Name:_____________________

Class:_____________________

Trial HSC Examination 2024

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes.
- Working time – 120 minutes.
- Only NESA-approved calculators may be used.
- Write using blue or black pen.
- All necessary working should be shown in all questions.
- Write your student number at the top of every answer sheet.

Total marks – 70

Attempt all questions.

Section A – Answer on the Multiple-Choice Answer Sheet.

Section B - Start each question on a new sheet of paper.

NESA Reference Sheet is provided.

Z table for unit normal distribution is provided.

This paper MUST OT be removed from the examination room

Section A Multiple-Choice (10 Marks)

QUESTION 1

X is defined as a random variable such that $X \sim \mathrm{Bin}(30, 0.4)$. Which of the following is $\mathrm{E}(X)$ and $\mathrm{Var}(X)$?

A. $\mathrm{E}(X) = 12,\ \mathrm{Var}(X) = 7.2$

B. $\mathrm{E}(X) = 18,\ \mathrm{Var}(X) = \sqrt{7.2}$

C. $\mathrm{E}(X) = 18,\ \mathrm{Var}(X) = 7.2$

D. $\mathrm{E}(X) = 12,\ \mathrm{Var}(X) = \sqrt{7.2}$

QUESTION 2

Which of the following is the range of $\tan^{-1}(\sin x)$?

A. $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

B. $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

C. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

D. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

QUESTION 3

Which of the following gives the number of values of x in the interval $[0, 5\pi]$ that will satisfy the following equation: $3\sin^2 x - 7\sin x + 2 = 0$?

A. 0

B. 5

C. 6

D. 10

QUESTION 4

Let $f(x) = x^3$ where $x \in \{0, 1, 2, 3\}$. Which of the following is the domain of $f^{-1}(x)$?

A. $\{0, 1, \sqrt[3]{8}, \sqrt[3]{27}\}$

B. $\left\{0, 1, \frac{1}{\sqrt[3]{8}}, \frac{1}{\sqrt[3]{27}}\right\}$

C. $\{0, 1, 8, 27\}$

D. $\left\{0, 1, \frac{1}{8}, \frac{1}{27}\right\}$

QUESTION 5

Which of the following has the same solution as that of $\frac{3}{2-x} > 2$?

A. $2x - 1 \geq 0$

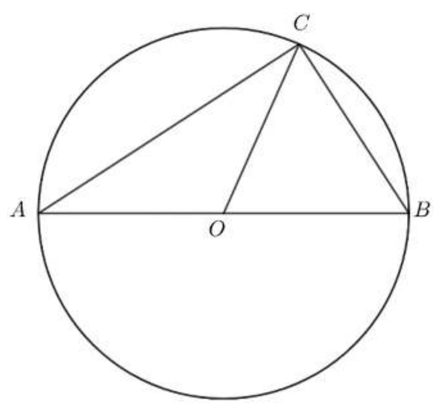
B. $(x-2)(2x-1) > 0$

C. $(x-2)(2x-1) < 0$

D. None of the above.

QUESTION 6

In the diagram below, AOB is a diameter of the circle ABC with centre O. Point C lies on the circumference of the circle.



If $\overrightarrow{OC} = \mathbf{r}$ and $\overrightarrow{BC} = \mathbf{s}$, to which of the following is $\overrightarrow{AC}$ equal?

A. $\mathbf{r} + 2\mathbf{s}$

B. $\mathbf{r} - 2\mathbf{s}$

C. $2\mathbf{r} + \mathbf{s}$

D. $2\mathbf{r} - \mathbf{s}$

QUESTION 7

Which of the following is the unit vector perpendicular to $\boldsymbol{p} = -6\mathbf{i} + 2\mathbf{j}$?

A. $\frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}$

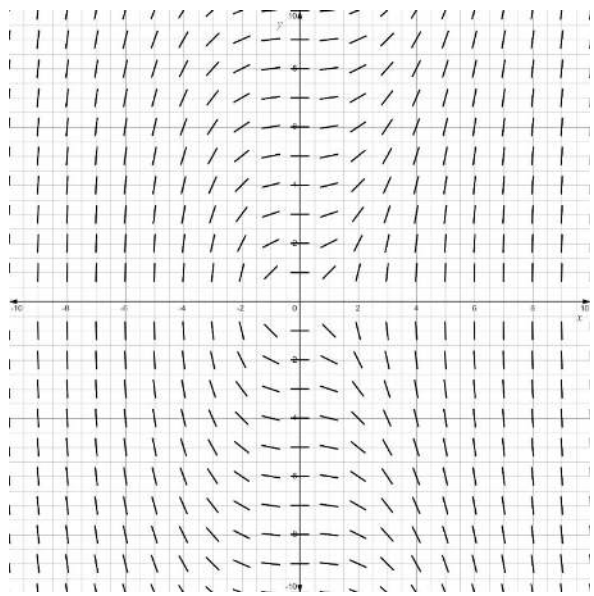
B. $\frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$

C. $\frac{1}{\sqrt{10}}\mathbf{i} + \frac{-3}{\sqrt{10}}\mathbf{j}$

D. $\frac{-3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}$

QUESTION 8

Which of the following differential equations could represent the slope field below?



A. $\frac{dy}{dx} = \frac{x}{y^2}$

B. $\frac{dy}{dx} = \frac{x^2}{y}$

C. $\frac{dy}{dx} = -\frac{x^2}{y}$

D. $\frac{dy}{dx} = -\frac{x}{y^2}$

QUESTION 9

There are 11 points in a plane such that 4 of the points are collinear. Which of the following gives the number of lines that may be formed such that those lines pass through at least two of the 11 points?

A. 55

B. 49

C. 50

D. 52

QUESTION 10

For the binomial expansion of $(2 + kx)^7$, where $k > 0$ is a constant, it is given that the coefficient of x^2 is six times the coefficient of x. Which of the following is the value of k?

A. $\frac{1}{144}$

B. $\frac{1}{4}$

C. 4

D. 144

End of Section A

SECTION B

QUESTION 11 Start a new page (9 marks) **Marks**

a) Given $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + \underset{\sim}{j}$, find:

i. $\underset{\sim}{a} \cdot \underset{\sim}{b}$. **1**

ii. $\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a}$ and express your answer in the form, $x\underset{\sim}{i} + y\underset{\sim}{j}$. **2**

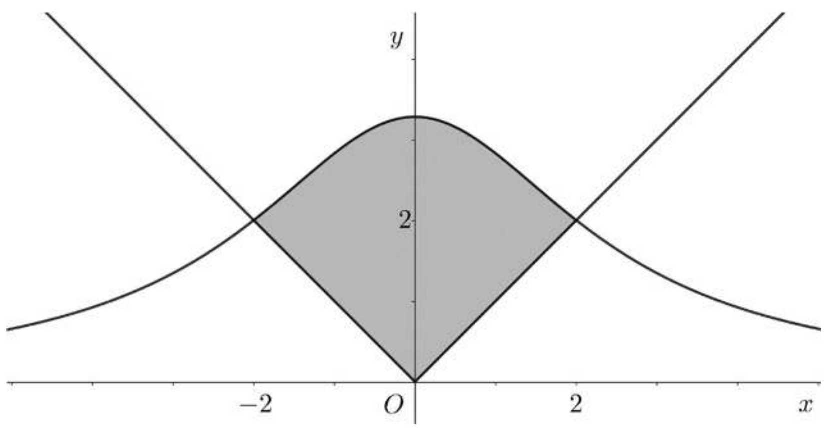
b) Ten unbiased, six-sided dice are tossed simultaneously. Write an expression for the probability of exactly three of them landing with the number *5* facing up. **1**

c) Given $P(x) = 3x^3 - 2x^2 + x - 3$ has zeroes α, β and γ:

i. Write down the value of $\alpha\beta\gamma$. **1**

ii. Hence, or otherwise, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **1**

d) It is given that the curves $y = |x|$ and $y = \frac{82}{25+4x^2}$ intersect at the points $(2, 2)$ and $(-2, 2)$. Find the area bounded by the curves, as indicated in the diagram below. Give your answer correct to one decimal place. **3**



QUESTION 12 Start a new page (9 marks) **Marks**

a) Three adults and five children go to the cinema and are seated next to each other in a row of eight seats. How many ways can these eight people sit so that at least two of the adults sit next to each other? **2**

b) Find the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^5\left(2x^2-\frac{3}{x}\right)^6$. **3**

c) A thermometer, reading 24 °C, is brought into a room whose temperature is 5 °C. At five minutes, the thermometer registers 18 °C. Assume that the temperature T of the thermometer decreases at a rate proportional to the difference between the temperature on the thermometer and the temperature of the room; that is:

$$\frac{dT}{dt}=k(T-5)$$

i. Show that $T=5+Ae^{kt}$ is a solution to the differential equation above. **1**

ii. How long will it take for the thermometer to read 10°C? **3**
Give your answer correct to the nearest minute.

QUESTION 13 Start a new page (9 marks) **Marks**

a) Find the values of s and t in the following sum of binomial coefficients: **2**

$$\binom{2022}{146} + \binom{2022}{147} + \binom{2023}{1875} = \binom{s}{t}$$

b) An unbiased, regular coin is tossed 30 times. Let the random variable $\hat{p}$ be the proportion of heads obtained amongst the 30 tosses.

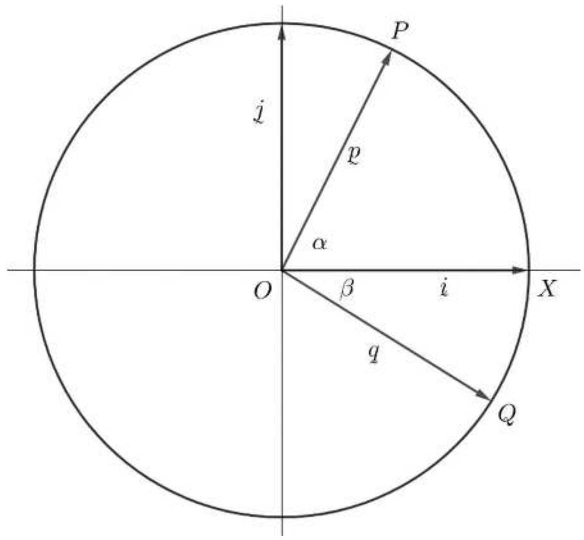
i. Justify mathematically why this distribution may be approximated using the normal distribution. **1**

ii. Hence approximate the value of $P\left(\frac{12}{30} \leq \hat{p} \leq \frac{16}{30}\right)$. **3**

c) Prove, by mathematical induction, that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all $n \geq 1, n \in \mathbb{Z}$. **3**

QUESTION 14 Start a new page (8 marks) **Marks**

a) Consider the unit circle around the origin O in the diagram below, with $\underset{\sim}{i}$ and $\underset{\sim}{j}$ representing the standard basis unit vectors in the horizontal and vertical directions respectively. The points P and Q lie on the unit circle such that P is in the first quadrant and Q is in the fourth quadrant. The angles POX and QOX have measures α and β respectively, where X is the point (1,0). Let $\overrightarrow{OP} = \underset{\sim}{p}$, $\overrightarrow{OQ} = \underset{\sim}{q}$.



i. Find $\underset{\sim}{p}$ in terms of α, and $\underset{\sim}{q}$ in terms of β. **2**

ii. Hence, by considering $\underset{\sim}{p} \cdot \underset{\sim}{q}$, show that **2**

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

b) By using an appropriate t-formula substitution, solve the equation **4**

$$3\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) = -3 \ \text{ for } \ 0 \leq \theta \leq 2\pi$$

QUESTION 15 Start a new page (9 marks) **Marks**

a)

i. Sketch the graphs of $y = |x^2 - 3x + 2|$ and $y = 2$ on the same plane. **2**

ii. Hence, or otherwise, solve $|x^2 - 3x + 2| > 2$. **1**

b) Let θ be the measure of an acute angle.

i. Using a suitable expansion of $\sin 6\theta$, show that **2**

$$(\sin 2\theta)^3 - \frac{3}{4}\sin 2\theta + \frac{1}{4}\sin 6\theta = 0$$

ii. If $x = 4\sin 2\theta$ and $x^3 - 12x + 8 = 0$, show that $\sin 6\theta = \frac{1}{2}$. **1**

iii. Use your result in (ii) to find the value of **3**

$$\left(\sin\frac{\pi}{18}\right)^2 + \left(\sin\frac{13\pi}{18}\right)^2 + \left(\sin\frac{25\pi}{18}\right)^2$$

QUESTION 16 Start a new page (8 marks) **Marks**

a) Liquid is poured into a large vertical circular cylinder at a constant rate of $1600\ \text{cm}^3\text{s}^{-1}$. At the same time, water is leaking from a hole in the base of the cylinder at a rate that is proportional to the square root of the height of the liquid present in the cylinder. It is given that the area of the circular cross-section of the cylinder is $4000\ \text{cm}^2$.

i. Show that at time t seconds, the height h cm of liquid in the cylinder satisfies the differential equation **1**

$$\frac{dh}{dt} = 0.4 - c\sqrt{h}$$

where c is a constant.

When $h = 25$ cm, water is leaking from the hole at a rate of $400\ \text{cm}^3\text{s}^{-1}$.

ii. Show that $c = 0.02$. **1**

iii. Show that the time taken to fill the cylinder from being empty to having a height of 100 cm is given by the integral: **1**

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh$$

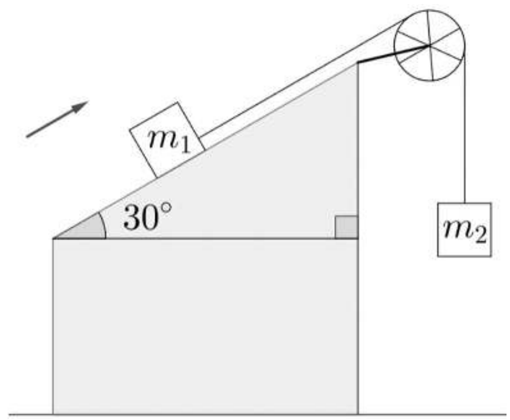
iv. Using the substitution $\sqrt{h} = 20 - x$, or otherwise, evaluate the integral in (iii). Give your answer correct to the nearest second. **3**

b) By use of a product-to-sum identity, find: **2**

$$\int \cos 2x \sin 3x\, dx$$

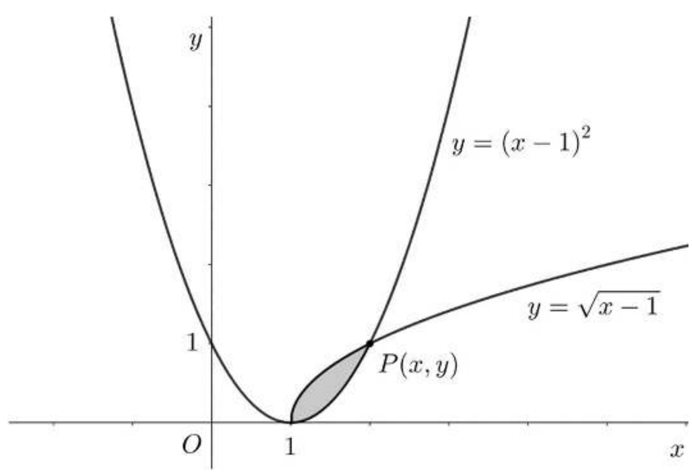
QUESTION 17 Start a new page (8 marks) **Marks**

a) Consider the two-body construction shown in the diagram below. A crate, having mass $m_1 = 2500$ kg, lies on a smooth, inclined plane. It is connected by a light, inextensible cable through a smooth pulley to a second crate having mass $m_2 = 4000$ kg. The plane has an angle of inclination of 30°. **2**



Taking the upward direction of the incline as positive, find the acceleration of m_1 in terms of g in its simplest form.

b) The graphs of $y = (x-1)^2$ and $y = \sqrt{x-1}$ intersect at $(1,0)$ and $P(x,y)$ as shown in the diagram below.



i. Write down the point of intersection $P(x,y)$. **1**

ii. Hence find the volume of the solid of revolution formed by rotating the region bounded by $y = \sqrt{x-1}$ and $y = (x-1)^2$ about the y-axis. **2**

c) Show that $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = 3\tan^{-1}\left(\frac{x}{a}\right)$, where $a > 0, -\frac{a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$. **3**

NB: Any identity used that is not listed on the reference sheet must be derived.

End of paper

MATHEMATICS Extension 1: Question …		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
11. a) i $\underset{\sim}{a}\cdot\underset{\sim}{b} = (3)(-2)+(2)(1) = -6+2 = -4$	1	a) i) Correct or not
ii $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \left(\dfrac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\right)\underset{\sim}{b} = \dfrac{-4}{5}(-2\underset{\sim}{i}+\underset{\sim}{j}) = \dfrac{8}{5}\underset{\sim}{i} - \dfrac{4}{5}\underset{\sim}{j}$	1	ii) First mark for using projection formula correctly. 2/2 must be in $\underset{\sim}{i}+\underset{\sim}{j}$ form
b) $P(X=3) = \binom{10}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^7$	1	b) overall well done many forgot ${}^{10}C_3$
c) i $\alpha\beta\gamma = -\left(\frac{-3}{3}\right) = 1$	1	c) i) overall well done
ii $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma} = \dfrac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma} = \dfrac{\frac{1}{3}}{1} = \dfrac{1}{3}$	1	ii) Needed to show $\alpha\beta+\alpha\gamma+\beta\gamma = \frac{1}{3}$
d) $A = 2\times\displaystyle\int_0^2 \frac{82}{25+4x^2}\,dx - 2\int_0^2 x\,dx$ $= \displaystyle\int_0^2 \frac{41}{\frac{25}{4}+x^2}\,dx - \left[x^2\right]_0^2$ $= \dfrac{2\times 41}{5}\displaystyle\int_0^2 \frac{\frac{5}{2}}{\frac{25}{4}+x^2}\,dx - 4$ $= \dfrac{82}{5}\left[\tan^{-1}\left(\frac{2x}{5}\right)\right]_0^2 - 4$ $= \dfrac{82}{5}\tan^{-1}\left(\frac{4}{5}\right) - 4$ $= 7.1\,u^2$	1 1 1	d) - First mark to correctly set up integral - 2nd mark to integrate to $\tan^{-1}\left(\frac{ax}{b}\right)$ correctly. - 3rd mark to put in calculator (must be radian mode) * $\frac{2}{3}$ if left as $\tan^{-1}\left(\frac{ax}{b}\right)$ * no marks lost for transcription error, ie writing 85 not 82.

Student Number SOLUTIONS

QUESTION 12

a) 8 people can sit in $8!$ ways. ✓

Sit the 5 children. There are 6 spaces : $5!$

_ C _ C _ C _ C _ C _

If no adults sit together they can choose spaces in $6\times5\times4$ ways.

$\therefore$ At least 2 adults sit together in $8! - 6\times5\times4 \times 5!$

$= 25\,920$ ways.

✓

b) $\left(x+\frac{1}{x}\right)^5\left(2x^2-\frac{3}{x}\right)^6$

$= \left({}^5C_0\, x^5 + {}^5C_1\, x^4 . x^{-1} + \ldots\ {}^5C_k\, x^{5-k} x^{-k} + \ldots + {}^5C_5\, x^{-5}\right)$ ✓ one binomial expansion

$\times\left({}^6C_0 (2x^2)^6 + {}^6C_1 (2x^2)^5(-3x^{-1}) + \ldots + {}^6C_n (2x^2)^{6-n}(-3x^{-1})^n + \ldots\right)$

$= \left(\ldots\ {}^5C_1\, x^3 + \ldots + {}^5C_4\, x^{-3} + \ldots\right)\left(\ldots\ {}^6C_3 (2)^3(-3)^3 x^3 + \ldots + {}^6C_5 (2)^1(-3)^5 x^{-3} + \ldots\right)$

$= \ldots + {}^5C_4\, {}^6C_3\, 2^3(-3)^3 + {}^5C_1\, {}^6C_5\, 2^1(-3)^5 + \ldots$

$\therefore$ Term independent of x is -36180

Powers of x

( )	( )
5	12
3	9
1	6
−1	$\boxed{3}$
$\boxed{-3}$	−3
−5	−6
	−9
	−12

c) i) LHS $= \frac{dT}{dt}$

$= \frac{d}{dt}\left(5 + Ae^{kt}\right)$

$= kAe^{kt}$ ✓

$= k\left(5 + Ae^{kt} - 5\right)$

$= k\,(T-5)$

$=$ RHS

You CANNOT do this:

$\frac{dt}{T-5} = k dt$

$\ln|T-5| = kt + c$

$|T-5| = e^c e^{kt}$

$T-5 = \pm e^c e^{kt}$

Let $A = \pm e^c$

$T-5 = Ae^{kt}$

Student Number SOLUTIONS

QUESTION 12 cont.

c) ii) $\left.\begin{array}{l} t=0 \\ T=24 \end{array}\right\} \Rightarrow 24 = 5 + Ae^{\cancel{0}\,1}$

$A = 19$

$\therefore T = 5 + 19e^{kt}$

$\left.\begin{array}{l} t=5 \\ T=18 \end{array}\right\} \Rightarrow 18 = 5 + 19e^{5k}$

$\frac{13}{19} = e^{5k}$

$k = \frac{1}{5}\ln\frac{13}{19}$

$\therefore T = 5 + 19e^{\frac{t}{5}\ln\frac{13}{19}}$

$\left.\begin{array}{l} T=10 \\ t=? \end{array}\right\} \quad 10 = 5 + 19e^{\frac{t}{5}\ln\frac{13}{19}}$

$\frac{5}{19} = e^{\frac{t}{5}\ln\frac{13}{19}}$

$t\ln\frac{13}{19} = 5\ln\frac{5}{19}$

$t = \dfrac{5\ln\frac{5}{19}}{\ln\frac{13}{19}}$

$= 18$ min (nearest min)

Question 13:

(a)

$$\binom{2022}{146}+\binom{2022}{147}+\binom{2023}{1875}$$

$$=\binom{2022}{146}+\binom{2022}{147}+\binom{2023}{148}$$

$$=\binom{2023}{147}+\binom{2023}{148}$$

$$=\binom{2024}{148}$$

$s = 2024$ 1 mark for s

$t = 148$ 1 mark for t

(b) (i) $n=30$, $p=q=\frac{1}{2}$

$np = 15 > 10$ and $nq = 15 > 10$

$\therefore$ distribution can be approximated using the normal distribution.

1 mark for np and $nq > 10$

(ii) $\mu = \frac{1}{2}$

$$\sigma = \sqrt{\frac{(\frac{1}{2})(\frac{1}{2})}{30}} = \sqrt{\frac{1/4}{30}} = \sqrt{\frac{1}{120}} = \frac{1}{2\sqrt{30}}$$

one mark for mean and standard deviation

$$\hat{p} = \frac{12}{30} \Rightarrow z = \frac{\frac{12}{30}-\frac{15}{30}}{\frac{1}{2\sqrt{30}}} = \frac{-\frac{1}{10}}{2\sqrt{30}} = \frac{-2\sqrt{30}}{10} = -1.095$$

$$\hat{p} = \frac{16}{30} \Rightarrow z = \frac{\frac{16}{30}-\frac{15}{30}}{\frac{1}{2\sqrt{30}}} = \frac{\frac{1}{30}}{2\sqrt{30}} = \frac{2}{\sqrt{30}} = 0.365$$

1 mark for finding the z-scores

$$P\left(\frac{12}{30} \le \hat{p} \le \frac{16}{30}\right) \doteq P(-1.095 \le z \le 0.365)$$

$$\doteq 0.6443 - 0.1357$$

$$= 0.5086$$

1 mark for correct working and correct final answer

(c) $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 $n \ge 1$

For $n=1$, $1^3 + (1+1)^3 + (1+2)^3$

$= 1 + 2^3 + 3^3$

$= 1 + 8 + 27$

$= 36 = 9 \times 4$ $\therefore$ divisible by 9

$\therefore$ True for $n=1$.

This mark for the base case

Assume it is true for $n = k$, $k \geq 1$, $k \in \mathbb{Z}$

$k^3 + (k+1)^3 + (k+2)^3$ is divisible by 9

$\therefore \exists\, m \in \mathbb{Z} \therefore k^3 + (k+1)^3 + (k+2)^3 = 9m$, $m \geq 1$, $m \in \mathbb{Z}$ $(*)$

prove it true for $n = k+1$

$(k+1)^3 + (k+2)^3 + (k+3)^3$ is divisible by 9

$$
\begin{aligned}
&(k+1)^3 + (k+2)^3 + (k+3)^3 \\
&= 9m - k^3 + (k+3)^3 \quad \text{using } (*) \\
&= 9m + (k+3)^3 - k^3 \\
&= 9m + (k+3-k)\left((k+3)^2 + (k+3)k + k^2\right) \\
&= 9m + 3\left(k^2 + 6k + 9 + k^2 + 3k + k^2\right) \\
&= 9m + 3(3k^2 + 9k + 9) \\
&= 9m + 9(k^2 + 3k + 3) \\
&= 9\underbrace{(m + k^2 + 3k + 3)} \\
&= 9L, \text{ where } L = m + k^2 + 3k + 3 \in \mathbb{Z}.
\end{aligned}
$$

$\therefore$ divisible by 9

This mark for using the assumption

To obtain this mark, you need to state that $m + k^2 + 3k + 3 \in \mathbb{Z}$

By mathematical induction,

$n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9

since it is true for $n=1$, Assume it is true for $n=k$ and proven it is true for $n=k+1$.

Question 14

a)

i.

$\underset{\sim}{p} = \cos\alpha\,\underset{\sim}{i} + \sin\alpha\,\underset{\sim}{j}$ or $\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}$ 1 mark

$\underset{\sim}{q} = \cos\beta\,\underset{\sim}{i} - \sin\beta\,\underset{\sim}{j}$ or $\begin{pmatrix}\cos\beta\\ -\sin\beta\end{pmatrix}$ 1 mark

ii.

$\underset{\sim}{p}\cdot\underset{\sim}{q} = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ 1 mark

$\underset{\sim}{p}\cdot\underset{\sim}{q} = \left|\underset{\sim}{p}\right|\left|\underset{\sim}{q}\right|\cos(\alpha+\beta)$

$= \cos(\alpha+\beta) \quad \left(\left|\underset{\sim}{p}\right| = \left|\underset{\sim}{q}\right| = 1\right)$ This must be acknowledged to get the mark!!!!

$\therefore \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ 1 mark

b)

$\theta \in [0, 2\pi]$

$\theta \in \left[0, \frac{\pi}{4}\right]$

let $t = \tan\frac{\theta}{4}$ 1 mark

$3\left(\frac{1-t^2}{1+t^2}\right) - \frac{2t}{1+t^2} = -3$

$3 - 3t^2 - 2t = -3 - 3t^2$

$2t = 6$

$t = 3$ 1 mark

$\tan\frac{\theta}{4} = 3$

$\frac{\theta}{4} \approx 1.24$

$\theta \approx 5.00$ 1 mark

Test for $\theta = 2\pi$

$\text{LHS} = 3\cos\pi - \sin\pi$

$= 3(-1) - 0$

$= -3$

$= \text{RHS}$

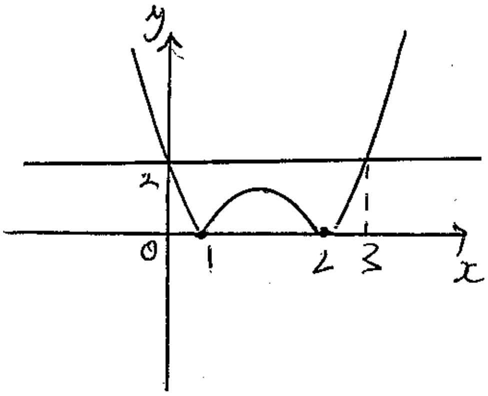
$\therefore \theta = 5.00$ or 2π 1 mark

Note:

– If you make the wrong substitution at the beginning, the maximum you can get is 2 marks

– If you leave your answer in terms of $4\tan^{-1}3$, you do not receive the maximum marks.

2024 Y12 Extension 1, Task 4 (Trial) Question 15

Suggested Solutions	Marks	Marker's Comments
Question 15 (a) (i)  (ii) $x < 0$ or $x > 3$		Aw 1. $y = 2$ and correct intercepts of transformed parabola. Aw2 correct solution Aw1 correct solution
(b) (i) $\sin 6\theta = \sin(4\theta + 2\theta)$ $= \sin 4\theta \cos 2\theta + \cos 4\theta \sin 2\theta$ $= 2\sin 2\theta \cos^2 2\theta + (1 - 2\sin^2 2\theta)\sin 2\theta$ $= 2\sin 2\theta(1 - \sin^2 2\theta) + \sin 2\theta - 2\sin^3 2\theta$ $= 3\sin 2\theta - 4\sin^3 2\theta$ $\therefore$ L.H.S $= (\sin 2\theta)^3 - \frac{3}{4}\sin 2\theta + \frac{1}{4}\sin 6\theta$ $= \sin^3 2\theta - \frac{3}{4}\sin 2\theta + \frac{1}{4}(3\sin 2\theta - 4\sin^3 2\theta)$ $= \cancel{\sin^3 2\theta} - \cancel{\frac{3}{4}\sin 2\theta} + \cancel{\frac{3}{4}\sin 2\theta} - \cancel{\sin^3 2\theta}$ $= 0$ as required.		Aw1 correct use of compound angle formula for $\sin(4\theta + 2\theta)$ or correct double angle formula for $\sin(2 \times 3\theta)$ Aw2 Correct progress to the result required.
ii) $x^3 - 12x + 8 = 0$ and $x = 4\sin 2\theta$ $\therefore 4^3\sin^3 2\theta - 48\sin 2\theta + 8 = 0$ $\div 64$: $\sin^3 2\theta - \frac{3}{4}\sin 2\theta + \frac{1}{8} = 0$ $\therefore -\frac{1}{4}\sin 6\theta = -\frac{1}{8}$ $\therefore \sin 6\theta = \frac{1}{2}$		Aw1 correct substitution and simplification from part i).

2024 Y12 Extension 1, Task 4 (Trial) Question 15

Suggested Solutions	Marks	Marker's Comments
(iii) If $\sin 6\theta = \frac{1}{2}$ $6\theta = n\pi + (-1)^n \cdot \sin^{-1}(\frac{1}{2}) \quad n \in \mathbb{Z}$ $\therefore 2\theta = n\frac{\pi}{3} + (-1)^n \cdot \frac{\pi}{18}$ $= (6n + (-1)^n) \cdot \frac{\pi}{18}$ 3 Distinct solutions when $n = 0, 1, 4$ (all others are repititions) i.e. $x = 4\sin\frac{\pi}{18}, 4\sin\frac{13\pi}{18}$ and $4\sin\frac{25\pi}{18}$		AW1. Finding the distinct roots of the given cubic via solving $\sin 6\theta = \frac{1}{2}$
$\Sigma\alpha$: $4\sin\frac{\pi}{18} + 4\sin\frac{13\pi}{18} + 4\sin\frac{25\pi}{18} = -\frac{b}{a}$ $= \frac{-0}{1}$ $= 0$ $\div 4$: $\therefore \sin\frac{\pi}{18} + \sin\frac{13\pi}{18} + \sin\frac{25\pi}{18} = 0$ $\Sigma\alpha\beta$: $16\sin\frac{\pi}{18}\sin\frac{13\pi}{18} + 16\sin\frac{\pi}{18}\sin\frac{25\pi}{18} + 16\sin\frac{13\pi}{18}\sin\frac{25\pi}{18} = \frac{-12}{1}$ $\therefore \sin\frac{\pi}{18}\sin\frac{13\pi}{18} + \sin\frac{\pi}{18}\sin\frac{25\pi}{18} + \sin\frac{13\pi}{18}\sin\frac{25\pi}{18}$ $= \frac{-12}{16}$ $= -\frac{3}{4}$		AW2 find expressions for both the sum of roots <u>and</u> sum of roots taken two at time.
$\therefore \sin^2\frac{\pi}{18} + \sin^2\frac{13\pi}{18} + \sin^2\frac{25\pi}{18}$ $= \left(\sin\frac{\pi}{18} + \sin\frac{13\pi}{18} + \sin\frac{25\pi}{18}\right)^2$ $- 2\left(\sin\frac{\pi}{18}\sin\frac{13\pi}{18} + \sin\frac{\pi}{18}\sin\frac{25\pi}{18} + \sin\frac{13\pi}{18}\sin\frac{25\pi}{18}\right)$ $= 0^2 - 2(-\frac{3}{4})$ $= \frac{3}{2}$		AW3 Correct solution. (must use part ii)

Trials 2024 - Ext 1 - Q16

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments

a) i) Show that at time t seconds, the ht h cm of liquid in the cylinder satisfies the differential eqn.

$\frac{dV}{dt}_{in} = 1600\ cm^3/s$ poured in. $A = 4000\ cm^2$

Water leaking out $\propto \sqrt{h}$ (direct variation)

$\therefore \frac{dV}{dt}_{out} = k\sqrt{h}$

Show $\frac{dh}{dt} = ?$

$V = \pi r^2 h$

$\frac{dV}{dh} = \pi r^2 = 4000\ cm^2$

$\frac{dh}{dt}$ = water in - water out

$= 1600 - k\sqrt{h}$

$\frac{dV}{dt} = \frac{dh}{dt} \times \frac{dV}{dh}$

$\frac{dV}{dt} = (1600 - k\sqrt{h}) \times 4000$

$= \left(0.4 - \frac{k}{4000}\sqrt{h}\right)$

$= 0.4 - c\sqrt{h}$ where $c = \frac{k}{4000}$

ii) Show $c = 0.02$

($h = 25$ $\frac{dh}{dt}_{out} = 400\ cm^3/s$) given

$\frac{dV}{dt} = V_{in} - V_{out}$

$= 1600 - 400$

$= 1200\ cm^3/s$

$\frac{dh}{dV} = \frac{1}{4000}$

$\therefore \frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$= 1200 \times \frac{1}{4000}$

$= 0.3$

Marker's Comments

Poorly done!

Many students did $\frac{dV}{dt} = \frac{1600}{4000} = \frac{cm^3/s}{cm^2} = 0.4$

$\therefore 0.4 - c\sqrt{h}$

→ unless you show $c = \frac{k}{4000}$ and chain rule of something, no marks awarded

incorrect { ✗ $\frac{dh}{dt} = \frac{\text{amt of water in}}{\text{area of base}}$

✗ change of dt in water $\frac{1}{A} \times 1600$

Also poorly done! Using the same concept from (ai) Difficult decision ∴ but I gave CFE unless you just can't do the maths for this question!!

(It's related rate of change !!!)

you have to show. NOT!! just state 0.3 Many students wrote 0.3. No explanation

But still awarded mark because you have done the same mistake as pt(i)

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments

mistakes or fudging for a ii) that was awarded CFE mark.

1) $c \times 4000\sqrt{25} = 400$

2) $c\sqrt{h} = \dfrac{400}{4000}$

$5c = 0.1$

3) $\dfrac{dh}{dt} = 0.4 - c\sqrt{h}$

$0.3 = 0.4 - c\sqrt{25}$

NOT AWARDED MARKS

1) $\dfrac{dh}{dt} = \dfrac{1}{10h}$

$\dfrac{1}{10(25)} = 0.4 - c\sqrt{25} = 0.08$

2) $c = \dfrac{k}{4000}$

$1600 - k\sqrt{25} = 400$

$k = 240$

$(c = 0.02)$ somehow got 0.02

3) $400 = 1600\, c\sqrt{25} = 0.05$

4) $\dfrac{400 - 1600}{-5} = k$

$c = \dfrac{80}{4000}$ which gives 0.02 but not sure how k becomes c

5) $c = \dfrac{400 - 0.4}{-5}$ $= -79.92$

6) $-0.1 = 0.4 - 5c$

$5c = 0.5$

$c = \dfrac{0.5}{0.25} = 2$

7) $400 = 4000(0.4 - c\sqrt{h}) = 0.1$

8) $400 = 0.4 - 5c = -79.92$ $c = \dfrac{k}{4000}$

9) $c\sqrt{h} = 400 = 80$

10) $\dfrac{400 - 1600}{-5} = k$ gives 240

$c = \dfrac{80}{4000}$ not sure

11) winner of fudging 🙇🤦

$4000 \times 0.4 - 5c = 1600 - 4000c \times 5$ (if solved, $c = 0.02$)

$1600 - 5c = 1600 - 20000c$

$0 = -20000c + 5c$ Really?? mind blowing!!

MATHEMATICS Extension 1 : Question……..

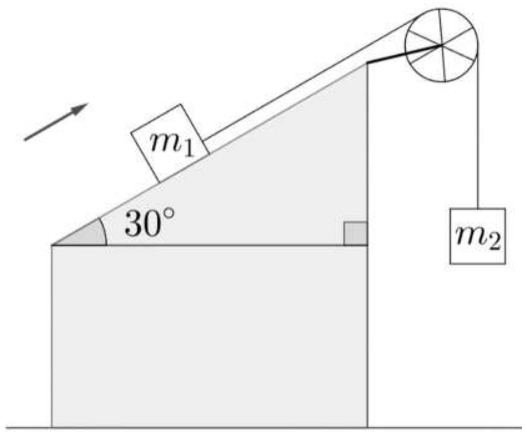
Suggested Solutions	Marks	Marker's Comments
a iii) Show $\int_0^{100} \frac{50}{20-\sqrt{h}}\,dh$		
$\frac{dh}{dt} = 0.4 - 0.02\sqrt{h}$	ie	$h = 100,\ t = T$
$dt = \frac{1}{0.4 - 0.02\sqrt{h}}\,dh$		$h = 0,\ t = 0$
$\int dt = \int \frac{1}{0.4 - 0.02\sqrt{h}}\,dh$		
$\int_0^{t/T} dt = \int_0^{100} \frac{50}{20-\sqrt{h}}\,dh$		have to show this to get 1 mark
$T/t = \int_0^{100} \frac{50}{20-\sqrt{h}}\,dh$		
a iv) use $\sqrt{h} = 20 - x$ or otherwise, to evaluate the integral in (iii) (nearest second)		
$\int_{20}^{10} \frac{50}{20-(20-x)} \times (-2(20-x))\,dx$ $\sqrt{h} = 20 - x$ $x = 20 - \sqrt{h}$ $\frac{dx}{dh} = -\frac{1}{2}h^{-1/2}$		when $h = 100,\ x = 10$; $h = 0,\ x = 20$ (1)
$\int_{20}^{10} \frac{50(2x-40)}{x}\,dx$ $= \frac{-1}{2\sqrt{h}} = \frac{-1}{2\sqrt{20-x}}$		
$= 50\int_{20}^{10} 2 - \frac{40}{x}\,dx$ $\therefore dh = -2(20-x)\,dx$		
$= (386.29 \text{ secs})$ Not many students got this!		Therefore awarded the final mark of getting 386 seconds.

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments
6) Using product to sum identity, find $\int \cos 2x \sin 3x \, dx$		The 1st mark was awarded if the product to sum was done correctly & show the next step of integration with plus a constant.
$= \frac{1}{2}\int \sin 5x - \sin(-x) \, dx$		
$= \frac{1}{2}\int \sin 5x + \sin x \, dx$		
$= \frac{1}{2}\left[-\frac{1}{5}\cos 5x - \cos x\right] + C$		
$= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$		1

<table>
<tr><th colspan="3">Extension 1, Trial HSC, 2024: Question 17</th></tr>
<tr><th>Suggested Solutions</th><th>Marks</th><th>Marker’s Comments</th></tr>
<tr><td>

(a)

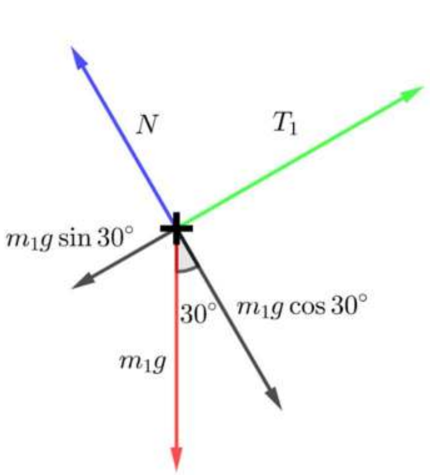


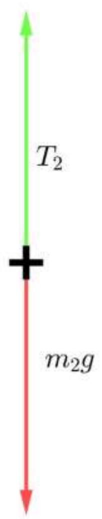
On m_1, we have:

- Tension
- Normal reaction force
- Gravitational effect

On m_2, we have :

- Tension
- Gravitational effect





For mass 1, resolve the force basis vectors as:

(I) parallel to incline, positive direction taken from mass 1 to pulley;

(II) perpendicular to incline, first basis vector rotated 90 degrees anticlockwise.

By Newton’s Second Law, resolving forces gives:

$$T_1 - m_1 g \sin 30° = m_1 a_{1_{\text{horiz}}} \quad ...(1)$$
$$N - m_1 g \cos 30° = m_1 a_{1\text{ver}} = 0 \rightarrow a_{1\text{vert}} = 0$$

where the second force resolution is zero since mass 1 remains on the surface.

</td><td>1</td><td>

First mark for at least one pair of resolved equations of motion correct.

Second mark for full solution. Full award also given to those who correctly solved the problem but did not substitute values for masses (they worked the problem correctly and gave the acceleration in terms of g).

Assumptions that were accepted:

(1) $T_1 = T_2 := T$

(2) Magnitude of accelerations in both free-body systems was the same.

That said, you should try to justify why these are the case.

Problems: Many students assumed the system was static, in least with respect to mass 2, and calculated tension as $T = m_2 g$. This

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For mass 2, basis vectors for forces will be as:

(I) Positive horizontal, left-to-right.
(II) Positive vertical, downward (since we are anticipating a certain direction given the setup, but it is not critical to make the downward direction positive).

Then, resolving forces on mass 2:

$$m_2 g - T_2 = m_2 a_{2\mathrm{ve}} \quad \ldots(2)$$
$$0 = m_2 a_{2_{\mathrm{horiz}}} \rightarrow a_{2\mathrm{horiz}} = 0$$

Now, since the cable is inextensible, the tension T_1 on mass 1 exerted by the cable is the same (in magnitude) as that exerted on mass 2, T_2. Hence

$$T_1 = T_2$$

and the accelerations (their magnitudes) of the masses must be the same (if not, the cable would compress or break, depending on which mass has the greater acceleration). This implies $a_{1_{\mathrm{horiz}}} = a_{2\mathrm{vert}} := a$.

Hence, we have from (1) and (2):

$$T - \frac{1}{2} m_1 g = m_1 a \quad \ldots(3)$$
$$m_2 g - T = m_2 a \quad \ldots(4)$$

Add (3) and (4):

$$\left(m_2 - \frac{1}{2} m_1\right) g = (m_1 + m_2) a$$

and so

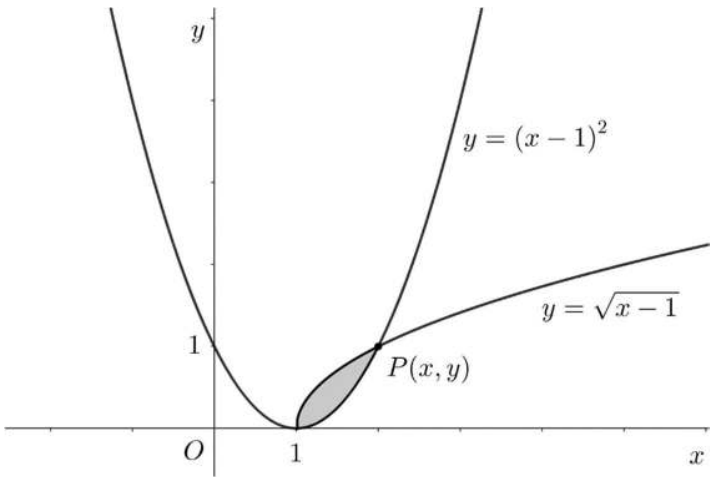
$$a = \frac{(2m_2 - m_1)}{2(m_1 + m_2)} g = \frac{11}{26} g$$

1

given $m_1 = 2500$ kg, $m_2 = 4000$ kg.

tension was then used for m_1 in a situation where a static system was not assumed…this is inconsistent with the information in the problem. It is also wrong to assume special physical situations without any information (nothing in the question implied that either of the masses were static relative to the incline, just the opposite).

Others established vectors on the free bodies of differing bases and then added those vectors. You can only add components of vectors if those vectors have the same basis.

(b) 		
(i) We see that $P(2,1)$ is the point of intersection (this did not have to be proved, but can be verified easily by substituting the coordinates into each relation).	1	Single mark available.
(ii) From $y_1 = (x_1 - 1)^2 \rightarrow x_1 = 1 \pm \sqrt{y_1}$. Now, $x_1 \geq 1$ and we only want a non-negative output x_1 for a given y_1, so we must take $x_1 = 1 + \sqrt{y_1}$. Next, $y_2 = \sqrt{x_2 - 1} \rightarrow x_2 = 1 + y_2^2$ is the second curve wanted. The volume about the y-axis is: $\int_0^1 \pi x_1^2 dy - \int_0^1 \pi x_2^2 dy = \pi \int_0^1 (x_1^2 - x_2^2) dy$ $= \pi \int_0^1 \left(1 + \sqrt{y}\right)^2 - (1 + y^2)^2 dy$ $= \pi \int_0^1 2y^{\frac{1}{2}} + y - 2y^2 - y^4 \, dy$ $= \dfrac{29\pi}{30}$ u^3	1 1	First mark for <u>establishing</u> integral correctly. Second mark for the evaluation. Common error: $\pi \int_0^1 (x_1 - x_2)^2 dx$ This integral does <u>not</u> give the volume wanted. We are effectively integrating ‘washers’ - slices of circular cylinders of ‘infinitesimal’ height. Removing a smaller cylinder from a larger one means calculating something of the order of $\pi R^2 \Delta y - \pi r^2 \Delta y$ where $r < R$. Many students did not integrate about the vertical axis, instead formulating a volume of revolution integral about the x-axis. No

Solutions continue, next page…		marks were awarded in this case for the following two reasons: (1) There was a fundamental misconception about what integration would give the appropriate volume. (2) There was no need to interpret the problem from the perspective of another axis and ∴ no need to conduct the extra work needed to express x as a function of y. It can also be argued that the integration is marginally simpler. Although you need to perform the integration to find the volume, you're not being assessed on the integration itself; the integral here, after correct setup, is a 2 Unit integral.

(c) We are required to prove that $$\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right) = 3\tan^{-1}\left(\frac{x}{a}\right)$$ where $a > 0, -\frac{a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}\cdot$ The equation suggests it's easiest to let $\theta = \tan^{-1}\left(\frac{x}{a}\right)$ and to then manipulate to see where we arrive. The problem also requires us to look at $3\tan^{-1}\frac{x}{a}$, so $3\tan^{-1}\frac{x}{a} = 3\theta$. Consider then $$\tan(3\theta) = \frac{(\tan(2\theta) + \tan\theta)}{1 - \tan(2\theta)\tan\theta}$$ Let $t = \tan\theta$. Then $$\tan(3\theta) = \frac{\frac{2t}{1-t^2} + t}{1 - \left(\frac{2t}{1-t^2}\right)t} = \frac{3t - t^3}{1 - 3t^2} \quad \dots (1)$$ Now, $$t = \tan\theta = \tan\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{x}{a}$$ so, substituting into (1), we have $$\tan(3\theta) = \frac{3\frac{x}{a} - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2} = \frac{3a^2x - x^3}{a^3 - 3ax^2}$$ Now, in order to release 3θ from tan by applying arctan directly, we need to ensure that $-\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$. We have: $$-\frac{a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}} \quad (\because a > 0, \text{direction of inequality maintained})$$ $$\Rightarrow \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) < \tan^{-1}\left(\frac{x}{a}\right) < \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad (\because \text{arctan is monotonic increasing on its domain})$$ $$\Rightarrow -\frac{\pi}{6} < \tan^{-1}\left(\frac{x}{a}\right) < \frac{\pi}{6}$$ $$\Rightarrow -\frac{\pi}{2} < 3\tan^{-1}\left(\frac{x}{a}\right) < \frac{\pi}{2}$$ $$\Leftrightarrow -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$$	1 1	First mark for logical, productive start. This typically included picking one side of the identity to be proved and working with tangent + compound angle formula. Second mark for proving the relation or equivalent. The third mark required more than simply applying arctan to both sides of the last expression. Candidates needed to prove that $3\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ in order for it to follow that $\tan^{-1}(\tan 3\theta) = 3\theta$. For example, although it is the case that $$\tan\frac{\pi}{3} = \tan\frac{4\pi}{3}$$ it is **not** the case that $$\tan^{-1}\tan\frac{\pi}{3} = \tan^{-1}\tan\frac{4\pi}{3}$$ implies

Hence we may write: $$\tan^{-1}(\tan(3\theta)) = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$$ $$\Leftrightarrow 3\theta = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$$ $$\Leftrightarrow 3\tan^{-1}\left(\frac{x}{a}\right) = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$$ □	1	$$\frac{\pi}{3} = \frac{4\pi}{3}$$ Hence, students must have used the information about a to **prove** that $$-\frac{\pi}{2} < 3\tan^{-1}\frac{x}{a} < \frac{\pi}{2}$$ **follows**, permissioning the final moves here.