CSSA

2024

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

DO NOT REMOVE PAPER FROM EXAMINATION ROOM

Centre Number

Student Number

Mathematics Extension 1

Afternoon Session
Friday 16 August 2024

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- A normal cumulative distribution function table is provided
- Use the Multiple-Choice Answer Sheet provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 70

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6–11)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1–10

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1 What is the coefficient of x^3 in the binomial expansion $(2x-3)^7$?

A. $-22\,680$

B. $-15\,120$

C. $15\,120$

D. $22\,680$

2 Which of the following expressions is equivalent to $\frac{d}{dx}\left(2\arcsin\frac{x}{2}\right)$?

A. $\frac{1}{\sqrt{1-x^2}}$

B. $\frac{2}{\sqrt{1-x^2}}$

C. $\frac{1}{\sqrt{4-x^2}}$

D. $\frac{2}{\sqrt{4-x^2}}$

3 The recent Australian Census found that 30% of Australians live in rented housing. A random sample of 50 people is taken.

What is the standard deviation of the proportion of people who rent in this sample?

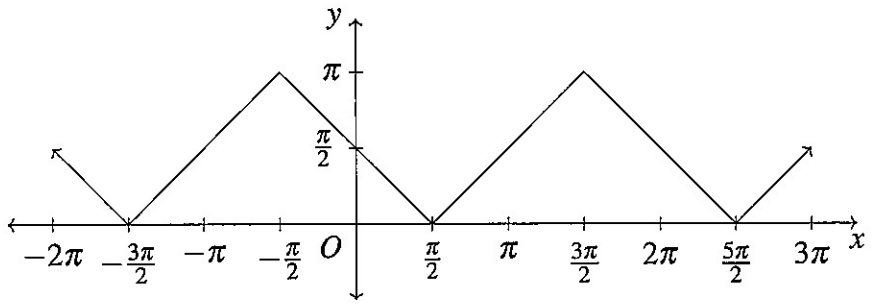
A. 0.00420

B. 0.0648

C. 1.48

D. 3.24

4 A function $y = f(x)$ is graphed below.



Which of the following is the equation of the function?

A. $y = \cos^{-1}(\sin x)$

B. $y = \sin^{-1}(\cos x)$

C. $y = \cos(\sin^{-1} x)$

D. $y = \sin(\cos^{-1} x)$

5 A polynomial function $P(x)$ has a remainder of $2x+3$ when divided by $4x^2 - 1$.

What is the remainder when $P(x)$ is divided by $2x - 1$?

A. $-\frac{5}{2}$

B. -2

C. $\frac{7}{2}$

D. 4

6 Eight cards, each marked with a different digit from 1 to 8, are randomly placed in two rows of four.

What is the probability that the odd numbered cards are in one row?

A. $\frac{1}{70}$

B. $\frac{1}{35}$

C. $\frac{1}{24}$

D. $\frac{1}{12}$

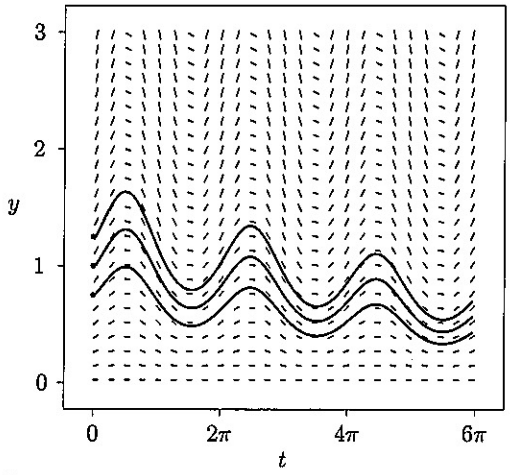
7 A differential equation is defined for $t \geqslant 0$ and $y > 0$ as

$$\frac{dy}{dt} = (\cos t + k)y,$$

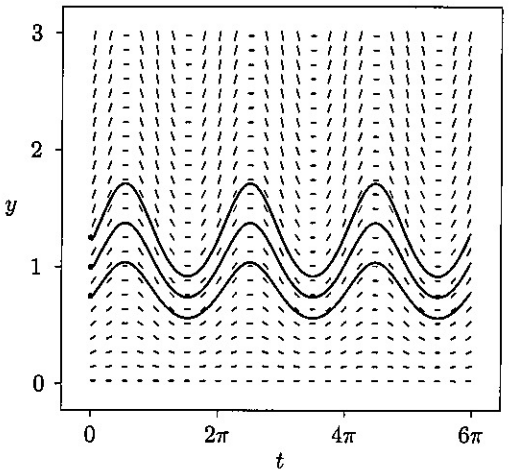
where k is a constant.

The diagrams below each show a direction field (slope field) with three solution curves. Which direction field best represents the differential equation when $k > 0$?

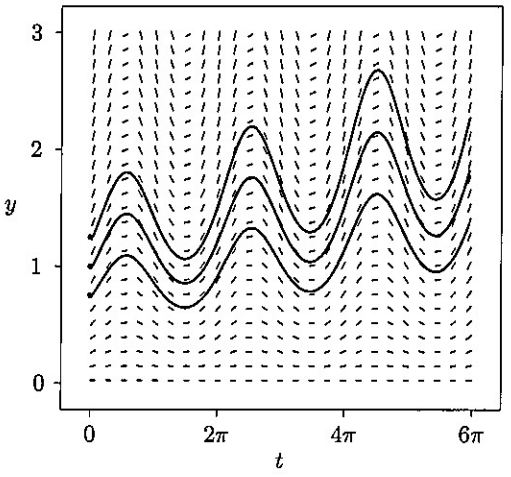
A.



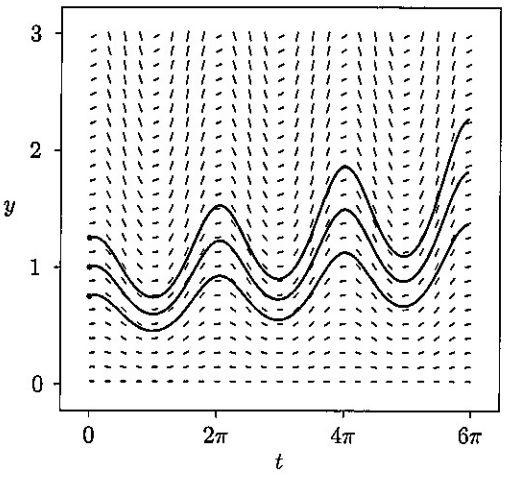
B.



C.



D.



8 Given the function $f(x) = e^{-x} + ex,\ x \geq -1$.

What is the gradient of the inverse function $y = f^{-1}(x)$ at $(1,0)$?

A. $e-1$

B. $\dfrac{1}{e-1}$

C. $\dfrac{e^2-1}{e}$

D. $\dfrac{e}{e^2-1}$

9 The parametric equations of a function are given below.

$$x = 2at$$
$$y = at^2$$

The function is dilated horizontally by 2 and dilated vertically by 3.

What is the gradient of this function at $x = a$?

A. $\dfrac{3}{8}$

B. $\dfrac{3}{2}$

C. 3

D. 6

10 Louis has a large set of two types of cards, one showing the direction vector $\begin{pmatrix}1\\0\end{pmatrix}$ and the other showing the direction vector $\begin{pmatrix}0\\1\end{pmatrix}$.

How many unique combinations of cards can be formed to describe a sequence of moves from the origin to the point (m,n) where m and n are positive integers?

A. $m!+n!$

B. $\dfrac{m!n!}{(m+n)!}$

C. $\dfrac{(m+n)!}{m!n!}$

D. $m!n!$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. EXTRA writing booklets are available.

Your responses for Questions 11–14 should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Solve $\sin 2\theta + \cos\theta = 0$ for $0 \leqslant \theta \leqslant 2\pi$. **3**

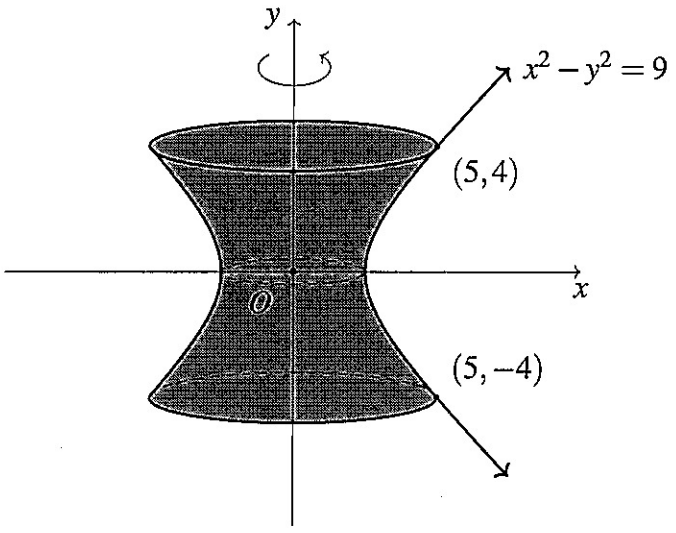
(b) Find the angle, to the nearest degree, between the two position vectors **3**

$$\underset{\sim}{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

(c) Solve $\dfrac{1}{x-1} \geqslant 2$. **3**

(d) Use the substitution $u = 1-x$ to evaluate $\displaystyle\int_0^1 x(1-x)^{10}\,dx$. **3**

(e) Find the exact volume of the hyperboloid created when the curve $x^2 - y^2 = 9$, between $(5,4)$ and $(5,-4)$, is rotated about the y-axis. **3**

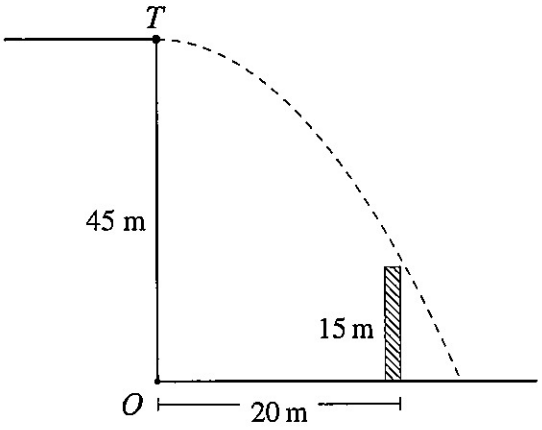


Question 12 (14 marks)

(a) The polynomial $P(x) = x^3 + x^2 - mx - 3$ has two of its roots equal in magnitude and opposite in sign. Find the value of m. **2**

(b) Write $3\sin x + \sqrt{3}\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(c) An object is launched horizontally from the point T at the top of a 45 metre tower. A 15 metre high wall is 20 metres from O.



The vector displacement function for this object, with initial horizontal velocity V m/s, is given by

$$\underset{\sim}{r}(t) = \begin{bmatrix} Vt \\ 45 - 5t^2 \end{bmatrix}$$

where t is the time in seconds.

(i) Show that the value of V such that the object clears the wall must be greater than $\frac{20}{\sqrt{6}}$ m/s. **2**

(ii) The object is fired from the wall with a horizontal velocity of 9 m/s. Find the exact impact speed as it hits the ground. **2**

(d) Using polynomial division, or otherwise, evaluate $\int_0^1 \frac{x^3}{x+1}\,dx$. **3**

(e) A bag contains blue and red balls in the ratio 2 : 3. A ball is drawn from the bag, its colour recorded, and it is returned to the bag. This is repeated 25 times. **3**

Let X be the random variable representing the number of red balls.

Use the standard normal table on page 13 to estimate the probability that there are more red balls drawn from the bag.

Question 13 (16 marks)

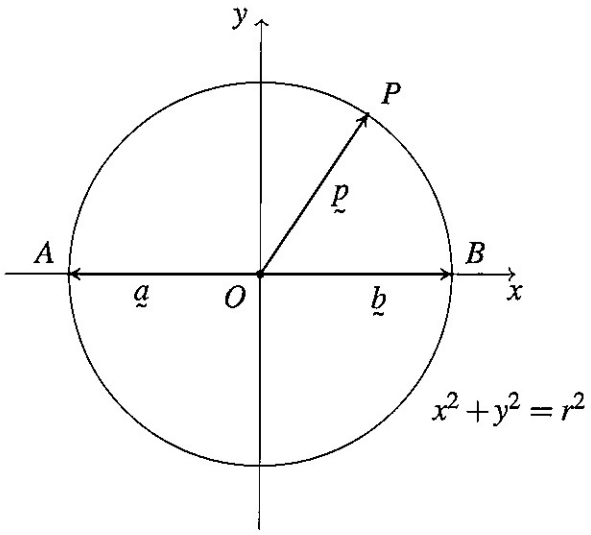
(a) Find $\int \sin x \sin 2x \, dx$. **2**

(b) Prove by mathematical induction that, for all integers, $n \geqslant 1$ **3**

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n}.$$

(c) The point P with position vector $\underset{\sim}{p}=\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ lies on the circle $x^2+y^2=r^2$. **3**

Points A and B are the x-intercepts of the circle with position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, respectively.



Show that $|\underset{\sim}{p}-\underset{\sim}{a}|^2+|\underset{\sim}{p}-\underset{\sim}{b}|^2=4r^2$.

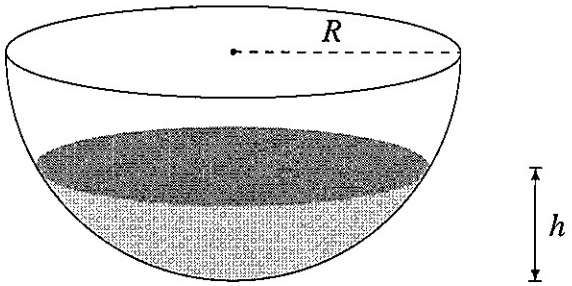
Question 13 continues on page 9

Question 13 (continued)

(d) The volume of water in a hemispherical bird bath of radius R metres is given by **3**

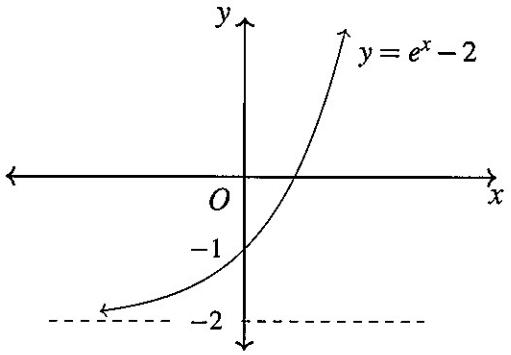
$$V = \frac{1}{3}\pi h^2(3R - h)$$

where h is height of the water in metres. (Do NOT prove this.)



A bird bath of radius 0.5 m is being filled with water at a constant rate of $0.03\ \text{m}^3/\text{minute}$. Calculate the exact rate at which the height of water is increasing when $h = 0.25$ m.

(e) Graphed below is the function $f(x) = e^x - 2$.



(i) Sketch the function $y = f(|x|)$. **1**

(ii) On the same diagram, sketch the function $y = \dfrac{1}{f(|x|)}$, clearly identifying all significant features (expressed in exact form). **2**

(iii) Hence, or otherwise, find the exact solutions to $f(|x|) = \dfrac{1}{f(|x|)}$. **2**

End of Question 13

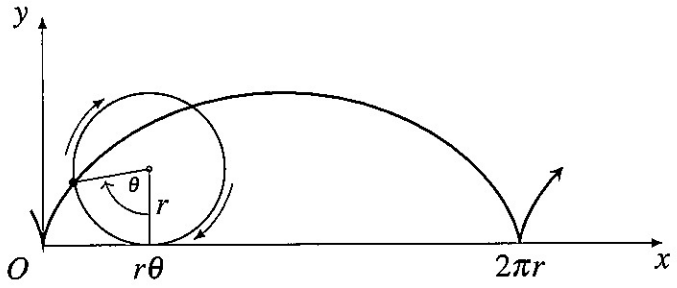
Question 14 (15 marks)

(a) Points A and B have position vectors $\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j}$ and $\overrightarrow{OB} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$.

(i) Find the projection of $\overrightarrow{AO}$ onto $\overrightarrow{AB}$. **2**

(ii) Hence, or otherwise, find the perpendicular distance of $\overrightarrow{AB}$ from O. **2**

(b) A cycloid is the path traced by a point on the circumference of a circle as it rolls along the x-axis. **5**



For a circle of radius r, it is described by the parametric equations

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$

where θ is the angle the radius to the particular point on the circumference makes with a vertical line.

The arc length ℓ can be calculated by evaluating

$$\ell = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Find the distance, in terms of r, a point on the circumference travels in one complete rotation.

Question 14 continues on page 11

Question 14 (continued)

(c) A team of ecologists is re-introducing a penguin colony to an island in Antarctica. From observations on other islands, the ecologists know that populations of more than 1000 are unsustainable due to lack of fish. They propose the following logistic model of population growth

$$\frac{dP}{dt} = rP(1000 - P),$$

where P is the number of penguins on the island, t is time in years and r is a constant.

(i) In an ideal breeding season, when the maximum growth rate occurs, all penguins in the colony find a mate and half the penguin pairs successfully raise one chick. **2**

Show that $r = \dfrac{1}{2000}$.

(ii) The team of ecologists initially releases 200 penguins on the island. **3**

Given that

$$\frac{1}{P(1000-P)} = \frac{1}{1000}\left(\frac{1}{P} + \frac{1}{1000-P}\right),$$

show that

$$P(t) = \frac{1000}{1+4e^{-0.5t}}.$$

(iii) How long, to the nearest year, will it take for the penguin population to reach 90% of the maximum sustainable population? **1**

End of Examination

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NORMAL CUMULATIVE DISTRIBUTION FUNCTION

Entries represent $P(Z \leqslant z)$. The value of z to the first decimal place is given in the left column. The second decimal place is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

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EXAMINERS

Stephen Ewington (Convenor)	Ascham School, Edgecliff
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Catheryn Gray	School of Mathematics and Statistics, UNSW
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Rebekah Johnson	Loreto Kirribilli, Kirribilli
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2024
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MARKING GUIDELINES

Mathematics Extension 1

Section I
10 marks

Multiple Choice Answer Key

Question	Answer	Outcomes Assessed	Targeted Performance Bands
1	D	ME11-5	E1-E2
2	D	ME12-1	E1-E2
3	B	ME12-5	E1-E2
4	A	ME11-3	E2-E3
5	D	ME11-2	E2-E3
6	B	ME11-5	E2-E3
7	C	ME12-4	E3-E4
8	B	ME12-4	E3-E4
9	A	ME11-2	E3-E4
10	C	ME11-5	E4

Question 1 (1 mark)
Outcomes Assessed: *ME11-5*
Targeted Performance Bands: *E1-E2*

Solution	Mark
The term containing x^3 is $\binom{7}{4}(2x)^3(-3)^4 = 22\,680x^3$ $\therefore$ coefficient of x^3 is 22680 Hence D	1

Question 2 (1 mark)
Outcomes Assessed: *ME12-1*
Targeted Performance Bands: *E1-E2*

Solution	Mark
$\frac{d}{dx}\left(2\sin^{-1}\frac{x}{2}\right) = 2\times\frac{1}{2}\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$ $= \frac{2}{\sqrt{4-x^2}}$ Hence D	1

Question 3 (1 mark)
Outcomes Assessed: *ME12-5*
Targeted Performance Bands: *E1-E2*

Solution	Mark
$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$ $= \sqrt{\frac{0.3\times 0.7}{50}}$ ≈ 0.0648 Hence B	1

Question 4 (1 mark)
Outcomes Assessed: *ME11-3*
Targeted Performance Bands: *E2-E3*

Solution	Mark
The graphs of $y = \cos(\sin^{-1}x)$ and $y = \sin(\cos^{-1}x)$ both have their domain restricted to $-1 \leq x \leq 1$, so this eliminates options C and D. Considering $y = \sin^{-1}(\cos x)$, we know that $y = \cos x$ is an even function so the graph in $-\pi \leq x \leq 0$ is the reflection of the graph in $0 \leq x \leq \pi$; this eliminates option B. Also the range of $y = \cos^{-1}x$ is $0 \leqslant y \leqslant \pi$. Hence A	1

Question 5 (1 mark)
Outcomes Assessed: *ME11-2*
Targeted Performance Bands: *E2-E3*

Solution	Mark
$P(x) = (4x^2-1)Q(x)+2x+3$ $= (2x-1)(2x+1)Q(x)+2x+3$ Substitute $x = 0.5$ to find remainder when divided by $(2x-1)$ $P(0.5) = (2(0.5)-1)(2(0.5)+1)Q(0.5)+2(0.5)+3$ $P(0.5) = 2(0.5)+3$ $= 4$ Hence D	1

Question 6 (1 mark)
Outcomes Assessed: *ME11-5*
Targeted Performance Bands: *E2-E3*

Solution	Mark
Place an odd number in any of the eight possible positions. Then the next odd number can be placed in any of the three remaining positions in the same row, and so on for the next two odd numbers. This leaves four positions for the even numbers, giving: $\frac{8\times3\times2\times1\times4!}{8!} = \frac{1152}{40320} = \frac{1}{35}$ Hence B	1

Question 7 (1 mark)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E3-E4*

Solution	Mark
The function $f(t) = \cos t + k$ oscillates about an average value of k. If $k > 0$, then over the course of one period (2π), the solution will increase, which eliminates A and B. Furthermore, $\cos t + k > 0$ for $t < \frac{\pi}{2}$, which eliminates D. Consequently, the answer is C. Hence C	1

Question 8 (1 mark)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E3-E4*

Solution	Mark
$y = e^{-x} + ex, \quad x \geq -1$ interchanging x and y, $\quad x = e^{-y} + ey, \quad y \geq -1 \quad$ (inverse function) $\frac{dy}{dx}$ of inverse function $= \frac{1}{dx/dy}$ $\therefore \quad \frac{dx}{dy} = -e^{-y} + e$ when $x = 1,\ y = 0$ $\frac{dx}{dy} = -1 + e$ $\therefore \quad \frac{dy}{dx}$ of inverse function at $x = 1$ is $\frac{1}{e-1}$ Hence B	1

Question 9 (1 mark)
Outcomes Assessed: *ME11-2*
Targeted Performance Bands: *E3-E4*

Solution	Mark
After dilations $x = 2at$ becomes $x = 4at$ and $y = at^2$ becomes $y = 3at^2$. Solving simultaneously to eliminate t gives $y = \frac{3x^2}{16a}$ $\frac{dy}{dx} = \frac{3x}{8a}$ at $x = a$, $\frac{dy}{dx} = \frac{3}{8}$ Hence A	1

Question 10 (1 mark)
Outcomes Assessed: *ME11-5*
Targeted Performance Bands: *E4*

Solution	Mark
To move from the origin to (m,n) we need $m \times \begin{pmatrix}1\\0\end{pmatrix}$ and $n \times \begin{pmatrix}0\\1\end{pmatrix}$ cards. So there is a total of $m+n$ cards, where $\begin{pmatrix}1\\0\end{pmatrix}$ is repeated m times and $\begin{pmatrix}0\\1\end{pmatrix}$ is repeated n times. Total number of unique combinations is $\frac{(m+n)!}{m!n!}$ Hence C	1

Section II

60 marks

Question 11 (15 marks)

Question 11(a) (3 marks)

Outcomes Assessed: *ME12-3*

Targeted Performance Bands: *E1-E2*

Criteria	Marks
• provides correct solution	3
• factorises the equation correctly and finds two correct solutions	2
• correctly uses the double angle formula for $\sin 2\theta$	1

Sample Answer:

$$\sin 2\theta + \cos\theta = 0$$
$$2\sin\theta\cos\theta + \cos\theta = 0$$
$$\cos\theta(2\sin\theta + 1) = 0$$

So either $\cos\theta = 0$ or $\sin\theta = -\frac{1}{2}$

If $\cos\theta = 0, \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ If $\sin\theta = -\frac{1}{2}, \theta = \frac{7\pi}{6}$ or $\theta = \frac{11\pi}{6}$

$$\therefore \quad \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$$

Question 11(b) (3 marks)

Outcomes Assessed: *ME12-2*

Targeted Performance Bands: *E1-E2*

Criteria	Marks
• provides correct solution	3
• obtains $-1 = \sqrt{50}\cos\theta$	2
• evaluates $\underset{\sim}{a}\cdot\underset{\sim}{b} = -1$ correctly or finds correct magnitude of both $\underset{\sim}{a}$ and $\underset{\sim}{b}$	1

Sample Answer:

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$
$$-3+2 = \sqrt{10}\sqrt{5}\cos\theta$$
$$\cos\theta = \frac{-1}{\sqrt{50}}$$
$$\therefore \quad \theta \approx 98^\circ$$

Question 11(c) (3 marks)
Outcomes Assessed: *ME11-2*
Targeted Performance Bands: *E2*

Criteria	Marks
• provides correct solution	3
• correctly finds the two critical points of the solution	2
• multiplies by the square of the denominator or recognises that $x \neq 1$	1

Sample Answer:

$$\frac{1}{x-1} \geqslant 2, \quad x \neq 1$$
$$x-1 \geqslant 2(x-1)^2$$
$$2(x-1)^2-(x-1) \leqslant 0$$
$$(x-1)(2x-3) \leqslant 0$$
$$\therefore \quad 1 < x \leqslant \frac{3}{2}$$

Question 11(d) (3 marks)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	3
• integrates correctly or work of equivalent merit	2
• finds correct limits or integrand in terms of u or work of equivalent merit	1

Sample Answer:

$$\int_0^1 x(1-x)^{10}\,dx \qquad \text{let } u = 1-x \quad \text{when } x = 0,\ u = 1$$
$$du = -dx \quad \text{when } x = 1,\ u = 0$$

$$\int_0^1 x(1-x)^{10}\,dx = -\int_1^0 (1-u)u^{10}\,du$$
$$= \int_0^1 \left(u^{10}-u^{11}\right)\,du$$
$$= \left[\frac{u^{11}}{11}-\frac{u^{12}}{12}\right]_0^1$$
$$= \frac{1}{11}-\frac{1}{12}$$
$$= \frac{1}{132}$$

Question 11(e) (3 marks)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	3
• integrates correctly or work of equivalent merit	2
• obtains correct integral expression for the volume or work of equivalent merit	1

Sample Answer:

The volume of the hyperboloid can be found using symmetry:

$$\begin{aligned} V &= 2\pi \int_0^4 x^2\,dy \\ &= 2\pi \int_0^4 \left(9+y^2\right)\,dy \\ &= 2\pi \left[9y+\frac{y^3}{3}\right]_0^4 \\ &= 2\pi \left(9(4)+\frac{4^3}{3}-(0-0)\right) \\ \therefore \quad V &= \frac{344\pi}{3} \text{ units}^3 \end{aligned}$$

Question 12 (14 marks)

Question 12(a) (2 marks)
Outcomes Assessed: *ME11-2*
Targeted Performance Bands: *E2*

Criteria	Marks
• provides correct solution	2
• finds one of the roots or equivalent merit	1

Sample Answer:

Let the roots be α, $-\alpha$ and β.

$$\begin{aligned} \text{sum of roots}: \alpha-\alpha+\beta &= -1 \\ \beta &= -1 \\ \text{by the factor theorem} \quad P(-1) &= 0 \\ 0 &= (-1)^3+(-1)^2-m(-1)-3 \\ \therefore \quad m &= 3 \end{aligned}$$

Question 12(b) (2 marks)
Outcomes Assessed: *ME12-3*
Targeted Performance Bands: *E2*

Criteria	Marks
• provides correct solution	2
• finds correct R or α or equivalent merit	1

Sample Answer:

Using the compound angle formula to expand,

$$\begin{aligned} R\sin(x+\alpha) &= R(\sin x\cos\alpha+\cos x\sin\alpha) \\ &= R\cos\alpha\sin x+R\sin\alpha\cos x \\ &= 3\sin x+\sqrt{3}\cos x \end{aligned}$$

Equating coefficients, $R\cos\alpha=3$ and $R\sin\alpha=\sqrt{3}$

$$R^2\cos^2\alpha+R^2\sin^2\alpha=9+3$$

$$\therefore \quad R^2=12$$

$$R=2\sqrt{3} \quad \text{and} \quad \frac{R\sin\alpha}{R\cos\alpha}=\frac{\sqrt{3}}{3}$$

$$\tan\alpha=\frac{1}{\sqrt{3}}$$

$$\alpha=\frac{\pi}{6} \quad (\alpha \text{ is acute})$$

$$\therefore \quad 3\sin x+\sqrt{3}\cos x=2\sqrt{3}\sin\left(x+\frac{\pi}{6}\right)$$

Question 12(c) (i) (2 marks)

Outcomes Assessed: *ME12-2*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	2
• finds t in terms of V, or equivalent merit	1

Sample Answer:

The object clears the wall if $y > 15$, when $x = 20$.

$$\text{i.e.} \quad Vt = 20$$

$$\therefore \quad t = \frac{20}{V}$$

$$\text{So} \quad 45 - 5\left(\frac{20}{V}\right)^2 > 15$$

$$V^2 > \frac{400}{6}$$

$$V > \frac{20}{\sqrt{6}} \quad (V > 0), \quad \text{as required}$$

Question 12(c) (ii) (2 marks)
Outcomes Assessed: *ME12-2*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	2
• differentiates to find velocity components or finds $t = 3$	1

Sample Answer:

$$\underset{\sim}{v}(t) = \begin{bmatrix} V \\ -10t \end{bmatrix}$$

object hits ground when $y = 0$, i.e. $45 - 5t^2 = 0$

$$t = 3 \quad (t > 0)$$

$$\text{so}, \quad \underset{\sim}{v}(3) = \begin{bmatrix} 9 \\ -30 \end{bmatrix}$$

$$\therefore \quad \text{impact speed} = \sqrt{9^2 + (-30)^2}$$

$$= \sqrt{981} \text{ m/s}$$

Question 12(d) (3 marks)

Outcomes Assessed: *ME11-2, ME12-4*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	3
• correctly integrates	2
• correct division or work of equivalent merit	1

Sample Answer:

Using polynomial division:

$$\begin{array}{r} x^2-x+1 \\ x+1\overline{)\,x^3} \\ \underline{x^3+x^2} \\ -x^2 \\ \underline{-x^2-x} \\ x \\ \underline{x+1} \\ -1 \end{array}$$

$$\begin{aligned} \text{so } \int_0^1 \frac{x^3}{x+1}\,dx &= \int_0^1 x^2-x+1-\frac{1}{x+1}\,dx \\ &= \left[\frac{x^3}{3}-\frac{x^2}{2}+x-\ln|x+1|\right]_0^1 \\ &= \left(\frac{1}{3}-\frac{1}{2}+1-\ln 2\right)-(0-\ln 1) \\ &= \frac{5}{6}-\ln 2 \end{aligned}$$

Question 12(e) (3 marks)
Outcomes Assessed: *ME12-5, ME12-7*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	3
• correct expression to find the probability in terms of Z	2
• finds both $E(X)$ and $\text{Var}(X)$	1

Sample Answer:

$X \sim \text{Bin}(25,0.6)$ $\quad\therefore\quad$ $\text{E}(X) = np = (25)(0.6), = 15$ and $\text{Var}(X) = npq = (25)(0.6)(0.4) = 6$

Since this (discrete) binomial distribution can be approximated by the (continuous) normal distribution, $np = 15$ and $nq = 10$, and applying the continuity correction,

$$\begin{aligned} P(\text{more red balls}) &\approx P(X > 12.5) \\ &\approx P\left(Z > \frac{12.5-15}{\sqrt{6}}\right) \\ &\approx P(Z < 1.02) \\ &\approx 0.8461 \end{aligned}$$

Without continuity correction,

$$\begin{aligned} P(\text{more red balls}) &\approx P(X \geqslant 13) \\ &\approx P\left(Z > \frac{13-15}{\sqrt{6}}\right) \\ &\approx P(Z < 0.82) \\ &\approx 0.7393 \end{aligned}$$

Question 13 (16 marks)

Question 13(a) (2 marks)

Outcomes Assessed: *ME11-3, ME12-3, ME12-4*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides either correct solution	2
• uses products to sums or double angle formula to simplify the integral	1

Sample Answer:

$$\begin{aligned}\int \sin x \sin 2x\,dx &= \frac{1}{2}\int \cos(-x) - \cos 3x\,dx \\ &= \frac{1}{2}\int \cos x - \cos 3x\,dx \\ &= \frac{1}{2}\left[\sin x - \frac{\sin 3x}{3}\right] + c\end{aligned}$$

Alternate solution

$$\begin{aligned}\int \sin x \sin 2x\,dx &= \int 2\sin^2 x \cos x\,dx \\ &= \frac{2\sin^3 x}{3} + c\end{aligned}$$

Question 13(b) (3 marks)
Outcomes Assessed: *ME12-1, ME12-7*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct proof	3
• proves the inductive step, or equivalent merit	2
• establishes the base case, or equivalent merit	1

Sample Answer:

Step 1: Prove the result true for $n = 1$.

$$\text{LHS} = \frac{1}{2}$$
$$\text{RHS} = 2 - \frac{3}{2} = \frac{1}{2}$$

so the statement is true for $n = 1$

Step 2: Suppose the statement is true for some integer $n = k$.

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$$

Step 3: Prove the result is true for $n = k+1$.

i.e. Prove that $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$

$$\begin{aligned}
\text{LHS} &= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} \\
&= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \qquad \text{By Step 2} \\
&= 2 - \frac{2k+4}{2^{k+1}} + \frac{k+1}{2^{k+1}} \\
&= 2 - \frac{2k+4-k-1}{2^{k+1}} \\
&= 2 - \frac{k+3}{2^{k+1}} \\
&= \text{RHS}
\end{aligned}$$

Since the statement is true for $n = k$ and $n = k+1$, by mathematical induction, the statement is true for all integers $n \geq 1$.

Question 13(c) (3 marks)
Outcomes Assessed: *ME12-2, ME12-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• provides correct solution, recognising that $p_1^2 + p_2^2 = r^2$	3
• finds both $\underset{\sim}{p} - \underset{\sim}{a}$ and $\underset{\sim}{p} - \underset{\sim}{b}$ and expands $\lvert\underset{\sim}{p} - \underset{\sim}{a}\rvert^2 + \lvert\underset{\sim}{p} - \underset{\sim}{b}\rvert^2$ or equivalent merit	2
• finds $\underset{\sim}{p} - \underset{\sim}{a}$ or $\underset{\sim}{p} - \underset{\sim}{b}$ or equivalent merit	1

Sample Answer:

$$\underset{\sim}{p} - \underset{\sim}{a} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \begin{pmatrix} -r \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} p_1 + r \\ p_2 \end{pmatrix}$$

Similarly, $$\underset{\sim}{p} - \underset{\sim}{b} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \begin{pmatrix} r \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} p_1 - r \\ p_2 \end{pmatrix}$$

Thus $$\lvert\underset{\sim}{p} - \underset{\sim}{a}\rvert^2 + \lvert\underset{\sim}{p} - \underset{\sim}{b}\rvert^2 = \begin{pmatrix} p_1 + r \\ p_2 \end{pmatrix} \cdot \begin{pmatrix} p_1 + r \\ p_2 \end{pmatrix} + \begin{pmatrix} p_1 - r \\ p_2 \end{pmatrix} \cdot \begin{pmatrix} p_1 - r \\ p_2 \end{pmatrix}$$
$$= (p_1 + r)^2 + p_2^2 + (p_1 - r)^2 + p_2^2$$
$$= p_1^2 + 2rp_1 + r^2 + p_1^2 - 2rp_1 + r^2 + p_2^2$$
$$= 2\left(p_1^2 + p_2^2\right) + 2r^2$$

Since $\underset{\sim}{p}$ lies on the circle, $p_1^2 + p_2^2 = r^2$

$$\therefore \quad \lvert\underset{\sim}{p} - \underset{\sim}{a}\rvert^2 + \lvert\underset{\sim}{p} - \underset{\sim}{b}\rvert^2 = 4r^2, \quad \text{as required.}$$

Question 13(d) (3 marks)

Outcomes Assessed: *ME11-4*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	3
• correct equation for chain rule	2
• finds $\frac{dV}{dh}$ or equal merit	1

Sample Answer:

$$\frac{dV}{dt} = 0.03\,\text{m}^3/\text{min}$$

since $R = 0.5$, $\quad V = \frac{\pi h^2}{2} - \frac{\pi h^3}{3}$ $\quad$ and $\quad \frac{dV}{dh} = \frac{2\pi h}{2} - \frac{3\pi h^2}{3} = \pi h - \pi h^2$

$$\text{so} \quad \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{\pi h - \pi h^2} \times \frac{3}{100}$$

$$\text{at} \quad h = 0.25, \quad \frac{dh}{dt} = \frac{1}{\pi(0.25) - \pi(0.25)^2} \times \frac{3}{100}$$

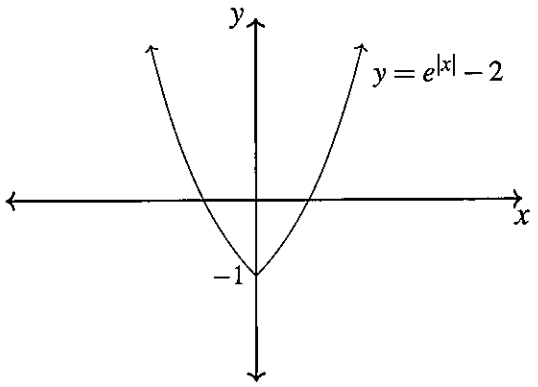
$$= \frac{4}{25\pi}$$

$$\therefore \quad \frac{dh}{dt} = \frac{4}{25\pi} \text{ m/minute.}$$

Question 13(e) (i) (1 mark)
Outcomes Assessed: *ME11-2, ME11-7*
Targeted Performance Bands: *E2*

Criteria	Mark
• provides correct sketch of $y = f(\|x\|)$ with cusp clearly indicated	1

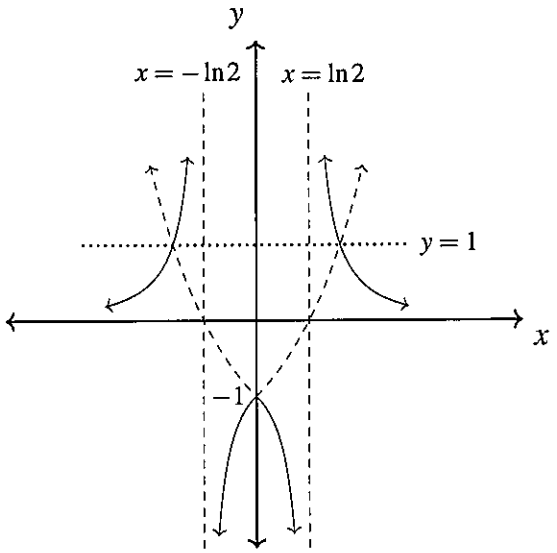
Sample Answer:



Question 13(e) (ii) (2 marks)
Outcomes Assessed: *ME11-2, ME11-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• provides a correct sketch of $y = \dfrac{1}{f(\|x\|)}$, with important features indicated	2
• indicates the asymptotes correctly, or equivalent merit	1

Sample Answer:



Question 13(e) (iii) (2 marks)
Outcomes Assessed: *ME11-2*
Targeted Performance Bands: *E4*

Criteria	Marks
• provides all solutions	2
• finds at least one solution or some progress towards an algebraic solution	1

Sample Answer:

The functions intersect at $y = \pm 1$ and since they are even functions we consider the case $x > 0$.

$$\text{Solving} \quad e^x - 2 = \pm 1$$
$$e^x = 3,\ 1$$
$$\therefore \quad x = 0 \text{ or } \ln 3$$
$$\text{so by symmetry} \quad x = 0, \pm \ln 3$$

Alternate algebraic solution: $\text{for } x \geqslant 0, \quad e^x - 2 = \dfrac{1}{e^x - 2}$

$$e^{2x} - 4e^x + 3 = 0$$
$$(e^x - 1)(e^x - 3) = 0$$
$$e^x = 1,\ 3$$
$$\therefore \quad x = 0,\ \ln 3$$
$$\text{for } x < 0, \quad e^{-x} - 2 = \frac{1}{e^{-x} - 2}$$
$$e^{-2x} - 4e^{-x} + 3 = 0$$
$$3e^{2x} - 4e^x + 3 = 0$$
$$(3e^x - 1)(e^x - 1) = 0$$
$$3e^x = 1, \text{or } e^x = 1$$
$$\therefore \quad x = \ln \frac{1}{3} \quad (x = 0 \text{ is not in this domain})$$
$$x = -\ln 3$$
$$\therefore \quad \text{solutions are} \quad x = 0,\ \pm \ln 3$$

Question 14 (15 marks)

Question 14(a) (i) (2 marks)

Outcomes Assessed: *ME12-2*

Targeted Performance Bands: *E2*

Criteria	Marks
• provides correct solution	2
• finds $\overrightarrow{AB}$ or gives the correct formula for $\text{proj}_{\overrightarrow{AB}}\overrightarrow{AO}$	1

Sample Answer:

$$\overrightarrow{AO} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad \text{and} \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$\text{proj}_{\overrightarrow{AB}}\overrightarrow{AO} = \frac{\overrightarrow{AO}\cdot\overrightarrow{AB}}{|\overrightarrow{AB}|^2}\overrightarrow{AB}$$

$$= \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Question 14(a) (ii) (2 marks)

Outcomes Assessed: *ME12-2*

Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	2
• correct vector description to obtain the distance (or distance squared), or equivalent merit	1

Sample Answer:

Let the perpendicular distance of $\overrightarrow{AB}$ from O be d

$$d^2 = |\overrightarrow{OA}|^2 - |\text{proj}_{\overrightarrow{AB}}\overrightarrow{AO}|^2$$

$$= (\sqrt{5})^2 - \frac{1}{25}(\sqrt{25})^2$$

$$= 4$$

$$\therefore \quad d = 2$$

Question 14(b) (5 marks)

Outcomes Assessed: *ME11-3, ME12-2*

Targeted Performance Bands: *E4*

Criteria	Marks
• correct solution	5
• integrates correctly	4
• correctly recognises that a double angle formula is required	3
• substitutes and simplifies correctly into the formula for ℓ	2
• identifies the correct limits of integration or finds both $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$	1

Sample Answer:

$$\frac{dx}{d\theta} = r(1-\cos\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = r\sin\theta$$

$$\therefore \quad \ell = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2-2\cos\theta}\, d\theta$$

$$= r\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\tfrac{\theta}{2}}\, d\theta$$

$$= 2r\int_0^{2\pi} \sin\tfrac{\theta}{2}\, d\theta$$

$$= 4r\left[-\cos\tfrac{\theta}{2}\right]_0^{2\pi}$$

$$= 4r[-(-1)-(-(1))]$$

$$= 8r$$

Question 14(c) (i) (2 marks)
Outcomes Assessed: *ME12-4, ME12-7*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correctly justifies the value of r	2
• identifies the maximum growth rate is when $P = 500$	1

Sample Answer:

The maximum of $\frac{dP}{dt}$ is when $P = 500$.

$$\begin{aligned}\frac{dP}{dt} &= r(500)(1000-500)\\ &= 250000r\end{aligned}$$

Under ideal conditions the population increases by 125 $\left(250 \text{ parent pairs} \times \frac{1}{2}\right)$.

$$\begin{aligned}125 &= 250000r\\ r &= \frac{1}{2000} \quad \text{(as required)}\end{aligned}$$

Question 14(c) (ii) (3 marks)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E4*

Criteria	Marks
• provides correct solution	3
• integrates both sides correctly, or equivalent merit	2
• attempts to separate the variables in the differential equation, or equivalent merit	1

Sample Answer:

$$\text{If} \quad \frac{dP}{dt} = \frac{P}{2000}(1000-P),$$

$$\int \frac{1}{P(1000-P)}\,dP = \int \frac{dt}{2000}$$

$$\int \frac{1}{1000}\left(\frac{1}{P}+\frac{1}{1000-P}\right)dP = \int \frac{1}{2000}\,dt$$

$$\int \left(\frac{1}{P}+\frac{1}{1000-P}\right)dP = \int \frac{dt}{2}$$

$$\ln|P| - \ln|1000-P| = \frac{t}{2}+C$$

$$\ln\left|\frac{P}{1000-P}\right| = \frac{t}{2}+C$$

Since $0 < P(0) < 1000$,we may drop the absolute values and thus

$$\frac{P}{1000-P} = Ae^{0.5t}, \quad \text{where } A = e^C.$$

When $t = 0$, $P = 200$, so $A = \frac{200}{800} = \frac{1}{4}$

$$\therefore \quad \frac{P}{1000-P} = \frac{e^{0.5t}}{4}$$

$$4P = (1000-P)e^{0.5t}$$

$$P(4+e^{0.5t}) = 1000e^{0.5t}$$

$$P = \frac{1000e^{0.5t}}{4+e^{0.5t}} \times \frac{e^{-0.5t}}{e^{-0.5t}}$$

$$= \frac{1000}{1+4e^{0.5t}}$$

Question 14(c) (iii) (1 mark)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E3*

Criteria	Mark
• correct solution	1

Sample Answer:

$$\begin{aligned}
\text{If} \quad P &= 900, \\
900 &= \frac{1000}{1+4e^{-0.5t}} \\
1+4e^{-0.5t} &= \frac{10}{9} \\
e^{-0.5t} &= \frac{1}{36} \\
-\frac{t}{2} &= \ln\frac{1}{36} \\
t &= 2\ln 36 \\
&\approx 7.167038
\end{aligned}$$

It will take approximately 7 years.