CSSA

2022
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

DO NOT REMOVE PAPER FROM EXAMINATION ROOM

Centre Number

Student Number

Mathematics Extension 2

Morning Session
Monday, 8 August 2022

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- Use the Multiple-Choice Answer Sheet provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 100

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks

Attempt Questions 1–10
Allow about 15 minutes for this section
Use the Multiple-Choice Answer Sheet for Questions 1–10

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1 What is the smallest positive value for n so that $\left(\sqrt{3}+i\right)^n$ is real?

A. 0

B. 3

C. 6

D. 12

2 The displacement x metres of a particle undergoing simple harmonic motion at time t seconds is given by $x = 3\sin\left(2t + \frac{\pi}{3}\right) + 1$. Which of the following statements is true?

A. The period is π and the amplitude is 3.

B. The period is π and the amplitude is 4.

C. The period is $\frac{\pi}{3}$ and the amplitude is 3.

D. The period is $\frac{\pi}{3}$ and the amplitude is 4.

3 What is the remainder when $17z^4 - 5z + 2$ is divided by $z + i$?

A. $-15 - 5i$

B. $-15 + 5i$

C. $19 - 5i$

D. $19 + 5i$

4 Consider the statement:

'If it is sunny, then Jamie wears a hat'.

Which of the following is the converse of this statement?

A. If Jamie wears a hat, then it is sunny.

B. If Jamie wears a hat, then it is not sunny.

C. If Jamie does not wear a hat, then it is sunny.

D. If Jamie does not wear a hat, then it is not sunny.

5 Given that $z = 2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$, which expression is equal to $(\bar{z})^{-1}$?

A. $\frac{1}{2}(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5})$

B. $2(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5})$

C. $\frac{1}{2}(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$

D. $2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$

6 Which expression is equal to $\displaystyle\int \frac{2x+4}{x^2+16}\,dx$?

A. $2\ln\left|x^2+16\right| + 4\tan^{-1}\left(\frac{x}{4}\right) + c$

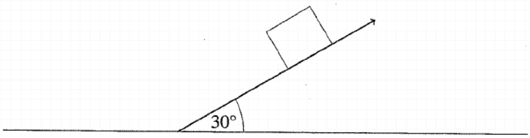
B. $\ln\left|x^2+16\right| + \tan^{-1}\left(\frac{x}{4}\right) + c$

C. $\ln\left|x^2+16\right| + 4\tan^{-1}\left(\frac{x}{4}\right) + c$

D. $2\ln\left|x^2+16\right| + \tan^{-1}\left(\frac{x}{4}\right) + c$

7 A 10 kg box on a plane inclined at an angle of 30° to the horizontal is undergoing uniform acceleration of $1{\cdot}5\ \text{m/s}^2$.

Take the acceleration g due to gravity to be $9{\cdot}8\ \text{m/s}^2$.



What is the magnitude of the frictional force resisting the motion of the box?

A. 34 N

B. 64 N

C. 70 N

D. 100 N

8 Consider the lines $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ a \end{pmatrix}$ and $\underset{\sim}{s} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$, where $\lambda, \mu \in \mathbb{R}$.

For what value of a will the lines $\underset{\sim}{r}$ and $\underset{\sim}{s}$ intersect at a point?

A. $a = -6$

B. $a = -1$

C. $a = 1$

D. $a = 6$

9 A particle of mass m moves horizontally through a medium with velocity v at time t. Initially, the particle is at the origin O moving with speed v_0. The resistance on the particle due to the medium is proportional to the square of the speed.

If k is a constant of proportionality, which expression gives the correct velocity of the particle?

A. $v = \frac{k}{m}t + \frac{1}{v_0}$

B. $v = \frac{mv_0}{ktv_0 + m}$

C. $v = v_0 e^{-\frac{k}{m}t}$

D. $v = -\frac{k}{m}t + \ln v_0$

10 The position vector of the point P is given by $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ where $\lambda \in \mathbb{R}$.

The point Q has coordinates $(2, -2, -5)$.
Which of the following gives the correct expression for $|\overrightarrow{QP}|$ in terms of λ?

A. $\sqrt{5\lambda^2 + 18\lambda + 18}$

B. $\sqrt{5\lambda^2 + 10\lambda + 66}$

C. $\sqrt{5\lambda^2 + 8\lambda + 9}$

D. $\sqrt{5\lambda^2 + 6\lambda + 18}$

Section II

90 marks

Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

Your responses for Questions 11-16 should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Write the contrapositive of the following statement. **1**

'If you have measured your size correctly then your clothes fit you well'.

(b) Find $\displaystyle\int \frac{7x-11}{(x-1)(x-3)}\,dx$. **3**

(c) The complex numbers $z = 2+3i$ and $w = 3-2i$ are given.

(i) Find the value of $z + 2\overline{w}$ in the form $x+iy$. **1**

(ii) Find the value of $\dfrac{w}{z}$ in the form $x+iy$. **2**

(d) A particle moves in one dimension such that its acceleration $a\,\text{ms}^{-2}$ is inversely proportional to its velocity $v\,\text{ms}^{-1}$ as given by the equation $a = \dfrac{72}{v}$. When the time t seconds is $t = 1$ its displacement x metres will be $x = 8$ and also $v = 12$. **3**

Given that $t > 0$ show that $x = 8t^{3/2}$.

(e) Find $\displaystyle\int \frac{1}{4x^2+8x+13}\,dx$. **3**

(f) Prove by contradiction that $\log_{10} 7$ is an irrational number. **2**

Question 12 (14 marks) Use a SEPARATE writing booklet.

(a) Consider the equation $z^3 + 15z^2 + cz + 34 = 0$ where c is a real number. One of the roots of the equation is $1 + i$.

(i) Find the real root of the equation. **1**

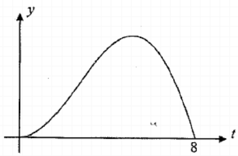
(ii) Determine the value of c. **1**

(b) A complex number z satisfies the inequation $|z - 4i| \le 2$.

(i) Sketch the region of z on an Argand diagram. **2**

(ii) Find the range of possible values for the principal argument of z. **2**

(c) The instantaneous rate of energy production of a solar panel, y megajoules per hour, during an 8 hour period is given by the equation $y = t\sin\left(\frac{\pi t}{8}\right)$ as shown in the diagram below. **3**



By finding the area under the curve, calculate the number of megajoules produced by the solar panel over the 8 hour period. Give your answer correct to 2 decimal places.

(d) Consider the line $\underset{\sim}{l} = \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 7 \\ 8 \end{pmatrix}$ where $\lambda \in \mathbb{R}$, and the line $\underset{\sim}{m} = \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ where $\mu \in \mathbb{R}$.

(i) Show that $\underset{\sim}{l}$ and $\underset{\sim}{m}$ intersect at right angles. **2**

(ii) Find the equation of a line that intersects both $\underset{\sim}{l}$ and $\underset{\sim}{m}$ at right angles. **3**

Question 13 (16 marks) Use a SEPARATE writing booklet.

(a) The nth term T_n of a sequence is defined such that $T_n = 2T_{n-1} - n^2$, and $T_1 = 10$. Prove by mathematical induction that $T_n = n^2 + 4n + 6 - 2^{n-1}$ for all positive integers n. **3**

(b) (i) Given $z = e^{i\theta}$, show that $2\cos(k\theta) = z^k + z^{-k}$. **1**

(ii) Expand $\left(z - z^{-1}\right)^4$. Hence, or otherwise, show that **3**

$$\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3).$$

(c) (i) Show that $\dfrac{d}{dx}\sec x = \sec x \tan x$. **1**

(ii) A constant k satisfies $\displaystyle\int_0^{\frac{\pi}{3}} \left(k\cos^2 x - \sec^2 x\right)\sin x\,dx = \frac{11}{24}$. Evaluate k. **3**

(d) A particle moving in one dimension has position x m and its velocity v m/s is given by

$$\tfrac{1}{2}v^2 = 2 - 4x - 2x^2.$$

(i) Show that the motion of the particle is simple harmonic. **2**

(ii) Given the range of motion is $x_1 \le x \le x_2$, determine the values of x_1 and x_2. **2**

(iii) At time $t = 0$, $x = 0$ and $v > 0$. Find when the particle is next at the origin. **1**

Question 14 (16 marks) Use a SEPARATE writing booklet.

(a) (i) If a and b are real numbers, and $\underset{\sim}{p} = 3a\underset{\sim}{i} + b\underset{\sim}{j}$ show that $|\underset{\sim}{p}| = \sqrt{9a^2 + b^2}$. **1**

(ii) By choosing an appropriate vector $\underset{\sim}{q}$, use the triangle inequality, or otherwise, to prove for all real numbers a and b, that **3**

$$\sqrt{a^2+b^2} \leq \frac{\sqrt{9a^2+b^2}+\sqrt{a^2+9b^2}}{4}.$$

(b) Let $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1+x^2}}\,dx$.

(i) Show when $n \geq 2$, that $I_n = \dfrac{\sqrt{2}}{n} - \dfrac{n-1}{n} I_{n-2}$. **3**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx$. **2**

(c) Prove that the double of the sum of the squares of two distinct positive integers can be written as the sum of two distinct non-zero square integers. **2**

(d) Let $z = a + ib$, where $a > 0$ and $b > 0$, be represented by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

(i) Find the vector representation for $\dfrac{1}{z}$. **1**

(ii) Let the angle between the two vectors represented by z and $\dfrac{1}{z}$ be θ. **2**

By using the dot product, show $\theta = \cos^{-1}\left(\dfrac{a^2-b^2}{a^2+b^2}\right)$.

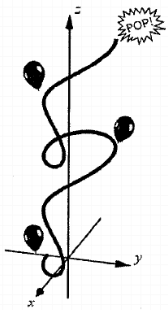
(iii) Hence show that $\cos^{-1}\left(\dfrac{a^2-b^2}{a^2+b^2}\right) = 2\tan^{-1}\left(\dfrac{b}{a}\right)$. **2**

Question 15 (13 marks) Use a SEPARATE writing booklet.

(a) By considering the roots of the equation $z^9 + 1 = 0$, or otherwise, show that **3**

$$\cos\left(\frac{\pi}{9}\right) = \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right).$$

(b) A helium balloon is released from the ground and floats upwards for 10 seconds before bursting as shown in the diagram below.



The position in metres of the balloon after t seconds is given by the vector

$$\underset{\sim}{r} = \begin{pmatrix} 4\sin t \\ -\cos 2t \\ 2t - \sin 2t \end{pmatrix}.$$

(i) Find an expression for the velocity $\underset{\sim}{v}$ of the balloon at time t. **2**

(ii) Show that the speed of the balloon $|\underset{\sim}{v}|$ is a constant 4 m/s. **2**

(iii) Hence find the length of the path the balloon took from when it was released to when it burst at $t = 10$. **1**

(c) (i) Show that $\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \mathrm{Re}\left(e^{i\theta}\frac{1-e^{in\theta}}{1-e^{i\theta}}\right)$. **2**

(ii) Hence, or otherwise, show that **3**

$$\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \cos\left((n+1)\tfrac{\theta}{2}\right) \times \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

Question 16 (16 marks) Use a SEPARATE writing booklet.

(a) (i) Given that p and q are two positive integers, show that **2**

$$\int_0^1 x^p(1-x)^q\,dx = \frac{q}{p+1}\int_0^1 x^{p+1}(1-x)^{q-1}dx.$$

(ii) Hence, show that $\displaystyle\int_0^1 x^p(1-x)^q\,dx = \frac{p!q!}{(p+q+1)!}$. **4**

(b) By considering the concavity of $y=\sqrt[3]{x}$, prove that if $a>b>0$, then **3**

$$\sqrt[3]{a-b}+\sqrt[3]{a+b}<2\sqrt[3]{a}.$$

(c) A falling object of mass m kg experiences acceleration due to gravity of g m/s^2 and air resistance of magnitude kv^2 newtons where v is the object's velocity in m/s at time t seconds.

(i) Assuming that the upwards direction is positive, show that the velocity v of a dropped object is given by **4**

$$v=\sqrt{\frac{mg}{k}}\left(\frac{e^{-t\sqrt{gk/m}}-e^{t\sqrt{gk/m}}}{e^{-t\sqrt{gk/m}}+e^{t\sqrt{gk/m}}}\right).$$

(ii) Andre steps from a plane at an altitude of 5000 metres and must open his parachute at an altitude of 1500 metres to land safely. His coefficient k of air resistance is 0·25, his mass is 100 kg, and the acceleration due to gravity is 10 m/s^2. After how many seconds must Andre open his parachute? **3**

End of Examination