Student's Name: ______________________ Teacher's Code: ______________

Saint Ignatius' College, Riverview

Mathematics Assessment Task

2024

Year 12
Mathematics (Extension One)
Task 4
Trial HSC Examination
Date: 21st August 2024

General Instructions:

- **Reading time: 10 minutes**
- **Time Allowed: 2 hours.**
- Write using a black pen.
- Calculators approved by NESA may be used.
- Attempt all questions in the booklets provided.
- Write **your name** and **your teacher's code** in the positions indicated.
- Marks may not be awarded for missing or carelessly arranged work.

Teachers:

- Mr R Maxwell **REM**
- Mr D Reidy **DPR**
- Mr N Mushan **NHM**
- Mr J Newey **JPN**

Topics Examined:

Section A Multiple Choice	**10 Marks**
Section B Short Answer	
Question 11	**15 Marks**
Question 12	**15 Marks**
Question 13	**15 Marks**
Question 14	**15 Marks**
Total	**70 Marks**

SECTION I: Multiple Choice Questions 1 - 10:

1. How many integer solutions are there for the inequality $\frac{1}{|x-2|} > \frac{1}{4}$?

A. 7

B. 6

C. 5

D. 4

2. Consider the equation $3x^3 + 2x^2 - x + 5 = 0$.

Therefore $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ is equal to (*i.e* sum of reciprocal of the roots.

A. $\frac{1}{5}$

B. $\frac{2}{5}$

C. $\frac{5}{2}$

D. -5

3. What is the maximum value of $15sin\theta - 8cos\theta + 2$?

A. 9

B. 17

C. 19

D. 23

4. The cartisian equation of the curve represented by the parametric equations:

$x = 4 - t$ and $y = 3t^3$ is.

A. $y = 64 - 48x + 12x^2 - x^3$

B. $y = 64 + 48x + 12x^2 + x^3$

C. $y = 192 + 144x + 36x^2 - 3x^3$

D. $y = 192 - 144x + 36x^2 - 3x^3$

5. A particle is initially at $x = -1$. Its motion has a velocity of $v = \dfrac{1}{t+e}$.

Which of the folling will be true when the particle is at the origin?

A. $t = e^{-1}$, $v = 1$

B. $t = 1$, $v = e^{-1}$

C. $t = e^{-2}$, $v = -1$

D. $t = e^{2} - e$, $v = e^{-2}$

6. What is the domain and range of $y = 4cos^{-1}\dfrac{3x}{2}$?

A. Domain: $-\dfrac{2}{3} \leq x \leq \dfrac{2}{3}$
Range: $-4\pi \leq y \leq 4\pi$.

B. Domain: $-\dfrac{3}{2} \leq x \leq \dfrac{3}{2}$
Range: $-4\pi \leq y \leq 4\pi$.

C. Domain: $-\dfrac{2}{3} \leq x \leq \dfrac{2}{3}$
Range: $0 \leq y \leq 4\pi$.

D. Domain: $-\dfrac{2}{3} \leq x \leq \dfrac{2}{3}$
Range: $0 \leq y \leq 4$.

7. The angle between two vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is approximately.

A. $cos^{-1}(.5)$

B. $cos^{-1}(-.5)$

C. $cos^{-1}(1)$

D. $cos^{-1}(-1)$

8. X, Y, and Z are collinear points, with position vectors $\underset{\sim}{\mathbf{x}}$, $\underset{\sim}{\mathbf{y}}$ and $\underset{\sim}{\mathbf{z}}$ respectively. Y lies between X and Z.

Given that $|\overrightarrow{YZ}| = \frac{1}{2}|\overrightarrow{XY}|$, which of the following expressions is equal to $\underset{\sim}{\mathbf{z}}$?

A. $\frac{1}{2}\underset{\sim}{\mathbf{x}} - \frac{3}{2}\underset{\sim}{\mathbf{y}}$

B. $\frac{3}{2}\underset{\sim}{\mathbf{x}} - \frac{1}{2}\underset{\sim}{\mathbf{y}}$

C. $\frac{3}{2}\underset{\sim}{\mathbf{y}} - \frac{3}{2}\underset{\sim}{\mathbf{x}}$

D. $\frac{3}{2}\underset{\sim}{\mathbf{y}} - \frac{1}{2}\underset{\sim}{\mathbf{x}}$

9. Which of the following is the derivatve of $y = x^2 sin^{-1}(2x)$?

A. $2xsin^{-1}(2x) + \frac{2x^2}{\sqrt{1-4x^2}}$

B. $2xsin^{-1}(2x) + \frac{x^2}{\sqrt{1-4x^2}}$

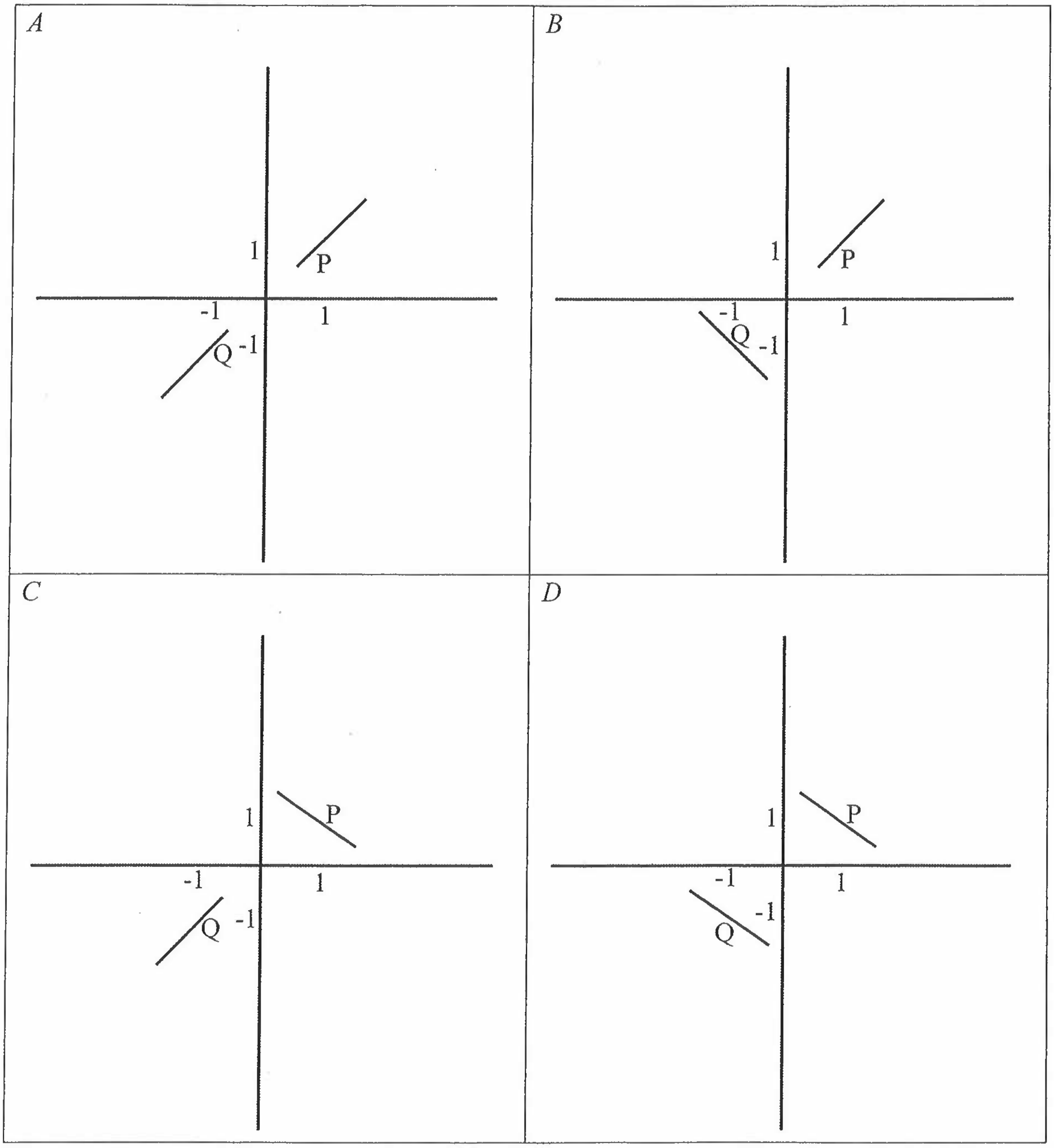
C. $2xsin^{-1}(2x) + \frac{2x^2}{\sqrt{1-2x^2}}$

D. $2xsin^{-1}(2x) + \frac{x^2}{\sqrt{1-2x^2}}$

10. A directional field is to be drawn for the differential equation,

$$\frac{dy}{dx} = 2x + \frac{y}{x}.$$

Which of the following graphs shows the correct slope lines at the points $P(1,1)$ and $Q(-1,-1)$?



End of Section 1

Section II **60 marks** **Attempt Questions 11-14**

Answer each question in the appropriate writing booklet, and number your answer booklet correctly. Extra exam writing booklets are available.

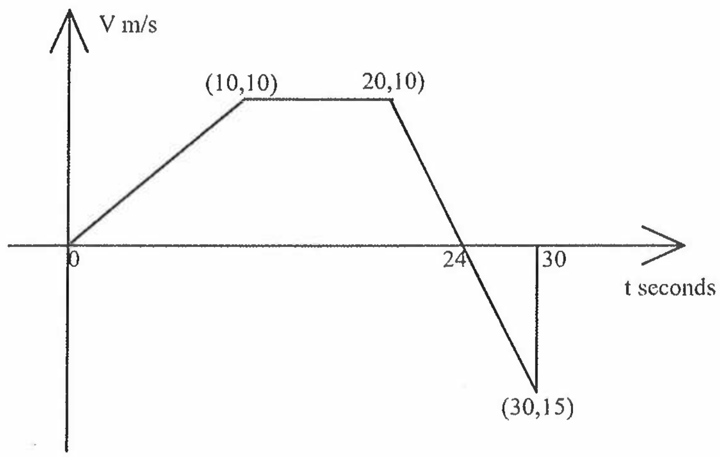
Question 11 **Start on a new answer booklet** **15 Marks.**

(*a*) Solve $|1 - 2x| \geq 15$. 2

(*b*) Given the velocity time graph below find:

(*i*) The acceleration for the first 10 seconds 1

(*ii*) The total distance travelled in the first 30 seconds. 1



(*c*) Use the substitution $u = e^{2x}$ to find $\displaystyle\int \frac{e^{2x}}{9 + e^{4x}}\, dx$. 3

(*d*) Water is poured into a container at a rate of 8 cm^3/s. If the volume of the water in the container is given by $V = \frac{3}{2}(h^2 + 8h)\ cm^3$ where h is the depth of the water, find the rate of change of the depth of the water when $V = 72\ cm^3$. 4

(*e*) (*i*) Express $cosx - \sqrt{3}\,sinx$ in the form $Rcos(x + \alpha)$ where $R > 0$ and $0 \leq x \leq 2\pi$. 2

(*ii*) Find the co-ordinates of the points of intersection of the graph $y = cosx - \sqrt{3}\,sinx$ with the x and y axis, in the interval $0 \leq x \leq 2\pi$. 2

End of Question 11

Question 12 **Start on a new answer booklet** **15 Marks.**

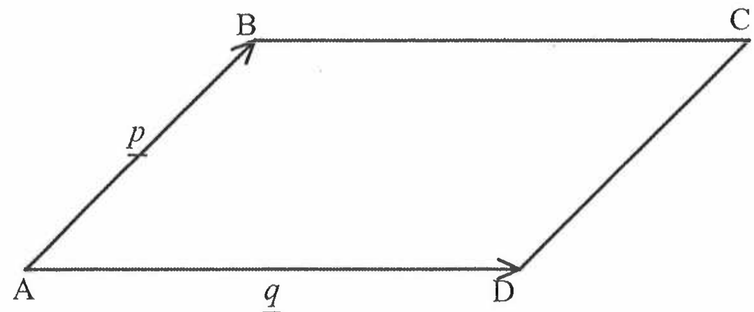
(*a*) Prove that $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2.$ 3

(*b*) Use the substitution $x = 2\sin\theta$ to find the exact value of $\int_0^1 \sqrt{4 - x^2}\, dx.$ 3

(*c*) The vectors $\underset{\sim}{\mathbf{u}} = \begin{pmatrix} 3 \\ \sin 2\alpha \end{pmatrix}$ and $\underset{\sim}{\mathbf{v}} = \begin{pmatrix} \cos\alpha \\ -3 \end{pmatrix}$, where $0 < \alpha < \pi$,
are perpendicular. Find possible values of α. 3

(*d*) Let $P(x) = x^5 - 6x^3 - 8x^2 - 3x.$
Show that $x = -1$ is a root of $P(x)$ of multiplicity three. 3

(*e*) $ABCD$ is a parallelogram, where $\overrightarrow{AB} = \underset{\sim}{\mathbf{p}}$ and $\overrightarrow{AD} = \underset{\sim}{\mathbf{q}}$.



Prove, using vectors, that the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of all the four sides. 3

End of Question 12

Question 13 **Start on a new answer booklet** **15 Marks.**

(*a*) (*i*) Show that $sin(x - y)sin(x + y) = sin^2x - sin^2y$. 1

(*ii*) Hence or otherwise,
Solve $sin^2 3x - sin^2x = sin\,4x$. $x = [0, \pi]$ 2

(*b*) (*i*) Sketch the graph of $y = f(x) = x^2 - 1$. 1

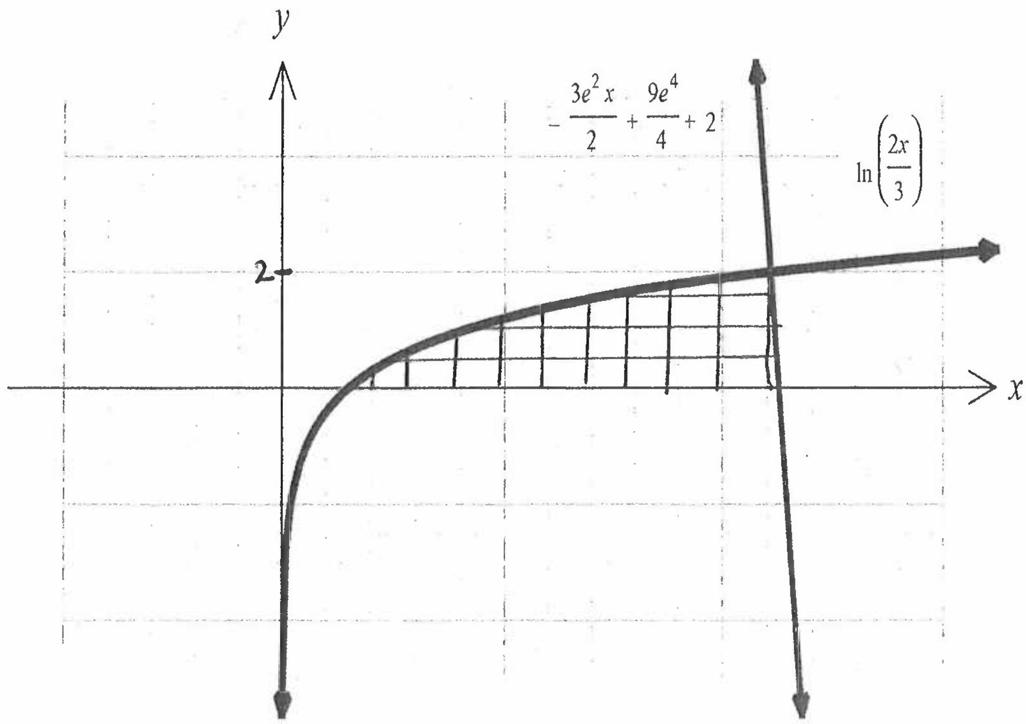
(*ii*) On the same graph sketch $\dfrac{1}{x^2 - 1} + 2$.
Clearly identifying asymptotes and intercepts. 3

(*c*) Given $|\underset{\sim}{\mathbf{p}}| = 3$, $|\underset{\sim}{\mathbf{q}}| = 5$ and $\underset{\sim}{\mathbf{p}} \cdot \underset{\sim}{\mathbf{q}} = 6$.
Calculate the length of $2\underset{\sim}{\mathbf{p}} - 3\underset{\sim}{\mathbf{q}}$. 3

(*d*) Let $f(x) = ln\left(\dfrac{2x}{3}\right)$.

(*i*) Show that the equation of the normal to the graph of $y = f(x)$ at the point where $x = \dfrac{3e^2}{2}$ has the equation $y = -\dfrac{3e^2x}{2} + \dfrac{9e^4}{4} + 2$. 2

(*ii*) Find the exact area enclosed by the graph of $y = f(x)$, the line $y = -\dfrac{3e^2x}{2} + \dfrac{9e^4}{4} + 2$ and the x axis. 3



You may use the diagram above to answer *d (ii)*.

End of Question 13

Question 14 **Start on a new answer booklet** **15 Marks.**

(*a*) (*i*) Show that $\frac{d}{dx}(tan^3x) = 3sec^4 x - 3sec^2x.$ 1

(*ii*) Hence or otherwise, evaluate $\int_0^{\frac{\pi}{4}} sec^4xdx.$ 3

(*b*) Prove by mathematical induction for all integers $n \geq 1$

that $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}.$ 3

(*c*) Given that $\frac{dy}{dx} = \frac{1}{4}(y - 1)^2$ when $x = 0$, $y = 0$.

What is the value of x when $y = 2$? 3

(*d*) Solve the differential equation $\frac{dy}{dx} = \frac{2}{x^3e^y}$ given $x = 1, y = 0.$

Express your solution in the form $y = f(x)$. 3

(e) Let $\underset{\sim}{\mathbf{g}} = |\underset{\sim}{\mathbf{e}}|\underset{\sim}{\mathbf{f}} + |\underset{\sim}{\mathbf{f}}|\underset{\sim}{\mathbf{e}}$ where $\underset{\sim}{\mathbf{e}}$, $\underset{\sim}{\mathbf{f}}$ and $\underset{\sim}{\mathbf{g}}$ are non-zero vectors.

Show that $\underset{\sim}{\mathbf{g}}$ bisects the angle between $\underset{\sim}{\mathbf{e}}$ and $\underset{\sim}{\mathbf{f}}$. 2

End of Question 14

END OF TASK

Student's Name: ______________________ Teacher's Code: _______________

SOLUTIONS

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2024 Trial HSC Examination
Mathematics Extension 1 Course

Name ______________________________ Teacher **REM DPR NHM JPN**

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider the correct answer, indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

1.	A ●	B ○	C ○	D ○
2.	A ●	B ○	C ○	D ○
3.	A ○	B ○	C ●	D ○
4.	A ○	B ○	C ○	D ●
5.	A ○	B ○	C ○	D ●
6.	A ○	B ○	C ●	D ○
7.	A ●	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ●
9.	A ●	B ○	C ○	D ○
10.	A ○	B ●	C ○	D ○

Start your answer here.

1. $\frac{1}{\lvert x-2\rvert} > \frac{1}{4}$ $\quad\therefore \lvert x-2\rvert < 4$	
$-4 < x-2 < 4$	
$-2 < x < 6$	
Integer Solutions : $\{-1, 0, 1, 2, 3, 4, 5\}$	
$\therefore$ 7 Integer Solutions	A
2. $3x^3 + 2x^2 - x + 5 = 0$	
$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = \frac{\frac{c}{a}}{-\frac{d}{a}}$	
$= -\frac{c}{a} = \frac{-(-1)}{5} = \frac{1}{5}$	A
3. $15\sin\theta - 8\cos\theta + 2$ $\quad \sqrt{15^2+8^2} = 17$	
$\therefore 17(\cos\theta + \alpha) + 2$	
Max : $17 + 2 = 19$	C
4. $x = 4 - t$ $\quad \therefore t = 4 - x$	
$y = 3t^3$ $\quad \therefore y = 3(4-x)^3$	
$y = 3(4^3 + 3(4)(-x)(4-x) - x^3)$	
$= 3(64 - 12x(4-x) - x^3)$	
$= 3(64 - 48x + 12x^2 - x^3)$	
$= 192 - 144x + 36x^2 - 3x^3$	D.

5. $v = \frac{dx}{dt} = \frac{1}{t+e}$ $\quad \therefore \int dx = \int \frac{1}{t+e}\,dt$

$x = \ln(t+e) + c$ $\quad$ at $t=0,\ x=-1$

$-1 = \ln e + c$ $\quad \therefore c = -2$

$x = \ln(t+e) - 2$ $\quad$ at $x = 0$

$2 = \ln(t+e)$

$e^2 = t + e$ $\quad \underline{t = e^2 - e}$

Subt t in $v = \frac{1}{e^2 - e + e} = \frac{1}{e^2} = \underline{e^{-2}}$ $\quad$ D.

6. $y = 4\cos^{-1}\frac{3x}{2}$

D: $-1 \le \frac{3x}{2} \le 1$ $\quad$ R: $0 \le \frac{y}{4} \le \pi$

$\therefore -\frac{2}{3} \le x \le \frac{2}{3}$ $\quad 0 \le y \le 4\pi$ $\quad$ C

7. $\cos\theta = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \frac{2(-1) + 3(2)}{\sqrt{2^2+3^2} \times \sqrt{(-1)^2+2^2}}$

$= \frac{4}{\sqrt{13} \times \sqrt{5}} = 0.496 \doteqdot .5$

$\therefore \cos\theta = 0.5$ $\quad \theta \doteqdot \cos^{-1}(0.5)$ $\quad$ A.

8. $|\overline{YZ}| = \frac{1}{2}|\overline{XY}|$ $\quad \therefore \underset{\sim}{z} - \underset{\sim}{y} = \frac{1}{2}(\underset{\sim}{y} - \underset{\sim}{x})$

$\underset{\sim}{z} = \frac{1}{2}\underset{\sim}{y} + \underset{\sim}{y} - \frac{1}{2}\underset{\sim}{x} = \frac{3\underset{\sim}{y}}{2} - \frac{1}{2}\underset{\sim}{x}$ $\quad$ D

Additional writing space on back page.

9. $y = x^2 \sin^{-1}(2x)$

$\frac{dy}{dx} = 2x\sin^{-1}(2x) + x^2\left(\frac{1}{\sqrt{1-(2x)^2}} \times 2\right)$

$= 2x\sin^{-1}(2x) + \frac{2x^2}{\sqrt{1-4x^2}}$ A.

10. Gradient

	P	Q
A	+	+
B	+	−
C	−	+
D	−	−

Check for values P(1,1) and Q(−1,−1) in the gradient eqnt.

Can be easily done by setting the equation in the calculator. $\frac{dy}{dx} = 2x + \frac{y}{x}$

A : P $2(1) + \frac{1}{1} = 3$ + ✓

Q $2(-1) + \frac{-1}{-1} = -1$ − X Not A.

B : P $2(+1) + \frac{(+1)}{(+1)} = 3$ + ✓

Q $2(-1) + \frac{(-1)}{(-1)} = -1$ − ✓ ∴ B.

You may ask for an extra Writing Booklet if you need more space to answer this question.

Start your answer here. Question 11

Many did not understand how to manipulate the inequalities

ie $1-2x \geq 15$ or $-1+2x \geq 15$

(a) Solve $|1-2x| \geq 15$

$1-2x = -15$ $\qquad$ $1-2x = 15$

$-2x = -16$ $\qquad$ $-2x = 14$

$x = 8$ $\qquad$ $x = -7$

$\therefore \boxed{x \leq -7} \cup \boxed{x \geq 8}$ ✓ ✓

1 mark for each correct solution

(b) (i) $a = \frac{v}{t} = \frac{10}{10} = \boxed{1 \text{ m/s}^2}$ ✓

1 mark

(ii) Distance travelled is Area A+B

Distance not displacement
∴ total area above and below axis

$A = \text{trapezium} = \frac{1}{2}(10+24) \times 10$

$= \frac{1}{2}\, 34 \times 10 = 170$

$B: \text{Triangle} = \frac{1}{2} \times 6 \times 15 = 45$

∴ Distance travelled is $170+45$

$= \boxed{215 \text{ m.}}$ ✓

1 mark

(c) $u = e^{2x}$ $\quad du = 2e^{2x}\,dx$

$\frac{1}{2}du = e^{2x}dx$ ✓ correct organisation

$\therefore \int \frac{e^{2x}\,du}{9+e^{2x}} = \frac{1}{2}\int \frac{du}{9+u^2}$ ✓ correct integral in terms of u

$= \frac{1}{2}\int \frac{du}{3^2+u^2} = \frac{1}{6}\int \frac{3\,du}{3^2+u^2}$

some did not realise that it involved inverse tan

$= \frac{1}{6}\tan^{-1}\left(\frac{u}{3}\right) + c$

$= \frac{1}{6}\tan^{-1}\left(\frac{e^{2x}}{3}\right) + c.$ ✓ correct answer

(d) Given $\frac{dv}{dt} = 8\text{ cm}^3/\text{s}$ $\quad V = \frac{3}{2}(h^2+8h)\text{ cm}^3$

Find $\frac{dh}{dt}$, when $V = 72\text{ cm}^3$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $\qquad \frac{dV}{dh} = \frac{3}{2}(2h+8)$

$= 3h+12$ ✓ Finding correct expression for $\frac{dV}{dh}$

$\therefore 8 = (3h+12) \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{8}{3(h+4)}$ ✓ correct Expression for $\frac{dh}{dt}$

To find 'h' $\quad V = \frac{3}{2}(h^2+8h) = 72$

$h^2+8h = 48$ $\qquad h^2+8h-48 = 0$ finding value of h

$(h-4)(h+12) = 0$ $\qquad h = 4, \; h = -12$ $\therefore h = 4$ ✓

$\therefore \frac{dh}{dt} = \frac{8}{3(4+8)} = \boxed{\frac{1}{3}\text{ cm/s.}}$ ✓ calculating value of $\frac{dh}{dt}$

Additional writing space on back page.

(e)(i) Express $\cos x - \sqrt{3}\sin x \equiv R\cos(x+\alpha)$

$1\cos x - \sqrt{3}\sin x$ $\quad R = \sqrt{1^2 + (\sqrt{3})^2} = 2$ | 1 finding r

$\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \frac{\pi}{3}$ (acute angle only)

$\therefore \cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$ | 1 finding α

(ii)

For points of intersection:

at $x = 0$ $\quad y = 2\cos\left(\frac{\pi}{3}\right) = 2 \times \frac{1}{2} = 1$ $\quad \boxed{(0,1)}$

finding y intercept

at $y = 0$ $\quad 2\cos\left(x + \frac{\pi}{3}\right) = 0$

$\cos\left(x + \frac{\pi}{3}\right) = 0 \qquad 0 \le x \le 2\pi$

$x + \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$x = \frac{\pi}{2} - \frac{\pi}{3}, \frac{3\pi}{2} - \frac{\pi}{3}, \frac{5\pi}{2} - \frac{\pi}{3}, \frac{7\pi}{2} - \frac{\pi}{3}, \ldots$

$x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}, \ldots$ out of domain

$\therefore x = \frac{\pi}{6}, \frac{7\pi}{6}$

$\therefore \boxed{\left(\frac{\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right)}$

finding co-ords of x intercepts

You may ask for an extra Writing Booklet if you need more space to answer this question.

Start your answer here. Question 12

(a) $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x}$

$= \dfrac{\sin 3x \cos x - \cos 3x \sin x}{\sin x \cos x} = \dfrac{\sin(3x-x)}{\frac{1}{2}\sin 2x}$

$= \dfrac{2\sin 2x}{\sin 2x} = 2$

$\sin x \cos x$

$= \frac{1}{2} \times 2\sin x \cos x$

$= \frac{1}{2}\sin 2x$

* Well Answered

* Most students split $\sin 3x$ and $\cos 3x$ into $\sin(2x+x)$ and $\cos(2x+x)$ and use appropriate Trig identities

(b) $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$ when $x = 0$, $2\sin\theta = 0$, $\theta = 0$

$x = 1$, $2\sin\theta = 1$ $\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

$\therefore \int_0^1 \sqrt{4-x^2}\, dx = \int_0^{\pi/6} \sqrt{4-4\sin^2\theta} \times 2\cos\theta\, d\theta$

$= \int_0^{\pi/6} 2\cos\theta \times 2\cos\theta\, d\theta$

$= 4\int_0^{\pi/6} \cos^2\theta\, d\theta = 4 \times \frac{1}{2}\int_0^{\pi/6} 1+\cos 2\theta\, d\theta$

$\therefore 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/6} = 2\left[\frac{\pi}{6} + \frac{\sqrt{3}}{2\times 2} - 0 - 0\right]$

$= 2\left[\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right] = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

* Poorly answered.

* Many students did not express dx in terms of $d\theta$

* Many students did not change the limits.

(c) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ for perpendicular.

$$\begin{pmatrix} 3 \\ \sin 2\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha \\ -3 \end{pmatrix} = 0$$

$3\cos\alpha - 3\sin 2\alpha = 0$

$3\cos\alpha - 6\sin\alpha\cos\alpha = 0$

$3\cos\alpha(1 - 2\sin\alpha) = 0$

$\cos\alpha = 0 \quad , \quad 1 - 2\sin\alpha = 0$

$\cos\alpha = 0 \qquad \sin\alpha = \frac{1}{2}$

$$\left.\begin{array}{l} \text{For } \cos\alpha = 0 \quad \alpha = 0, \pi/2 \\ \text{For } \sin\alpha = \frac{1}{2} \quad \alpha = \frac{\pi}{6}, \frac{5\pi}{6} \end{array}\right\} 0 < \alpha < \pi$$

$\therefore \alpha = \pi/6 \; , \; \pi/2 \; , \; 5\pi/6$

* Well answered.

* some students included $\frac{3\pi}{2}$ in their solutions and forget the limit for α was $0 < \alpha < \pi$

* Many students forget to include $\frac{5\pi}{6}$ in the solution set.

(d) $P(x) = x^5 - 6x^3 - 8x^2 - 3x$

$P'(x) = 5x^4 - 18x^2 - 16x - 3$

$P''(x) = 20x^3 - 36x - 16$

To prove multiplicity of roots.

$P(x) = P'(x) = P''(x)$ at $x = -1$ (Given).

$\therefore P(-1) = -1 + 6 - 8 + 3 = 0$

$P'(-1) = 5 - 18 + 16 - 3 = 0$

$P''(-1) = -20 + 36 - 16 = 0$

$\therefore x = -1$ is a root of multiplicity three.

or $(x+1)^3$ is a factor of $P(x)$.

* Reasonably well answered.

* Some students divided $P(x)$ by $x+1$ and then divided the divided by $x+1$ and repeated the process until such time as they were able to express in the form $(x+1)^3 Q(x)$

If you did this correctly you get full marks.

Additional writing space on back page.

(e) $\overline{AD} = \overline{BC} = \underset{\sim}{q}$

$\overline{AB} = \overline{DC} = \underset{\sim}{p}$

$\overline{AC} = \underset{\sim}{p} + \underset{\sim}{q}$, $\overline{BD} = \underset{\sim}{p} - \underset{\sim}{q}$

$\therefore$ Sum of squares of diagonal

$|\underset{\sim}{p}+\underset{\sim}{q}|^2 + |\underset{\sim}{p}-\underset{\sim}{q}|^2$

$= (\underset{\sim}{p}+\underset{\sim}{q}) \cdot (\underset{\sim}{p}+\underset{\sim}{q}) + (\underset{\sim}{p}-\underset{\sim}{q})(\underset{\sim}{p}-\underset{\sim}{q})$

$= \underset{\sim}{p}^2 + 2\underset{\sim}{p}\underset{\sim}{q} + \underset{\sim}{q}^2 + \underset{\sim}{p}^2 - 2\underset{\sim}{p}\underset{\sim}{q} + \underset{\sim}{q}^2$

$= \underset{\sim}{p}^2 + \underset{\sim}{q}^2 + \underset{\sim}{p}^2 + \underset{\sim}{q}^2$

= Sum of the squares of all four sides.

* Well answered.

You may ask for an extra Writing Booklet if you need more space to answer this question.

Start your answer here. Question 13

(a) (i) $\sin(x-y)\sin(x+y)$

$= \frac{1}{2}\left[\cos[(x-y)-(x+y)] - \cos[(x-y)+(x+y)]\right]$

$= \frac{1}{2}\left[\cos(-2y) - \cos(2x)\right]$

$= \frac{1}{2}\left(\cos 2y - \cos 2x\right)$

$= \frac{1}{2}\left[1 - 2\sin^2 y - (1 - 2\sin^2 x)\right]$

$= \frac{1}{2}\left[-2\sin^2 y + 2\sin^2 x\right]$

$= \sin^2 x - \sin^2 y.$

(ii) From (i) $\sin^2 3x - \sin^2 x$

$= \sin(3x-x)\sin(3x+x)$

$= \sin 2x \sin 4x$

$\therefore \quad \sin^2 3x - \sin^2 x = \sin 4x$

ie $\quad \sin 2x \sin 4x = \sin 4x$ from (i)

$\sin 2x \sin 4x - \sin 4x = 0$

$\sin 4x\left[\sin 2x - 1\right] = 0$

ie $\sin 4x = 0 \qquad \sin 2x = 1$

For $\sin 4x = 0 \qquad 4x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi \qquad 0 \le x \le \pi.$

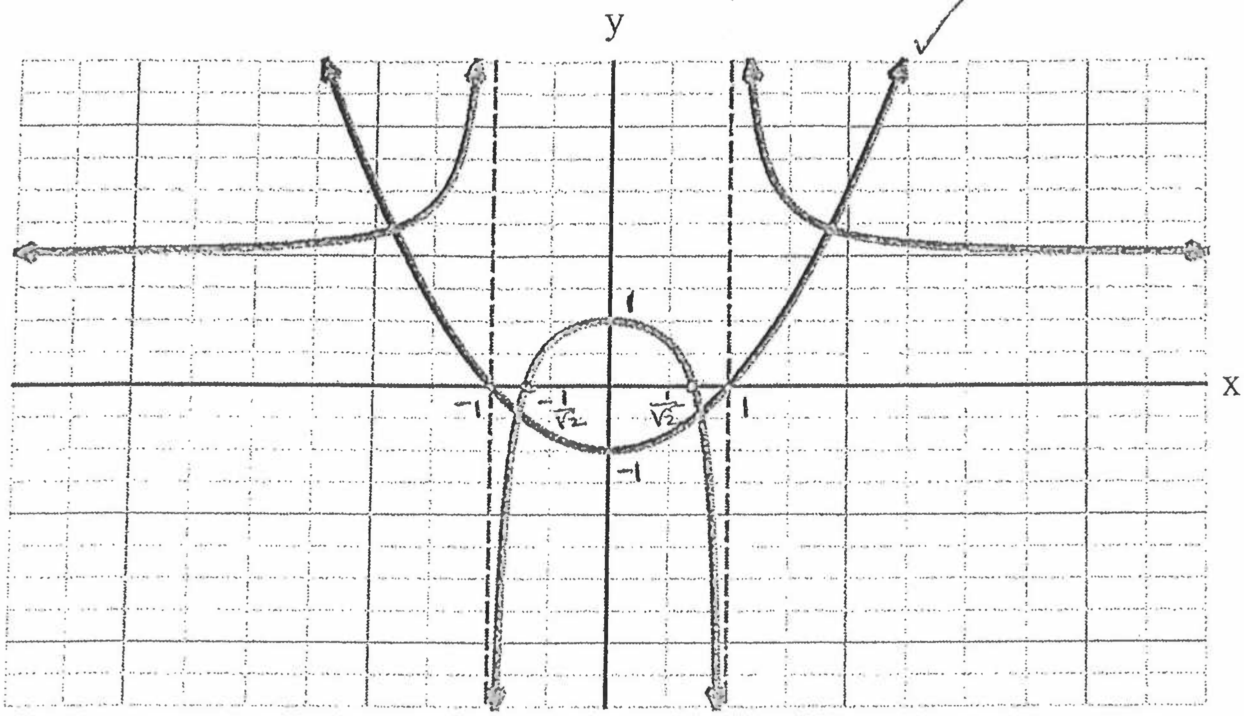
$\sin 2x = 1, \quad 2x = \pi/2, \quad x = \frac{\pi}{4}$

$\therefore$ Solution: $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi.$

Solutions to 13 (b) (i) and (ii)

(*i*) Sketch the graph of $y = f(x) = x^2 - 1$.

(*ii*) On the same graph sketch $\dfrac{1}{x^2-1} + 2$.



for x-intercepts.

$$\frac{1}{x^2-1} + 2 = 0$$

$$\frac{1}{x^2-1} = -2$$

$$1 = -2(x^2-1)$$

$$1 = -2x^2 + 2$$

$$\therefore 2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}}$$

NOTE:
Most students lost a mark for not getting the x intercept

1 Mark Correct Shape

1 " Asymptotes

$x = \pm 1$

$y = 2$

1 Mark correct " intercepts

(c) $|p| = 3$, $|q| = 5$ $\quad p \cdot q = 6$

Formula $\quad |a - b| = \sqrt{(a-b)(a-b)}$

$\therefore |2p - 3q| = \sqrt{(2p-3q)\cdot(2p-3q)}$

$= \sqrt{2p\cdot 2p - 2p\cdot 3q - 3q\cdot 2p + 3q\cdot 3q}$

$= \sqrt{4(p\cdot p) - 6(p\cdot q) - 6(q\cdot p) + 9(q\cdot q)}$

$p\cdot p = |p|^2 = 3^2 = 9$

$q\cdot q = |q|^2 = 5^2 = 25$

$p\cdot q = 6$

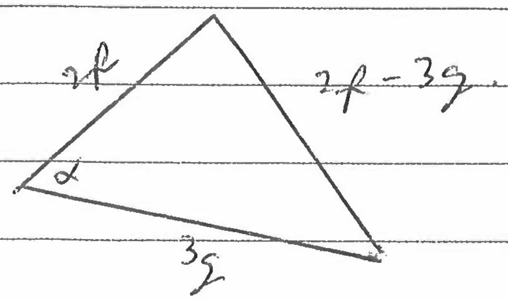
$= \sqrt{4(9) - 12(6) + 9(25)}$

$= \sqrt{189}$

$\therefore |2p - 3q| = \sqrt{189} = \sqrt{9\times 21}$

$= 3\sqrt{21}$

OR



$|2p-3q|^2 = |2p|^2 + |3q|^2 - 2|2p||3q| \cos\alpha$

$= 4|p|^2 + 9|q|^2 - 2\times 2|p|\times 3\times |q| \dfrac{2p\cdot 3q}{|2p||3q|}$

$= 4(3)^2 + 9(5)^2 - 2\times 2\times (3)\times 3\times (5) \left(\dfrac{6\times \overset{12}{6}}{2\times 3\times 3\times 5}\right)$

Additional writing space on back page.

$= 189$

length $2p - 3q = \sqrt{189}$

$= 3\sqrt{21}$

(d) $f(x) = \ln\left(\frac{2x}{3}\right)$

(i) $y = \ln 2x - \ln 3$

$y' = \frac{1}{2x} \times 2 = \frac{1}{x}$

$y = \ln\left(\frac{2x}{3}\right)$

@ $x = \frac{3e^2}{2}$

$y = \ln\left(\frac{2}{3} \times \frac{3}{2}e^2\right)$

$= \ln e^2$

$= 2\ln e = 2$

$m_t = \frac{1}{x} = \frac{2}{3e^2}$

$\therefore m_n = -\frac{3e^2}{2}$

/ Mark

Equn. of normal at $\left(\frac{3e^2}{2}, 2\right)$ is:

$y - y_1 = m_n(x - x_1)$

$y - 2 = -\frac{3e^2}{2}\left(x - \frac{3e^2}{2}\right)$

/ Mark

$y = -\frac{3e^2}{2}x + \frac{9e^4}{4} + 2$

(ii) Area of Shaded region

$= A(\text{Trapezium OABC}) - \int_0^2 \frac{3}{2}e^y\,dy$

See diagram.

$y = -\frac{3e^2}{2}x + \frac{9e^4}{4} + 2$

$0 = -\frac{3e^2}{2}x + \frac{9e^4}{4} + 2$

$\frac{3e^2}{2}x = \frac{9e^4}{4} + 2$

$x = \frac{2}{3e^2} \times \frac{9e^4}{4} + \frac{2 \times 2}{3e^2}$

$x = \frac{3e^2}{2} + \frac{4}{3e^2}$

$A(\text{Trap}) = \frac{1}{2} \times \left(\frac{3e^2}{2} + \frac{3e^2}{2} + \frac{4}{3e^2}\right) \times 2$

$= 3e^2 + \frac{4}{3e^2}$

Additional writing space on back page.

$y = \ln\left(\frac{2x}{3}\right)$

$e^y = \frac{2x}{3}$

$\frac{3}{2}e^y = x$

A curve $= \int_0^2 \frac{3}{2}e^y \, dy$

$= \left[\frac{3}{2}e^y\right]_0^2 = \frac{3}{2}e^2 - \frac{3}{2}$

$\therefore$ Area Shaded $= 3e^2 + \frac{4}{3e^2} - \left(\frac{3}{2}e^2 - \frac{3}{2}\right)$

$= 3e^2 + \frac{4}{3e^2} - \frac{3e^2}{2} + \frac{3}{2}$

$= \frac{3e^2}{2} + \frac{4}{3e^2} + \frac{3}{2} \; u^2.$

You may ask for an extra Writing Booklet if you need more space to answer this question.

$y = \ln\left(\frac{2x}{3}\right)$

$y' = \frac{1 \times 3 \times 2}{2x \times 3} = \frac{1}{x}$

$m_t = \frac{1}{x} = \frac{2}{3e^2}$

$m_n = -\frac{3e^2}{2}$

$y = \ln\left(\frac{2x}{3}\right)$

$y = \ln\left(\frac{2}{3} \times \frac{3}{2} e^2\right)$

$y = \ln e^2$

$y = 2$

$y - 2 = -\frac{3e^2}{2}\left(x - \frac{3e^2}{2}\right)$

$y = \frac{-3e^2 x}{2} + \frac{9e^4}{4} + 2$

✓✓

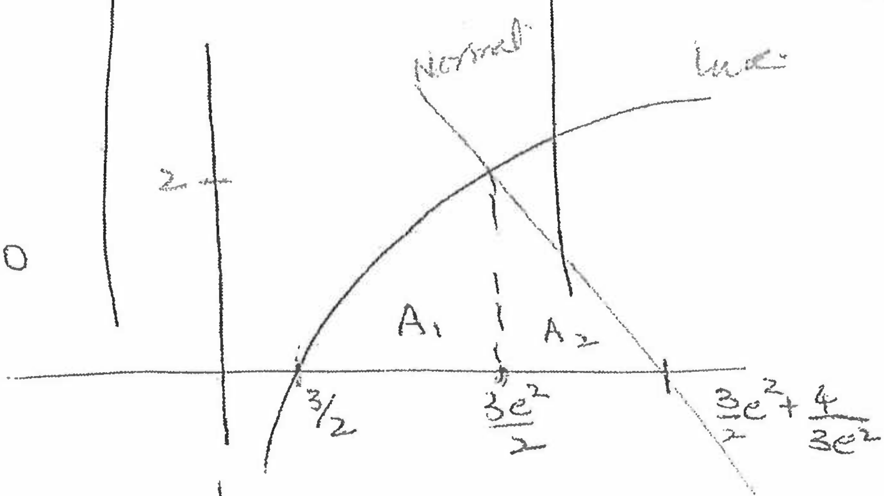
Area

at $y=0$ Curve.

$\ln\left(\frac{2x}{3}\right) = 0$

$e^0 = \frac{2x}{3}$

$x = \frac{3}{2}$



at $y=0$ Normal.

$\frac{3e^2 x}{2} = \frac{9e^4}{4} + 2$

$x = \frac{\cancel{9}^{3} \times 2}{\cancel{3} \times \cancel{4}_{2}} \times \frac{e^4}{e^2} + \frac{2 \times 2}{3e^2}$

$x = \frac{3e^2}{2} + \frac{4}{3e^2}$

at $y=2$ Curve.

$2 = \ln\left(\frac{2x}{3}\right)$

$e^2 = \frac{2x}{3}$

$x = \frac{3e^2}{2}$

Area = A_1 $\int_{3/2}^{\frac{3e^2}{2}} \ln\left(\frac{2x}{3}\right) dx + \frac{1}{2} \times b \times h$

$+ \frac{1}{\cancel{2}}\left(\cancel{\frac{3e^2}{2}} + \frac{4}{3e^2} - \cancel{\frac{3e^2}{2}}\right) \times \cancel{2}$

$+ \frac{4}{3e^2}$

↓

Int. by Parts

Next Page.

$$A_1 = \int_{3/2}^{\frac{3e^2}{2}} \ln\left(\frac{2x}{3}\right) dx$$

$$= \int \ln 2x - \ln 3 \, dx$$

$$= \int (\ln x + \ln 2 - \ln 3) dx$$

$$\therefore \left[x \ln x - x + x \ln 2 - x \ln 3\right]_{3/2}^{\frac{3e^2}{2}}$$

$$\frac{3e^2}{2}\ln\left(\frac{3e^2}{2}\right) - \frac{3e^2}{2} + \frac{3e^2}{2}\ln 2 - \frac{3e^2}{2}\ln 3$$

$$-\frac{3}{2}\ln\left(\frac{3}{2}\right) + \frac{3}{2} - \frac{3}{2}\ln 2 + \frac{3}{2}\ln 3$$

$$= \frac{3e^2}{2}(\ln 3 + 2 - \ln 2) - \frac{3e^2}{2} + \frac{3e^2}{2}\ln 2 - \frac{3e^2}{2}\ln 3$$

$$-\frac{3}{2}(\ln 3 - \ln 2) + \frac{3}{2} - \frac{3}{2}\ln 2 + \frac{3}{2}\ln 3$$

$$= \cancel{\frac{3e^2}{2}\ln 3} + \frac{3e^2}{2} - \cancel{\frac{3e^2}{2}\ln 2} - \frac{3e^2}{2} + \cancel{\frac{3e^2}{2}\ln 2} - \cancel{\frac{3e^2}{2}\ln 3}$$

$$-\cancel{\frac{3}{2}\ln 3} + \cancel{\frac{3}{2}\ln 2} + \frac{3}{2} - \cancel{\frac{3}{2}\ln 2} + \cancel{\frac{3}{2}\ln 3}$$

$$= \frac{3e^2}{2} + \frac{3}{2}$$

$$= A_1 + A_2$$

$$= \frac{3e^2}{2} + \frac{3}{2} + \frac{4}{3e^2}$$

$$\int \ln x \, dx = x \ln x - x$$

$$\int \ln 2 = x \ln 2$$

$$\int \ln 3 = x \ln 3$$

$$\ln\left(\frac{3e^2}{2}\right)$$

$$= \ln 3e^2 - \ln 2$$

$$= \ln 3 + \ln e^2 - \ln 2$$

$$= \ln 3 + 2 - \ln 2$$

QUESTION 14

a) i) $\frac{d}{dx}(\tan^3 x) = 3\tan^2 x(\sec^2 x)$

$= 3(\sec^2 x - 1)(\sec^2 x)$

$= 3\sec^4 x - 3\sec^2 x$

Surprisingly, many mocked this up.

1mk

(ii) $\int_0^{\frac{\pi}{4}} \sec^2\theta = ??$

Poorly done

Many students "lost" the 3.

From (i)

$3\sec^4 x - 3\sec^2 x = \frac{d}{dx}(\tan^3 x)$

$3\sec^4 x = \frac{d}{dx}(\tan^3 x) + 3\sec^2 x$ ← 1mk

$\sec^4 x = \frac{1}{3}\left[\frac{d}{dx}(\tan^3 x)\right] + \sec^2 x$

$\therefore \int_0^{\frac{\pi}{4}} \sec^4 x\,dx = \left[\frac{1}{3}\tan^3 x + \tan x\right]_0^{\frac{\pi}{4}}$ ← 1mk

$= \left(\frac{1}{3}\tan^3\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - (0)$

$= \frac{1}{3} + 1$

$= \frac{4}{3}$ ← 1mk

b) Prove $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

for $n \geq 1$

Step 1. Prove true for $n = 1$.

$LHS = \frac{1}{2}$

$RHS = 2 - \frac{3}{2}$

$= \frac{1}{2}$

$\therefore$ true for $n = 1$

1mk for correct setting out

Step 2. Assume true for $n = k$.

ie $\frac{1}{2} + \frac{2}{2^2} + \dots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$

Now Prove true for $n = k+1$

ie $\frac{1}{2} + \frac{2}{2^2} + \dots + \frac{k+1}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$

1mk for correct setting out

Proof.

$LHS = \frac{1}{2} + \dots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}}$

$= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$ From assumption

$= \frac{2(2^{k+1}) - 2(k+2) + k+1}{2^{k+1}}$

$= \frac{2(2^{k+1})}{2^{k+1}} - \frac{k+3}{2^{k+1}}$

$= 2 - \frac{k+3}{2^{k+1}}$ etc by MI

1mk for use of assumption & correct working to soln.

c) $\frac{dy}{dx} = \frac{1}{4}(y-1)^2$

METHOD 1

$\frac{dy}{(y-1)^2} = \frac{1}{4}dx$

$(y-1)^{-2}dy = \frac{1}{4}dx$

$\frac{(y-1)^{-1}}{-1} = \frac{x}{4} + c$

$-\frac{1}{y-1} = \frac{x}{4} + c$

$x = 0 \quad y = 0$

$1 = c$

$-\frac{1}{y-1} = \frac{x}{4} + 1$

$y = 2 \quad x = ???$

$-\frac{1}{1} = \frac{x}{4} + 1$

$x = -8$

METHOD 2

$\frac{dx}{dy} = 4(y-1)^{-2}$

$x = -4(y-1)^{-1} + c$

$x = -\frac{4}{y-1} + c$

$x = 0 \quad y = 0$

$0 = -\frac{4}{-1} + c$

$c = -4$

$x = -\frac{4}{y-1} - 4$

$y = 2 \quad x = ??$

$x = -\frac{4}{1} - 4$

$= -8$

1mk for either correct method

1mk for correct constant. (many forgot to allow for constant)

1mk for correct solution

d) $\frac{dy}{dx} = \frac{2}{x^3 e^y}$ $x = 1$ $y = 0$

$e^y \, dy = \frac{2}{x^3} \, dx$

1mk for correct method.

$e^y \, dy = 2x^{-3} \, dx$

$e^y = \frac{2x^{-2}}{-2} + c$

$e^y = -\frac{1}{x^2} + c$

Again, many forgot the "c"

$x = 1$ $y = 0$

$e^0 = -1 + c$

$c = 2$

1mk for value of "c".

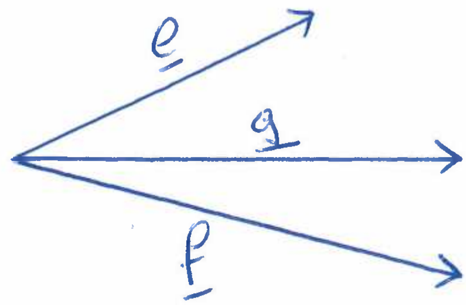
$\therefore e^y = -\frac{1}{x^2} + 2$

$y = \ln\left(2 - \frac{1}{x^2}\right)$

1mk for correct solution

e) One path to a correct solution is using the dot product

$\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}| \cos\theta$



$\underline{g} = |\underline{e}|\underline{f} + |\underline{f}|\underline{e}$

1mk awarded for substantial effort

$$\underline{e} \cdot \underline{g} = \underline{e} \cdot \left[|\underline{e}|\underline{f} + |\underline{f}|\underline{e} \right]$$
$$= |\underline{e}|\, \underline{e}\cdot\underline{f} + |\underline{f}|\, \underline{e}\cdot\underline{e}$$
$$= |\underline{e}|\, \underline{e}\,\underline{f} + |\underline{f}|\, |\underline{e}|^2$$
$$= |\underline{e}| \left[\underline{e}\,\underline{f} + |\underline{f}|\,|\underline{e}| \right]$$

$$\therefore |\underline{e}| . |\underline{g}| \cos\alpha = |\underline{e}| \left[\underline{e}\,\underline{f} + |\underline{f}||\underline{e}| \right]$$

OR $|\underline{g}| \cos\alpha = \underline{e} . \underline{f} + |\underline{f}||\underline{e}|$ *

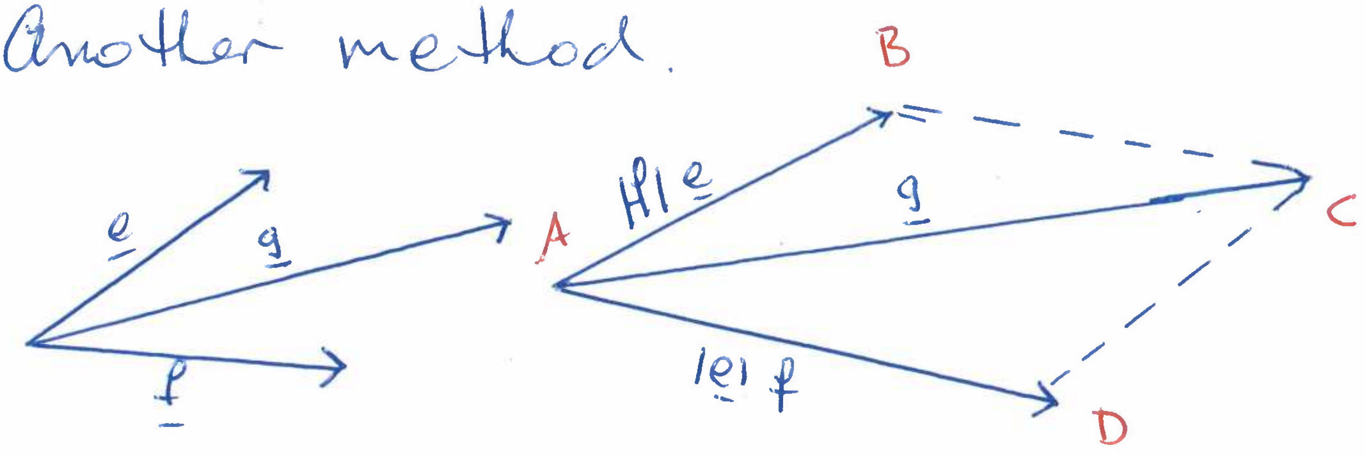
Similarly, $\underline{f} \cdot \underline{g} = \underline{f} \left[|\underline{e}|\underline{f} + |\underline{f}|\underline{e} \right]$
$$= |\underline{e}||\underline{f}|^2 + |\underline{f}|\, \underline{e}\,\underline{f}$$
$$= |\underline{f}| \left[|\underline{e}||\underline{f}| + \underline{e} . \underline{f} \right]$$

$$\therefore |\underline{f}||\underline{g}| \cos\beta = |\underline{f}| \left[|\underline{e}||\underline{f}| + \underline{e}\,\underline{f} \right]$$

$|\underline{g}| \cos\beta = |\underline{e}||\underline{f}| + \underline{e}\,\underline{f}$ *

$\therefore \cos\alpha = \cos\beta$

e) Another method.



$\underline{g} = |\underline{f}|\,\underline{e} + |\underline{e}|\,\underline{f}$

Now $\big|\,|\underline{f}|\,\underline{e}\,\big| = \big|\,|\underline{e}|\,\underline{f}\,\big|$ * MUST BE REFERENCED IN SOLUTION

$\therefore$ ABCD is a Rhombus

If ABCD is a Rhombus

then $\underline{g}$ bisects the angle between

$|\underline{f}|\,\underline{e}$ and $|\underline{e}|\,\underline{f}$.

Hence, it bisects the angle between

$\underline{e}$ and $\underline{f}$

1 mk awarded for substantial effort