NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6–14)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the length of the vector $-\underset{\sim}{i} + 18\underset{\sim}{j} - 6\underset{\sim}{k}$?

A. 5

B. 19

C. 25

D. 361

2 Given that $z = 3 + i$ is a root of $z^2 + pz + q = 0$, where p and q are real, what are the values of p and q?

A. $p = -6$, $q = \sqrt{10}$

B. $p = -6$, $q = 10$

C. $p = 6$, $q = \sqrt{10}$

D. $p = 6$, $q = 10$

3 What is the Cartesian equation of the line $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \end{pmatrix}$?

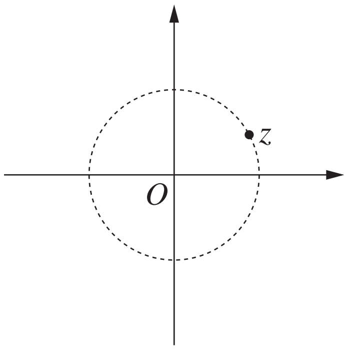
A. $2y + x = 7$

B. $y - 2x = -5$

C. $y + 2x = 5$

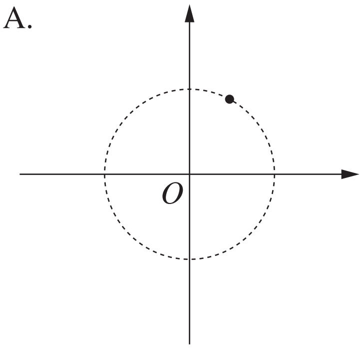
D. $2y - x = -1$

4 The diagram shows the complex number z on the Argand diagram.

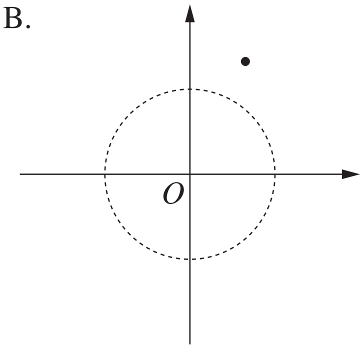


Which of the following diagrams best shows the position of $\dfrac{z^2}{|z|}$?

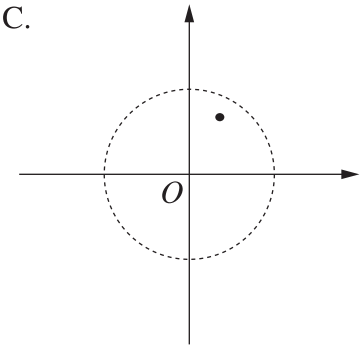
A.



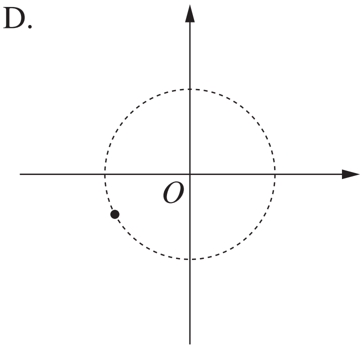
B.



C.



D.



5 A particle undergoing simple harmonic motion has a maximum acceleration of 6 m/s^2 and a maximum velocity of 4 m/s.

What is the period of the motion?

A. π

B. $\dfrac{2\pi}{3}$

C. 3π

D. $\dfrac{4\pi}{3}$

6 Which expression is equal to $\displaystyle\int \frac{1}{x^2+4x+10}\,dx$?

A. $\dfrac{1}{\sqrt{6}}\tan^{-1}\left(\dfrac{x+2}{\sqrt{6}}\right)+c$

B. $\tan^{-1}\left(\dfrac{x+2}{\sqrt{6}}\right)+c$

C. $\dfrac{1}{2\sqrt{6}}\ln\left|\dfrac{x+2-\sqrt{6}}{x+2+\sqrt{6}}\right|+c$

D. $\ln\left|\dfrac{x+2-\sqrt{6}}{x+2+\sqrt{6}}\right|+c$

7 Consider the proposition:

‘If $2^n - 1$ is not prime, then n is not prime’.

Given that each of the following statements is true, which statement disproves the proposition?

A. $2^5 - 1$ is prime

B. $2^6 - 1$ is divisible by 9

C. $2^7 - 1$ is prime

D. $2^{11} - 1$ is divisible by 23

8 Consider the statement:

'If n is even, then if n is a multiple of 3, then n is a multiple of 6'.

Which of the following is the negation of this statement?

A. n is odd and n is not a multiple of 3 or 6.

B. n is even and n is a multiple of 3 but not a multiple of 6.

C. If n is even, then n is not a multiple of 3 and n is not a multiple of 6.

D. If n is odd, then if n is not a multiple of 3 then n is not a multiple of 6.

9 What is the maximum value of $\left|e^{i\theta}-2\right|+\left|e^{i\theta}+2\right|$ for $0 \le \theta \le 2\pi$?

A. $\sqrt{5}$

B. 4

C. $2\sqrt{5}$

D. 10

10 Which of the following is equal to $\int_0^{2a} f(x)\,dx$?

A. $\int_0^a f(x)-f(2a-x)\,dx$

B. $\int_0^a f(x)+f(2a-x)\,dx$

C. $2\int_0^a f(x-a)\,dx$

D. $\int_0^a \frac{1}{2}f(2x)\,dx$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) Consider the complex numbers $w = -1 + 4i$ and $z = 2 - i$.

(i) Evaluate $|w|$. **1**

(ii) Evaluate $w\overline{z}$. **2**

(b) Use integration by parts to evaluate $\displaystyle\int_1^e x \ln x \, dx$. **3**

(c) A particle starts at the origin with velocity 1 and acceleration given by **3**

$$a = v^2 + v,$$

where v is the velocity of the particle.

Find an expression for x, the displacement of the particle, in terms of v.

(d) Consider the two vectors $\underset{\sim}{u} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{v} = p\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$. **3**

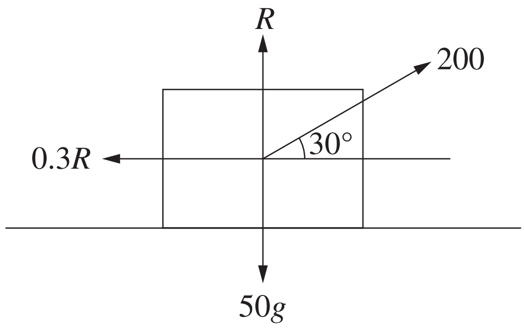
For what values of p are $\underset{\sim}{u} - \underset{\sim}{v}$ and $\underset{\sim}{u} + \underset{\sim}{v}$ perpendicular?

(e) Solve $z^2 + 3z + (3 - i) = 0$, giving your answer(s) in the form $a + bi$, where a and b are real. **4**

Question 12 (14 marks) Use the Question 12 Writing Booklet

(a) A 50-kilogram box is initially at rest. The box is pulled along the ground with a force of 200 newtons at an angle of 30° to the horizontal. The box experiences a resistive force of $0.3R$ newtons, where R is the normal force, as shown in the diagram.

Take the acceleration g due to gravity to be 10 m/s^2.



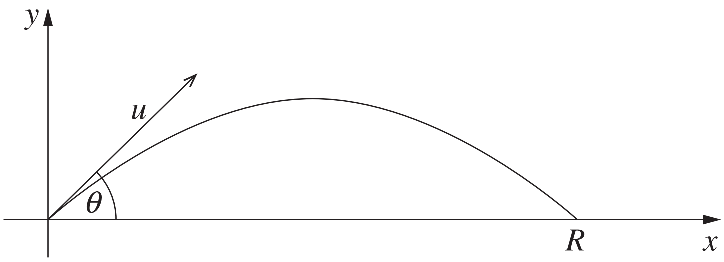
(i) By resolving the forces vertically, show that $R = 400$. **2**

(ii) Show that the net force horizontally is approximately 53.2 newtons. **2**

(iii) Find the velocity of the box after the first three seconds. **2**

Question 12 continues on page 8

(b) A particle is projected from the origin with initial velocity u m/s at an angle θ to the horizontal. The particle lands at $x = R$ on the x-axis. The acceleration vector is given by $\underset{\sim}{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$, where g is the acceleration due to gravity. (Do NOT prove this.)



(i) Show that the position vector $\underset{\sim}{r}(t)$ of the particle is given by **3**

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}.$$

(ii) Show that the Cartesian equation of the path of flight is given by **3**

$$y = \frac{-gx^2}{2u^2}\left(\tan^2\theta - \frac{2u^2}{gx}\tan\theta + 1\right).$$

(iii) Given $u^2 > gR$, prove that there are 2 distinct values of θ for which the particle will land at $x = R$. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) A particle is undergoing simple harmonic motion with period $\frac{\pi}{3}$. The central point of motion of the particle is at $x = \sqrt{3}$. When $t = 0$ the particle has its maximum displacement of $2\sqrt{3}$ from the central point of motion. **3**

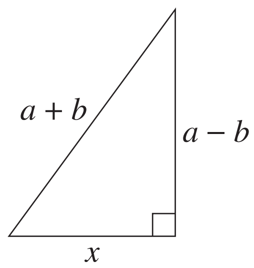
Find an equation for the displacement, x, of the particle in terms of t.

(b) Consider the two lines in three dimensions given by **3**

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}.$$

By equating components, find the point of intersection of the two lines.

(c) (i) By considering the right-angled triangle below, or otherwise, prove that $\frac{a+b}{2} \geq \sqrt{ab}$, where $a > b \geq 0$. **2**



(ii) Prove that $p^2 + 4q^2 \geq 4pq$. **1**

(d) (i) Show that for any integer n, $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$. **1**

(ii) By expanding $\left(e^{i\theta} + e^{-i\theta}\right)^4$, show that **3**

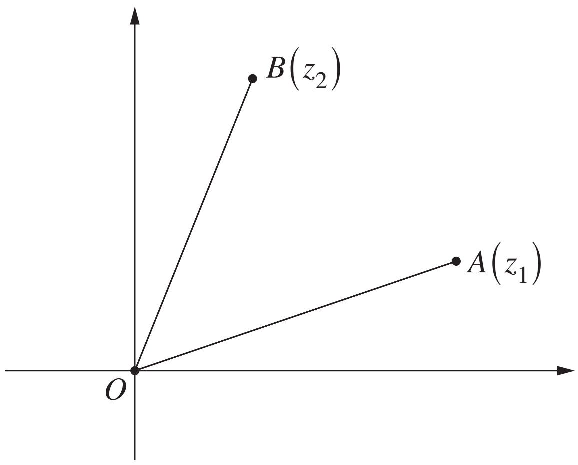
$$\cos^4\theta = \frac{1}{8}\left(\cos(4\theta) + 4\cos(2\theta) + 3\right).$$

(iii) Hence, or otherwise, find $\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$. **2**

Question 14 (16 marks) Use the Question 14 Writing Booklet

(a) Let z_1 be a complex number and let $z_2 = e^{\frac{i\pi}{3}} z_1$.

The diagram shows points A and B which represent z_1 and z_2, respectively, in the Argand plane.



(i) Explain why triangle OAB is an equilateral triangle. **2**

(ii) Prove that $z_1^2 + z_2^2 = z_1 z_2$. **3**

(b) A particle starts from rest and falls through a resisting medium so that its acceleration, in m/s^2, is modelled by **4**

$$a = 10\left(1 - (kv)^2\right),$$

where v is the velocity of the particle in m/s and $k = 0.01$.

Find the velocity of the particle after 5 seconds.

(c) Prove by mathematical induction that, for $n \geq 2$, **4**

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}.$$

(d) Prove that for any integer $n > 1$, $\log_n(n+1)$ is irrational. **3**

Question 15 (13 marks) Use the Question 15 Writing Booklet

(a) In the set of integers, let P be the proposition:

'If $k + 1$ is divisible by 3, then $k^3 + 1$ is divisible by 3'.

(i) Prove that the proposition P is true. **2**

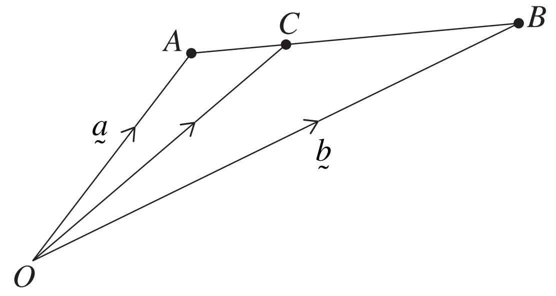
(ii) Write down the contrapositive of the proposition P. **1**

(iii) Write down the converse of the proposition P and state, with reasons, whether this converse is true or false. **3**

Question 15 continues on page 12

Question 15 (continued)

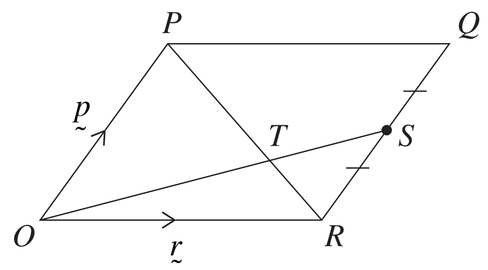
(b) The point C divides the interval AB so that $\frac{CB}{AC} = \frac{m}{n}$. The position vectors of A and B are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, as shown in the diagram.



(i) Show that $\overrightarrow{AC} = \frac{n}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$. **2**

(ii) Prove that $\overrightarrow{OC} = \frac{m}{m+n}\underset{\sim}{a} + \frac{n}{m+n}\underset{\sim}{b}$. **1**

Let $OPQR$ be a parallelogram with $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OR} = \underset{\sim}{r}$. The point S is the midpoint of QR and T is the intersection of PR and OS, as shown in the diagram.



(iii) Show that $\overrightarrow{OT} = \frac{2}{3}\underset{\sim}{r} + \frac{1}{3}\underset{\sim}{p}$. **3**

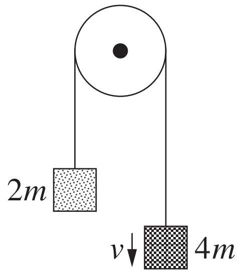
(iv) Using parts (ii) and (iii), or otherwise, prove that T is the point that divides the interval PR in the ratio $2:1$. **1**

End of Question 15

Question 16 (16 marks) Use the Question 16 Writing Booklet

(a) Two masses, $2m$ kg and $4m$ kg, are attached by a light string. The string is placed over a smooth pulley as shown.

The two masses are at rest before being released and v is the velocity of the larger mass at time t seconds after they are released.



The force due to air resistance on each mass has magnitude kv, where k is a positive constant.

(i) Show that $\dfrac{dv}{dt} = \dfrac{gm - kv}{3m}$. **2**

(ii) Given that $v < \dfrac{gm}{k}$, show that when $t = \dfrac{3m}{k}\ln 2$, the velocity of the larger mass is $\dfrac{gm}{2k}$. **3**

Question 16 continues on page 14

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\, d\theta,\ n = 0, 1, \ldots .$

(i) Prove that $I_n = \dfrac{2n}{2n+1} I_{n-1},\ n \geq 1.$ **3**

(ii) Deduce that $I_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}.$ **3**

Let $J_n = \int_0^1 x^n (1-x)^n\, dx,\ n = 0, 1, 2, \ldots .$

(iii) Using the result of part (ii), or otherwise, show that $J_n = \dfrac{(n!)^2}{(2n+1)!}.$ **3**

(iv) Prove that $\left(2^n n!\right)^2 \leq \left(2n+1\right)!.$ **2**

End of paper

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2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}\left(a + b\right)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$\left(x - h\right)^2 + \left(y - k\right)^2 = r^2$$

Financial Mathematics

$$A = P\left(1 + r\right)^n$$

Sequences and series

$$T_n = a + \left(n - 1\right)d$$

$$S_n = \frac{n}{2}\left[2a + \left(n - 1\right)d\right] = \frac{n}{2}\left(a + l\right)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r} = \frac{a\left(r^n - 1\right)}{r - 1}, \; r \neq 1$$

$$S = \frac{a}{1 - r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

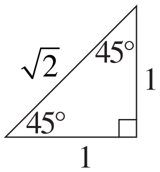
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

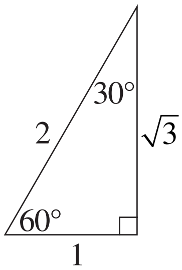
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

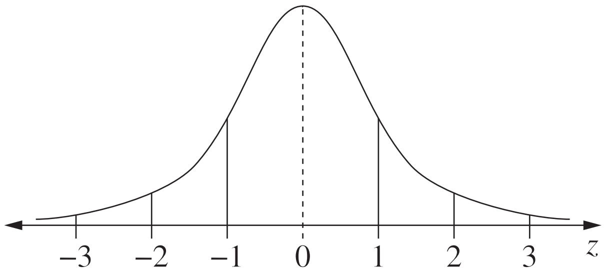
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \text{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x}p^x(1-p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\text{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^ne^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

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2020 HSC Mathematics Extension 2
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	B
2	B
3	C
4	A
5	D
6	A
7	D
8	B
9	C
10	B

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$w = -1 + 4i$$

$$|w| = \sqrt{17}$$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains $\overline{z}$ OR • Correctly evaluates wz	1

Sample answer:

$$\begin{aligned} w\overline{z} &= (-1+4i)(2+i) \\ &= -2-4+8i-i \\ &= -6+7i \end{aligned}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly applies integration by parts, or equivalent merit	2
• Correctly identifies the two functions to be used in integration by parts, or equivalent merit	1

Sample answer:

$$\int_1^e x \ln x \, dx$$

$$u = \ln x, \qquad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{x}, \qquad v = \frac{x^2}{2}$$

$$= \frac{x^2}{2} \ln x \Big|_1^e - \int_1^e \frac{x}{2} \, dx$$

$$= \frac{e^2}{2} - \frac{x^2}{4} \Big|_1^e$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	3
• Correctly separates variables, or equivalent merit	2
• Uses $a = v\dfrac{dv}{dx}$, or equivalent merit	1

Sample answer:

$$a = v^2 + v$$

$$\therefore \quad v \times \frac{dv}{dx} = v^2 + v$$

$$\therefore \quad \frac{dx}{dv} = \frac{1}{v+1}$$

$$x = \ln(v+1) + C$$

When $x = 0, \quad v = 1 \quad \Rightarrow \quad C = -\ln 2$

$$\therefore \quad x = \ln(v+1) - \ln 2$$

Question 11 (d)

Criteria	Marks
• Provides correction solution	3
• Correctly obtains $(\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v})$, or equivalent merit OR • Attempts to apply $(\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = 0$ using correct $(\underset{\sim}{u} - \underset{\sim}{v})$ and $(\underset{\sim}{u} + \underset{\sim}{v})$	2
• Obtains $(\underset{\sim}{u} - \underset{\sim}{v})$ and $(\underset{\sim}{u} + \underset{\sim}{v})$, or equivalent merit	1

Sample answer:

$$\underset{\sim}{u} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$

$$\underset{\sim}{v} = p\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$$

$$\underset{\sim}{u} - \underset{\sim}{v} = (-2-p)\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$$

$$\underset{\sim}{u} + \underset{\sim}{v} = (p-2)\underset{\sim}{i} + 5\underset{\sim}{k}$$

$$(\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = 0$$

$$-(p+2)(p-2) + 5 = 0$$

$$p^2 - 4 = 5$$

$$p = \pm 3$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	4
• Obtains the square root of the discriminant, or equivalent merit	3
• Makes progress towards finding the square root of the discriminant	2
• Correctly obtains the discriminant, or equivalent merit OR • Attempts to use quadratic formula	1

Sample answer:

$$z^2 + 3z + (3 - i) = 0$$

$$\begin{aligned}\Delta &= 9 - 4(3 - i)\\ &= -3 + 4i\end{aligned}$$

Write $-3 + 4i = (x + iy)^2$

$$\therefore \quad x^2 - y^2 = -3 \quad \text{(real parts)}$$
$$x^2 + y^2 = 5 \quad \text{(moduli)}$$

$$\therefore \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$$
$$\Rightarrow \quad y = \pm 2$$

$$\therefore \quad -3 + 4i = (1 + 2i)^2$$

Hence, $z = \dfrac{-3 \pm (1 + 2i)}{2}$

$$= -1 + i, -2 - i$$

Question 12 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains the vertical component of the 200 newton force, or equivalent merit	1

Sample answer:

Resolving vertically,

$$200\sin 30° + R = mg$$

$$\therefore\ R = 50 \times 10 - 200 \times \frac{1}{2}$$

$$= 400 \text{ newtons}$$

Question 12 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains the horizontal component of the 200 newton force, or equivalent merit	1

Sample answer:

Horizontally,

$$\begin{aligned}\text{Net horizontal force } &= 200\cos 30° - 0.3R \\ &= 200 \times \frac{\sqrt{3}}{2} - 0.3 \times 400 \\ &\simeq 53.2 \text{ newtons}\end{aligned}$$

Question 12 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains the acceleration, or equivalent merit	1

Sample answer:

From part (b), using $F = ma$,

$50a = 53.2$

$\therefore \dfrac{dV}{dt} = 1.064$

Hence the velocity after 3 seconds is

$$
\begin{aligned}
V &= \int_0^3 1.064\, dt \\
&= 1.064 \times 3 \\
&= 3.192 \text{ m/s}
\end{aligned}
$$

Question 12 (b) (i)

Criteria	Marks
• Provides correct solution	3
• Correctly obtains $\underset{\sim}{v}(t) = \begin{pmatrix} u\cos\theta \\ u\sin\theta - gt \end{pmatrix}$, or equivalent merit OR • Correctly obtains one component of $\underset{\sim}{r}(t)$	2
• Correctly obtains one component of $\underset{\sim}{v}(t)$, or equivalent merit	1

Sample answer:

$$\underset{\sim}{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$\underset{\sim}{v} = \begin{pmatrix} c_1 \\ -gt + c_2 \end{pmatrix}$$

Now at $t = 0$, $\underset{\sim}{v} = \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix}$

$$\therefore \quad \underset{\sim}{v} = \begin{pmatrix} u\cos\theta \\ u\sin\theta - gt \end{pmatrix}$$

$$\therefore \quad \underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta + c_3 \\ ut\sin\theta - \frac{1}{2}gt^2 + c_4 \end{pmatrix}$$

$$\underset{\sim}{r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad c_3 = c_4 = 0$$

$$\therefore \quad \underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$

Question 12 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains a correct Cartesian equation of flight, or equivalent merit	2
• Attempts to eliminate t, or equivalent merit	1

Sample answer:

$$x = ut\cos\theta \quad \Rightarrow \quad t = \frac{x}{u\cos\theta}$$

$$\begin{aligned} y &= ut\sin\theta - \frac{1}{2}gt^2 \\ &= \frac{ux\sin\theta}{u\cos\theta} - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta} \\ &= x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2}\left(1 + \tan^2\theta\right) \\ &= -\frac{gx^2}{2u^2}\left(\tan^2\theta - \frac{2u^2}{gx}\tan\theta + 1\right) \end{aligned}$$

Question 12 (b) (iii)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the discriminant $\Delta = \dfrac{4u^4}{g^2x^2} - 4$, or equivalent merit	1

Sample answer:

From part (ii), we require the quadratic to have two distinct real roots, when $x = R$.

$$\Delta = \frac{4u^4}{g^2R^2} - 4$$

$$= \frac{4\left(u^4 - g^2R^2\right)}{g^2R^2}$$

Given $u^2 > gR \quad \Rightarrow \quad u^4 > g^2R^2$

$\therefore\ \Delta > 0$

Hence there are two distinct values for $\tan\theta$, hence two distinct values for θ, as $\tan\theta$ is increasing.

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Correctly finds any two of amplitude, n, function with consistent phase	2
• Correctly finds one of amplitude, n, function with consistent phase	1

Sample answer:

$$-\sqrt{3} \qquad 0 \qquad \sqrt{3} \qquad\qquad 3\sqrt{3}$$

The equation has the form

$x(t) = A\cos(nt) + \sqrt{3}$

Now $A = 2\sqrt{3}$ and $\dfrac{2\pi}{n} = \dfrac{\pi}{3}$, so $n = 6$

$\therefore\ x(t) = 2\sqrt{3}\cos(6t) + \sqrt{3}$

Question 13 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly finds either λ_1 or λ_2, or equivalent merit	2
• Finds one equation linking λ_1 and λ_2, or equivalent merit	1

Sample answer:

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

Equating components,

$$\begin{aligned} 3 + \lambda_1 &= 3 - 2\lambda_2 \quad (1) \\ -1 + 2\lambda_1 &= -6 + \lambda_2 \quad (2) \\ 7 + \lambda_1 &= 2 + 3\lambda_2 \quad (3) \end{aligned}$$

(1) $\Rightarrow \lambda_1 = -2\lambda_2$

(2) $\Rightarrow -1 - 4\lambda_2 = -6 + \lambda_2 \Rightarrow \lambda_2 = 1 \Rightarrow \lambda_1 = -2.$

As a check, using equation (3),

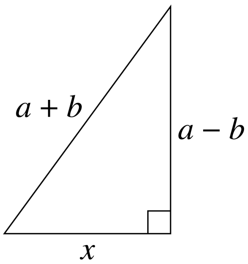
Left hand side = 5, Right hand side = 5

$\therefore$ Point of intersection is $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Applies Pythagoras to find x, or equivalent merit OR • Observes that $x \le a + b$	1

Sample answer:



By Pythagoras,

$$(a+b)^2 = x^2 + (a-b)^2$$
$$\therefore x = 2\sqrt{ab}$$

Now the hypotenuse is the longest side so

$$a + b \ge 2\sqrt{ab}$$
$$\therefore \frac{a+b}{2} \ge \sqrt{ab}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Put $a = p^2,\ b = (2q)^2$

Then $\dfrac{p^2 + 4q^2}{2} \ge \sqrt{4p^2q^2}$

So $\quad p^2 + 4q^2 \ge 4pq$

Question 13 (d) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$e^{in\theta} + e^{-in\theta}$

$= (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$

$= 2\cos(n\theta)$

Question 13 (d) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly uses binomial theorem and groups conjugate pairs, or equivalent merit	2
• Uses part (a) to obtain $\left(e^{i\theta} + e^{-i\theta}\right)^4 = 16\cos^4\theta$ OR • Attempts to use binomial theorem, or equivalent merit	1

Sample answer:

$$\begin{aligned}\left(e^{i\theta} + e^{-i\theta}\right)^4 &= e^{4i\theta} + 4e^{3i\theta} \cdot e^{-i\theta} + 6e^{2i\theta} \cdot e^{-2i\theta} + 4e^{i\theta} \cdot e^{-3i\theta} + e^{-4i\theta} \\ &= \left(e^{4i\theta} + e^{-4i\theta}\right) + 4\left(e^{2i\theta} + e^{-2i\theta}\right) + 6 \\ &= 2\cos 4\theta + 8\cos 2\theta + 6\end{aligned}$$

Also $\left(e^{i\theta} + e^{-i\theta}\right)^4 = 2^4\cos^4\theta$

$\therefore \quad \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

Question 13 (d) (iii)

Criteria	Marks
• Provides correct solution	2
• Uses part (a) and attempts to integrate, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{\pi}{2}} \cos^4 \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right) d\theta$$

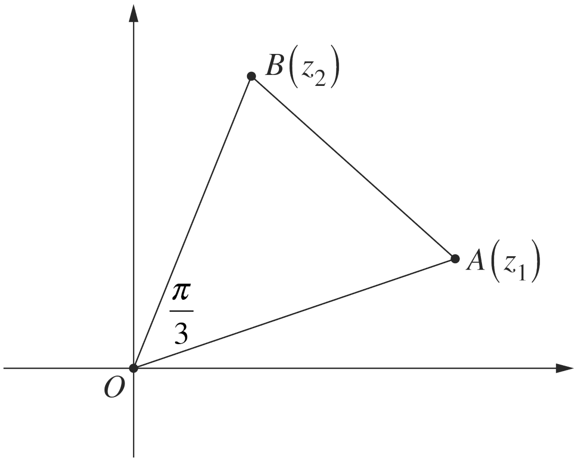
$$= \frac{1}{8}\left(\frac{\sin 4\theta}{4} + 2\sin 2\theta + 3\theta\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left(\frac{3\pi}{2}\right) = \frac{3\pi}{16}$$

Question 14 (a) (i)

Criteria	Marks
• Provides correction solution	2
• Observes either that $\lvert z_2 \rvert = \lvert z_1 \rvert$, or that multiplying by $e^{\frac{i\pi}{3}}$ rotates through an angle of $\frac{\pi}{3}$	1

Sample answer:



Multiplying z_1 by $e^{\frac{i\pi}{3}}$ rotates it anticlockwise through an angle $\frac{\pi}{3}$, giving z_2.

Also $\lvert z_1 \rvert = \lvert z_2 \rvert$.

So $\triangle OAB$ is isosceles giving all angles equal to $\frac{\pi}{3}$ and hence $\triangle OAB$ is equilateral.

Question 14 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Factors $z_2^3 + z_1^3$, or equivalent merit OR • Simplifies either side of $z_1^2 + z_2^2 = z_1 z_2$ using $z_2 = e^{\frac{i\pi}{3}} z_1$	2
• Observes that $z_2^3 = e^{i\pi} z_1^3$, or equivalent merit OR • Uses $z_2 = e^{\frac{i\pi}{3}} z_1$ in both sides of $z_1^2 + z_2^2 = z_1 z_2$	1

Sample answer:

$$z_2 = e^{\frac{i\pi}{3}} z_1$$

$$\therefore \quad z_2^3 = e^{i\pi} z_1^3 = -z_1^3$$

$$\therefore \quad z_1^3 + z_2^3 = 0$$

$$(z_1 + z_2)(z_1^2 + z_2^2 - z_1 z_2) = 0$$

Now $z_1 \neq -z_2$ so $z_1^2 + z_2^2 = z_1 z_2$.

OR

$$\begin{aligned} \text{LHS} = z_1^2 + z_2^2 &= \left(e^{\frac{2\pi i}{3}} + 1 \right) z_1^2 \\ &= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + 1 \right) z_1^2 \\ &= \left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) z_1^2 \end{aligned}$$

$$\begin{aligned} \text{RHS} = z_1 z_2 &= e^{\frac{i\pi}{3}} z_1^2 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \right) z_1^2 \\ &= \left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) z_1^2 \end{aligned}$$

Therefore $z_1^2 + z_2^2 = z_1 z_2$.

Question 14 (b)

Criteria	Marks
• Provides correct solution	4
• Integrates to find v as a function of t, evaluating the constant, or equivalent merit	3
• Correctly applies partial fractions and attempts to integrate, or equivalent merit	2
• Correctly separates variables and attempts to apply partial fractions, or equivalent merit	1

Sample answer:

$$\frac{dv}{dt} = 10\left(1-(kv)^2\right)$$

$$t = \frac{1}{10}\int \frac{1}{(1-kv)(1+kv)}\,dv$$

$$= \frac{1}{20}\int \frac{1}{1-kv} + \frac{1}{1+kv}\,dv \qquad \text{(partial fractions)}$$

$$= \frac{1}{20} \times \frac{1}{k}\left[\ln(1+kv) - \ln(1-kv)\right] + C$$

$$= \frac{1}{20k}\ln\left(\frac{1+kv}{1-kv}\right) + C$$

When $t = 0$, $v = 0$, so $C = 0$.

When $t = 5$, we have

$$\ln\left(\frac{1+kv}{1-kv}\right) = 5 \times 20 \times 0.01 = 1$$

$$\therefore \quad 1 + 0.01v = e(1 - 0.01v)$$

$$v = 100\left(\frac{e-1}{e+1}\right) \simeq 46.2 \text{ m/s}$$

Question 14 (c)

Criteria	Marks
• Provides correct solution	4
• Correctly proves the inductive step OR • Verifies initial case, $n = 2$, AND assumes true for k and attempts to verify true for $k + 1$	3
• Assumes true for k and attempts to verify true for $k + 1$	2
• Verifies initial case, $n = 2$	1

Sample answer:

Let $P(n)$ be the given proposition.

$P(2)$ is true since $\dfrac{1}{2^2} < \dfrac{1}{2}$

Let k be an integer for which $P(k)$ is true.

Thus we assume that

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < \frac{k-1}{k}$$

Consider $\dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{k^2} + \dfrac{1}{(k+1)^2}$

$$< \frac{1}{(k+1)^2} + \frac{k-1}{k}$$

$$= \frac{k + (k+1)^2(k-1)}{(k+1)^2 k}$$

$$< \frac{(k+1) + (k+1)^2(k-1)}{(k+1)^2 k}$$

$$= \frac{1 + k^2 - 1}{(k+1)k} = \frac{k}{k+1}$$

$\therefore\ P(k+1)$ is true.

$\therefore\ P(n)$ true for all integers $n \geq 2$ by induction.

Question 14 (d)

Criteria	Marks
• Provides correct solution	3
• Assumes $\log_n(n+1)$ is rational and attempts to eliminate the logarithm, or equivalent merit	2
• Attempts to use proof by contradiction, or equivalent merit	1

Sample answer:

Suppose $\log_n(n+1)$ is rational.

Then $\log_n(n+1) = \dfrac{p}{q}$, where p, q are integers and $q \neq 0$.

$\therefore\ n+1 = n^{\frac{p}{q}}$

so $(n+1)^q = n^p$

If n is even then RHS is even, LHS is odd – a contradiction.

If n is odd then RHS is odd, LHS is even – a contradiction.

Hence in either case we have a contradiction, so $\log_n(n+1)$ is irrational.

Question 15 (a) (i)

Criteria	Marks
• Provides correct proof	2
• Factors $k^3 + 1$ OR • Writes $k + 1 = 3j$, or equivalent merit	1

Sample answer:

Suppose $k + 1$ is divisible by 3, then $k + 1 = 3j$, for some integer j.

Now $k^3+1 = \left(k+1\right)\left(k^2 - k + 1\right)$

$$= 3j\left(k^2 - k + 1\right)$$

Hence $k^3 + 1$ is divisible by 3.

Question 15 (a) (ii)

Criteria	Marks
• Provides correct statement	1

Sample answer:

If $k^3 + 1$ is not divisible by 3, then $(k + 1)$ is not divisible by 3.

Question 15 (a) (iii)

Criteria	Marks
• Provides correct solution	3
• Considers cases to justify their statement, or equivalent merit	2
• Correctly states the converse or equivalent merit	1

Sample answer:

The converse states that:

'If $k^3 + 1$ is divisible by 3, then $(k + 1)$ is divisible by 3'.

This statement is true.

The integer k must be of the form $3j$, $3j + 1$ or $3j - 1$, for some integer j.

Then $k^3 + 1 = 27j^3 + 1$ or $(3j + 1)^3 + 1$ or $(3j - 1)^3 + 1$.

Only the third case gives $k^3 + 1$ divisible by 3.

OR

Using proof by contradiction, suppose $k^3 + 1$ is divisible by 3.

But $(k + 1)$ is not divisible by 3.

$\therefore$ $k^2 - k + 1$ is divisible by 3

$\therefore$ $(k + 1)^2 - 3k$ is divisible by 3

$\therefore$ $(k + 1)^2$ and hence $(k + 1)$ is divisible by 3.

This is a contradiction.

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Recognises that $\overrightarrow{CB} = \frac{m}{n}\overrightarrow{AC}$, or equivalent merit	1

Sample answer:

$$\overrightarrow{OA} + \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{OB}$$

$$\therefore \underset{\sim}{a} + \overrightarrow{AC} + \frac{m}{n}\overrightarrow{AC} = \overrightarrow{OB}$$

$$\therefore \overrightarrow{AC}\left(1+\frac{m}{n}\right) = \underset{\sim}{b} - \underset{\sim}{a}$$

$$\therefore \overrightarrow{AC} = \frac{n}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$$

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \underset{\sim}{a} + \frac{n}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)\\ &= \underset{\sim}{a}\left(1-\frac{n}{m+n}\right) + \frac{n}{m+n}\left(\underset{\sim}{b}\right)\\ &= \frac{m}{m+n}\underset{\sim}{a} + \frac{n}{m+n}\underset{\sim}{b}\end{aligned}$$

Question 15 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Finds equations of lines OS and PR and attempts to solve for T OR • Writes $\overrightarrow{OT}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{r}$ in two ways and attempts to solve	2
• Finds equation of line OS OR • Writes $\overrightarrow{OT}$ as a multiple of $\overrightarrow{OS} = \underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}$, or equivalent merit	1

Sample answer:

$$\overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{SR}$$

$$= \underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}$$

Equation of line OS is $\underset{\sim}{x} = \left(\underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}\right)\lambda$.

Also equation of line PR is $\underset{\sim}{x} = \underset{\sim}{p} + \left(\underset{\sim}{r} - \underset{\sim}{p}\right)\mu$.

Equating, we have

$$\underset{\sim}{p}\left(\frac{1}{2}\lambda\right) + \underset{\sim}{r}\lambda = \underset{\sim}{p}(1-\mu) + \underset{\sim}{r}\mu$$

$$\therefore \ \frac{1}{2}\lambda = 1-\mu \text{ and } \lambda = \mu \Rightarrow \lambda = \frac{2}{3}$$

Hence point of intersection is

$$\overrightarrow{OT} = \frac{2}{3}\left(\underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}\right) = \frac{2}{3}\underset{\sim}{r} + \frac{1}{3}\underset{\sim}{p}$$

Question 15 (b) (iv)

Criteria	Marks
• Provides correct solution	1

Sample answer:

By part (ii) the point which divides PR in the ratio $2:1$ is given by $\frac{2}{3}\underset{\sim}{r} + \frac{1}{3}\underset{\sim}{p}$, but this is $\overrightarrow{OT}$.

$\therefore$ T divides the interval PR in the ratio $2:1$.

Question 16 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to balance forces, or equivalent merit	1

Sample answer:

Let a be the acceleration of the masses. Comparing forces,

$$4gm - kv - 2gm - kv = 2ma + 4ma$$

$$\therefore\ 2gm - 2kv = 6am$$

$$\therefore\ a = \frac{gm - kv}{3m}$$

Question 16 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains $t = \frac{3m}{k}\ln\left(\frac{gm}{gm-kv}\right)$, or equivalent merit	2
• Integrates $\frac{dt}{dv}$, or equivalent merit	1

Sample answer:

From part (i), $\frac{dv}{dt} = \frac{gm-kv}{3m}$

$$\therefore t = \int_0^v \frac{3m}{gm-kv}\,dv$$

$$= -\frac{3m}{k}\ln\left(gm-kv\right)\Big|_0^v$$

$$= -\frac{3m}{k}\left[\ln\left(gm-kv\right) - \ln\left(gm\right)\right]$$

$$= \frac{3m}{k}\ln\left(\frac{gm}{gm-kv}\right)$$

When $t = \frac{3m}{k}\ln 2$

We have $\frac{gm}{gm-kv} = 2$

$\Rightarrow 1 - \frac{kv}{gm} = \frac{1}{2}$

$\Rightarrow v = \frac{gm}{2k}$

Question 16 (b) (i)

Criteria	Marks
• Provides correct proof	3
• Correctly applies integration by parts and attempts to reduce to integrals involving only $\sin(2\theta)$, or equivalent merit	2
• Attempts to use integration by parts, or equivalent merit	1

Sample answer:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin(2\theta)\,\sin^{2n}(2\theta)\,d\theta$$

$$u = \sin^{2n}(2\theta) \qquad \frac{dv}{d\theta} = \sin 2\theta$$

$$\frac{du}{d\theta} = 4n\sin^{2n-1}(2\theta)\cos 2\theta$$

$$v = -\frac{\cos(2\theta)}{2}$$

$$= \frac{1}{2}\sin^{2n}(2\theta) \times -\cos(2\theta)\Big|_0^{\frac{\pi}{2}} + 2n\int_0^{\frac{\pi}{2}} \sin^{2n-1}(2\theta)\,\cos^2(2\theta)\,d\theta$$

$$\therefore\ I_n = 2n\left[\int_0^{\frac{\pi}{2}} \sin^{2n-1}(2\theta)\,d\theta - \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\,d\theta\right]$$

$$I_n(2n+1) = 2n\,I_{n-1}$$

$$\therefore I_n = \frac{2n}{(2n+1)} I_{n-1}.$$

Question 16 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Writes I_n in terms of I_0 and evaluates I_0, or equivalent merit	2
• Calculates I_0 OR • Writes I_n in terms of I_{n-2} OR • Equivalent merit	1

Sample answer:

Applying the reduction formula,

$$I_n = \frac{2n}{2n+1} \times I_{n-1}$$

$$= \frac{2n(2n-2)}{(2n+1)(2n-1)} \times I_{n-2}$$

$$= \frac{2n(2n-2)(2n-4)\ldots\ldots}{(2n+1)(2n-1)(2n-3)\ldots\ldots} \times \frac{2}{3} I_0$$

$$= \frac{[2n(2n-2)\ldots 2][2n(2n-2)\ldots 2] \times I_0}{(2n+1)(2n)(2n-1)(2n-2)\ldots\ldots 3 \times 2}$$

$$= \frac{2^{2n}(n!)^2 \times I_0}{(2n+1)!}$$

Now $I_0 = \int_0^{\frac{\pi}{2}} \sin(2\theta)\,d\theta$

$$= -\frac{1}{2}\cos(2\theta)\Big|_0^{\frac{\pi}{2}}$$

$$= 1$$

$$\therefore\ I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

Question 16 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Obtains an integral of $\sin^{2n+1} 2\theta$, or equivalent merit	2
• Attempts the substitution $x = \sin^2\theta$, or equivalent merit	1

Sample answer:

$$J_n = \int_0^1 x^n (1-x)^n \, dx$$

Put $x = \sin^2\theta$

$$\frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$\therefore \; J_n = \int_0^{\frac{\pi}{2}} 2\sin^{2n+1}(\theta)\cos^{2n+1}(\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n}}\left(2\sin\theta\cos\theta\right)^{2n+1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n}}\sin^{2n+1}(2\theta)\, d\theta$$

$$= \frac{1}{2^{2n}} I_n$$

$$= \frac{(n!)^2}{(2n+1)!}$$

Question 16 (b) (iv)

Criteria	Marks
• Provides correct solution	2
• Finds the maximum value of $x(1-x)$, for $0 \le x \le 1$, or equivalent merit	1

Sample answer:

The quadratic expression $x(1-x)$ has maximum value of $\frac{1}{4}$ when $x = \frac{1}{2}$.

$$\therefore\ J_n = \int_0^1 x^n (1-x)^n \, dx \le \int_0^1 \left(\frac{1}{4}\right)^n dx$$

$$= \frac{1}{2^{2n}}$$

$$\therefore\ \frac{(n!)^2}{(2n+1)!} \le \frac{1}{2^{2n}}$$

$$\Rightarrow \left(2^n n!\right)^2 \le \left(2n+1\right)!$$

2020 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	MEX-V1 Further Work with Vectors	MEX12–3
2	1	MEX-N2 Using Complex Numbers	MEX12–4
3	1	MEX-V1 Further Work with Vectors	MEX12–3
4	1	MEX-N1 Introduction to Complex Numbers	MEX12–4
5	1	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
6	1	MEX-C1 Further Integration	MEX12–5
7	1	MEX-P1 The Nature of Proof	MEX12–2
8	1	MEX-P1 The Nature of Proof	MEX12–2
9	1	MEX-N1 Introduction to Complex Numbers	MEX12–4
10	1	MEX-C1 Further Integration	MEX12–8

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	1	MEX-N1 Introduction to Complex Numbers	MEX12–4
11 (a) (ii)	2	MEX-N1 Introduction to Complex Numbers	MEX12–4
11 (b)	3	MEX-C1 Further Integration	MEX12–5
11 (c)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
11 (d)	3	MEX-V1 Further Work with Vectors	MEX12–3
11 (e)	4	MEX-N2 Using Complex Numbers	MEX12–4
12 (a) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (a) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (a) (iii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (b) (i)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (b) (ii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (b) (iii)	2	MEX-M1 Applications of Calculus to Mechanics 4	MEX12–7
13 (a)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
13 (b)	3	MEX-V1 Further Work with Vectors	MEX12–3
13 (c) (i)	2	MEX-P1 The Nature of Proof	MEX12–2

Question	Marks	Content	Syllabus outcomes
13 (c) (ii)	1	MEX-P1 The Nature of Proof	MEX12–2
13 (d) (i)	1	MEX-N1 Introduction To Complex Numbers	MEX12–4
13 (d) (ii)	3	MEX-N1 Introduction To Complex Numbers	MEX12–4
13 (d) (iii)	2	MEX-N1 Introduction To Complex Numbers	MEX12–5
14 (a) (i)	2	MEX-N2 Using Complex Numbers	MEX12–4
14 (a) (ii)	3	MEX-N2 Using Complex Numbers	MEX12–4
14 (b)	4	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
14 (c)	4	MEX-P2 Further Proof by Mathematical Induction	MEX12–2, MEX12–8
14 (d)	3	MEX-P1 The Nature of Proof	MEX12–2
15 (a) (i)	2	MEX-P1 The Nature of Proof	MEX12–2
15 (a) (ii)	1	MEX-P1 The Nature of Proof	MEX12–2
15 (a) (iii)	3	MEX-P1 The Nature of Proof	MEX12–2
15 (b) (i)	2	MEX-V1 Further Work with Vectors	MEX12–3
15 (b) (ii)	1	MEX-V1 Further Work with Vectors	MEX12–3
15 (b) (iii)	3	MEX-V1 Further Work with Vectors	MEX12–3
15 (b) (iv)	1	MEX-V1 Further Work with Vectors	MEX12–3
16 (a) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
16 (a) (ii)	3	MEX-C1 Further Integration	MEX12–5
16 (b) (i)	3	MEX-C1 Further Integration	MEX12–5
16 (b) (ii)	3	MEX-C1 Further Integration	MEX12–5
16 (b) (iii)	2	MEX-C1 Further Integration	MEX12–5
16 (b) (iv)	2	MEX-P1 The Nature of Proof	MEX12–2

Mathematics Extension 2

HSC Marking Feedback 2020

Question 11

Part (a) (i)

Students should:

- convert complex numbers to modulus-argument form.

In better responses, students were able to:

- calculate the modulus of a complex number.

Areas for students to improve include:

- finding the modulus of a complex number.

Part (a) (ii)

Students should:

- carry out complex number arithmetic involving a conjugate.

In better responses, students were able to:

- find the conjugate of a complex number
- perform complex number multiplication.

Areas for students to improve include:

- being familiar with finding the conjugate of a complex number
- correctly multiplying two complex numbers.

Part (b)

Students should:

- know the techniques needed to apply integration by parts to a function involving $\ln x$.

In better responses, students were able to:

- choose the correct functions for integration by parts
- apply integration by parts and substitute the limits correctly.

Areas for students to improve include:

- using the correct choice for $u, \frac{du}{dx}, v$ and $\frac{dv}{dx}$ in integration by parts questions
- taking care when applying the integration by parts formula

- showing the substitution of limits clearly.

Part (c)

Students should:

- use $v\frac{dv}{dx}$ to find an expression for x and evaluate the constant using the given conditions.

In better responses, students were able to:

- convert acceleration to $v\frac{dv}{dx}$
- manipulate and simplify $v\frac{dv}{x} = v^2 + v$ to gain $\frac{dx}{dv} = \frac{1}{v+1}$
- integrate and gain an expression in $\ln(v+1)$
- interpret the initial conditions to evaluate the constant.

Areas for students to improve include:

- using the most appropriate expression for acceleration
- simplifying an algebraic expression before integration
- calculating the constant of integration using the given information.

Part (d)

Students should:

- manipulate vectors using vector arithmetic.

In better responses, students were able to:

- add and subtract vectors
- know the condition for two vectors to be perpendicular
- multiply two vectors involving pronumerals then solve the equation equal to zero.

Areas for students to improve include:

- taking care with basic number and algebra skills when using vector arithmetic, in particular, expanding brackets involving negatives
- reading questions carefully.

Part (e)

Students should:

- solve a complex quadratic equation by finding the square root of a complex number.

In better responses, students were able to:

- find the discriminant of quadratic equation
- find the square root of a complex number showing clear working of how it was obtained

  find the value of z in $a + ib$ form.

Areas for students to improve include

- knowing the most efficient technique to solve a quadratic equation
- knowing how to algebraically find the square root of a complex number

- using brackets correctly to avoid careless errors in obtaining z
- taking more care with basic arithmetic and algebraic calculations.

Question 12

Part (a) (i)

Students should:

- draw a force diagram and add their own annotations to their copy, for example, addition force components
- show working that actually gets the given result
- clearly show how they arrived at their calculation.

In better responses, students were able to:

- write an equation in which the sum of the vertical forces is equal to zero
- ensure that their calculation was the same as the given result
- resolve the vertical and horizontal forces separately.

Areas for students to improve include:

- not getting confused with the $50g$ (thinking 50 grams) when g was given in the question to be 10 m/s^2
- writing down relevant calculations.

Part (a) (ii)

Students should:

- clearly show how they arrived at their calculation
- understand the difference between vertical and horizontal forces and resolving a force.

In better responses, students were able to:

- write an equation in which the sum of the horizontal forces is equal to F_{net}.

Areas for students to improve include:

- writing with clarity.

Part (a) (iii)

Students should:

- know that $F_{net} = ma$ and that the mass was given and the acceleration was from part (ii)
- recognise that the motion is only in the horizontal direction.

In better responses, students were able to:

- write an integration statement
- understand that when $t = 3$, this implies to use $\frac{dv}{dt}$ for acceleration
- find the acceleration and integrate to find v.

Areas for students to improve include:

- understanding that when the box was moving there was no vertical velocity

- understanding there is no diagonal velocity
- knowing when to use $\ddot{x} = \frac{dv}{dt}$ and/or $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
- knowing $F_{net} = ma$ and so $a = \frac{F}{m}$
- knowing when to include resistive motion.

Part (b) (i)

Students should:

- derive the equations from the acceleration vector or using components
- write vectors clearly if using this technique
- realise that for three marks, significant working must be shown.

In better responses, students were able to:

- justify all constants with reference to $t = 0$
- use a diagram to assist them find initial velocity components
- perform vector calculations so that both components were solved simultaneously
- use definite integrals as a means of defining constants.

Areas for students to improve include:

- using vector notation properly, if they choose to go that way
- improving vector notation and understanding, that if used correctly, was more succinct and clearer
- refraining from simply quoting $\dot{x} = u\ \cos\theta$ with no justification or making errors in notation such as $t = 0, \dot{x} = 0, \therefore \dot{x} = u\cos\theta$
- refraining from the use of rote learnt responses and leaving out critical details
- finding velocity and the various constants rather than writing acceleration and displacement components/vectors which were given in the question
- knowing what is important to include in a 'show that' question in order to score the allocated marks.

Part (b) (ii)

Students should:

- write clearly – often u and x, g and 5 looked the same
- keep their working logical, without the need for multiple arrows to indicate where to find working.

In better responses, students were able to:

- substitute t into the equation for y and simplify the resulting expression with clarity and precision
- factor out the common factor, $-\frac{gx^2}{2u^2}$ very clearly, without losing squares on the u or x.

Areas for students to improve include:

- taking care with details, ensuring that squares do not go missing, negative signs appear as required, and pronumerals are currently included or omitted
- improving handwriting for ease of reading

- showing sufficient steps to justify progression
- making changes to working clear.

Part (b) (iii)

Students should:

- use clear mathematical arguments to justify their answer
- find the discriminant ($\Delta > 0$) when looking for distinct solutions.

In better responses, students were able to:

- use the simplest quadratic in $\tan\theta$ to find the discriminant
- simplify the discriminant into a form which made it easier to apply the given condition $u^2 > gR$
- construct a logically ordered proof that showed 'If $u^2 > gR$ then $\Delta > 0$'.

Areas for students to improve include:

- taking care to be precise with their notation. It was common to see $\Delta \geq 0$ or Δ is real
- reducing algebraic errors
- realising that substituting $u^2 > gR$ into the quadratic does not result in the quadratic expression being positive
- writing larger to avoid error in powers and fractions
- reading the clues in the question: this part required only the discriminant and so time and effort spent on simplifying the quadratic created opportunities for more errors
- keeping powers in fractions legible resulting in less omissions or unintended sign changes.

Question 13

Part (a)

Students should:

- determine the equation for the displacement of a particle undergoing simple harmonic motion.

In better responses, students were able to:

- use the equation found on the Reference Sheet and replace each of the variables to suit this question
- interpret the information in the question correctly
- use a diagram to assist in finding all the missing parts to the equation.

Areas for students to improve include:

- knowing what is on the Reference Sheet
- knowing that simple harmonic motion questions can be asked in Mathematics Extension 2
- knowing the difference between when to use the $x = a\cos nt + x_0$ or $x = a\cos(nt + \alpha) + x_0$ forms and when to use the sine equations, what the phase change means, how to find the amplitude, and how the period gives the constant, n.

Part (b)

Students should:

- understand and use the vector equation $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\, \underset{\sim}{b}$
- find the point of intersection of two lines.

In better responses, students were able to:

- equate the components in order to then find the point of intersection.

Areas for students to improve include:

- remembering to find the point of intersection once the variables are found
- checking the solution to make sure the same point of intersection can be found using either one of the variables.

Part (c) (i)

Students should:

- be able to prove the arithmetic mean is greater than the geometric mean.

In better responses, students were able to:

- use the diagram and show the connection between the diagram and the inequality
- use another method to prove the result.

Areas for students to improve include:

- learning a standard method to prove this result.

Part (c) (ii)

Students should:

- use the result in part (i) to prove an additional result through substitution or use another method.

In better responses, students were able to:

- substitute into the result in part (i) to prove the additional inequality
- use another method to prove the result.

Areas for students to improve include:

- understanding a variety of methods that can be used to prove inequalities. These could be substitution into a known result, using $LHS - RHS$ approach and proving positive, using proof by contradiction, or starting with a true positive inequality and building an argument from there.

Part (d) (i)

Students should:

- be able to prove a known relationship in complex numbers.

In better responses, students were able to:

- show the relationship between the complex number and its conjugate.

Areas for students to improve include:

- showing the result by writing the complex number in modulus-argument form.

Part (d) (ii)

Students should:

- use binomial expansion and the result in part (i) to show the desired relationship.

In better responses, students were able to:

- expand carefully and then use the result in the part (i) to show the desired result.

Areas for students to improve include:

- being careful with the use of the result from part (i) to ensure that the final answer was correct
- being careful not to make simple errors such as stating that $2^4 = 8$.

Part (d) (iii)

Students should:

- integrate correctly.

In better responses, students were able to:

- use the result in part (ii) and then integrate correctly.

Areas for students to improve include:

- making use of formulae given in examinations, such as the Reference Sheet results
- being aware that this question could have been attempted even if the previous part was not attempted. Students should look out for the types of questions where a previous result can be used.

Question 14

Part (a) (i)

Students should:

- be able to interpret the expression $z_2 = e^{\frac{i\pi}{3}} z_1$ geometrically and link it to relevant properties of an equilateral triangles.

In better responses, students were able to:

- clearly state that z_2 is the rotation of z_1 by $\frac{\pi}{3}$ with no change of length, so that the included angle is $\frac{\pi}{3}$ and the moduli are equal
- state that the triangle is isosceles with an apex angle of $\frac{\pi}{3}$ so the triangle is equilateral.

Areas for students to improve include:

- understanding the geometrical significance of a complex number in polar form without converting to other form
- giving a clear explanation of why moduli are equal and angle is $\frac{\pi}{3}$
- practising questions where they need to explain using a mixture of words, diagrams and mathematical language
- knowing the most appropriate form of the complex number to use in particular situations
- knowing the properties of an equilateral triangle.

Part (a) (ii)

Students should:

- be able to prove a relationship involving z_1 using result from part (i).

In better responses, students were able to:

- cube both sides of $z_2 = e^{\frac{i\pi}{3}} z_1$, rearrange and factorise to gain the result
- take out a factor of z_1^2 from LHS, convert $1 + e^{\frac{i2\pi}{3}}$ into mod-arg form then convert back to $e^{\frac{i\pi}{3}}$, and rearrange to form the RHS
- take out a factor of $z_1^2 e^{\frac{i\pi}{3}}$ from the LHS then use conjugate results.

Areas for students to improve include:

- choosing which complex number form is most appropriate depending on the type of calculation to be performed
- making sure all necessary steps are included in a proof
- understanding that z_1 and z_2 are complex not real numbers so using the cosine rule is invalid.

Part (b)

Students should:

- take the reciprocal of $\frac{dv}{dt}$ and integrate, either finding c or with limits then substitute to solve.

In better responses, students were able to:

- substitute $k = 0.01$ at the start and rearrange the equation of motion into a simpler form such as $\frac{dv}{dt} = \frac{10000-v^2}{1000}$
- invert the fraction and form the partial fractions
- integrate the partial fractions including evaluating c then rearrange to make v the subject and substitute given conditions

Areas for students to improve include:

- recognising appropriate and most efficient integration techniques to use for $\int \frac{dx}{1-a^2x^2}$
- integrating expressions involving a variable in the denominator with an unknown constant coefficient

- recognising when to use partial fractions in applied questions, rather than standalone integration questions
- avoiding the use of memorised formulas for integrating partial fractions
- taking care with the primitive of $\int \frac{1}{1-kv} dv$ to not forget the $-\frac{1}{k}$ factor in front of $\ln|1-kv|$
- thinking about whether the answer they gain is reasonable.

Part (c)

Students should:

- understand the process of mathematical induction and manipulate the LHS to match RHS involving an inequality.

In better responses, students were able to:

- prove the base case for $n = 2$
- clearly show the inductive step using the correct assumption for $n = k + 1$ case
- work from the true result (LHS) to gain the result to be proved (RHS) with clear working and appropriate use of $<$ and $=$ signs.

Areas for students to improve include:

- practising induction where $n = 1$ is not the base case
- taking care with algebra manipulation involving algebraic fractions
- being able to write clear and logical reasoning for an inequality proof
- understanding when an inequality rather than an equality sign is required

Part (d)

Students should:

- be able to complete a proof by contradiction that involves irrational numbers.

In better responses, students were able to:

- let $\log_n(n+1) = \frac{p}{q}$ as the contradiction
- remove $\log_n$ to gain $n^p = (n+1)^q$
- clearly state that if n was even then n^p was even, but that $n+1$ must be odd and so $(n+1)^q$ must be odd or vice versa which is a contradiction
- take out a factor of n on one side and on the other do the same by expanding $(n+1)^q$ and factorising to leave a remainder of 1 to prove the contradiction.

Areas for students to improve include:

- taking care when choosing variables to avoid using the same pronumeral for two different variables eg $\log_n(n+1) = \frac{m}{n}$
- working with logarithms being equal to fractions
- working with logarithms with bases other than e or 10
- creating clear and succinct explanations of why a statement is a contradiction.

Question 15

Part (a) (i)

Students should:

- know the correct expansions for $(A + B)^3$
- know how to set out a basic proof.

In better responses, students were able to:

- utilise the factorisation of $k^3 + 1 = (k + 1)(k^2 - k + 1)$ requiring a simple substitution.

Areas for students to improve include:

- being careful with factorisation, particularly towards the end of an algebraic expression – quite often the result $3(9M^3 - 9M^2 + M)$ was obtained.

Part (a) (ii)

Students should:

- understand a broad range of techniques for proving results, including direct methods and the indirect methods which use contrapositive, converse, contradiction and counter-example.

In better responses, students were able to:

- write the contrapositive clearly and concisely
- understand truth tables and implications.

Areas for students to improve include:

- knowing the different methods of proof.

Part (a) (iii)

Students should:

- understand a broad range of techniques for proving results, including direct methods and the indirect methods which use contrapositive, converse, contradiction and counter-example
- learn how to use the language and write the notation of proof correctly.

In better responses, students were able to:

- use algebraic techniques effectively: $k^3 + 1 = 3M \rightarrow 3M - k^3 = 1$ or $k + 1 = k - k^3 + 3M \rightarrow k(1 - k)(1 + k) + 3M$ and used the fact that one of three consecutive integers is a multiple of three
- use the contrapositive and identify cases to test where $k + 1 \neq 3M$.

Areas for students to improve include:

- understand a broad range of techniques for proving results, including direct methods and the indirect methods which use contrapositive, converse, contradiction and counter-example
- writing concisely and knowing when to finish their solution.

Part (b) (i)

Students should:

- understand vector notation, knowing the difference between a scalar and a vector
- know how to 'show' and 'prove' without starting and finishing with the same result.

In better responses, students were able to:

- use similar triangles theory to provide reasons for the proof and show parts
- use strong vector theory associated with the point of intersection of two vectors
- use internal division theory effectively.

Areas for students to improve include:

- learning how to use the information given in a question to work towards a mathematical solution
- using proper vector notations, including the difference between $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$
- understanding vector equations of lines to solve problems.

Part (b) (ii)

Students should:

- use the vector content taught in the course
- know how to 'show' and 'prove' without starting and finishing with the same result
- understand vector notation, knowing the difference between a scalar and a vector.

In better responses, students were able to:

- write solutions very clearly and logically, with explanations where required
- make connections between the parts of the question.

Areas for students to improve include:

- realising that a question of one mark should not take a page of vector notation
- understanding that $\frac{m}{m+n}$ is a ratio which indicates the proportion of the vector, eg $\frac{m}{m+n}\overrightarrow{AB}$.

Part (b) (iii)

Students should:

- draw enlarged diagrams to help facilitate and visualise their working of the problem
- annotate diagrams with labels for all vectors that might be useful
- extract triangles that may be proven to be similar and label correctly.

In better responses, students were able to:

- use equations of lines using vector notation and solve simultaneously
- use similar triangles to establish ratios and deduce the relative lengths of other vectors.

Areas for students to improve include:

- avoiding substitution into previous answers without further explanation
- making sure that vector notations are correct.

Part (b) (iv)

Students should:

- use previous parts, or otherwise, to prove the result
- make the proof clear and the connections obvious.

In better responses, students were able to:

- provide clear reasoning for the result to be proven
- use the previous results in parts (ii) and (iii) with clear explanation and connections.

Areas for students to improve include:

- ensuring that previous results are applied accurately, especially with respect to the order of ratios
- understanding that 'or otherwise' may mean there are other easier ways, they just have not been thought of at this time
- using the answers provided can guide your direction in solving a problem.

Question 16

Part (a) (i)

Students should:

- draw free body diagrams
- realise that the resultant force is not an actual force acting on an object but the 'result' of all of the forces acting upon an object
- choose the positive direction to be the direction the object is travelling
- write an expression for the net force on the whole system
- write the forces (including T) acting on both masses separately, then add them together.

In better responses, students were able to:

- handle the resolution of forces when more than one object (mass) is involved
- write the net force on the system in one expression
- calculate the overall mass as being $6m$.

Areas for students to improve include:

- using the linking force, in this case tension, to solve problems involving more than one object
- calculating the overall mass of the system to be $6m$ and not 'averaging' the masses.

Part (a) (ii)

Students should:

- use the techniques of differential equations as seen in Mathematics Extension 1 to solve dynamics problems
- remember the constant when using indefinite integrals
- write the expression for $\frac{dv}{dt}$ from part (i).

In better responses, students were able to:

- use definite integrals and separation of variables to efficiently solve the problem with minimal working
- obtain an expression for t in terms of v, or v in terms of t and then substitute $t = \frac{3m}{k}\ln 2$ to the derived expression for v
- execute the appropriate integration technique successfully and simplify the expression using logarithm laws.

Areas for students to improve include:

- using general integration techniques
- demonstrating knowledge of standard integrals
- using the log laws correctly
- remembering to add an arbitrary constant ($+\, c$) in the primitive.

Part (b) (i)

Students should:

- recognise the need to use integration by parts to solve this recurrence relation
- know techniques to split an expression into the parts that are required
- explicitly state all four parts involved in an integration by parts
- avoid doing too many calculations in one step of a proof: whilst it may seem obvious to the writer it may not be to the reader of the proof.

In better responses, students were able to:

- explain the logic behind their proof when it may not be immediately obvious what they had done moving from one line to another in their proof
- indicate clearly the values for u, u', v and v'.

Areas for students to improve include:

- focusing on the setting out of solutions using integration by parts
- using the correct choices for u and $\frac{dv}{dx}$ when performing integration by parts
- crossing out and rewriting a correct solution, rather than trying to change a pronumeral or symbol by overwriting the incorrect part
- writing the given integral as $\int \sin^{2n}(2\theta)\sin(2\theta)d\theta$ or $\int \sin^{2n-1}(2\theta)\sin^2(2\theta)d\theta$
- using the chain rule to differentiate $\sin^{2n}(2\theta)$ correctly.

Part (b) (ii)

Students should:

- telescope their reduction formula including at least three terms at the beginning of the sequence, the second last term and find the last iterant of the formula I_0
- use the recurrence relation to write a full expansion of I_n

In better responses, students were able to:

- find I_0 correctly
- show clearly the final step to get the denominator $(2n + 1)$.

Areas for students to improve include:

- understanding that deduction and induction are two different types of proof
- explaining each step
- calculating the value for I_0.

Part (b) (iii)

Students should:

- look at the previous parts of the question and attempt to find a way of using them, particularly when the question instructs them to do so, eg compare the expression to be shown with expressions previously found and look for what is different
- use the substitution $x = \sin^2 \theta$ or $x = \cos^2 \theta$.

In better responses, students were able to:

- notice the connection between this question and previous parts of the question
- recognise the need for an appropriate substitution and take out a factor of $\frac{1}{2^{2n}}$.

Areas for students to improve include:

- clearly showing any substitutions being used, including any limits of integration
- looking for a suitable trigonometric substitution.

Part (b) (iv)

Students should:

- understand that over a specified domain the maximum value of a function is different to the maximum value of the area under the curve of the function
- justify the statements made in a proof
- realise that there are other forms of proof than 'by induction'
- choose one of the many possible arguments to prove the result
- prove or justify their steps, not just state them.

In better responses, students were able to:

- explain why $I_n \leq 1$, and not simply state it
- prove each important step
- clearly explain that a product of fractions and 1 is less than 1, or that powers of numbers between 0 and 1 get smaller.
- use diagrams and/or words to explain each step
- state I_0 from part (ii)

Areas for students to improve include:

- solving inequalities using graphs
- considering the values of the integrals, not just the function.