Neap

HSC Trial Examination 2020

Mathematics Extension 1

General Instructions	• Reading time – 5 minutes • Working time – 2 hours • Write using black pen • NESA approved calculators may be used • A reference sheet is provided at the back of this paper • In Questions 11–14, show relevant mathematical reasoning and/or calculations
Total marks: 70	**Section I – 10 marks** (pages 2–5) • Attempt Questions 1–10 • Allow about 15 minutes for this section **Section II – 60 marks** (pages 6–12) • Attempt Questions 11–14 • Allow about 1 hour and 45 minutes for this section

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2020 HSC Mathematics Extension 1 Examination.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. Let $P(x) = x^2 + bx + c$ where b and c are constants. The zeros of $P(x)$ are α and $\alpha + 1$.

What are the correct expressions for b and c in terms of α?

(A) $b = -(2\alpha + 1)$ and $c = \alpha^2 + \alpha$

(B) $b = 2\alpha + 1$ and $c = \alpha^2 + \alpha$

(C) $b = \alpha^2 + \alpha$ and $c = -(2\alpha + 1)$

(D) $b = \alpha^2 + \alpha$ and $c = 2\alpha + 1$

2. What is the derivative of $\tan^{-1}(2x - 1)$?

(A) $\dfrac{1}{4x^2 - 4x + 2}$

(B) $\dfrac{2x - 1}{2x^2 - 2x + 1}$

(C) $\dfrac{2}{2x^2 - 2x + 1}$

(D) $\dfrac{1}{2x^2 - 2x + 1}$

3. An experiment consisted of tossing a biased coin three times and recording the number of tails obtained. This experiment was repeated 1000 times and the results are shown in the table.

Number of tails	*Frequency*
0	219
1	427
2	292
3	62

Based on these results, what is the probability that the coin shows tails when tossed?

(A) 0.3
(B) 0.4
(C) 0.5
(D) 0.6

4. Which of the following expressions is equal to $\cos(x) + \sin(x)$?

(A) $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

(B) $2\sin\left(x + \frac{\pi}{4}\right)$

(C) $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

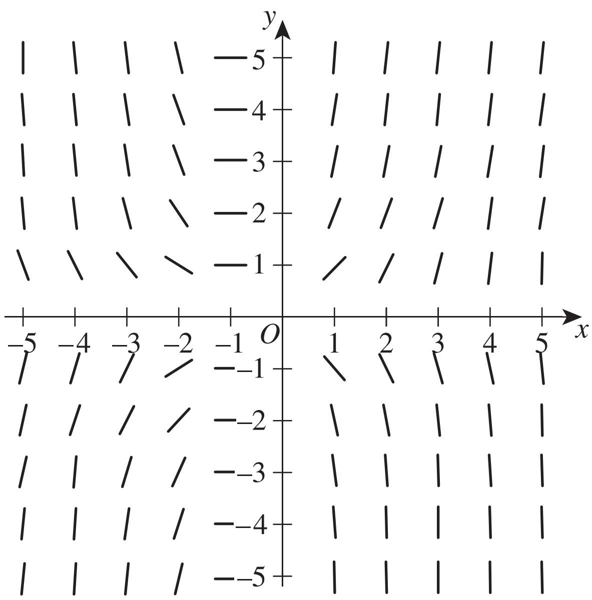
(D) $2\sin\left(x - \frac{\pi}{4}\right)$

5. Four males and four females are to sit around a table.

In how many ways can this be done if the males and females alternate?

(A) 144
(B) 2880
(C) 5040
(D) 40 320

6. The direction (slope) field for a first order differential equation is shown.



Which of the following could be the differential equation represented?

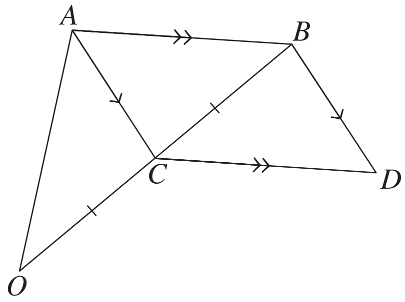
(A) $\frac{dy}{dx} = (x + 1)^3$

(B) $\frac{dy}{dx} = x(y + 1)$

(C) $\frac{dy}{dx} = (x + 1)y$

(D) $\frac{dy}{dx} = (x - 1)y$

7. The position vectors of points A and B are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Point C is the midpoint of OB and point D is such that $ABDC$ is a parallelogram.



Which of the following is the position vector of D?

(A) $\frac{3}{2}\underset{\sim}{b} + \underset{\sim}{a}$

(B) $\frac{3}{2}\underset{\sim}{b} - \underset{\sim}{a}$

(C) $\frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

(D) $\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}$

8. Which of the following functions is a primitive of $\frac{1}{\sqrt{4-9x^2}}$?

(A) $\frac{1}{3}\sin^{-1}\frac{2x}{3}$

(B) $\frac{1}{9}\sin^{-1}\frac{3x}{2}$

(C) $\frac{1}{9}\sin^{-1}\frac{2x}{3}$

(D) $\frac{1}{3}\sin^{-1}\frac{3x}{2}$

9. A curve C has parametric equations $x = \cos^2 t$ and $y = 4\sin^2 t$ for $t \in R$.

What is the Cartesian equation of C?

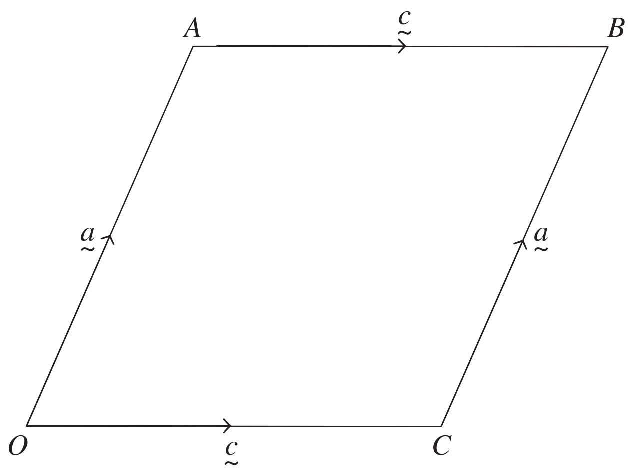
(A) $y = 1 - x$ for $0 \le x \le 1$

(B) $y = 4 - 4x$ for $x \in R$

(C) $y = 4 - 4x$ for $0 \le x \le 1$

(D) $y = 1 - x$ for $x \in R$

10. The diagram shows $OABC$, a rhombus in which $\overrightarrow{OA} = \overrightarrow{CB} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \underset{\sim}{c}$.



To prove that the diagonals of $OABC$ are perpendicular, it is required to show that

(A) $(\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{c}) = 0$.

(B) $(\underset{\sim}{a} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = 0$.

(C) $(\underset{\sim}{a} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{c}) = 0$.

(D) $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$.

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the function $f(x) = x^2 - 4x + 6$.

(i) Explain why the domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function. **1**

(ii) Given that the domain of $f(x)$ is restricted to $x \le 2$, find an expression for $f^{-1}(x)$. **2**

(iii) Given the restriction in part (a) (ii), state the domain and range of $f^{-1}(x)$. **2**

(iv) The curve $y = f(x)$ with its restricted domain and the curve $y = f^{-1}(x)$ intersect at the point P. **2**

Find the coordinates of P.

(b) Use the substitution $u = 1 + 2\tan x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{(1 + 2\tan x)^2 \cos^2 x}\,dx$. **2**

(c) Use t-formulae to solve the equation $\cos x - \sin x = 1$, where $0 \le x \le 2\pi$. **3**

(d) The work done, W, by a constant force, $\underset{\sim}{F}$, in moving a particle through a displacement, $\underset{\sim}{s}$, is defined by the formula $W = \underset{\sim}{F} \cdot \underset{\sim}{s}$. A force described by the vector $\underset{\sim}{F} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ moves a particle along the line l from $P(-1, 2)$ to $Q(2, -2)$.

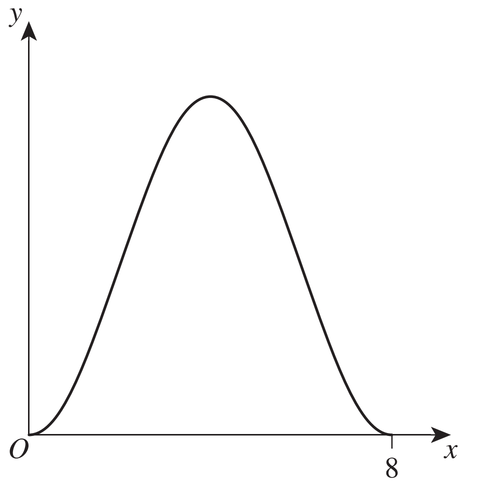
(i) Find $\underset{\sim}{s} = \overrightarrow{PQ}$ and hence find the value of W. **1**

(ii) Hence, verify that W is also given by $W = (\underset{\sim}{F} \cdot \hat{\underset{\sim}{s}})|\underset{\sim}{s}|$. **1**

(iii) Find the component of $\underset{\sim}{F}$ in the direction of l. **1**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A proposed plan for a garden is shown in the diagram. The curved boundary of the garden is modelled by the function $f(x) = 6\sin^2\left(\frac{\pi x}{8}\right)$, $0 \le x \le 8$.



(i) Use the identity $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ to show that **2**

$\sin^2\left(\frac{\pi x}{8}\right) = \frac{1}{2}\left(1 - \cos\frac{\pi x}{4}\right)$.

(ii) Use the result from part (a) (i) to find the area, A, of the garden. **2**

Question 12 continues on page 8

Question 12 (continued)

(b) A state-wide housing study found that 36% of adults in NSW have a mortgage.

(i) A random sample of 25 adults in NSW is to be taken to determine the proportion of those who have a mortgage. **2**

Show that the mean and standard deviation for the distribution of sample proportions of such random samples are 0.36 and 0.096 respectively.

(ii) Part of a table of $P(Z \leq z)$ values, where Z is a standard normal variable, is shown. **2**

z	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964

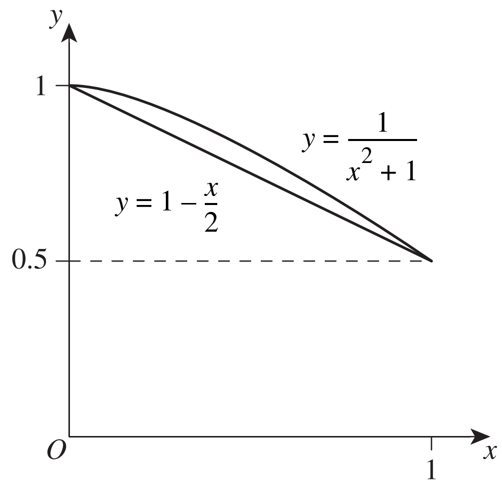
Of a random sample of 25 adults in NSW, use the table to estimate the probability that at most three will have a mortgage. Give your answer correct to four decimal places.

(iii) If a random sample of 25 adults in NSW is taken, find the probability that the sample proportion is equal to the population proportion. Give your answer correct to four decimal places. **2**

Question 12 continues on page 9

Question 12 (continued)

(c) The diagram shows the graph of $y = \dfrac{1}{x^2 + 1}$ and the graph of $y = 1 - \dfrac{x}{2}$ for $0 \le x \le 1$.



(i) Find the exact volume of the solid of revolution formed when the region bounded by the graph of $y = \dfrac{1}{x^2 + 1}$, the y-axis and the line $y = \dfrac{1}{2}$ is rotated 360° about the y-axis. **2**

(ii) Find the exact volume of the solid of revolution formed when the region bounded by the graph of $y = 1 - \dfrac{x}{2}$, the y-axis and the line $y = \dfrac{1}{2}$ is rotated 360° about the y-axis. **2**

(iii) Use the results from parts (c) (i) and (ii) to show that $\ln 2 > \dfrac{2}{3}$. **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is projected from a point O on level horizontal ground with a speed of 21 m s^{-1} at an angle θ to the horizontal. At time T seconds, the particle passes through the point $B(12, 2)$.

Neglecting the effects of air resistance, the equations describing the motion of the particle are:

$$x = Vt\cos\theta$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2$$

where t is the time in seconds after projection, g m s^{-2} is the acceleration due to gravity where $g = 9.8$ m s^{-2} and x and y are measured in metres. Do NOT prove these equations.

(i) By considering the horizontal component of the particle's motion, show that $T = \frac{4}{7}\sec\theta$. **1**

(ii) By considering the vertical component of the particle's motion and, using the result from part (a) (i), show that $4\tan^2\theta - 30\tan\theta + 9 = 0$. **2**

(iii) Find the particle's least possible flight time from O to B. Give your answer correct to two decimal places. **2**

(b) Prove by mathematical induction that, for all integers $n \geq 1$, **3**

$$\frac{2}{1\times 3} + \frac{2}{2\times 4} + \frac{2}{3\times 5} + \ldots + \frac{2}{n(n+2)} = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}.$$

(c) (i) Prove the trigonometric identity $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$. **3**

(ii) Use the identity from part (c) (i) to find the roots of the cubic equation $x^3 - 3x^2 - 3x + 1 = 0$ and hence find the exact value of $\tan\frac{\pi}{12}$. **4**

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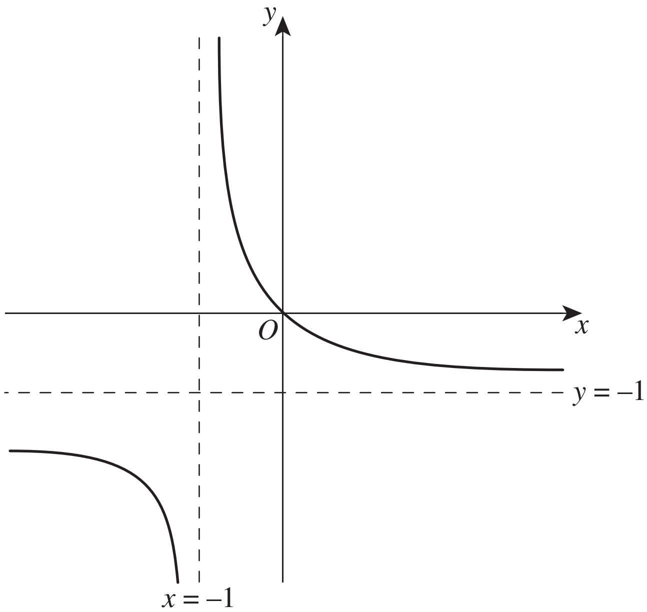
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below is a sketch of the graph of the function $f(x) = -\dfrac{x}{x+1}$.



(i) Sketch the graph of $y = (f(x))^2$, showing all asymptotes and intercepts. **2**

(ii) Sketch the graph of $y = x + f(x)$, showing all asymptotes and intercepts. **2**

(iii) Solve the equation $(f(x))^2 = f(x)$. **1**

Question 14 continues on page 12

Question 14 (continued)

(b) The area A cm^2 is occupied by a bacterial colony. The colony has its growth modelled by the logistic equation $\frac{dA}{dt} = \frac{1}{25}A(50 - A)$ where $t \geq 0$ and t is measured in days. At time $t = 0$, the area occupied by the bacteria colony is $\frac{1}{2}$ cm^2.

(i) Show that $\frac{1}{A(50-A)} = \frac{1}{50}\left(\frac{1}{A} + \frac{1}{50-A}\right)$. **1**

(ii) Using the result from part (b) (i), solve the logistic equation and hence show that $A = \frac{50}{1 + 99e^{-2t}}$. **3**

(iii) According to this model, what is the limiting area of the bacteria colony? **1**

(iv) Find the exact time when the rate of change in the area occupied by the bacterial colony is at its maximum. **2**

(c) The table shows selected values of a one-to-one differentiable function $g(x)$ and its derivative $g'(x)$. **3**

x	-1	0
$g(x)$	-5	-1
$g'(x)$	3	$\frac{1}{2}$

Let $f(x)$ be a function such that $f(x) = g^{-1}(x)$.

Find the value of $f'(-1)$.

End of paper

 TENME1_QA_20.FM

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\text{and } \alpha\beta\gamma = -\frac{d}{a}$$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

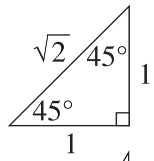
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

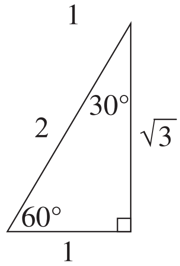
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A\cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A\sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

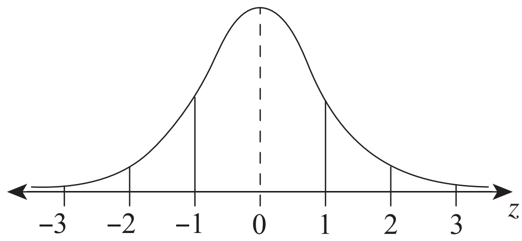
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\text{Var}(X) = E[(X - \mu)^2] = E(X^2) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r\,p^r(1 - p)^{n - r}$$

$$X \sim \text{Bin}(n, p)$$
$$\Rightarrow P(X = x)$$
$$= \binom{n}{x}p^x(1 - p)^{n - x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\text{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1 - [f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1 - [f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1 + [f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\{f(a) + f(b) + 2[f(x_1) + \ldots + f(x_{n-1})]\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \ldots + \binom{n}{r}x^{n-r}a^{r} + \ldots + a^{n}$$

Vectors

$$|\underset{\sim}{u}| = |x\underset{\sim}{i} + y\underset{\sim}{j}| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$[r(\cos\theta + i\sin\theta)]^{n} = r^{n}(\cos n\theta + i\sin n\theta)$$
$$= r^{n}e^{in\theta}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^{2}(x - c)$$

Neap

HSC Trial Examination 2020

Mathematics Extension 1

Solutions and marking guidelines

Section I

Sample answer	Syllabus content, outcomes and targeted performance bands
Question 1 **A** If α and $\alpha + 1$ are zeros of $P(x)$, then $P(x) = x^2 - (2\alpha + 1)x + (\alpha^2 + \alpha)$. Equating coefficients gives $b = -(2\alpha + 1)$ and $c = \alpha^2 + \alpha$.	ME-F2 Polynomials ME11–1 Bands E2–E3
Question 2 **D** $\frac{d}{dx}(\tan^{-1} f(x)) = \frac{f'(x)}{1 + (f(x))^2}$ $f(x) = 2x - 1$ and $f'(x) = 2$ $\frac{d}{dx}(\tan^{-1}(2x - 1)) = \frac{2}{1 + (2x - 1)^2}$ $= \frac{2}{4x^2 - 4x + 2}$ $= \frac{1}{2x^2 - 2x + 1}$	ME-C2 Further Calculus Skills ME12–1 Bands E2–E3
Question 3 **B** Let X represent the number of tails where $X \sim \text{Bin}(3, p)$ and let p represent the probability of obtaining tails. From the frequency distribution, it is clear that $p < 0.5$. Consider: $\{1000P(X = 0), 1000P(X = 1), 1000P(X = 2), 1000P(X = 3)\}$ For $p = 0.3$, the theoretical frequency distribution is $\{343, 441, 189, 27\}$, and for $p = 0.4$ it is $\{216, 432, 288, 64\}$. Compared to the given experimental frequency distribution, the closest theoretical distribution is for $p = 0.4$.	ME-S1 The Binomial Distribution ME12–5 Bands E2–E3
Question 4 **A** $R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$ $\sin x + \cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$ Equating coefficients of $\sin x$ gives $R\cos\alpha = 1$. (1) Equating coefficients of $\cos x$ gives $R\sin\alpha = 1$. (2) Squaring both (1) and (2) and adding gives $R^2 = 2 \Rightarrow R = \sqrt{2}$ (> 0). Substituting into (1) and (2) gives $\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$. So $\alpha = \frac{\pi}{4}$ and hence $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.	ME-T3 Trigonometric Equations ME12–3 Bands E2–E3

Sample answer	Syllabus content, outcomes and targeted performance bands
Question 5 **A** The table outlines the possible seating arrangements. M1: 1, M2: 3, M3: 2, M4: 1, F1: 4, F2: 3, F3: 2, F4: 1 Therefore the number of possible seating arrangements is $3! \times 4! = 144$.	ME-A1 Working with Combinatorics ME11–5, ME11–7 Bands E2–E3
Question 6 **C** At $(0, 0)$, $\frac{dy}{dx} = 0$ and so **A** is incorrect. At $(-1, 1)$, $\frac{dy}{dx} = 0$ and so **B** and **D** are incorrect.	ME-C3 Applications of Calculus ME12–4 Bands E2–E3
Question 7 **B** $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$ $= \frac{1}{2}\overrightarrow{OB} + \overrightarrow{AB}$ $= \frac{1}{2}\overrightarrow{OB} + \overrightarrow{AO} + \overrightarrow{OB}$ $= \frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a} + \underset{\sim}{b}$ $= \frac{3}{2}\underset{\sim}{b} - \underset{\sim}{a}$	ME-V1 Introduction to Vectors ME12–2 Bands E2–E3
Question 8 **D** $\int \frac{1}{\sqrt{4-9x^2}}dx = \int \frac{1}{\sqrt{9\left(\frac{4}{9}-x^2\right)}}dx$ Consider integrals of the form $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$ with $a^2 = \frac{4}{9} \Rightarrow a = \frac{2}{3}$ (> 0). $\frac{1}{3}\sin^{-1}\frac{x}{\frac{2}{3}} = \frac{1}{3}\sin^{-1}\frac{3x}{2}$	ME-C2 Further Calculus Skills ME12–1 Bands E2–E3
Question 9 **C** The parametric equations are: $x = \cos^2 t$ (1) $y = 4\sin^2 t$ (2) $\frac{(2)}{4}$ gives $\frac{y}{4} = \sin^2 t$. (3) (1) + (3) and using $\cos^2 t + \sin^2 t = 1$ gives $x + \frac{y}{4} = 1 \Rightarrow 4x + y = 4$. $0 \le \cos^2 t \le 1$ and so $0 \le x \le 1$. Therefore, $y = 4 - 4x$ for $0 \le x \le 1$.	ME-F1 Further Work with Functions ME11–2 Bands E2–E3

Sample answer	Syllabus content, outcomes and targeted performance bands
Question 10 **C** The diagonals of $OABC$ are given by $\overrightarrow{OB}$ and $\overrightarrow{CA}$. To prove they are perpendicular, form $\overrightarrow{CA} \cdot \overrightarrow{OB}$ and show that it equals zero. $\overrightarrow{CA} = \underset{\sim}{a} - \underset{\sim}{c}$ and $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$. Therefore $\overrightarrow{CA} \cdot \overrightarrow{OB} = 0$ if $(\underset{\sim}{a} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{c}) = 0$.	ME-V1 Introduction to Vectors ME12–2 Bands E2–E3

Section II

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 11	
(a) (i) $f(x) = x^2 - 4x + 6$ is a parabola. Excluding the turning point at (2, 2), for each value of $f(x)$ in the range there are two x-values. Geometrically, this corresponds to a horizontal line intersecting the graph twice. If x and y are swapped, each x-value in the domain will have two y-values. Hence the inverse will not be a function.	ME-F1 Further Work with Functions ME11–1 Bands E2–E3 • Explains using the horizontal line test OR equivalent merit 1
(ii) Use the completing the square method to express $f(x)$ in turning point form: $f(x) = x^2 - 4x + 6 \qquad (x \le 2)$ $= (x-2)^2 + 2$ Swap x and y, then make y the subject. $x = (y-2)^2 + 2$ $x - 2 = (y-2)^2$ $y - 2 = -\sqrt{x-2}$ ($\sqrt{x-2}$ is discarded as $y \le 2$) $y = -\sqrt{x-2} + 2$ $f^{-1}(x) = -\sqrt{x-2} + 2$	ME-F1 Further Work with Functions ME11–1 Bands E2–E3 • Gives the correct solution . . . . .2 • Swaps x and y OR equivalent merit. 1
(iii) The domain is $x \ge 2$ as $x - 2 \ge 0$. The range is $y \le 2$ as $-\sqrt{x-2} \le 0$.	ME-F1 Further Work with Functions ME11–1 Bands E2–E3 • States correct domain AND range. . . . . . .2 • States correct domain OR range 1
(iv) The curves $y = f(x)$ and $y = f^{-1}(x)$ have a common intersection with the line $y = x$. For example, attempting to solve $f(x) = x$ for x: $x^2 - 4x + 6 = x$ $x^2 - 5x + 6 = 0$ $x = 2, 3$ When $x = 2$, $y = 2$ and so $(2, 2)$ lies on the line $y = x$. When $x = 3$, $y = 1$ and so $(3, 1)$ does not lie on the line $y = x$. Therefore the coordinates of P are $(2, 2)$.	ME-F1 Further Work with Functions ME11–1 Bands E2–E3 • Gives the correct solution . . . . .2 • Attempts to solve $f(x) = x$ for x OR equivalent merit. .1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) Let $u = 1 + 2\tan x$. $\frac{du}{dx} = 2\sec^2 x = \frac{2}{\cos^2 x} \Rightarrow dx = \frac{\cos^2 x}{2}du$ When $x = 0$, $u = 1$ and when $x = \frac{\pi}{4}$, $u = 3$. $\int_0^{\frac{\pi}{4}} \frac{1}{(1+2\tan x)^2\cos^2 x}dx = \int_1^3 \frac{1}{2u^2}du$ $= -\left[\frac{1}{2u}\right]_1^3$ $= -\left(\frac{1}{6} - \frac{1}{2}\right)$ $= \frac{1}{3}$	ME-C2 Further Calculus Skills ME12–1 Bands E2–E3 • Gives the correct solution. . . . . 2 • Finds an expression for the integral in terms of u OR equivalent merit 1
(c) Substituting $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ where $t = \tan\frac{1}{2}x$ into $\cos x - \sin x = 1$ and expressing $1 = \frac{1+t^2}{1+t^2}$ gives: $\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} = \frac{1+t^2}{1+t^2}$ $\frac{1-t^2-2t-1-t^2}{1+t^2} = 0$ $\frac{-2(t^2+t)}{1+t^2} = 0$ $t^2 + t = 0$ $t(t+1) = 0$ $t = -1, 0$ $\tan\frac{1}{2}x = -1, 0$ $\tan\frac{1}{2}x = 0 \Rightarrow \frac{1}{2}x = 0, \pi$ $\tan\frac{1}{2}x = -1$ tan is negative in the second quadrant and the related angle is $\frac{\pi}{4}$. $\tan\frac{1}{2}x = -1 \Rightarrow \frac{x}{2} = \frac{3\pi}{4}$ So $x = 0, \frac{3\pi}{2}, 2\pi$.	ME-T3 Trigonometric Equations ME12–3 Bands E2–E3 • Gives the correct solution. . . . . 3 • Determines that $\tan\frac{1}{2}x = -1, 0$ . . . . 2 • Attempts to form a quadratic equation in t 1

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Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(d) (i) Substituting $\underset{\sim}{F} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\underset{\sim}{s} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ into $W = \underset{\sim}{F} \cdot \underset{\sim}{s}$ gives: $W = \underset{\sim}{F} \cdot \underset{\sim}{s}$ $= \begin{pmatrix} 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ $= 20$	ME-V1 Introduction to Vectors ME12–2 Bands E3–E4 • Gives the correct solution 1
(ii) A unit vector in the direction of $\overrightarrow{PQ}$ is $\hat{\underset{\sim}{s}} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Substituting $\underset{\sim}{F} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\hat{\underset{\sim}{s}} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\lvert\underset{\sim}{s}\rvert = 5$ into $W = (\underset{\sim}{F} \cdot \hat{\underset{\sim}{s}})\lvert\underset{\sim}{s}\rvert$ gives: $W = \left(\begin{pmatrix} 4 \\ -2 \end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right)5$ $= 20$	ME-V1 Introduction to Vectors ME12–2 Bands E3–E4 • Gives the correct solution 1
(iii) The component of $\underset{\sim}{F}$ in the direction of l is given by $\left(\frac{\underset{\sim}{F} \cdot \underset{\sim}{s}}{\underset{\sim}{s} \cdot \underset{\sim}{s}}\right)\underset{\sim}{s}$. Substituting $\underset{\sim}{F} \cdot \underset{\sim}{s} = 20$, $\underset{\sim}{s} \cdot \underset{\sim}{s} = 25$ and $\underset{\sim}{s} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ into $\left(\frac{\underset{\sim}{F} \cdot \underset{\sim}{s}}{\underset{\sim}{s} \cdot \underset{\sim}{s}}\right)\underset{\sim}{s}$ gives: $\left(\frac{\underset{\sim}{F} \cdot \underset{\sim}{s}}{\underset{\sim}{s} \cdot \underset{\sim}{s}}\right)\underset{\sim}{s} = \frac{20}{25}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ $= \frac{4}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ $= \begin{pmatrix} 2.4 \\ -3.2 \end{pmatrix}$ Alternatively, the component of $\underset{\sim}{F}$ in the direction of l is $(\underset{\sim}{F} \cdot \hat{\underset{\sim}{s}})\hat{\underset{\sim}{s}}$.	ME-V1 Introduction to Vectors ME12–2 Bands E3–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 12	
(a) (i) Using $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ with $A = B = \frac{\pi x}{8}$ gives: LHS $= \sin\frac{\pi x}{8}\sin\frac{\pi x}{8}$ $= \sin^2\frac{\pi x}{8}$ RHS $= \frac{1}{2}\left[\cos\left(\frac{\pi x}{8} - \frac{\pi x}{8}\right) - \cos\left(\frac{\pi x}{8} + \frac{\pi x}{8}\right)\right]$ $= \frac{1}{2}\left(\cos 0 - \cos\frac{\pi x}{4}\right)$ $= \frac{1}{2}\left(1 - \cos\frac{\pi x}{4}\right)$ So $\sin^2\left(\frac{\pi x}{8}\right) = \frac{1}{2}\left(1 - \cos\frac{\pi x}{4}\right)$.	ME-T2 Further Trigonometric Identities ME11–3 Bands E2–E3 • Demonstrates that LHS $= \sin^2\frac{\pi x}{8}$. AND • Demonstrates that RHS $= \frac{1}{2}\left(1 - \cos\frac{\pi x}{4}\right)$ . . . . 2 • Demonstrates that LHS $= \sin^2\frac{\pi x}{8}$. OR • Demonstrates that RHS $= \frac{1}{2}\left(1 - \cos\frac{\pi x}{4}\right)$ 1
(ii) $A = 6\int_0^8 \sin^2\left(\frac{\pi x}{8}\right)dx$ $= 3\int_0^8 1 - \cos\frac{\pi x}{4}dx$ $= 3\left[x - \frac{4}{\pi}\sin\frac{\pi x}{4}\right]_0^8$ $= 3\left(8 - \frac{4}{\pi}\sin 2\pi - (0 - \sin 0)\right)$ $= 3(8 - 0)$ $= 24$	ME-C2 Further Calculus Skills ME12–1, 12–4 Bands E2–E3 • Gives the correct solution. . . . . 2 • Uses the part (a) (i) result to form a definite integral 1
(b) (i) $\mathrm{E}(\hat{P}) = p$ $= 0.36$ $\mathrm{sd}(\hat{P}) = \sqrt{\frac{0.36 \times 0.64}{25}}$ $= 0.096$	ME-S1 The Binomial Distribution ME12–5 Bands E2–E3 • Correctly shows the mean AND standard deviation. . . . . 2 • Correctly shows the mean OR standard deviation. 1
(ii) Transforming to a standard normal variable, Z, gives: $P\left(Z < \frac{0.12 - 0.36}{0.096}\right) = P(Z < -2.5)$ $= 1 - P(Z < 2.5)$ $= 1 - 0.9938$ $= 0.0062$	ME-S1 The Binomial Distribution ME12–5 Bands E2–E3 • Gives the correct solution. . . . . 2 • Calculates $z = \frac{0.12 - 0.36}{0.096}$. OR • Uses the table appropriately with an incorrect value for z. 1

<table>
<tr><th></th><th>Sample answer</th><th>Syllabus content, outcomes, targeted performance bands and marking guide</th></tr>
<tr><td>(iii)</td><td>The number of adults in the sample who have a mortgage is 25 × 0.36 = 9.
Let X represent the number of adults who have a mortgage and $X \sim \text{Bin}(25, 0.36)$.
$P(X = 9) = \binom{25}{9}(0.36)^9(1 - 0.36)^{16}$
$= 0.1644$</td><td>ME-S1 The Binomial Distribution
ME12–5 Bands E2–E3
• Gives the correct solution 2
• Attempts to find $P(X = 9)$ where $X \sim \text{Bin}(25, 0.36)$ .... 1</td></tr>
<tr><td>(c) (i)</td><td>Rearranging $y = \frac{1}{x^2 + 1}$ to express x^2 in terms of y gives $x^2 = \frac{1}{y} - 1$.
$V = \pi\int_{\frac{1}{2}}^{1}\left(\frac{1}{y} - 1\right)dy$
$= \pi\Big[\ln\lvert y\rvert - y\Big]_{\frac{1}{2}}^{1}$
$= \pi\left(\ln 1 - 1 - \left(\ln\frac{1}{2} - \frac{1}{2}\right)\right)$
$= \pi\left(\ln 2 - \frac{1}{2}\right)$</td><td>ME-C3 Applications of Calculus
ME12–4 Bands E2–E4
• Gives the correct solution 2
• Provides correct integrand for volume of revolution .... 1</td></tr>
<tr><td>(ii)</td><td>Rearranging $y = 1 - \frac{x}{2}$ to express x in terms of y gives $x = 2(1 - y)$.
$V = \pi\int_{\frac{1}{2}}^{1}(4(1 - y)^2)dy$
$= -\frac{4\pi}{3}\Big[(1 - y)^3\Big]_{\frac{1}{2}}^{1}$
$= -\frac{4\pi}{3}\left(0 - \frac{1}{8}\right)$
$= \frac{\pi}{6}$
Alternatively:
The solid formed is a cone of radius 1 and height $\frac{1}{2}$.
Substituting these values into $V = \frac{1}{3}\pi r^2 h$ gives:
$V = \frac{1}{3} \times \pi \times 1^2 \times \frac{1}{2}$
$= \frac{\pi}{6}$</td><td>ME-C3 Applications of Calculus
ME12–4 Bands E2–E4
• Gives the correct solution 2
• Provides correct integrand for volume of revolution OR equivalent merit....... 1</td></tr>
<tr><td>(iii)</td><td>From the diagram, it can be reasoned that $\pi\left(\ln 2 - \frac{1}{2}\right) > \frac{\pi}{6}$.
So $\ln 2 - \frac{1}{2} > \frac{1}{6} \Rightarrow \ln 2 > \frac{2}{3}$.</td><td>ME-C3 Applications of Calculus
ME12–4 Bands E2–E4
• Gives the correct solution .... 1</td></tr>
</table>

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 13	
(a) (i) Substituting $x = 12$, $V = 21$ and $t = T$ into $x = Vt\cos\theta$ gives $12 = 21T\cos\theta \Rightarrow T = \frac{12}{21\cos\theta}$. Cancelling and using $\frac{1}{\cos\theta} = \sec\theta$ gives $T = \frac{4}{7}\sec\theta$.	ME-V1 Introduction to Vectors ME12–2 Bands E2–E3 • Gives the correct solution. 1
(ii) Substituting $y = 2$, $V = 21$ and $t = T$ into $y = Vt\sin\theta - \frac{1}{2}gt^2$ gives $2 = 21T\sin\theta - 4.9T^2$. Substituting $T = \frac{4}{7}\sec\theta$ into $2 = 21T\sin\theta - 4.9T^2$ gives: $2 = 21\left(\frac{4}{7}\sec\theta\right)\sin\theta - 4.9\left(\frac{4}{7}\sec\theta\right)^2$ $= 12\tan\theta - \frac{8}{5}\sec^2\theta$ $= 12\tan\theta - \frac{8}{5}(1 + \tan^2\theta)$ $10 = 60\tan\theta - 8(1 + \tan^2\theta)$ $0 = 8\tan^2\theta - 60\tan\theta + 18$ So $4\tan^2\theta - 30\tan\theta + 9 = 0$.	ME-V1 Introduction to Vectors ME12–2 Bands E2–E3 • Gives the correct solution. . . . . 2 • Substitutes $T = \frac{4}{7}\sec\theta$ into $2 = 21T\sin\theta - 4.9T^2$ and attempts to form a quadratic in $\tan\theta$ 1
(iii) Using the quadratic formula to solve $4\tan^2\theta - 30\tan\theta + 9 = 0$ for $\tan\theta$ gives $\tan\theta = \frac{15 \pm 3\sqrt{21}}{4} (= 0.3130\ldots, 7.1869\ldots)$. The shortest flight time occurs for $\theta = \tan^{-1}\left(\frac{15 - 3\sqrt{21}}{4}\right) (= 0.3130\ldots)$. Substituting $\theta = \tan^{-1}\left(\frac{15 - 3\sqrt{21}}{4}\right) (= 0.3130\ldots)$ into $T = \frac{4}{7}\sec\theta$ gives $T = 0.60$ (s).	ME-V1 Introduction to Vectors ME12–2, 12–6 Bands E2–E3 • Gives the correct solution. . . . . 2 • Correctly solves quadratic equation to obtain two values for $\tan\theta$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) Consider $n = 1$. $\text{LHS} = \frac{2}{1 \times 3} = \frac{2}{3}$ and $\text{RHS} = \frac{3}{2} - \frac{2(1)+3}{(1+1)(1+2)} = \frac{4}{6} = \frac{2}{3} = \text{LHS}$. The statement is true when $n = 1$. Suppose true for $n = k$. So $\frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} = \frac{3}{2} - \frac{2k+3}{(k+1)(k+2)}$. Show it is true for $n = k + 1$; that is, $\frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} + \frac{2}{(k+1)(k+3)} = \frac{3}{2} - \frac{2(k+1)+3}{((k+1)+1)((k+1)+2)}$ $\text{LHS} = \frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} + \frac{2}{(k+1)(k+3)}$ $= \frac{3}{2} - \frac{2k+3}{(k+1)(k+2)} + \frac{2}{(k+1)(k+3)}$ $= \frac{3}{2} - \frac{(2k+3)(k+3) - 2(k+2)}{(k+1)(k+2)(k+3)}$ $= \frac{3}{2} - \frac{2k^2 + 7k + 5}{(k+1)(k+2)(k+3)}$ $= \frac{3}{2} - \frac{(2k+5)(k+1)}{(k+1)(k+2)(k+3)}$ $= \frac{3}{2} - \frac{2k+5}{(k+2)(k+3)}$ $= \frac{3}{2} - \frac{2(k+1)+3}{((k+1)+1)((k+1)+2)}$ $= \text{RHS}$ If true for $n = k$, then true for $n = k + 1$. Hence, by mathematical induction, true for $n \geq 1$.	ME-P1 Proof by Mathematical Induction ME12–1 Bands E2–E4 • Gives the correct proof . . . . .3 • Establishes the inductive step OR equivalent merit. . . . . .2 • Establishes the $n = 1$ case OR equivalent merit. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) Use of $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ with $A = B = \theta$. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$ Use of $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ with $A = 2\theta$ and $B = \theta$. $\tan 3\theta = \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$ $= \dfrac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \dfrac{2\tan\theta}{1-\tan^2\theta}\tan\theta}$ $= \dfrac{\dfrac{2\tan\theta + \tan\theta(1-\tan^2\theta)}{1-\tan^2\theta}}{\dfrac{(1-\tan^2\theta) - 2\tan^2\theta}{1-\tan^2\theta}}$ $= \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$	ME-T3 Trigonometric Equations ME12–3 Bands E2–E4 • Gives the correct proof . . . . 3 • Obtains a correct unsimplified expression for $\tan 3\theta$ involving only $\tan^2\theta$ and $\tan\theta$ OR equivalent merit . . . . 2 • Obtains a correct expression for $\tan 3\theta$ involving only $\tan 2\theta$ and $\tan\theta$ OR equivalent merit. 1
(ii) Consider $x^3 - 3x^2 - 3x + 1 = 0$ with $x = \tan\theta$. $\tan^3\theta - 3\tan^2\theta - 3\tan\theta + 1 = 0$ $\dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 1$ So $\tan 3\theta = 1$ and finding the roots of $\tan 3\theta = 1$ corresponds to finding the roots of the cubic equation where $x = \tan\theta$. $3\theta = \tan^{-1}1 + k\pi$ where k is an integer $\theta = \dfrac{\pi}{12} + \dfrac{k\pi}{3}$ $= \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{9\pi}{12}$ $\tan\dfrac{3\pi}{4} = -1$ and so one factor of the cube is $x + 1$. So $x^3 - 3x^2 - 3x + 1 = (x+1)(x^2 - 4x + 1)$. So $\tan\dfrac{\pi}{12}$ and $\tan\dfrac{5\pi}{12}$ are the roots of $x^2 - 4x + 1 = 0$. Solving the quadratic equation $x^2 - 4x + 1 = 0$ for x gives $x = 2 \pm \sqrt{3}$. Since $\tan\dfrac{\pi}{12} < \tan\dfrac{5\pi}{12}$, $\tan\dfrac{\pi}{12}$ is the smaller root and $x = 2 - \sqrt{3}$.	ME-T3 Trigonometric Equations ME12–3 Bands E2–E4 • Gives correct exact value of $\tan\dfrac{\pi}{12}$ . . . . 4 • Gives correct solutions to the cubic equation. . . . . 3 • Deduces that $\theta = \dfrac{\pi}{12} + \dfrac{k\pi}{3}$ where k is an integer OR equivalent merit . . . . . 2 • Deduces that $\tan 3\theta = 1$ 1

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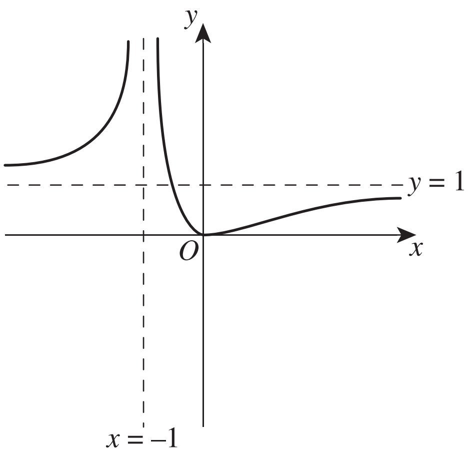
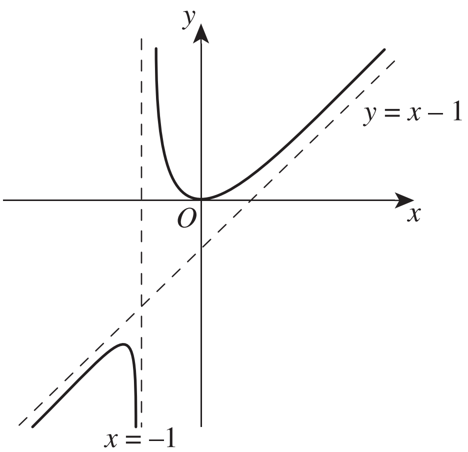
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 14	
(a) (i) 	ME-F1 Further Work with Functions ME11–2, 11–7 Bands E2–E4 • Sketches correct graph with asymptotes at $x = -1$ and $y = 1$ . . . . .2 • Shows minimum turning point at origin OR equivalent merit 1
(ii) 	ME-F1 Further Work with Functions ME11–2, 11–7 Bands E2–E4 • Sketches correct graph with asymptotes at $x = -1$ and $y = x - 1$ . . . . .2 • Shows minimum turning point at origin OR equivalent merit 1
(iii) $(f(x))^2 = f(x) \Rightarrow f(x)(f(x) - 1) = 0$ So $f(x) = 1$ or $f(x) = 0$. $-\frac{x}{x+1} = 1 \Rightarrow x = -\frac{1}{2}$ Hence $x = -\frac{1}{2}$ or $x = 0$. **OR** The graphs of $y = f(x)$ and $y = (f(x))^2$ intersect at O, where $x = 0$. The graphs of $y = f(x)$ and $y = (f(x))^2$ intersect on the line $y = 1$, where $x = -\frac{1}{2}$.	ME-F1 Further Work with Functions ME11–2, 11–7 Bands E2–E4 • Gives the correct solution 1
(b) (i) Start with the RHS and show that it equals the LHS. $\text{RHS} = \frac{1}{50}\left(\frac{(50 - A) + A}{A(50 - A)}\right)$ $= \frac{1}{A(50 - A)}$ $= \text{LHS}$	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) This is a differential equation of the form $\frac{dA}{dt} = g(A)$. Attempt to separate variables and integrate both sides. $\int 1\,dt = \int \frac{25}{A(50-A)}dA$ $t = \frac{1}{2}\int\left(\frac{1}{A} + \frac{1}{50-A}\right)dA$ (using the part (i) result) $= \frac{1}{2}(\ln\lvert A\rvert - \ln\lvert 50-A\rvert) + c$ $= \frac{1}{2}\ln\left\lvert\frac{A}{50-A}\right\rvert + c$ Rearranging gives $A_0 e^{2t} = \frac{A}{50-A}$ where $A_0 = e^{-2c}$ and hence $A_0 > 0$. When $t = 0$, $A = \frac{1}{2}$ and so $A_0 = \frac{1}{99}$. *Note: There are various possible ways to find the value of the constant.* $e^{2t} = \frac{99A}{50-A}$ $99Ae^{-2t} = 50 - A$ $A(1 + 99e^{-2t}) = 50$ So $A = \frac{50}{1 + 99e^{-2t}}$.	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives correct solution. . . . . 3 • Correctly applies initial condition . . . . . . 2 • Uses the part (b) (i) result and separation of variables to find t in terms of A 1
(iii) As $t \to \infty$, $1 + 99e^{-2t} \to 1$ and so $A \to \frac{50}{1} = 50$. The limiting area of the bacteria colony is 50 cm^2.	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives the correct solution. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iv) The graph of $\frac{dA}{dt}$ versus A (inverted parabola) has a maximum at $A = 25$. It requires us to find the value of t such that $25 = \frac{50}{1+99e^{-2t}}$. $25(1+99e^{-2t}) = 50$ $1+99e^{-2t} = 2$ $e^{-2t} = \frac{1}{99}$ $e^{2t} = 99$ $t = \frac{1}{2}\ln 99$ (days) The rate of change of the area is at its maximum at $t = \frac{1}{2}\ln 99$ (days). *Note: There are other valid but more time-consuming methods of determining this solution.* *Method 1:* *Finding* $\frac{d^2A}{dt^2} = \frac{1}{25^2}A(50-A)(50-2A)$, *determining that* $\frac{dA}{dt}$ *is a maximum when* $A = 25$ *and then solving for t as above.* *Method 2:* *Determining the value of t when the (non-stationary) point of inflection occurs by finding* $\frac{d^2A}{dt^2}$ *in terms of t and then finding the value of t such that* $\frac{d^2A}{dt^2} = 0$.	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives the correct solution . . . . .2 • Recognises that the graph of $\frac{dA}{dt}$ versus A has a maximum at $A = 25$ OR equivalent merit. 1
(c) From the table, $f(x) = g^{-1}(x)$ and so $f(-1) = g^{-1}(-1) = 0$. $f'(-1) = \frac{1}{g'(f(-1))}$ $= \frac{1}{g'(0)}$ $= \frac{1}{\frac{1}{2}}$ $= 2$	ME-C2 Further Calculus Skills ME12–1 Bands E2–E4 • Gives the correct solution . . . . .3 • Determines $f(-1) = g^{-1}(-1) = 0$ AND $f'(-1) = \frac{1}{g'(f(-1))}$ . . . . .2 • Determines $f(-1) = g^{-1}(-1) = 0$ OR $f'(-1) = \frac{1}{g'(f(-1))}$ 1