**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1 – 10.

1 Which expression is equivalent to $\int \sin^3 x \, dx$?

A. $\frac{1}{3}\cos^3 x + \cos x + c$

B. $\frac{1}{3}\cos^3 x - \cos x + c$

C. $\frac{1}{3}\sin^3 x + \sin x + c$

D. $\frac{1}{3}\sin^3 x - \sin x + c$

2 A particle moves along a straight line with an acceleration of $\frac{2}{V}$ m/s$^2$, where $V$ is the velocity at any instant in m/s.

The initial velocity of the particle is $-1$ m/s. The particle will move to the:

A. left, increasing in speed

B. left, stop, then move to the right

C. right, increasing in speed

D. right, stop, then move to the left

3 What are the values of the real numbers $p$ and $q$ such that $1 - i$ is a root of the equation $z^3 + pz + q = 0$?

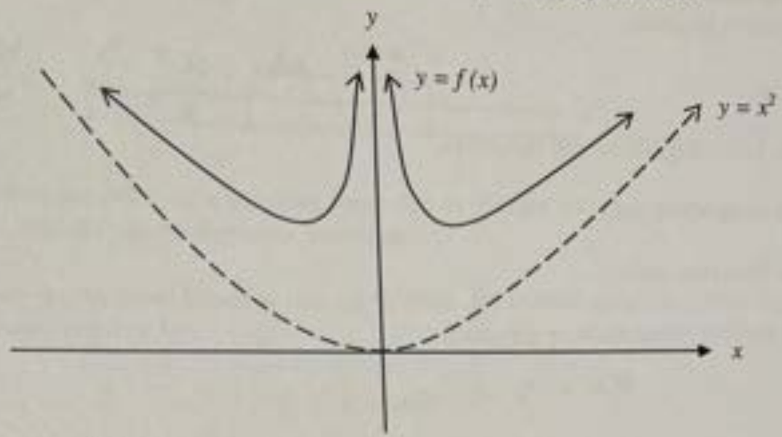
A. $p = 2$ *and* $q = 4$

B. $p = -2$ *and* $q = 4$

C. $p = 2$ *and* $q = -4$

D. $p = -2$ *and* $q = -4$

For the graph shown in the diagram below, a possible equation is:



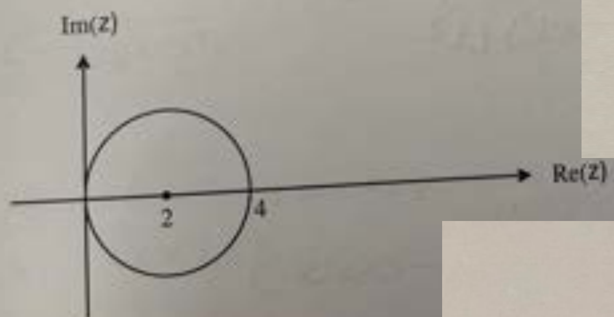

A. $y = \dfrac{x^3 + k}{x}, k > 0$

B. $y = \dfrac{x^3 + k}{x}, k < 0$

C. $y = \dfrac{x^4 + k}{x^2}, k > 0$

D. $y = \dfrac{x^4 + k}{x^2}, k < 0$

5 Which of the following is the equation of the circle below?




A. $(z-2)(\bar{z}-2i) = 4$

B. $(z-2)(\bar{z}-2) = 4$

C. $(z+2)(\bar{z}-2) = 4$

D. $(z+2i)(\bar{z}-2i) = 4$

6 The position of a moving object is given by the cartesian coordinates $(3t, e^t)$ where $t$ is the time in seconds.

Its acceleration is:

A. constant in both magnitude and direction.

B. constant in magnitude only.

C. constant in direction only.

D. constant in neither magnitude or direction.

7 Using an appropriate substitution, $\int e^{2x}\sqrt{e^x - 1}\,dx$ is equivalent to:

A. $\int\left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$

B. $\int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$

C. $\int\left(u^3 + u\right)du$

D. $\int\left(u^3 - u\right)du$

8 Let $z = \cos\theta + i\sin\theta$, where $\theta$ is acute.

The value of $\arg\left(z^2 - z\right)$ is:

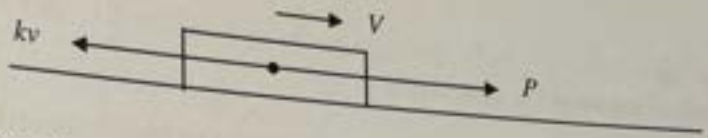
A. $\frac{\pi}{2} - \frac{\theta}{2}$

B. $\frac{\pi}{2} + \frac{\theta}{2}$

C. $\frac{\pi}{2} - \frac{3\theta}{2}$

D. $\frac{\pi}{2} + \frac{3\theta}{2}$

The diagram below shows a box of mass $m$ being pushed by a constant force $P$ across a table.




The box experiences a resistive force due to friction which is proportional to its velocity, that is $kv$, and it is in the opposite direction.

There are no other forces acting on the box. By considering the forces on the box, its limiting velocity is given by:

A. $0$

B. $\dfrac{P}{mk}$

C. $\dfrac{P}{k}$

D. $\sqrt{\dfrac{P}{k}}$

10 $J = \displaystyle\int_0^1 \sqrt{1-x^4}\,dx$

$K = \displaystyle\int_0^1 \sqrt{1+x^4}\,dx$

$L = \displaystyle\int_0^1 \sqrt{1-x^8}\,dx$

Which of the following statements is true for the definite integrals shown above?

A. $J < L < 1 < K$

B. $J < L < K < 1$

C. $L < J < 1 < K$

D. $L < J < K < 1$

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 - 16 your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) **[START A NEW BOOKLET]**

(a) What is the cartesian equation of the line: **2**

$$\underset{\sim}{r} = \underset{\sim}{i} + 2\underset{\sim}{j} + \lambda\left(-3\underset{\sim}{i} + 6\underset{\sim}{j}\right)\ ?$$

(b) Find $r$ and $\theta$ such that: **2**

$$e^{i\pi} - e^{i\frac{\pi}{2}} = re^{i\theta}$$

(c) Find $\int x \sin 2x \, dx$ **3**

(d) For integer $m$, prove by contrapositive that if $m^2$ is not divisible by 4, then $m$ is odd. **2**

(e) Prove that a particle with $x = 3\cos^2 t$ is in simple harmonic motion. State its centre of motion and its amplitude. **3**

(f) Three vertices of a parallelogram are $O(0,0,0)$, $A(2,2,1)$ and $B(1,2,2)$. **3**

Find **<u>all</u>** the possible positions of $C$, the fourth vertex.

**Question 13** (15 marks) **[START A NEW BOOKLET]**

(a) Let $I_n = \int_0^1 \frac{x^n}{(1+x)^2}\,dx$ for $n \geq 0$.

(i) For $n \geq 2$, show that $I_n = \frac{1}{2(n-1)} - \frac{n}{n-1} I_{n-1}$ **4**

(ii) Hence find $\int_0^1 \frac{x^3}{(1+x)^2}\,dx$. **2**

(b) Find the shortest distance from the origin to the line through $A(1, 3, 1)$ and $B(0, 1, -1)$. **3**

(c) A body of unit mass is projected vertically upwards, under gravity, from the ground in a medium. This produces a resistance force of $kv^2$, where $v$ is the velocity and $k$ is a positive constant. The acceleration due to gravity is $g$.

(i) If the initial velocity is $v_0$, prove that the maximum height, $H$, of the body above the ground is:

$$H = \frac{1}{2k}\log_e\left(1 + \frac{kv_0^2}{g}\right)$$ **3**

(ii) In a second projection vertically upwards of the body, it is noticed that the new maximum height reached is $2H$.

Show that the initial velocity of the second projection was $v_0\left(e^{2kH}+1\right)^{\frac{1}{2}}$ **3**

...14 (15 marks) [START A NEW BOOKLET]

(a) (i) Prove that $\sqrt{10}$ is irrational. 2

(ii) Hence prove that $\sqrt{2}+\sqrt{5}$ is irrational. 2

(b) (i) Prove that $\sqrt{ab} \leq \frac{a+b}{2}$ where $a \geq 0$ and $b \geq 0$. 1

(ii) Prove that $\sqrt{ab} \leq \sqrt{\frac{a^2+b^2}{2}}$ 1

(iii) Prove that $\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$ 2

(iv) If $1 \leq x \leq y$, show that $x(y-x+1) \geq y$ 2

(v) Let $n$ and $k$ be positive integers with $1 \leq k \leq n$.

Prove that $\sqrt{n} \leq \sqrt{k(n-k+1)} \leq \frac{n+1}{2}$. 2

(vi) For integers $n \geq 1$ prove that $\left(\sqrt{n}\right)^n \leq n! \leq \left(\frac{n+1}{2}\right)^n$. 3

**Question 15** (15 marks) **[START A NEW BOOKLET]**

(a) (i) Expand and simplify $(x+1)(x^2-x+1)$. 1

(ii) Hence evaluate $\displaystyle\int_0^{\infty} \frac{1}{1+x^3}\,dx$. 4

(b) A stone is projected from a point on the ground and it just clears a fence $d$ metres away. The height of the fence is $h$ metres.

The angle of projection is $\theta$ and the speed of projection is $v$ m/s. Air resistance is negligible. The displacement equations are $x = vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + vt\sin\theta$.

(i) Show that $v^2 = \dfrac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$ 2

(ii) Show that the maximum height reached is $\dfrac{d^2\tan^2\theta}{4(d\tan\theta - h)}$ 2

(iii) Show that the stone, when at its maximum height, will just clear the fence if $\tan\theta = \dfrac{2h}{d}$ 2

(c) (i) Show that $\displaystyle\int_{-a}^{a} \frac{x^4}{1+e^x}\,dx = \int_{-a}^{a} \frac{x^4 e^x}{1+e^x}\,dx$. 2

(ii) Hence, or otherwise, evaluate $\displaystyle\int_{-2}^{2} \frac{x^4}{1+e^x}\,dx$ 2

Question 16 (15 marks) [START A NEW BOOKLET]

(a) Relative to a fixed origin $O$, the horizontal unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are pointing due east and north respectively. A particle $P$, of mass 2kg, is moving under the action of a single constant force of $\underset{\sim}{F}$ newtons.

When $t=0$ seconds, the velocity of $P$ is $\left(3\underset{\sim}{i}-5\underset{\sim}{j}\right)$ m/s and

when $t=4$ seconds, the velocity of $P$ is $\left(11\underset{\sim}{i}+7\underset{\sim}{j}\right)$ m/s.

You may assume that $\underset{\sim}{v}=\underset{\sim}{u}+\underset{\sim}{a}t$, where $\underset{\sim}{v}$ is the velocity at time $t$ with initial velocity $\underset{\sim}{u}$

(i) Calculate the speed of the particle when $t=0$. 1

(ii) Determine the vector $\underset{\sim}{F}$. 1

(iii) Find the value of $t$ at the moment the particle is moving due east. 1

(b) Consider two spheres $S_1$ and $S_2$.

$$S_1:(x-1)^2+(y+1)^2+(z-2)^2=64$$

$$S_2:(x+1)^2+(y-3)^2+(z+2)^2=4$$

(i) Show that $S_1$ and $S_2$ are tangential (that is, they touch at only one point). 2

(ii) Find the coordinates of the point of contact. 2

(c) (i) Show that $e^{in\theta} - e^{-in\theta} = 2i\sin(n\theta)$

(ii) Consider the expression $S_n = 1 + e^{i\theta} + e^{i2\theta} + e^{i3\theta} + \ldots\ldots + e^{in\theta}$

Show that $S_n = \dfrac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$

(iii) Show that $S_n = e^{i\left(\frac{n}{2}\right)\theta} \times \dfrac{\sin\left(\frac{n+1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}$

(iv) Deduce that:

$$\sin\theta + \sin 2\theta + \ldots\ldots \sin n\theta = \frac{\sin\left(\frac{n}{2}\right)\theta \,\sin\left(\frac{n+1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}$$

**End of Paper**

Year 12 Extension 2 Mathematics Trial Examination 2024.

Q1. $\int \sin^3 x \, dx$

$= \int (1-\cos^2 x)\sin x \, dx$

$= \int \sin x - \cos^2 x \sin x \, dx$

$= -\cos x + \frac{1}{3}\cos^3 x + c$

$= \frac{1}{3}\cos^3 x - \cos x + c$ (B)

Q2. The initial velocity is $-1$ m/s. So it is moving to left. Notice that $\ddot{x} = -2$ m/s initially. So acceleration is in the same direction, so the particle will not stop, but move to the left increasing in speed. (A)

Q3. $(1-i)$ is a root of $z^3 + pz + q = 0$, so

$(1-i)^3 + p(1-i) + q = 0$

$-2i(1-i) + p - ip + q = 0$

$-2i - 2 + p - ip + q = 0$

$(p+q-2) + i(-p-2) = 0$

$(p+q-2) - i(p+2) = 0$

$\therefore p = -2$

$q = 4$

(B)

Q4. The curve approaches the parabola as an asymptote from above. Hence $k > 0$. Eliminate (B) and (D). Notice that $y = f(x)$ is an even function.

(A) is not an even function.

(C) is an even function.

Hence (C).

Q5. The Cartesian equation of the circle is $(x-2)^2 + y^2 = 4$.

Now if

$(z-2)(\bar{z}-2) = 4$ then

$((x-2)+iy)((x-2)-iy) = 4$

$(x-2)^2 - iy(x-2) + iy(x-2) + y^2 = 4$

$(x-2)^2 + y^2 = 4$

Hence (B).

Q6. $x = 3t \qquad y = e^t$

$\dot{x} = 3 \qquad \dot{y} = e^t$

$\ddot{x} = 0 \qquad \ddot{y} = e^t$

So no acceleration in $x$, only acceleration in $y$. Acceleration is constant in direction only. Hence (C)

Q7. $\int e^{2x}\sqrt{e^x - 1} \, dx$

Let $u = e^x - 1$

$du = e^x \, dx$

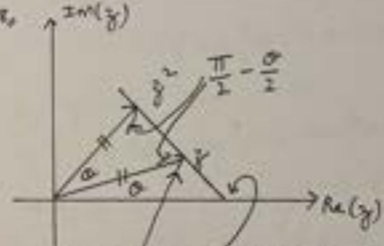
$I = \int e^x \sqrt{e^x - 1}\,(e^x \, dx)$

$= \int (u+1)\sqrt{u} \, du$

$= \int u^{3/2} + u^{1/2} \, du$

(A)

Q8.




$\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$ $\quad \arg(z^2 - z) = \theta + \left(\frac{\pi}{2} + \frac{\theta}{2}\right)$

$= \frac{\pi}{2} + \frac{3\theta}{2}$

(D)

Q9. $m\ddot{x} = P - kV$

$\ddot{x} = \frac{P}{m} - \frac{kV}{m}$

Solve $\ddot{x} = 0$

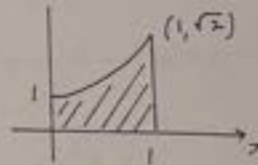
$\frac{P}{m} = \frac{kV}{m}$

$V = \frac{P}{k}$

(C)

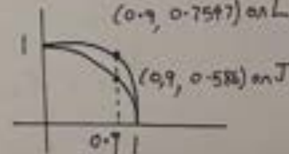
Q10. Is $K < 1$ or $k > 1$?

The graph of $y = \sqrt{1 + x^4}$ between $x = 0$ and $x = 4$ looks like this:




K is the shaded area, so $K > 1$. Eliminate (B) and (D).

which is bigger L or J?

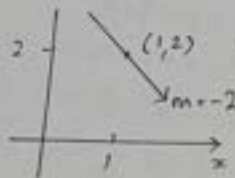



So $L > J$.

Hence (A)

Q1b (a) The Cartesian equation of the line

$\underset{\sim}{r} = \underset{\sim}{i} + 2\underset{\sim}{j} + \lambda(-3\underset{\sim}{i} + 6\underset{\sim}{j})$ is




$y - 2 = -2(x-1)$
$y = -2x + 4$

(b) $e^{i\pi} - e^{i\frac{\pi}{2}}$

$= \cos\pi + i\sin\pi - (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$= -1 - i$

$= \sqrt{2}\, e^{-i\frac{3\pi}{4}}$

Hence $r = \sqrt{2}$ and $\theta = \frac{-3\pi}{4}$

(c) $\int x \sin 2x \, dx$

$= -\frac{1}{2}\cos 2x . x - \int -\frac{1}{2}\cos 2x\,(1)\, dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x \, dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

(d) The contrapositive is:

"If m is not odd, then $m^2$ is divisible by 4."

Suppose that m is even.
Let $m = 2k$ (k an integer).

Then $m^2 = 4k^2$,
which is divisible by 4.

So if m is even, then $m^2$ is divisible by 4.
Hence, if $m^2$ is not divisible by 4, then m is odd.

(e) $x = 3\cos^2 t$

$\dot{x} = -6\cos t \sin t$

$= -3\sin 2t$

$\ddot{x} = -6\cos 2t$

But $\cos 2t = 2\cos^2 t - 1$ so

$\ddot{x} = -6(2\cos^2 t - 1)$

$= -12\cos^2 t + 6$

$= -4(3\cos^2 t) + 6$

$= -4x + 6$

$= -4(x - \frac{3}{2})$

which is of the form

$\ddot{x} = -n^2(x - a)$

The centre of motion is $x = \underline{3}$ and the amplitude is $3/2$.

(f) There are 3 possible positions for the fourth vertex, D, to make a parallelogram.

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB}$

$\overrightarrow{OD} = \overrightarrow{OA} - \overrightarrow{OB}$

$\overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$

Hence $\overrightarrow{OD} = (2,2,1) + (1,2,2) = (3,4,3)$ or

$\overrightarrow{OD} = (2,2,1) - (1,2,2) = (1,0,-1)$ or

$\overrightarrow{OD} = (1,2,2) - (2,2,1) = (-1,0,1)$.

(a) (i) By substitution,

$(\sqrt{2}-i)^3-(\sqrt{2}-i)(\sqrt{2}-i)^3+8(\sqrt{2}-i)-8\sqrt{2}+8i$

$=0+8\sqrt{2}-8i-8\sqrt{2}+8i$

$=0$

Hence $\sqrt{2}-i$ is a root of the equation.

(ii) The sum of the roots is $-\frac{b}{a}=\sqrt{2}-i$

Hence $\alpha+\beta+\sqrt{2}-i=\sqrt{2}-i$

$\alpha+\beta=0$

$\alpha=-\beta$

and the product of the roots is $-\frac{d}{a}=8\sqrt{2}-8i$

Hence $\alpha\beta(\sqrt{2}-i)=8\sqrt{2}-8i$

$\alpha\beta=\frac{8\sqrt{2}-8i}{\sqrt{2}-i}\times\frac{\sqrt{2}+i}{\sqrt{2}+i}$

$\alpha(-\alpha)=\frac{16-8\sqrt{2}i+8\sqrt{2}i+8}{3}$

$-\alpha^2=8$

$\alpha^2=-8$

$\alpha=\pm2\sqrt{2}i$

If $\alpha=2\sqrt{2}i$ then $\beta=-2\sqrt{2}i$, or vice versa.

(b) Now $(\cos\theta+i\sin\theta)^5=\cos5\theta+i\sin5\theta$ by de Moivre

(i) By the Binomial Theorem

$(\cos\theta+i\sin\theta)^5$

$=\cos^5\theta+5\cos^4\theta(i\sin\theta)+10\cos^3\theta(i\sin\theta)^2+10\cos^2\theta(i\sin\theta)^3+$
$5\cos\theta(i\sin\theta)^4+(i\sin\theta)^5$

$=\cos^5\theta+5\cos^4\theta(i\sin\theta)-10\cos^3\theta\sin^2\theta-10\cos^2\theta(i\sin^3\theta)$
$+5\cos\theta\sin^4\theta+i\sin^5\theta$

$=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta+$
$i(5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta)$

By equating Real parts

$\cos5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$

and by equating imaginary parts

$\sin5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$

(ii) $\tan5\theta=\frac{\sin5\theta}{\cos5\theta}$

$=\frac{5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta}{\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta}$

(Now divide top and bottom by $\cos^5\theta$)

$=\frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-\tan^2\theta+5\tan^4\theta}$

$=\frac{5t-10t^3+t^5}{1-10t^2+5t^4}$, where $t=\tan\theta$.

(iii) Consider solving $\tan 5\theta = 0$.

Then $5t - 10t^3 + t^5 = 0$

$t(5 - 10t^2 + t^4) = 0$

So $t = 0$ or $t^4 - 10t^2 + 5 = 0$

From $\tan 5\theta = 0$

$5\theta = 0, \pi, 2\pi, 3\pi, 4\pi$

$\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

Hence the four roots of $t^4 - 10t^2 + 5 = 0$

are $\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$

and the product of the roots is $\frac{e}{a} = 5$

Hence $\tan\frac{\pi}{5} \times \tan\frac{2\pi}{5} \times \tan\frac{3\pi}{5} \times \tan\frac{4\pi}{5} = 5$,

as required.

(c) $\begin{pmatrix}5\\5\\5\end{pmatrix} = \lambda\begin{pmatrix}2\\3\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix} + v\begin{pmatrix}-1\\2\\-1\end{pmatrix}$

$\begin{pmatrix}5\\5\\5\end{pmatrix} = \begin{pmatrix}2\lambda + \mu - v\\3\lambda - \mu + 2v\\\lambda + 2\mu - v\end{pmatrix}$

$2\lambda + \mu - v = 5$ — ①

$3\lambda - \mu + 2v = 5$ — ②

$\lambda + 2\mu - v = 5$ — ③

+② gives $5\lambda + v = 10$ — ④

+2×② gives $7\lambda + 3v = 15$ — ⑤

3×④ gives

$15\lambda + 3v = 30$ — ⑥

⑥ − ⑤ gives

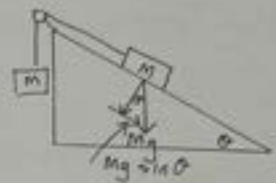
$8\lambda = 15$

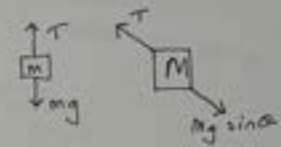
$\therefore \lambda = \frac{15}{8}$

$\therefore v = \frac{5}{8}$

$\therefore \mu = \frac{15}{8}$

(d)




For block M,

$Mg\sin\theta - T = M\ddot{x}$ — ①

For block m,

$T - mg = m\ddot{x}$ — ②

Add ① and ② to get

$Mg\sin\theta - mg = \ddot{x}(m + M)$

$\ddot{x} = \dfrac{Mg\sin\theta - mg}{m + M}$

Substitute this into ② to get

$T = mg + m\left(\dfrac{Mg\sin\theta - mg}{m + M}\right)$

$= \dfrac{mg(m+M) + mMg\sin\theta - m^2g}{m + M}$

$= \dfrac{mMg + mMg\sin\theta}{m + M}$

$= \dfrac{mMg(1 + \sin\theta)}{m + M}$.

(i)

$$I_n = \int_0^1 \frac{x^n}{(1+x)^2}\,dx$$

$$= \int_0^1 x^n (1+x)^{-2}\,dx$$

$$= \left[-(1+x)^{-1}x^n\right]_0^1 - \int_0^1 -(1+x)^{-1} n x^{n-1}\,dx$$

$$= -\tfrac{1}{2} + n\int_0^1 \frac{x^{n-1}}{(1+x)}\,dx$$

$$= -\tfrac{1}{2} + n\left[\int_0^1 \frac{x^{n-1}(1+x)}{(1+x)^2}\,dx\right]$$

$$= -\tfrac{1}{2} + n\left[\int_0^1 \frac{x^{n-1}}{(1+x)^2}\,dx + \int_0^1 \frac{x^n}{(1+x)^2}\,dx\right]$$

$$= -\tfrac{1}{2} + n I_{n-1} + n I_n$$

$$I_n - nI_n = -\tfrac{1}{2} + n I_{n-1}$$

$$I_n(1-n) = -\tfrac{1}{2} + n I_{n-1}$$

$$I_n = \frac{-1}{2(1-n)} + \frac{n}{1-n} I_{n-1}$$

$$= \frac{1}{2(n-1)} - \frac{n}{n-1} I_{n-1}, \text{ as required.}$$

(ii) $\int_0^1 \frac{x^3}{(1+x)^2}\,dx = I_3$

$I_3 = \frac{1}{4} - \frac{3}{2} I_2$

$I_2 = \frac{1}{2} - 2 I_1$

$I_1 = \int_0^1 \frac{x}{(1+x)^2}\,dx$

Let $u = 1+x$

$du = dx$

when $x=1$, $u=2$

when $x=0$, $u=1$

$I_1 = \int_1^2 \frac{u-1}{u^2}\,du$

$= \int_1^2 \frac{1}{u} - \frac{1}{u^2}\,du$

$= \left[\ln u + \frac{1}{u}\right]_1^2$

$= \ln 2 - \frac{1}{2}$

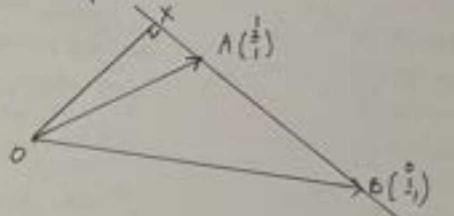
$I_2 = \frac{3}{2} - 2\ln 2$

$I_3 = -2 + 3\ln 2$

Q13 (b) Now $A(1,3,1)$ and $B(0,1,-1)$.

so $\vec{AB} = \vec{OB} - \vec{OA}$

$= \begin{pmatrix}0\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\3\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$




$\vec{AB} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$

The shortest vector from the origin to the line $\vec{AB}$ is the vector $\vec{OX}$ where $\vec{OX} = \vec{OA} + \vec{AX}$ and $\vec{AX}$ is the projection of $\vec{AO}$ onto the line AB.

Now $\text{proj}_{\vec{AB}} \vec{AO}$

$$= \frac{\begin{pmatrix}-1\\-3\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\-2\end{pmatrix}}{1^2+2^2+2^2} \times \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$

$$= \frac{1+6+2}{9} \times \begin{pmatrix}-1\\-2\\-2\end{pmatrix} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$

$\vec{OX} = \vec{OA} + \vec{AX}$

$= \begin{pmatrix}1\\3\\1\end{pmatrix} + \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$ and $|\vec{OX}| = \sqrt{2}$ units

$= \begin{pmatrix}0\\1\\-1\end{pmatrix}$

(c) (i) $v\dfrac{dv}{dx} = -g - kv^2$

$$\frac{dv}{dx} = \frac{-(g+kv^2)}{v}$$

$$\frac{dx}{dv} = \frac{-v}{g+kv^2}$$

Max height $= \displaystyle\int_{V_0}^{0} \frac{-v\,dv}{g+kv^2}$

$$H = \frac{-1}{2k}\int_{V_0}^{0} \frac{2kv}{g+kv^2}\,dv$$

$$= \frac{1}{2k}\int_{0}^{V_0} \frac{2kv}{g+kv^2}\,dv$$

$$= \frac{1}{2k}\Big[\ln(g+kv^2)\Big]_0^{V_0}$$

$$= \frac{1}{2k}\ln(g+kV_0^2) - \frac{1}{2k}\ln(g)$$

$$= \frac{1}{2k}\ln\left(\frac{g+kV_0^2}{g}\right) = \frac{1}{2k}\ln\left(1+\frac{kV_0^2}{g}\right)$$

as required.

(ii) Let the new initial velocity be $V_1$.

$$2H = \frac{1}{2k}\ln\left(1+\frac{kV_1^2}{g}\right)$$

$$4kH = \ln\left(1+\frac{kV_1^2}{g}\right)$$

$$e^{4kH} = 1+\frac{kV_1^2}{g}$$

$$V_1^2 = \frac{g}{k}\left(e^{4kH}-1\right)$$

So $V_1 = \sqrt{\frac{g}{k}\left(e^{4kH}-1\right)}$

Now $V_0 = \sqrt{\frac{g}{k}\left(e^{2kH}-1\right)}$

So $V_1 = \sqrt{\frac{g}{k}\left(e^{2kH}-1\right)\left(e^{2kH}+1\right)}$

$$= \sqrt{\frac{g}{k}\left(e^{2kH}-1\right)} \times \sqrt{e^{2kH}+1}$$

$$= V_0 \times \sqrt{e^{2kH}+1}$$

$$= V_0\left(e^{2kH}+1\right)^{\frac{1}{2}},$$

as required.

Q14 (a)

(i) Proof by contradiction.

Suppose that $\sqrt{10}$ is a rational number and can be written in the form $\sqrt{10} = \frac{p}{q}$ where $p$ and $q$ are integers with no common factors.

Then $10 = \dfrac{p^2}{q^2}$

$$p^2 = 10q^2 \quad \text{— } (1)$$

so $p^2$ is even

so $p$ is even

Let $p = 2m$. Sub into (1).

$$(2m)^2 = 10q^2$$

$$4m^2 = 10q^2$$

$$2m^2 = 5q^2$$

$5q^2$ is even

so $q^2$ is even

so $q$ is even

Contradiction!

$p$ and $q$ cannot both be even as they have no common factors.

Hence $\sqrt{10}$ is not a rational number.

$\sqrt{10}$ is irrational.

(ii) Proof by contradiction.

Suppose that $\sqrt{2}+\sqrt{5}$ is a rational number.

Let $\sqrt{2}+\sqrt{5} = \frac{p}{q}$ where $p$ and $q$ are integers.

Then $(\sqrt{2}+\sqrt{5})^2 = \dfrac{p^2}{q^2}$

$$7+2\sqrt{10} = \frac{p^2}{q^2}$$

$$2\sqrt{10} = \frac{p^2-7q^2}{q^2}$$

$$\sqrt{10} = \frac{p^2-7q^2}{2q^2}$$

Contradiction!

The RHS is an integer $(p^2-7q^2)$ divided by another integer $(2q^2)$, and therefore rational.

But the LHS is $\sqrt{10}$ which we proved in (i) to be irrational.

Hence $\sqrt{2}+\sqrt{5}$ is an irrational number.

a)

(i) Now $(a-b)^2 \geq 0$

$$a^2 - 2ab + b^2 \geq 0$$
$$a^2 + b^2 \geq 2ab$$
$$ab \leq \frac{a^2+b^2}{2}$$

Sub $\sqrt{a}$ for $a$ and $\sqrt{b}$ for $b$

$$\sqrt{a}\cdot\sqrt{b} \leq \frac{(\sqrt{a})^2 + (\sqrt{b})^2}{2}$$
$$\sqrt{ab} \leq \frac{a+b}{2}$$

(ii) Now $ab \leq \frac{a^2+b^2}{2}$ from (i) above

$$\therefore \sqrt{ab} \leq \sqrt{\frac{a^2+b^2}{2}}$$

(iii) From (ii) above

$$\sqrt{ab} \leq \sqrt{\frac{a^2+b^2}{2}}$$

So $\sqrt{cd} \leq \sqrt{\frac{c^2+d^2}{2}}$

$$\sqrt{\sqrt{ab} \cdot \sqrt{cd}} \leq \sqrt{\left(\sqrt{\frac{a^2+b^2}{2}}\right)^2 + \left(\sqrt{\left(\frac{c^2+d^2}{2}\right)^2}\right)^2} \div \sqrt{2}$$

$$\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$$

(iv) If $1 \leq x \leq y$, show that

$$x(y-x+1) \geq y$$

Now $y \geq x$

so $y(x-1) \geq x(x-1)$

because $x \geq 1$

so $xy - y \geq x^2 - x$

$$xy - x^2 + x \geq y$$
$$\therefore x(y-x+1) \geq y$$

Alternatively:

LHS − RHS

$= xy - x^2 + x - y$

$= y(x-1) - x(x-1)$

$= (y-x)(x-1)$

this is positive or zero

this is positive or zero

$\therefore$ LHS − RHS $\geq 0$

$\therefore x(y-x+1) \geq y$

(v) Let $n$ and $k$ be positive integers with $1 \leq k \leq n$.

Prove that $\sqrt{n} \leq \sqrt{k(n-k+1)} \leq \frac{n+1}{2}$.

From (iv) $x(y-x+1) \geq y$

substitute $x = k$ and $y = n$, then

$$k(n-k+1) \geq n$$
$$\therefore \sqrt{k(n-k+1)} \geq \sqrt{n}$$
$$\therefore \sqrt{n} \leq \sqrt{k(n-k+1)} \quad — \text{②}$$

From (i) $\sqrt{ab} \leq \frac{a+b}{2}$ so

$$\sqrt{k(n-k+1)} \leq \frac{k + (n-k+1)}{2}$$

ie, $\sqrt{k(n-k+1)} \leq \frac{n+1}{2}$

Hence $\sqrt{n} \leq \sqrt{k(n-k+1)} \leq \frac{n+1}{2}$

(vi) For integers $n \geq 1$, to prove that

$$(\sqrt{n})^n \leq n! \leq \left(\frac{n+1}{2}\right)^n$$

Substitute $k=1$ into $\alpha$ above gives $\sqrt{n} \leq \sqrt{1(n)} \leq \frac{n+1}{2}$

" $k=2$ " " " $\sqrt{n} \leq \sqrt{2(n-1)} \leq \frac{n+1}{2}$

" $k=3$ " " " $\sqrt{n} \leq \sqrt{3(n-2)} \leq \frac{n+1}{2}$

etc

" $k=n$ " " " $\sqrt{n} \leq \sqrt{n} \leq \frac{n+1}{2}$

Now multiply all the inequalities above together to get

$$(\sqrt{n})^n \leq \sqrt{(n(n-1)(n-2)\dots 3\times 2\times 1)^2} \leq \left(\frac{n+1}{2}\right)^n$$

Hence $(\sqrt{n})^n \leq n! \leq \left(\frac{n+1}{2}\right)^n$, as required.

Q15 (a)

(i) $(x+1)(x^2-x+1)$

$= x^3 - x^2 + x + x^2 - x + 1$

$= x^3 + 1$

(ii) $\int_0^\infty \frac{1}{1+x^3}\,dx$

Let $\frac{1}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{x^2-x+1}$

$1 = A(x^2-x+1) + (Bx+C)(1+x)$

Let $x=-1$, $\quad 1 = 3A$ so $A = \frac{1}{3}$

Let $x=0$, $\quad 1 = A + C$ so $C = \frac{2}{3}$

Let $x=1$, $\quad 1 = A + 2(B+C)$

$1 = \frac{1}{3} + 2B + \frac{4}{3}$ so $B = -\frac{1}{3}$

$$\therefore \int_0^\infty \frac{dx}{1+x^3} = \int_0^\infty \frac{\frac{1}{3}}{1+x} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}\,dx$$

$$= \left[\tfrac{1}{3}\ln|1+x|\right]_0^\infty - \tfrac{1}{3}\int_0^\infty \frac{x-2}{x^2-x+1}\,dx$$

$$= \left[\tfrac{1}{3}\ln|1+x|\right]_0^\infty - \tfrac{1}{3}\int_0^\infty \frac{\frac{1}{2}(2x-1)-\frac{3}{2}}{x^2-x+1}\,dx$$

$$= \left[\tfrac{1}{3}\ln|1+x|\right]_0^\infty - \tfrac{1}{6}\int_0^\infty \frac{2x-1}{x^2-x+1}\,dx + \tfrac{1}{2}\int_0^\infty \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

$$= \left[\tfrac{1}{6}\ln(1+x)^2\right]_0^\infty - \left[\tfrac{1}{6}\ln|x^2-x+1|\right]_0^\infty + \left[\tfrac{1}{2}\cdot\tfrac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right)\right]_0^\infty$$

$$= \left[\tfrac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}\right]_0^\infty + \left[\tfrac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^\infty$$

This term approaches zero.

Is $K<1$ or $K>1$?

$= \frac{1}{\sqrt3}\tan^{-1}\left(\frac{\infty-1}{\sqrt3}\right) - \frac{1}{\sqrt3}\tan^{-1}\left(\frac{-1}{\sqrt3}\right)$

$= \frac{1}{\sqrt3}\cdot\frac{\pi}{2} - \frac{1}{\sqrt3}\left(\frac{-\pi}{6}\right)$

$= \frac{\pi}{2\sqrt3} + \frac{\pi}{6\sqrt3}$

$= \frac{4\pi}{6\sqrt3}$

$= \frac{2\pi}{3\sqrt3}$

(b) Now $t = \frac{x}{V\cos\theta}$

(i) $y = -\frac12 g\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$

$y = \frac{-gx^2}{2V^2}\sec^2\theta + x\tan\theta$

Let $x = d$ and $y = h$, then

$h = \frac{-gd^2}{2V^2}\sec^2\theta + d\tan\theta$

$\frac{gd^2}{2V^2}\sec^2\theta = d\tan\theta - h$

$V^2 = \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$

as required

(ii) Max height occurs on the axis of symmetry.

$t = \frac{V\sin\theta}{g}$

$y_{max} = -\frac12 g\left(\frac{V\sin\theta}{g}\right)^2 + V\left(\frac{V\sin\theta}{g}\right)\sin\theta$

$= -\frac{V^2\sin^2\theta}{2g} + \frac{V^2\sin^2\theta}{g}$

$= \frac{V^2\sin^2\theta}{2g}$

$= V^2\left(\frac{\sin^2\theta}{2g}\right)$

$= \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)} \times \frac{\sin^2\theta}{2g}$

$= \frac{d^2\tan^2\theta}{4(d\tan\theta - h)}$

as required.

(iii) $h = \frac{d^2\tan^2\theta}{4(d\tan\theta - h)}$

$4dh\tan\theta - 4h^2 = d^2\tan^2\theta$

$d^2\tan^2\theta - 4dh\tan\theta + 4h^2 = 0$

$(d\tan\theta - 2h)^2 = 0$

$d\tan\theta - 2h = 0$

$\tan\theta = \frac{2h}{d}$

as required.

[illegible] (c)

(i) To show that

$I = \int_{-a}^{a}\frac{x^4\,dx}{1+e^{x}} = \int_{-a}^{a}\frac{x^4e^x\,dx}{1+e^x}$

Let $x = -u$

$dx = -du$

when $x = a$, $u = -a$

when $x = -a$, $u = a$

$I = \int_{a}^{-a}\frac{(-u)^4(-du)}{1+e^{-u}}$

$= \int_{-a}^{a}\frac{u^4\,du}{1+e^{-u}}$

$= \int_{-a}^{a}\frac{u^4e^u\,du}{1+e^u}$

$= \int_{-a}^{a}\frac{x^4e^x\,dx}{1+e^x}$

(change of variables)

(ii) $\int_{-2}^{2}\frac{x^4\,dx}{1+e^x} = \int_{-2}^{2}\frac{x^4e^x\,dx}{1+e^x}$

$\therefore 2I = \int_{-2}^{2}\frac{x^4(1+e^x)}{(1+e^x)}\,dx$

$= \int_{-2}^{2}x^4\,dx$

$= \left[\frac15 x^5\right]_{-2}^{2}$

$= 64/5$

$\therefore I = \frac{32}{5}$

Q15 (a) (i) when $t = 0$, $\underline{v} = 3\underline{i} - 5\underline{j}$ m/s

Hence speed $= |\underline{v}| = |3\underline{i} - 5\underline{j}|$

$= \sqrt{3^2 + 5^2}$

$= \sqrt{34}$ m/s

(ii) Using $\underline{v} = \underline{u} + \underline{a}t$

$11\underline{i} + 7\underline{j} = (3\underline{i} - 5\underline{j}) + \underline{a} \times 4$

$8\underline{i} + 12\underline{j} = 4\underline{a}$

$\underline{a} = 2\underline{i} + 3\underline{j}$ m/s$^2$

Hence $F = ma$

$= 2(2\underline{i} + 3\underline{j})$

$= 4\underline{i} + 6\underline{j}$ N

(iii) $\underline{v} = \underline{u} + \underline{a}t$

$= (3\underline{i} - 5\underline{j}) + (2\underline{i} + 3\underline{j})t$

$= (2t + 3)\underline{i} + (3t - 5)\underline{j}$

when the particle is moving east, $\underline{j} = 0$ so

$3t - 5 = 0$

$t = \frac{5}{3}$ seconds.

(b)

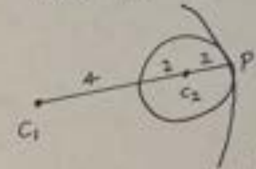
(i) The centre of $S_1$ is at $C_1(1, -1, 2)$ and its radius is 8.

The centre of $S_2$ is at $C_2$ $(-1, 3, -2)$ and its radius is 2.

The distance between $C_1$ and $C_2$ is

$d = \sqrt{(1 - -1)^2 + (-1 - 3)^2 + (2 - -2)^2} = \sqrt{36} = 6$ units

Notice that $S_2$ is inside $S_1$.




The two spheres touch at P.

$C_1C_2 = 6$

$C_2P = \frac{2}{}$

$\overline{8}$

The two spheres are tangential.

(ii) To find the coordinates of the point of contact, we need to extend the direction vector $\overrightarrow{C_1C_2}$ by $\frac{1}{3}$.

Consider the $\underline{i}$ component: from 1 to $-1$ extended by $\frac{1}{3}$ takes us to $-\frac{5}{3}$.

Consider the $\underline{j}$ component: from $-1$ to 3 extended by $\frac{1}{3}$ takes us to $\frac{13}{3}$.

Consider the $\underline{k}$ component: from 2 to $-2$ extended by $\frac{1}{3}$ takes us to $\frac{-10}{3}$.

Hence $P = \left(\frac{-5}{3}, \frac{13}{3}, \frac{-10}{3}\right)$.