NSW Education Standards Authority

Centre Number

Student Number

2023 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 70

Section I – 10 marks (pages 2–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8–16)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 The temperature $T(t)$°C of an object at time t seconds is modelled using Newton's Law of Cooling,

$$T(t) = 15 + 4e^{-3t}.$$

What is the initial temperature of the object?

A. -3

B. 4

C. 15

D. 19

2 A standard six-sided die is rolled 12 times.

Let $\hat{p}$ be the proportion of the rolls with an outcome of 2.

Which of the following expressions is the probability that at least 9 of the rolls have an outcome of 2?

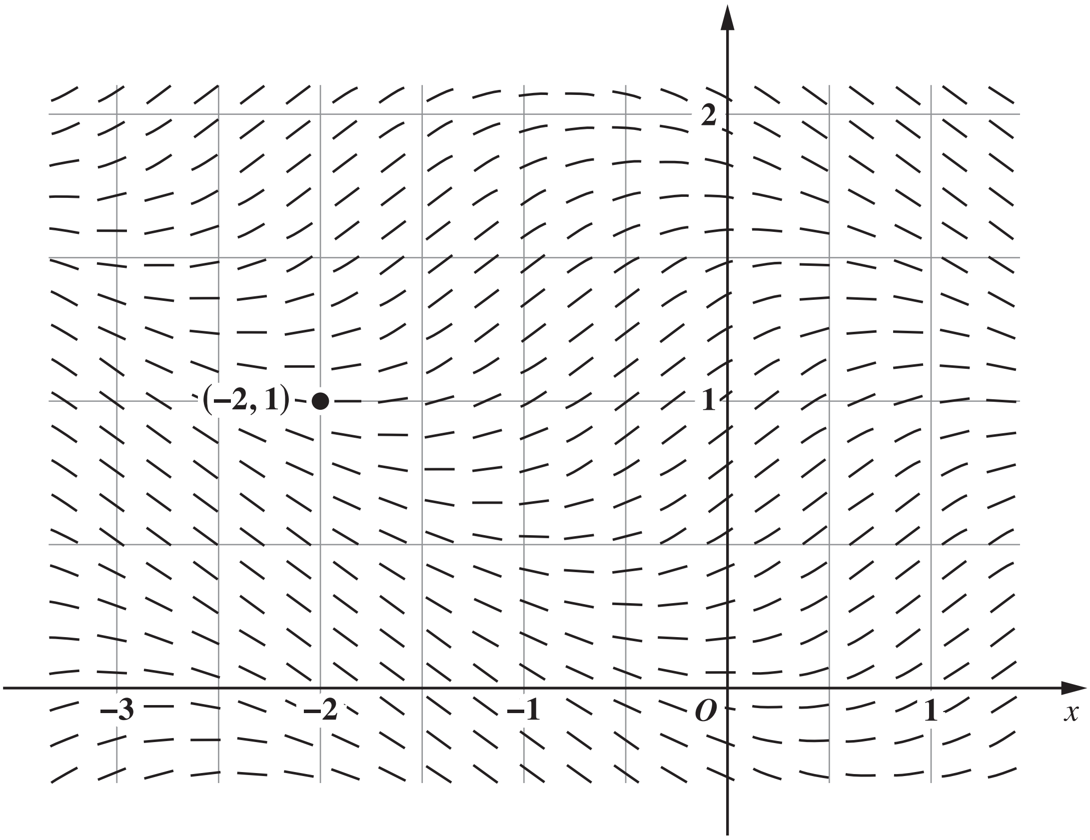
A. $P\left(\hat{p} \geq \frac{3}{4}\right)$

B. $P\left(\hat{p} \geq \frac{1}{6}\right)$

C. $P\left(\hat{p} \leq \frac{3}{4}\right)$

D. $P\left(\hat{p} \leq \frac{1}{6}\right)$

3 The diagram shows the direction field of a differential equation. A particular solution to the differential equation passes through $(-2, 1)$.



Where does the solution that passes through $(-2, 1)$ cross the y-axis?

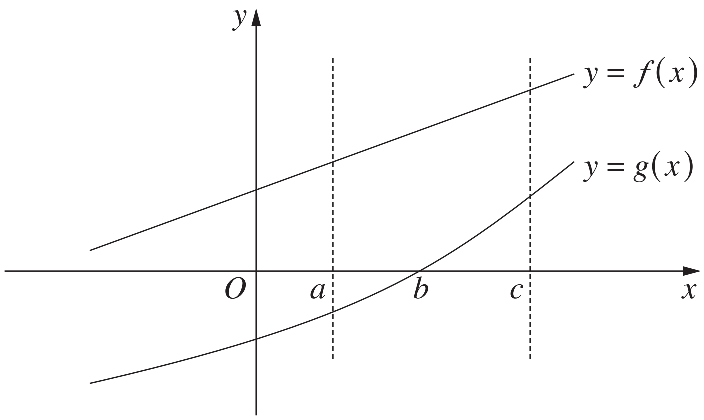
A. $y = 1.12$

B. $y = 1.34$

C. $y = 1.56$

D. $y = 1.78$

4 The diagram shows the graphs of the functions $f(x)$ and $g(x)$.



It is known that

$$\int_a^c f(x)\,dx = 10$$

$$\int_a^b g(x)\,dx = -2$$

$$\int_b^c g(x)\,dx = 3\,.$$

What is the area between the curves $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = c$?

A. 5

B. 7

C. 9

D. 11

5 Which of the following is the value of $\sin^{-1}(\sin a)$ given that $\pi < a < \frac{3\pi}{2}$?

A. $a - \pi$

B. $\pi - a$

C. a

D. $-a$

6 Given the two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, let $\underset{\sim}{c}$ be the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

What is the projection of $10\underset{\sim}{a}$ onto $2\underset{\sim}{b}$?

A. $2\underset{\sim}{c}$

B. $5\underset{\sim}{c}$

C. $10\underset{\sim}{c}$

D. $20\underset{\sim}{c}$

7 Which statement is always true for real numbers a and b where $-1 \leq a < b \leq 1$?

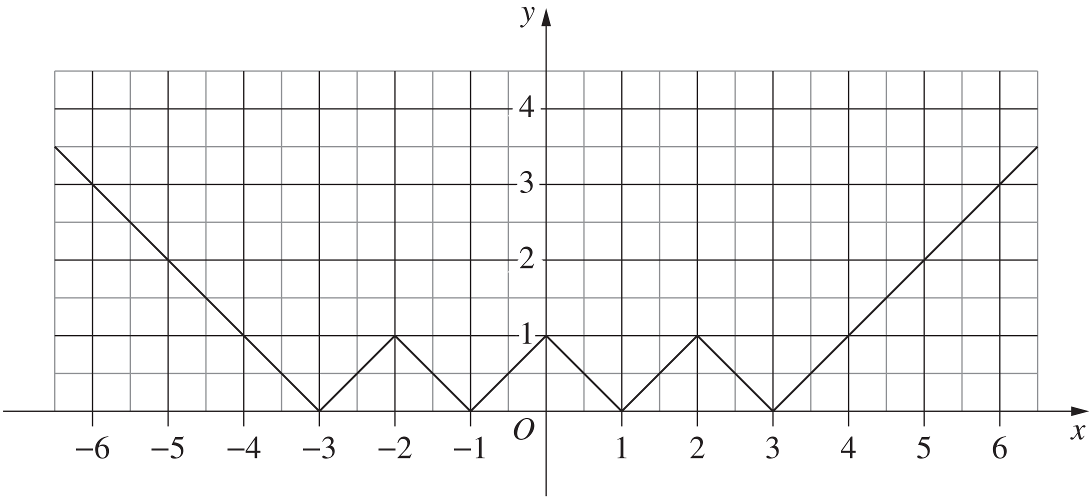
A. $\sec a < \sec b$

B. $\sin^{-1} a < \sin^{-1} b$

C. $\arccos a < \arccos b$

D. $\cos^{-1} a + \sin^{-1} a < \cos^{-1} b + \sin^{-1} b$

8 The diagram shows the graph of a function.



Which of the following is the equation of the function?

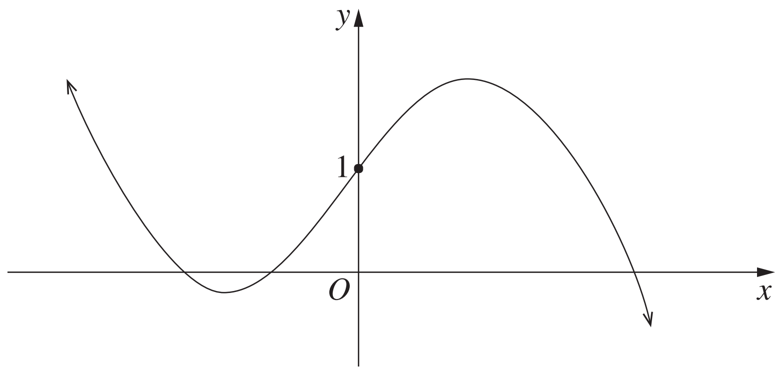
A. $y = \left|1 - \left||x| - 2\right|\right|$

B. $y = \left|2 - \left||x| - 1\right|\right|$

C. $y = \left|1 - |x - 2|\right|$

D. $y = \left|2 - |x - 1|\right|$

9 The graph of a cubic function, $y = f(x)$, is given below.



Which of the following functions has an inverse relation whose graph has more than 3 points with an x-coordinate of 1?

A. $y = \sqrt{f(x)}$

B. $y = \dfrac{1}{f(x)}$

C. $y = f(|x|)$

D. $y = |f(x)|$

10 A group with 5 students and 3 teachers is to be arranged in a circle.

In how many ways can this be done if no more than 2 students can sit together?

A. $4! \times 3!$

B. $5! \times 3!$

C. $2! \times 5! \times 3!$

D. $2! \times 2! \times 2! \times 3!$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) The parametric equations of a line are given below. **2**

$$x = 1 + 3t$$
$$y = 4t$$

Find the Cartesian equation of this line in the form $y = mx + c$.

(b) In how many different ways can all the letters of the word CONDOBOLIN be arranged in a line? **2**

(c) Consider the polynomial **3**

$$P(x) = x^3 + ax^2 + bx - 12,$$

where a and b are real numbers.

It is given that $x + 1$ is a factor of $P(x)$ and that, when $P(x)$ is divided by $x - 2$, the remainder is -18.

Find a and b.

Question 11 continues on page 9

(d) Find $\int \frac{1}{\sqrt{4-9x^2}}\,dx$. **2**

(e) Solve $\cos\theta + \sin\theta = 1$ for $0 \le \theta \le 2\pi$. **3**

(f) A recent census found that 30% of Australians were born overseas.

A sample of 900 randomly selected Australians was surveyed.

Let $\hat{p}$ be the sample proportion of surveyed people who were born overseas.

A normal distribution is to be used to approximate $P(\hat{p} \le 0.31)$.

(i) Show that the variance of the random variable $\hat{p}$ is $\frac{7}{30\,000}$. **2**

(ii) Use the standard normal distribution and the information on page 16 to approximate $P(\hat{p} \le 0.31)$, giving your answer correct to two decimal places. **2**

End of Question 11

Please turn over

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Evaluate $\int_3^4 (x+2)\sqrt{x-3}\,dx$ using the substitution $u = x - 3$. **3**

(b) Use mathematical induction to prove that **3**

$$\left(1 \times 2\right) + \left(2 \times 2^2\right) + \left(3 \times 2^3\right) + \cdots + \left(n \times 2^n\right) = 2 + (n-1)2^{n+1}$$

for all integers $n \geq 1$.

(c) A gym has 9 pieces of equipment: 5 treadmills and 4 rowing machines.

On average, each treadmill is used 65% of the time and each rowing machine is used 40% of the time.

(i) Find an expression for the probability that, at a particular time, exactly 3 of the 5 treadmills are in use. **2**

(ii) Find an expression for the probability that, at a particular time, exactly 3 of the 5 treadmills are in use and no rowing machines are in use. **1**

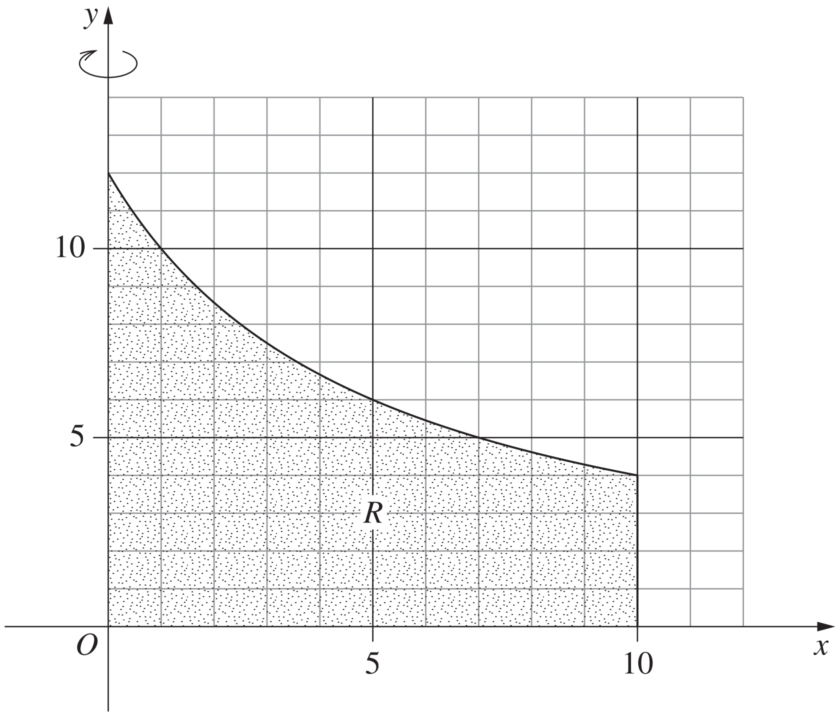
(d) It is known that ${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$ for all integers such that $1 \leq r \leq n - 1$. (Do NOT prove this.) **2**

Find ONE possible set of values for p and q such that

$${}^{2022}C_{80} + {}^{2022}C_{81} + {}^{2023}C_{1943} = {}^{p}C_{q}.$$

Question 12 continues on page 11

(e) The region, R, bounded by the hyperbola $y = \dfrac{60}{x+5}$, the line $x = 10$ and the coordinate axes is shown. **4**



Find the volume of the solid of revolution formed when the region R is rotated about the y-axis. Leave your answer in exact form.

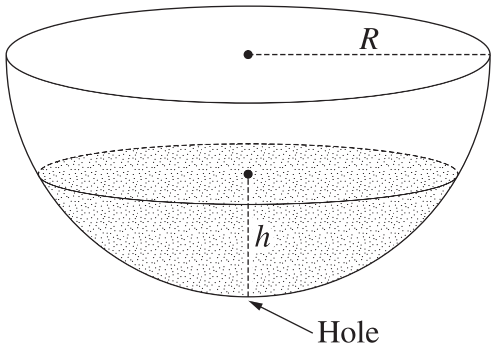
End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) A hemispherical water tank has radius R cm. The tank has a hole at the bottom which allows water to drain out.

Initially the tank is empty. Water is poured into the tank at a constant rate of $2kR\ \text{cm}^3\ \text{s}^{-1}$, where k is a positive constant.

After t seconds, the height of the water in the tank is h cm, as shown in the diagram, and the volume of water in the tank is $V\ \text{cm}^3$.



It is known that $V = \pi\left(Rh^2 - \dfrac{h^3}{3}\right)$. (Do NOT prove this.)

While water flows into the tank and also drains out of the bottom, the rate of change of the volume of water in the tank is given by $\dfrac{dV}{dt} = k(2R - h)$.

(i) Show that $\dfrac{dh}{dt} = \dfrac{k}{\pi h}$. **2**

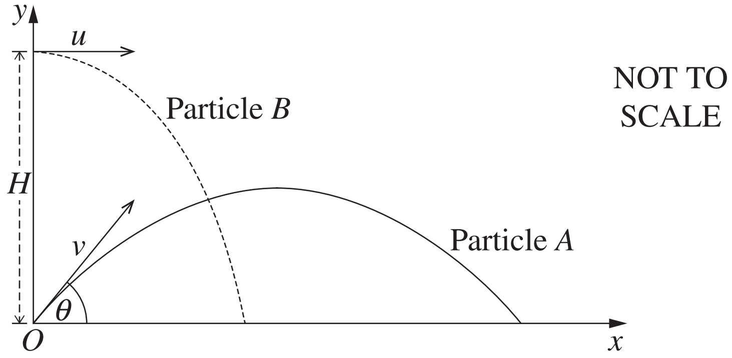
(ii) Show that the tank is full of water after $T = \dfrac{\pi R^2}{2k}$ seconds. **2**

(iii) The instant the tank is full, water stops flowing into the tank, but it continues to drain out of the hole at the bottom as before. **3**

Show that the tank takes 3 times as long to empty as it did to fill.

Question 13 continues on page 13

(b) Particle A is projected from the origin with initial speed v m s^{-1} at an angle θ with the horizontal plane. At the same time, particle B is projected horizontally with initial speed u m s^{-1} from a point that is H metres above the origin, as shown in the diagram.



The position vector of particle A, t seconds after it is projected, is given by

$$\mathbf{r}_A(t) = \begin{pmatrix} vt\cos\theta \\ vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

The position vector of particle B, t seconds after it is projected, is given by

$$\mathbf{r}_B(t) = \begin{pmatrix} ut \\ H - \frac{1}{2}gt^2 \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

The angle θ is chosen so that $\tan\theta = 2$.

The two particles collide.

(i) By first showing that $\cos\theta = \dfrac{1}{\sqrt{5}}$, verify that $v = \sqrt{5}u$. **2**

(ii) Show that the particles collide at time $T = \dfrac{H}{2u}$. **1**

When the particles collide, their velocity vectors are perpendicular.

(iii) Show that $H = \dfrac{2u^2}{g}$. **3**

(iv) Prior to the collision, the trajectory of particle A was a parabola. (Do NOT prove this.) **2**

Find the height of the vertex of that parabola above the horizontal plane. Give your answer in terms of H.

End of Question 13

Question 14 (14 marks) Use the Question 14 Writing Booklet

(a) Let $f(x) = 2x + \ln x$, for $x > 0$.

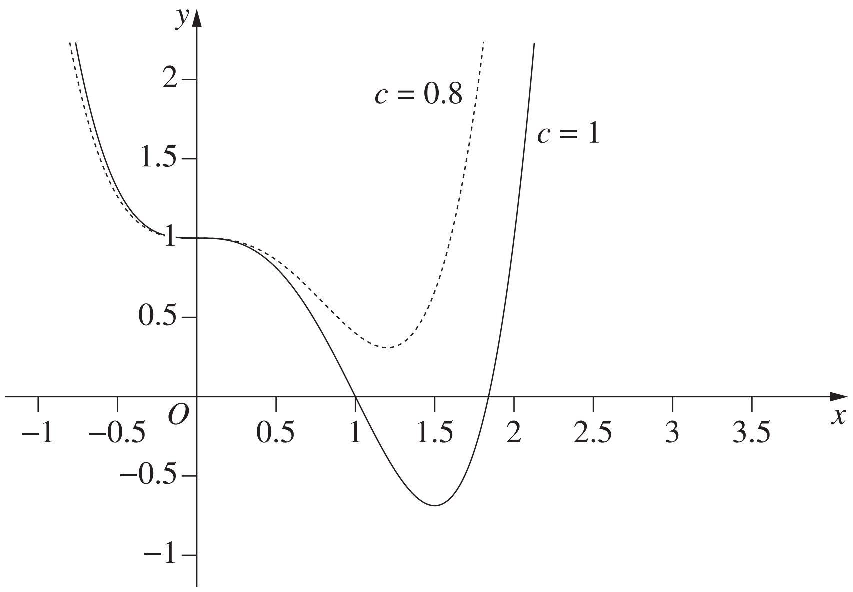
(i) Explain why the inverse of $f(x)$ is a function. **1**

(ii) Let $g(x) = f^{-1}(x)$. By considering the value of $f(1)$, or otherwise, evaluate $g'(2)$. **2**

(b) Consider the hyperbola $y = \dfrac{1}{x}$ and the circle $(x - c)^2 + y^2 = c^2$, where c is a constant.

(i) Show that the x-coordinates of any points of intersection of the hyperbola and circle are zeros of the polynomial $P(x) = x^4 - 2cx^3 + 1$. **1**

(ii) The graphs of $y = x^4 - 2cx^3 + 1$ for $c = 0.8$ and $c = 1$ are shown. **3**



By considering the given graphs, or otherwise, find the exact value of $c > 0$ such that the hyperbola $y = \dfrac{1}{x}$ and the circle $(x - c)^2 + y^2 = c^2$ intersect at only one point.

Question 14 continues on page 15

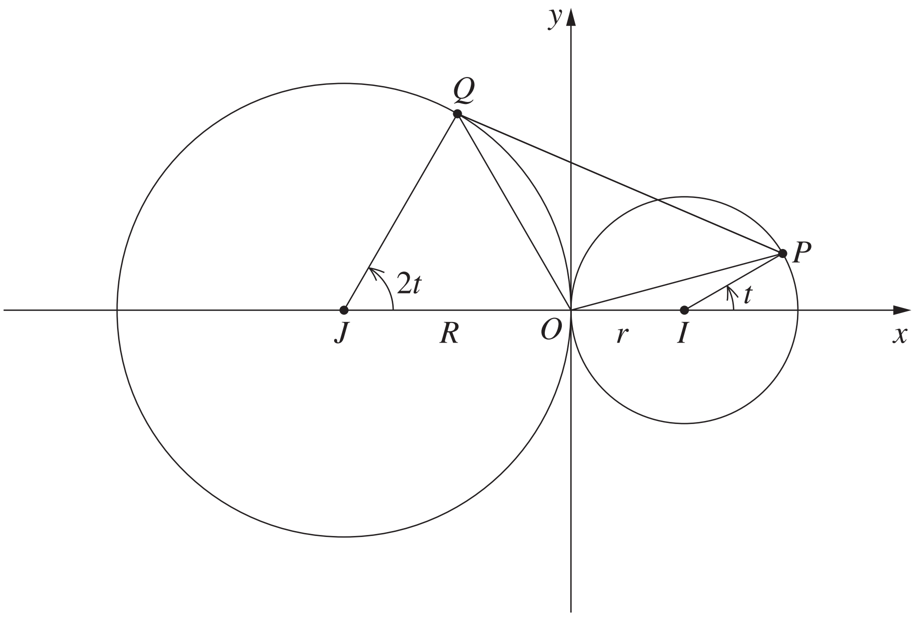
(c) (i) Given a non-zero vector $\begin{pmatrix} p \\ q \end{pmatrix}$, it is known that the vector $\begin{pmatrix} q \\ -p \end{pmatrix}$ is perpendicular to $\begin{pmatrix} p \\ q \end{pmatrix}$ and has the same magnitude. (Do NOT prove this.) **3**

Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, respectively.

Using the given information, or otherwise, show that the area of triangle OAB is $\dfrac{1}{2}\left|a_1b_2 - a_2b_1\right|$.

(ii) The point P lies on the circle centred at $I(r, 0)$ with radius $r > 0$, such that $\overrightarrow{IP}$ makes an angle of t to the horizontal. **4**

The point Q lies on the circle centred at $J(-R, 0)$ with radius $R > 0$, such that $\overrightarrow{JQ}$ makes an angle of $2t$ to the horizontal.



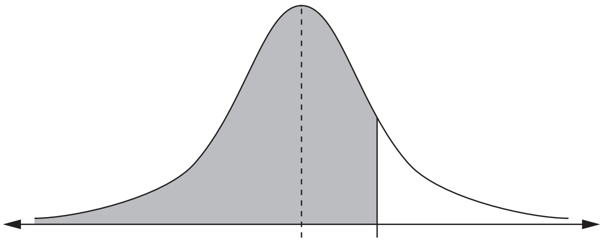
Note that $\overrightarrow{OP} = \overrightarrow{OI} + \overrightarrow{IP}$ and $\overrightarrow{OQ} = \overrightarrow{OJ} + \overrightarrow{JQ}$.

Using part (i), or otherwise, find the values of t, where $-\pi \le t \le \pi$, that maximise the area of triangle OPQ.

End of paper

Use the information below to answer Question 11 (f) (ii).

Table of values $P(Z \le z)$ for the normal distribution $N(0, 1)$



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995

NSW Education Standards Authority

2023 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1$$

$$S = \frac{a}{1 - r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

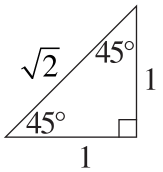
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

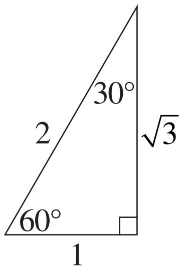


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

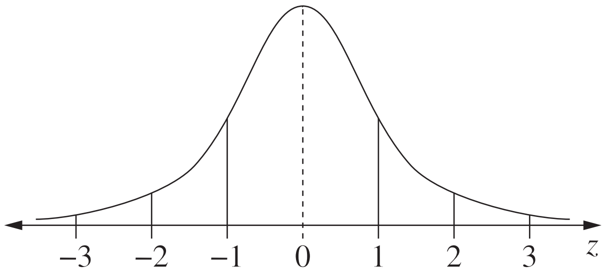
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

2023 HSC Mathematics Extension 1 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	D
2	A
3	C
4	C
5	B
6	C
7	B
8	A
9	D
10	B

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	2
• Obtains *t* in terms of *x*, or equivalent merit	1

Sample answer:

$x = 1 + 3t$

$\therefore \quad t = \dfrac{x-1}{3}$

$\therefore \quad y = 4\left(\dfrac{x-1}{3}\right)$

$\quad\quad y = \dfrac{4}{3}x - \dfrac{4}{3}$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Divides 10! by a relevant number, or equivalent merit	1

Sample answer:

10 letters, 3 Os and 2 Ns

$\therefore \quad \dfrac{10!}{3!2!} = 302\,400$

Question 11 (c)

Criteria	Marks
• Provides correct solution	3
• Obtains two equations for *a* and *b*, or equivalent merit	2
• Observes that $P(-1) = 0$ or $P(2) = -18$, or equivalent merit	1

Sample answer:

$P(-1) = 0$ by factor theorem

$$\begin{aligned}(-1)^3 + a(-1)^2 + b(-1) - 12 &= 0\\ -1 + a - b - 12 &= 0\\ a - b &= 13 \qquad (1)\end{aligned}$$

$P(2) = -18$ by remainder theorem

$$\begin{aligned}2^3 + a(2)^2 + b(2) - 12 &= -18\\ 8 + 4a + 2b - 12 &= -18\\ 4a + 2b &= -14\\ 2a + b &= -7 \qquad (2)\end{aligned}$$

(1) + (2): $3a = 6$
$a = 2$

Using this in (1) $2 - b = 13$
$b = -11$

Question 11 (d)

Criteria	Marks
• Provides correct solution	2
• Attempts to rewrite the integrand in a suitable form, or equivalent merit	1

Sample answer:

$$\begin{aligned}\int \frac{1}{\sqrt{4-9x^2}}\,dx &= \frac{1}{3}\int \frac{3}{\sqrt{2^2-(3x)^2}}\,dx\\ &= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C\end{aligned}$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	3
• Obtains an equation or graph involving a single trigonometric function OR • Obtains a quadratic equation in *t*, and attempts to solve it OR • equivalent merit	2
• Uses the *t*-substitution, attempts to write $\cos\theta + \sin\theta$ as a single trigonometric function, or equivalent merit	1

Sample answer:

Let $t = \tan\dfrac{\theta}{2}$

$$\therefore \quad \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = 1$$

$$1 - t^2 + 2t = 1 + t^2$$
$$2t^2 - 2t = 0$$
$$2t(t-1) = 0$$

$\therefore \quad t = 0$ or $t = 1$

$\tan\dfrac{\theta}{2} = 0$

$\dfrac{\theta}{2} = 0,\ \pi,\ \cancel{2\pi},\ \ldots$

$\theta = 0,\ 2\pi$

$\tan\dfrac{\theta}{2} = 1$

$\dfrac{\theta}{2} = \dfrac{\pi}{4},\ \cancel{\dfrac{5\pi}{4}},\ \ldots$

$\theta = \dfrac{\pi}{2}$

$$\therefore \quad \theta = 0,\ \frac{\pi}{2},\ 2\pi$$

Question 11 (e) (continued)

Alternative solution

$$\cos\theta + \sin\theta = 1 \qquad \theta \in [0, 2\pi]$$

Let $\cos\theta + \sin\theta = R\sin(\theta + \alpha)$

$$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

$$\therefore \quad \left.\begin{aligned} R\sin\alpha &= 1 \\ R\cos\alpha &= 1 \end{aligned}\right\}$$

$$\therefore \quad R = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\tan\alpha = 1 \qquad \therefore \quad \alpha = \frac{\pi}{4}$$

$$\therefore \quad \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = 1$$

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\theta + \frac{\pi}{4} = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{9\pi}{4},\ \ldots$$

$$\theta = 0,\ \frac{\pi}{2},\ 2\pi$$

Question 11 (f) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains p = 0.3, or equivalent merit	1

Sample answer:

$P(\text{born overseas}) = 0.3$

Let $p = 0.3 \quad \therefore\ 1 - p = 0.7$

$n = 900$

$\therefore$ Distribution of $\hat{p}$ is approximately normal with mean $\mu = 0.3$ and

$$\begin{aligned}\sigma^2 &= \frac{0.3 \times 0.7}{900} \\ &= \frac{0.21}{900} \\ &= \frac{7}{30\,000}\end{aligned}$$

Question 11 (f) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains the correct z-score, or equivalent merit	1

Sample answer:

$$\begin{aligned}P(\hat{p} \le 0.31) &\approx P\left(Z \le \frac{0.31 - 0.3}{\sqrt{\frac{7}{30\,000}}}\right) \\ &= P(X \le 0.6546\ldots) \\ &\approx P(Z \le 0.65) \\ &= 0.7422 \qquad \text{from table} \\ &= 0.74 \qquad \text{2 decimal places}\end{aligned}$$

Question 12 (a)

Criteria	Marks
• Provides correct solution	3
• Obtains correct anti-derivative in terms of *u*, or equivalent merit	2
• Obtains correct integrand in terms of *u*, or equivalent merit	1

Sample answer:

$$\int_3^4 (x+2)\sqrt{x-3}\,dx \qquad u = x-3 \quad \begin{cases} x=4 \to u=1 \\ x=3 \to u=0 \end{cases}$$

$$du = dx$$

$$= \int_0^1 (u+5)\sqrt{u}\,du$$

$$= \int_0^1 u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\,du$$

$$= \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{5\times 2}{3}u^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{5} + \frac{10}{3}$$

$$= \frac{56}{15}$$

Question 12 (b)

Criteria	Marks
• Provides correct solution	3
• Establishes the inductive step, or equivalent merit	2
• Establishes the base case, or equivalent merit	1

Sample answer:

When $n = 1$

$$\text{LHS} = 1 \times 2 = 2$$

$$\text{RHS} = 2 + (1-1)2^{1+1} = 2$$

$$\text{LHS} = \text{RHS}$$

$\therefore$ True when $n = 1$

Assume true when $n = k$

ie $\quad (1\times 2)+\left(2\times 2^2\right)+\cdots+\left(k\times 2^k\right)=2+(k-1)2^{k+1}$

Consider $n = k + 1$

$$\begin{aligned}\text{RHS} &= 2 + (k+1-1)2^{k+1+1} \\ &= 2 + k \,.\, 2^{k+2}\end{aligned}$$

$$\begin{aligned}\text{LHS} &= (1\times 2)+\left(2\times 2^2\right)+\cdots+\left(k\times 2^k\right)+\left((k+1)\times 2^{k+1}\right) \\ &= 2 + (k-1)2^{k+1} + (k+1)2^{k+1} \qquad \text{by assumption} \\ &= 2 + 2^{k+1}(k-1+k+1) \\ &= 2 + 2^{k+1}(2k) \\ &= 2 + k \,.\, 2^{k+2} \\ &= \text{RHS}\end{aligned}$$

Therefore, by mathematical induction, it is true for all integers $n \geq 1$.

Question 12 (c) (i)

Criteria	Marks
• Provides correct answer	2
• Demonstrates some understanding of binomial probability, or equivalent merit	1

Sample answer:

$${}^{5}C_{3}(0.65)^{3}(0.35)^{2}$$

Question 12 (c) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$${}^{5}C_{3}(0.65)^{3}(0.35)^{2} \times (0.6)^{4}$$

Question 12 (d)

Criteria	Marks
• Provides correct solution	2
• Uses given result to combine the first two terms on the left-hand side, or equivalent merit	1

Sample answer:

$$\begin{aligned}
&{}^{2022}C_{80} + {}^{2022}C_{81} + {}^{2023}C_{1943} \\
&= {}^{2023}C_{81} + {}^{2023}C_{1943} \\
&= {}^{2023}C_{81} + {}^{2023}C_{80} \\
&= {}^{2024}C_{81}
\end{aligned}$$

So $p = 2024$ and $q = 81$ is a possible solution

Question 12 (e)

Criteria	Marks
• Provides correct solution	4
• Obtains the volume formed by revolving the hyperbola about the *y*-axis, or equivalent merit	3
• Obtains correct integral expression for the volume formed by revolving the hyperbola about the *y*-axis, or equivalent merit	2
• Writes *x* in terms of y, or recognises that the volume is the sum of two simpler volumes, or equivalent merit	1

Sample answer:

The total volume is the sum of the volume when the given hyperbola is revolved about the y-axis, V_1, and the volume of a cylinder, V_2.

$$y = \frac{60}{x+5}$$

$$x + 5 = \frac{60}{y}$$

$$x = \frac{60}{y} - 5$$

$$V_1 = \pi \int_4^{12} x^2 \, dy$$

$$= \pi \int_4^{12} \left(\frac{60}{y} - 5 \right)^2 dy$$

$$= \pi \int_4^{12} \frac{3600}{y^2} - \frac{600}{y} + 25 \, dy$$

$$= \pi \left[-\frac{3600}{y} - 600 \ln y + 25y \right]_4^{12}$$

$$= \pi \left[\left(-\frac{3600}{12} - 600 \ln 12 + 25 \times 12 \right) - \left(-\frac{3600}{4} - 600 \ln 4 + 25 \times 4 \right) \right]$$

$$= \pi (-600 \ln 12 + 800 + 600 \ln 4)$$

$$= \pi (800 - 600 \ln 3)$$

$$V_2 = \pi \times 10^2 \times 4$$

$$= 400\pi$$

Total = $V_1 + V_2$

$\therefore$ Total = $\pi(1200 - 600 \ln 3)$ units^3

Question 13 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains $\frac{dV}{dh}$, or equivalent merit	1

Sample answer:

$$V = \pi\left(Rh^2 - \frac{h^3}{3}\right)$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

So $\quad \frac{dV}{dt} = \pi\left(2Rh - h^2\right)\frac{dh}{dt}$

$$= \pi h(2R - h)\frac{dh}{dt}$$

But we are given

$$\frac{dV}{dt} = k(2R - h)$$

So $\quad \pi h(2R - h)\frac{dh}{dt} = k(2R - h)$

$\pi h\frac{dh}{dt} = k \qquad$ as $0 \le h \le R$ so $2R - h \ne 0$

$$\frac{dh}{dt} = \frac{k}{\pi h}$$

Question 13 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Separates the variables from the differential equation in part (i), or equivalent merit	1

Sample answer:

$$\frac{dh}{dt} = \frac{k}{\pi h}$$

When $t = 0,\ h = 0$
$t = T,\ h = R$

So $\displaystyle\int_0^R \pi h \, dh = \int_0^T k \, dt$

$$\left[\frac{\pi h^2}{2}\right]_0^R = \left[kt\right]_0^T$$

$$\frac{\pi R^2}{2} - 0 = kT - 0$$

$$T = \frac{\pi R^2}{2k}$$

Question 13 (a) (iii)

Criteria	Marks
• Provides correct solution	3
• Finds correct expression for $\frac{dh}{dt}$, or equivalent merit	2
• Models the situation to find the new differential equation $\frac{dV}{dt} = -kh$, or equivalent merit	1

Sample answer:

When the tank is full the inflow of water stops. This means that the rate of change in volume is only dependent on h.

ie $\quad -kh = \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

So $\quad -kh = \pi h(2R - h)\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{-k}{\pi(2R - h)}$$

When $t = 0, \quad h = R$
$t = T_E, \quad h = 0$

$$\int_R^0 \pi(2R - h)\,dh = \int_0^{T_E} -k\,dt$$

$$\left[\pi\left(2Rh - \frac{h^2}{2}\right)\right]_R^0 = -kT_E$$

$$-\pi\left(2R^2 - \frac{R^2}{2}\right)^2 = -kT_E$$

$$\pi\frac{3R^2}{2} = kT_E$$

$$T_E = 3\frac{\pi R^2}{2k} = 3T$$

So the tank takes 3 times as long to empty.

Question 13 (b) (i)

Criteria	Marks
• Provides correct proof	2
• Equates *x*-coordinates, or equivalent merit	1

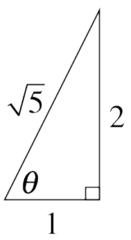
Sample answer:

The particles collide when their x-coordinates are the same at the same time.

$$vt\cos\theta = ut$$

So $\quad u = v\cos\theta$

θ in 1st quad. and $\tan\theta = 2$



So $\quad \cos\theta = \dfrac{1}{\sqrt{5}} \quad$ and $\quad \sin\theta = \dfrac{2}{\sqrt{5}}$

$$u = v\cos\theta$$

$$u = \frac{v}{\sqrt{5}}$$

$$v = \sqrt{5}\,u$$

Question 13 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Particles collide at time T so y-coordinates are the same.

$$vT\sin\theta - \frac{1}{2}gT^2 = H - \frac{1}{2}gT^2$$

$$\sqrt{5}uT\frac{2}{\sqrt{5}} = H$$

$$T = \frac{H}{2u}$$

Question 13 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Evaluates the dot product of the velocity vectors and equates to 0, or equivalent merit	2
• Observes that the dot product of the velocity vectors is 0, or equivalent merit	1

Sample answer:

At $t = T$ the vectors $\underset{\sim}{v}_A$ and $\underset{\sim}{v}_B$ are perpendicular, so $\underset{\sim}{v}_A \cdot \underset{\sim}{v}_B = 0$.

$$\begin{pmatrix} v\cos\theta \\ v\sin\theta - gT \end{pmatrix} \cdot \begin{pmatrix} u \\ -gT \end{pmatrix} = 0$$

$$uv\cos\theta - gT\left(v\sin\theta - gT\right) = 0$$

$$u\sqrt{5}u\frac{1}{\sqrt{5}} - \sqrt{5}ugT\frac{2}{\sqrt{5}} + g^2T^2 = 0$$

$$u^2 - 2ugT + g^2T^2 = 0$$

$$\left(u - gT\right)^2 = 0$$

$$u = gT = \frac{gH}{2u}$$

$$H = \frac{2u^2}{g}$$

Question 13 (b) (iv)

Criteria	Marks
• Provides correct solution	2
• Identifies that $\dot{y} = 0$ for particle *A*, or equivalent merit	1

Sample answer:

Vertex of $y_A(t) = vt\sin\theta - \frac{1}{2}gt^2$ occurs when $\dot{y} = 0$

$$t = \frac{-v\sin\theta}{2\left(-\frac{1}{2}g\right)} \qquad \text{[axis of symmetry]}$$

$$= \frac{\sqrt{5}u\frac{2}{\sqrt{5}}}{g}$$

$$= \frac{2u}{g}$$

Height of particle *A* at that time is

$$y_A = vt\sin\theta - \frac{1}{2}gt^2$$

$$= \sqrt{5}u \times \frac{2u}{g} \times \frac{2}{\sqrt{5}} - \frac{g}{2} \times \frac{4u^2}{g^2}$$

$$= \frac{4u^2}{g} - \frac{2u^2}{g}$$

$$= \frac{2u^2}{g}$$

$$= H$$

Question 14 (a) (i)

Criteria	Marks
• Provides correct explanation	1

Sample answer:

$f'(x) = 2 + \frac{1}{x} > 0$ for $x > 0$

f is always increasing.

Therefore since f is one-to-one for $x > 0$, f has an inverse function.

Question 14 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds $g'(x)$ or $f'(1)$, or equivalent merit	1

Sample answer:

$f(1) = 2$ therefore $g(2) = 1$

$$g'(x) = \frac{1}{f'(g(x))}$$

So $\quad g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(1)} = \frac{1}{3}$

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

The points of intersection of the hyperbola and the circle satisfy

$$\begin{cases} y = \dfrac{1}{x} & (1) \\ (x-c)^2 + y^2 = c^2 & (2) \end{cases}$$

Substitute (1) into (2)

$$(x-c)^2 + \frac{1}{x^2} = c^2 \qquad \times x^2$$

$$x^2\left(x^2 - 2cx + c^2\right) + 1 = c^2x^2$$

$$x^4 - 2cx^3 + 1 = 0$$

Question 14 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Finds the *x*-coordinate of the double root in terms of *c*, or equivalent merit	2
• Recognises that the circle and hyperbola are tangent when the polynomial has a double root and so is tangent to the *x*-axis, or equivalent merit	1

Sample answer:

Let $f(x) = x^4 - 2cx^3 + 1$

The hyperbola and the circle have exactly one point of intersection if there exists a value of x such that $f(x) = 0$ and $f'(x) = 0$.

$$f'(x) = 4x^3 - 6cx^2$$

$$f'(x) = 2x^2(2x - 3c)$$

$$f'(x) = 0 \Leftrightarrow x = 0 \quad \text{or} \quad x = \frac{3c}{2}$$

Note that $f(0) \neq 0$

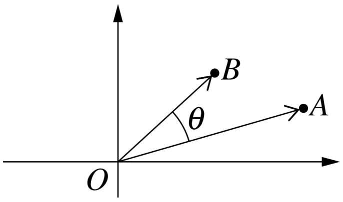
$$\begin{aligned}
f\left(\frac{3c}{2}\right) = 0 \quad &\Leftrightarrow \left(\frac{3c}{2}\right)^4 - 2c\left(\frac{3c}{2}\right)^3 + 1 = 0 \qquad \times 2^4 \\
&\Leftrightarrow 3^4c^4 - 3^3 \times 2^2c^4 + 2^4 = 0 \\
&\Leftrightarrow -27c^4 = -16 \\
&\Leftrightarrow c = \frac{2}{\sqrt[4]{27}}
\end{aligned}$$

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Recognise that the dot product of $\overline{OB}$ with the vector perpendicular to $\overline{OA}$, using the given information, is related to the area, or vice versa, or equivalent merit	2
• Attempts to use a vector perpendicular to $\overline{OA}$ or $\overline{OB}$, or equivalent merit	1

Sample answer:

Let θ be the angle between $\overline{OA}$ and $\overline{OB}$



Area of triangle = $\frac{1}{2}ab\sin\theta$, where $a = \left|\overline{OA}\right|$ and $b = \left|\overline{OB}\right|$

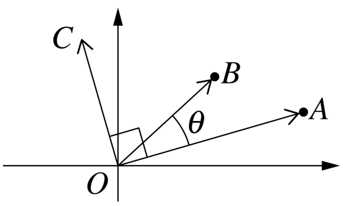
Let $\overrightarrow{OC} = \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}$ be the vector perpendicular to $\overline{OA}$ using the given information.

We are told that $\overline{OC}$ is perpendicular to $\overline{OA}$ and with the same magnitude.

$$\overrightarrow{OB} \cdot \overrightarrow{OC} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} = a_2 b_1 - a_1 b_2 \qquad \text{——— (1)}$$

We can also express $\overline{OB} \cdot \overline{OC}$ in terms of θ

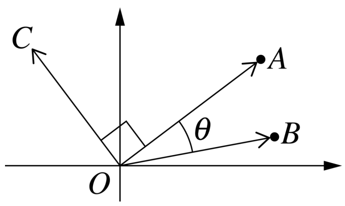
$$\begin{aligned} \overline{OB} \cdot \overline{OC} &= \left|\overline{OB}\right| \cdot \left|\overline{OC}\right| \cos(\angle BOC) \\ &= b \times a \times \cos(\angle BOC) \end{aligned}$$



Case 1:

$\angle BOC = \frac{\pi}{2} - \theta$

so $\overline{OB} \cdot \overline{OC} = ab\sin\theta$



Case 2:

$\angle BOC = \frac{\pi}{2} + \theta$

so $\overline{OB} \cdot \overline{OC} = -ab\sin\theta$

In both cases $\left|\overline{OB} \cdot \overline{OC}\right| = \left|ab\sin\theta\right|$ = 2 × (Area of triangle)

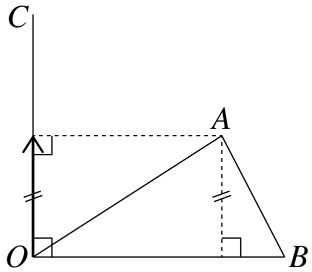
Therefore, area of triangle = $\frac{1}{2}\left|\overrightarrow{OB} \cdot \overrightarrow{OC}\right| = \frac{1}{2}\left|a_2 b_1 - a_1 b_2\right|$ [From (1)]

$$= \frac{1}{2}\left| a_1b_2 - a_2b_1 \right|$$

Question 14 (c) (i) (continued)

Alternative solution

Projection of $\left|\overline{AC}\right|$ onto $\left|\overline{OC}\right|$



Area triangle $= \frac{1}{2}\left|\overrightarrow{OB}\right| \times$ perpendicular height of triangle OAB

$= \frac{1}{2}\left|\overrightarrow{OB}\right| \times \left|\text{projection of } \overrightarrow{OA} \text{ onto } \overrightarrow{OC}\right|$

where $\overline{OC} \perp \overline{OB}$ and $\left|\overline{OC}\right| = \left|\overline{OB}\right|$

$$\overrightarrow{OC} = \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix}$$

$$\therefore \quad \left|\overrightarrow{OA} \text{ projected onto } \overrightarrow{OC}\right| = \frac{\left|\overline{OA} \cdot \overline{OC}\right|}{\left|\overrightarrow{OC}\right|}$$

$$= \frac{\left|a_1b_2 - a_2b_1\right|}{\left|\overrightarrow{OB}\right|}$$

$$\therefore \quad \text{Area of triangle } = \frac{1}{2}\left|\overrightarrow{OB}\right| \times \frac{\left|a_1b_2 - a_2b_1\right|}{\left|\overrightarrow{OB}\right|}$$

$$= \frac{1}{2}\left|a_1b_2 - a_2b_1\right|$$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	4
• Finds the values of *t* at the stationary points of the area, or equivalent merit	3
• Obtains the area of triangle *OPQ* in terms of *t*, or equivalent merit	2
• Obtains the components of $\overrightarrow{OP}$ or $\overrightarrow{OQ}$ in terms of *t*, or equivalent merit	1

Sample answer:

$$\overrightarrow{OP} = \overrightarrow{OI} + \overrightarrow{IP} = \begin{pmatrix} r + r\cos t \\ r\sin t \end{pmatrix}$$

$$\overrightarrow{OQ} = \overrightarrow{OJ} + \overrightarrow{JQ} = \begin{pmatrix} -R + R\cos 2t \\ R\sin 2t \end{pmatrix}$$

Using part (i), we get that the area of triangle *OPQ* is

$$\begin{aligned}
\text{Area of triangle } &= \frac{1}{2}\left|(r + r\cos t)(R\sin 2t) - (r\sin t)(-R + R\cos 2t)\right| \\
&= \frac{1}{2}\left|Rr(1+\cos t)\sin 2t - Rr\sin t(-1+\cos 2t)\right| \\
&= \frac{1}{2}Rr\left|(\sin 2t\cos t - \sin t\cos 2t) + \sin 2t + \sin t\right| \\
&= \frac{1}{2}Rr\left|\sin t + \sin 2t + \sin t\right| \\
\text{Area of triangle } &= \frac{1}{2}Rr\left|\sin 2t + 2\sin t\right|
\end{aligned}$$

Let $f(t) = \sin 2t + 2\sin t$, so Area triangle $= Rr\left|f(t)\right|$

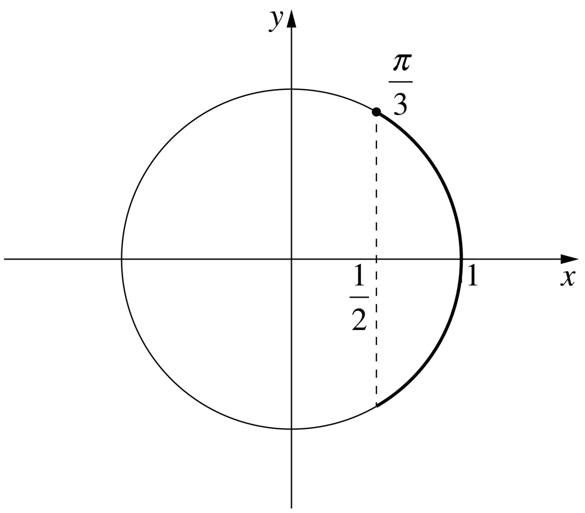
$$\begin{aligned}
f'(t) &= 2\cos 2t + 2\cos t \\
&= 2\left(2\cos^2 t + \cos t - 1\right) \\
&= 2(2\cos t - 1)(\cos t + 1)
\end{aligned}$$

For all t, $\cos t + 1 \geq 0$ therefore $f'(t)$ has the same sign as $2\cos t - 1$

$f(t)$ is odd so $\left|f(t)\right|$ is even and we only need to study $\left|f(t)\right|$ on $[0, \pi]$ where area of triangle $= \frac{1}{2}Rr\left|f(t)\right|$, so $f(t) \geq 0$ on $[0, \pi]$

$f'(t) \geq 0 \quad \Leftrightarrow \quad 2\cos t - 1 \geq 0 \quad \Leftrightarrow \quad \cos t \geq \frac{1}{2}$

which will happen when $-\frac{\pi}{3} \leq t \leq \frac{\pi}{3}$



t	0		$\frac{\pi}{3}$		π
$f'(t)$		$+$	0	$-$	
	0	$\nearrow$		$\searrow$	0

By symmetry, since $|f(t)|$ is even, we see that the area of the triangle is maximum when $t = -\frac{\pi}{3}$ or $t = \frac{\pi}{3}$

2023 HSC Mathematics Extension 1 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	ME-C1 Rates of Change	ME11-1
2	1	ME-S1 The Binomial Distribution	ME12-5
3	1	ME-C3 Applications of Calculus	ME12-1
4	1	ME-C3 Applications of Calculus	ME12-4
5	1	ME-T1 Inverse Trigonometric Functions	ME11-3
6	1	ME-V1 Introduction to Vectors	ME12-2
7	1	ME-T1 Inverse Trigonometric Functions	ME11-3
8	1	ME-F1 Further Work with Functions	ME11-1
9	1	ME-F1 Further Work with Functions	ME11-1
10	1	ME-A1 Working with Combinatorics	ME11-5

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	2	ME-F1 Further Work with Functions	ME11-2
11 (b)	2	ME-A1 Working with Combinatorics	ME11-5
11 (c)	3	ME-F2 Polynomials	ME11-2
11 (d)	2	ME-C2 Further Calculus Skills	ME12-1
11 (e)	3	ME-T3 Trigonometric Equations	ME12-3
11 (f) (i)	2	ME-S1 The Binomial Distribution	ME12-5
11 (f) (ii)	2	ME-S1 The Binomial Distribution	ME12-5
12 (a)	3	ME-C2 Further Calculus Skills	ME12-1
12 (b)	3	ME-P1 Proof by Mathematical Induction	ME12-1
12 (c) (i)	2	ME-S1 The Binomial Distribution	ME12-5
12 (c) (ii)	1	ME-S1 The Binomial Distribution	ME12-5
12 (d)	2	ME-A1 Working with Combinatorics	ME11-5
12 (e)	4	ME-C3 Applications of Calculus	ME12-4
13 (a) (i)	2	ME-C1 Rates of Change	ME11-4
13 (a) (ii)	2	ME-C3 Applications of Calculus	ME12-1, ME12-4
13 (a) (iii)	3	ME-C3 Applications of Calculus	ME12-1, ME12-7
13 (b) (i)	2	ME-V1 Introduction to Vectors	ME12-2
13 (b) (ii)	1	ME-V1 Introduction to Vectors	ME12-2
13 (b) (iii)	3	ME-V1 Introduction to Vectors	ME12-2
13 (b) (iv)	2	ME-V1 Introduction to Vectors	ME12-2

Question	Marks	Content	Syllabus outcomes
14 (a) (i)	1	ME-F1 Further Work with Functions	ME11-1
14 (a) (ii)	2	ME-C2 Further Calculus Skills	ME12-1
14 (b) (i)	1	ME-F2 Polynomials	ME11-6
14 (b) (ii)	3	ME-F2 Polynomials	ME11-2, ME11-6
14 (c) (i)	3	ME-V1 Introduction to Vectors	ME12-2
14 (c) (ii)	4	ME-V1 Introduction to Vectors ME-T3 Trigonometric Equations	ME12-3

Mathematics Extension 1

HSC Marking Feedback 2023

General feedback

Students should:

- show relevant mathematical reasoning and/or calculations
- read the question carefully to ensure that they do not miss important components of the question
- have a clear understanding of key words in the question and recognise the intent of the question and its requirements, such as show, solve, evaluate, hence, calculate, derive
- use the Reference Sheet where appropriate
- ensure the solution is legible and follows a clear sequence
- engage with any stimulus material provided and refer to it in their response when required by the question
- check their solution answers the question
- round off numerical solutions only at the final step of the solution
- construct graphs neatly, with precision and display all relevant information as required by the question
- interpret information presented in graphs across a range of contexts
- understand when to use relevant calculator functions
- carefully note any information in the questions which supplies units of measurement.

Section II

Question 11 (a)

In better responses, students were able to:

- change the subject of an equation
- eliminate the parameter and find the Cartesian equation.

Areas for students to improve include:

- solving simultaneous equations.

Question 11 (b)

In better responses, students were able to:

- determine the number of arrangements using factorial notation
- eliminate the repeated letters by division.

Areas for students to improve include:

- identifying repeated letters and eliminating these from the number of arrangements.

Question 11 (c)

In better responses, students were able to:

- apply both the factor theorem and remainder theorem to form equations
- solve simultaneous equations.

Areas for students to improve include:

- applying the remainder theorem correctly
- applying the factor theorem correctly.

Question 11 (d)

In better responses, students were able to:

- use knowledge of inverse trigonometric functions
- rewrite the integrand in a suitable form.

Areas for students to improve include:

- rewriting the integrand to match the Reference Sheet
- recognising the integrand as an inverse trigonometric function.

Question 11 (e)

In better responses, students were able to:

- use the compound angle formula to transform the sum of sin- and cos
- use the relevant t formula to transform the sum of sin and cos-
- use the relevant double angle formula to solve a trigonometric equation.

Areas for students to improve include:

- nominating the correct compounding formula for the equation
- solving a quadratic equation from a t formula substitution
- solving a trigonometric equation over a given domain.

Question 11 (f) (i)

In better responses, students were able to:

- recognise the values needed to calculate $Var(x)$.

Areas for students to improve include:

- connecting the information supplied to a relevant formula.

Question 11 (f) (ii)

In better responses, students were able to:

- calculate the relevant $\boldsymbol{z}$ -score use the $\boldsymbol{z}$ -score table to find the required probability.

Areas for students to improve include:

- determining the values to substitute into the $\boldsymbol{z}$ -score formula
- knowing how to read from the $\boldsymbol{z}$ -score table.

Question 12 (a)

In better responses, students were able to:

- obtain the correct integrand in terms of $\boldsymbol{u}$
- change the limits in terms of $\boldsymbol{u}$
- substitute correctly in the anti-derivative to arrive at the correct solution.

Areas for students to improve include:

- using brackets around their factors, so that a correct simplification of terms can occur
- changing the limits with variable $\boldsymbol{x}$ to limits with variable $\boldsymbol{u}$
- integrating fractional powers correctly.

Question 12 (b)

In better responses, students were able to:

- establish the base assumption steps
- use the exponential laws correctly in the LHS of the inductive step
- arrive at the RHS from the LHS and write a correct conclusion.

Areas for students to improve include:

- testing for $n = k + 1$ on both sides of the equation
- using exponential laws correctly
- equating the LHS to the RHS.

Question 12 (c) (i)

In better responses, students were able to:

- demonstrate the understanding of binomial probability correctly.

Areas for students to improve include:

- applying binomial probability correctly.

Question 12 (c) (ii)

In better responses, students were able to:

- demonstrate the understanding of binomial probability taking into consideration one of the probabilities were not in use.

Areas for students to improve include:

- knowing how to apply the complimentary event in binomial probability.

Question 12 (d)

In better responses, students were able to:

- use the ${}^nC_r = {}^nC_{n-r}$ notation successfully to combine the first two terms of the LHS.

Areas for students to improve include:

- using the symmetry of Pascals Triangle
- applying ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$ successfully.

Question 12 (e)

In better responses, students were able to:

- obtain the volume of the hyperbola correctly
- obtain the volumes of the cylinder correctly
- calculate the sum of both volumes correctly.

Areas for students to improve include:

- analysing the question by combining volumes as the sum of 2 simpler volumes
- knowing logarithmic laws in terms of ln 0 are undefined
- combining logarithmic laws to obtain a simpler fraction
- knowing how to expand a binomial that has fractional powers
- knowing how to solve for x if it is in the denominator.

Question 13 (a) (i)

In better responses, students were able to:

- find a correct expression for $\frac{dV}{dh}$
- use their knowledge of related rates to link $\frac{dV}{dh}$ and $\frac{dV}{dt}$ to find $\frac{dh}{dt}$
- simplify the algebraic expression to obtain the required result.

Areas for students to improve include:

- showing all steps of their processing.

Question 13 (a) (ii)

In better responses, students were able to:

- solve the differential equation from part (i) to obtain an expression relating time and the height of water in the tank
- clearly explain that when the tank is full, the height of the water will equal the radius of the tank.

Areas for students to improve include:

- recognising the need to use the result generated in part (i) of the question
- separating the variables of the differential equation correctly so that the terms of integration match the variable being integrated
- finding the value of any constants generated in their integration
- showing all steps of their processing.

Question 13 (a) (iii)

In better responses, students were able to:

- model the situation to develop a new differential equation representing the rate at which water was draining from the tank
- use their knowledge of related rates to link $\frac{dV}{dh}$ and their new equation $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$
- integrate the rate to obtain the required result.

Areas for students to improve include:

- using the information in the question to determine a new expression for $\frac{dV}{dt}$ when the tank is draining
- separating the variables of the differential equation correctly so that the terms of integration matched the variable being integrated
- finding the value of any constants generated in their integration
- showing all steps of their processing.

Question 13 (b) (i)

In better responses, students were able to:

- clearly show that $cos\theta = \frac{1}{\sqrt{5}}$
- use the result $cos\theta = \frac{1}{\sqrt{5}}$ and the horizontal displacements of the two projectiles to achieve the required result.

Areas for students to improve include:

- using a diagram or trigonometric identities to find a value for $cos\theta$ in exact form
- recognising that the horizontal displacements must be equal when the projectiles collide
- showing all steps of their processing.

Question 13 (b) (ii)

In better responses, students were able to:

- use information from part (i) and the vertical displacements of the projectile to achieve the required result.

Areas for students to improve include:

- recognising that the vertical displacements must be equal when the projectiles collide
- finding an exact value for $sin\theta$ using the methods of part (i)
- showing all steps of their processing.

Question 13 (b) (iii)

In better responses, students were able to:

- find expressions for the velocities of each projectile
- use the $\underset{\sim}{v}_A . \underset{\sim}{v}_B = 0$ result for the projectiles having perpendicular paths to obtain an equation which can be solved
- solve the resulting quadratic equation to obtain the required result.

Areas for students to improve include:

- differentiating with respect to time in order to find the velocities of each projectile
- recognising that H is a constant, its derivative is 0 and that it does not feature as part of the expression for the vertical velocity of projectile *B*
- knowing that the dot product of the velocities of the projectiles will be 0 since their paths are perpendicular at the point of collision
- using algebraic skills to simplify the resulting quadratic equation formed from the dot product
- factorising or using the quadratic formula to solve the equation
- showing all steps of their processing.

Question 13 (b) (iv)

In better responses, students were able to:

- find the time to maximum height for projectile A
- use time to find the maximum height
- express their maximum height in terms of H.

Areas for students to improve include:

- recognising that maximum height will occur when the vertical velocity is 0
- using the correct expression for maximum height
- substituting their value for time and using algebraic skills to simplify the resulting expression
- understanding the intent of the phrase 'Give your answer in terms of H'.

Question 14 (a) (i)

In better responses, students were able to:

- provide clear explanations
- express answers clearly and unambiguously using appropriate mathematical notations.

Areas for students to improve include:

- knowing what the term 'explain' means in the context of the problem
- using proper mathematical terms.

Question 14 (a) (ii)

In better responses, students were able to:

- correctly use the chain rule to find an expression for the gradient of the inverse function
- relate the tangents to corresponding points on a function and its inverse function.

Areas for students to improve include:

- relating the coordinates of a point on a function to the coordinates of a corresponding point on the inverse of that function
- finding derivatives of inverse functions using algebraic methods as well as the geometrical approach.

Question 14 (b) (i)

In better responses, students were able to:

- substitute and use the correct algebraic techniques to arrive at the given formula.

Areas for students to improve include:

- showing full working before arriving at the required equation
- performing algebraic manipulations with directional signs correctly

- realising that finding points of intersection involves solving simultaneous equations.

Question 14 (b) (ii)

In better responses, students were able to:

- correctly interpret information given in graphical form
- understand that the question required locating a double root
- recognise from the graph the constraints on the variables *x* and *c*.

Areas for students to improve include:

- knowing how and when to use the concept of multiple roots in polynomials
- demonstrating how the interaction of geometric shapes can be depicted on a graph
- estimating graphically the values of variables to meet specific conditions.

Question 14 (c) (i)

In better responses, students were able to:

- understand the significance of the given property and use it to arrive at the required proof
- demonstrate skills in working with vectors and arrive at the required result using a variety of equivalent proofs.

Areas for students to improve include:

- constructing and clearly labelling diagrams with all the relevant information
- using correct vector notation
- knowing that in 'show that...' questions the last line should be what they are asked to show
- knowing how to find the projection of a vector and its application in unfamiliar situations.

Question 14 (c) (ii)

In better responses, students were able to:

- make effective use of the information in part (i) to facilitate the solution of the problem
- use a combination of skills in trigonometry, calculus, and vectors to solve the problem.

Areas for students to improve include:

- writing down all they know about the question in the hope that may trigger some key thought processes that they may earlier not have realised
- improving problem-solving skills by practising more complex questions in vector applications
- knowing how the area and its rate of changes, varies with the angle *t*.