Bayes Classifier and Naïve Bayes

Probabilistic Classification

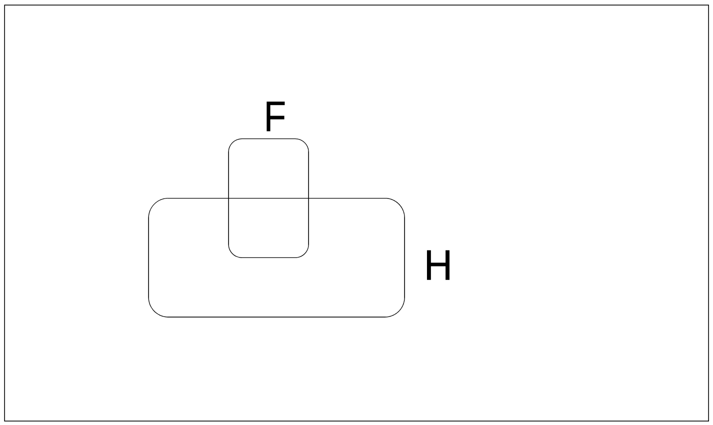
- Credit scoring:
  - Inputs are income and savings
  - Output is low-risk vs high-risk
- Input: $\boldsymbol{x} = [x_1, x_2]^T$,Output: $C \in \{0,1\}$
- Prediction:

$$\text{choose} \begin{cases} C = 1 \text{ if } P(C = 1 \mid x_1, x_2) > 0.5 \\ C = 0 \text{ otherwise} \end{cases}$$

or equivalently

$$\text{choose} \begin{cases} C = 1 \text{ if } P(C = 1 \mid x_1, x_2) > P(C = 0 \mid x_1, x_2) \\ C = 0 \text{ otherwise} \end{cases}$$

A side note: probabilistic inference



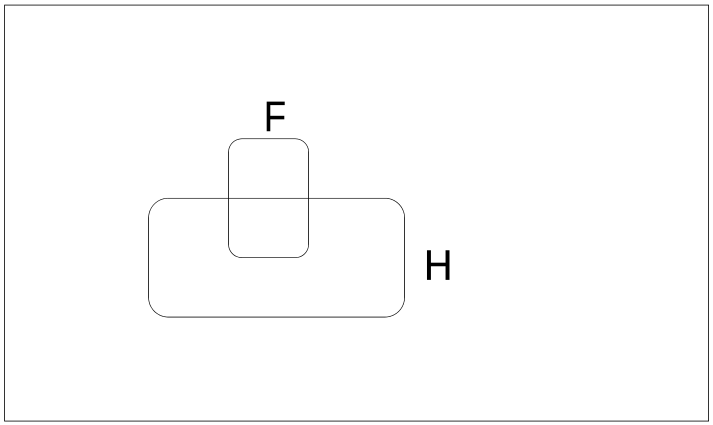
H = “Have a headache”
F = “Coming down with Flu”

$P(H) = 1/10$
$P(F) = 1/40$
$P(H|F) = 1/2$

One day you wake up with a headache. You think: “Drat! 50% of flus are associated with headaches so I must have a 50-50 chance of coming down with flu”

Is this reasoning good?

Probabilistic Inference



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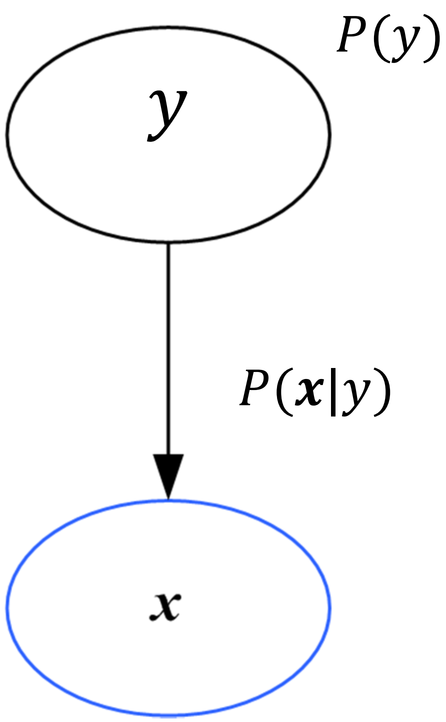
P(F ^ H) = $P(F)P(H \mid F) = \frac{1}{40} * \frac{1}{2} = \frac{1}{80}$

P(F|H) = $\frac{P(F \wedge H)}{P(H)} = \frac{1}{8}$

Bayes classifier



A simple bayes net

prior

posterior

$$\boxed{P(y \mid \mathbf{x}) = \frac{P(y)\, p(\mathbf{x} \mid y)}{p(\mathbf{x})}}$$

Given a set of training examples, to build a Bayes classifier, we need to

1. Estimate P(y) from data
2. Estimate P(x|y) from data

Given a test data point x, to make prediction

1. Apply bayes rule: $P(y \mid \mathbf{x}) \propto P(y) P(\mathbf{x} \mid y)$
2. Predict $\arg\max_y P(y \mid \mathbf{x})$

Bayes Classifiers in a nutshell

1. Estimate P(x_1, x_2, … x_m | $y=v_i$) for each value v_i
3. Estimate P($y=v_i$) as fraction of records with $y=v_i$.

learning

4. For a new prediction:

$$y^{\text{predict}} = \underset{v}{\operatorname{argmax}} P(y = v \mid x_1 = u_1 \cdots x_m = u_m)$$
$$= \underset{v}{\operatorname{argmax}} P(x_1 = u_1 \cdots x_m = u_m \mid y = v) P(y = v)$$

Estimating the joint distribution of x_1, x_2, … x_m given y can be problematic!

Joint Density Estimator Overfits

- Typically we don't have enough data to estimate the joint distribution accurately
- It is common to encounter the following situation:
  - If no training examples have the exact $\mathbf{x}=(u_1, u_2, \ldots.\ u_m)$, then $P(x|y=v_i) = 0$ for all values of Y.

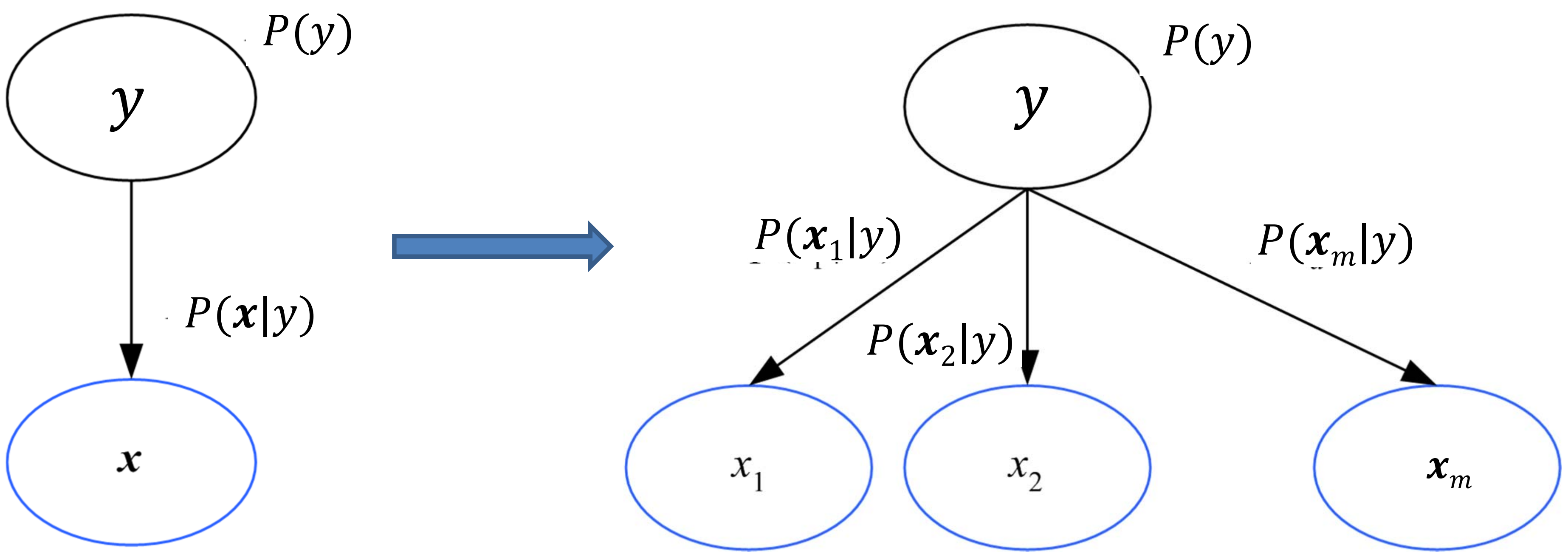
- In that case, what can we do?
  - we might as well guess a random y based on the prior, i.e., p(y)

$$\boxed{P(y \mid \mathbf{x}) = \frac{P(y)\, p(\mathbf{x} \mid y)}{p(\mathbf{x})}}$$

The Naïve Bayes Assumption

- Assume that each attribute is independent of any other attributes given the class label

$$\boxed{\begin{aligned}&P(x_1 = u_1 \cdots x_m = u_m \mid y = v_i)\\ &= P(x_1 = u_1 \mid y = v_i) \cdots P(x_m = u_m \mid y = v_i)\end{aligned}}$$



Conditional Independence

- $P(x_1|x_2,y) = P(x_1|y)$
  - X_1 is independent of x_2 given y
  - x_1 and x_2 are conditionally independent given y
- If X_1 and X_2 are conditionally independent given y, then we have
  - $P(X_1,X_2|y) = P(X_1|y)\,P(X_2|y)$

Naïve Bayes Classifier

- By assuming that each attribute is independent of any other attributes given the class label, we now have a *Naïve* Bayes Classifier
- Instead of learning a joint distribution of all features, we learn $p(x_i|y)$ separately for each feature x_i
- Everything else remains the same

Naïve Bayes Classifier

- Assume you want to predict output *y* which has n_Y values v_1, v_2, … v_{ny}.
- Assume there are *m* input attributes called **x**=(x_1, x_2, … x_m)
- Learn a conditional distribution of p(**x**|y) for each possible y value, y = v_1, v_2, … v_{ny}, we do this by:
  - Break training set into n_Y subsets called S_1, S_2, … S_{ny} based on the y values, i.e., S_i contains examples in which $y=v_i$
  - For each S_i, learn p(y=v_i)= $|S_i|/|S|$
  - For each S_i , learn the conditional distribution each input features, e.g.:

$$\boxed{P(x_1 = u_1 \mid y = v_i), \cdots, P(x_m = u_m \mid y = v_i)}$$

$$y^{\text{predict}} = \underset{v}{\operatorname{argmax}}\, P(x_1 = u_1 \mid y = v) \cdots P(x_m = u_m \mid y = v) P(y = v)$$

Example

X_1	X_2	X_3	Y
1	1	1	0
1	1	0	0
0	0	0	0
0	1	0	1
0	0	1	1
0	1	1	1

Apply Naïve Bayes, and make prediction for (1,0,1)?

1. Learn the prior distribution of y.
   P(y=0)=1/2, **P**(y=1)=1/2
2. Learn the conditional distribution of x_i given y for each possible y values
   p(X_1|y=0), **p**(X_1|y=1)
   p(X_2|y=0), **p**(X_2|y=1)
   p(X_3|y=0), **p**(X_3|y=1)

For example, **p**(X_1|y=0):
P(X_1=1|y=0)=2/3, **P**(X_1=1|y=1)=0
…

To predict for (1,0,1):

P(y=0|(1,0,1)) = P((1,0,1)|y=0)P(y=0)/P((1,0,1))

P(y=1|(1,0,1)) = P((1,0,1)|y=1)P(y=1)/P((1,0,1))

Laplace Smoothing

- With the Naïve Bayes Assumption, we can still end up with zero probabilities
- E.g., if we receive an email that contains a word that has never appeared in the training emails
  - P(**x**|y) will be 0 for all y values
  - We can only make prediction based on p(y)
- This is bad because we ignored all the other words in the email because of this single rare word
- Laplace smoothing can help

  $P(X_1=1|y=0)$

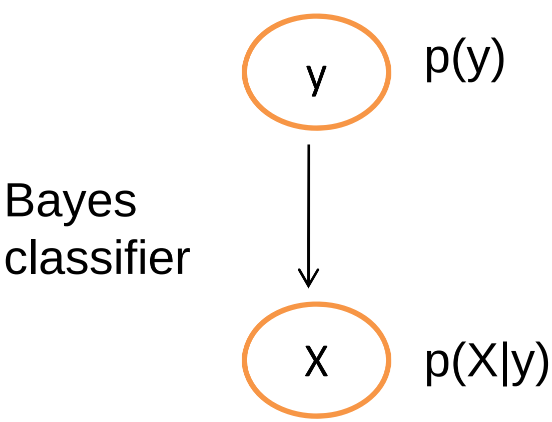
  = (**1+** # of examples with $y=0, X_1=1$) /(**k+** # of examples with $y=0$)

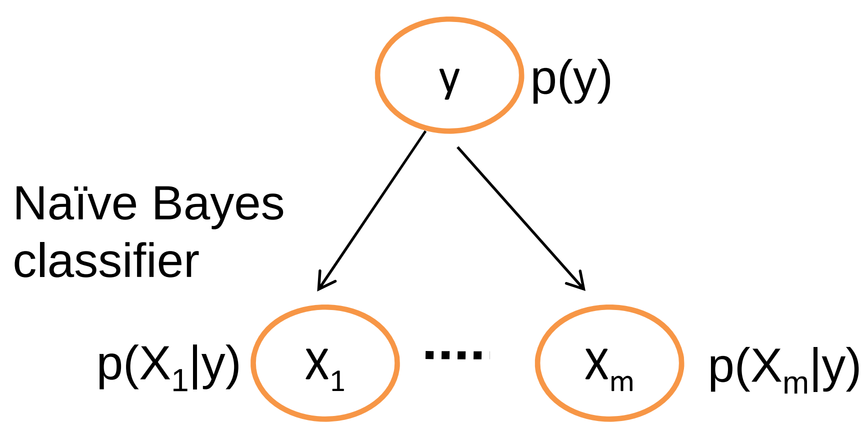
  k = the total number of possible values of x

- For a binary feature like above, p(x|y) will not be 0

Bayes Classifier is a ***Generative Approach***

- Generative approach:
  - Learn p(y), p(**x**|y), and then apply bayes rule to compute p(y|**x**) for making predictions
  - This is equivalent to assuming that each data point is generated following a ***generative process*** governed by p(y) and p(X|y)





- Generative approach is just one type of learning approaches used in machine learning
  - Learning a correct generative model is difficult
  - And sometimes unnecessary
- KNN and DT are both what we call discriminative methods
  - They are not concerned about any generative models
  - They only care about finding a good discriminative function
  - For KNN and DT, these functions are deterministic, not probabilistic
- One can also take a probabilistic approach to learning discriminative functions
  - i.e., Learn $p(y|X)$ directly without assuming X is generated based on some particular distribution given y (i.e., $p(X|y)$)
  - Logistic regression is one such approach