Project Report – Yahav Ben Shimon

1) Introduction

Micro-electromechanical systems (MEMS) hold a key role in a variety of fields as sensors and actuators. Resonators are one of MEMS's main applications, which are typically excited near their natural frequency where the dynamic response is amplified. Actuation of resonators can be achieved by diverse means while the most typical one is electrostatic force. Despite the relatively easy implementation of electrostatic force, its nature is non-linear.

This report will examine the influence of a time-dependent external electrostatic force on a damped parallel plate capacitor. The different chapters of the report will discuss the dynamical model of the problem, different methods for equilibrium analysis, asymptotic approximation analysis, and different aspects of the dynamical behavior via numerical solution.

2) Model

Simple capacitor actuator can be modeled as follows:

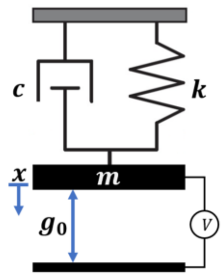

Fig 1: schematic drawing of parallel -plate resonator.

By using newtons laws, the governing equation of motion could be formulated:

(2.1)
$$m\ddot{x} + c\dot{x} + kx = F_{es}$$

Where m is the mass, c is the dimensional damping, k is the dimensional spring constant and F_{es} is the electrostatic force.

The electrostatic force can be derived using electrical potential energy:

(2.2)
$$F_{es} = \frac{dU}{dx} = \frac{1}{2} \cdot \frac{\epsilon_0 A}{(g_0 - x)^2} \cdot V^2$$

Where ϵ_0 is the vacuum permittivity, A is the surface of the plate, V is the voltage and g_0 is the initial gap.

Dividing the m from equation (1) (for a matter of convenience I merged the constant parameter to one, $C = \frac{\epsilon_0 A}{2m}$):

(2.3)
$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \frac{CV^2}{(g_0 - x)^2}$$

Where ζ is the damping ratio and ω_0 is the undamped angular frequency which is equal to: $\sqrt{\frac{k}{m}}$

inserting a time dependent voltage, $V(t) = V_{AC}\sin(\omega t) + V_{DC}$, (assuming $V_{DC} \gg V_{AC}$) to equation (3):

(2.4)
$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \frac{C}{(g_0 - x)^2} \cdot (V_{DC}^2 + 2V_{AC}V_{DC}\sin(\omega t))$$

Equation (2.4) indicates on two features of this given system. First, $2\zeta\omega_0 > 0$ what means that this is dissipative system. Second, at this form the driven force is external.

For the nondimensional form of equation (4) ill normalized the following parameters:

(2.5) $$\hat{x} = \frac{x}{g_0} \qquad \hat{t} = \omega_0 t \qquad \Omega = \frac{\omega}{\omega_0}$$

Their time derivative will be:

(2.6) $$\dot{x} = \dot{\hat{x}} \cdot g_0 \cdot \omega_0 \qquad \ddot{x} = \ddot{\hat{x}} \cdot g_0 \cdot \omega_0^2$$

Substituting in equation (4) will provide the following equation:

(2.7) $$g_0\omega_0^2\ddot{\hat{x}} + 2\zeta\omega_0^2 g_0\dot{\hat{x}} + \omega_0^2 g_0\hat{x} = \frac{C}{(g_0 - \hat{x}g_0)^2} \cdot (V_{DC}^2 + 2V_{AC}V_{DC}\sin(\omega t))$$

Dividing equation (7) in $g_0 \cdot \omega_0^2$:

(2.8) $$\ddot{\hat{x}} + 2\zeta\dot{\hat{x}} + \hat{x} = \frac{\alpha}{(1-\hat{x})^2} \cdot (V_{DC}^2 + 2V_{AC}V_{DC}\sin(\Omega\hat{t}))$$

Where $\alpha = \frac{C}{g_0^3 \cdot \omega_0^2}$

3) Equilibrium Analysis

The DC part of the voltage induced a constant force which moves the equilibrium point while the AC part induce oscillations. For an equilibrium analysis there is need to discard the oscillatory part of the electrostatic force. Let's define phase-space variable:

(3.1)
$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

Therefor the time derivatives of the phase-space variables will looks as follow:

(3.2)
$$\dot{\bar{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{CV_{DC}^2}{(g_0 - x_1)^2} - 2\zeta\omega_0 x_2 - \omega_0^2 x_1 \end{pmatrix}$$

The equilibrium points can be found by setting $\dot{\bar{x}} = 0$:

(3.3)
$$\omega_0^2 x_{st} = \frac{CV_{DC}^2}{(g_0 - x_{st})^2}$$

Ill define two assistance functions:

(3.5)
$$f_1(x_{st}) = x_{st}^3 - 2g_0 x_{st}^2 + g_0^2 x_{st}$$
$$f_2(x_{st}) = \frac{CV_{DC}^2}{\omega_0^2}$$

For $g_0 > 0$ the extremum points of $f_1(x_{st})$ is on the right side of the plane and their equal to $\left(\frac{1}{3}g, \frac{4}{27}g^3\right)$ and $(g, 0)$. It's easy to see that the intersection between $f_1(x_{st})$ and $f_2(x_{st})$for $x_{static} > g$ is not a physical solution (the capacitor cannot penetrate the substrate). At $0 < x_{st} < g$ there can be 3 possible conditions. For the first case $f_2(x_{st})$ isn't transect $f_1(x_{st})$ which equivalent to the lack of equilibrium points. for this case the capacitor will crush into the substrate, effect named as "Pull-in". for the second case $f_2(x_{st})$ is interface $f_1(x_{st})$ at $x_{st} = \frac{1}{3}g$. at this value, the voltage is equal to its maximum level before the pull-in effect. The magnitude of the critical voltage can be calculated by comparing $f_2(x_{st})$to the local maximum of $f_1(x_{st})$:

(3.6)
$$V_{cr} = \sqrt{\frac{4\omega_0^2 g^3}{27C}}$$

For the third case, y_2 is smaller than the local maximum of y_1 and there is two intersection which indicates for two equilibrium points.

On one hand, the method above is beneficial to find the equilibrium points. On another hand, there is still a lack of information like the stability of the points. For that, there is a need to use the potential energy function. By examine the behavior of the potential energy along space, the equilibrium points can be classified easily as stable (valley), semi-stable (inflections) or unstable-one (hill).

Integrating equation over x (3.3) will give the potential energy equation:

(3.7) $$U(x) = \frac{CV_{DC}^2}{(x_{st} - g_0)} + \frac{\omega_0^2}{2} x_{st}^2$$

For matter of convenience let's set the variables to numbers:

(3.8) $$V_{DC} = 1 - 2 \ \ ; \ \ C = 1 \ \ ; \ \ \omega_0 = 1 \ \ ; \ \ g_0 = 3 \ \ ; \ \ x_{st} = 0 - 3$$

Using MATLAB, I have plotted the potential for different voltages:

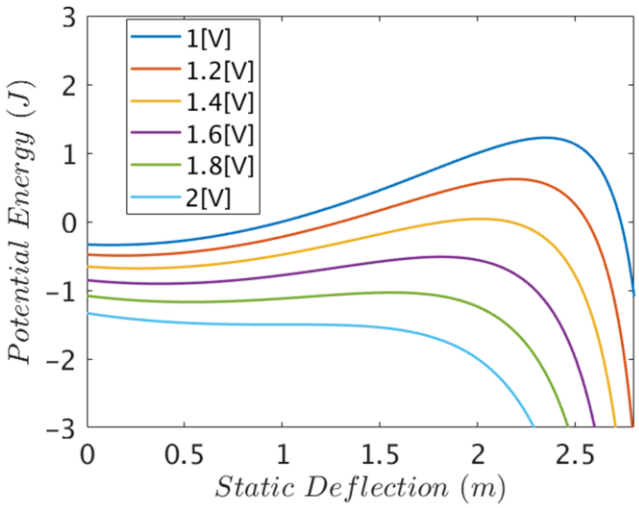

Fig 2: Potential energy function for different DC voltages.

Notice that it's easy to see from Fig. 2 that there is 2 points of equilibrium for voltages below $2[V]$. The maximum point of potential energy is the unstable one while the minimum point is the stable one. As the DC voltage is increasing those points getting closer to the point which they merged to the bifurcation point.

Fig 3. Is the calculation of the equilibrium points using equation (3.8) parameters. It's easy to see that the value that equation (3.6) predict is suitable to the bifurcation point ($V_{pull-in} = 2V$).

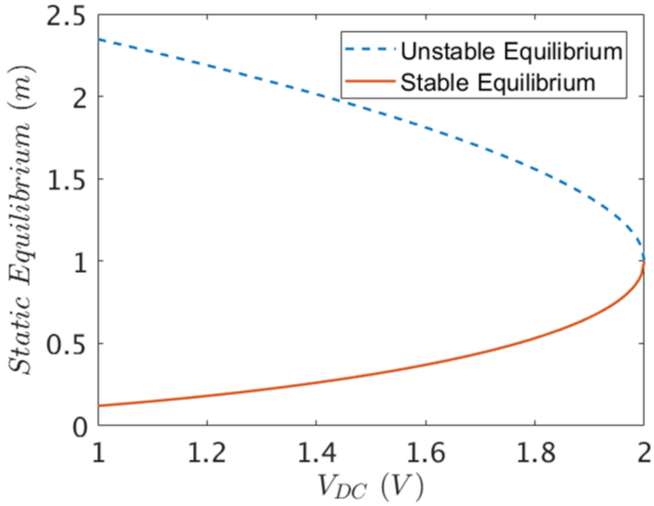

Fig 3: Bifurcation diagram.

Another way of determining the stability of the equilibrium points is by using the Jacobine of the phase space variables

(3.8)
$$\bar{\bar{J}} = \begin{bmatrix} 0 & 1 \\ \frac{2CV_{DC}^2}{(g_0 - x_1)^3} - \omega_0^2 & -2\zeta\omega_0 \end{bmatrix}$$

The trace of the Jacobine is:

(3.9)
$$\tau(\bar{\bar{J}}) = -2\zeta\omega_0$$

The determinant of the Jacobine is:

(3.9)
$$\Delta(\bar{\bar{J}}) = \omega_0^2 - \frac{2CV_{DC}^2}{(g_0 - x_1)^3}$$

For the case that $\Delta(\bar{\bar{J}}) \neq 0$ and $\Delta(\bar{\bar{J}}) \neq (\zeta\omega_0)^2$ the Jacobine can predict the stability of the equilibrium point reliably:

- For $\Delta\left(\bar{\bar{J}}\big|_{\bar{x}=x_{st}}\right) < 0$ the equilibrium is an unstable saddle.
- For $\Delta\left(\bar{\bar{J}}\big|_{\bar{x}=x_{st}}\right) > 0$ and $\Delta(\bar{\bar{J}}) > (\zeta\omega_0)^2$ the equilibrium is a stable spiral.
- For $\Delta\left(\bar{\bar{J}}\big|_{\bar{x}=x_{st}}\right) > 0$ and $\Delta(\bar{\bar{J}}) < (\zeta\omega_0)^2$ the equilibrium is a stable cross section.

4) Weakly Nonlinear Asymptotic Analysis

The first step is to eliminate the DC term of the voltage, for that ill multiply equation (2.4) in $(g_0 - x)^2$:

(4.1) $$(g_0 - x)^2(\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x) = C \cdot V_{DC}^2 + 2C \cdot V_{AC}V_{DC}\sin(\omega t)$$

Let's define a new degree of freedom which include the static deflection:

(4.2) $$x = u + x_{st} \ ; \ \dot{x} = \dot{u} \ ; \ddot{x} = \ddot{u}$$

Substituting equation (4.2) to (4.1):

(4.3) $$(g_0 - x_{st} - u)^2\big(\ddot{u} + 2\zeta\omega_0\dot{u} + \omega_0^2(u + x_{st})\big) = C \cdot V_{DC}^2 + 2C \cdot V_{AC}V_{DC}\sin(\omega t)$$

*For the sake of simplicity Ill define new parameters $\beta = (g_0 - x_{st})\ ; \Gamma = \zeta\omega_0$

After some mathematical manipulations on equation (4.3) (the full process is in the appendix) we will get:

(4.4) $$\begin{aligned}&\epsilon(\ddot{v} + \omega_0^2 v) = \\ &\epsilon[2\omega_0^2 x_{st}\beta^{-1}v - 2\Gamma\dot{v}] + \\ &\epsilon^2\big[2\beta^{-1}v\ddot{v} + 2\omega_0^2\beta^{-1}v^2 + 4\Gamma\beta^{-1}v\dot{v} + \omega_0^2 x_{st}\beta^{-2}v^2\big] + \\ &\epsilon^3[v^2\ddot{v}\beta^{-2} + 2\Gamma v^2\dot{v}\beta^{-2} + \omega_0^2 v^3\beta^{-2}] + \\ &2C\beta^{-2} \cdot V_{AC}V_{DC}\sin(\omega t)\end{aligned}$$

let's assume a solution of the following form:

(4.5) $$\begin{aligned}&A = \frac{a}{2}e^{i\varphi}\ ;\ \dot{A} = \frac{\dot{a}}{2}e^{i\varphi} + i\dot{\varphi}\frac{a}{2}e^{i\varphi}\ ;\ |A|^2 = \frac{a^2}{4} \\ &v(t) = Ae^{i\omega t} + \bar{A}e^{-i\omega t} \\ &\dot{v}(t) = i\omega Ae^{i\omega t} - i\omega\bar{A}e^{-i\omega t} + \dot{A}e^{i\omega t} + \dot{\bar{A}}e^{-i\omega t}\end{aligned}$$

By forcing $\dot{A}e^{i\omega t} + \dot{\bar{A}}e^{-i\omega t} = 0$:

(4.6) $$\begin{aligned}&\dot{v}(t) = i\omega Ae^{i\omega t} - i\omega\bar{A}e^{-i\omega t} \\ &\ddot{v}(t) = -\omega^2(Ae^{i\omega t} + \bar{A}e^{-i\omega t}) + i\omega\dot{A}e^{i\omega t} - i\omega\dot{\bar{A}}\,e^{-i\omega t} \\ &\ddot{v}(t) = -\omega^2 v + 2i\omega\dot{A}e^{i\omega t}\end{aligned}$$

Using Euler identity:

(4.7) $$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

By substituting the assumed solution to the left-side of the equation:

(4.8) $$(\omega_0^2 - \omega^2)(Ae^{i\omega t} + \bar{A}e^{-i\omega t}) + 2i\omega\dot{A}e^{i\omega t}$$

Neglecting the non-resonative terms from equation (4.8):

(4.9) $$(\omega_0^2 - \omega^2)Ae^{i\omega t} + 2i\omega\dot{A}e^{i\omega t}$$

substituting the assumed solution to the right-side of the equation for terms of order ϵ:

$$(4.10)\quad 2\omega_0^2 x_{st}\beta^{-1}(Ae^{i\omega t}+\bar{A}e^{-i\omega t})-2\Gamma(i\omega Ae^{i\omega t}-i\omega\bar{A}e^{-i\omega t})$$

Neglecting the non-resonative terms from equation (4.10):

$$(4.11)\quad 2\omega_0^2 x_{st}\beta^{-1}Ae^{i\omega t}-2\Gamma i\omega Ae^{i\omega t}$$

substituting the assumed solution to the right-side of the equation for terms of order ϵ^2:

$$(4.12)\quad \begin{aligned} &2\beta^{-1}v\ddot{v}=2\beta^{-1}\left[(Ae^{i\omega t}+\bar{A}e^{-i\omega t})\left(-\omega^2Ae^{i\omega t}-\omega^2\bar{A}e^{-i\omega t}+2i\omega\dot{A}e^{i\omega t}\right)\right]\\ &2\omega_0^2\beta^{-1}v^2=\ 2\omega_0^2\beta^{-1}[(A^2e^{2i\omega t}\ +2|A|^2+\bar{A}^2e^{-2i\omega t})]\\ &4\Gamma\beta^{-1}v\dot{v}=4\Gamma\beta^{-1}[(Ae^{i\omega t}+\bar{A}e^{-i\omega t}\ )(i\omega Ae^{i\omega t}-i\omega\bar{A}e^{-i\omega t})]\\ &\omega_0^2x_{st}\beta^{-2}v^2=\omega_0^2x_{st}\beta^{-2}[(A^2e^{2i\omega t}\ +2|A|^2+\bar{A}^2e^{-2i\omega t})] \end{aligned}$$

By examine equation (4.12) one can conclude that there is no resoantive terms.

Inserting the assumed solution to the right-side of the equation for terms of order ϵ^3:

$$(4.13)\quad \begin{aligned} &\beta^{-2}v^2\ddot{v}=\beta^{-2}\left[\left(A^2e^{2i\omega t}\ +2|A|^2+\bar{A}^2e^{-2i\omega t}\right)\left(-\omega^2Ae^{i\omega t}-\omega^2\bar{A}e^{-i\omega t}+2i\omega\dot{A}e^{i\omega t}\right)\right]\\ &2\Gamma\beta^{-2}v^2\dot{v}=2\Gamma\beta^{-2}\left[\left(A^2e^{2i\omega t}\ +2|A|^2+\bar{A}^2e^{-2i\omega t}\right)\left(i\omega Ae^{i\omega t}-i\omega\bar{A}e^{-i\omega t}\right)\right]\\ &\omega_0^2\beta^{-2}v^3=\omega_0^2\beta^{-2}\left(A^3e^{3i\omega t}+3A|A|^2e^{i\omega t}+3\bar{A}|A|^2e^{-i\omega t}+\bar{A}^3e^{-3i\omega t}\right) \end{aligned}$$

Neglecting the non-resonative terms from equation (4.13):

$$(4.14)\quad -3\beta^{-2}A|A|^2\omega^2e^{i\omega t}+4\beta^{-2}i\omega|A|^2\dot{A}e^{i\omega t}+6\Gamma\beta^{-2}A|A|^2i\omega e^{i\omega t}+3\omega_0^2\beta^{-2}A|A|^2e^{i\omega t}$$

Using equation (4.7):

$$(4.15)\quad 2C\beta^{-2}\cdot V_{AC}V_{DC}\sin(\omega t)=-iC\beta^{-2}\cdot V_{AC}V_{DC}e^{i\omega t}$$

After neglecting all the non-resonative terms in equation (4.4), ill divide $e^{i\omega t}$ from the equation (4.4):

$$(4.16)\quad \begin{aligned} &(\epsilon2\omega-\epsilon^3 4\beta^{-2}\omega|A|^2)i\dot{A}=-\epsilon(\omega_0^2-\omega^2)A+\epsilon2\omega_0^2x_{st}\beta^{-1}A-\epsilon2\Gamma i\omega A\\ &(\epsilon^3 3\omega_0^2\beta^{-2}-\epsilon^3 3\beta^{-2}\omega^2)A|A|^2+\epsilon^3 6\Gamma\beta^{-2}A|A|^2i\omega+\\ &-iC\beta^{-2}\cdot V_{AC}V_{DC} \end{aligned}$$

By using the identity of A from equation (4.5):

$$(4.17)\quad \begin{gathered}\left(\epsilon 2\omega-\epsilon^3 4\beta^{-2}\omega\frac{a^2}{4}\right)i\left(\frac{\dot{a}}{2}+i\dot{\varphi}\frac{a}{2}\right)= \\ -\epsilon(\omega_0^2-\omega^2)\frac{a}{2}+\epsilon 2\omega_0^2 x_{st}\beta^{-1}\frac{a}{2}-\epsilon 2\Gamma i\omega\frac{a}{2}+ \\ (\epsilon^3 3\omega_0^2\beta^{-2}-\epsilon^3 3\beta^{-2}\omega^2)\frac{a^3}{8}+\epsilon^3 6\Gamma\beta^{-2}\frac{a^3}{8}i\omega- \\ -C\beta^{-2}\cdot V_{AC}V_{DC}\cos i(\varphi)-C\beta^{-2}\cdot V_{AC}V_{DC}\sin(\varphi)\end{gathered}$$

Combining like terms:

$$(4.18)\quad \begin{gathered}\left(\epsilon 2\omega-\epsilon^3 4\beta^{-2}\omega\frac{a^2}{4}\right)\left(\frac{\dot{a}}{2}+i\dot{\varphi}\frac{a}{2}\right)= \\ \left[-\epsilon 2\Gamma\omega\frac{a}{2}+\epsilon^3 6\Gamma\beta^{-2}\frac{a^3}{8}\omega-C\beta^{-2}\cdot V_{AC}V_{DC}\cos(\varphi)\right]+ \\ \left[-(\epsilon^3 3\omega_0^2\beta^{-2}-\epsilon^3 3\beta^{-2}\omega^2)\frac{a^3}{8}+\epsilon(\omega_0^2-\omega^2)\frac{a}{2}-\epsilon 2\omega_0^2 x_{st}\beta^{-1}\frac{a}{2}+C\beta^{-2}\cdot V_{AC}V_{DC}\sin(\varphi)\right]i\end{gathered}$$

The equilibrium points ($\dot{a}=0$, $\dot{\varphi}=0$) need to be found (because of the lack of computational abilities and time ill omit terms of $O(\epsilon^2)$):

$$(4.19)\quad \begin{gathered}I)-\epsilon 2\Gamma\omega\frac{a}{2}=C\beta^{-2}\cdot V_{AC}V_{DC}\cos(\varphi) \\ II)-\epsilon((\omega_0^2-\omega^2)-2\omega_0^2 x_{st}\beta^{-1})\frac{a}{2}=C\beta^{-2}\cdot V_{AC}V_{DC}\sin(\varphi)\end{gathered}$$

Summarizing the quadratic power of (4.19) will give the first equation while dividing (4.19) will lead to the second one:

$$(4.20)\quad \begin{gathered}I)\epsilon^2[(2\Gamma\omega)^2+((\omega_0^2-\omega^2)-2\omega_0^2 x_{st}\beta^{-1})^2]\frac{a^2}{4}=C^2\beta^{-4}\cdot V_{AC}^2V_{DC}^2 \\ II)\tan(\varphi)=\frac{((\omega_0^2-\omega^2)-2\omega_0^2 x_{st}\beta^{-1})}{2\Gamma\omega}\end{gathered}$$

Isolating a and φ leaves us with:

$$(4.21)\quad \begin{gathered}I)a^2=\frac{4C^2\beta^{-4}\cdot V_{AC}^2V_{DC}^2}{(2\Gamma\omega)^2+\left((1-2x_{st}\beta^{-1})\omega_0^2-\omega^2\right)^2} \\ II)\tan(\varphi)=\frac{((\omega_0^2-\omega^2)-2\omega_0^2 x_{st}\beta^{-1})}{2\Gamma\omega}\end{gathered}$$

Let's define new parameters for equation (4.20) $C^2\beta^{-4}\cdot V_{AC}^2V_{DC}^2=\eta$; $x_{st}\beta^{-1}=\mu$

$$(4.22)\quad a=\frac{2\,\eta^{0.5}}{\sqrt{(2\Gamma\omega)^2+((1-2\mu)\omega_0^2-\omega^2)^2}}$$

It's easy to see that the maximum amplitude occurs when $\omega=\omega_0$.

5) <u>Numerical Analysis</u>

For the numerical analysis there is need to set the parameters for chosen values:

(5.1) $$C = 1\,; V_{AC} = 0.1\,; \omega_0 = 10\,; \zeta = 0.2\,; g_0 = 3$$

Inserting those parameters to the governing equation of motion (2.4):

(5.2) $$\ddot{x} + 4\dot{x} + \omega_0^2 x = \frac{1}{(3-x)^2} \cdot (V_{DC}^2 + 0.2 \cdot V_{DC} \sin{(\omega t)})$$

Now, the order of equation (5.2) needs to be reduced:

(5.3) $$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$
$$\dot{\bar{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{1}{(3-x_1)^2} \cdot (V_{DC}^2 + 0.2 \cdot V_{DC} \sin(\omega t)) - 4\dot{x}_2 - \omega_0^2 x_1 \end{pmatrix}$$

Finally, the solver of MATLAB that have been used is ODE45 for initial condition of:

(5.4) $$\bar{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The parameters of the asymptotic approximation (4.22) are affected by the static deflection. Because of that, they will be calculated for every simulation based on equation (3.3).

The first simulation is the frequency response of the approximation and the numerical solution. The amplitude of the numerical solution acquired by calculating half of the time signal magnitude at steady state. By examine fig 4. One can clearly notice the softening effect of the electrostatic force. As the

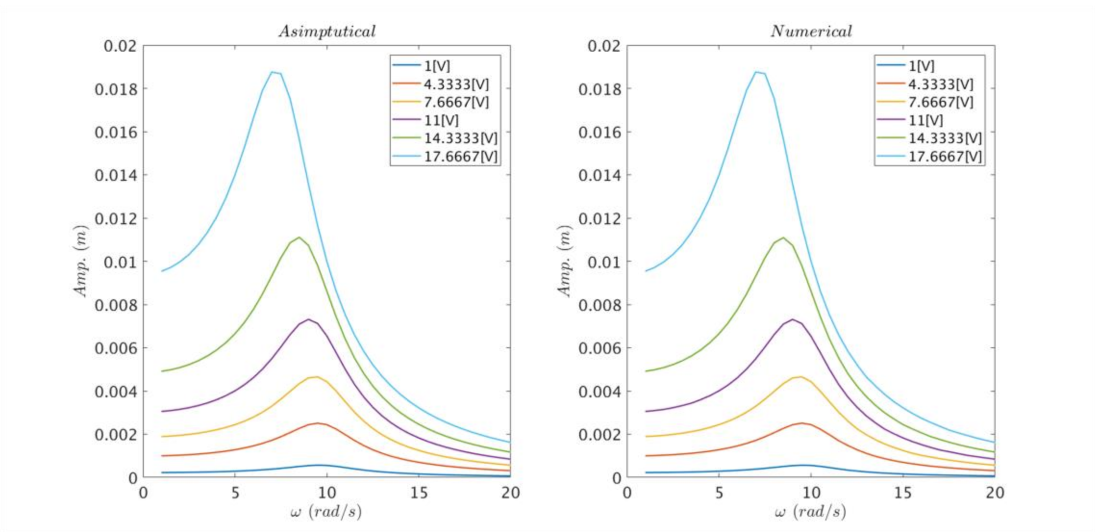

Fig 4: Frequency response for given parameters.

amplitude is increasing the resonance frequency decrease. The shifting of the resonance frequency can be describe using effective frequency:

$$\omega_{eff} = (1 - 2x_{st}\beta^{-1})\omega_0 \tag{5.5}$$

Increasing the DC voltage influence the system in 2 ways. on one hand, it is increasing the amplitude (equation (4.22)). On other hand, it is increasing the static deflection which in turn decreasing the effective resonance frequency. Looking at fig.5 one can see that the frequency is tend to 0 around V_{cr} (where the pull-in effects take place, and the capacitor is moved towards the substrate) and to its natural frequency for low V_{DC}.

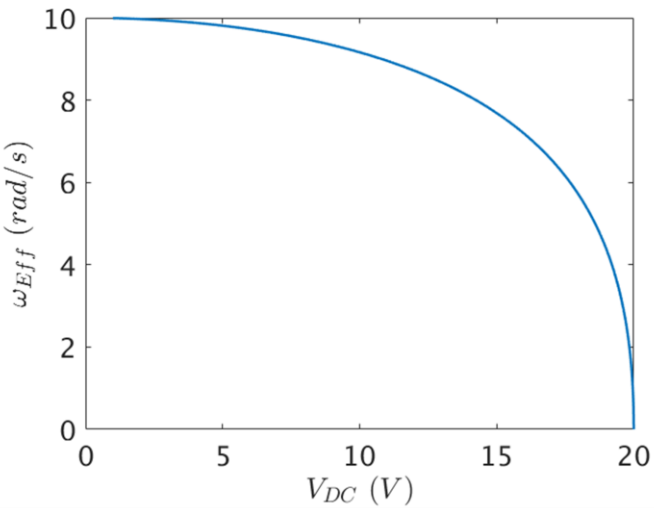

Fig 5: Effective frequency as function of Dc voltage.

Another aspect of the simulation is the difference between the asymptotical approximation to the numerical solution. By looking at fig.6 it is noticeable that for range of $V_{AC} \ll V_{DC} < V_{cr}$ the approximation gives a good indication to the frequency response $\left(Error \sim O(0.1\%)\right)$. The problem of the asymptotic method appears out of this range. For DC voltages that are at the same order of magnitude of the AC voltage there is new dominant terms that influence the frequency response (for example a resonance at the half of the natural frequency). Alternatively, for DC voltage around V_{cr} both solution does not describe the behaviour of the actuator well.

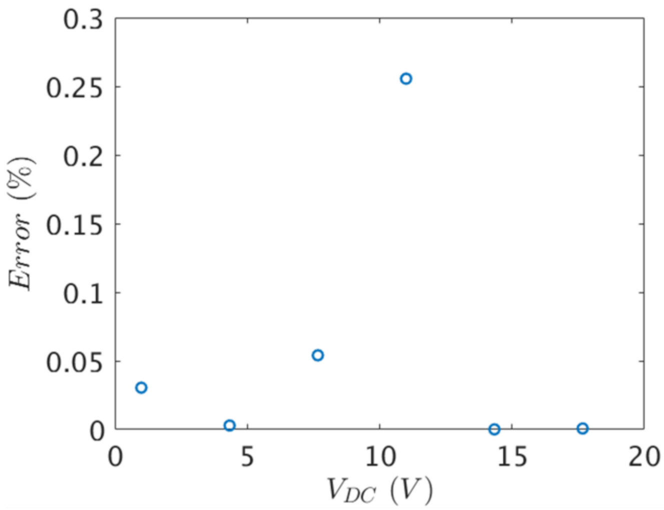

Fig 6: Relative error of the asymptotical approximation as function of DC voltage.

Another simulation that has been made is the phase space for a different DC voltage. There are 2 main conclusions from the simulation. Firstly, it is noticeable that for different DC voltages the system oscillates around different equilibrium point. If the over shot does not pass the critical static deflection the system will converge around some equilibrium point. Secondly, by examine the radius of the equilibrium circles, one can notice the influence of the DC voltage on the amplitude of the velocity and displacement of the system.

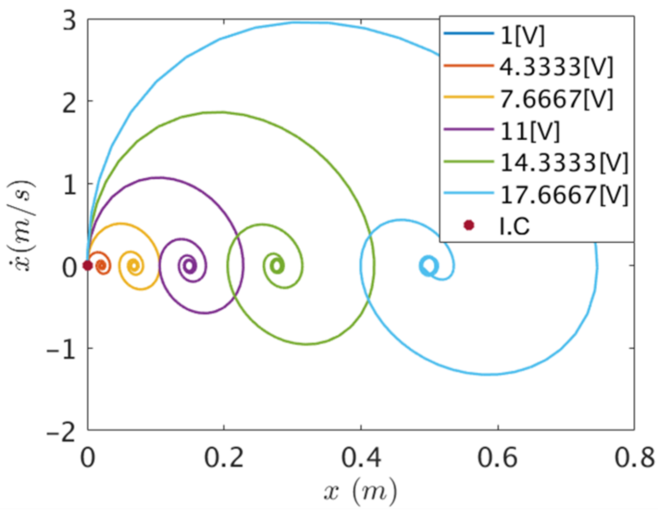

Fig 7: The phase-space for different DC voltages.

another simulation demonstrated in fig 8. And its shows the time response for different voltages around V_{cr}. For $V_{DC} < 19.5V$ the system converges to a steady state response while when V_{DC} pass this value the system diverges. Noticeably, the system diverges where the over shots get the value of the critical static deflection (1 for this given problem).

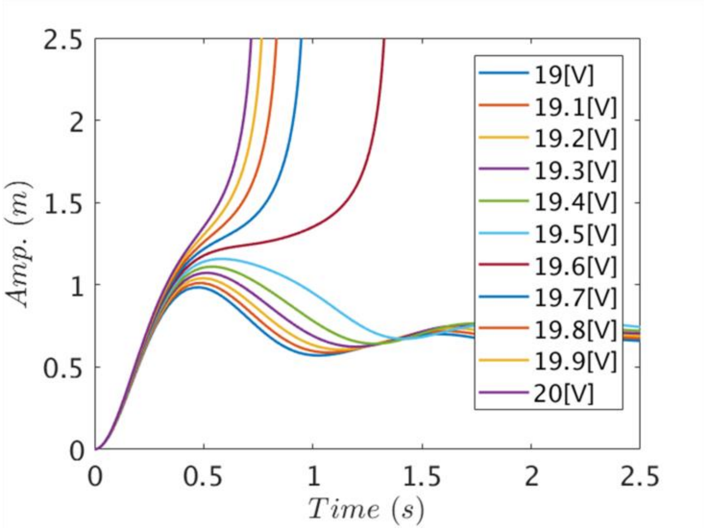

Fig 8: Time response near the critical voltage

For the case of V_{AC} which is comparable to V_{DC}, $\left(V(t)\right)^2$ takes the following form:

$$\left(V(t)\right)^2 = V_{DC}^2 + \frac{1}{2}V_{AC}^2 + 2V_{DC}V_{AC}\sin(\omega t) - \frac{1}{2}V_{AC}^2\cos(2\omega t) \tag{5.4}$$

From equation (5.4) one can notice that there is an independent time terms for the AC and DC part. Moreover, another oscillating term appear at twice of the actuation

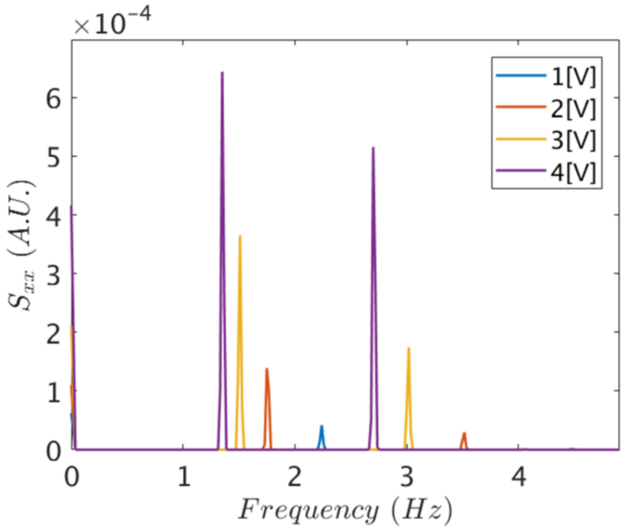

Fig 8: PSD for different AC voltages, the DC voltage equal to 2(V).

frequency. At fig 8 there is a peak at $0(Hz)$ (equivalent to a static deflection) which implies for a movement of the mass to another equilibrium point. The peak at $0\ (Hz)$ is affected by the free AC and DC terms. Hence, as the AC and DC terms increase so the peak at $0(Hz)$.

When the exaction frequency equal to the half - resonance frequency we get two peaks, one at the half of the resonance frequency (the actuation frequency) while the other one is at the resonance frequency. Those peaks are equivalent to the time dependent terms in equation (5.4) where the DC part is proportional to the actuation frequency and the AC part is proportional to twice of the actuation frequency.

Another phenomenon that one can be notice in fig 8 is the drift of the peaks for different AC voltages. The decreasing at the resonance frequency for increasing AC voltages is happening because of the softening effect of electrostatic force. in that case, the force act like a negative spring.

Fig 9 shows the evolution of the phase space for different AC voltages. It seems that between $2(V)$ to $3(V)$ another stable solution is forming. As the AC term increase the "half-resonance" term that is proportional to the V_{AC}^2 became more dominant, hence the other loop at the phase space.

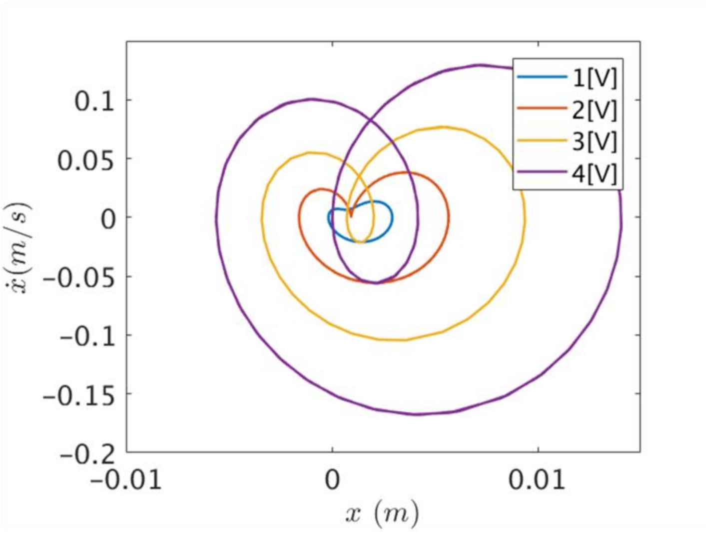

Fig 9: Phase Space for comparable AC DC terms

I tried to use the Poincare map for the analysis but there was a problem for the solver to use those specific time intervals (even for the case of non-dimensional form). If the analysis were succeeded, I was expected for a two different point at the Poincare map, one for each equilibrium.

6) Closing Remarks

In this project a non-conservative, driven electrostatic actuator have been studied. It has been shown that for the case of a DC voltage solely the system has 2 different equilibrium points, stable and unstable, and that for a DC voltage that is $V_{DC} < \sqrt{\frac{4\omega_0^2 g^3}{27C}}$. for the case of DC voltage that equal to this value bifurcation occurs and the two points merged to one. Later, approximate solution has been compared to a numerical one. The relative error of the approximate solution for the problem conditions was $Err < 1\%$ and it describe the frequency response in a good way. Finally, the dynamical behaviour of the system was investigated near the critical voltage and the time response for different voltages was plotted.

7) References

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[2] Meirovitch, Leonard, and R. G. Parker. "Fundamentals of vibrations." Appl. Mech. Rev. 54.6 (2001): B100-B101.

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8) Appendix

Development of equation (4.3):

$(g_0 - x_{st} - u)^2 = [(g_0 - x_{st})^2 - 2u(g_0 - x_{st}) - u^2]$

$[(g_0 - x_{st})^2 - 2u(g_0 - x_{st}) - u^2](\ddot{u} + 2\zeta\omega_0\dot{u} + \omega_0^2 u + \omega_0^2 x_{st}) =$

$\ddot{u}(g_0 - x_{st})^2 - 2u\ddot{u}(g_0 - x_{st}) - u^2\ddot{u} + 2\zeta\omega_0\dot{u}(g_0 - x_{st})^2 - 4\zeta\omega_0 u\dot{u}(g_0 - x_{st}) -$
$2\zeta\omega_0 u^2\dot{u} + \omega_0^2 u(g_0 - x_{st})^2 - 2\omega_0^2 u^2(g_0 - x_{st}) - \omega_0^2 u^3 + \omega_0^2 x_{st}(g_0 - x_{st})^2 -$
$2u\omega_0^2 x_{st}(g_0 - x_{st}) - \omega_0^2 x_{st} u^2 = C \cdot V_{DC}^2 + 2C \cdot V_{AC}V_{DC}\sin(\omega t)$

By using the static solution, the DC part of the voltage can be eliminated:

$\boldsymbol{\ddot{u}(g_0 - x_{st})^2} - 2u\ddot{u}(g_0 - x_{st}) - u^2\ddot{u} + 2\zeta\omega_0\dot{u}(g_0 - x_{st})^2 - 4\zeta\omega_0 u\dot{u}(g_0 - x_{st}) -$
$2\zeta\omega_0 u^2\dot{u} + \boldsymbol{\omega_0^2 u(g_0 - x_{st})^2} - 2\omega_0^2 u^2(g_0 - x_{st}) - \omega_0^2 u^3 - 2u\omega_0^2 x_{st}(g_0 - x_{st}) -$
$\omega_0^2 x_{st} u^2 = 2C \cdot V_{AC}V_{DC}\sin(\omega t)$

By combining like terms ill rearrange the equation in the form of a weekly nonlinear equation:

$(g_0 - x_{st})^2 \cdot (\ddot{u} + \omega_0^2 u) = 2u\ddot{u}(g_0 - x_{st}) + u^2\ddot{u} - 2\zeta\omega_0\dot{u}(g_0 - x_{st})^2$

$+4\zeta\omega_0 u\dot{u}(g_0 - x_{st}) + 2\zeta\omega_0 u^2\dot{u} + 2\omega_0^2 u^2(g_0 - x_{st}) + \omega_0^2 u^3 + 2u\omega_0^2 x_{st}(g_0 - x_{st})$

$+\omega_0^2 x_{st} u^2 + 2C \cdot V_{AC}V_{DC}\sin(\omega t) /: (g_0 - x_{st})^2$

After division:

$\ddot{u} + \omega_0^2 u = 2u\ddot{u}(g_0 - x_{st})^{-1} + u^2\ddot{u}(g_0 - x_{st})^{-2} - 2\zeta\omega_0\dot{u}$

$+4\zeta\omega_0 u\dot{u}(g_0 - x_{st})^{-1} + 2\zeta\omega_0 u^2\dot{u}(g_0 - x_{st})^{-2} + 2\omega_0^2 u^2(g_0 - x_{st})^{-1}$

$+\omega_0^2 u^3(g_0 - x_{st})^{-2} + 2u\omega_0^2 x_{st}(g_0 - x_{st})^{-1}$

$+\omega_0^2 x_{st} u^2(g_0 - x_{st})^{-2} + 2C \cdot V_{AC}V_{DC}\sin(\omega t)(g_0 - x_{st})^{-2}$

Let's define new parameter for small deflection:

$v = \epsilon u\ ;\ \dot{v} = \epsilon\dot{u}\ ;\ \ddot{v} = \epsilon\ddot{u}\ ; \epsilon \ll 1$

Therefore, the equation will look:

$\epsilon(\ddot{v} + \omega_0^2 v) = \epsilon^2 2v\ddot{v}(g_0 - x_{st})^{-1} + \epsilon^3 v^2\ddot{v}(g_0 - x_{st})^{-2} - \epsilon 2\zeta\omega_0\dot{v}$

$+\epsilon^2 4\zeta\omega_0 v\dot{v}(g_0 - x_{st})^{-1} + \epsilon^3 2\zeta\omega_0 v^2\dot{v}(g_0 - x_{st})^{-2} + \epsilon^2 2\omega_0^2 v^2(g_0 - x_{st})^{-1}$

$$+\epsilon^3\omega_0^2 v^3(g_0 - x_{st})^{-2} + \epsilon 2v\omega_0^2 x_{st}(g_0 - x_{st})^{-1}$$

$$+\epsilon^2\omega_0^2 x_{st} v^2(g_0 - x_{st})^{-2} + 2C \cdot V_{AC}V_{DC}\sin(\omega t)\,(g_0 - x_{st})^{-2}$$

Now ill combine like terms based on ϵ

$$\epsilon(\ddot{v} + \omega_0^2 v) =$$

$$\epsilon[2v\omega_0^2 x_{st}(g_0 - x_{st})^{-1} - 2\zeta\omega_0\dot{v}]$$

$$\epsilon^2[2v\ddot{v}(g_0 - x_{st})^{-1} + 2\omega_0^2 v^2(g_0 - x_{st})^{-1} + 4\zeta\omega_0 v\dot{v}(g_0 - x_{st})^{-1} + \omega_0^2 x_{st} v^2(g_0 - x_{st})^{-2}]$$

$$\epsilon^3[v^2\ddot{v}(g_0 - x_{st})^{-2} + 2\zeta\omega_0 v^2\dot{v}(g_0 - x_{st})^{-2} + \omega_0^2 v^3(g_0 - x_{st})^{-2}]$$

$$2C \cdot V_{AC}V_{DC}\sin(\omega t)\,(g_0 - x_{st})^{-2}$$