NSW GOVERNMENT

NSW Education Standards Authority

2021 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number on the Question 12 Writing Booklet attached

Total marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–12)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Given that $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, what is $\overrightarrow{PQ}$?

A. $\begin{pmatrix} 1 \\ -6 \end{pmatrix}$

B. $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$

C. $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

D. $\begin{pmatrix} -5 \\ -4 \end{pmatrix}$

2 Which of the following integrals is equivalent to $\int \sin^2 3x \, dx$?

A. $\int \frac{1+\cos 6x}{2} \, dx$

B. $\int \frac{1-\cos 6x}{2} \, dx$

C. $\int \frac{1+\sin 6x}{2} \, dx$

D. $\int \frac{1-\sin 6x}{2} \, dx$

3 What is the remainder when $P(x) = -x^3 - 2x^2 - 3x + 8$ is divided by $x + 2$?

A. -14

B. -2

C. 2

D. 14

4 Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$.

Which of the following equations best represents this relationship between x and y?

A. $y^2 = x^2 + c$

B. $y^2 = \dfrac{x^2}{2} + c$

C. $y = x \ln|y| + c$

D. $y = \dfrac{x^2}{2} \ln|y| + c$

5 For the two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ it is known that

$$\overrightarrow{OA} \cdot \overrightarrow{OB} < 0.$$

Which of the following statements MUST be true?

A. Either, $\overrightarrow{OA}$ is negative and $\overrightarrow{OB}$ is positive, or, $\overrightarrow{OA}$ is positive and $\overrightarrow{OB}$ is negative.

B. The angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is obtuse.

C. The product $\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|$ is negative.

D. The points O, A and B are collinear.

6 The random variable X represents the number of successes in 10 independent Bernoulli trials. The probability of success is $p = 0.9$ in each trial.

Let $r = P(X \geq 1)$.

Which of the following describes the value of r?

A. $r > 0.9$

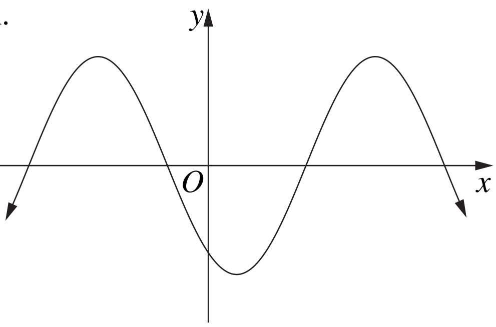
B. $r = 0.9$

C. $0.1 < r < 0.9$

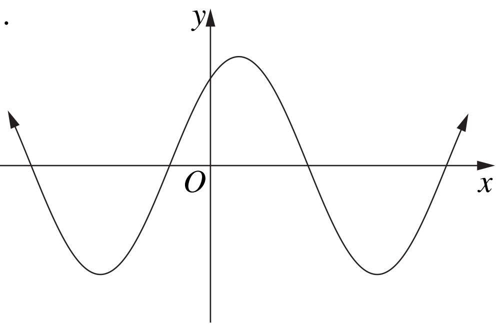
D. $r \leq 0.1$

7 Which curve best represents the graph of the function $f(x) = -a\sin x + b\cos x$ given that the constants a and b are both positive?

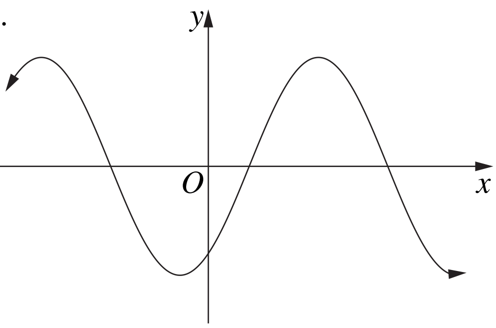
A.



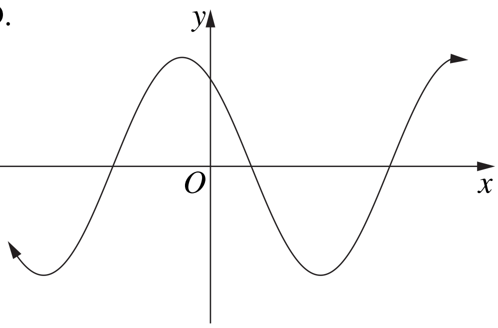
B.



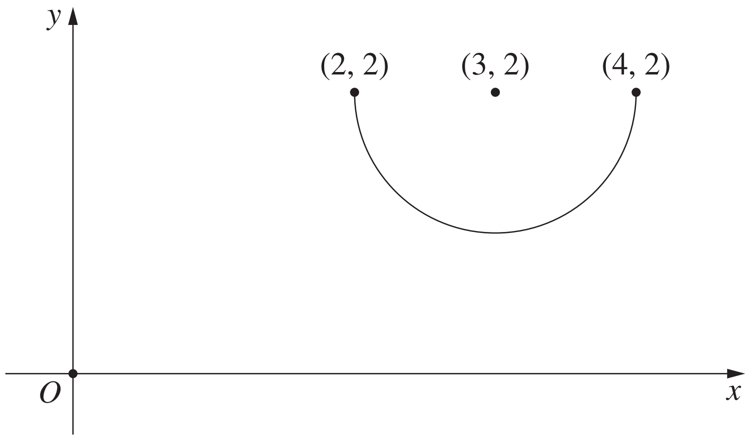
C.



D.



8 The diagram shows a semicircle.



Which pair of parametric equations represents the semicircle shown?

A. $\begin{cases} x = 3 + \sin t \\ y = 2 + \cos t \end{cases}$ for $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$

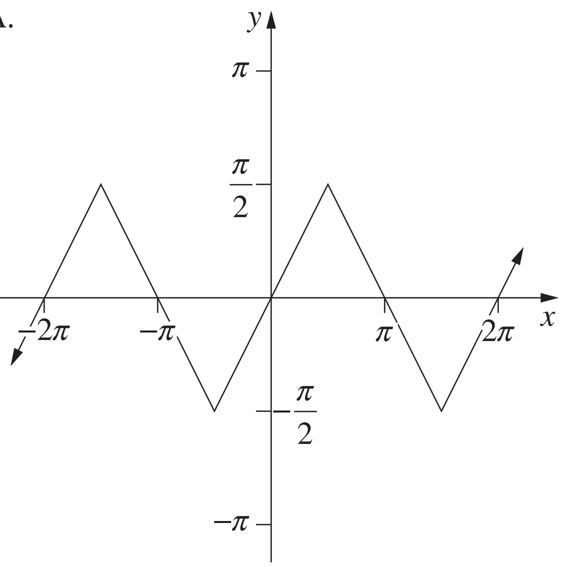
B. $\begin{cases} x = 3 + \cos t \\ y = 2 + \sin t \end{cases}$ for $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$

C. $\begin{cases} x = 3 - \sin t \\ y = 2 - \cos t \end{cases}$ for $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$

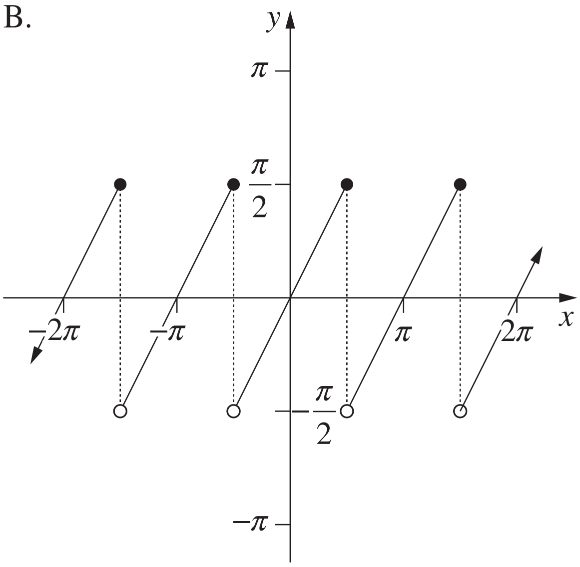
D. $\begin{cases} x = 3 - \cos t \\ y = 2 - \sin t \end{cases}$ for $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$

9 Which graph represents the function $y = \sin^{-1}(\sin x)$?

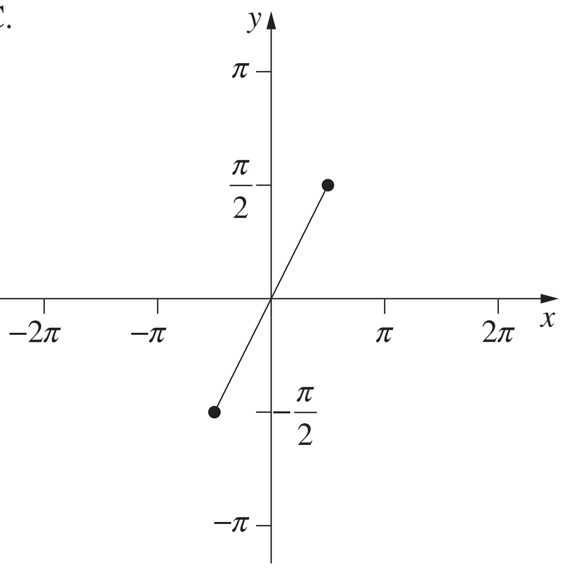
A.



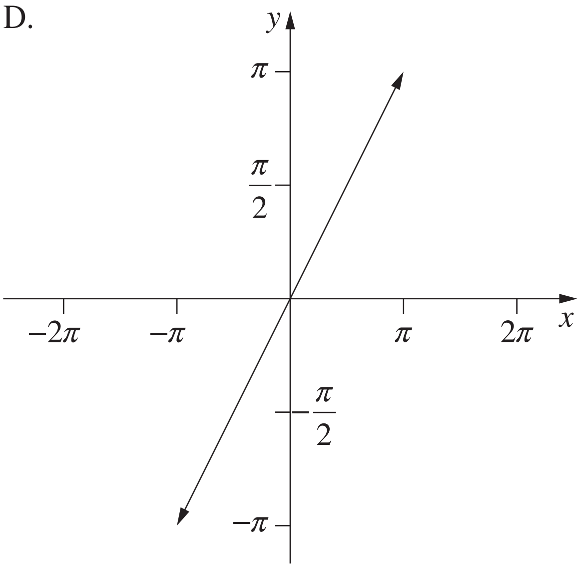
B.



C.



D.



10 The members of a club voted for a new president. There were 15 candidates for the position of president and 3543 members voted. Each member voted for one candidate only.

One candidate received more votes than anyone else and so became the new president.

What is the smallest number of votes the new president could have received?

A. 236

B. 237

C. 238

D. 239

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) Find $\left(\underset{\sim}{i}+6\underset{\sim}{j}\right)+\left(2\underset{\sim}{i}-7\underset{\sim}{j}\right)$. **1**

(b) Expand and simplify $(2a-b)^4$. **2**

(c) Use the substitution $u=x+1$ to find $\int x\sqrt{x+1}\,dx$. **3**

(d) A committee containing 5 men and 3 women is to be formed from a group of 10 men and 8 women. **1**

In how many different ways can the committee be formed?

(e) A spherical bubble is moving up through a liquid. As it rises, the bubble gets bigger and its radius increases at the rate of 0.2 mm/s. **2**

At what rate is the volume of the bubble increasing when its radius reaches 0.6 mm? Express your answer in mm^3/s rounded to one decimal place.

(f) Evaluate $\displaystyle\int_0^{\sqrt{3}}\frac{1}{\sqrt{4-x^2}}\,dx$. **2**

(g) By factorising, or otherwise, solve $2\sin^3 x+2\sin^2 x-\sin x-1=0$ for $0\le x\le 2\pi$. **3**

(h) The roots of $x^4-3x+6=0$ are α, β, γ and δ. **2**

What is the value of $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}+\dfrac{1}{\delta}$?

Question 12 (14 marks) Use the Question 12 Writing Booklet

(a) The direction field for a differential equation is given on page 1 of the Question 12 Writing Booklet. **1**

The graph of a particular solution to the differential equation passes through the point P.

On the diagram provided in the writing booklet, sketch the graph of this particular solution.

(b) A bottle of water, with temperature 5°C, is placed on a table in a room. The temperature of the room remains constant at 25°C. After t minutes, the temperature of the water, in degrees Celsius, is T.

The temperature of the water can be modelled using the differential equation

$$\frac{dT}{dt} = k(T - 25) \qquad \text{(Do NOT prove this.)}$$

where k is the growth constant.

(i) After 8 minutes, the temperature of the water is 10°C. **3**

By solving the differential equation, find the value of t when the temperature of the water reaches 20°C. Give your answer to the nearest minute.

(ii) Sketch the graph of T as a function of t. **1**

(c) Use mathematical induction to prove that **3**

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

for all integers $n \geq 1$.

(d) A function is defined by $f(x) = 4 - \left(1 - \frac{x}{2}\right)^2$ for x in the domain $(-\infty, 2]$.

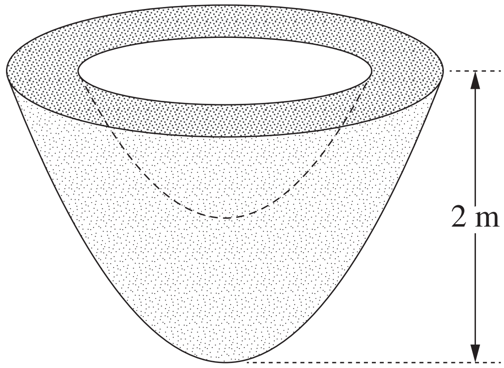
(i) Sketch the graph of $y = f(x)$ showing the x- and y-intercepts. **2**

(ii) Find the equation of the inverse function, $f^{-1}(x)$, and state its domain. **3**

(iii) Sketch the graph of $y = f^{-1}(x)$. **1**

Question 13 (14 marks) Use the Question 13 Writing Booklet

(a) A 2-metre-high sculpture is to be made out of concrete. The sculpture is formed by rotating the region between $y = x^2$, $y = x^2 + 1$ and $y = 2$ around the y-axis. **3**



Find the volume of concrete needed to make the sculpture.

(b) When an object is projected from a point h metres above the origin with initial speed V m/s at an angle of $\theta°$ to the horizontal, its displacement vector, t seconds after projection, is **4**

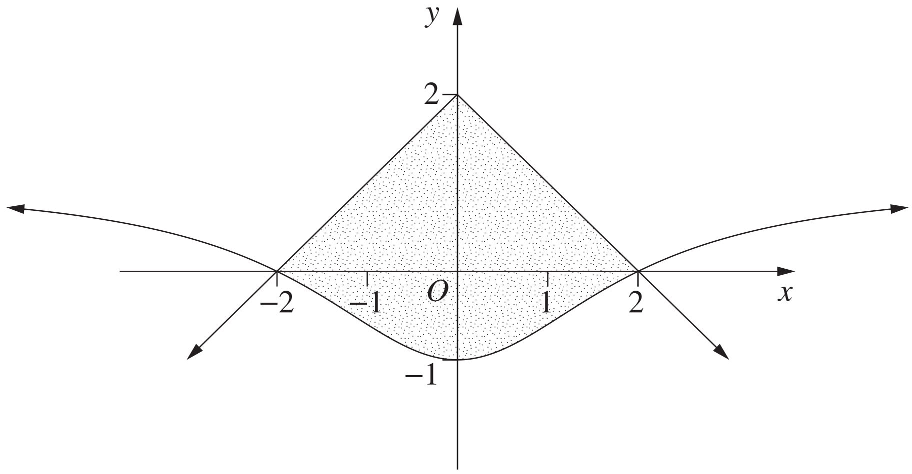
$$\underset{\sim}{r}(t) = \left(Vt\cos\theta\right)\underset{\sim}{i} + \left(-5t^2 + Vt\sin\theta + h\right)\underset{\sim}{j}.$$
(Do NOT prove this.)

A person, standing in an empty room which is 3 m high, throws a ball at the far wall of the room. The ball leaves their hand 1 m above the floor and 10 m from the far wall. The initial velocity of the ball is 12 m/s at an angle of 30° to the horizontal.

Show that the ball will NOT hit the ceiling of the room but that it will hit the far wall without hitting the floor.

Question 13 continues on page 10

(c) The region enclosed by $y = 2 - |x|$ and $y = 1 - \dfrac{8}{4 + x^2}$ is shaded in the diagram. **3**



Find the exact value of the area of the shaded region.

(d) (i) The numbers A, B and C are related by the equations $A = B - d$ and $C = B + d$, where d is a constant. **2**

Show that $\dfrac{\sin A + \sin C}{\cos A + \cos C} = \tan B$.

(ii) Hence, or otherwise, solve $\dfrac{\sin\frac{5\theta}{7} + \sin\frac{6\theta}{7}}{\cos\frac{5\theta}{7} + \cos\frac{6\theta}{7}} = \sqrt{3}$, for $0 \le \theta \le 2\pi$. **2**

End of Question 13

Question 14 (16 marks) Use the Question 14 Writing Booklet

(a) A plane needs to travel to a destination that is on a bearing of 063°. The engine is set to fly at a constant 175 km/h. However, there is a wind from the south with a constant speed of 42 km/h. **3**

On what constant bearing, to the nearest degree, should the direction of the plane be set in order to reach the destination?

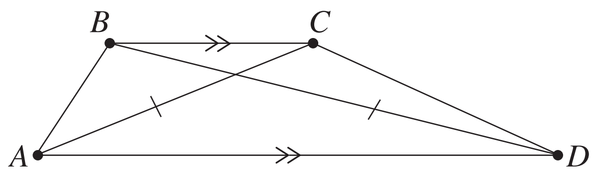
(b) In a certain country, the population of deer was estimated in 1980 to be 150 000. The population growth is given by the logistic equation $\dfrac{dP}{dt} = 0.1P\left(\dfrac{C-P}{C}\right)$ where t is the number of years after 1980 and C is the carrying capacity. **4**

In the year 2000, the population of deer was estimated to be 600 000.

Use the fact that $\dfrac{C}{P(C-P)} = \dfrac{1}{P} + \dfrac{1}{C-P}$ to show that the carrying capacity is approximately 1 130 000.

(c) (i) For vector $\underset{\sim}{v}$, show that $\underset{\sim}{v} \cdot \underset{\sim}{v} = |\underset{\sim}{v}|^2$. **1**

(ii) In the trapezium $ABCD$, BC is parallel to AD and $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BD}\right|$. **3**



NOT TO SCALE

Let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$ and $\overrightarrow{AD} = k\,\overrightarrow{BC}$, where $k > 0$.

Using part (i), or otherwise, show $2\underset{\sim}{a} \cdot \underset{\sim}{b} + (1-k)\,|\underset{\sim}{b}|^2 = 0$.

Question 14 continues on page 12

(d) At a certain factory, the proportion of faulty items produced by a machine is $p = \frac{3}{500}$, which is considered to be acceptable. To confirm that the machine is working to this standard, a sample of size n is taken and the sample proportion $\hat{p}$ is calculated. **3**

It is assumed that $\hat{p}$ is approximately normally distributed with $\mu = p$ and $\sigma^2 = \frac{p(1-p)}{n}$.

Production by this machine will be shut down if $\hat{p} \geq \frac{4}{500}$.

The sample size is to be chosen so that the chance of shutting down the machine unnecessarily is less than 2.5%.

Find the approximate sample size required, giving your answer to the nearest thousand.

(e) The polynomial $g(x) = x^3 + 4x - 2$ passes through the point $(1, 3)$. **2**

Find the gradient of the tangent to $f(x) = xg^{-1}(x)$ at the point where $x = 3$.

End of paper

Centre Number

Student Number

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2021 **HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 1

Writing Booklet

Question 12

Instructions

- Use this Writing Booklet to answer Question 12.
- Write the number of this booklet and the total number of booklets that you have used for this question (eg: 1 of 3).
- Write your Centre Number and Student Number at the top of this page.
- Write using black pen.
- You may ask for an extra writing booklet if you need more space.
- If you have not attempted the question(s), you must still hand in the writing booklet, with 'NOT ATTEMPTED' written clearly on the front cover.
- You may NOT take any writing booklets, used or unused, from the examination room.

12

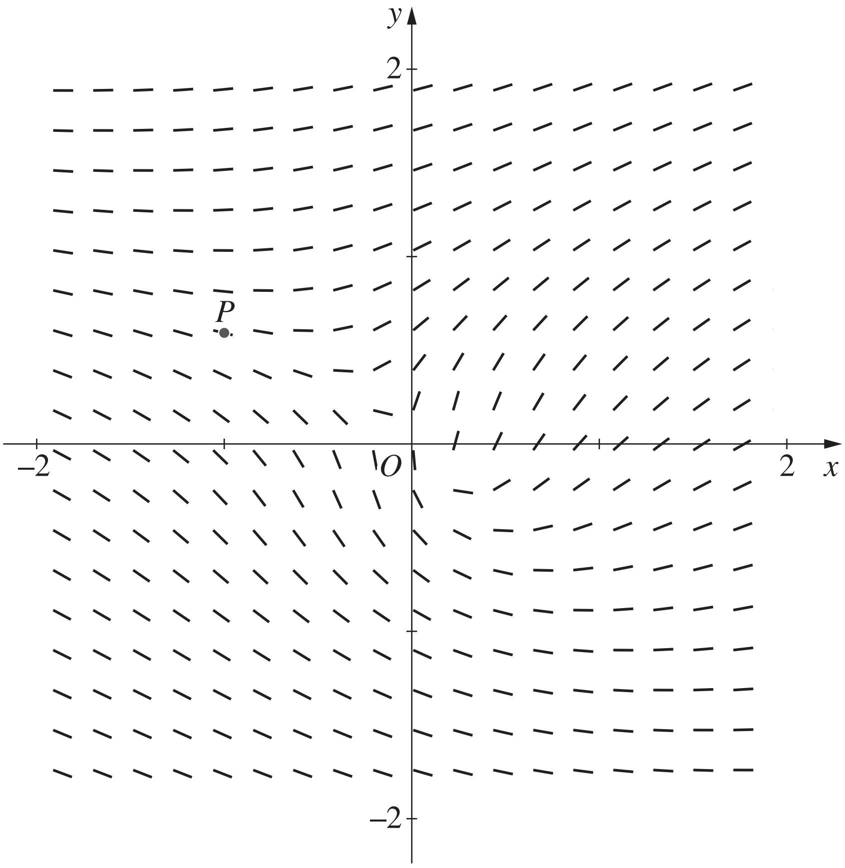
of

this booklet

number of booklets for this question

Start here for Question Number: **12**

(a) Sketch the graph of the particular solution that passes through the point P.



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Additional writing space on back page.

Tick this box if you have continued this answer in another writing booklet.

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2021 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1$$

$$S = \frac{a}{1 - r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

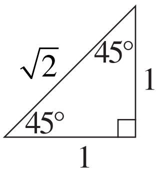
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

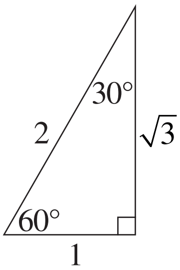
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

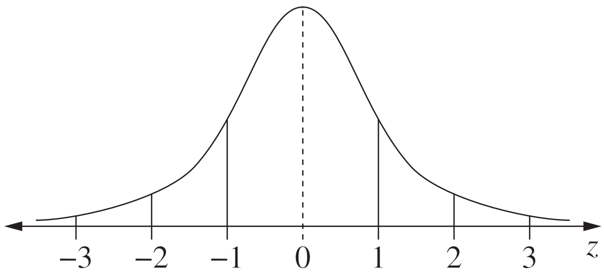
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1-p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1-p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2\,,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

2021 HSC Mathematics Extension 1 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	B
3	D
4	A
5	B
6	A
7	D
8	C
9	A
10	C

Section II

Question 11 (a)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\left(\underset{\sim}{i}+6\underset{\sim}{j}\right)+\left(2\underset{\sim}{i}-7\underset{\sim}{j}\right)=3\underset{\sim}{i}-\underset{\sim}{j}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Expands using the binomial expansion, or equivalent merit	1

Sample answer:

$$\begin{aligned}(2a-b)^4 &= (2a)^4-4(2a)^3b+6(2a)^2b^2-4(2a)b^3+b^4\\ &= 16a^4-32a^3b+24a^2b^2-8ab^3+b^4\end{aligned}$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	3
• Obtains correct primitive in terms of u	2
• Obtains the integrand in terms of u, or equivalent merit	1

Sample answer:

$$\int x\sqrt{x+1}\,dx \qquad\qquad u = x+1 \qquad \therefore x = u-1$$
$$du = dx$$

$$= \int (u-1)\sqrt{u}\,du$$

$$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c$$

$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$$

Question 11 (d)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$^{10}C_5 \times {}^{8}C_3 = 14\ 112$

Question 11 (e)

Criteria	Marks
• Provides correct solution	2
• Uses the chain rule $\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$ OR • Obtains $\frac{dv}{dr}$ from the volume of a sphere	1

Sample answer:

$$\frac{dr}{dr} = 0.2 \text{ mm/s} \quad \text{and} \quad \frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt}$$

$$v = \frac{4}{3}\pi r^3 \qquad \therefore \frac{dv}{dr} = 4\pi^2 \text{ and } \frac{dv}{dt} = 4\pi^2 . \frac{dr}{dt}$$

$$\frac{dv}{dt} = 4\pi \times (0.6)^2 \times (0.2) \simeq 0.9 \text{ mm}^3\text{/s} \qquad \text{(1 decimal place)}$$

Question 11 (f)

Criteria	Marks
• Provides correct solution	2
• Integrates to obtain an arcsin function, or equivalent merit	1

Sample answer:

$$\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^{\sqrt{3}}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1} 0$$

$$= \frac{\pi}{3} - 0$$

$$= \frac{\pi}{3}$$

Question 11 (g)

Criteria	Marks
• Provides correct solution	3
• Finds all three possible values of $\sin x$ OR • Finds at least two possible values of x, or equivalent merit	2
• Notes that $\sin x = -1$ is a solution of the polynomial, or equivalent merit OR • Takes a factor of $\sin^2 x$ OR $2\sin^2 x$ out of first two terms	1

Sample answer:

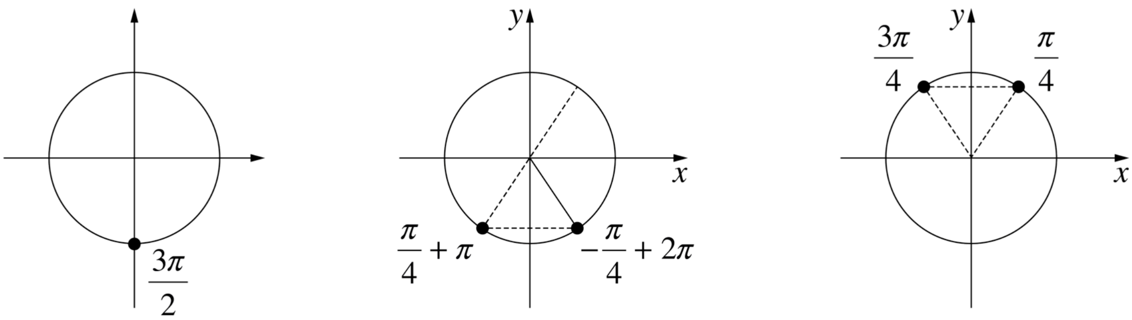
$$2\sin^3 x + 2\sin^2 x - \sin x - 1 = 0$$

$$2\sin^2 x(\sin x + 1) - (\sin x + 1) = 0$$

$$(\sin x + 1)\left(2\sin^2 x - 1\right) = 0$$

$$\sin x = -1 \qquad \text{or} \qquad \sin^2 x = \frac{1}{2}$$

$$\sin x = -1 \qquad \text{or} \qquad \sin x = -\frac{1}{\sqrt{2}} \qquad \text{or} \qquad \sin x = \frac{1}{\sqrt{2}}$$



The solutions in $[0, 2\pi]$ are: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$.

Question 11 (h)

Criteria	Marks
• Provides correct solution	2
• Writes $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}$ with a common denominator OR • Applies one formula relating roots and coefficients	1

Sample answer:

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma}{\alpha\beta\gamma\delta}$$

$x^4-3x+6=0 \qquad \therefore\ a=1,\ b=0,\ c=0,\ d=-3,\ e=6$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\delta\gamma=\frac{-d}{a} \qquad \text{(coeff of } x\text{)}$$

$$=3$$

and $\alpha\beta\gamma\delta=\frac{e}{a}$ (constant term)

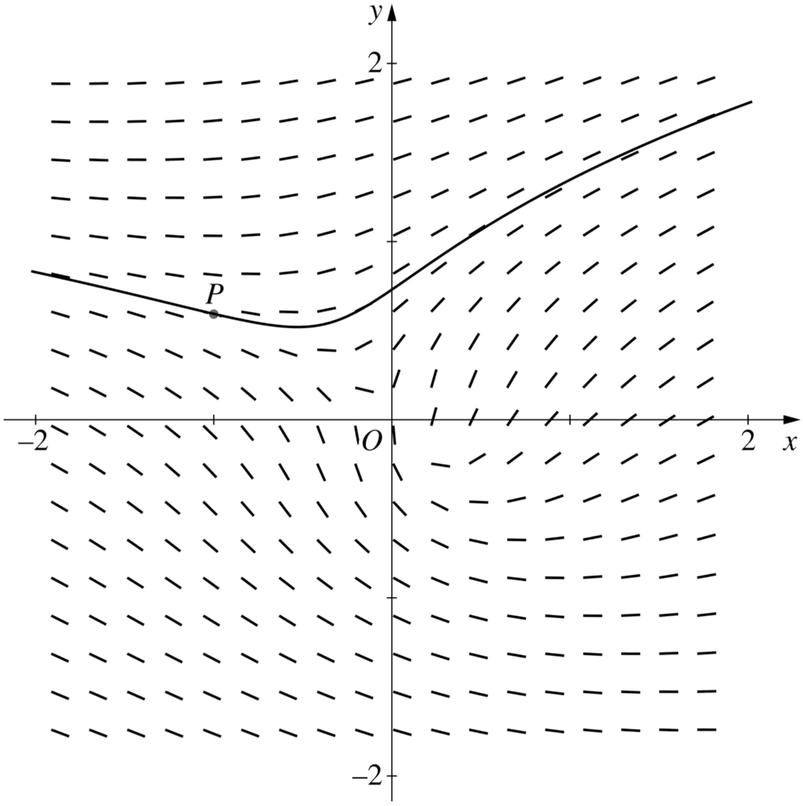
$$=6$$

Therefore $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{3}{6}=\frac{1}{2}$.

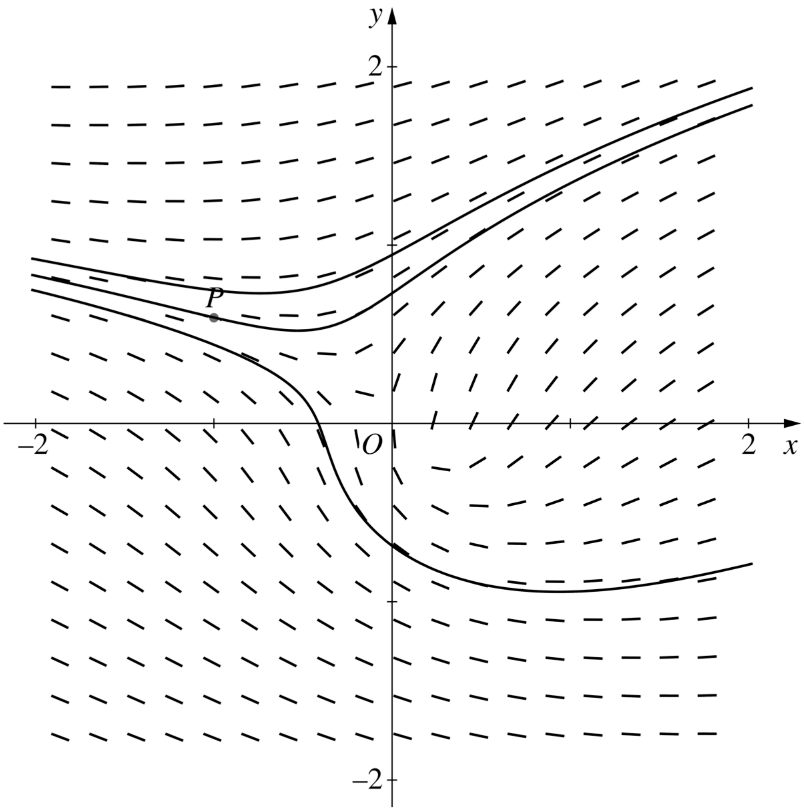
Question 12 (a)

Criteria	Marks
• Provides correct sketch	1

Sample answer:



Answers could include:



Note: An acceptable solution curve does not cross any tangent line in the direction field.

Question 12 (b) (i)

Criteria	Marks
• Provides correct solution	3
• Finds the value of k, or equivalent merit	2
• Obtains a solution to the differential equation, that is, $T = 25 + Ae^{kt}$, or equivalent merit OR • Finds the value of A	1

Sample answer:

$$\int \frac{dT}{T-25} = \int k\ dt$$

$$\therefore kt = \ln(T-25) + c$$

$$\therefore T - 25 = Ae^{kt}$$

when $t = 0,\ T = 5$

$$\therefore -20 = A$$

$$\therefore T = 25 - 20e^{kt}$$

when $t = 8,\ T = 10$

$$\therefore 10 = 25 - 20e^{8k}$$

$$-15 = -20e^{8k}$$

$$e^{8k} = \frac{3}{4}$$

$$8k = \ln\left(\frac{3}{4}\right)$$

$$k = \frac{1}{8}\ln\left(\frac{3}{4}\right)$$

when $T = 20$

$$20 = 25 - 20e^{\frac{1}{8}\ln\left(\frac{3}{4}\right)t}$$

$$-5 = -20e^{\frac{1}{8}\ln\left(\frac{3}{4}\right)t}$$

$$\frac{1}{4} = e^{\frac{1}{8}\ln\left(\frac{3}{4}\right)t}$$

$$\frac{1}{8}\ln\left(\frac{3}{4}\right)t = \ln\left(\frac{1}{4}\right)$$

$$\therefore t = \frac{8\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{3}{4}\right)}$$

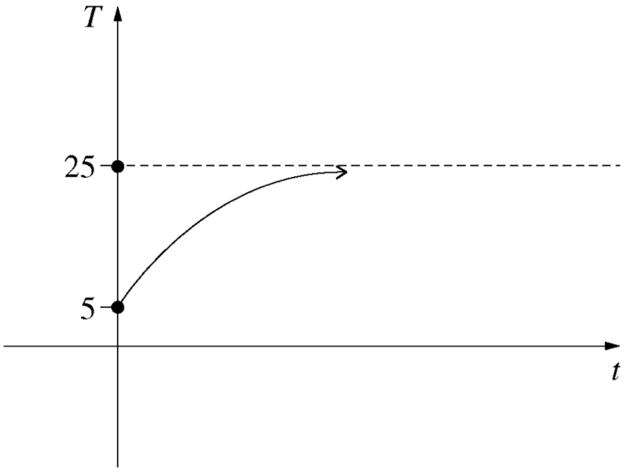
$$= 38.55\ldots$$

$$\simeq 39 \text{ minutes.}$$

Question 12 (b) (ii)

Criteria	Marks
• Provides correct sketch	1

Sample answer:



Question 12 (c)

Criteria	Marks
• Provides correct solution	3
• Proves that $p(k)\ true \Rightarrow p(k+1)$ is true, or equivalent merit	2
• Verifies the initial case, or equivalent merit	1

Sample answer:

When $n = 1$

$$\text{LHS} = \frac{1}{1(1+1)(1+2)} = \frac{1}{1\times 2\times 3} = \frac{1}{6}$$

$$\text{RHS} = \frac{1}{4} - \frac{1}{2(1+1)(1+2)} = \frac{1}{4} - \frac{1}{2\times 2\times 3} = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$$

$\therefore$ statement is true when $n = 1$.

Assume statement is true when $n = k$, some integer $k \geq 1$, that is

$$\frac{1}{1\times 2\times 3} + \cdots + \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$$

Consider $n = k + 1$:

$$\frac{1}{1\times 2\times 3} + \cdots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+1+1)(k+1+2)}$$

$$= \frac{1}{4} - \frac{1}{2\big((k+1)+1\big)\big((k+1)+2\big)} = \frac{1}{4} - \frac{1}{2(k+2)(k+3)}$$

$$\text{LHS} = \frac{1}{1\times 2\times 3} + \cdots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+1+1)(k+1+2)}$$

$$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \left\{\frac{1}{2(k+1)(k+2)} - \frac{1}{(k+1)(k+2)(k+3)}\right\}$$

$$= \frac{1}{4} - \left\{\frac{k+3-2}{2(k+1)(k+2)(k+3)}\right\}$$

$$= \frac{1}{4} - \frac{1}{2\big((k+1)+1\big)\big((k+1)+2\big)}$$

$$= \frac{1}{4} - \frac{1}{2(k+2)(k+3)}$$

$$= \text{RHS}$$

$\therefore$ statement is true for all integers $n \geq 1$, by mathematical induction.

Question 12 (d) (i)

Criteria	Marks
• Provides correct sketch	2
• Sketches the left half of a concave-down parabola OR • Sketches a full concave-down parabola and provides one intercept or the vertex	1

Sample answer:

$y = 4 - \left(1 - \frac{x}{2}\right)^2$ is a parabola.

Vertex when $\left(1 - \frac{x}{2}\right)^2 = 0 \qquad \therefore\ x = 2$

$$\begin{aligned} y &= 4 - 0 \\ &= 4 \end{aligned}$$

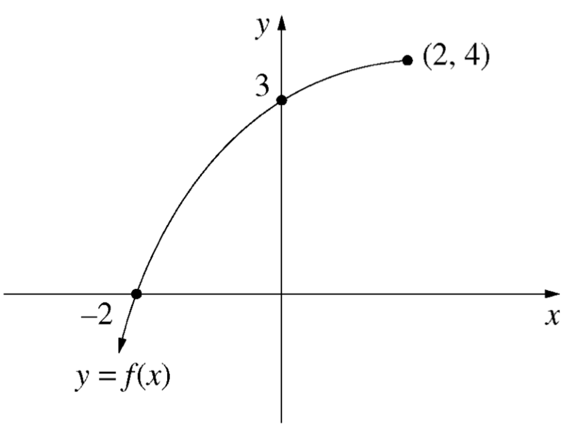
$\therefore$ vertex $(2, 4)$

y-intercept:
$$\begin{aligned} y &= 4 - \left(1 - \frac{0}{2}\right)^2 \\ &= 3 \end{aligned}$$

x-intercept:
$$\begin{aligned} 4 - \left(1 - \frac{x}{2}\right)^2 &= 0 \\ \left(1 - \frac{x}{2}\right)^2 &= 4 \\ 1 - \frac{x}{2} &= \pm 2 \end{aligned}$$

$$\therefore \quad \frac{x}{2} = 1 + 2 \quad \text{or} \quad \frac{x}{2} = 1 - 2$$
$$= 3 \qquad\qquad = -1$$
$$x = 6 \qquad\qquad x = -2$$

Coefficient of x^2 is negative, so concave down.
Domain $(-\infty, 2]$, so left half only.



Question 12 (d) (ii)

Criteria	Marks
• Provides correct solution	3
• Finds the expression for $f^{-1}(x)$ OR • States the domain and attempts to either – make x the subject, or – find the inverse by switching variables	2
• States the domain $(\infty, 4]$ OR • Attempts to make x the subject OR • Attempts to find the inverse by switching variables	1

Sample answer:

For $f(x)$, $\quad y = 4 - \left(1 - \dfrac{x}{2}\right)^2$, x in the domain $(-\infty, 2]$

For inverse, $\quad x = 4 - \left(1 - \dfrac{y}{2}\right)^2$, y in the domain $(-\infty, 2]$

$$\left(1 - \frac{y}{2}\right)^2 = 4 - x$$

$$\left(\frac{y}{2} - 1\right)^2 = 4 - x$$

$$\frac{y}{2} - 1 = \pm\sqrt{4 - x}$$

$$y = 2 \pm 2\sqrt{4 - x}$$

Range of $f(x)$ is $(-\infty, 4]$, so domain of $f^{-1}(x)$ is $(-\infty, 4]$.

$\therefore\ f^{-1}(x) = 2 - 2\sqrt{4 - x}$ for x in the domain $(-\infty, 4]$.

Alternative solution for Q12 (d) (ii)

$$y = 4 - \left(1 - \frac{x}{2}\right)^2 \quad \text{for} \quad x \le 2$$

For $f^{-1}(x)$, make x the subject:

$$\left(1 - \frac{x}{2}\right)^2 = 4 - y$$

$$x \le 2 \quad \therefore 1 - \frac{x}{2} \ge 0$$

$$\therefore\ 1 - \frac{x}{2} = \sqrt{4 - y}$$

$$\frac{x}{2} = 1 - \sqrt{4 - y}$$

$$x = 2 - 2\sqrt{4 - y}$$

$$\therefore f^{-1}(x) = 2 - 2\sqrt{4 - x}$$

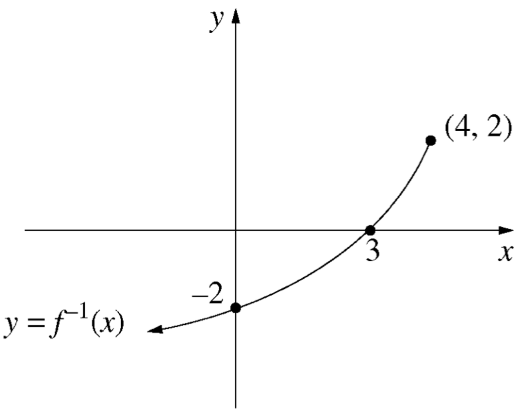
From the graph, range of $f(x)$ is $(-\infty, 4]$

$\therefore$ domain of $f^{-1}(x)$ is $(-\infty, 4]$.

Question 12 (d) (iii)

Criteria	Marks
• Provides correct sketch	1

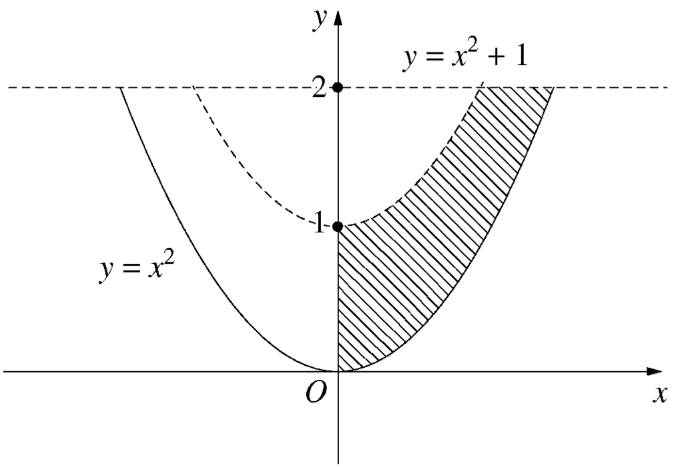
Sample answer:



Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Evaluates one volume OR • Obtains a correct expression for the volume as a difference of two integrals that are only in terms of y	2
• Obtains an integral for one relevant volume OR • Writes the required volume as the difference of two relevant volumes OR • Finds the limits of integration for both volumes	1

Sample answer:



The volume of the garden sculpture is the difference between the outer volume and the inner one.

$$\text{Outer volume} = \pi\int_0^2 x^2\,dy$$

$$= \pi\int_0^2 y\,dy \quad \text{since } y = x^2$$

$$\text{Inner volume} = \pi\int_1^2 x^2\,dy$$

$$= \pi\int_1^2 y - 1\,dy \quad \text{since } y = x^2 + 1$$

$$\therefore V = \int_0^2 \pi y\,dy - \int_1^2 \pi(y-1)\,dy$$

$$V = \pi\left[\frac{y^2}{2}\right]_0^2 - \pi\left[\frac{y^2}{2} - y\right]_1^2$$

$$= \pi\left(\frac{4}{2} - 0\right) - \pi\left(\left[\frac{4}{2} - 2\right] - \left[\frac{1}{2} - 1\right]\right)$$

$$= \frac{3\pi}{2}$$

The volume of the sculpture is $\dfrac{3\pi}{2}$ m^3.

Question 13 (b)

Criteria	Marks
• Provides correct solution	4
• Finds the maximum height and the time of flight, or equivalent merit.	3
• Finds the maximum height, or equivalent merit	2
• Finds an expression for the vertical velocity, or equivalent merit	1

Sample answer:

$\underset{\sim}{v}(t) = V\cos\theta\underset{\sim}{i} + (-10t + V\sin\theta)\underset{\sim}{j}$

The height is greatest when $\dot{y} = 0$ ie when $-10t + V\sin\theta = 0$

$$t = \frac{V\sin\theta}{10} = \frac{12\sin 30°}{10} = \frac{6}{10} = 0.6 \text{ s}$$

$$\therefore y = -5\times(0.6)^2 + 12\times(0.6)\times\sin 30° + 1$$
$$= 2.8 \text{ m}$$

The room is 3 metres high and maximum height reached by the object is 2.8 m so the object will not hit the ceiling.

To know if the object hits the far wall or not, it is enough to determine if the unrestricted horizontal range would be more or less than 10 metres.

$y = 0$ if $-5t^2 + Vt\sin\theta + h = 0$

$$-5t^2 + 12\times\frac{1}{2}t + 1 = 0$$
$$-5t^2 + 6t + 1 = 0$$

$\triangle = 6^2 - 4(-5) = 56$

$$\therefore t = \frac{-6+\sqrt{56}}{-10} < 0 \quad \text{or} \quad \frac{-6-\sqrt{56}}{-10} > 0$$

We need $t > 0$, so the object, in the absence of a far wall, would hit the floor after $t = \dfrac{6+\sqrt{56}}{10} \simeq 1.348$ s.

Substitute in $x(t)$:

$$x(t) = Vt\cos\theta = 12\times 1.348\cos 30°$$
$$\approx 14 \text{ m}$$

Since the far wall is only 10 m away, the object will hit the wall.

Alternative solution for second part

When $x = 10$ $\quad 12t\cos 30° = 10$

$$6\sqrt{3}t = 10$$

$$t = \frac{10}{6\sqrt{3}} = 0.9622 \text{ seconds}$$

When $t = 0.9622$

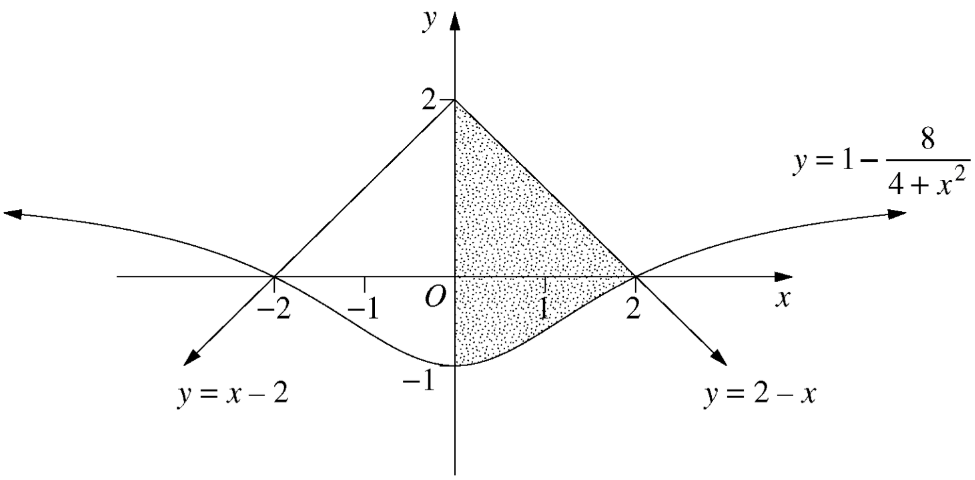
$$y = -5(0.9622\ldots)^2 + 12(0.9622\ldots)\sin 30° + 1$$
$$= 2.144 \text{ m}$$

$\therefore$ Ball is still above the floor when $x = 10$ m and so will hit the far wall without hitting the floor.

Question 13 (c)

Criteria	Marks
• Provides correct solution	3
• Obtains the (signed) area OR the area of either side of the y-axis between the x-axis and the curve $y = 1 - \dfrac{8}{4+x^2}$, or equivalent merit	2
• Finds an integral expression for the area, or equivalent merit OR • Recognises that the total area is twice the area on one side of the y-axis OR • Finds the area of a relevant triangle	1

Sample answer:



By symmetry, area required is double the shaded region above

$$\frac{\text{Area}}{2} = \int_0^2 (2-x) - \left(1 - \frac{8}{4+x^2}\right) dx$$

$$= \int_0^2 1 - x + \frac{8}{4+x^2}\, dx$$

$$= \left[x - \frac{x^2}{2} + \frac{8}{2}\tan^{-1}\left(\frac{x}{2}\right) \right]_0^2$$

$$= \left(2 - 2 + 4\tan^{-1}1\right) - 0$$

$$= \pi$$

So Area $= 2\pi$

Question 13 (d) (i)

Criteria	Marks
• Provides correct solution	2
• Uses $A = B - d$ and $C = B + d$ in the left hand side and attempts to use a suitable trigonometric identity, or equivalent merit	1

Sample answer:

Given $A = B - d$ and $C = B + d$

$$\therefore \frac{\sin A + \sin C}{\cos A + \cos C} = \frac{\sin(B-d) + \sin(B+d)}{\cos(B-d) + \cos(B+d)}$$

$$= \frac{2\sin B \cos d}{2\cos B \cos d}$$

$$= \tan B$$

Question 13 (d) (ii)

Criteria	Marks
• Provides correct solution	2
• Identifies the value of B in order to use as in part (i), or equivalent merit	1

Sample answer:

$$A = \frac{5\theta}{7}, \quad C = \frac{6\theta}{7} \quad \therefore B = \frac{\frac{5\theta}{7} + \frac{6\theta}{7}}{2} = \frac{11\theta}{14}$$

$$\therefore \text{ LHS} = \tan\frac{11\theta}{14}$$

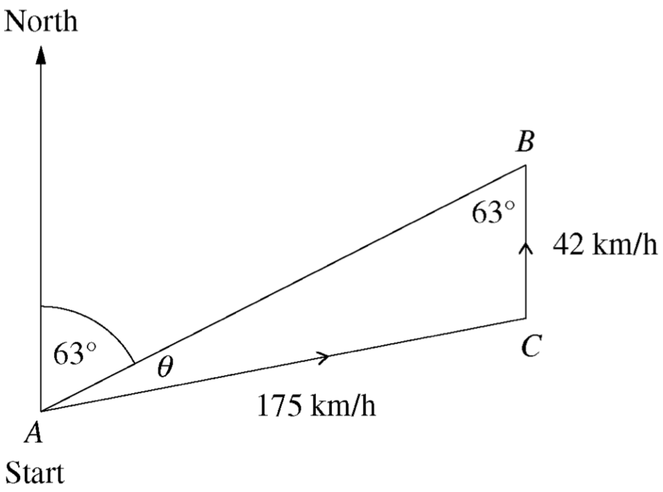
$$\tan\frac{11\theta}{14} = \sqrt{3}$$

$$\theta = \frac{14\pi}{33} \quad \text{or} \quad \frac{56\pi}{33}$$

Question 14 (a)

Criteria	Marks
• Provides correct solution	3
• Attempts to use the sine rule in the correct triangle, or equivalent merit	2
• Sketches a suitable diagram, or equivalent merit	1

Sample answer:



$$\angle ABC = 63°$$

Let $\angle BAC = \theta$

$$\therefore \quad \frac{\sin\theta}{42} = \frac{\sin 63°}{175}$$

$$\sin\theta = 0.2138\ldots$$

$$\theta = 12.34\ldots$$

$$\approx 12°$$

$$\therefore \text{ required bearing } = 63 + 12$$

$$= 075°$$

Question 14 (b)

Criteria	Marks
• Provides correct solution	4
• Uses the given information to obtain two equations in A and C, or equivalent merit	3
• Integrates both sides correctly, or equivalent merit	2
• Attempts to separate the variables in the differential equation, or equivalent merit	1

Sample answer:

$$\frac{dP}{dt} = 0.1P\left(\frac{C-P}{C}\right)$$

$$\int \frac{C}{P(C-P)}\,dP = \int 0.1\,dt$$

$$\int \left(\frac{1}{P} + \frac{1}{C-P}\right)dP = \int 0.1\,dt$$

$\ln|P| - \ln|C-P| = 0.1t + k$ where k is a constant

$$\ln\left|\frac{P}{C-P}\right| = 0.1t + k$$

$\dfrac{P}{C-P} = Ae^{0.1t}$ where $A = e^k$ is a constant

When $t = 0$, $P = 150\,000$ so $\dfrac{150\,000}{C-150\,000} = A$ ①

When $t = 20$, $P = 600\,000$ so $\dfrac{600\,000}{C-600\,000} = Ae^2$ ②

Substituting ① into ②: $\dfrac{600\,000}{C-600\,000} = \dfrac{150\,000}{C-150\,000}e^2$

Taking the reciprocal of both sides

$$\frac{C-600\,000}{600\,000} = \frac{C-150\,000}{150\,000}e^{-2}$$

$$150\,000(C-600\,000) = 600\,000(C-150\,000)e^{-2}$$

$$C\left(150\,000 - 600\,000e^{-2}\right) = 150\,000 \times 600\,000\left(1-e^{-2}\right)$$

$$C = \frac{150\,000 \times 600\,000\left(1-e^{-2}\right)}{150\,000 - 600\,000e^{-2}} \simeq 1\,131\,121$$

$$\simeq 1\,131\,000$$

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Let $\underset{\sim}{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ be a vector

$$\underset{\sim}{v} \cdot \underset{\sim}{v} = x \times x + y \times y = x^2 + y^2 = |\underset{\sim}{v}|^2$$

Alternative:

$$\begin{aligned} \underset{\sim}{v} \cdot \underset{\sim}{v} &= |\underset{\sim}{v}|.|\underset{\sim}{v}|\cos\theta \\ &= |\underset{\sim}{v}|^2 \cos\theta \end{aligned}$$

where θ is the angle between and $\underset{\sim}{v}$ and $\underset{\sim}{v}$

$\therefore \quad \theta = 0 \quad$ and $\quad \cos\theta = 1$

$\therefore \quad \underset{\sim}{v} \cdot \underset{\sim}{v} = |\underset{\sim}{v}|^2$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Equates $\overrightarrow{AC} \cdot \overrightarrow{AC}$ with $\overrightarrow{BD} \cdot \overrightarrow{BD}$, in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, or equivalent merit	2
• Expresses AC or BD in terms of a and b, or equivalent merit	1

Sample answer:

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\underset{\sim}{a} + k\underset{\sim}{b}$

By part (i) $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BD}\right|$ implies

$$\overrightarrow{AC} \cdot \overrightarrow{AC} = \overrightarrow{BD} \cdot \overrightarrow{BD}$$

$$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = (-\underset{\sim}{a} + k\underset{\sim}{b}) \cdot (-\underset{\sim}{a} + k\underset{\sim}{b})$$

$$\cancel{\underset{\sim}{a} \cdot \underset{\sim}{a}} + 2\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b} = \cancel{\underset{\sim}{a} \cdot \underset{\sim}{a}} - 2k\underset{\sim}{a} \cdot \underset{\sim}{b} + k^2\underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$2(1+k)\underset{\sim}{a} \cdot \underset{\sim}{b} = \left(k^2 - 1\right)\underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$2(1+k)\underset{\sim}{a} \cdot \underset{\sim}{b} = (k+1)(k-1)\underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$1 + k \neq 0 \quad \text{since } k > 0$$

$$\therefore \quad 2\underset{\sim}{a} \cdot \underset{\sim}{b} = (k-1)\underset{\sim}{b} \cdot \underset{\sim}{b}$$

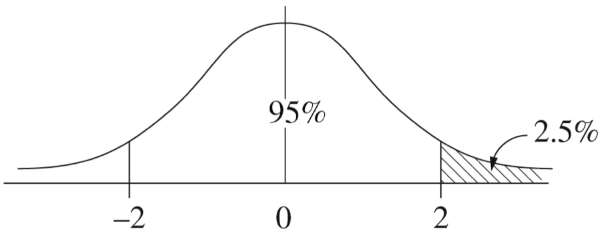
$$\therefore \quad 2\underset{\sim}{a} \cdot \underset{\sim}{b} = (k-1)\left|\underset{\sim}{b}\right|^2$$

$$2\underset{\sim}{a} \cdot \underset{\sim}{b} + (1-k)\left|\underset{\sim}{b}\right|^2 = 0$$

Question 14 (d)

Criteria	Marks
• Provides correct solution	3
• Recognises 2σ is important AND has σ in terms of n OR • Recognises 2σ is important AND has the value of σ OR • Obtains $2 = \dfrac{\sqrt{n}}{c}$ where c is a constant, or equivalent merit	2
• Finds σ in terms of n OR • Sketches a normal distribution and shades the correct region OR • Writes $P\left(\hat{p} \geq \dfrac{4}{500}\right) < 0.025$, or explains in words, or equivalent merit	1

Sample answer:



A tail with area 2.5% means $\hat{p} = \dfrac{4}{500}$ is two standard deviations above the mean

so $$\frac{4}{500} = \mu + 2\sigma$$

$$\frac{4}{500} = \frac{3}{500} + 2\sqrt{\frac{\frac{3}{500} \times \left(1 - \frac{3}{500}\right)}{n}}$$

$$\frac{1}{500} = 2\frac{\sqrt{\frac{3}{500}\left(\frac{497}{500}\right)}}{\sqrt{n}}$$

$$\sqrt{n} = 2 \times 500\sqrt{\frac{3}{500} \times \frac{497}{500}}$$

$$= 2 \times \cancel{500}\,\frac{\sqrt{3 \times 497}}{\cancel{500}}$$

$$n = 4 \times 3 \times 497 = 5964$$

The sample size to be chosen so that the chances of shutting down unnecessarily is less than 2.5% is $n = 5964$, ie approximately 6000.

Question 14 (e)

Criteria	Marks
• Provides correct solution	2
• Obtains an expression for the derivative of $g^{-1}(x)$ in terms of x, or equivalent merit	1

Sample answer:

$g(1) = 3 \quad \therefore \; g^{-1}(3) = 1$

Using product rule:

$$f'(x) = g^{-1}(x).1 + x.\frac{d}{dx}\left(g^{-1}(x)\right)$$

Now if $g^{-1}(x) = y$ then $x = g(y)$

$$\therefore \quad \frac{dx}{dy} = g'(y)$$

and $\dfrac{dy}{dx} = \dfrac{1}{g'(y)}$

$$\therefore \quad \frac{d}{dx}\left(g^{-1}(x)\right) = \frac{1}{g'\left(g^{-1}(x)\right)}$$

$$\therefore \quad f'(x) = g^{-1}(x) + \frac{x}{g'\left(g^{-1}(x)\right)}$$

$$\therefore \quad f'(3) = g^{-1}(3) + \frac{3}{g'\left(g^{-1}(3)\right)}$$

$$= 1 + \frac{3}{g'(1)}$$

Now, $g(x) = x^3 + 4x - 2$

$$\therefore \quad g'(x) = 3x^2 + 4$$

$$\therefore \quad f'(3) = 1 + \frac{3}{3(1)^2 + 4}$$

$\therefore$ gradient of tangent $= \dfrac{10}{7}$.

2021 HSC Mathematics Extension 1 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	ME V1 Introduction to vectors	ME 12–2
2	1	ME C2 Further calculus skills	ME 12–1
3	1	ME F2 Polynomials	ME 11–2
4	1	ME C3 Applications of calculus	ME 12–4
5	1	ME V1 Introduction to vectors	ME 12–2
6	1	ME S1 The binomial distribution	ME 12–5
7	1	ME T3 Trigonometric equations	ME 12–3
8	1	ME F1 Further work with functions	ME 11–2
9	1	ME T1 Inverse trigonometric functions	ME 11–3
10	1	ME A1 Working with combinations	ME 11–5

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	1	ME V1 Introduction to vectors	ME 12–2
11 (b)	2	ME A1 Working with combinations	ME 11–5
11 (c)	3	ME C2 Further calculus skills	ME 12–4
11 (d)	1	ME C1 Applications of calculus	ME 11–4
11 (e)	2	ME A1 Working with combinations	ME 11–5
11 (f)	2	ME C2 Further calculus skills	ME 12–4
11 (g)	3	ME T3 Trigonometric equations	ME 12–3
11 (h)	2	ME F2 Polynomials	ME 11–2
12 (a)	1	ME C3 Applications of calculus	ME 12–4
12 (b) (i)	3	ME C1 Rates of change	ME 11-4, ME 12–4
12 (b) (ii)	1	ME C1 Rates of change	ME 11–4
12 (c)	3	ME P1 Proof by mathematical induction	ME 12–1
12 (d) (i)	2	ME F1 Further work with functions	ME 11–2
12 (d) (ii)	3	ME F1 Further work with functions	ME 11–1
12 (d) (iii)	1	ME F1 Further work with functions	ME 11–1

Question	Marks	Content	Syllabus outcomes
13 (a)	3	ME C3 Applications of calculus	ME 12–4
13 (b)	4	ME V1 Introduction to vectors	ME 12–2
13 (c)	3	ME C2 Further calculus skills ME C3 Applications of calculus	ME 12–1
13 (d) (i)	2	ME T2 Further trigonometric identities	ME 11–3
13 (d) (ii)	2	ME T3 Trigonometric equations	ME 12–3
14 (a)	3	ME V1 Introduction to vectors	ME 12–2
14 (b)	4	ME C3 Applications of calculus	ME 12–4
14 (c) (i)	1	ME V1 Introduction to vectors	ME 12–2
14 (c) (ii)	3	ME V1 Introduction to vectors	ME 12–2
14 (d)	3	ME S1 The binomial distribution	ME 12–5
14 (e)	2	ME C2 Further calculus skills	ME 12–1

Mathematics Extension 1

HSC Marking Feedback 2021

Question 11

Part (a)

Students should:

- add and subtract simple vectors.

In better responses, students were able to:

- demonstrate their knowledge of course content.

Areas for students to improve include:

- having the knowledge and application of adding and subtracting like vectors.

Part (b)

Students should:

- use and engage with the coefficients of the binomial expansion using Pascal's triangle
- square a binomial twice, in this instance $(2a-b)^2(2a-b)^2$, and collect like terms to arrive at the simplified expression for the expansion.

In better responses, students were able to:

- successfully use Pascal's Triangle to find the coefficients with the correct signs
- successfully expand the squares of the two binomials and arrive at the correct expansion.

Areas for students to improve include:

- using Pascal's Triangle effectively, especially considering the negative sign in $(2a-b)^4$
- expanding a simple binomial $(2a-b)^2(2a-b)^2$, by itself, ensuring the negative sign is considered.

Part (c)

Students should:

- use the substitution of u to arrive at $x=u-1$ and $dx=du$
- obtain the integrand in terms of u
- obtain the correct primitive in terms of u
- replace u with x as their final solution.

In better responses, students were able to:

- obtain the correct substitution in terms of u
- integrate successfully to arrive at the correct primitive in terms of x
- resubstitute u for x in their final solution.

Areas for students to improve include:

- integrating expressions with fractional indices correctly
- realising the need to replace the substituted u by x in their final solution
- realising it is an indefinite integral and hence will need a constant in their final solution.

Part (d)

Students should:

- use combinatorics knowledge to arrive at the correct solution.

In better responses, students were able to:

- effectively demonstrate their knowledge in applying combinatorics to the problem.

Areas for students to improve include:

- recognising that when calculating the number of ways the committee can be formed, they need to use multiplication of the outcomes rather than addition.

Part (e)

Students should:

- recall the formula for the volume of a sphere
- find the derivative of the volume of the sphere with respect to r correctly
- obtain the correct chain rule for the expression $\frac{dv}{dt}$
- substitute correctly into the $\frac{dv}{dt}$ expression to arrive at the answer, correct to 1 decimal place.

In better responses, students were able to:

- obtain the correct $\frac{dv}{dt}$ expression
- substitute correctly to arrive at the answer correct to 1 decimal place.

Areas for students to improve include:

- recalling the formula for the volume of a sphere and correctly finding the derivative with respect to r
- obtaining and using the chain rule
- evaluating correct to 1 decimal place.

Part (f)

Students should:

- integrate to obtain the correct inverse trigonometric primitive

- evaluate the limits correctly to obtain the correct solution
- ensure their answer is in radians.

In better responses, students were able to:

- apply their knowledge of inverse trigonometric functions
- substitute the limits of integration correctly to arrive at $\frac{\pi}{3}$.

Areas for students to improve include:

- familiarising themselves with the use of the Reference Sheet, using it for integration leading to inverse trigonometric functions
- correctly evaluating limits to arrive at an answer in radians.

Part (g)

Students should:

- recognise a polynomial in terms of $\sin x$
- factorise the polynomial by removing the highest common factor
- solve $\sin^2 x = \frac{1}{2}$, ensuring they obtain $\sin x = \pm\frac{1}{\sqrt{2}}$
- find all the possible solutions in the required domain.

In better responses, students were able to:

- apply their knowledge of trigonometric functions and polynomials by fully factorising a trigonometric polynomial equation
- solve for all the angles, in radians, in the given domain.

Areas for students to improve include:

- recognising the highest common factor when factorising a polynomial
- recognising that the square root must have both the plus and minus signs
- avoiding dividing by a factor because this results in the elimination of possible solutions
- finding trigonometric solutions in radians, where required.

Part (h)

Students should:

- apply the relevant formulae relating to roots and coefficients of a quartic equation
- recognise the coefficients of b and c are 0 in this quartic equation
- write the four terms as a single fraction with a common denominator.

In better responses, students were able to:

- apply their knowledge of the relationships between the roots and coefficients of polynomial equations, finding the product and sum of roots taken three at a time correctly to obtain a simplified answer of $\frac{1}{2}$.

Areas for students to improve include:

- learning the formulae relating roots and coefficients of quartic equations
- obtaining a single algebraic fraction from four algebraic terms
- evaluating algebraic fractions correctly.

Question 12

Part (a)

Students should:

- use the slope or direction fields to sketch a possible solution through a particular point.

In better responses, students were able to:

- apply their knowledge of slope or direction fields
- follow the tangent lines to sketch a possible solution passing through a given point.

Areas for students to improve include:

- demonstrating their understanding of direction fields and their use
- following the tangent lines to produce a sketch.

Part (b) (i)

Students should:

- solve a differential equation of the form $\frac{dT}{dt} = k(T - 25)$ resulting in a logarithmic function
- use initial conditions to find the value of the integration constant
- use given values to find the value of an unknown
- rearrange a logarithmic function to an exponential function
- substitute a given value and hence solve the equation to find the required value.

In better responses, students were able to:

- successfully use integration to solve the differential equation
- use given values to find the particular solution $T = 25 - 20e^{kt}$ where $k = \frac{1}{8} ln\left(\frac{3}{4}\right)$ or equivalent
- complete the question by finding the desired time correct to the nearest minute.

Areas for students to improve include:

- knowing how to solve differential equations
- using the Reference Sheet, in particular the integral resulting in a logarithmic function
- changing a logarithmic function to an exponential function
- using initial conditions to find the integration constant
- using the calculator correctly when calculating expressions that involve one or more logarithms
- using logarithmic laws correctly, for example, $\ln a - \ln b = \ln \frac{a}{b}$

- using the degree and minutes function correctly on a calculator, for example, $38.55..$ minutes $\neq$ $38°33'$.

Part (b) (ii)

Students should:

- graph logarithmic and exponential functions
- recognise important features such as intercepts and asymptotes
- rearrange functions, for example, rearrange $t = f(T)$ to $T = F(t)$.

In better responses, students were able to:

- graph $T = F(t)$ showing both the y-intercept and the asymptote.

Areas for students to improve include:

- finding intercepts and asymptotes of a given function
- realising that time cannot be negative, hence the graph should commence at $t = 0$.

Part (c)

Students should:

- work through the steps of an induction proof
- show the substitution when verifying $n = 1$ case
- clearly articulate the inductive hypothesis for $n = k$
- state what they are required to prove for $n = k + 1$
- perform algebraic manipulation to prove the left-hand side is equivalent to the right-hand side for the $n = k + 1$ case.

In better responses, students were able to:

- produce a proof by mathematical induction showing all necessary steps
- clearly showed the inductive step using the correct assumption in the $n = k + 1$ case
- simplify algebraic fractions
- display sufficient working to indicate a correct proof and obtain full marks.

Areas for students to improve include:

- clearly setting out the proof in a logical sequence
- explaining the use of the assumption for $n = k$, not just stating $T(k)$ and $T(k + 1)$
- ensuring that each step of the proof is algebraically correct, especially the inductive step
- working with algebraic fractions
- working from one side to the other when attempting proofs, and not manipulating both sides simultaneously.

Part (d) (i)

Students should:

- recognise the function as a concave down parabola
- sketch a parabola with a restricted domain
- show all necessary features on the graphs, in particular the intercepts in this case
- consider the vertex of the parabola to assist with sketching the parabola.

In better responses, students were able to:

- correctly sketch the correct left-hand side of the concave down parabola for the restricted domain
- as requested, indicate the x- and y-intercepts of the function
- indicate the end point of the function for the restricted domain, that is, where $x = 2$.

Areas for students to improve include:

- recognising the function as the left-hand side of a concave down parabola given the restricted domain
- sketching the parabola $y = 4 - \left(1 - \frac{x}{2}\right)^2$ and then indicating the required part of the parabola
- finding x- and y-intercepts and displaying them as requested on the graph.

Part (d) (ii)

Students should:

- find the inverse function either by interchanging x and y and then making y the subject, or by rearranging and making x the subject and then expressing this as the inverse function
- state the domain and range of the original function $y = 4 - \left(1 - \frac{x}{2}\right)^2$
- state the domain and range of the inverse function $y = 2 - 2\sqrt{4 - x}$.

In better responses, students were able to:

- correctly find the inverse relation of $y = 4 - \left(1 - \frac{x}{2}\right)^2$ as $y = 2 \pm 2\sqrt{4 - x}$
- use the domain to correctly recognise the inverse function as $y = 2 - 2\sqrt{4 - x}$
- correctly state the domain as $(-\infty, 4]$.

Areas for students to improve include:

- understanding that finding $f^{-1}(x)$ requires more work than just interchanging x and y
- understanding the relationship between the domain and range of $y = f(x)$ and the range and domain of its inverse
- moving algebraically from $x = 4 - \left(1 - \frac{y}{2}\right)^2$ to $y = 2 \pm 2\sqrt{4 - x}$ then determining $y = f^{-1}(x)$ by considering the restriction on the domain.

Part (d) (iii)

Students should:

- sketch $y = f^{-1}(x)$ from the equation or by reflecting $y = f(x)$ through $y = x$, taking into consideration the restricted domain.

In better responses, students were able to:

- demonstrate their knowledge of inverse functions and restricted domain by accurately sketching $y = 2 - 2\sqrt{4 - x}$ for the restricted domain $(-\infty, 4]$.

Areas for students to improve include:

- understanding the definition of a function – for each x there is only one y
- displaying important features such as intercepts and endpoints
- understanding how an inverse can be sketched by reflecting the curve through $y = x$
- understanding how a graph is affected by restricted domains.

Question 13

Part (a)

Students should:

- recognise the need to evaluate volumes of revolution about the y-axis.

In better responses, students were able to:

- determine that the required volume was the result of the difference between an outer and an inner volume
- correctly write an expression for a volume which is created by rotating about the y-axis
- choose the correct limits of integration for each of the different volumes.

Areas for students to improve include:

- recognising which of the two curves created the outer and the inner volumes
- drawing a simple diagram which could help to choose correct limits of integration
- taking note of their evaluations and questioning a result in which the inner volume was calculated to be zero.

Part (b)

Students should:

- use their knowledge of projectile motion to show the required results.

In better responses, students were able to:

- find the time taken to reach maximum height and use this to show that the maximum height was less than 3 metres
- find either the time of flight for the ball to reach a horizontal displacement of 10 metres and show that the ball was still above ground level at this time or find the time of flight for maximum unrestricted range and show that this range is greater than 10 metres.

Areas for students to improve include:

- reading the question carefully to gain a full understanding of the situation, in particular, the relevance of the ball leaving the hand at 1 metre to the h term in the supplied equation
- taking care to ensure that the correct value for time is used with the relevant vertical or horizontal displacement expression
- understanding that, as the ball had left the hand at a height of 1 metre, the range would not be found by doubling the time to maximum height
- taking note of their results and acting on contradictory statements. For example, finding a height at 10 metres displacement which was higher than their previously stated maximum height
- showing all necessary working as opposed to making a statement which had not been fully proven
- using a few words to explain the relevance of a particular calculation
- noting that there was no need to derive equations from acceleration in this question.

Part (c)

Students should:

- use integration and, possibly, area formulae to evaluate the shaded area.

In better responses, students were able to:

- identify that the area bounded by the straight lines would be best evaluated as a triangle
- explain that, by symmetry or even functions, the areas on either side of the y-axis are equal
- understand that the area below the x-axis would be evaluated as a negative value and take the necessary actions to deal with this
- recognise that $|4 - 2\pi| = 2\pi - 4$, or similar
- correctly integrate the $\frac{8}{4+x^2}$ term and then evaluate the resulting substitution in radians.

Areas for students to improve include:

- understanding how to approach integrals involving absolute values
- recognising that their workload can be simplified through use of symmetry
- identifying the axis along which the limits of integration should be chosen
- developing methods to deal with areas which lie below the x-axis
- checking whether terms have positive or negative values and understanding how, and when, absolute value signs can be removed
- using the Reference Sheet for integrations which result in inverse trigonometric functions
- understanding that inverse trigonometric functions should only be evaluated in radians

Part (d) (i)

Students should:

- use trigonometric identities to show a relationship.

In better responses, students were able to:

- follow the hints in the question to substitute for A and C
- use the trigonometric identities on the Reference Sheet to simplify the expression which resulted from the substitution
- simplify with correct algebraic working.

Areas for students to improve include:

- taking the trigonometric identities from the Reference Sheet
- taking care algebraically to simplify correctly
- ensuring that sufficient working is shown to provide a convincing argument that the supplied result has been achieved.

Part (d) (ii)

Students should:

- use simultaneous equations to find an expression for B
- solve the resulting trigonometric equation for θ within the given domain.

In better responses, students were able to:

- recognise the connection to part (i) and let $A = \frac{5\theta}{7}$ and $C = \frac{6\theta}{7}$
- solve the resulting simultaneous equations to obtain $B = \frac{11\theta}{14}$
- solve the resulting trigonometric equation to obtain the two relevant values of θ.

Areas for students to improve include:

- identifying the connections between parts of questions
- recognising that the equation to solve had no B term and so the equation was to be solved for θ
- understanding that many trigonometric equations have more than one solution and ensuring that all relevant solutions are found
- understanding that, for trigonometric equations, the domain given indicates the form in which the answer should be given.

Question 14

Part (a)

Students should:

- read the worded question carefully and attempt to interpret it appropriately
- draw a neat diagram clearly labelling all the given information
- use appropriate trigonometric formulae to find the values of the required angles.

In better responses, students were able to:

- draw a clear and correct diagram, use the sine rule and their knowledge of bearings to arrive at the correct answer
- alternatively, use appropriate vector components and trigonometric ratios to find the correct bearing.

Areas for students to improve include:

- constructing and clearly labelling diagrams from the given information
- using the vector resolution method to solve questions involving motion.

Part (b)

Students should:

- separate variables in a differential equation and correctly integrate terms that lead to logarithmic expressions
- evaluate the constant of integration using the initial condition.

In better responses, students were able to:

- correctly solve the definite integral thereby removing the need for the constant of integration.

Areas for students to improve include:

- making effective use of the logarithmic and exponential laws to solve equations
- rearranging and manipulating terms to simultaneously solve two logarithmic equations
- solving differential equations.

Part (c) (i)

Students should:

- recognise that the angle between parallel vectors is zero
- use the correct notation for vectors and magnitude of vectors.

In better responses, students were able to:

- prove the given statement by using the dot product and either the fact that $\cos 0 = 1$ in this case or representing the two vectors in component form.

Areas for students to improve include:

- completing problems involving proofs
- showing all the steps in a proof
- not using specific values for variables when completing a proof.

Part (c) (ii)

Students should:

- be able to express a vector in terms of the sum of two other vectors
- effectively use the information given in the question, in this case, the fact that the magnitudes of the diagonals are equal

- use the result from part (i) as an indication to square the magnitude of each diagonal.

In better responses, students were able to:

- recognise the link between the result shown in part (i) and used it effectively as a key to the completing the proof for this part.

Areas for students to improve include:

- recognising the importance of using the information given in the question
- carefully considering the implications of the interconnectivity of parts within a question
- distinguishing between vectors and the magnitude of vectors by using the correct notation for each
- practising vector problems involving proofs of geometrical properties of polygons.

Part (d)

Students should:

- identify the relevant statistical descriptors from the information given
- understand the importance of finding the z-score of the standardised normal distribution.

In better responses, students were able to:

- find the correct z-score using the empirical rule
- show a clear understanding of the relationship between a score, the mean, the standard deviation and the z-score.

Areas for students to improve include:

- understanding the difference between variance and standard deviation and where each is used
- using the empirical rule to determine how many standard deviations is the score away from the mean
- analysing statistical data that exhibit binomial distribution.

Part (e)

Students should:

- use their knowledge of the product rule of differentiation to find expression for $f'(x)$
- realise that if $g(1) = 3$ then $g^{-1}(3) = 1$
- apply their knowledge of inverse functions to determine the gradient of the tangent at the given point.

In better responses, students were able to:

- correctly find the value of $\frac{d}{dx}[g^{-1}(3)]$ and used the product rule to arrive at the correct solution
- use implicit differentiation to correctly solve the question.

Areas for students to improve include:

- reviewing the application of the product rule of differentiation

- understanding the fundamentals of inverse functions
- not confusing gradients of a tangent and normal with gradients of a function and its inverse
- understanding the relationship between the derivative of a function at point (a, b) and the derivative of its inverse function at the point (b, a).