question 13

STUDENT NAME/NUMBER......................................

Marks

Question 14 (15 marks) **Use a separate writing booklet.**

(a) In the Argand diagram, PQR is an equilateral triangle inscribed in a circle with centre at the origin O. The point P represents the complex number $\frac{4i}{1+i}$. Find in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$, the complex numbers represented by Q and R. **3**

(b) Use the substitution $u^2=6-x,\ u>0$ to evaluate in simplest exact form **4**

$$\int_2^5 \frac{5}{x+\sqrt{6-x}}\,dx.$$

(c)(i) Use proof by contradiction to show that for all prime numbers $p>2$, the closed interval $[p\ ,2p]$ always contains at least one positive integer power of 2. **2**

(ii) Use proof by contradiction to show that for all prime numbers $p>2$, the closed interval $[p\ ,2p]$ always contains exactly one positive integer power of 2. **2**

(d) A body of mass m kg moves along the x axis with initial speed $49\,\text{ms}^{-1}$ subject to a resistance of magnitude $m\left(v+\sqrt{v}\right)$ Newtons, where $v\,\text{ms}^{-1}$ is its speed. Find the distance travelled by the body until its speed has dropped to $1\,\text{ms}^{-1}$. **4**

End of Question 14

STUDENT NAME/NUMBER..

Marks

Question 15 (15 marks) **Use a separate writing booklet.**

(a) With respect to a fixed origin O, two lines have equations
$L_1 : \underset{\sim}{r} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 7\underset{\sim}{k} + \lambda\left(\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right)$ and $L_2 : \underset{\sim}{r} = 6\underset{\sim}{i} + 5\underset{\sim}{j} + 2\underset{\sim}{k} + \mu\left(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$,
where λ and μ are scalar parameters. The vector $\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}$ is perpendicular to both lines L_1 and L_2.

(i) Find the point of intersection of the lines L_1 and L_2. **3**

(ii) Find the values of a and b. **3**

(b) The logarithmic mean of any two numbers $a > b > 0$ is given by $L(a\,,\,b) = \int_0^1 a^t b^{1-t}\,dt$.

(i) Show that $L(a\,,\,b) = \dfrac{a-b}{\log_e a - \log_e b}$. **2**

(ii) Use differentiation to show that $\dfrac{\log_e u}{2} > \dfrac{u-1}{u+1}$ for $u > 1$. **3**

(iii) By substituting $u = \dfrac{a}{b}$ in (ii), show that $L(a\,,\,b) < \dfrac{a+b}{2}$ for $a > b > 0$. **2**

(iv) Hence show that $\{L(a\,,\,b)\}^2 < L\left(a^2\,,\,b^2\right)$ for $a > b > 0$. **2**

End of Question 15

STUDENT NAME/NUMBER......................................

Marks

Question 16 (15 marks) **Use a separate writing booklet.**

(a) Use the Binomial expansion of $(1+\cos\theta+i\sin\theta)^{20}$ and De Moivre's Theorem to show that $\sum_{r=0}^{20} {}^{20}C_r \cos r\theta = 2^{20}\cos^{20}\left(\frac{\theta}{2}\right)\cos 10\theta$. **4**

(b)(i) Show that for $a \neq 0$, $\int_{\frac{1}{a}}^{a} \frac{f(x)}{(1+x)^2}\,dx = \frac{1}{2}\int_{\frac{1}{a}}^{a} \frac{f(x)+f\left(\frac{1}{x}\right)}{(1+x)^2}\,dx$. **2**

(ii) Hence show that for $a > 0$, $\int_{\frac{1}{a}}^{a} \frac{\tan^{-1}x + \log_e x}{(1+x)^2}\,dx = \frac{\pi}{4}\left(\frac{a-1}{a+1}\right)$. **3**

(c) A particle of mass m kg is projected from a point O with speed $50\,\text{ms}^{-1}$ at an angle $\tan^{-1}\frac{4}{3}$ above the horizontal. It moves in a vertical plane under gravity in a medium in which the resistance to motion has magnitude $\frac{1}{10}mv$ Newtons when its speed is $v\,\text{ms}^{-1}$. The acceleration due to gravity is $10\,\text{ms}^{-2}$. At time t seconds the particle has position vector $x\underset{\sim}{i} + y\underset{\sim}{j}$ relative to O, velocity vector $v_x\underset{\sim}{i} + v_y\underset{\sim}{j}$ and acceleration vector $-\frac{1}{10}v_x\underset{\sim}{i} - \left(\frac{1}{10}v_y + 10\right)\underset{\sim}{j}$.

(i) Show that $v_y = 140e^{-0\cdot1t} - 100$ and $y = 1400\left(1-e^{-0\cdot1t}\right) - 100t$. **4**

(ii) Hence find in simplest exact form the time to greatest height and the greatest height reached by the particle. **2**

END OF PAPER

Independent Trial HSC 2024 Mathematics Extension 2 Marking Guidelines

Section 1 Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1	C	Let $a=1,\ b=2$. Then $\frac{a}{1+a^2}=\frac{1}{2}>\frac{2}{5}=\frac{b}{1+b^2}$ $\therefore$ C is False If $a<0<b$, all 4 given inequalities are true. If $0<a<b$: $\frac{1+\lvert a\rvert}{a}=\frac{1}{a}+1>\frac{1}{b}+1=\frac{1+\lvert b\rvert}{b}$ $\therefore \frac{a}{1+\lvert a\rvert}<\frac{b}{1+\lvert b\rvert}$ (A) $a(1+\lvert a\rvert)<b(1+\lvert b\rvert)$ $\therefore \frac{a}{1+\lvert b\rvert}<\frac{b}{1+\lvert a\rvert}$ (B) $a(1+a^2)<b(1+b^2)$ $\therefore \frac{a}{1+b^2}<\frac{b}{1+a^2}$ (D) If $a<b<0$, then $0<-b<-a$. Replacing $b\to -a,\ a\to -b$ shows inequalities in A, B, D are true. Hence only false statement is C.	MEX12-2
2	A	$\frac{e^{iB}\times e^{iC}}{e^{iA}}=\frac{e^{i(B+C)}}{e^{iA}}=\frac{e^{i(\pi-A)}}{e^{iA}}=e^{i\pi}e^{i(-2A)}=-e^{i(-2A)}$ $=-\cos 2A+i\sin 2A$	MEX12-4
3	C	$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=6\underset{\sim}{i}+3\underset{\sim}{j}-2\underset{\sim}{k}$ $PQ=\sqrt{6^2+3^2+(-2)^2}=7$	MEX12-3
4	D	$\int\cos^3 x\,dx=\int(1-\sin^2 x)\cos x\,dx=\sin x-\frac{1}{3}\sin^3 x+c$	MEX12-5
5	B	$v^2=8-2(x+1)^2=2\{2^2-(x+1)^2\}$ Motion centred about $x=-1$	MEX12-6
6	D	$f(x),\ g(x)$ *both odd* $\Rightarrow f(g(-x))=f(-g(x))=-f(g(x))$ Hence the original statement and its contrapositive are true. Converse: $f(g(x))$ *is odd* $\Rightarrow f(x)$ *is odd and* $g(x)$ *is odd* Consider the functions $g(x)=x-1,\ f(x)=x+1.\ f(g(x))=f(x-1)=x$ Hence $f(g(x))$ is odd but neither $f(x)$ nor $g(x)$ is odd. This counter example shows the converse is false.	MEX12-2
7	B	$\lvert z\rvert^2=\cos^4\theta+\sin^4\theta=(\cos^2\theta+\sin^2\theta)^2-2\sin^2\theta\cos^2\theta=1-\frac{1}{2}\sin^2 2\theta$ $0\le\frac{1}{2}\sin^2 2\theta\le\frac{1}{2}$ $\therefore\frac{1}{2}\le\lvert z\rvert^2\le 1$ $\therefore\frac{1}{\sqrt{2}}\le\lvert z\rvert\le 1$	MEX12-4
8	A	For any point on the line, $2x+y+z=2(1+\lambda)+(1-\lambda)+(1-\lambda)=4$. Hence no point on the line lies in the plane.	MEX12-3
9	B	Let $u=a-x$. $\int_0^a f(a-x)dx=\int_a^0 f(u).-du=\int_0^a f(u)du=\int_0^a f(x)dx$ *Let* $v=2a-x$. $\int_0^a f(2a-x)dx=\int_{2a}^a f(v).-dv=\int_a^{2a} f(v)dv=\int_a^{2a} f(x)dx$ $\therefore \int_0^a\{f(a-x)+f(2a-x)\}dx=\int_0^a f(x)dx+\int_a^{2a} f(x)dx=\int_0^{2a} f(x)dx$	MEX12-5
10	C	Hits ground when $x=X,\ y=-H$. $\therefore \frac{1}{2}gT^2=H$ and $VT=X$ $\therefore\frac{gT}{2V}=\frac{H}{X}$ and $\tan\theta=\left\lvert\frac{v_y}{v_x}\right\rvert=\frac{gT}{V}=\frac{2H}{X}$ $\therefore\ \theta=\tan^{-1}\left(\frac{2H}{X}\right)$	MEX12-6

Section II

Question 11

a. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Uses the given substitution to find the anti-derivative as a function of x	2
Substantial progress eg. finds and simplifies the integrand of the transformed integral	1

Answer

$$t=\tan\frac{x}{2}$$

$$dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$$

$$dx=\frac{2}{1+t^2}dt$$

$$\frac{1-\tan\frac{x}{2}}{\cos x}=\frac{(1-t)(1+t^2)}{1-t^2}=\frac{1+t^2}{1+t}$$

$$\int\frac{1-\tan\frac{x}{2}}{\cos x}dx=\int\frac{1+t^2}{1+t}\cdot\frac{2}{1+t^2}dt$$

$$=2\log_e|1+t|+c$$

$$=2\log_e\left|1+\tan\frac{x}{2}\right|+c$$

b. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Evaluates the definite integral	2
Substantial progress eg. finds the anti-derivative	1

Answer

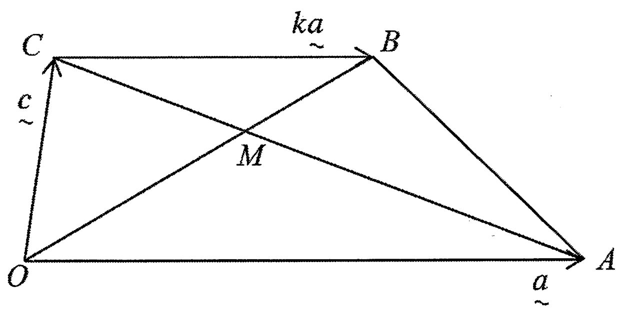
$$\int_0^1\frac{\tan^{-1}x}{1+x^2}dx=\frac{1}{2}\Big[(\tan^{-1}x)^2\Big]_0^1=\frac{1}{2}\left(\frac{\pi}{4}\right)^2=\frac{1}{32}\pi^2$$

c.i. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Uses vector methods to establish required result	2
Substantial progress eg. writes a vector equation in terms of $\underset{\sim}{a}$, $\underset{\sim}{c}$, λ, μ and k.	1

Answer



$$\overrightarrow{OM}=\underset{\sim}{a}+\mu\overrightarrow{AC}$$

$$\overrightarrow{OM}=\underset{\sim}{a}+\mu(\underset{\sim}{c}-\underset{\sim}{a})$$

$$\lambda\overrightarrow{OB}=\underset{\sim}{a}+\mu(\underset{\sim}{c}-\underset{\sim}{a})$$

$$\lambda(\underset{\sim}{c}+k\underset{\sim}{a})=\underset{\sim}{a}+\mu(\underset{\sim}{c}-\underset{\sim}{a})$$

$$(\lambda-\mu)\underset{\sim}{c}=(1-\mu-\lambda k)\underset{\sim}{a}$$

$$\therefore\ \lambda-\mu=0 \text{ and } 1-\mu-\lambda k=0 \quad (\text{since } \underset{\sim}{a}\nparallel\underset{\sim}{c})$$

$$\lambda=1-\lambda k$$

$$\lambda(1+k)=1$$

$$\therefore\ \lambda=\frac{1}{k+1} \text{ and } \overrightarrow{OM}=\frac{1}{k+1}\overrightarrow{OB}$$

ont.

c.ii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Explains why *OABC* is a parallelogram	1

Answer

If $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB}$, then $k = 1$ and OA, CB are equal in length and parallel so that *OABC* is a parallelogram.

d.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
States the remaining two roots	2
Substantial progress eg. identifies $5 - 2i$ as a second root	1

Answer

$z^3 - 12z^2 + cz + d = 0$ has roots $5+2i,\ 5-2i,\ \alpha$ (c, d real). $\quad \therefore\ 10 + \alpha = 12 \quad \therefore \alpha = 2$

d.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Evaluates both c and d	2
Substantial progress eg. evaluates one of c or d	1

Answer

$$c = 2(5+2i) + 2(5-2i) + (5+2i)(5-2i) = 49$$
$$d = -2(5+2i)(5-2i) = -58$$

e.i. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Finds an expression for v in terms of x by integration	2
Substantial progress eg. integrates to get v^2 in terms of x	1

Answer

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x - 10$$
$$\frac{1}{2}v^2 = \frac{1}{2}(x-10)^2 + c$$

$t = 0$, $x = 1$, $v = 9$ $\Rightarrow$ $\frac{1}{2}\times 9^2 = \frac{1}{2}\times(-9)^2 + c$, $c = 0$ $\quad \therefore \quad v^2 = (x-10)^2$

$t = 0 \Rightarrow x - 10 < 0$ and $v > 0 \quad \therefore\ v = 10 - x$

Q11 cont.

e.ii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Uses integration to find x as a function of t	2
Substantial progress eg. finds a relationship between $\ln(10-x)$ and t.	1

Answer

$$\frac{dx}{dt}=10-x$$

$$\int\frac{-1}{10-x}dx=-\int dt$$

$$\ln A(10-x)=-t\ ,\ A\ const$$

$$t=0, x=1 \Rightarrow A=\frac{1}{9}$$

$$\ln\frac{1}{9}(10-x)=-t$$

$$\frac{1}{9}(10-x)=e^{-t}$$

$$10-x=9e^{-t}$$

$$x=10-9e^{-t}$$

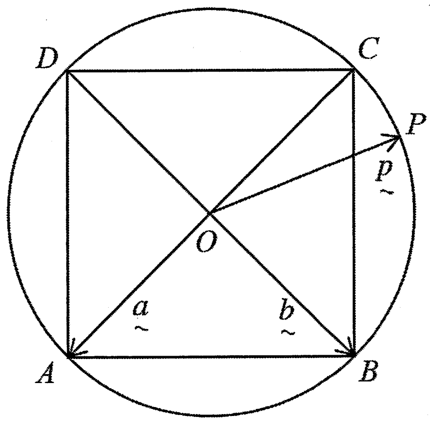
Question 12

a. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Use vector methods to establish the required results	3
Substantial progress eg. writes $\lvert\overrightarrow{PA}\rvert^2$, $\lvert\overrightarrow{PB}\rvert^2$ in terms of r and $\underset{\sim}{a}\cdot\underset{\sim}{p}$, $\underset{\sim}{b}\cdot\underset{\sim}{p}$	2
Some progress eg. writes $\lvert\overrightarrow{PA}\rvert^2=(\underset{\sim}{a}-\underset{\sim}{p})\cdot(\underset{\sim}{a}-\underset{\sim}{p})$ and expands	1

Answer



$$\lvert\overrightarrow{PA}\rvert^2=(\underset{\sim}{a}-\underset{\sim}{p})\cdot(\underset{\sim}{a}-\underset{\sim}{p})=\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{p}\cdot\underset{\sim}{p}-2\underset{\sim}{a}\cdot\underset{\sim}{p}=2r^2-2\underset{\sim}{a}\cdot\underset{\sim}{p}$$

Similarly, $\lvert\overrightarrow{PB}\rvert^2=2r^2-2\underset{\sim}{b}\cdot\underset{\sim}{p}$

The diagonals of a square bisect each other.

Hence $\overrightarrow{OC}=-\underset{\sim}{a}$ and $\overrightarrow{OD}=-\underset{\sim}{b}$.

$$\lvert\overrightarrow{PC}\rvert^2=2r^2-2(-\underset{\sim}{a})\cdot\underset{\sim}{p}=2r^2+2\underset{\sim}{a}\cdot\underset{\sim}{p}$$

$$\lvert\overrightarrow{PD}\rvert^2=2r^2-2(-\underset{\sim}{b})\cdot\underset{\sim}{p}=2r^2+2\underset{\sim}{b}\cdot\underset{\sim}{p}$$

Hence $\lvert\overrightarrow{PA}\rvert^2+\lvert\overrightarrow{PB}\rvert^2+\lvert\overrightarrow{PC}\rvert^2+\lvert\overrightarrow{PD}\rvert^2=8r^2$

b.i. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
States the contrapositive statement	1

Answer

Contrapositive: *If* $p\neq 6k\pm 1$ *for any positive integer* $k=1, 2, 3, \ldots$ *then* p *is not prime.*

Q12 cont.

b.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves the contrapositive statement is true	3
Substantial progress eg. correct procedure but explanation is incomplete or lacks clarity	2
Some progress eg. identifies the forms of $p>3$ modulo 6 and shows one of 0, 3, ±2 is not prime	1

Answer

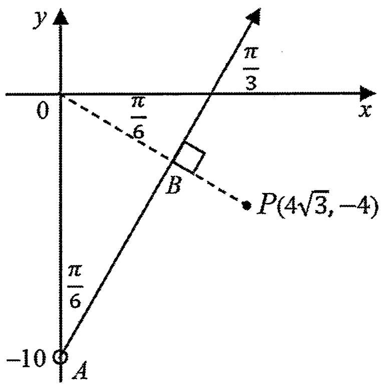
Any positive integer $p>3$ can be written in one of the forms $6k$, $6k\pm1$, $6k\pm2$, $6k+3$ for some positive integer $k=1, 2, 3, \ldots$.
For such k, $6k=2\times(3k)$, $6k\pm2=2\times(3k\pm1)$, $6k+3=3\times(2k+1)$ are not prime, since the factors $3k$, $(3k\pm1)$, $(2k+1)$ are all positive integers greater than or equal to 2.
Hence positive integers $p>3$ that are not of the form $6k\pm1$ where $k=1, 2, 3, \ldots$ are not prime.

c.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Sketches locus of z and point P showing sufficient detail	2
Substantial progress eg. shows one of locus of z or position of P correctly	1

Answer



$OP=|w|=8$

c.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Finds perpendicular distance from P to locus of z	2
Substantial progress eg. uses trigonometry to find OB	1

Answer

$OB=10\sin\frac{\pi}{6}=5$ and $PB=OP-OB$ $\therefore PB=8-5=3$

Q12 cont.

d. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Solves an appropriate set of 4 equations to find values of b, a and n	4
Substantial progress eg. writes an appropriate set of 4 equations and shows $a=b$ and $\cos n=0$	3
Moderate progress eg. writes an appropriate set of 4 equations and shows $a=b$	2
Some progress eg. writes an appropriate set of equations	1

Answer

$x = b + a\cos(nt+\pi) = b - a\cos nt$

$v = -na\sin(nt+\pi) = na\sin nt$

$t=0, x=0 \Rightarrow b-a=0 \quad (1)$

$t=1, x=6 \Rightarrow b-a\cos n=6 \quad (2)$

$t=2, x=12 \Rightarrow b-a\cos 2n=12 \quad (3)$

Max. speed is $3\pi\,\text{ms}^{-1}$

$na=3\pi \quad (4)$

From (1), $b=a$. Then $\frac{(3)}{(2)} \Rightarrow \frac{1-\cos 2n}{1-\cos n}=2$

$$\frac{2\sin^2 n}{1-\cos n}=2$$

$$\frac{2(1-\cos^2 n)}{1-\cos n}=2$$

$$1+\cos n=1$$

$\therefore \cos n=0$

Then from (2), $b=6$

$\therefore a=6$

Then from (4), $6n=3\pi$

$n=\frac{\pi}{2}$

Question 13

a. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses the process of Mathematical Induction to prove the required result	3
Substantial progress eg. correct procedure but some lack of clarity in explanation	2
Some progress eg. shows statement true for n=5	1

Answer

Let P_n, $n=5, 6, 7, \ldots$, be the sequence of statements P_n: ${}^{2n}C_n < 2^{2n-2}$.

Consider P_5: ${}^{10}C_5 = 252 < 256 = 2^{10-2}$. Hence P_5 is true.

If P_k is true: ${}^{2k}C_k < 2^{2k-2}$ $\quad *$

Consider P_{k+1}: ${}^{2(k+1)}C_{k+1} = \frac{(2(k+1))!}{(k+1)!(k+1)!}$

$$= \frac{(2k+2)(2k+1)(2k)!}{(k+1)^2 k!k!}$$

$$= \frac{2(2k+1)}{(k+1)}\,{}^{2k}C_k$$

$$< \frac{4\left(k+\frac{1}{2}\right)}{(k+1)}\,2^{2k-2} \quad \text{if } P_k \text{ is true, using } *.$$

$$< 2^{2k}$$

$$= 2^{2(k+1)-2}$$

Hence if P_k is true, then P_{k+1} is true. But P_5 is true. Hence by Mathematical Induction, P_n is true for all integers $n \geq 5$.

Q13 cont.

b. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Uses integration by parts to evaluate the definite integral	3
Substantial progress eg. correct procedure but minor error in evaluation by substitution of limits	2
Some progress eg. performs first step correctly	1

Answer

$$\int_1^e 2^{\ln x}dx = \Big[\, x2^{\ln x} \,\Big]_1^e - \int_1^e x\,(\ln 2)\,2^{\ln x}.\frac{1}{x}dx$$

$$= (2e-1) - \int_1^e (\ln 2)\,2^{\ln x}dx$$

$$(1+\ln 2)\int_1^e 2^{\ln x}dx = 2e-1$$

$$\int_1^e 2^{\ln x}dx = \frac{2e-1}{1+\ln 2}$$

c. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Proves the required result	3
Substantial progress eg. shows that each of $w+z$, $1+wz$ has argument $\frac{1}{2}(\alpha+\beta)$	2
Some progress eg. shows that one of $w+z$, $1+wz$ has argument $\frac{1}{2}(\alpha+\beta)$	1

Answer

Let $w = e^{i\alpha}$, $z = e^{i\beta}$ where $-\pi < \alpha \le \pi$, $-\pi < \beta \le \pi$.

Then $w+z = e^{i\alpha} + e^{i\beta} = e^{i\left(\frac{\alpha+\beta}{2}\right)}\left(e^{i\left(\frac{\alpha-\beta}{2}\right)} + e^{-i\left(\frac{\alpha-\beta}{2}\right)}\right) = 2\cos\left(\frac{\alpha-\beta}{2}\right)e^{i\left(\frac{\alpha+\beta}{2}\right)}$

and $1+wz = 1 + e^{i(\alpha+\beta)} = e^{i\left(\frac{\alpha+\beta}{2}\right)}\left(e^{-i\left(\frac{\alpha+\beta}{2}\right)} + e^{i\left(\frac{\alpha+\beta}{2}\right)}\right) = 2\cos\left(\frac{\alpha+\beta}{2}\right)e^{i\left(\frac{\alpha+\beta}{2}\right)}$

Hence for $wz \ne -1$, $\dfrac{w+z}{1+wz} = \dfrac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}$ which is real.

d.i. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Shows A lies on L	1

Answer

$\underset{\sim}{r} = 6\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k} + \lambda\left(3\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}\right)$ $\quad \lambda = -1 \rightarrow \underset{\sim}{r} = 3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} = \overrightarrow{OA}$ $\quad \therefore A$ lies on L

Q13 cont.

d. ii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Shows A, B, C are collinear	2
Substantial progress eg. finds one of the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{BC}$	1

Answer

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (7-3)\underset{\sim}{i} + (3-1)\underset{\sim}{j} + (4-2)\underset{\sim}{k} = 2\left(2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (13-3)\underset{\sim}{i} + (6-1)\underset{\sim}{j} + (7-2)\underset{\sim}{k} = 5\left(2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$

$\overrightarrow{AC} = \frac{5}{2}\overrightarrow{AB}$ $\quad \therefore A, B, C$ are collinear.

d.iii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Finds the acute angle between the lines	2
Substantial progress eg. finds an expression for the cosine of this angle	1

Answer

L has direction vector $\underset{\sim}{u} = 3\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}$. Line through A, B, C has direction vector $\underset{\sim}{v} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

$\cos\theta = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \dfrac{6+4+5}{\sqrt{50}\sqrt{6}} = \dfrac{\sqrt{3}}{2}$ $\quad \therefore \theta = \frac{\pi}{6}$ is the acute angle between lines L and ABC.

d.iv. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
States the size of the required angle	1

Answer

Let H be the plane through A that is $\perp$ to line L. Consider the plane K determined by the line L and the line ABC, these lines intersecting at A. Planes H and K meet at right angles and their line of intersection contains the projection of line ABC on plane H. Hence the acute angle between line ABC and plane H, being equal to the angle between line ABC and its projection on H, is $\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$.

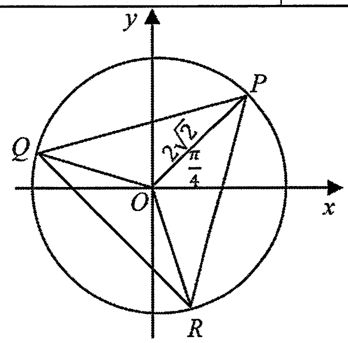
Question 14

a. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Finds both required complex numbers in stated form	3
Substantial progress eg. finds the radius of the circle and argument of the given complex number	2
Some progress eg. finds the radius of the circle	1

Answer



P represents the complex number $z=\dfrac{4i}{1+i}=\dfrac{4i(1-i)}{2}=2(1+i)=2\sqrt{2}e^{i\frac{\pi}{4}}$.

$\therefore$ Equilateral ΔPQR has vertices on the circle with centre O and radius $2\sqrt{2}$.

Then $OQ=OR=2\sqrt{2}$, and $\Delta OPQ \equiv \Delta OQR \equiv \Delta ORP$ (SSS).

$\therefore \angle POQ=\angle POR=\dfrac{2\pi}{3}$. Hence Q, R represent the complex numbers

$2\sqrt{2}e^{i\left(\frac{\pi}{4}\pm\frac{2\pi}{3}\right)}$, that is $2\sqrt{2}e^{i\frac{11\pi}{12}}$, $2\sqrt{2}e^{-i\frac{5\pi}{12}}$.

b. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Uses the given substitution to evaluate the definite integral in simplest exact form	4
Substantial progress eg. correct procedure, finding anti-derivative in terms of u	3
Moderate progress eg. finds integrand of transformed integral as sum of partial fractions in u	2
Some progress eg. writes transformed definite integral in terms of u	1

Answer

$u^2=6-x\,,\ u>0$

$2u\,du=-dx$

$x=2\Rightarrow u=2$

$x=5\Rightarrow u=1$

$x+\sqrt{6-x}=6-u^2+u$

$=(3-u)(2+u)$

$$\int_2^5 \frac{5}{x+\sqrt{6-x}}dx=\int_2^1 \frac{5}{(3-u)(2+u)}(-2u)du$$

$$=2\int_1^2 \frac{5u}{(3-u)(2+u)}du$$

$$=2\int_1^2\left(\frac{3}{3-u}-\frac{2}{2+u}\right)du$$

$$=2\Big[-3\ln(3-u)-2\ln(2+u)\Big]_1^2$$

$$=2\{-3(\ln1-\ln2)-2(\ln4-\ln3)\}$$

$$=2(2\ln3-\ln2)$$

$$=2\ln\frac{9}{2}$$

c.i. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses the method of proof by contradiction to establish the required result.	2
Substantial progress eg. correct procedure but some lack of clarity in argument	1

Answer

Consider the integer p, $p>2$. If $[p\,,2p]$ does not contain any positive integer powers of 2, then $\exists k>0$, k integral, such that $2^k<p<2p<2^{k+1}$. But $2^k<p \Rightarrow 2\times2^k<2p$ so that $2^{k+1}<2p$. Hence by contradiction $[p\,,2p]$ must contain at least one positive integer power of 2 for all integers $p>2$ and hence for all prime numbers $p>2$.

Q14 cont.

c.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses proof by contradiction and the result from i. to establish the required result	2
Substantial progress eg. correct procedure but some lack of clarity in argument	1

Answer

p prime, $p>2$. If $[p\ ,2p]$ contains more than one positive integer power of 2, then $\exists k>1$, k integral, such that $2<p\le 2^k<2^{k+1}\le 2p$. Then $2^{k+1}\le 2p\ \Rightarrow 2^k\le p$ so that $p=2^k$, $k>1$, and p has a factor 2.

Hence if $[p\ ,2p]$ contains more than one positive integer power of 2, then p is not prime.
Hence by contradiction, $[p\ ,2p]$ does not contain more than one positive integer power of 2.
Hence from i., $[p\ ,2p]$ contains exactly one positive integer power of 2.

d. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Writes and solves a first order differential equation in terms of v and x to find required distance	4
Substantial progress eg. correct process, finding an appropriate anti-derivative	3
Moderate progress eg. expresses the distance as an appropriate integral in terms of v	2
Some progress eg. writes a first order differential equation in terms of v and x	1

Answer

Taking the initial position as the origin, $x=0$ when $v=49$, and let $x=X$ when $v=1$.

$$v\frac{dv}{dx}=-\left(v+\sqrt{v}\right).\qquad \therefore\ \int_0^X dx=\int_{49}^{1}\frac{-v}{v+\sqrt{v}}dv$$

Make the substitution $u^2=v,\ u>0$.

$$u^2=v,\ u>0$$
$$2u\,du=dv$$

$$v=49\Rightarrow u=7$$
$$v=1\ \Rightarrow u=1$$

$$\frac{v}{v+\sqrt{v}}=\frac{\sqrt{v}}{\sqrt{v}+1}=\frac{u}{u+1}$$

$$X=\int_7^1\frac{-u}{u+1}\,2u\,du$$
$$=2\int_1^7\frac{u^2-1+1}{u+1}du$$
$$=2\int_1^7\left\{(u-1)+\frac{1}{u+1}\right\}du$$
$$=2\left[\tfrac{1}{2}(u-1)^2+\ln(u+1)\right]_1^7$$
$$=2\{18+\ln 4\}$$

Distance travelled is $(36+4\ln 2)$ metres.

Question 15

a.i. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Writes and solves a system of 3 equations in λ and μ to find the point of intersection	3
Substantial progress eg. correct process but neglects to check equations are consistent	2
Some progress eg. writes a system of 3 equations in λ and μ	1

Answer

$L_1 : \underset{\sim}{r} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 7\underset{\sim}{k} + \lambda\left(\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right)$ and $L_2 : \underset{\sim}{r} = 6\underset{\sim}{i} + 5\underset{\sim}{j} + 2\underset{\sim}{k} + \mu\left(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$. At any point of intersection

$$3 + \lambda = 6 + 2\mu \quad (1) \qquad (1) + (2) \Rightarrow 5 = 11 + 3\mu \qquad \textit{Substituting in } (3)$$
$$2 - \lambda = 5 + \mu \quad (2) \qquad \mu = -2 \qquad LHS = 7 + 3(-1) = 4$$
$$7 + 3\lambda = 2 - \mu \quad (3) \qquad \textit{Then } (2) \Rightarrow \lambda = -1 \qquad RHS = 2 - (-2) = 4$$

Hence L_1, L_2 intersect at $(2, 3, 4)$

a.ii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Writes and solves a pair of linear equations in a and b	3
Substantial progress eg. writes a pair of linear equations in a and b	2
Some progress eg. writes two dot products equal to 0	1

Answer

$\left(\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}\right) \cdot \left(\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right) = 0$ and $\left(\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}\right) \cdot \left(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right) = 0.$

$$1 - a + 3b = 0 \quad (1) \qquad (1) + (2) \Rightarrow 3 + 2b = 0 \quad \therefore b = -\frac{3}{2}$$
$$2 + a - b = 0 \quad (2) \qquad \textit{Then } (2) \Rightarrow a + \frac{7}{2} = 0 \quad \therefore a = -\frac{7}{2}$$

b.i. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Establishes the required result by integration	2
Substantial progress eg. correct process but error in evaluation	1

Answer

$$L(a, b) = b\int_0^1 \left(\frac{a}{b}\right)^t dt = \frac{b}{\ln\left(\frac{a}{b}\right)}\left[\left(\frac{a}{b}\right)^t\right]_0^1 = \frac{b}{\ln\left(\frac{a}{b}\right)}\left(\frac{a}{b} - 1\right) \qquad \therefore L(a, b) = \frac{a - b}{\log_e a - \log_e b}$$

Q15 cont.

b.ii Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses differentiation to prove the required inequality	3
Substantial progress eg. correct process but some lack of clarity or detail in explanation	2
Some progress eg. defines an appropriate function of u and finds its derivative	1

Answer

Let $f(u)=\dfrac{\ln u}{2}-\dfrac{u-1}{u+1}$, $u\ge 1$. Then $f(1)=0$.

$$f'(u)=\frac{1}{2u}-\frac{(u+1)-(u-1)}{(u+1)^2}$$

$$=\frac{(u+1)^2-4u}{2u(u+1)^2}$$

$$=\frac{(u-1)^2}{2u(u+1)^2}$$

$\therefore f'(u)>0$ for $u>1$

$\therefore f(u)$ is increasing for $u>1$

But $f(u)$ is continuous for $u\ge 1$ and $f(1)=0$

$\therefore f(u)>0$ for $u>1$.

$\therefore \dfrac{\log_e u}{2}>\dfrac{u-1}{u+1}$ for $u>1$.

b.iii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses ii. to deduce required inequality	2
Substantial progress eg. makes directed substitution in (ii) with some simplification	1

Answer

$$u=\frac{a}{b}\,,a>b>0 \quad \rightarrow \quad \frac{\log_e\left(\frac{a}{b}\right)}{2}>\frac{\frac{a}{b}-1}{\frac{a}{b}+1}$$

$$\frac{\log_e a-\log_e b}{2}>\frac{a-b}{a+b}$$

$$\frac{a+b}{2}>\frac{a-b}{\log_e a-\log_e b}$$

$$\therefore L(a\,,b)<\frac{a+b}{2}$$

Q15 cont.

b.iv. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses iii. to prove required inequality	2
Substantial progress eg. writes and simplifies expression for $L(a^2, b^2)$ in terms of a and b	1

Answer

$$\{L(a,b)\}^2 < L(a,b)\left(\frac{a+b}{2}\right)$$
$$= \frac{a-b}{\log_e a - \log_e b}\left(\frac{a+b}{2}\right)$$
$$= \frac{(a-b)(a+b)}{2\log_e a - 2\log_e b}$$
$$= \frac{a^2-b^2}{\log_e a^2 - \log_e b^2}$$
$$= L(a^2, b^2)$$

Question 16

a. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Proves required identity using Binomial expansion and De Moivre's Theorem	4
Substantial progress eg. finds the two appropriate expressions for $(1+\cos\theta+i\sin\theta)^{20}$	3
Moderate progress eg. finds $(1+\cos\theta+i\sin\theta)^{20}$ in terms of $\frac{\theta}{2}$ and 10θ	2
Some progress eg. uses the binomial theorem to expand $(1+\cos\theta+i\sin\theta)^{20}$ in terms of $r\theta$	1

Answer

$$(1+\cos\theta+i\sin\theta)^{20} = \sum_{r=0}^{20} {}^{20}C_r(\cos\theta+i\sin\theta)^r$$
$$= \sum_{r=0}^{20} {}^{20}C_r(\cos r\theta+i\sin r\theta)$$

$$1+\cos\theta+i\sin\theta = 2\cos^2\frac{\theta}{2} + i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$
$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$$
$$(1+\cos\theta+i\sin\theta)^{20} = 2^{20}\left(\cos\frac{\theta}{2}\right)^{20}(\cos 10\theta + i\sin 10\theta)$$

Equating real parts of the two expressions for $(1+\cos\theta+i\sin\theta)^{20}$:

$$\sum_{r=0}^{20} {}^{20}C_r\cos r\theta = 2^{20}\cos^{20}\left(\frac{\theta}{2}\right)\cos 10\theta$$

Q16 cont.

b. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Uses an appropriate substitution to prove the required result	2
Substantial progress eg. makes the substitution $u=\frac{1}{x}$ to obtain definite integral in terms of u	1

Answer

$$\frac{1}{u}=x$$

$$-\frac{1}{u^2}du=dx$$

$$x=\frac{1}{a}\rightarrow u=a$$

$$x=a\rightarrow u=\frac{1}{a}$$

$$\int_{\frac{1}{a}}^{a}\frac{f(x)}{(1+x)^2}dx=\int_{a}^{\frac{1}{a}}\frac{f\left(\frac{1}{u}\right)}{\left(1+\frac{1}{u}\right)^2}\left(-\frac{1}{u^2}\right)du$$

$$=\int_{\frac{1}{a}}^{a}\frac{f\left(\frac{1}{u}\right)}{\left(u+1\right)^2}du$$

$$=\int_{\frac{1}{a}}^{a}\frac{f\left(\frac{1}{x}\right)}{\left(1+x\right)^2}dx$$

$$\therefore 2\int_{\frac{1}{a}}^{a}\frac{f(x)}{(1+x)^2}dx=\int_{\frac{1}{a}}^{a}\frac{f(x)}{(1+x)^2}dx+\int_{\frac{1}{a}}^{a}\frac{f\left(\frac{1}{x}\right)}{\left(1+x\right)^2}dx$$

$$\therefore\int_{\frac{1}{a}}^{a}\frac{f(x)}{(1+x)^2}dx=\frac{1}{2}\int_{\frac{1}{a}}^{a}\frac{f(x)+f\left(\frac{1}{x}\right)}{\left(1+x\right)^2}dx$$

Q16 cont.

b.ii. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Uses result from i. and identities for $\tan^{-1}$ and log functions to establish required result	3
Substantial progress eg. uses i. and $\tan^{-1}$ and log identities to find anti-derivative	2
Some progress eg. uses i. and one of the $\tan^{-1}$ or log identities to partially simplify integrand	1

Answer

$\log_e\left(\frac{1}{x}\right) = -\log_e x$ and for $x > 0$, $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$. Hence

$$\int_{\frac{1}{a}}^{a} \frac{\tan^{-1}x + \log_e x}{\left(1+x\right)^2}dx = \frac{1}{2}\int_{\frac{1}{a}}^{a} \frac{\left(\tan^{-1}x + \tan^{-1}\frac{1}{x}\right) + \left(\log_e x + \log_e\left(\frac{1}{x}\right)\right)}{\left(1+x\right)^2}dx$$

$$= \frac{1}{2}\int_{\frac{1}{a}}^{a} \frac{\frac{\pi}{2}+0}{(1+x)^2}dx$$

$$= \frac{\pi}{4}\left[-\frac{1}{1+x}\right]_{\frac{1}{a}}^{a}$$

$$= \frac{\pi}{4}\left(\frac{-1}{1+a} + \frac{1}{1+\frac{1}{a}}\right)$$

$$= \frac{\pi}{4}\left(\frac{a}{a+1} - \frac{1}{a+1}\right)$$

$$= \frac{\pi}{4}\left(\frac{a-1}{a+1}\right)$$

Q16 cont.

c.i. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Establishes both required results by integration	4
Substantial progress eg. establishes result for the vertical velocity by integration	3
Moderate progress eg. establishes the result for the height by integration	2
Some progress eg. finds the anti-derivative for the integral involving v_y and t	1

Answer

$$\frac{dv_y}{dt} = -\left(\frac{1}{10}v_y + 10\right)$$

$$= -\frac{v_y + 100}{10}$$

$$\int \frac{1}{v_y + 100}\,dv_y = -\frac{1}{10}\int dt$$

$$\ln\left(A\left(v_y + 100\right)\right) = -\frac{1}{10}t\,, \quad A \text{ constant}$$

$$\left.\begin{array}{l} t = 0 \\ v_y = \frac{4}{5}\times 50 \end{array}\right\} \Rightarrow \quad A = \frac{1}{140}$$

$$\ln\left(\frac{v_y + 100}{140}\right) = -\frac{1}{10}t$$

$$\frac{v_y + 100}{140} = e^{-0\cdot 1t}$$

$$v_y = 140e^{-0\cdot 1t} - 100$$

$$\frac{dy}{dt} = 140e^{-0\cdot 1t} - 100$$

$$y = -1400e^{-0\cdot 1t} - 100t + c$$

$$\left.\begin{array}{l} t = 0 \\ y = 0 \end{array}\right\} \Rightarrow \quad \begin{array}{l} c = 1400 \\ y = 1400\left(1 - e^{-0\cdot 1t}\right) - 100t \end{array}$$

c.ii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Finds both exact time to greatest height and exact greatest height	2
Substantial progress eg. finds the exact time to greatest height	1

Answer

Greatest height is value of y when $v_y = 0$.

$$v_y = 0 \Rightarrow e^{-0\cdot 1t} = \frac{100}{140}$$

$$e^{0\cdot 1t} = \frac{7}{5}$$

$$\frac{1}{10}t = \ln\frac{7}{5}$$

$$t = 10\ln\frac{7}{5}$$

$$t = 10\ln\frac{7}{5} \Rightarrow y = 1400\left(1 - \frac{100}{140}\right) - 1000\ln\frac{7}{5}$$

$$= 400 - 1000\ln\frac{7}{5}$$

Hence greatest height is $\left(400 - 1000\ln\frac{7}{5}\right)$ metres

reached $10\ln\frac{7}{5}$ seconds after projection.

Independent HSC Trial Examinations 2024 Mathematics Extension 2 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1	1	The nature of proof	MEX12-2	E2-E3
2	1	Introduction to complex numbers	MEX12-4	E2-E3
3	1	Further work with vectors	MEX12-3	E2-E3
4	1	Further integration	MEX12-5	E2-E3
5	1	Applications of calculus to mechanics	MEX12-6	E2-E3
6	1	The nature of proof	MEX12-2	E3-E4
7	1	Introduction to complex numbers	MEX12-4	E3-E4
8	1	Further work with vectors	MEX12-3	E3-E4
9	1	Further integration	MEX12-5	E3-E4
10	1	Applications of calculus to mechanics	MEX12-6	E3-E4
11 a	2	Further integration	MEX12-5	E2-E3
b	2	Further integration	MEX12-5	E2-E3
c i	2	Further work with vectors	MEX12-3	E2-E3
ii	1	Further work with vectors	MEX12-3	E2-E3
d i	2	Introduction to complex numbers	MEX12-4	E2-E3
ii	2	Introduction to complex numbers	MEX12-4	E2-E3
e i	2	Applications of calculus to mechanics	MEX12-6	E2-E3
ii	2	Applications of calculus to mechanics	MEX12-6	E2-E3
12 a	3	Further work with vectors	MEX12-3	E2-E3
b i	1	The nature of proof	MEX12-2	E2-E3
ii	3	The nature of proof	MEX12-2	E2-E3
c i	2	Introduction to complex numbers	MEX12-4	E2-E3
ii	2	Introduction to complex numbers	MEX12-4	E2-E3
d	4	Applications of calculus to mechanics	MEX12-6	E3-E4
13 a	3	The nature of proof	MEX12-2	E3-E4
b	3	Further integration	MEX12-5	E3-E4
c	3	Introduction to complex numbers	MEX12-4	E3-E4
d i	1	Further work with vectors	MEX12-3	E2-E3
ii	2	Further work with vectors	MEX12-3	E2-E3
iii	2	Further work with vectors	MEX12-3	E2-E3
iv	1	Further work with vectors	MEX12-3	E2-E3
14 a	3	Introduction to complex numbers	MEX12-4	E2-E3
b	4	Further integration	MEX12-5	E3-E4
c i	2	The nature of proof	MEX12-2	E2-E3
ii	2	The nature of proof	MEX12-2	E2-E3
d	4	Applications of calculus to mechanics	MEX12-6	E3-E4
15 a i	3	Further work with vectors	MEX12-3	E2-E3
ii	3	Further work with vectors	MEX12-3	E2-E3
b i	2	Further integration	MEX12-5	E3-E4
ii	3	The nature of proof	MEX12-2	E3-E4
iii	2	The nature of proof	MEX12-2	E3-E4
iv	2	The nature of proof	MEX12-2	E3-E4
16 a	4	Using complex numbers	MEX12-4	E3-E4
b i	2	Further integration	MEX12-5	E3-E4
ii	3	Further integration	MEX12-5	E3-E4
c i	4	Applications of calculus to mechanics	MEX12-6	E3-E4
ii	2	Applications of calculus to mechanics	MEX12-6	E3-E4