2024
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your student name and/or number at the top of every page

Total marks – 70

Section I – 10 marks (pages 3 - 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6 - 9)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

This paper MUST NOT be removed from the examination room.

STUDENT NAME/NUMBER..

STUDENT NAME/NUMBER.......................................

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Select the alternative A, B, C, D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

STUDENT NAME/NUMBER.......................................

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

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1. What is the Cartesian equation of the straight line with parametric equations $x = 1 - 2t,\ y = 3t + 1$?

(A) $3x - 2y - 5 = 0$

(B) $3x + 2y - 5 = 0$

(C) $3x - 2y + 5 = 0$

(D) $3x + 2y + 5 = 0$

2. What is the remainder when the polynomial $P(x) = x^3 - 3ax$ is divided by $(x - a + 1)$?

(A) $a^3 + 1$

(B) $a^3 + 6a - 1$

(C) $a^3 - 6a^2 - 1$

(D) $a^3 - 6a^2 + 6a - 1$

3. A bag contains 6 blue balls, 7 green balls, 8 yellow balls, 9 orange balls and 10 red balls. Balls are drawn at random one at a time without replacement from the bag. What is the least number of balls that needs to be drawn from the bag in order to be certain that 8 balls of the same colour are amongst those selected ?

(A) 21

(B) 22

(C) 35

(D) 36

4. In $\triangle AOB$, $OA = 10\text{cm}$, $OB = 10\text{cm}$ and $\angle AOB = \theta$ radians. θ is increasing at a constant rate of $0{\cdot}01$ radians/sec. What, correct to 2 decimal places, is the rate at which the area of the triangle is increasing when $\theta = 1$?

(A) $0{\cdot}27\,\text{cm}^2/\text{s}$

(B) $0{\cdot}28\,\text{cm}^2/\text{s}$

(C) $0{\cdot}49\,\text{cm}^2/\text{s}$

(D) $0{\cdot}50\,\text{cm}^2/\text{s}$

5. What is the range of the function $f(x)=\cos^{-1}\sqrt{x}$?

(A) $\left[0\,,\,\frac{\pi}{2}\right]$

(B) $\left[\frac{\pi}{2}\,,\,\pi\right]$

(C) $(0\,,\,\pi]$

(D) $[0\,,\,\pi]$

6. What is the size of the angle between the vectors $\underset{\sim}{a}=3\underset{\sim}{i}+\underset{\sim}{j}$ and $\underset{\sim}{b}=2\underset{\sim}{i}-\underset{\sim}{j}$?

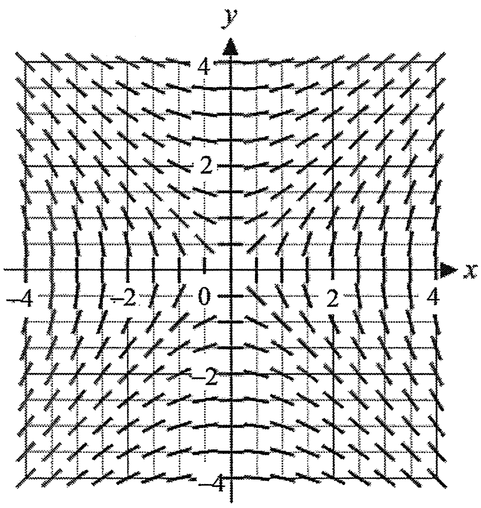
(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

7. Which differential equation is represented by the following slope field ?



(A) $\frac{dy}{dx}=-\frac{x}{y}$

(B) $\frac{dy}{dx}=-\frac{y}{x}$

(C) $\frac{dy}{dx}=\frac{x}{y}$

(D) $\frac{dy}{dx}=\frac{y}{x}$

8. If $y = \tan^{-1}\left(\dfrac{1}{x}\right)$ which of the following is an expression for $\dfrac{dy}{dx}$?

(A) $\dfrac{dy}{dx} = -\tan^2 y$

(B) $\dfrac{dy}{dx} = -\cot^2 y$

(C) $\dfrac{dy}{dx} = -\cos^2 y$

(D) $\dfrac{dy}{dx} = -\sin^2 y$

9. Which of the following is an expression for $\displaystyle\int \frac{1}{ax^2+a^2}dx$, where $a > 0$?

(A) $\dfrac{1}{\sqrt{a}}\tan^{-1}\left(\dfrac{x}{\sqrt{a}}\right)+c$

(B) $\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{\sqrt{a}}\right)+c$

(C) $\dfrac{1}{a\sqrt{a}}\tan^{-1}\left(\dfrac{x}{\sqrt{a}}\right)+c$

(D) $\dfrac{1}{a^2}\tan^{-1}\left(\dfrac{x}{\sqrt{a}}\right)+c$

10. A game consists of rolling together n fair coins where $n \geq 3$. The game is won if it results in rolling exactly one head or exactly one tail. How many games need to be played in order that the expected number of games won is n ?

(A) 2^{n-1}

(B) 2^n

(C) 2^{n+1}

(D) $n2^n$

STUDENT NAME/NUMBER.......................................

Marks

Section II

60 Marks
Attempt Questions 11-14.
Allow about 1 hour and 45 minutes for this section.

Answer the questions in writing booklets provided. Use a separate writing booklet for each question. In Questions 11-14 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

(a) Find the number of ways that the letters of the word TRIANGLE can be arranged in a line such that the vowels are all next to each other but the consonants are not all next to each other. **2**

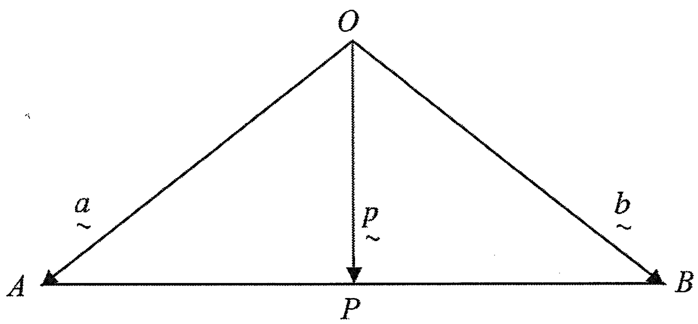
(b) Use the $t=\tan\frac{x}{2}$ results to show that $\sin x-\tan\frac{x}{2}=\tan\frac{x}{2}\cos x$. **2**

(c) Find in simplest exact form $\int_1^{\sqrt{3}}\frac{1}{\sqrt{4-x^2}}\,dx$. **2**

(d) Solve the inequality $8-|2x-1|\le 5$. **3**

(e) If $\alpha,\ \frac{\pi}{4},\ \beta$ are consecutive terms in an arithmetic sequence, find in simplest exact form the value of $(\cos\alpha+\cos\beta)^2+(\sin\alpha-\sin\beta)^2$. **3**

(f)



In the diagram, OAB is a triangle with $\overrightarrow{OA}=\underset{\sim}{a}$, $\overrightarrow{OB}=\underset{\sim}{b}$ and $OA=OB$. **3**
P is a point on AB such that $\overrightarrow{OP}=\underset{\sim}{p}$ and OP bisects $\angle AOB$.
Use vector methods to show that $OP\perp AB$.

End of Question 11

Marks

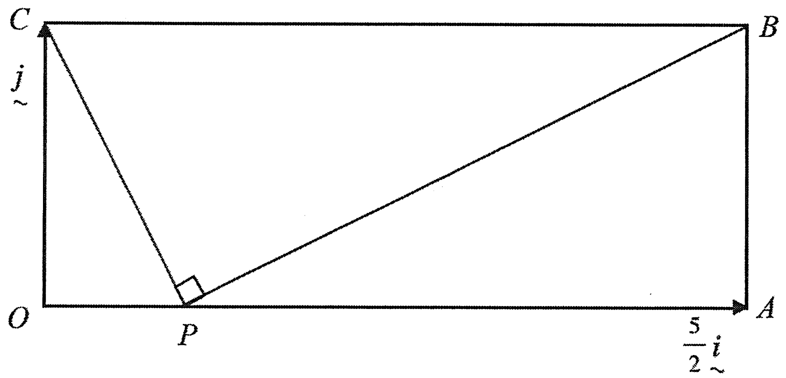
Question 12 (15 marks) **Use a separate writing booklet.**

(a) On the same diagram, sketch the curves $y = \dfrac{1}{\sin^{-1}x}$ and $y = \dfrac{1}{\cos^{-1}x}$, showing clearly any intercepts on the axes, the coordinates of any endpoints and the equations of any asymptotes. **3**

(b)(i) Show that $2\sin\left(x - \frac{\pi}{3}\right) + 2\sin x = 2\sqrt{3}\sin\left(x - \frac{\pi}{6}\right)$. **2**

(ii) Hence solve the equation $2\sin\left(x - \frac{\pi}{3}\right) + 2\sin x = 3$ for $0 \le x \le 2\pi$. **2**

(c)



In the diagram, $OABC$ is a rectangle in which $\overrightarrow{OA} = \frac{5}{2}\underset{\sim}{i}$ and $\overrightarrow{OC} = \underset{\sim}{j}$. P is a point on OA such that $\overrightarrow{OP} = \lambda\underset{\sim}{i}$ for some scalar parameter λ and $\angle CPB = 90°$.

Use vector methods to show that $2\lambda^2 - 5\lambda + 2 = 0$ and hence find any values of λ. **4**

(d) Use the substitution $x = \sin^2\theta$, $0 \le \theta \le \frac{\pi}{2}$, to evaluate $\int_0^1 \sqrt{x(1-x)}\, dx$. **4**

End of Question 12

STUDENT NAME/NUMBER...

Marks

Question 13 (15 marks) **Use a separate writing booklet.**

(a) Find, in simplest exact form, the coordinates of the stationary points on the curve $y = x^2 + \cos^{-1} x$. **3**

(b) Find the particular solution of the differential equation $\frac{dy}{dx} = e^{-(2x+y)}$ given that $y = 0$ when $x = 0$. **3**

(c)(i) Find the domain and range of the function $f(x) = \sin^{-1}\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}$. **2**

(ii) Show that the normal to the curve $y = \sin^{-1}\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}$ at the point where $x = -\frac{\pi}{2}$ passes through the origin. **2**

(d) Records show that 40% of residents of a town went to school in the town. A sample of 150 residents of the town is to be taken to determine the proportion who went to school in the town.

(i) Find, in simplest fraction form, the mean and standard deviation of the distribution of such sample proportions. **2**

(ii) Use the following extract of the table of values of $P(Z \le z)$, where Z has a standard normal distribution, to estimate correct to 2 decimal places the probability that a sample of 150 residents of the town contains at least 57 and at most 69 who went to school in the town. **3**

STANDARD NORMAL DISTRIBUTION TABLE

Entries represent $\Pr(Z \le z)$. The value of z to the first decimal is given in the left column. The second decimal is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

End of Question 13

STUDENT NAME/NUMBER......................................

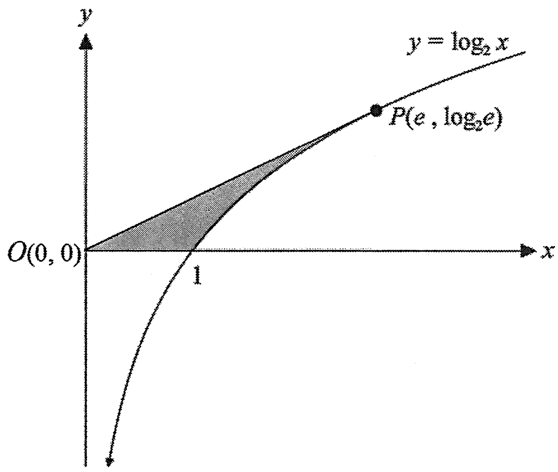
Marks

Question 14 (15 marks) **Use a separate writing booklet.**

(a) Use Mathematical Induction to show that for all integers $n \geq 1$ **3**

$$\left(1\times 1!+\frac{0}{1!}\right)+\left(2\times 2!+\frac{1}{2!}\right)+\left(3\times 3!+\frac{2}{3!}\right)+\ldots+\left(n\times n!+\frac{n-1}{n!}\right)=(n+1)!-\frac{1}{n!}.$$

(b)



The diagram shows the point $P\left(e, \log_2 e\right)$ on the curve $y=\log_2 x$.

(i) Show that the line OP is tangent to the curve at the point P. **2**

(ii) The shaded region bounded by the curve $y=\log_2 x$, the line OP and the x axis is rotated through one revolution about the y axis. Find, in simplest exact form, the volume of the solid formed. **4**

(c) A stone is projected from point O with speed $V\text{ms}^{-1}$ at an angle $\alpha=\tan^{-1}\frac{4}{3}$ above the horizontal. O is at the base of a ramp inclined at an angle $\beta=\tan^{-1}\frac{1}{2}$ above the horizontal. The stone moves in a vertical plane above the ramp under gravity where the acceleration due to gravity is $g\text{ms}^{-2}$. At time t seconds the position vector of the stone relative to O is $\underset{\sim}{r}(t)=(Vt\cos\alpha)\underset{\sim}{i}+\left(Vt\sin\alpha-\frac{1}{2}gt^2\right)\underset{\sim}{j}$.

(i) Show that the stone hits the ramp after time $T=\dfrac{V}{g}$ seconds. **3**

(ii) Use an appropriate trigonometric identity to show exactly that when the stone hits the ramp its direction of motion is inclined at an angle $\frac{\pi}{4}$ to the ramp. **3**

END OF PAPER

Section 1 Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1	B	$3x=3-6t$, $2y=2+6t$ $\quad\therefore 3x+2y=5 \quad 3x+2y-5=0$	ME11-1
2	D	Remainder is $P(a-1)=(a-1)^3-3a(a-1)=a^3-6a^2+6a-1$	ME11-2
3	C	Max. number chosen without having 8 of same colour is 34 comprising 6 blue and 7 each of green, yellow, orange and red. Then 35th ball is one of yellow, orange or red giving 8 of one colour.	ME11-5
4	A	Area $\triangle AOB=50\sin\theta \quad \frac{d}{dt}50\sin\theta=50\cos\theta\frac{d\theta}{dt}=0\cdot5\cos\theta$ Hence area is increasing at a rate $0\cdot5\cos1\approx0\cdot27$ cm²/s	ME11-4
5	A	Considering domain of $\cos^{-1}$, $0\le\sqrt{x}\le1$ and hence $0\le\cos^{-1}\sqrt{x}\le\frac{\pi}{2}$	ME11-3
6	B	$\cos\theta=\frac{\underset{\sim}{a}\,.\,\underset{\sim}{b}}{\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{a}\rvert}=\frac{6-1}{\sqrt{10}\sqrt{5}}=\frac{1}{\sqrt{2}} \quad \therefore\theta=\frac{\pi}{4}$	ME12-2
7	C	Slope is 0 for $x=0$ and vertical for $y=0$. Hence not B nor D. Isoclines $y=x$ (slope 1) and $y=-x$ (slope -1) are consistent with C but not A	ME12-4
8	D	$y=\tan^{-1}\left(\frac{1}{x}\right) \quad \frac{dy}{dx}=\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)=-\frac{\tan^2y}{1+\tan^2y}$ $\therefore \frac{dy}{dx}=-\frac{\sin^2y}{\cos^2y\sec^2y}=-\sin^2y$	ME12-4
9	C	$\int\frac{1}{ax^2+a^2}dx=\frac{1}{a\sqrt{a}}\int\frac{\sqrt{a}}{x^2+(\sqrt{a})^2}dx=\frac{1}{a\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)+c$	ME12-4
10	A	Probability game is won is $2\,{}^nC_1\,\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}=n\left(\frac{1}{2}\right)^{n-1}$ Expected number of wins with m plays is $mn\left(\frac{1}{2}\right)^{n-1}=n$ for $m=2^{n-1}$	ME12-5

Section II

Question 11

a. Outcomes assessed: ME11-5

Marking Guidelines

Criteria	Marks
Calculates the number of orders subject to the specified restriction	2
Substantial progress eg. correct procedure but neglects to account for different orders of vowels	1

Answer

Arrange [I, A, E] (in any order), T, R, N, G, L in a straight line so that the group of vowels is neither first nor last. This can be done in $4 \times 3! \times 5! = 2880$ ways.

b. Outcomes assessed: ME11-3

Marking Guidelines

Criteria	Marks
Uses t formulae to obtain required result	2
Substantial progress eg. correct process but lack of clarity in simplification of expression in t	1

Answer

$$t = \tan\frac{x}{2} \Rightarrow \sin x - \tan\frac{x}{2} = \frac{2t}{1+t^2} - t$$

$$= t\left(\frac{2-(1+t^2)}{1+t^2}\right)$$

$$= t\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \tan\frac{x}{2}\cos x$$

c. Outcomes assessed: ME12-4

Marking Guidelines

Criteria	Marks
Calculates the definite integral in simplest exact form	2
Substantial progress eg. substitutes limits into the correct anti-derivative	1

Answer

$$\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt{3}}$$

$$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}$$

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

Q11 cont.

d. Outcomes assessed: ME11-2

Marking Guidelines

Criteria	Marks
Solves the inequality for x, combining two inequalities correctly for the solution	3
Substantial progress eg. correct process with one error	2
Some progress eg. finds only one appropriate inequality for x	1

Answer

$$8-|2x-1|\le 5 \qquad 2x-1\le -3 \quad or \quad 2x-1\ge 3 \qquad \therefore x\le -1 \; or \; x\ge 2$$
$$|2x-1|\ge 3 \qquad 2x\le -2 \qquad 2x\ge 4$$

e. Outcomes assessed: ME11-3

Marking Guidelines

Criteria	Marks
Uses the definition of an AP to find $\alpha+\beta$, then trig identities to deduce required value	3
Substantial progress eg. expands and simplifies given expression and finds value of $\alpha+\beta$	2
Some progress eg. expands and simplifies given expression using trig identities	1

Answer

$$\beta-\frac{\pi}{4}=\frac{\pi}{4}-\alpha$$
$$\alpha+\beta=\frac{\pi}{2}$$
$$\sin\alpha=\cos\beta\,, \quad \cos\alpha=\sin\beta$$

$$\begin{aligned}(\cos\alpha+\cos\beta)^2+(\sin\alpha-\sin\beta)^2&=(\cos\alpha+\sin\alpha)^2+(\sin\alpha-\cos\alpha)^2\\&=2\left(\cos^2\alpha+\sin^2\alpha\right)\\&=2\end{aligned}$$

Alternatively,

$$\begin{aligned}(\cos\alpha+\cos\beta)^2+(\sin\alpha-\sin\beta)^2&=\cos^2\alpha+\sin^2\alpha+\cos^2\beta+\sin^2\beta+2(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\\&=2+2\cos(\alpha+\beta)\\&=2+2\cos\frac{\pi}{2}\\&=2\end{aligned}$$

f. Outcomes assessed: ME12-2

Marking Guidelines

Criteria	Marks
Applies vector methods to deduce required result	3
Substantial progress eg. shows $\underset{\sim}{a}\cdot\underset{\sim}{p}=\underset{\sim}{b}\cdot\underset{\sim}{p}$	2
Some progress eg. applies the vector formula for cosines of $\angle AOP$, $\angle BOP$	1

Answer

$\angle AOP=\angle BOP$ and $|\underset{\sim}{a}|=|\underset{\sim}{b}|$. $\therefore \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{p}}{|\underset{\sim}{a}||\underset{\sim}{p}|}=\dfrac{\underset{\sim}{b}\cdot\underset{\sim}{p}}{|\underset{\sim}{b}||\underset{\sim}{p}|}$ and hence

$$\underset{\sim}{a}\cdot\underset{\sim}{p}=\underset{\sim}{b}\cdot\underset{\sim}{p}$$
$$(\underset{\sim}{b}-\underset{\sim}{a})\cdot\underset{\sim}{p}=0$$

But $\underset{\sim}{b}-\underset{\sim}{a}=\overrightarrow{AB}$. $\therefore$ $OP\perp AB$.

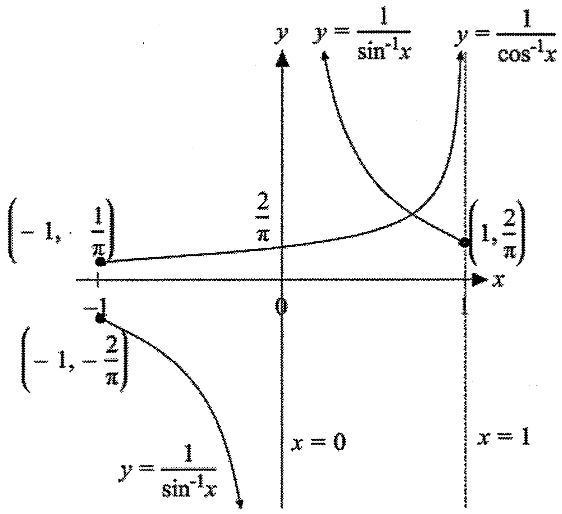
Question 12

a. Outcomes assessed: ME11-3

Marking Guidelines

Criteria	Marks
Sketches both curves showing all the required detail	3
Substantial progress eg. sketches both curves but some required detail missing or incorrect	2
Some progress eg. sketches one of the two curves showing most of the required detail	1

Answer



b.i. Outcomes assessed: ME11-3

Marking Guidelines

Criteria	Marks
Uses appropriate compound angle trig identities to establish required result	2
Substantial progress eg. expands and simplifies LHS using compound angle trig formulae	1

Answer

$$\begin{aligned}
2\sin\left(x-\frac{\pi}{3}\right)+2\sin x &= 2\left(\sin x\cos\frac{\pi}{3}-\cos x\sin\frac{\pi}{3}\right)+2\sin x \\
&= 2\left(\frac{1}{2}\sin x-\frac{\sqrt{3}}{2}\cos x\right)+2\sin x \\
&= 3\sin x-\sqrt{3}\cos x \\
&= 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x\right) \\
&= 2\sqrt{3}\left(\sin x\cos\frac{\pi}{6}-\cos x\sin\frac{\pi}{6}\right) \\
&= 2\sqrt{3}\sin\left(x-\frac{\pi}{6}\right)
\end{aligned}$$

Q12b cont.

b.ii. Outcomes assessed: ME12-3

Marking Guidelines

Criteria	Marks
Finds both exact solutions for x	2
Substantial progress eg. finds one exact solution for x	1

Answer

$$2\sin\left(x-\frac{\pi}{3}\right)+2\sin x=3\ ,\ 0\le x\le 2\pi$$

$$2\sqrt{3}\sin\left(x-\frac{\pi}{6}\right)=3$$

$$\sin\left(x-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$x-\frac{\pi}{6}=\frac{\pi}{3}\ ,\ \frac{2\pi}{3}$$

$$x=\frac{\pi}{2}\ ,\ \frac{5\pi}{6}$$

c. Outcomes assessed: ME12-2

Marking Guidelines

Criteria	Marks
Uses vector methods to establish required equation then solves to find both values of λ	4
Uses vector methods to establish required equation	3
Substantial progress eg. uses right angle to write appropriate dot product in terms of λ is 0	2
Some progress eg. writes vectors $\overrightarrow{PC}$, $\overrightarrow{PB}$ in terms of unit vectors and λ	1

Answer

$\overrightarrow{PC}=-\lambda \underset{\sim}{i}+\underset{\sim}{j}\ ,\ \ \overrightarrow{PB}=\left(\frac{5}{2}-\lambda\right)\underset{\sim}{i}+\underset{\sim}{j}$ since $\overrightarrow{AB}=\overrightarrow{OC}=\underset{\sim}{j}$ in rectangle $OABC$.

$\angle CPB=90°$ $\quad \therefore \left(-\lambda\underset{\sim}{i}+\underset{\sim}{j}\right)\cdot\left\{\left(\frac{5}{2}-\lambda\right)\underset{\sim}{i}+\underset{\sim}{j}\right\}=0$

$-\lambda\left(\frac{5}{2}-\lambda\right)+1=0$ since $\underset{\sim}{i}\cdot\underset{\sim}{i}=\underset{\sim}{j}\cdot\underset{\sim}{j}=1$ and $\underset{\sim}{i}\cdot\underset{\sim}{j}=0$

$$2\lambda^2-5\lambda+2=0$$

$$2\lambda^2-5\lambda+2=0\ ,\ \ 0<\lambda<\frac{5}{2}$$

$$(2\lambda-1)(\lambda-2)=0$$

$$\therefore\ \lambda=\frac{1}{2}\ \ or\ \ \lambda=2$$

Q12 cont.

d. Outcomes assessed: ME12-4

Marking Guidelines

Criteria	Marks
Evaluates the integral using the given substitution	4
Substantial progress eg. correct process with minor error in evaluation	3
Moderate progress eg. finds the transformed and simplified definite integral with limits for θ	2
Some progress eg. writes the new integrand after substitution	1

Answer

$x=\sin^2\theta\ ,\ \ 0\le\theta\le\frac{\pi}{2}$

$dx=2\sin\theta\cos\theta\,d\theta$

$x=0\Rightarrow\theta=0$

$x=1\Rightarrow\theta=\frac{\pi}{2}$

$\sqrt{x(1-x)}=\sqrt{\sin^2\theta\cos^2\theta}$
$=\sin\theta\cos\theta$

$$\int_0^1\sqrt{x(1-x)}\,dx=\int_0^{\frac{\pi}{2}}(\sin\theta\cos\theta).2\sin\theta\cos\theta\,d\theta$$
$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,d\theta$$
$$=\frac{1}{4}\int_0^{\frac{\pi}{2}}(1-\cos 4\theta)d\theta$$
$$=\frac{1}{4}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$$
$$=\frac{1}{4}\left(\frac{\pi}{2}-0\right)$$
$$=\frac{\pi}{8}$$

Question 13

a. Outcomes assessed: ME12-4

Marking Guidelines

Criteria	Marks
Uses differentiation to find coordinates of both stationary points	3
Substantial progress eg. finds and solves quartic equation in x	2
Some progress eg. differentiates and writes equation for x	1

Answer

$$y=x^2+\cos^{-1}x$$
$$\frac{dy}{dx}=2x-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx}=0\ \Rightarrow\ 2x=\frac{1}{\sqrt{1-x^2}}$$
$$4x^2(1-x^2)=1\quad and\quad x\ne\pm1$$
$$4x^4-4x^2+1=0$$
$$(2x^2-1)^2=0$$
$$x^2=\frac{1}{2}$$

Stationary points are

$$\left(\frac{1}{\sqrt{2}}\,,\,\frac{1}{2}+\frac{\pi}{4}\right)$$
$$\left(\frac{-1}{\sqrt{2}}\,,\,\frac{1}{2}+\frac{3\pi}{4}\right)$$

Q13 Cont.

b. Outcomes assessed: ME12-4

Marking Guidelines

Criteria	Marks
Finds the particular solution by integration	3
Substantial progress eg. finds general solution but not required particular solution	2
Some progress eg. uses variables separable and integrates	1

Answer

$$\frac{dy}{dx} = e^{-(2x+y)}$$

$$\int e^y dy = \int e^{-2x} dx$$

$$e^y = -\frac{1}{2}e^{-2x} + c$$

$$\left.\begin{matrix} x=0 \\ y=0 \end{matrix}\right\} \Rightarrow \quad 1 = -\frac{1}{2} + c \qquad c = \frac{3}{2}$$

$$e^y = \frac{1}{2}\left(3 - e^{-2x}\right)$$

$$y = \ln\left(\frac{3 - e^{-2x}}{2}\right)$$

c.i. Outcomes assessed: ME11-3

Marking Guidelines

Criteria	Marks
States both domain and range	2
Substantial progress eg. states exactly one of domain or range correctly	1

Answer

$f(x) = \sin^{-1}\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}$

Domain: $-1 \le x + \frac{\pi}{2} \le 1$ $\quad \left[-1 - \frac{\pi}{2}, 1 - \frac{\pi}{2}\right]$

Range: $-\frac{\pi}{2} \le \sin^{-1}\left(x + \frac{\pi}{2}\right) \le \frac{\pi}{2}$ $\quad [0, \pi]$

c.ii. Outcomes assessed: ME14-4

Marking Guidelines

Criteria	Marks
Uses differentiation to show specified normal passes through the origin	2
Substantial progress eg. finds the gradient of then required normal	1

Answer

$$y = \sin^{-1}\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(x + \frac{\pi}{2}\right)^2}}$$

$$x = -\frac{\pi}{2} \Rightarrow \begin{cases} y = \frac{\pi}{2} \\ \frac{dy}{dx} = 1 \end{cases}$$

Normal at $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has gradient -1

and equation $y - \frac{\pi}{2} = -1\left(x + \frac{\pi}{2}\right)$

$$y = -x$$

hence passes through the origin.

Q13 Cont.

d.i. Outcomes assessed: ME12-5

Marking Guidelines

Criteria	Marks
States mean and standard deviation in simplest fraction form	2
Substantial progress eg. states exactly one of mean or standard deviation in simplest fraction form	1

Answer

Distribution of sample proportion $\widehat{p}$ has $\mu = \frac{2}{5}$ and $\sigma = \sqrt{\frac{\frac{2}{5} \times \frac{3}{5}}{150}} = \frac{1}{25}$

d.ii. Outcomes assessed: ME12-5

Marking Guidelines

Criteria	Marks
Uses the standard normal table to estimate the required probability to 2 dec. pl.	3
Substantial progress eg. finds both limits for Z and reads one appropriate value from the table	2
Some progress eg. finds one limit for Z	1

Answer

Ignoring the continuity correction, $P\left(\frac{57}{150} \le \widehat{p} \le \frac{69}{150}\right) \approx P\left(\frac{\frac{19}{50} - \frac{2}{5}}{\frac{1}{25}} \le Z \le \frac{\frac{23}{50} - \frac{2}{5}}{\frac{1}{25}}\right) = P\left(-\frac{1}{2} \le Z \le \frac{3}{2}\right)$

If $\Phi(z) = P(Z \le z)$, $P\left(-\frac{1}{2} \le Z \le \frac{3}{2}\right) = \Phi(1 \cdot 5) - (1 - \Phi(0 \cdot 5)) = \Phi(1 \cdot 5) + \Phi(0 \cdot 5) - 1$

$\therefore P\left(\frac{57}{150} \le \widehat{p} \le \frac{69}{150}\right) \approx 0 \cdot 9332 + 0 \cdot 6915 - 1 \approx 0 \cdot 62$

Question 14

a. Outcomes assessed: ME12-1

Marking Guidelines

Criteria	Marks
Proves the required result by the process of Mathematical Induction	3
Substantial progress eg. correct procedure but some lack of clarity in deducing P_{k+1} from P_k	2
Some progress eg. defines a sequence of propositions and shows first is true	1

Answer

Let P_n, $n = 1, 2, 3, \ldots$ be the sequence of propositions defined by

$$P_n : \left(1\times 1! + \frac{0}{1!}\right) + \left(2\times 2! + \frac{1}{2!}\right) + \left(3\times 3! + \frac{2}{3!}\right) + \ldots + \left(n\times n! + \frac{n-1}{n!}\right) = \left(n+1\right)! - \frac{1}{n!}$$

Consider P_1: LHS $= 1 + 0 = 1$ and RHS $= 2! - 1 = 1$ $\therefore P_1$ is true.

If P_k is true: $\left(1\times 1! + \frac{0}{1!}\right) + \left(2\times 2! + \frac{1}{2!}\right) + \left(3\times 3! + \frac{2}{3!}\right) + \ldots + \left(k\times k! + \frac{k-1}{k!}\right) = (k+1)! - \frac{1}{k!}$ **

Consider P_{k+1}:

$$\text{LHS} = \left(1\times 1! + \frac{0}{1!}\right) + \left(2\times 2! + \frac{1}{2!}\right) + \left(3\times 3! + \frac{2}{3!}\right) + \ldots + \left(k\times k! + \frac{k-1}{k!}\right) + \left((k+1)\times(k+1)! + \frac{(k+1)-1}{(k+1)!}\right)$$

$= \left\{(k+1)! - \frac{1}{k!}\right\} + \left((k+1)\times(k+1)! + \frac{k}{(k+1)!}\right)$ if P_k is true using **.

$$= (k+1)!\{1+(k+1)\} - \frac{1}{(k+1)!}\{(k+1)-k\}$$

$$= \{(k+1)+1\}! - \frac{1}{(k+1)!}$$

$=$ RHS

Hence if P_k is true then P_{k+1} is true. But P_1 is true. Hence by Mathematical Induction, P_n is true for all integers $n \geq 1$.

b.i. Outcomes assessed: ME12-4

Marking Guidelines

Criteria	Marks
Compares gradients to establish required result	2
Substantial progress eg. evaluates derivative at $x = e$.	1

Answer

$y = \log_2 x$

$\frac{dy}{dx} = \frac{1}{x\ln 2}$

$x = e \Rightarrow \frac{dy}{dx} = \frac{1}{e\ln 2} = \frac{\ln e}{e\ln 2} = \frac{\log_2 e}{e}$ (Change of base rule)

$\therefore$ Gradient of tangent at $P\left(e, \log_2 e\right)$ is $\frac{\log_2 e}{e}$ which is also gradient of OP.

Hence OP is tangent to curve at P.

Q14b Cont.

b.ii. Outcomes assessed: ME12-4

Marking Guidelines

Criteria	Marks
Uses integration to evaluate the required volume in simplest exact form	4
Substantial progress eg. correct procedure but error in simplification	3
Moderate progress eg. writes expression for V and finds an appropriate anti-derivative	2
Some progress eg. writes an expression for V involving an appropriate definite integral	1

Answer

Volume is given by $V = \pi \int_0^{\log_2 e} 2^{2y}\,dy - \frac{1}{3}\pi e^2 \log_2 e$

$$= \frac{\pi}{2\ln 2}\Big[\, 2^{2y} \,\Big]_0^{\log_2 e} - \frac{\pi e^2}{3}\log_2 e$$

$$= \frac{\pi \ln e}{2\ln 2}\left(2^{2\log_2 e} - 1\right) - \frac{\pi e^2}{3}\log_2 e$$

$$= \pi \log_2 e\left\{\frac{1}{2}\left(e^2 - 1\right) - \frac{1}{3}e^2\right\}$$

$$= \pi \log_2 e\left(\frac{1}{6}e^2 - \frac{1}{2}\right)$$

Hence volume is $\frac{1}{6}\pi\left(e^2 - 3\right)\log_2 e$ cubic units.

c.i. Outcomes assessed: ME12-2

Marking Guidelines

Criteria	Marks
Writes and solves an appropriate equation for t in terms of V and g.	3
Substantial progress eg. correct procedure but some lack of clarity in explanation	2
Some progress eg. writes an appropriate equation for t in terms of V, g and α	1

Answer

The stone hits the ramp when the coordinates of the stone lie on the line $y = \frac{1}{2}x$. That is when

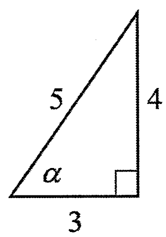
$$Vt\sin\alpha - \frac{1}{2}gt^2 = \frac{1}{2}Vt\cos\alpha\,, \quad t \neq 0$$

$$\frac{4}{5}Vt - \frac{1}{2} \times \frac{3}{5}Vt = \frac{1}{2}gt^2$$

$$\frac{1}{2}Vt = \frac{1}{2}gt^2$$

$$\frac{V}{g} = t$$

$$\therefore T = \frac{V}{g}$$



Q14c Cont.

c.ii. Outcomes assessed: ME12-2

Marking Guidelines

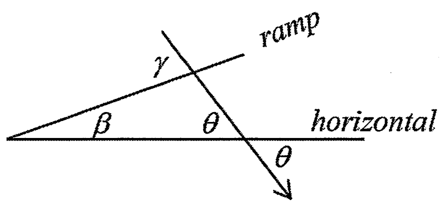
Criteria	Marks
Uses exact method to find required angle	3
Substantial progress eg. finds angle between direction of travel and horizontal when ramp hit	2
Some progress eg. finds vector for velocity when stone hits ramp	1

Answer

$$r'(t) = V\cos\alpha\, \underset{\sim}{i} + (V\sin\alpha - gt)\underset{\sim}{j} = \frac{3}{5}V\underset{\sim}{i} + \left(\frac{4}{5}V - gt\right)\underset{\sim}{j}$$

$$t = \frac{V}{g} \Rightarrow r'(t) = \frac{3}{5}V\underset{\sim}{i} + \left(\frac{4}{5}V - V\right)\underset{\sim}{j} = \frac{3}{5}V\underset{\sim}{i} - \frac{1}{5}V\underset{\sim}{j}$$

If stone hits ramp when direction of travel is at angle θ below the horizontal, $\tan\theta = \frac{1}{5} \div \frac{3}{5} = \frac{1}{3}$



$$\gamma = \beta + \theta$$

$$\tan\gamma = \frac{\tan\beta + \tan\theta}{1 - \tan\beta\tan\theta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}}$$

$$\tan\gamma = 1$$

$$\gamma = \frac{\pi}{4}$$

Stone hits ramp at angle $\frac{\pi}{4}$ to the ramp.

Independent HSC Trial Examinations 2024 Mathematics Extension 1 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1	1	Further work with functions	ME11-1	E2-E3
2	1	Polynomials	ME11-2	E2-E3
3	1	Working with combinatorics	ME11-5	E2-E3
4	1	Rates of change	ME11-4	E2-E3
5	1	Inverse trigonometric functions	ME11-3	E2-E3
6	1	Introduction to vectors	ME12-2	E3-E4
7	1	Applications of calculus	ME12-4	E3-E4
8	1	Further calculus skills	ME12-4	E3-E4
9	1	Further calculus skills	ME12-4	E3-E4
10	1	The binomial distribution	ME12-5	E3-E4
11 a	2	Working with combinatorics	ME11-5	E2-E3
b	2	Further trigonometric identities	ME11-3	E2-E3
c	2	Further calculus skills	ME12-4	E2-E3
d	3	Further work with functions	ME11-2	E2-E3
e	3	Further trigonometric identities	ME11-3	E2-E3
f	3	Introduction to vectors	ME12-2	E2-E3
12 a	3	Inverse trigonometric functions	ME11-3	E2-E3
b i	2	Further trigonometric identities	ME11-3	E2-E3
ii	2	Trigonometric equations	ME12-3	E2-E3
c	4	Introduction to vectors	ME12-2	E3-E4
d	4	Applications of calculus	ME12-4	E3-E4
13 a	3	Applications of calculus	ME12-4	E2-E3
b	3	Applications of calculus	ME12-4	E3-E4
c i	2	Inverse trigonometric functions	ME11-3	E2-E3
ii	2	Applications of calculus	ME12-4	E3-E4
d i	2	The binomial distribution	ME12-5	E2-E3
ii	3	The binomial distribution	ME12-5	E3-E4
14 a	3	Proof by Mathematical Induction	ME12-1	E3-E4
b i	2	Applications of calculus	ME12-4	E3-E4
ii	4	Applications of calculus	ME12-4	E3-E4
c i	3	Introduction to vectors	ME12-2	E3-E4
ii	3	Introduction to vectors	ME12-2	E3-E4