Student Number

PEM

2024 **TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using blue or black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- For questions in Section II show relevant mathematical reasoning and/or calculations

Total Marks – 70

Section I Questions 1 – 10 **10 marks**
Allow about 15 minutes for this section

Section II Questions 11 – 14 **60 marks**
Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Evaluate the definite integral $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-4x^2}}\, dx$.

A. $\frac{\pi}{2}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{6}$

2 Which of the following vectors are perpendicular?

A. $\underset{\sim}{u} = 3\underset{\sim}{i} - 7\underset{\sim}{j}$, $\underset{\sim}{v} = \frac{7}{3}\underset{\sim}{i} - \underset{\sim}{j}$

B. $\underset{\sim}{u} = \frac{2}{5}\underset{\sim}{i} - 2\underset{\sim}{j}$, $\underset{\sim}{v} = 5\underset{\sim}{i} + 3\underset{\sim}{j}$

C. $\underset{\sim}{u} = \frac{9}{2}\underset{\sim}{i} + \frac{7}{2}\underset{\sim}{j}$, $\underset{\sim}{v} = -\frac{2}{9}\underset{\sim}{i} - \frac{1}{7}\underset{\sim}{j}$

D. $\underset{\sim}{u} = 14\underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{v} = \frac{1}{2}\underset{\sim}{i} - 7\underset{\sim}{j}$

3 The success rate of a particular Bernoulli trial is 0.29. What is the variance of this trial?

A. 0.2059

B. 0.2149

C. 0.1927

D. 0.3299

4 The acute angle between the vectors $\underset{\sim}{u} = \begin{pmatrix} \sqrt{3} \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 9 \\ \sqrt{17} \end{pmatrix}$ is (to the nearest degree):

A. 24°

B. 51°

C. 46°

D. 48°

5 If $3\sin x + \sqrt{3}\cos x$ was expressed in the form $A\sin(x + \alpha)$, then the values of A and α would be:

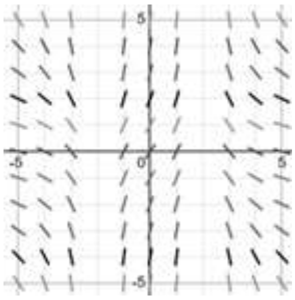
A. $A = 2, \alpha = \frac{\pi}{6}$

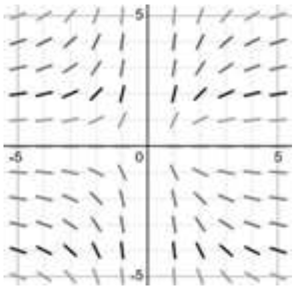
B. $A = 2, \alpha = \frac{\pi}{3}$

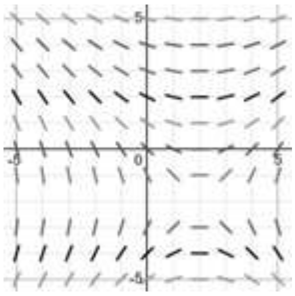
C. $A = 2\sqrt{3}, \alpha = \frac{\pi}{6}$

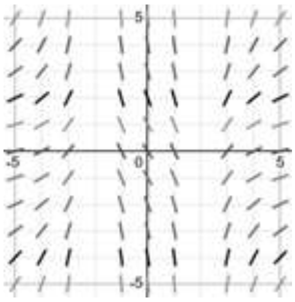
D. $A = 2\sqrt{3}, \alpha = \frac{\pi}{3}$

6 Which of the following slope fields represents $\frac{dy}{dx} = \frac{y^2+6}{x^2-4}$?

A. 

B. 

C. 

D. 

7 6 adults and 4 children need to be seated at a circular table. How many arrangements exist if the children must sit together?

A. 120960

B. 17280

C. 362880

D. 30240

8 If $f(x) = \frac{2x}{x-4}$, what is the domain of $f^{-1}(x)$?

A. $x \in [4, \infty)$

B. $x \in (-\infty, 2) \cup (2, \infty)$

C. $x \in (-\infty, \infty)$

D. $x \in (-\infty, 4) \cup (4, \infty)$

9 A die is rolled six times. Let N denote the number of times that the number 3 is shown on the uppermost face. Find, correct to four decimal places, the binomial probability $P(N < 2)$.

A. 0.7386

B. 0.7436

C. 0.7368

D. 0.7378

10 If $p(2) = 3$ and the polynomial $p(x)$ has a remainder of 4 when divided by $x + 1$, find the remainder when $p(x)$ is divided by $x^2 - x - 2$.

A. $-\frac{1}{3}x + \frac{11}{3}$

B. $\frac{1}{3}x + \frac{11}{3}$

C. $-\frac{2}{3}x + \frac{13}{3}$

D. $-\frac{1}{3}x + \frac{10}{3}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Consider the following vectors $\underset{\sim}{u} = 3\,\underset{\sim}{i} + 7\,\underset{\sim}{j}$ and $\underset{\sim}{v} = 5\,\underset{\sim}{i} - 6\,\underset{\sim}{j}$. Find $\text{proj}_{\underset{\sim}{u}}\,\underset{\sim}{v}$. **1**

(b) Solve the following inequality: $|2x - 7| + |x + 3| \geq 10$ **2**

(c) Evaluate $\cos\left[2\sin^{-1}\left(\frac{3}{8}\right)\right]$. **2**

(d) Integrate $\int_0^{\frac{1}{\sqrt{3}}} \frac{\sin(\tan^{-1} x)}{1+x^2}\,dx$ using the substitution $u = \tan^{-1} x$. **2**

(e) Find the Cartesian equation of the curves with the parametric equations given. **2**

$$\begin{cases} x = \dfrac{t}{1+t^2} \\ y = \dfrac{2}{1+t^2} \end{cases}$$

Question 11 continues on page 7

(f) Given polynomial $4x^3 + 3x^2 - 2x + 5 = 0$ has roots α, β, and γ, then find:

i. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

ii. $(\alpha\beta - \beta - \alpha + 1)(\gamma - 1)$ **2**

iii. $\alpha^2\beta + \alpha\beta^2 + \alpha\gamma^2 + \alpha^2\gamma + \beta^2\gamma + \beta\gamma^2$ **2**

Question 12 (14 marks) Use the Question 12 Writing Booklet

(a) A man stands on the edge of a vertical cliff 40 m above the ground and throws a ball horizontally with an initial velocity of u ms^{-1}. Assume that the position of the ball at time t is represented by the following parametric equations:

$$x = 30t$$

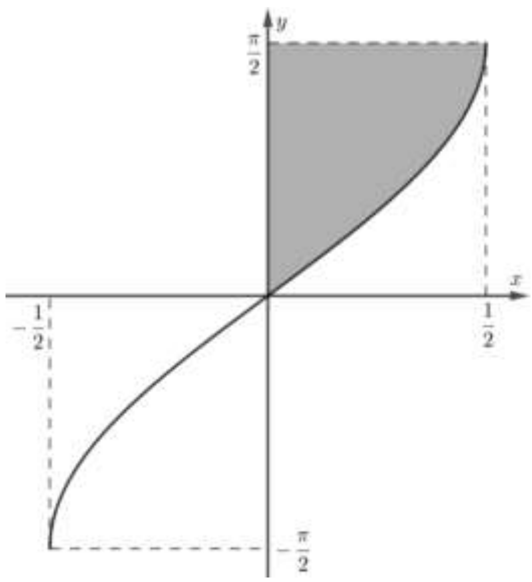
$$y = -\frac{1}{2}gt^2 + 40$$

where $g = 10$ ms^{-2}

i. Final the initial velocity. **1**

ii. Find the time it takes for the ball to hit the ground. **2**

(b) The diagram below shows the graph of $y = \sin^{-1} 2x$. The shaded area is that bounded by the curve, the y axis and the line $y = \frac{\pi}{2}$.



i. Find this shaded area **2**

ii. If this area is rotated about the y axis, find the volume of the solid thus formed. **3**

Question 12 continues on page 9

(c) The amount Q, measured in milligrams, of a substance present in a chemical reaction at time t minutes is given by the differential equation $\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 9Q = 0$.

i. Show that $Q = (A + Bt)e^{-3t}$ satisfies the differential equation. **2**

ii. Find A and B if Q is 2 initially and Q is $102e^{-15}$ after 5 min. **2**

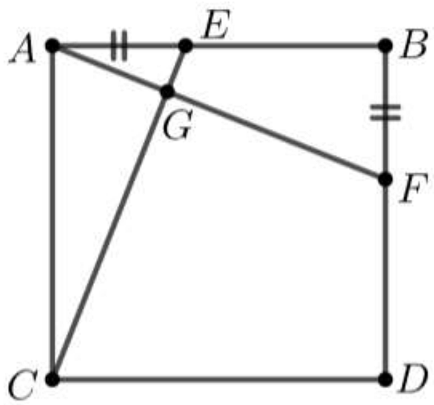
iii. Find the maximum value of Q and the time at which it occurs. **2**

End of Question 12

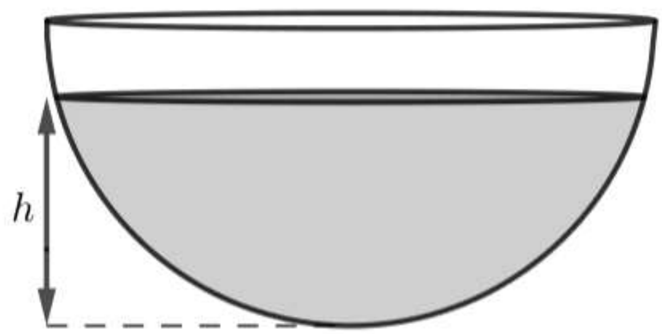
Question 13 (14 marks) Use the Question 13 Writing Booklet

(a) Given that $f(x) = \frac{2}{x^2+1}$. Let $x = N$ where $N < 0$. Find the value of $f^{-1}(f(N))$. **2**

(b) If $ABDC$ is a square and $AE = BF$, by using vector method prove that $\angle CGF = 90°$. **3**



(c) A hollow sphere with a diameter of 20 m is cut in half and its top removed. Water is poured into the remaining half of the sphere at a constant rate of 20 m^3/minute. After 1 minute, the water level is h cm above the base of the semi sphere.



i. Find an expression for the volume of the water in terms of h. **3**
Hint: consider the volume of rotation.

ii. Find the rate of change of the water level when the water is 6 m above the base. **1**

Question 13 continues on page 11

(d) By considering the expansion of $(1+x)^n$ in ascending power of x, where n is a positive integer. **3**

Find a simplified expression for $\binom{n}{1}+\frac{1}{2}\binom{n}{3}+\frac{1}{3}\binom{n}{5}+\cdots\frac{2}{n+1}\binom{n}{n}$, given that n is odd.

(e) How many arrangements of the word AABBBCDDD contain the subword BDA? **2**

End of Question 13

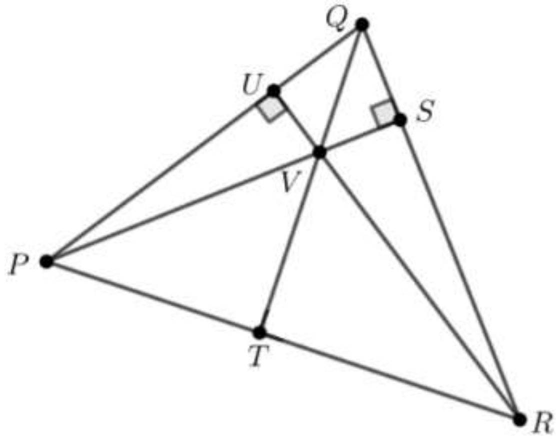
Question 14 (17 marks) Use the Question 14 Writing Booklet

(a) A company sells AI-generated "paintings". These paintings are then reviewed and graded by a quality control team. Each painting is classified as either "regular" or "exceptional". The company aims to ensure that the probability of an exceptional painting being produced each day is at least 98%. **3**

The probability that a painting is of exceptional quality is 10%. Calculate the minimum number of paintings that must be generated each day to achieve the company's aim.

(b) Prove by Mathematical Induction that $9(9^k - 1) - 8k$ is divisible by 64 for $k \in \mathbb{Z}^+$. **3**

(c) Given that $\vec{p}$, $\vec{q}$, and $\vec{r}$ are the position vectors of points P, Q, and R respectively. By using vector method, show that $QT \perp PR$. **3**



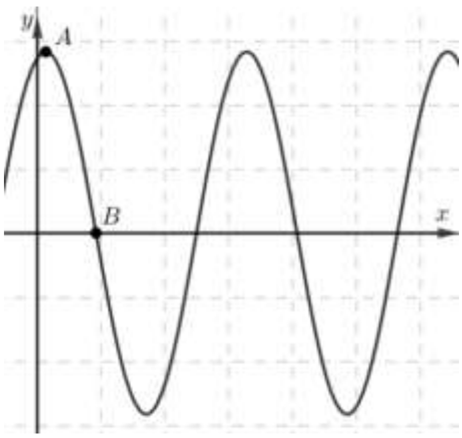
Question 14 continues on page 12

(d) A tiny pond is initially unpolluted and holds 450 litres of water. There is an input stream with a concentration of 5 g/L of pollutant flowing into the pond at a rate of 30 L/day. Moreover, there is an output stream with a volumetric flow rate of 30 L/day. Let m be the mass (in grams) of pollutant in the pond and t be the number of days after the input stream first begins to flow into the pond.

i. Construct a differential equation to model the mass of pollutant after t days. **1**

ii. What is the concentration of the pollutant in the pond after 10 days? **3**

(e) The graph of $y = \sqrt{3}(\cos 2x + \sin 2x) - (\sin 2x - \cos 2x)$ is shown.



i. Find the coordinates of A and B as labelled on the graph if A is the maximum point on the graph and B is where the graph intersects the x-axis without using calculus. **2**

ii. Hence or otherwise, solve $\sqrt{3}(\cos 2x + \sin 2x) - \sin 2x + \cos 2x = 2\sqrt{2}$ for $0 \leq x \leq 2\pi$ **2**

End of Paper

SUGGESTED SOLUTIONS PEM 2024 Mathematics Extension 1 Trial HSC Examination

Section 1

10 marks

Questions 1 – 10 (1 mark each)

Question 1 (1 mark)

Outcomes assessed: ME12-1

Targeted Performance Band: E2

Solution	Answer	Mark
$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-4x^2}}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-4\left(\frac{1}{4}\sin^2 u\right)}} \cdot \frac{1}{2}\cos u \ du$ $= \int_0^{\frac{\pi}{2}} \frac{1}{\cos u} \cdot \frac{1}{2}\cos u \ du$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} du$ $= \frac{\pi}{4}$	**C**	**1**

Question 2 (1 mark)

Outcomes Assessed: ME12-2

Targeted Performance Band: E2-E3

Solution	Answer	Mark
The perpendicular vectors have a zero dot product. The option D is correct because: $\underset{\sim}{u} \cdot \underset{\sim}{v} = \left(14\,\underset{\sim}{i} + \underset{\sim}{j}\right) \cdot \left(\frac{1}{2}\underset{\sim}{i} - 7\,\underset{\sim}{j}\right)$ $= \left(14 \cdot \frac{1}{2}\right) + (1 \cdot -7)$ $= 7 - 7 = 0$	**D**	**1**

Question 3 (1 mark)

Outcomes assessed: ME12-5

Targeted Performance Band: E2-E3

Solution	Answer	Mark
$p(1-p) = 0.29(1-0.29)$ $= 0.2059$	**A**	**1**

Question 4 (1 mark)

Outcomes Assessed: ME12-2

Targeted Performance Band: E3-E4

Solution	Answer	Mark
$\cos^{-1}\left(\frac{u \cdot v}{\lvert u\rvert \lvert v\rvert}\right) = \cos^{-1}\left(\frac{9\sqrt{3}+2\sqrt{17}}{\sqrt{7}\cdot\sqrt{98}}\right) \approx 24°$	**A**	**1**

Question 5 (1 mark)

Outcomes assessed: ME12-3

Targeted Performance Band: E3-E4

Solution	Answer	Mark
$A\sin(x+\alpha) = A\cos\alpha\sin x + A\sin\alpha\cos x$ Equating, we get: $A\cos\alpha = 3 \cdots\cdots (1)$ $A\sin\alpha = \sqrt{3} \cdots\cdots (2)$ $(1)^2 + (2)^2$: $A^2 = 12 \rightarrow A = 2\sqrt{3}$ $\frac{(2)}{(1)}$: $\tan\alpha = \frac{1}{\sqrt{3}}$ $\alpha = \frac{\pi}{6}$	**C**	**1**

Question 6 (1 mark)

Outcomes Assessed: ME12-4

Targeted Performance Bands: E3

Solution	Answer	Mark
The slope field is $\frac{dy}{dx} = \frac{y^2+6}{x^2-4} = \frac{y^2+6}{(x-2)(x+2)}$. So, the slope field is not defined at $x = \pm 2$ and has vertical asymptotes at these points. Next, $x^2 - 4 > 0$ for $x < -2$ or $x > 2$, and $x^2 - 4 < 0$ for $-2 < x < 2$. Therefore, the slope field has positive gradients on $x < -2$ or $x > 2$, and has negative gradients on $-2 < x < 2$.	**D**	**1**

Question 7 (1 mark)

Outcomes Assessed: ME11-5

Targeted Performance Band: E2-E3

Solution	Answer	Mark
First, consider the four children as a bundle so that with the other 6 adults there are initially 7 entities to be arranged in a circle. The number of ways to arrange 7 entities in a circle is $(7-1)! = 6!$. Then, the number of ways to arrange the four children in that bundle (in a line) is $4!$. By the multiplication rule of counting, the number of arrangements is $6! \times 4! = 17{,}280$	**B**	**1**

Question 8 (1 mark)

Outcomes Assessed: ME11-1, ME11-4

Targeted Performance Band: E3

Solution	**Answer**	**Mark**
Note that $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x}{x-4} = \lim_{x\to\pm\infty} \frac{2}{1-\frac{4}{x}} = 2$. So, the horizontal asymptote is $y = 2$, and the range of f is $\mathbb{R}/\{2\}$. Since the domain of the inverse function f^{-1} is the range of the function f, so the solution is $(-\infty, 2) \cup (2, \infty)$	**B**	**1**

Question 9 (1 mark)

Outcomes Assessed: ME12-5

Targeted Performance Band: E2

Solution	**Answer**	**Mark**
Given that $n = 6$ and $p = \frac{1}{6}$, so $$P(N < 2) = {}^6C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^6 + {}^6C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5 = 0.7368$$	**C**	**1**

Question 10 (1 mark)

Outcomes assessed: ME11-1, ME11-2

Targeted Performance Band: E4

Solution	**Answer**	**Mark**
From the remainder theorem, we have $p(2) = 3$ and $p(-1) = 4$. Noting that $x^2 - x - 2 = (x-2)(x+1)$, write: $$p(x) = (x-2)(x+1)Q(x) + ax + b$$ $p(2) = 3$: $$2a + b = 3 \cdots\cdots (1)$$ $p(-1) = 4$: $$-a + b = 4 \cdots\cdots (2)$$ Simultaneously solve (1) and (2) to get $a = -\frac{1}{3}$ and $b = \frac{11}{3}$. So, the remainder is $-\frac{1}{3}x + \frac{11}{3}$	**A**	**1**

Question 11 (15 marks)

(a) (1 mark)

Outcomes assessed: ME12-2

Targeted Performance Band: E2

Criteria	**Marks**
• Provides correct solution	**1**

Sample answer:

Note that:

$$\underset{\sim}{v} \cdot \underset{\sim}{u} = 15 - 42 = -27$$

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = 3^2 + 7^2 = 58$$

So, the projection of v onto u is:

$$\text{proj}_{\underset{\sim}{u}}\, \underset{\sim}{v} = \left(\frac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}}\right) \underset{\sim}{u} = -\frac{27}{58}\left(3\, \underset{\sim}{i} + 7\, \underset{\sim}{j}\right)$$

(b) (2 marks)

Outcomes assessed: ME11-2

Targeted Performance Band: E2

Criteria	**Marks**
• Provides correct answer	**2**
• Obtains one of the two inequalities that bound x	**1**

Sample answer:

Consider three intervals below:

i. $x < -3$

$$-(2x - 7) - (x + 3) \geq 10 \rightarrow x \leq -2$$

The intersection of $x < -3$ and $x \leq -2$ is $x < -3$

ii. $-3 \leq x < \frac{7}{2}$

$$-(2x - 7) + x + 3 \geq 10 \rightarrow x \leq 0$$

The intersection of $-3 \leq x < \frac{7}{2}$ and $x \leq 0$ is $-3 \leq x \leq 0$

iii. $x \geq \frac{7}{2}$

$$(2x - 7) + x + 3 \geq 10 \rightarrow x \geq \frac{14}{3}$$

The intersection of $x \geq \frac{7}{2}$ and $x \geq \frac{14}{3}$ is $x \geq \frac{14}{3}$

The solution is the union of the results above, that is:

$$(x < -3) \cup (-3 \leq x \leq 0) \cup \left(x \geq \frac{14}{3}\right) \equiv (x \leq 0) \cup \left(x \geq \frac{14}{3}\right)$$

(c) (2 marks)

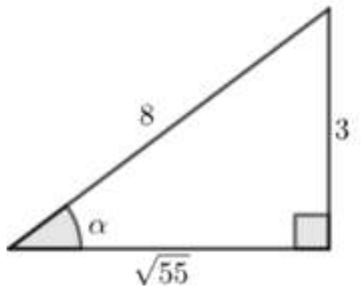
Outcomes assessed: ME11-3

Targeted Performance Bands: E2

Criteria	Marks
• Provides correct answer	**2**
• Applies the double angle formula to expand $\cos\left[2\sin^{-1}\left(\frac{3}{8}\right)\right]$ OR • Finds the correct values of $\cos\left[\sin^{-1}\left(\frac{3}{8}\right)\right]$ and $\sin\left[\sin^{-1}\left(\frac{3}{8}\right)\right]$	**1**

Sample answer:

Let $\alpha = \sin^{-1}\left(\frac{3}{8}\right)$ so that $\sin\alpha = \frac{3}{8}$ which corresponds to the following triangle:



From the triangle above, we have $\cos\alpha = \frac{\sqrt{55}}{8}$.

By using the double-angle formula of cosine, it follows:

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(\frac{\sqrt{55}}{8}\right)^2 - \left(\frac{3}{8}\right)^2 = \frac{23}{32}$$

(d) (2 marks)

Outcomes Assessed: ME12-1

Targeted Performance Band: E3

Criteria	Marks
• Provides correct solution	**2**
• Correctly expresses the integral in terms of u	**1**

Sample answer:

$$\int_0^{\frac{1}{\sqrt{3}}} \frac{\sin(\tan^{-1} x)}{1+x^2}\,dx = \int_0^{\frac{\pi}{6}} \sin u\,du$$

$$= -\cos\frac{\pi}{6} - (-\cos 0)$$

$$= 1 - \frac{\sqrt{3}}{2}$$

(e) (2 mark)

Outcomes assessed: ME11-1, ME11-2

Targeted Performance Band: E2

Criteria	Marks
• Provides correct solution	**2**
• Obtains a Cartesian equation that resembles a circle	**1**

Sample answer:

Notice that:

$$y - 1 = \frac{1-t^2}{1+t^2} \quad \text{and} \quad 2x = \frac{2t}{1+t^2}$$

Then

$$(2x)^2 + (y-1)^2 = \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2$$

$$4x^2 + (y-1)^2 = 1$$

(f)(i) (2 marks)

Outcomes assessed: ME11-2

Targeted Performance Band: E3

Criteria	Marks
• Provides correct answer	**2**
• Writes $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ as a fraction with a common denominator OR • Correctly applies a formula that relates the roots of a polynomial with its coefficients	**1**

Sample answer:

From the given cubic polynomial, we conjecture that:

$$\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{1}{2}, \alpha\beta\gamma = -\frac{5}{4}$$

Then, it follows:

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$$

$$= \frac{\left(-\frac{1}{2}\right)}{\left(-\frac{5}{4}\right)} = \frac{2}{5}$$

(f)(ii) (2 marks)

Outcomes Assessed: ME11-2

Targeted Performance Band: E3-E4

Criteria	Marks
• Provides correct answer	**2**
• Correctly expands the expression and applies the appropriate root-coefficient formulae	**1**

Sample answer:

From the given cubic polynomial, we conjecture that:

$$\alpha+\beta+\gamma=-\frac{3}{4}, \alpha\beta+\alpha\gamma+\beta\gamma=-\frac{1}{2}, \alpha\beta\gamma=-\frac{5}{4}$$

Then, it follows:

$$(\alpha\beta-\beta-\alpha+1)(\gamma-1)=\alpha\beta\gamma-(\alpha\beta+\alpha\gamma+\beta\gamma)+(\alpha+\beta+\gamma)-1$$

$$=-\frac{5}{4}-\left(-\frac{1}{2}\right)+\left(-\frac{3}{4}\right)-1=-\frac{5}{2}$$

(f)(iii) (2 marks)

Outcomes Assessed: ME11-2

Targeted Performance Band: E3-E4

Criteria	Marks
• Provides correct answer	**2**
• Performs the correct factorisation	**1**

Sample answer:

From the given cubic polynomial, we conjecture that:

$$\alpha+\beta+\gamma=-\frac{3}{4}, \alpha\beta+\alpha\gamma+\beta\gamma=-\frac{1}{2}, \alpha\beta\gamma=-\frac{5}{4}$$

Then, observe that $\alpha^2\beta+\alpha\beta^2+\alpha\gamma^2+\alpha^2\gamma+\beta^2\gamma+\beta\gamma^2$ is equivalent to:

$$(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)-3\alpha\beta\gamma=-\frac{3}{4}\left(-\frac{1}{2}\right)-3\left(-\frac{5}{4}\right)=\frac{33}{8}$$

Question 12 (14 marks)

(a)(i) (1 mark)

Outcomes assessed: ME12-2

Targeted Performance Band: E2

Criteria	Marks
• Provides correct solution	**1**

Sample answer:

From the given parametric equations, we observe that horizontal component of the initial velocity is 30 ms^{-1} while the vertical component is 0 ms^{-1}. So, the magnitude of the initial velocity is:

$$\sqrt{30^2 + 0^2} = 30 \text{ m/s}$$

(a) (ii) (2 marks)

Outcomes assessed: ME12-2

Targeted Performance Bands: E3

Criteria	Marks
• Provides correct solution	**2**
• Correctly substitutes the relevant values into the equation	**1**

Sample answer:

Substitute $y = 0$ and $g = 10$ into $y = -\frac{1}{2}gt^2 + 40$ and solve for t as follows:

$$-5t^2 + 40 = 0$$

$$t^2 = 8$$

$$\therefore t = 2\sqrt{2}$$

(b) (i) (2 marks)

Outcomes assessed: ME12-4

Targeted Performance Bands: E2

Criteria	Marks
• Provides correct solution	**2**
• Writes the integral that represents the shaded area	**1**

Sample answer:

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin y \ dy = \left[-\frac{1}{2}\cos y\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2}(0-1) = \frac{1}{2}u^2$$

(b) (ii) (3 marks)

Outcomes assessed: ME12-4

Targeted Performance Band: E3

Criteria	Marks
• Provides correct answer	**3**
• Integrates $\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin y\right)^2 dy$ and obtains a result involving a trigonometric ratio	**2**
• Finds an integral expression of the volume	**1**

Sample answer:

$$\pi\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin y\right)^2 dy = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2y)\,dy$$

$$= \frac{\pi}{8}\left[y - \frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{8}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{16}$$

(c) (i) (2 marks)

Outcomes assessed: ME12-4

Targeted Performance Band: E3

Criteria	Marks
• Provides correct solution	**2**
• Finds $\frac{d^2Q}{dt^2}$ and $\frac{dQ}{dt}$	**1**

Sample answer:

First, find the first and second derivatives of Q

$$\frac{dQ}{dt} = (-3A + B - 3Bt)e^{-3t}$$

$$\frac{d^2Q}{dt^2} = (9A - 6B + 9Bt)e^{-3t}$$

Substitute the function Q and its derivatives into the differential equation to get:

$$\begin{aligned}\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 9Q &= (9A - 6B + 9Bt)e^{-3t} + 6(-3A + B - 3Bt)e^{-3t} + 9(A + Bt)e^{-3t} \\ &= [(9A - 18A + 9A) + (-6B + 6B) + (9B - 18B + 9B)t]e^{-3t} \\ &= 0 \text{ (shown)}\end{aligned}$$

(c) (ii) (2 marks)

Outcomes assessed: ME12-4

Targeted Performance Band: E3

Criteria	**Marks**
• Provides correct solution	**2**
• Obtains the correct value for A OR • Obtains the correct value for B	**1**

Sample answer:

First condition: $Q(0) = 2$

$$[A + (B \cdot 0)]e^{-3(0)} = 2$$

$$\therefore A = 2$$

Second condition: $Q(5) = 102e^{-15}$

$$(A + 5B)e^{-15} = 102e^{-15}$$

$$2 + 5B = 102$$

$$\therefore B = 20$$

12(c) (iii) (2 marks)

Outcomes assessed: ME12-4

Targeted Performance Bands: E3

Criteria	**Marks**
• Provides correct solution	**2**
• Recognises that maximum occurs at $\frac{dQ}{dt} = 0$	**1**

Sample answer:

$$\frac{dQ}{dt} = (14 - 60t)e^{-3t}$$

$$(14 - 60t)e^{-3t} = 0$$

$$t = \frac{7}{30}$$

Proving that there is a relative maximum at the critical point above using a table of values:

t	0	$\frac{7}{30}$	1
$\frac{dQ}{dt}$	$14 > 0$	0	$-\frac{46}{e^3} < 0$
	$\nearrow$	$\rightarrow$	$\searrow$

The maximum value of Q is:

$$Q = \left(2 + 20 \cdot \frac{7}{30}\right) e^{-3\left(\frac{7}{30}\right)} = \frac{20}{3} e^{-\frac{7}{10}}$$

Question 13 (15 marks)

(a) (2 marks)

Outcomes Assessed: ME11-1, ME11-2

Targeted Performance Band: E3-E4

Criteria	**Marks**
• Provides correct solution	**2**
• Expresses $f^{-1}(f(N))$ in terms of N and makes attempts to express it in its simplest form	**1**

Sample answer:

$$f^{-1}(f(N)) = \sqrt{\frac{2}{\left(\frac{2}{N^2+1}\right)} - 1} = |N|$$

Since $N < 0$, so

$$f^{-1}(f(N)) = |N| = -N$$

(b) (3 marks)

Outcomes assessed: ME12-2

Targeted Performance Band: E3-E4

Criteria	**Marks**
• Correct proof	**3**
• Utilises angle between vectors or equivalent progress	**2**
• Utilises expansion of dot product or equivalent progress	**1**

Sample answer:

Note that:

$$\overrightarrow{CE} = \overrightarrow{AE} - \overrightarrow{AC}$$

$$\overrightarrow{AF} = \overrightarrow{BF} - \overrightarrow{BA}$$

Then

$$\overrightarrow{CE} \cdot \overrightarrow{AF} = \left(\overrightarrow{AE} - \overrightarrow{AC}\right)\left(\overrightarrow{BF} - \overrightarrow{BA}\right)$$

$$\overrightarrow{CE} \cdot \overrightarrow{AF} = \overrightarrow{AE} \cdot \overrightarrow{BF} - \overrightarrow{AE} \cdot \overrightarrow{BA} - \overrightarrow{AC} \cdot \overrightarrow{BF} + \overrightarrow{AC} \cdot \overrightarrow{BA}$$

$$\overrightarrow{CE} \cdot \overrightarrow{AF} = 0 - \overrightarrow{AE} \cdot \overrightarrow{BA} - \overrightarrow{AC} \cdot \overrightarrow{BF} + 0 \ \left(\because \overrightarrow{AE} \perp \overrightarrow{BF},\ \ \overrightarrow{AC} \perp \overrightarrow{BA}\right)$$

$$\overrightarrow{CE} \cdot \overrightarrow{AF} = -\left|\overrightarrow{AE}\right|\left|\overrightarrow{BA}\right| \cos 180° - \left|\overrightarrow{AC}\right|\left|\overrightarrow{BF}\right| \cos 0°$$

$$\overrightarrow{CE} \cdot \overrightarrow{AF} = \left|\overrightarrow{AE}\right|\left|\overrightarrow{BA}\right| - \left|\overrightarrow{AC}\right|\left|\overrightarrow{BF}\right|$$

$$\overrightarrow{CE} \cdot \overrightarrow{AF} = 0 \ \left(\because \left|\overrightarrow{AE}\right| = \left|\overrightarrow{BF}\right|, \left|\overrightarrow{BA}\right| = \left|\overrightarrow{AC}\right|\right)$$

Hence, $\overrightarrow{CE} \perp \overrightarrow{AF}$. Then, since $\overrightarrow{CG} \parallel \overrightarrow{CE}$ and $\overrightarrow{GF} \parallel \overrightarrow{AF}$, so $\overrightarrow{CG} \perp \overrightarrow{GF} \rightarrow \angle CGF = 90°$

(c) (i) (1 mark)

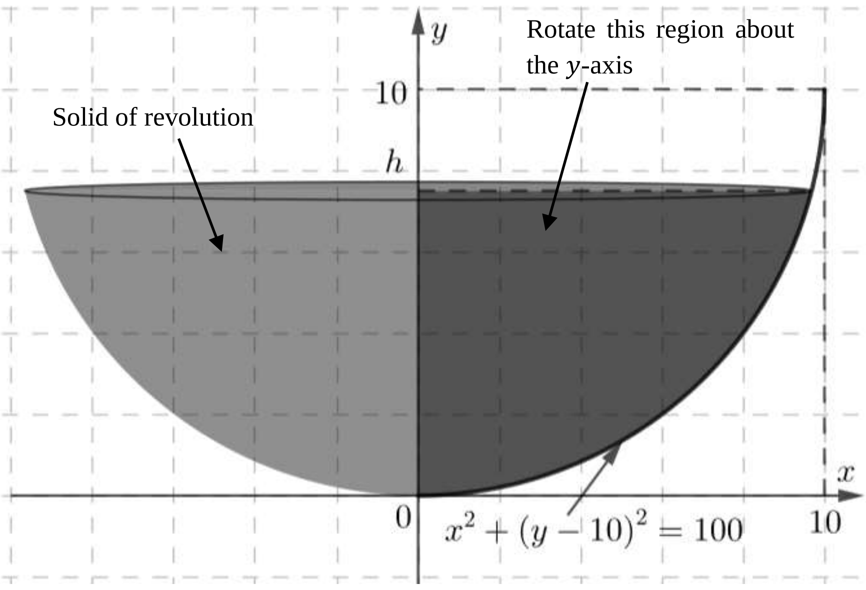
Outcomes assessed: ME11-4

Targeted Performance Bands: E4

Criteria	Marks
• Obtains the correct expression	**3**
• Correct integral or equivalent progress	**2**
• Considers volume of revolution or equivalent method	**1**

Sample answer:

Note that the required volume is the volume of revolution of part of a circle centred at $(0, 10)$ with radius 10, given by $x^2 + (y - 10)^2 = 100$, over the interval $0 \leq y \leq h$ about the y-axis.



Rearrange the equation of the circle into:

$$x^2 = 100 - (y - 10)^2$$

$$x^2 = 20y - y^2$$

Then, the required volume V of revolution is given by:

$$V = \pi \int_0^h x^2 \, dy$$

$$= \pi \int_0^h (20y - y^2) \, dy$$

$$= \pi \left[10y^2 - \frac{1}{3}y^3\right]_0^h$$

$$= \frac{1}{3}\pi h^2(30 - h)$$

(c) (ii) (3 marks)

Outcomes assessed: ME11-4

Targeted Performance Bands: E3-4

Criteria	Marks
• Provides correct solution	**1**

Sample answer:

Differentiate the expression of volume found in part (c) (i) with respect to h:

$$\frac{dV}{dh} = \pi \frac{d}{dh}\left(10h^2 - \frac{1}{3}h^3\right)$$

$$\frac{dV}{dh} = \pi(20h - h^2)$$

When $h = 6$, we have:

$$\frac{dV}{dh} = \pi(20 \cdot 6 - 6^2) = 84\pi$$

Given that $\frac{dV}{dt} = 20$ and by using the chain rule, it follows:

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$20 = 84\pi \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{5}{21\pi} m/minute$$

(d) (3 marks)

Outcomes assessed: ME11-5

Targeted Performance Bands: E4

Criteria	Marks
• Provides correct or equivalent simplified expression	**3**
• Makes 1 correct substitution or similar progress	**2**
• Integrates	**1**

Sample answer:

Use binomial expansion of $(1+x)^n$ to get:

$${}^nC_0x^0(1^n) + {}^nC_1x^1(1^{n-1}) + {}^nC_2x^2(1^{n-2}) + {}^nC_3x^3(1^{n-3}) + \cdots + {}^nC_nx^n(1^0) = (1+x)^n$$

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n = (1+x)^n$$

Then, integrate both sides with respect to x:

$$\int \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n\right] dx = \int (1+x)^n\, dx$$

$$\binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \frac{1}{4}\binom{n}{3}x^4 + \cdots + \frac{1}{n+1}\binom{n}{n}x^{n+1} = \frac{(1+x)^{n+1}}{n+1} \cdots\cdots (1)$$

Substitute $x = 1$ into (1):

$$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}}{n+1} \cdots\cdots (2)$$

Substitute $x = -1$ into (1):

$$-\binom{n}{0} + \frac{1}{2}\binom{n}{1} - \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} - \frac{1}{5}\binom{n}{4} + \frac{1}{6}\binom{n}{5} + \cdots + \frac{1}{n+1}\binom{n}{n} = 0 \cdots\cdots (3)$$

Add (2) to (3):

$$\binom{n}{1} + \frac{1}{2}\binom{n}{3} + \frac{1}{3}\binom{n}{5} + \cdots + \frac{2}{n+1}\binom{n}{n} = \frac{2^{n+1}}{n+1}$$

(e) (2 marks)

Outcomes assessed: ME11-5

Targeted Performance Band: E4

Criteria	**Marks**
• Provides correct solution	**2**
• Finds the total number of unique arrangements possible	**1**

Sample answer:

$$\frac{7!}{2! \cdot 2!} - \frac{5!}{2!} = 1200$$

Question 14 (16 marks)

(a) (3 marks)

Outcomes assessed: ME12-5

Targeted Performance Band: E3

Criteria	**Marks**
• Provides correct answer	**3**
• Uses logarithm laws to write an expression with n outside of the exponent	**2**
• Deduces that $(1-p)^n \leq 0.02$	**1**

Sample answer:

Since the probability of producing an exceptional drawing is $p = 0.1$, so the probability of not producing an exceptional drawing (complementary event) is:

$$1 - p = 1 - 0.1 = 0.9$$

Since the probability of producing an exceptional drawing from n trials is at least 98%, so the probability of producing all regular painting from n trials must be below 2%:

$$(1-p)^n \leq 1 - 0.98$$

$$(1-p)^n \leq 0.02$$

$$0.9^n \leq 0.02$$

$$\ln(0.9^n) \leq \ln 0.02$$

$$n \ln 0.9 \leq \ln 0.02$$

$$n \geq \frac{\ln 0.02}{\ln 0.9}$$

$$n \geq 37.13$$

So, at least 38 drawings have to be generated.

(b) (3 marks)

Outcomes assessed: ME12-1

Targeted Performance Bands: E3

Criteria	Marks
• Provides correct answer	**3**
• Proves that if the statement is true for $n = k$, then it must be true for $n = k + 1$	**2**
• Verifies the initial case	**1**

Sample answer:

Initial Step:

When $n = 1$, $9(9^n - 1) - 8n = 64$ is divisible by 64.

So, it is true for $n = 1$

Inductive Step:

Assume true for $n = k$:

$9(9^k - 1) - 8k = 64m, \qquad m \in \mathbb{Z} \quad (*)$

Final Step:

RTP: Prove to be true for $n = k + 1$:

(**Note:** rearrange the equation $(*)$ to get $9^{k+1} = 64m + 9 + 8k$)

$$\begin{aligned} 9(9^{k+1} - 1) - 8(k + 1) &= 9(64m + 9 + 8k - 1) - 8(k + 1) \\ &= 64(9m) + 64 + 64k \\ &= 64(9m + 1 + k) \\ &= 64M, \qquad M \in \mathbb{Z} \end{aligned}$$

So, it is true for $n = k + 1$

(c) (3 marks)

Outcomes assessed: ME12-2

Targeted Performance Band: E3-E4

Criteria	Marks
• Provides correct answer	**3**
• Attempts to show that $\overrightarrow{QT} \cdot \overrightarrow{PR} = 0$	**2**
• Expresses $\overrightarrow{QT}$ in terms of $\overrightarrow{PQ}$ and $\overrightarrow{PS}$	**1**

Sample answer:

Note that:

$$\overrightarrow{QT} = 3\overrightarrow{QV}$$

$$\overrightarrow{QT} = 3\left(-\overrightarrow{PQ} + \overrightarrow{PV}\right)$$

$$\overrightarrow{QT} = 3\left(-\overrightarrow{PQ} + \frac{2}{3}\overrightarrow{PS}\right)$$

$$\overrightarrow{QT} = -3\overrightarrow{PQ} + 2\overrightarrow{PS}$$

Then, observe that:

$$\overrightarrow{QT} \cdot \overrightarrow{PR} = \left(-3\overrightarrow{PQ} + 2\overrightarrow{PS}\right) \cdot \overrightarrow{PR}$$

$$\overrightarrow{QT} \cdot \overrightarrow{PR} = -3\overrightarrow{PQ} \cdot \overrightarrow{PR} + 2\overrightarrow{PS} \cdot \overrightarrow{PR}$$

$$\overrightarrow{QT} \cdot \overrightarrow{PR} = -3\overrightarrow{PQ} \cdot \overrightarrow{PR} + 2\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} + \overrightarrow{QR}\right)$$

$$\overrightarrow{QT} \cdot \overrightarrow{PR} = -3\overrightarrow{PQ} \cdot \overrightarrow{PR} + 2\overrightarrow{PS} \cdot \overrightarrow{PQ} + 2\overrightarrow{PS} \cdot \overrightarrow{QR}$$

$$\overrightarrow{QT} \cdot \overrightarrow{PR} = -3\overrightarrow{PQ} \cdot \overrightarrow{PR} + 2\overrightarrow{PS} \cdot \overrightarrow{PQ} + 2(0) \quad \left(\because \overrightarrow{PS} \perp \overrightarrow{QR}\right)$$

$$\overrightarrow{QT} \cdot \overrightarrow{PR} = 3\overrightarrow{PQ} \cdot \left(\frac{2}{3}\overrightarrow{PS} - \overrightarrow{PR}\right)$$

$$\overrightarrow{QT} \cdot \overrightarrow{PR} = 3\overrightarrow{PQ} \cdot \overrightarrow{RV}$$

$$\overrightarrow{QT} \cdot \overrightarrow{PR} = 3\overrightarrow{PQ} \cdot \frac{2}{3}\overrightarrow{RU}$$

$$\overrightarrow{QT} \cdot \overrightarrow{PR} = 2\overrightarrow{PQ} \cdot \overrightarrow{RU}$$

$$\overrightarrow{QT} \cdot \overrightarrow{PR} = 2(0) \quad \left(\because \overrightarrow{PQ} \perp \overrightarrow{RU}\right)$$

$$\therefore \overrightarrow{QT} \cdot \overrightarrow{PR} = 0 \rightarrow QT \perp PR$$

(d) (i) (1 mark)

Outcomes assessed: ME12-4

Targeted Performance Band: E3

Criteria	Marks
• Correctly constructs the differential equation	**1**

Sample answer:

$$\frac{dm}{dt} = (5 \cdot 30) - 30\left(\frac{m}{450}\right)$$

$$\therefore \frac{dm}{dt} = 150 - \frac{1}{15}m, m(0) = 0$$

(d) (ii) (3 marks)

Outcomes assessed: ME12-4

Targeted Performance Band: E3

Criteria	Marks
• Provides correct solution	**3**
• Correctly substitutes the initial condition	**2**
• Correctly integrates	**1**

Sample answer:

$$\frac{dm}{dt} = 150 - \frac{1}{15}m$$

$$\frac{1}{150 - \frac{m}{15}}dm = dt$$

$$\int \frac{1}{150 - \frac{m}{15}}dm = \int dt$$

$$-15 \ln\left|150 - \frac{1}{15}m\right| = t + C$$

With the initial condition $m(0) = 0$, we have:

$$C = -15 \ln 150$$

Then, the particular solution of the differential equation is:

$$-15\ln\left|150-\frac{1}{15}m\right| = t - 15\ln 150$$

$$15\ln 150 - 15\ln\left|150-\frac{1}{15}m\right| = t$$

$$15\ln\left|\frac{150}{150-\frac{m}{15}}\right| = t$$

$$\ln\left|\frac{150}{150-\frac{m}{15}}\right| = \frac{1}{15}t$$

$$\frac{150}{150-\frac{m}{15}} = e^{\frac{1}{15}t}$$

$$150-\frac{m}{15} = 150e^{-\frac{1}{15}t}$$

$$\frac{m}{15} = 150\left(1-e^{-\frac{1}{15}t}\right)$$

$$m = 2250\left(1-e^{-\frac{1}{15}t}\right)$$

Find the mass (in grams) of pollutant in the tank after $t = 10$ days:

$$m = 2250\left(1-e^{-\frac{1}{15}t}\right) = 2250\left(1-e^{-\frac{2}{3}}\right)$$

Since the volume of the pond is constant at $V = 450$ litres, the concentration of the pollutant in the pond after $t = 10$ days is:

$$\frac{m}{V} = \frac{2250\left(1-e^{-\frac{2}{3}}\right)}{450} = 5\left(1-e^{-\frac{2}{3}}\right) \approx 2.43\text{ g/L}$$

(e) (i) (2 marks)

Outcomes assessed: ME12-3

Targeted Performance Band: E4

Criteria	Marks
• Provides correct solution	**2**
• Recognises that y is in the form $R\cos(2x-\alpha)$.	**1**

Sample answer:

Re-factorise the trigonometric function as follows:

$$y = \left(\sqrt{3}+1\right)\cos 2x + \left(\sqrt{3}-1\right)\sin 2x$$

Consider the following compound angle formula:

$$R\cos(2x-\alpha) = R\cos\alpha\cos 2x + R\sin\alpha\sin 2x$$

Comparing the compound angle formula to the trigonometric function, we get:

$$R\cos\alpha = \sqrt{3}+1 \;\cdots\cdots\; (1)$$

$$R\sin\alpha = \sqrt{3}-1 \;\cdots\cdots\; (2)$$

$(1)^2 + (2)^2$:

$$R^2 = 8$$

$$R = 2\sqrt{2}$$

$\frac{(2)}{(1)}$:

$$\tan\alpha = 2-\sqrt{3}$$

$$\alpha = \frac{\pi}{12}$$

Hence, the trigonometric function is also equal to:

$$y = 2\sqrt{2}\cos\left(2x-\frac{\pi}{12}\right)$$

The maximum value is $2\sqrt{2}(1) = 2\sqrt{2}$ because the maximum of cosine function is 1.

To get the x-coordinate of the first maximum point on the right of the y-axis, solve the following:

$$2\sqrt{2}\cos\left(2x-\frac{\pi}{12}\right)=2\sqrt{2}$$

$$\cos\left(2x-\frac{\pi}{12}\right)=1$$

$$2x-\frac{\pi}{12}=0$$

$$\therefore x=\frac{\pi}{24}$$

Hence, the coordinates of A are $\left(\frac{\pi}{24},2\sqrt{2}\right)$.

To get the x-intercept, solve the following:

$$2\sqrt{2}\cos\left(2x-\frac{\pi}{12}\right)=0$$

$$\cos\left(2x-\frac{\pi}{12}\right)=0$$

$$2x-\frac{\pi}{12}=\frac{\pi}{2}$$

$$\therefore x=\frac{7\pi}{24}$$

Hence, the coordinates of B are $\left(\frac{7\pi}{24},0\right)$.

(e) (ii) (2 marks)

Outcomes assessed: ME12-3

Targeted Performance Band: E4

Criteria	Marks
• Provides correct solution	**2**
• Has one correct value of x	**1**

Sample answer:

$$2\sqrt{2}\cos\left(2x-\frac{\pi}{12}\right)=2\sqrt{2}$$

$$\cos\left(2x-\frac{\pi}{12}\right)=1$$

$$2x-\frac{\pi}{12}=0, 2\pi$$

$$\therefore x=\frac{\pi}{24},\frac{25\pi}{24}$$