Mathematical model

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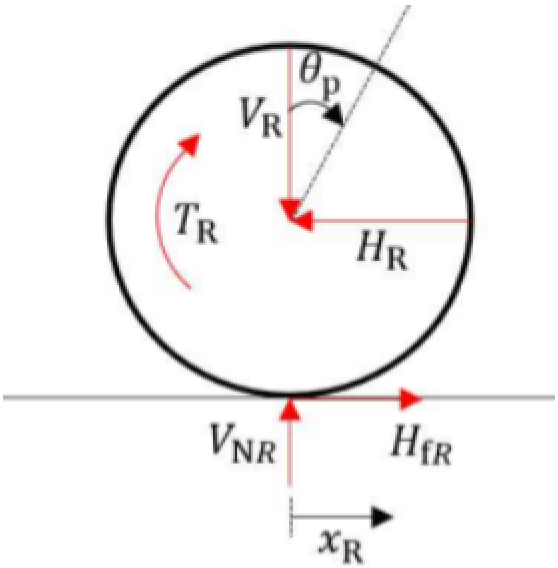
The structure of the balance car mainly consists of two parts: the body and the two wheels, which can be regarded as a moving inverted pendulum.

Wheel model

The movement of the balance car is achieved by the rotation of the wheels. We choose a pair of coaxially installed wheels with the same parameters (mass, moment of inertia, radius).

Take the right wheel as an example for force analysis:



The movement of the wheel is decomposed into translation and rotation. According to Newton's second law, we can get:

$$m\ddot{x}_R = H_{fR} - H_R (1)$$

According to the law of rigid body fixed axis rotation, we can get:

$$I\dot{\omega}_R = T_R - H_{fR}r (2)$$

> **Explanation**

Physical quantity	Description	Unit
m	Wheel quality	kg
r	Wheel radius	m
x_R	Right wheel horizontal displacement	m
H_{fR}	The magnitude of the ground friction force on the right wheel	N
H_R	The magnitude of the horizontal component of the force exerted by the vehicle body on the right wheel	N
T_R	The torque output by the right wheel motor	$N \cdot m$
I	Moment of inertia of the wheel	$kg \cdot m^2$
ω_R	Angular velocity of the right wheel	rad/s

Combining (1) and (2), we get:

$$m\ddot{x_R} = \frac{T_R - I\dot{\omega_R}}{r} - H_R (3)$$

When the wheel does not slip, the speed of the wheel is proportional to the speed of rotation, that is,

$$\begin{cases} \omega_R = \frac{\dot{x_R}}{r} \\ \dot{\omega_R} = \frac{\ddot{x_R}}{r} \end{cases} (4)$$

Combining (3) and (4), we get:

$$(m + \frac{I}{r^2})\ddot{x_R} = \frac{T_R}{r} - H_R (5)$$

Since the parameters of the left and right wheels are the same, the left wheel has a similar result, that is,

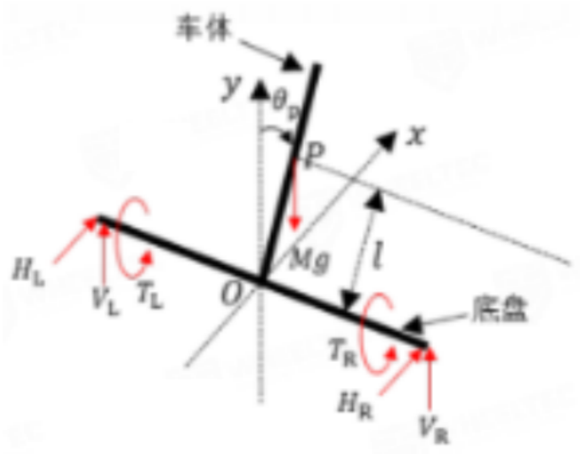
$$(m + \frac{I}{r^2})\ddot{x_L} = \frac{T_L}{r} - H_L (6)$$

Car body model

The movement of the car body can also be decomposed into forward movement (forward, pitch) and lateral movement (steering, yaw): yaw movement can be regarded as a special case of steering movement.

Forward motion

The forward motion of the car can be decomposed into forward motion and relative rotation (pitch) around the center of mass P of the car body:



The horizontal displacement of the center O of the car chassis is:

$$x = \frac{x_L + x_R}{2} \; (7)$$

Add equations (5) and (6) together, and divide both sides of the equation by 2 to get:

$$(m + \frac{I}{r^2})\frac{\ddot{x_L} + \ddot{x_R}}{2} = \frac{T_L + T_R}{2r} - \frac{H_L + H_R}{2} \; (8)$$

$$(m + \frac{I}{r^2})\ddot{x} = \frac{T_L + T_R}{2r} - \frac{H_L + H_R}{2} \; (9)$$

Apply Newton's second law to the car body, in the horizontal direction:

$$M\frac{d(x + lsin\theta_P)}{dt^2} = H_L + H_R(10)$$

Apply Newton's second law to the car body, in the vertical direction:

$$M\frac{d(lcos\theta_P)}{dt^2} = V_L + V_R - Mg(11)$$

Apply the law of rigid body fixed axis rotation to the car body and we get:

$$J_P\ddot{\theta_P} = (V_L + V_R)lsin\theta_P - (H_L + H_R)lcos\theta_P - (T_L + T_R)(12)$$

> **Explanation**

物理量	描述	单位
M	车体的质量	kg
l	质心距底盘中心的距离	m
J_P	车体绕质心转动时的转动惯量	$kg \cdot m^2$
θ_P	车体与竖直方向所成的夹角	rad
V_L	车体受到左轮作用力的竖直分力的大小	N

Simultaneous equations (9) and (10) yield:

$$(M + 2m + \frac{2I}{r^2})\ddot{x} - \frac{T_L + T_R}{r} + Ml\ddot{\theta_P}\,cos\theta_P - Ml\dot{\theta_P}^2 sin\theta_P = 0(13)$$

Among them:

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The equation contains nonlinear terms, so it is linearized. Considering that the inclination angle of the vehicle body is relatively small, it can be considered that:

$$\begin{cases} cos\theta_P = 1 \\ sin\theta_P = \theta_P \\ \dot{\theta_P}^2 = 0 \end{cases}$$

Therefore, equation (13):

$$\ddot{x} = \frac{T_L + T_R}{(M + 2m + \frac{2I}{r^2})r} - \frac{Ml}{(M + 2m + \frac{2I}{r^2})}\ddot{\theta_P}\,(14)$$

Substitute equations (10) and (11) into equation (12):

$$(\frac{J_P}{Ml} + l)\ddot{\theta_P} + \ddot{x}cos\theta_P - gsin\theta_P + \frac{T_L + T_R}{Ml} = 0(15)$$

Linearize equation (15):

$$\ddot{\theta_P} = \frac{Mlg}{(J_P + Ml^2)}\theta_P - \frac{Ml}{(J_P + Ml^2)}\ddot{x} - \frac{T_L + T_R}{(J_P + Ml^2)}(16)$$

Substitute equation (16) into equation (14):

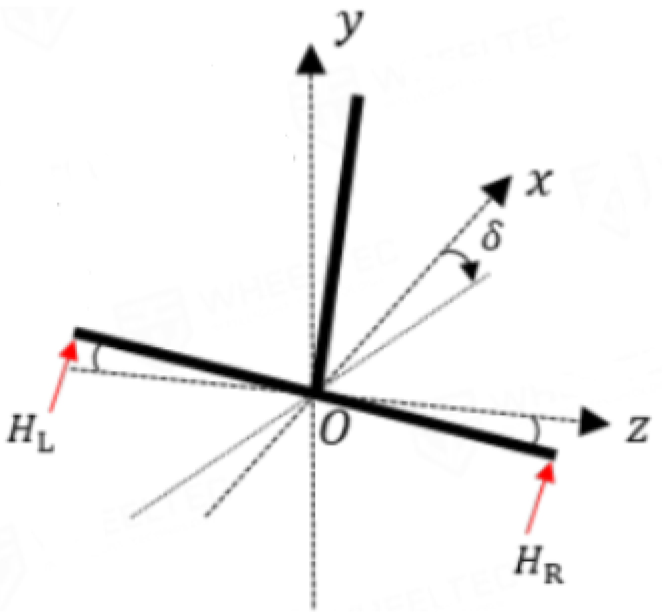
$$\ddot{x} = -\frac{M^2 l^2 g}{Q_{eq}} \theta_P + \frac{J_P + Ml^2 + Mlr}{Q_{eq} r} (T_L + T_R) (17)$$

$$Q_{eq} = J_P M + (J_P + Ml^2)(2m + \frac{2I}{r^2})$$

In summary, for forward motion:

$$\begin{cases} \ddot{x} = -\frac{M^2 l^2 g}{Q_{eq}} \theta_P + \frac{J_P + Ml^2 + Mlr}{Q_{eq} r} (T_L + T_R) \\ \ddot{\theta_P} = \frac{Mlg(M + 2m + \frac{2I}{r^2})}{Q_{eq}} \theta_P - \frac{(\frac{Ml}{r} + M + 2m + \frac{2I}{r^2})}{Q_{eq}} (T_L + T_R) \end{cases} \quad (19)$$

Steering motion



The steering motion is caused by the unequal magnitudes of the reaction forces HL and HR applied to the vehicle body by the left and right wheels in the horizontal direction. According to the law of rigid body fixed axis rotation, we can get:

$$J_\delta \ddot{\delta} = \frac{d}{2} (H_L + H_R) (20)$$

> **Explanation**

物理量	描述	单位
d	轮距	m
J_δ	车体绕y轴转动时的转动惯量	$kg \cdot m^2$
δ	小车的偏航角	rad

Subtract equations (5) and (6) to get:

$$(m+\frac{I}{r^2})(\ddot{x_L}-\ddot{x_R})=\frac{T_L-T_R}{r}-(H_L-H_R)(21)$$

When the left and right wheels move at different speeds, the vehicle body turns:

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According to the geometric relationship, we can get:

$$\begin{cases} \dot{x_L} = \dot{\delta} r_L \\ \dot{x_R} = \dot{\delta} r_R \\ r_L = r_R + d \end{cases} \quad (22)$$

$$\dot{\delta}=\frac{\dot{x_L}-\dot{x_R}}{d}(23)$$

$$\ddot{\delta}=\frac{\ddot{x_L}-\ddot{x_R}}{d}(24)$$

$$\ddot{\delta}=\frac{T_L-T_R}{r(md+\frac{Id}{r^2}+\frac{2J_\delta}{d})}(25)$$

From equations (19) and (25), we can get the state equation of the system:

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta_P} \\ \ddot{\theta_P} \\ \dot{\delta} \\ \ddot{\delta} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & A_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ \theta_P \\ \dot{\theta_P} \\ \delta \\ \dot{\delta} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \\ 0 & 0 \\ B_{41} & B_{42} \\ 0 & 0 \\ B_{61} & B_{62} \end{pmatrix} \begin{pmatrix} T_L \\ T_R \end{pmatrix} (26)$$

The state variables represent the displacement, forward speed, inclination angle, angular velocity, steering angle and steering speed of the car.

Since the magnitude of the motor output torque is difficult to control directly, it is converted into the acceleration of the two wheels through the law of rigid body fixed axis rotation:

$$\begin{cases} \dot{v_{LO}} = \frac{rT_L}{I} \\ \dot{v_{RO}} = \frac{rT_R}{I} \end{cases} \quad (27)$$

The left side of equation (27) is the magnitude of the linear velocity of the left/right wheel when there is no friction (in rad/s).

Therefore, the state space expression of the system becomes:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad (28)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & A_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \frac{I}{r} \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \\ 0 & 0 \\ B_{41} & B_{42} \\ 0 & 0 \\ B_{61} & B_{62} \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad D = 0$$

$$x = \begin{pmatrix} x \\ \dot{x} \\ \theta_P \\ \dot{\theta_P} \\ \delta \\ \dot{\delta} \end{pmatrix} \quad u = \begin{pmatrix} \dot{v_{LO}} \\ \dot{v_{RO}} \end{pmatrix}$$