Student Number

PEM

2024 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen
- NESA-approved Calculators may be used
- A reference sheet is provided.
- In Questions 11 – 16 show relevant mathematical reasoning and/or calculations

Total Marks – 100

Section I Questions 1 – 10 **10 marks**
Allow about 15 minutes for this section

Section II Questions 11 – 16 **90 marks**
Allow about 2 hour and 45 minutes for this section

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Section 1

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 Let $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

Which of the following is the value of $\underset{\sim}{a} \cdot \left(\underset{\sim}{a} - 3\underset{\sim}{b}\right)$?

(A) -7

(B) 0

(C) 3

(D) 6

2 In which quadrant does the complex number $2e^{-5i\pi/12} + 2e^{i\pi/12}$ lie?

(A) I

(B) II

(C) III

(D) IV

3 Which statement is true about the following integrals?

$$I_1 = \int \frac{d\theta}{2+\cos\theta} \quad \text{and} \quad I_2 = \int \frac{d\theta}{1+2\cos\theta}$$

(A) Neither I_1 nor I_2 requires partial fractions.

(B) Only I_1 requires partial fractions.

(C) Only I_2 requires partial fractions.

(D) Both I_1 and I_2 require partial fractions.

4 Let $n \in \mathbb{N}$ and P be the statement

"If n is even then $5n+3$ is odd".

Which of the following is the contrapositive of P?

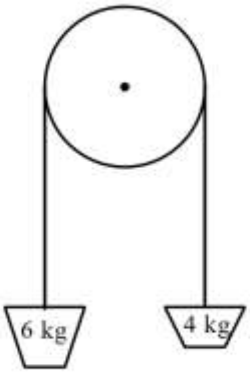
(A) If n is odd then $5n+3$ is even

(B) If $5n+3$ is odd then n is even

(C) If $5n+3$ is even then n is odd

(D) If $5n+3$ is even then n is even

5 Particles of mass 6 kg and 4 kg are attached to each end of a light inextensible string. The string passes over a smooth pulley as shown in the diagram below.



Which expression gives the acceleration of the 6 kg mass as it moves downwards?

(A) $\frac{1}{5}g$

(B) $\frac{2}{5}g$

(C) $2.5g$

(D) $5g$

6 For how many integer values of n, where $i^2 = -1$, is $n^4 + (n+i)^4$ an integer?

(A) 1

(B) 2

(C) 3

(D) 4

7 The complex number $z = a + ib$, where $0 < a < b$.

Which of the following best describes the complex number z^4?

(A) $\mathrm{Re}\left(z^4\right) < 0$

(B) $\mathrm{Re}\left(z^4\right) \le 0$

(C) $\mathrm{Im}\left(z^4\right) < 0$

(D) $\mathrm{Im}\left(z^4\right) \le 0$

8 Consider the two statements:

$$P: \quad \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad y^3 = x$$

$$Q: \quad \exists y \in \mathbb{R}, \forall x \in \mathbb{R}, \quad y^3 = x$$

Which statement best represents the truth of each of P and Q?

(A) P is true and Q is true

(B) P is true and Q is false

(C) P is false and Q is true

(D) P is false and Q is false

9 A particle travels in a line such that the velocity, v ms^{-1}, is given by $v = 16 - x^2$, where x is the displacement. What is the acceleration when $x = 3$?

(A) -42 ms^{-2}

(B) -7 ms^{-2}

(C) 7 ms^{-2}

(D) 42 ms^{-2}

10 A complex number ω satisfies

$$\omega^2 + \frac{1}{1+\omega^2} = \omega .$$

Which is the correct statement about ω^{2024}?

(A) $\omega^{2024} = \omega$

(B) $\omega^{2024} = \omega^2$

(C) $\omega^{2024} = \omega^3$

(D) $\omega^{2024} = \omega^4$

END OF SECTION I

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer the questions in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) The complex numbers z_1 and z_2 are given by **2**

$$z_1 = 3 - i \text{ and } z_2 = 1 - 2i .$$

Determine the possible values of the real constant k if

$$\left| \frac{z_1}{z_2} + k \right| = \sqrt{k+2} .$$

(b)

(i) Find real numbers A and B such that: **2**

$$\frac{1}{(x-2)(x-6)} \equiv \frac{A}{(x-2)} + \frac{B}{(x-6)}$$

(ii) Hence find $\displaystyle\int_3^5 \frac{dx}{(x-2)(x-6)}$. **2**

QUESTION 11 CONTINUES ON THE NEXT PAGE

Question 11 (Continued)

(c) Find:

(i) $\int_1^9 \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)}\,dx$ **2**

(ii) $\int \frac{\sin^{-1}\theta}{\sqrt{1-\theta^2}}\,d\theta$ **2**

(iii) $\int \frac{\cos 2x}{\sin x \cos x}\,dx$ **2**

(d) The position vectors of two points, A and B, are given by $\overrightarrow{OA} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\overrightarrow{OB} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

(i) Determine the exact distance between A and B. **1**

(ii) Show that there is no real value of m such that $\overrightarrow{OC} = m\mathbf{i} + 2\mathbf{j} - m^2\mathbf{k}$ is perpendicular to $\overrightarrow{OA}$. **2**

END OF QUESTION 11

Question 12 (17 marks) Use the Question 12 Writing Booklet.

(a) Determine the complex solutions to the equation $z^2-(1-2i)z=7+i$. **3**

(b) Prove that $\sqrt[3]{4}$ is irrational. **3**

(c)

(i) Prove that if $z=\overline{z}$, then z is real. **1**

(ii) The complex numbers z and w are such that $|z|=|w|=1$. **2**

Prove that $\dfrac{z+w}{1+zw}$ is real.

(d) A particle moves along the $x-$axis with velocity v and acceleration a according to the equation $a=v^3+4v$. The particle starts at the origin with velocity 2.

Find an expression for x, the displacement of the particle, in terms of v. **3**

QUESTION 12 CONTINUES ON THE NEXT PAGE

Question 12 (Continued)

(e) Consider the lines

$$l_1: \qquad x = y = z$$

$$l_2: \qquad r = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

where t is a parameter.

(i) Show that the lines are skew. **3**

(ii) Determine the angle between l_1 and l_2. **2**

END OF QUESTION 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) Prove or disprove the assertion that $|2x+1| \le 5 \Rightarrow |x| \le 2$. **2**

(b)

(i) Sketch, on a single Argand diagram, the region $\Re$ representing both of the following conditions: **4**

$$\frac{\pi}{4} \le \arg(iz+1) \le \frac{\pi}{2} \quad \text{and} \quad |z-1| \le 1.$$

(ii) Determine, $\forall z \in \Re$, the range of values of $|z+i|$. **2**

(c)

(i) Prove that, if $x > 1$, then $\dfrac{x}{\sqrt{x-1}} \ge 2$. **2**

(ii) Hence, or otherwise, prove that, for $a > 1, b > 1$ the following inequality holds **2**

$$\frac{a^2}{b-1} + \frac{b^2}{a-1} \ge 8.$$

(d) A particle undergoing simple harmonic motion has maximum acceleration at $x = -2\sqrt{2}$, zero acceleration at $x = \sqrt{2}$ and has period $\dfrac{\pi}{6}$. The particle starts at $x = -2\sqrt{2}$.
Find an equation for the displacement, x, of the particle in terms of t. **3**

END OF QUESTION 13

Question 14 (14 marks) Use the Question 14 Writing Booklet.

(a) The numbers a_n, for integers $n \geq 1$, are defined as $a_1 = 2$, $a_2 = 56$ and $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 3$.

Use mathematical induction to prove that, for all integers $n \geq 1$, **3**

$$a_n = 5(-2)^n + 4 \times 3^n.$$

(b) Let l be the line of intersection of the following two planes:

$$\pi_1 \quad : \quad ax + y + z = a$$

$$\pi_2 \quad : \quad x - ay + az = -1$$

where $a \in \mathbb{R}$.

(i) Show that $\begin{pmatrix} -1 \\ a \\ a \end{pmatrix}$ lies on both planes. **1**

(ii) Show that $\begin{pmatrix} -2a \\ a^2 - 1 \\ a^2 + 1 \end{pmatrix}$ is the direction vector of l. **2**

(iii) Show that, $\forall a \in \mathbb{R}$, where $a \neq 0, \pm 1$, an equation for l is **2**

$$\frac{x+1}{-2a} = \frac{y-a}{a^2-1} = \frac{z-a}{a^2+1}.$$

(iv) Show that, as a varies, the line l intersects the plane $z = t$ in a circle, **3**

centred at $\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$ and determine the radius of the circle.

QUESTION 14 CONTINUES ON THE NEXT PAGE

Question 14 (Continued)

(c) Consider the statement that:

for every positive real number x there is a real number y such that $y(y+1)=x$.

(i) Write the statement using the formal language of proof. **1**

(ii) Give a direct proof of the statement. **2**

END OF QUESTION 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) Let $I_n = \int_0^1 x^{2n}\sqrt{1-x^2}\,dx$ where $n = 0, 1, 2, \ldots$.

(i) State the value of I_0. **1**

(ii) Use integration by parts to show that for $n = 1, 2, 3, \ldots$ **3**

$$I_n = \frac{2n-1}{2n+2} I_{n-1} .$$

(iii) Hence calculate $\int_0^1 x^6\sqrt{1-x^2}\,dx$. **2**

(b) A particle of mass m is projected vertically upwards under gravity, g. The particle has initial speed U and the air resistance to the particle's motion has magnitude mkv^2 where v is the speed of the particle and k is a constant.

(i) Show that the greatest height attained is $\frac{1}{2k}\log_e\left(\frac{g+kU^2}{g}\right)$. **2**

The particle has speed v when it has fallen a distance y from the maximum height.

(ii) Show that $y = \frac{1}{2k}\log_e\left(\frac{g}{g-kv^2}\right)$. **2**

When the particle returns to its point of projection it has speed V .

(iii) Show that $\frac{k}{g} = \frac{1}{V^2} - \frac{1}{U^2}$. **1**

QUESTION 15 CONTINUES ON THE NEXT PAGE

Question 15 (Continued)

(c) A particle is projected from the origin with initial speed v_0 at an angle of inclination of θ. Gravity and air resistance, proportional to the velocity, act on the particle and the acceleration in the horizontal (x) and vertical (y) directions are given by

$$\ddot{x} = -k\dot{x}$$
$$\ddot{y} = -g - k\dot{y}$$

(i) Show that the horizontal displacement is $x = \dfrac{v_0 \cos\theta}{k}\left(1 - e^{-kt}\right)$. **3**

(ii) Hence determine the limiting horizontal displacement. **1**

END OF QUESTION 15

Question 16 (14 marks) Use the Question 16 Writing Booklet.

(a) A particle's movement is represented by the vector $\underset{\sim}{r}(t) = \begin{pmatrix} t + t^{-1} \\ t^3 + t^{-3} \end{pmatrix}$, where $t \in \mathbb{R}^+$. **3**

Sketch the Cartesian graph that shows that path the particle can move along.

(b)

(i) Determine real numbers A and B such that **1**

$$\cos x \equiv A(3\cos x + 4\sin x) + B(4\cos x - 3\sin x)$$

(ii) Hence determine $\displaystyle\int \frac{e^{x/2}\cos x}{\sqrt[3]{3\cos x + 4\sin x}}\,dx$ **3**

QUESTION 16 CONTINUES ON THE NEXT PAGE

Question 16 (Continued)

(c) Consider $p(z) = az^2 + bz + c$ where $a, b, c \in \mathbb{C}$. It is given that

$$p(0) = u, p(1) = v, \text{and } p(i) = w$$

(i) Show that **3**

$$a = -iu + \left(\frac{1+i}{2}\right)v + \left(\frac{-1+i}{2}\right)w$$

$$b = (-1+i)u + \left(\frac{1-i}{2}\right)v + \left(\frac{1-i}{2}\right)w$$

$$c = u\ .$$

(ii) Show that $f(2) = (-1-2i)u + (3+i)v + (-1+i)w$. **1**

(iii) If $1 \le u \le 2, 1 \le v \le 2,$ and $1 \le w \le 2$ sketch the region of the Argand diagram which contains $f(2)$. **3**

End of paper

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TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Multiple-Choice Answer Sheet

Select the alternative A, B, C, or D that best answers the question by

placing a **X** in the box.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

SUGGESTED SOLUTIONS PEM 2024 Mathematics Extension 2 Trial HSC Examination

Section I

Multiple Choice Answer Key

Question	Answer
1	A
2	D
3	C
4	C
5	A
6	C
7	C
8	B
9	A
10	D

Detailed Solutions for Section I

Question 1

$$\underset{\sim}{a}\cdot\left(\underset{\sim}{a}-3\underset{\sim}{b}\right)=\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}-8\\5\\-3\end{pmatrix}=-8+10-9=-7$$

Question 2

$$2e^{-5i\pi/12}+2e^{i\pi/12}=2e^{-i\pi/6}\left(e^{-i\pi/4}+e^{i\pi/4}\right)=2e^{-i\pi/6}2\,\mathrm{Re}\left(e^{i\pi/4}\right)$$

which is in quadrant IV

Question 3

Using $t-$substitutions :

$$I_1=\int\frac{d\theta}{2+\cos\theta}\quad\text{and}\quad I_2=\int\frac{d\theta}{1+2\cos\theta}$$

$$=\int\frac{2dt}{3+t^2}\qquad\qquad =\int\frac{2dt}{3-t^2}$$

$\Rightarrow$ inverse tan $\qquad\qquad \Rightarrow$ partial fractions

Question 4

The contrapositive of $P\Rightarrow Q$ is $\neg Q\Rightarrow\neg P$

Question 5

Let T be the tension in the string.

Forces on 6 kg mass: $6g - T = 6a$

Forces on 6 kg mass: $T - 4g = 4a$

Adding gives $2g = 10a \Rightarrow a = \frac{1}{5}g$

Question 6

$n^4 + (n+i)^4 = 2n^4 - 6n^2 + 1 + i(4n^3 - 4n)$

The imaginary part is zero for $n = 0, \pm 1$, hence three values.

Question 7

Let $\arg z = \theta$.

Then $0 < a < b \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow \pi < 4\theta < 2\pi$

i.e. $\pi < \arg(z^4) < 2\pi \Rightarrow \text{Im}(z^4) < 0$

Question 8

Only P is true.

P is true since there exists a (unique) real cube root for every real number

Q is false since each real number (y) can be the real cube root of only one real number (x)

Question 9

$v = 16 - x^2$

$a = v\frac{dv}{dx} = (16 - x^2) \times -2x = -42$ when $x = 3$

Question 10

$$\omega^2 + \frac{1}{1+\omega^2} = \omega$$

$$\omega^2(1+\omega^2) + 1 = \omega(1+\omega^2)$$

$$\omega^4 - \omega^3 + \omega^2 - \omega + 1 = 0$$

$$\omega^5 + 1 = 0 \quad (\text{where } \omega \neq -1)$$

$$\omega^5 = -1$$

Hence $\omega^{2024} = (\omega^5)^{404}\omega^4 = (-1)^{404}\omega^4 = \omega^4$

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$\frac{3-i}{1-2i}=\frac{3-i}{1-2i}\times\frac{1+2i}{1+2i}=1+i$$

$$\therefore\left|\frac{3-i}{1-2i}+k\right|=|k+1+i|=\sqrt{(k+1)^2+1}$$

$$\therefore\sqrt{(k+1)^2+1}=\sqrt{k+2}\Rightarrow k^2+2k+2=k+2$$

$$\therefore k^2+k=0$$

$$\Rightarrow k=0\text{ or }-1$$

Question 11(b) (i)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$\frac{1}{(x-2)(x-6)}\equiv\frac{A}{(x-2)}+\frac{B}{(x-6)}$$

$$1\equiv A(x-6)+B(x-2)$$

$$x=2\Rightarrow 1=-4A$$

$$x=6\Rightarrow 1=4B$$

$$\therefore A=-\frac{1}{4}\text{ and }B=\frac{1}{4}$$

Question 11 (b) (ii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$\begin{aligned}\int_3^5 \frac{dx}{(x-2)(x-6)} &= \frac{1}{4}\int_3^5 \left(\frac{1}{x-6}-\frac{1}{x-2}\right)dx \\ &= \frac{1}{4}\int_3^5 \left(\frac{-1}{6-x}-\frac{1}{x-2}\right)dx \\ &= \frac{1}{4}\Big[\ln(6-x)-\ln(x-2)\Big]_3^5 \\ &= \frac{1}{4}\big(\ln 1-\ln 3-(\ln 3-\ln 1)\big) \\ &= -\frac{1}{2}\ln 3\end{aligned}$$

Question 11(c) (i)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

Let $u=\sqrt{x} \Rightarrow du = \dfrac{1}{2\sqrt{x}}\,dx$

$x=1 \Rightarrow u=1$

$x=9 \Rightarrow u=3$

$$\begin{aligned}\int_1^9 \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)}\,dx &= 2\int_1^9 \frac{1}{\left(1+\sqrt{x}\right)}\frac{1}{2\sqrt{x}}\,dx \\ &= 2\int_1^3 \frac{1}{1+u}\,du \\ &= 2\Big[\ln(1+u)\Big]_1^3 \\ &= 2\ln 4-2\ln 2 \\ &= 2\ln 2\end{aligned}$$

Question 11(c) (ii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

Let $u = \sin^{-1}\theta \Rightarrow du = \dfrac{1}{\sqrt{1-\theta^2}}\,d\theta$

$$\begin{aligned}\int \frac{\sin^{-1}\theta}{\sqrt{1-\theta^2}}\,d\theta &= \int \sin^{-1}\theta \times \frac{1}{\sqrt{1-\theta^2}}\,d\theta \\ &= \int u\,du \\ &= \frac{1}{2}u^2 + c \\ &= \frac{1}{2}\left(\sin^{-1}\theta\right)^2 + c\end{aligned}$$

Question 11 (c) (iii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$\begin{aligned}\int \frac{\cos 2x}{\sin x \cos x}\,dx &= \int \frac{\cos^2 x - \sin^2 x}{\sin x \cos x}\,dx \\ &= \int \frac{\frac{d}{dx}(\sin x \cos x)}{\sin x \cos x}\,dx \\ &= \ln\left|\sin x \cos x\right| + c\end{aligned}$$

Question 11 (d) (i)

Criteria	Marks
• Provides correct solution	**1**

Sample answer:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \left(-\mathbf{i} + \mathbf{j} + 2\mathbf{k}\right) - \left(2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}\right)$$
$$\overrightarrow{AB} = -3\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$
$$\therefore \left|\overrightarrow{AB}\right| = \sqrt{9+9+25} = \sqrt{43}$$

Question 11 (d) (ii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$\overrightarrow{OA} \cdot \overrightarrow{OC} = \left(2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}\right) \cdot \left(m\mathbf{i} + 2\mathbf{j} - m^2\mathbf{k}\right)$$
$$= 2m + 8 + 3m^2$$
$$= 3\left(m + \frac{1}{3}\right)^2 + \frac{23}{3}$$
$$\neq 0$$

Hence there is no real value for m such that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OC}$.

Question 12(a)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$z^2 - (1-2i)z - 7 - i = 0$

$$z = \frac{(1-2i) \pm \sqrt{(1-2i)^2 + 4(7+i)}}{2}$$

$$= \frac{1-2i \pm \sqrt{1-4i-4+28+4i}}{2}$$

$$= \frac{1-2i \pm \sqrt{25}}{2}$$

$\therefore z = 3-i$ or $-2-i$

Question 12(b)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

Assume $\sqrt[3]{4}$ is rational, i.e. that $\sqrt[3]{4} = \frac{p}{q}$ where $p, q \in \mathbb{R}$ and have no common factor.

Then $p^3 = 4q^3 \Rightarrow p^3$ is even, hence p is even.

Say $p = 2r$

Then $8r^3 = 4q^3$, i.e. $q^3 = 2r^3 \Rightarrow q^3$ is even, hence q is even.

But this contradicts p, q having no common factors, hence $\sqrt[3]{4}$ cannot be rational.

Question 12(c) (i)

Criteria	Marks
• Provides correct solution	**1**

Sample answer:

Let $z = x + iy$, where x, y are real

Then $z = \overline{z}$ gives

$$x + iy = \overline{x + iy}$$
$$x + iy = x - iy$$

Equating imaginary parts gives $y = -y \Rightarrow y = 0$

i.e. z is real.

Question 12(c) (ii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$|z| = 1 \Rightarrow z\overline{z} = 1 \Rightarrow \overline{z} = \frac{1}{z}$$

Similarly, $\overline{w} = \dfrac{1}{w}$

$$\overline{\left(\frac{z+w}{1+zw}\right)} = \frac{\overline{(z+w)}}{\overline{(1+zw)}}$$
$$= \frac{\overline{z}+\overline{w}}{1+\overline{zw}}$$
$$= \frac{\overline{z}+\overline{w}}{1+\overline{z}\times\overline{w}}$$
$$= \frac{\frac{1}{z}+\frac{1}{w}}{1+\frac{1}{z}\times\frac{1}{w}}$$
$$= \frac{w+z}{zw+1}$$
$$= \frac{z+w}{1+zw}$$

By part (i), $\dfrac{z+w}{1+zw}$ is real.

Question 12(d)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$v\frac{dv}{dx} = v^3 + 4v \Rightarrow \frac{dv}{dx} = v^2 + 4$$

$$\therefore \frac{dx}{dv} = \frac{1}{v^2+4} \Rightarrow x = \frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) + c$$

$$\begin{cases} x = 0 \\ v = 2 \end{cases} \Rightarrow c = -\frac{\pi}{8} \Rightarrow x = \frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) - \frac{\pi}{8}$$

Question 12(e) (i)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

We need to show both that the lines are not parallel and do not intersect.

l_1 has direction $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and l_2 has direction $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Hence l_1 and l_2 are not parallel.

Assume the lines intersect, i.e. that $\exists$ real s, t such that

$$s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

From the x and z components, $s = t$ and $s = t + 2$, a contradiction. Hence the lines do not meet.

Since the lines do not meet and are not parallel, they are skew.

Question 12(e) (ii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

Since $\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=1-2+1=0,$ the angle between the lines is 90°.

Question 13(a)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

Consider $x=-3$:

then $|2(-3)+1|=5$ but $|x|>2$

Hence the assertion is false.

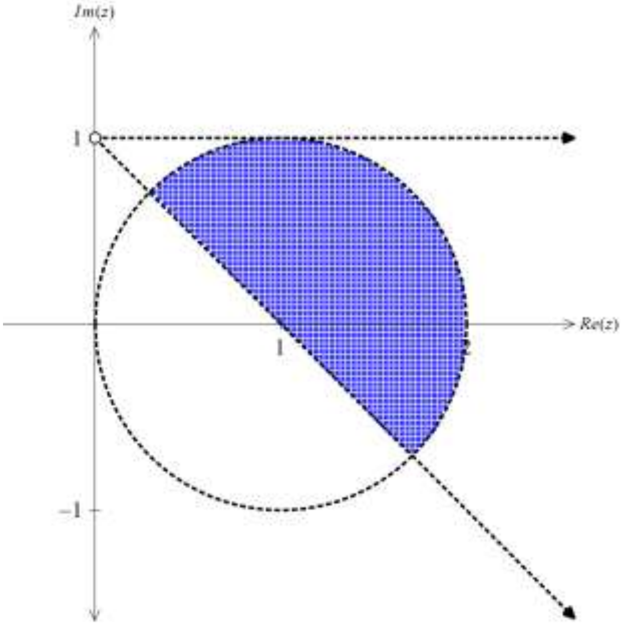
Question 13(b) (i)

Criteria	Marks
• Provides correct solution	**4**
• Makes progress towards sketching both regions	**3**
• Correctly identifies both of the regions OR • Correctly sketches one of the two regions	**2**
• Correctly identifies one of the two regions	**1**

Sample answer:

$$\frac{\pi}{4}\le\arg\left(iz+1\right)\le\frac{\pi}{2}\Rightarrow-\frac{\pi}{4}\le\arg\left(z-i\right)\le 0$$

The region is inside the sector starting at (0,1) between $y=1$ and $y=1-x$ and also on/inside the circle $\left(x-1\right)^2+y^2=1$.



Question 13(b) (ii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

The point closest to (0,-1) is the centre of $(x-1)^2+y^2=1$, hence the minimum value of $|z+i|$ is the distance from (0,-1) to (1,0) which is $\sqrt{2}$.

For the maximum modulus, z must be on the boundary of $|z-1|\le 1$, i.e. that $|z-1|=1$, hence $|z+i|=|(z-1)+(1+i)|\le|z-1|+|1+i|=1+\sqrt{2}$.

So the range of values is $\sqrt{2}\le|z+i|\le 1+\sqrt{2}$.

Question 13(c) (i)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

Method 1:

$$\left(\sqrt{x-1}-1\right)^2 \ge 0$$

$$x-1-2\sqrt{x-1}+1 \ge 0$$

$$x \ge 2\sqrt{x-1}$$

$$\frac{x}{\sqrt{x-1}} \ge 2, \quad \text{since } \sqrt{x-1} > 0$$

Method 2: using the AM-GM inequality,

$$\begin{aligned}\frac{x}{\sqrt{x-1}} &= \frac{x-1+1}{\sqrt{x-1}} \\ &= \sqrt{x-1}+\frac{1}{\sqrt{x-1}} \quad . \\ &\ge 2\sqrt{\sqrt{x-1}\times\frac{1}{\sqrt{x-1}}} \\ &= 2\end{aligned}$$

Question 13(c) (ii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

Using the AM-GM inequality,

$$\begin{aligned}\frac{a^2}{b-1}+\frac{b^2}{a-1} &\ge 2\sqrt{\frac{a^2}{b-1}\times\frac{b^2}{a-1}} \\ &= 2\sqrt{\frac{a^2}{a-1}}\sqrt{\frac{b^2}{b-1}} \\ &= 2\frac{a}{\sqrt{a-1}}\frac{b}{\sqrt{b-1}}, \quad \text{since } a,b>0 \\ &\ge 2\times 2\times 2, \qquad \text{using part (i)} \\ &= 8\end{aligned}$$

Question 13(d)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$\frac{2\pi}{n} = \frac{\pi}{6} \Rightarrow n = 12$

Amplitude is $\sqrt{2} - \left(-2\sqrt{2}\right) = 3\sqrt{2}$

$\therefore x = 3\sqrt{2}\sin\left(12t + \alpha\right) + \sqrt{2}$ for some α

Since the particle starts at $x = -2\sqrt{2}, \sin\alpha = -1 \Rightarrow \alpha = -\frac{\pi}{2}$

$\therefore x = 3\sqrt{2}\sin\left(12t - \frac{\pi}{2}\right) + \sqrt{2}$

Alternative answers include $3\sqrt{2}\cos\left(12t + \pi\right) + \sqrt{2}$

$x = -2\sqrt{2}$, zero acceleration at $x = \sqrt{2}$ and has period $\frac{\pi}{6}$.

Question 14(a)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

Let P_n be the proposition that $a_n = 5(-2)^n + 4\times 3^n$ for $n \in \mathbb{Z}^+$.

Since $5(-2)^1 + 4\times 3^1 = -10+12 = 2$ and $5(-2)^2 + 4\times 3^2 = 20+36 = 56$, both P_1 and P_2 are true.

Assume both P_{k-2} and P_{k-1} are true, where $k-2 \ge 1$, *i.e.* $k \ge 3$, that is

$$a_{k-2} = 5(-2)^{k-2} + 4\times 3^{k-2} \text{ and } a_{k-1} = 5(-2)^{k-1} + 4\times 3^{k-1}$$

Then:

$$\begin{aligned}
a_k &= a_{k-1} + 6a_{k-2} \\
&= 5(-2)^{k-1} + 4\times 3^{k-1} + 6\left(5(-2)^{k-2} + 4\times 3^{k-2}\right) \\
&= 5(-2)^{k-1} + 30(-2)^{k-2} + 4\times 3^{k-1} + 24\times 3^{k-2} \\
&= (-2)^{k-2}[-10+30] + 3^{k-2}[12+24] \\
&= 20\times(-2)^{k-2} + 36\times 3^{k-2} \\
&= 5(-2)^k + 4\times 3^k
\end{aligned}$$

$\therefore P_k$ is true whenever both P_{k-2} and P_{k-1} are true.

Since P_1 and P_2 are true, then P_k is true by mathematical induction.

Question 14(b) (i)

Criteria	Marks
• Provides correct solution	**1**

Sample answer:

Since $a(-1)+a+a = a$, $\begin{pmatrix}-1\\a\\a\end{pmatrix}$ lies on $ax+y+z=a$

Also, $-1-a\times a + a\times a = -1$, $\begin{pmatrix}-1\\a\\a\end{pmatrix}$ lies on $x-ay+az=-1$. So $\begin{pmatrix}-1\\a\\a\end{pmatrix}$ lies on both planes.

Question 14(b) (ii)

Criteria	Marks
• Provides correct solution	**3**
• Makes some progress towards a correct solution	**2**

Sample answer:

The normal direction to π_1 is $\begin{pmatrix} a \\ 1 \\ 1 \end{pmatrix}$ and to π_2 is $\begin{pmatrix} 1 \\ -a \\ a \end{pmatrix}$. Let $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ be the direction vector of l.

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}\begin{pmatrix} a \\ 1 \\ 1 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} p \\ q \\ r \end{pmatrix}\begin{pmatrix} 1 \\ -a \\ a \end{pmatrix} = 0$$

Then i.e. $ap + q + r = 0$ (1)

$p - aq + ar = 0$ (2)

Treating (1) and (2) as simultaneous equations between variables p and q**:**

$ap + q = -r$ (3)

$p - aq = -ar$ (4)

$a \times (3) + (4) \Rightarrow \left(a^2 + 1\right) p = -2ar$

$(3) - a \times (4) \Rightarrow \left(1 + a^2\right) q = \left(a^2 - 1\right) r$

$\therefore\ p = \dfrac{-2ar}{a^2 + 1}, q = \dfrac{\left(a^2 - 1\right) r}{a^2 + 1}$

Taking $r = a^2 + 1, p = -2a, q = a^2 - 1$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -2a \\ a^2 - 1 \\ a^2 + 1 \end{pmatrix}$$

Question 14(b) (iii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$r = \begin{pmatrix} -1 \\ a \\ a \end{pmatrix} + \lambda \begin{pmatrix} -2a \\ a^2-1 \\ a^2+1 \end{pmatrix}$$

l has equation $\Rightarrow x = -1 - 2a\lambda \Rightarrow \lambda = \dfrac{x+1}{-2a}$

Similarly, $\lambda = \dfrac{y-a}{a^2-1}$ and $\lambda = \dfrac{z-a}{a^2+1}$

Hence $\dfrac{x+1}{-2a} = \dfrac{y-a}{a^2-1} = \dfrac{z-a}{a^2+1} \quad (=\lambda)$

Question 14(b) (iv)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

From $\dfrac{x+1}{-2a} = \dfrac{y-a}{a^2-1} = \dfrac{z-a}{a^2+1}$

when $z = t$:

$$\frac{x+1}{-2a} = \frac{t-a}{a^2+1}$$

$$\Rightarrow x+1 = \frac{-2a}{a^2+1}(t-a)$$

$$x = \frac{2a^2 - 2at - (a^2+1)}{a^2+1}$$

$$x = \frac{-2at + a^2 - 1}{a^2+1}$$

$$\frac{y-a}{a^2-1} = \frac{t-a}{a^2+1}$$

$$\Rightarrow y - a = \frac{(t-a)(a^2-1)}{a^2+1}$$

$$y = \frac{(a^2-1)t - a^3 + a + a(a^2+1)}{a^2+1}$$

$$y = \frac{(a^2-1)t + 2a}{a^2+1}$$

$$x^2+y^2=\left(\frac{-2at+a^2-1}{a^2+1}\right)^2+\left(\frac{\left(a^2-1\right)t+2a}{a^2+1}\right)^2$$

$$=\frac{\left(-2at+a^2-1\right)^2+\left(\left(a^2-1\right)t+2a\right)^2}{\left(a^2+1\right)^2}$$

$$=\frac{(-2at)^2-4at\left(a^2-1\right)+\left(a^2-1\right)^2+\left(\left(a^2-1\right)t\right)^2+4a\left(a^2-1\right)t+4a^2}{\left(a^2+1\right)^2}$$

$$=\frac{4a^2t^2-\cancel{4at\left(a^2-1\right)}+a^4-2a^2+1+\left(a^2-1\right)^2t^2+\cancel{4a\left(a^2-1\right)t}+4a^2}{\left(a^2+1\right)^2}$$

$$=\frac{\left(4a^2+\left(a^2-1\right)^2\right)t^2+a^4+2a^2+1}{\left(a^2+1\right)^2}$$

$$=\frac{\left(a^2+1\right)^2t^2+\left(a^2+1\right)^2}{\left(a^2+1\right)^2}$$

$$x^2+y^2=t^2+1$$

Hence, when $z=t,(x,y)$ satisfy the equation of a circle, centre $(0,0)$, radius $\sqrt{t^2+1}$ that is, l intersects the plane $z=t$ in a circle, centre $(0,0,t)$ with radius $\sqrt{t^2+1}$

Question 14(c) (i)

Criteria	Marks
• Provides correct solution	**1**

Sample answer:

$\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R},\ y(y+1) = x$

Question 14(c) (ii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

Let x be an arbitrary positive real number, and let $y = \dfrac{-1+\sqrt{1+4x}}{2}$.

Then

$$\begin{aligned}
y(y+1) &= \left(\frac{-1+\sqrt{1+4x}}{2}\right)\left(\frac{-1+\sqrt{1+4x}}{2}+1\right) \\
&= \left(\frac{-1+\sqrt{1+4x}}{2}\right)\left(\frac{1+\sqrt{1+4x}}{2}\right) \\
&= \frac{1}{4}\left(\sqrt{1+4x}-1\right)\left(\sqrt{1+4x}+1\right) \\
&= \frac{1}{4}(1+4x-1) \\
&= x
\end{aligned}$$

Hence, for every positive real number x there is a real number y such that $y(y+1) = x$.

Question 15(a) (i)

Criteria	Marks
• Provides correct solution	**1**

Sample answer:

$I_0 = \int_0^1 x^{2\times 0}\sqrt{1-x^2}\,dx = \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$ as it represents the area inside the unit circle in the first quadrant.

Question 15(a) (ii)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$
\begin{aligned}
I_n &= \int_0^1 x^{2n}\sqrt{1-x^2}\,dx \\
&= \int_0^1 x^{2n-1} \times x\sqrt{1-x^2}\,dx \\
&= \int_0^1 \underbrace{x^{2n-1}}_{u} \times \underbrace{\frac{d}{dx}\left(-\frac{1}{3}\left(1-x^2\right)^{3/2}\right)}_{v'} dx \\
&= \left[\underbrace{x^{2n-1} \times -\frac{1}{3}\left(1-x^2\right)^{3/2}}_{uv}\right]_0^1 - \int_0^1 \underbrace{(2n-1)x^{2n-2}}_{u'} \times \underbrace{-\frac{1}{3}\left(1-x^2\right)^{3/2}}_{v} dx \\
&= 0 + \frac{(2n-1)}{3}\int_0^1 x^{2n-2}\left(1-x^2\right)^{3/2} dx \\
&= \frac{(2n-1)}{3}\int_0^1 x^{2n-2}\left(1-x^2\right)\sqrt{1-x^2}\,dx \\
&= \frac{(2n-1)}{3}\left(I_{n-1} - I_n\right)
\end{aligned}
$$

$$
\begin{aligned}
I_n &= \frac{(2n-1)}{3}\left(I_{n-1} - I_n\right) \\
3I_n &= (2n-1)I_{n-1} - (2n-1)I_n \\
(2n+2)I_n &= (2n-1)I_{n-1} \Rightarrow I_n = \frac{2n-1}{2n+2}I_{n-1}
\end{aligned}
$$

Question 15(a) (iii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$I_3 = \int_0^1 x^6 \sqrt{1-x^2}\, dx$$

$$I_0 = \frac{\pi}{4} \text{ and } I_n = \frac{2n-1}{2n+2} I_{n-1}$$

$$I_1 = \frac{1}{4} I_0 \quad \Rightarrow \quad I_1 = \frac{\pi}{16}$$

$$I_2 = \frac{3}{6} I_1 \quad \Rightarrow \quad I_2 = \frac{\pi}{32}$$

$$I_3 = \frac{5}{8} I_2 \quad \Rightarrow \quad I_3 = \frac{5\pi}{256}$$

$$\int_0^1 x^6 \sqrt{1-x^2}\, dx = \frac{5\pi}{256}$$

Question 15(b) (i)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$m\ddot{x} = -mg - kmv^2$, $\ddot{x} = -(g + kv^2)$

$v\dfrac{dv}{dx} = -(g + kv^2)$, $\dfrac{dx}{dv} = -\dfrac{v}{g + kv^2}$, $x = -\dfrac{1}{2k}\log_e(g + kv^2) + c_1$

When $x = 0$, $v = U$ $\therefore c_1 = \dfrac{1}{2k}\log_e(g + kU^2)$, $x = \dfrac{1}{2k}\log_e\left(\dfrac{g + kU^2}{g + kv^2}\right)$

Sub $v = 0$: Greatest height is $\dfrac{1}{2k}\log_e\left(\dfrac{g + kU^2}{g}\right)$

Question 15(b) (ii)

Criteria	Marks
• Provides correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$m\ddot{y} = mg - mkv^2$, $\ddot{y} = g - kv^2$, $v\dfrac{dv}{dy} = g - kv^2$, $\dfrac{dy}{dv} = \dfrac{v}{g - kv^2}$

$y = -\dfrac{1}{2k}\log_e(g - kv^2) + c_2$

When $y = 0$, $v = 0$:

$\therefore c_2 = \dfrac{1}{2k}\log_e g$, $y = \dfrac{1}{2k}\log_e\left(\dfrac{g}{g - kv^2}\right)$

Question 15(b) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

When the particle returns to the point of projection:

$$\frac{1}{2k}\log_e\left(\frac{g+kU^2}{g}\right)=\frac{1}{2k}\log_e\left(\frac{g}{g-kV^2}\right)$$

$$\frac{g+kU^2}{g}=\frac{g}{g-kV^2}$$

$$(g+kU^2)(g-kV^2)=g^2$$

$$g^2-gkV^2+gkU^2-k^2U^2V^2=g^2$$

$$gkU^2-gkV^2=k^2U^2V^2$$

$$\frac{gk\left(U^2-V^2\right)}{U^2V^2}=k^2$$

$$\frac{\left(U^2-V^2\right)}{U^2V^2}=\frac{k}{g}$$

$$\frac{1}{V^2}-\frac{1}{U^2}=\frac{k}{g}$$

Question 15(c) (i)

Criteria	Marks
• Provides correct solution	3
• Makes significant progress towards a correct solution	2
• Makes some progress towards a correct solution	1

Sample answer:

$$\ddot{x}=-k\dot{x}$$

$$\Rightarrow\frac{d}{dt}\left(\dot{x}\right)=-k\dot{x}$$

$$\Rightarrow\frac{1}{\dot{x}}\frac{d}{dt}\left(\dot{x}\right)=-k$$

$$\Rightarrow\int\frac{1}{\dot{x}}d\dot{x}=-\int kdt$$

$$\Rightarrow\ln\left(\dot{x}\right)=-kt+c_1$$

$$t=0,\dot{x}=v_0\cos\theta\Rightarrow c_1=\ln\left(v_0\cos\theta\right)\Rightarrow -kt=\ln\left(\dot{x}\right)-\ln\left(v_0\cos\theta\right)=\ln\left(\frac{\dot{x}}{v_0\cos\theta}\right)$$

$$\therefore\dot{x}=v_0\cos\theta\, e^{-kt}$$

$$\frac{dx}{dt} = v_0 \cos\theta\, e^{-kt}$$

$$\Rightarrow x = \int v_0 \cos\theta\, e^{-kt} dt$$

$$\Rightarrow x = -\frac{v_0 \cos\theta}{k} e^{-kt} + c_2$$

$$t = 0, x = 0 \Rightarrow c_2 = \frac{v_0 \cos\theta}{k}$$

$$\therefore x = \frac{v_0 \cos\theta}{k}\left(1 - e^{-kt}\right)$$

Question 15(c) (ii)

Criteria	Marks
• Provides correct solution	**1**

Sample answer:

$$x = \frac{v_0 \cos\theta}{k}\left(1 - e^{-kt}\right)$$

Since $k > 0$, as $t \to \infty, e^{-kt} \to 0 \Rightarrow x \to \dfrac{v_0 \cos\theta}{k}$

Question 16(a)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

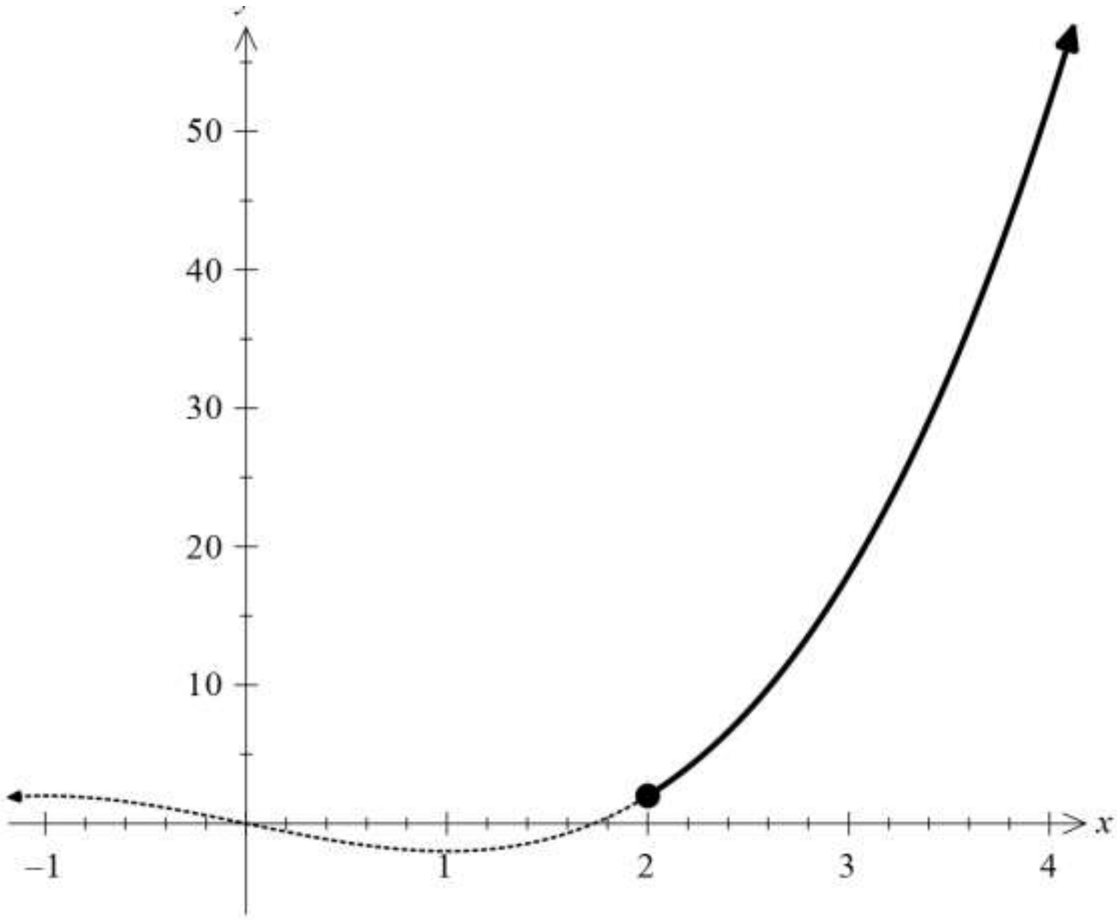
Sample answer:

$$\underset{\sim}{r}(t) = \begin{pmatrix} t + t^{-1} \\ t^3 + t^{-3} \end{pmatrix}$$

$x = t + \frac{1}{t} \Rightarrow x^3 = t^3 + 3t + \frac{3}{t} + \frac{1}{t^3} = y + 3x$

$\therefore\ y = x^3 - 3x$ is the Cartesian equation

Since $t > 0, x = t + \frac{1}{t} \geq 2\sqrt{t \times \frac{1}{t}} = 2,$ so the path is restricted to $y = x^3 - 3x$ for $x \geq 2.$



Question 16(b) (i)

Criteria	Marks
• Provides correct solution	**1**

Sample answer:

$$\cos x \equiv A\left(3\cos x + 4\sin x\right) + B\left(4\cos x - 3\sin x\right)$$
$$\cos x \equiv \left(3A + 4B\right)\cos x + \left(4A - 3B\right)\sin x$$
$$x = 0 \Rightarrow 1 = 3A + 4B \quad (1)$$
$$x = \frac{\pi}{2} \Rightarrow 0 = 4A - 3B \quad (2)$$
$$3\times(1) + 4\times(2) \Rightarrow A = \frac{3}{25} \text{ and } 4\times(1) - 3\times(2) \Rightarrow B = \frac{4}{25}$$

Question 16(b) (ii)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$\int \frac{e^{x/2}\cos x}{\sqrt[3]{3\cos x + 4\sin x}}\,dx = \int \frac{e^{x/2}\left[\frac{3}{25}\left(3\cos x + 4\sin x\right) + \frac{4}{25}\left(4\cos x - 3\sin x\right)\right]}{\sqrt[3]{3\cos x + 4\sin x}}\,dx$$
$$= \frac{3}{25}\int\left(\frac{\left(3\cos x + 4\sin x\right)e^{x/2}}{\sqrt[3]{3\cos x + 4\sin x}} + \frac{\frac{4}{3}\left(4\cos x - 3\sin x\right)e^{x/2}}{\sqrt[3]{3\cos x + 4\sin x}}\right)dx$$
$$= \frac{6}{25}\int\left(\underbrace{\left(3\cos x + 4\sin x\right)^{2/3}}_{u}\underbrace{\left(\frac{1}{2}e^{x/2}\right)}_{\frac{dv}{dx}} + \underbrace{\frac{2}{3}\left(3\cos x + 4\sin x\right)^{-1/3}\left(-3\sin x + 4\cos x\right)}_{\frac{du}{dx}}\underbrace{e^{x/2}}_{v}\right)dx$$
$$= \frac{6}{25}\int\left(uv' + u'v\right)dx$$
$$= \frac{6}{25}\int\left(uv\right)'\,dx$$
$$= \frac{6}{25}uv + c$$
$$= \frac{6}{25}\left(3\cos x + 4\sin x\right)^{2/3}e^{x/2} + c$$

Question 16(c) (i)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:

$$p(z) = az^2 + bz + c$$

$$p(0) = u \Rightarrow \underline{c = u}$$

$$p(1) = v \Rightarrow a + b + u = v, \qquad \text{i.e.} \quad a + b = v - u \quad (1)$$

$$p(i) = w \Rightarrow -a + bi + u = w \quad \text{i.e.} - a + bi = w - u \quad (2)$$

$$(1) + (2) \Rightarrow (1+i)b = v + w - 2u$$

$$\begin{aligned} b &= \frac{1}{1+i}(v + w - 2u) \\ &= \frac{1-i}{2}(v + w - 2u) \\ &\underline{b = (-1+i)u + \left(\frac{1-i}{2}\right)v + \left(\frac{1-i}{2}\right)w} \end{aligned}$$

$$(1) + i \times (2) \Rightarrow (1-i)a = v - u + i(w-u) = -(1+i)u + v + iw$$

$$\begin{aligned} a &= \frac{1}{1-i}\left(-(1+i)u + v + iw\right) \\ &= \frac{1+i}{2}\left(-(1+i)u + v + iw\right) \\ &= -\frac{(1+i)^2}{2}u + \left(\frac{1+i}{2}\right)v + i\left(\frac{1+i}{2}\right)w \\ &\underline{a = -iu + \left(\frac{1+i}{2}\right)v + \left(\frac{-1+i}{2}\right)w} \end{aligned}$$

Question 16(c) (ii)

Criteria	Marks
• Provides correct solution	**1**

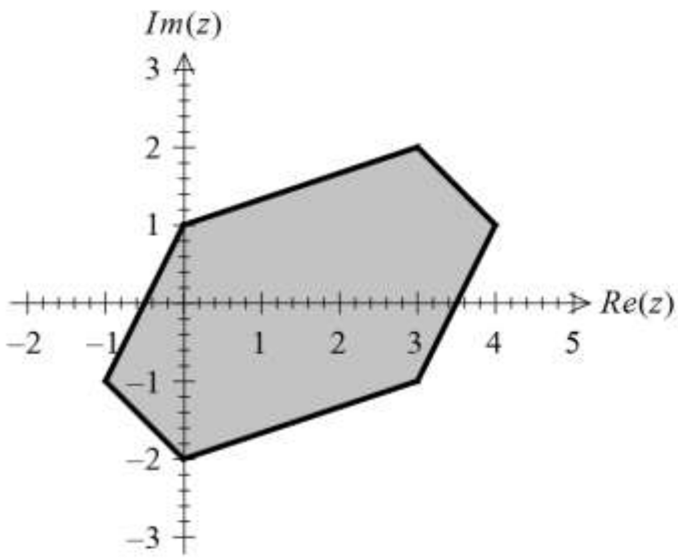
Sample answer:

$$f(2) = 4a + 2b + c$$
$$= 4\left(-iu + \left(\frac{1+i}{2}\right)v + \left(\frac{-1+i}{2}\right)w\right) + 2\left((-1+i)u + \left(\frac{1-i}{2}\right)v + \left(\frac{1-i}{2}\right)w\right) + u$$
$$= -4iu + 2(-1+i)u + u + 2(1+i)v + (1-i)v + 2(-1+i)w + (1-i)w$$
$$f(2) = (-1-2i)u + (3+i)v + (-1+i)w$$

Question 16(c) (iii)

Criteria	Marks
• Provides correct solution	**3**
• Makes significant progress towards a correct solution	**2**
• Makes some progress towards a correct solution	**1**

Sample answer:



2024 PEM Mathematics Extension 2

Mapping Grid

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Section II

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12 (c) (i)	1	MEX-N1 Introduction to Complex Numbers	MEX12-1
12 (c) (ii)	2	MEX-N1 Introduction to Complex Numbers	MEX12-1
12 (d)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
12 (e) (i)	3	MEX-V1 Further Work with Vectors	MEX12-3
12 (e) (ii)	2	MEX-V1 Further Work with Vectors	MEX12-3

Question	Marks	Content	Syllabus Outcomes
13 (a)	2	MEX-P1 The Nature of Proof	MEX12-2
13 (b) (i)	4	MEX-N2 Using Complex Numbers	MEX12-4
13 (b) (ii)	2	MEX-N2 Using Complex Numbers	MEX12-4
13(c) (i)	2	MEX-P1 The Nature of Proof	MEX12-2
13 (c) (ii)	2	MEX-P1 The Nature of Proof	MEX12-2
13 (d)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
14 (a)	3	MEX-P2 Further Proof by Mathematical Induction	MEX12-2
14 (b) (i)	1	MEX-V1 Further Work with Vectors	MEX12-3
14 (b) (ii)	2	MEX-V1 Further Work with Vectors	MEX12-3
14 (b) (iii)	2	MEX-V1 Further Work with Vectors	MEX12-3
14 (b) (iv)	3	MEX-V1 Further Work with Vectors	MEX12-3
14 (c) (i)	1	MEX-P1 The Nature of Proof	MEX12-2
14 (c) (ii)	2	MEX-P1 The Nature of Proof	MEX12-2
15 (a) (i)	1	MEX-C1 Further Integration	MEX12-5
15 (a) (ii)	3	MEX-C1 Further Integration	MEX12-5
15 (a) (iii)	2	MEX-C1 Further Integration	MEX12-5
15 (b) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
15 (b) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
15 (b) (iii)	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
15 (c) (i)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
15 (c) (ii)	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
16 (a)	3	MEX-V1 Further Work with Vectors	MEX12-3
16 (b) (i)	1	MEX-C1 Further Integration	MEX12-5
16 (b) (ii)	3	MEX-C1 Further Integration	MEX12-5
16 (c) (i)	3	MEX-N1 Introduction to Complex Numbers	MEX12-1
16 (c) (ii)	1	MEX-N1 Introduction to Complex Numbers	MEX12-1
16 (c) (iii)	3	MEX-N2 Using Complex Numbers	MEX12-4