The Harmony

of

Harmony

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Introduction

Music is much more than the sound waves that are made from the interaction of musicians and singers. The organized sound waves transferred from performer to listener holds much more information than harmonies and melodies. Rhythms, pulse, mood and not least personal feelings are major elements that contribute to the mysterious form of communication that music is. In no way do I believe that taking one of the elements, analyzing and defining fundamental properties of that element, that we will come to a deeper truth of the art of music. The whole is much more than the sum of the parts.

In the following I wish to investigate the mathematical properties of Just intonation and the harmonic series. It is not my intention to imply that music written and played by just tuning is the true way of expressing music. My intention is to see if there is a way to explain and understand the harmonies and scales we find so natural to use. Are the scales we use (my personal experience is through western music tradition) constructs that we have learned to understand, or are there fundamental and natural grounds that lead us to use the scales and harmonies as a universal language?

More specifically I wish to show a method of defining notes and intervals and thus use these to define scales and harmonies. Most of what I shall present is not new. I gather that most readers will already understand the principles so I will not go into the basics. In that much that I will go through some basics, it is to build up a mindset to explain my method.

My background for writing this is because I have not seen this principle explained anywhere. I am a self learned musician with a interest in maths. I do not have an academic knowledge of the theoretical base from which todays understanding of music theory is anchored, so it is possible that this method is known. Most of my references are found through the Internet. I hope I can in some way contribute to the discussion.

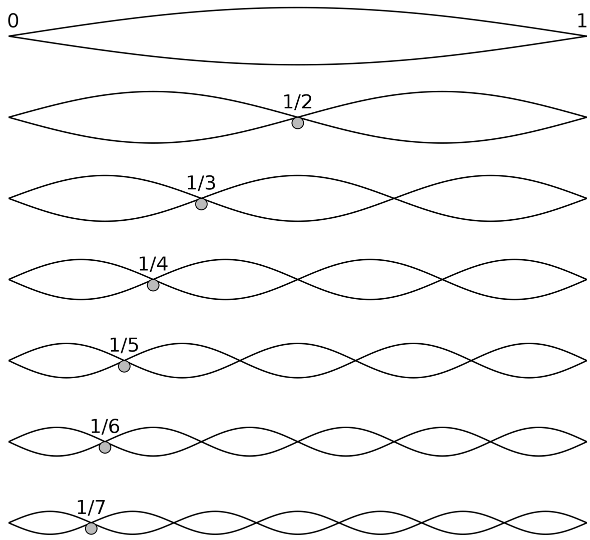
The harmonic series

> *«Just intonation is any musical tuning in which the frequencies of notes are related by ratios of whole numbers.» [wiki/Just_intonation]*

This definition of Just intonation, taken from Wikipedia, is a precise explanation, where the basis of the tuning system is a mathematical term. Most tuning systems are just that. Many tuning systems, like the Equal tempered intonation, are a human construct, a model for tuning instruments to achieve an appropriate basis for composing and performing music.

Just intonation is different in that the mathematical terms are based on the law of physics. “The ratio of whole numbers” is a mathematical way of explaining the physical properties of a “standing wave”, [wiki/Harmonic_series_(music)] like a vibrating string or a column of air in a trumpet. To understand Just intonation one has to understand the principles of the Harmonic series.

Take a guitar, and pluck a string. On a thicker string (a bass guitar is especially appropriate) you can see the string vibrating along the whole length of the string. The note the string produces is the fundamental frequency of the harmonic series this string produces. Take your finger and place it lightly on the string so that you divide the string in two equal lengths and pluck the string. The note now produced is an octave higher than the first note. Lifting your finger you can see the string vibrating on both sides of the node where your finger was. Essentially we have doubled the frequency of the vibrating string as the string is now vibrating on half of the length of the string, though it is vibrating on both half lengths. We can now determine that the 1^{st} harmonic has two times the frequency of the fundamental and this ratio can be written as 2:1 or 2/1 and we say the 1^{st} harmonic is the first octave of the fundamental.



Illustrasjon 1: Harmonic series of a string (Ill. Geoff Richards)

We can now place our finger on one of the two nodes that divide the string into three equal lengths. Plucking the string we can see that the string is vibrating in three parts. Lifting your finger and placing it on the other node does not stop the string from vibrating. (placing it any other place on the string will) We have now tripled the frequency of the string, giving a note that we understand as an octave plus a fifth above the fundamental. The 2^{nd} harmonic is written as the ratio 3/1.

Playing around with our guitar and applying this principle by increasing the number of divisions you will experience producing higher and higher notes as the frequencies are risen by ratio of the division. And thus we get the harmonic series.

1/1	Fundamental	
2/1	1^{st} harmonic	1 octave
3/1	2^{nd} harmonic	1 octave + 5^{th}
4/1	3^{rd} harmonic	2 octaves
5/1	4^{th} harmonic	2 octaves + 3^{rd}
6/1	5^{th} harmonic	2 octaves + 5^{th}
7/1	6^{th} harmonic	2 octaves + harmonic 7^{th}
8/1	7^{th} harmonic	3 octaves
9/1	8^{th} harmonic	3 octaves + 2^{nd}
10/1	9^{th} harmonic	3 octaves + 3^{rd}
11/1	10^{th} harmonic	3 octaves + ?4^{th}
12/1	11^{th} harmonic	3 octaves + 5^{th}
13/1	12^{th} harmonic	3 octaves + ?6^{th}
14/1	13^{th} harmonic	3 octaves + harmonic 7^{th}
15/1	14^{th} harmonic	3 octaves + major 7^{th}
16/1	15^{th} harmonic	4 octaves

We quickly notice that when we multiply a ratio with 2 we get a harmonic that is 1 octave higher. Multiplying the 1^{st} harmonic 2/1 with 2 we get the 3^{rd} harmonic 4/1. The practical application of this on the guitar string is dividing the half of the string that we got when we produced the 1^{st} harmonic into 2 once more. The string will have 3 nodes that don't vibrate and it will vibrate on 4 different segments of the string, producing a frequency that is four times the fundamental. We see that applying this principle to the 4^{th} , 5^{th} and 6^{th} harmonics we produce the 9^{th} ,11^{th} and 13^{th} harmonics respectively.

Physical properties of the standing wave

Now it should be noted that when we pluck the string with its fundamental configuration, in reality the string vibrates more or less in all the harmonics as well as the fundamental frequency. This is one of the reasons a violin, a guitar and a piano sound different from each other. The fundamental has the strongest amplitude. The harmonics have less amplitude and as a rule have lesser amplitude as the ratio rises. Of coarse soundboard, string and method of sounding the string will contribute to the relative amplitudes of the harmonics. The guitar sounds like a guitar because its fingerprint of amplitudes among harmonics is unique for the guitar.

Another reason for pointing out the physical property of the string that vibrates in all of it's harmonic frequencies, is to show why we conceive two notes (or an interval) as being harmonious. It is easy to show that the harmonic series of one fundamental frequency is a set of the harmonic series of another fundamental frequency, where the first frequency is in the harmonic series of the second. (rule 1)

Taken to a practical level, lets take our A string on the guitar and tune the highest E string to it. We can do this by playing the 2^{nd} harmonic on the A string, which produces an E, one octave + a fifth higher than the A. This matches the frequency of the E string we want to tune. (note that we have tuned it according to just intonation, but this is a regular method of tuning a guitar) Now all harmonics we can play on the E string are available on the A string (in theory – The harmonic level of the harmonics on the A string that compare to the higher harmonics of the E string are so high that it is impractical to play them.) The first harmonic of the E string is equivalent to the 3^{rd} harmonic of the A string, the 2^{nd} harmonic of the E string is equivalent to the 8^{th} harmonic of the A string and so on.

E string	A string	
1/1 fundamental	3/1 2^{nd} harmonic	1/1 * 3/1 = 3/1
2/1 1^{st} harmonic	6/1 5^{th} harmonic	2/1 * 3/1 = 6/1
3/1 2^{nd} harmonic	9/1 8^{th} harmonic	3/1 * 3/1 = 9/1
4/1 3^{rd} harmonic	12/1 11^{th} harmonic	4/1 * 3/1 = 12/1
5/1 4^{th} harmonic	15/1 14^{th} harmonic	5/1 * 3/1 = 15/1
6/1 5^{th} harmonic	18/1 17^{th} harmonic	6/1 * 3/1 = 18/1
7/1 6^{th} harmonic	21/1 20^{th} harmonic	7/1 * 3/1 = 21/1
8/1 7^{th} harmonic	24/1 23^{rd} harmonic	8/1 * 3/1 = 24/1

Now considering the fact that when we play the A string all the harmonics are present, when we play the E string simultaneously we do do not, in effect, add any frequencies, we only change the amplitude of already existing frequencies. Especially the A strings 2^{nd} harmonic is amplified so producing our perception of a open fifth. The open fifth is conceived as being the most fundamental interval, beside the octave, at least in western harmonic tradition. It is easy to see why, as the fifth is the first and closest non-octave interval to the fundamental.

Of coarse this does not just apply to the 5^{th}. This applies to all harmonics of the fundamental. The next interval that is not an octave of a previous interval is the major 3^{rd}. Together these three notes, the fundamental (1/1), the 5^{th} (3/1) and the 3^{rd} (5/1), make the major triad. I would highly recommend to try the following. By justly tuning some of the guitar strings, we will be able to experience hearing this harmony. And, because of some practical tuning issues, we can experience how off-tuned the equal-tempered 3^{rd} actually is. It is important not to use a tuner, as tuners are equal-tempered.

We will use the lowest E-string as the fundamental. By playing the 2^{nd} harmonic on the E-string, we can tune the B string. This is the same as what we did with the A and E string in the previous example. We have tuned the B-string an octave plus a 5^{th} above the fundamental. It is essential that we take care to tune the strings accurately to get the right effect. Equal-tempered tuned strings will naturally be a little out of tune. We experience beats between two equal-tempered strings. By beats I mean small vibrations that are the result of the inference between to almost identical tuned strings. We want the strings to be so tuned that these beats disappear. (The snare on the snare drum will stop vibrating.) Playing these two notes gives a satisfying feeling of harmony. The next note we want to play is the G#, two octaves and a 3^{rd} above the fundamental. If you tried to tune your E-string that high, you would probably break it. Try playing on the 4^{th} fret on the E string. You can hear that the G# is out of tune. It is actually very high. This is because the frets of a guitar are equal tempered. We need to tune down the string. Play on the 4^{th} harmonic on the fundamental string. Tune the 4^{th} fret of the highest E-string to this, as accurately as you can. When you now play these three notes, you will experience a major triad defined by the laws of nature.

Again, playing the B and the G# does not add any frequencies to the fundamental. They will only add to the amplitude of existing frequencies. The result is a harmonious harmony, just as nature intended it.

Prime number 2 – The octave

When we are going to construct scales and chords, its is impractical to use the ratios we use to determine the harmonic series as the intervals are spread over many octaves. A typical scale will have its fundamental as 1/1 and the octave as 2/1. All other intervals will then be written with ratios that are greater then 1 and less than 2. We have already determined that the 2^{nd} harmonic, 3/1 is an octave + a fifth above the harmonic. We have also shown that by multiplying a ratio with 2 we get the same interval one octave higher. To lower an interval one octave, all we have to do is the opposite. By dividing a ratio with 2, 3/1 : 2 = 3/2, we will lower our fifth by one octave, resulting in a fifth above the fundamental. So in our scale we will state that the perfect fifth is 3/2. We can lower the interval by an octave by dividing our ratio by 2. (rule 2)

We have now constructed an interval that is not a part of the harmonic series. (though its octave is) How can we relate to this in practical terms. How can I play both notes on my guitar using only harmonics. Lets take a look at what the ratios are telling us. In the ratio 3/1, the numeral 1 is the fundamental, and the numeral 3 is stating that we shall play the 2^{nd} harmonic, thus giving us an interval of 1 octave + 1 fifth. In the ratio 3/2, the numeral 2 is saying that we should lower the fundamental by 1 octave and then play its 2^{nd} harmonic, now giving us an interval of a fifth above the original fundamental.

On the guitar we can do this by using our highest E string as the fundamental. We can now tune our D string up to an E, an octave lower than our fundamental. (Take care you electric-guitarists. You might want to tune down the E string to D) To tune the lower string correctly (justly) we take our ratio 3/2, the numeral 2 states that we play the 1st harmonic on the lower string and tune it to the fundamental. Since we are playing the 1st harmonic, our tuned string will be an octave below the fundamental. Now playing the 2nd harmonic on the lower string we can produce a perfect fifth above the fundamental.

In mathematical terms we have used rule 1 which states that tuning a string in such a way that one of its harmonics is equal to the fundamental, then all of the harmonics of the one string is found as a set of the harmonics of the other string. All other harmonics of the tuned string relate to fundamental string as ratios of whole numbers, the definition of just intonation.

We can apply this method to our two next intervals, 5/1 and 7/1. (4/1 and 6/1 are octaves of already defined intervals) Both of these intervals are 2 octaves plus their respective interval, major 3rd and harmonic 7th, above the fundamental. To adjust these intervals we need to lower them by two octaves, mathematically we divide the ratios by 2*2 or 4, giving us 5/4 for the major 3rd and 7/4 for the harmonic 7th.

On the guitar this one is easy. The lower E string is two octaves lower than our fundamental. We can tune it justly by playing the 3rd harmonic and tuning it to the fundamental. Now the 4th and 6th harmonics will produce the major 3rd and harmonic 7th respectively. For those of you who have caught on will have discovered that the 5th harmonic is equal to the perfect 5th, as 6/4 = 3/2.

On our next octave of harmonics we need to lower the intervals by three octaves, or dividing the ratios with 2*2*2=8. This gives us following ratios (octaves of existing intervals omitted) :
9/8, 11/8, 13/8 and 15/8.

On the guitar this gets tricky as it is impractical to lower the lowest E string one octave. You could use the 12th fret of the highest E string producing the fundamental as an octave higher (or playing its 1st harmonic) or you can get your bassist to tune his E string justly to your highest E string by tuning with his 7th harmonic. In both cases it gets hard to produce notes above the 8th or 9th harmonics.

Now we can construct a scale based solely on the harmonic series. Ratios ordered by ascending value.

1/1 – 9/8 – 5/4 – 11/8 – 3/2 – 13/8 – 7/4 – 15/8 – 2/1

Three of these ratios you will seldom see in just intonation scales. 7/4, the harmonic 7th is noticeably lower than a regular minor 7th. 11/8 and 13/8 are close to a 4th and 6th but as with the harmonic 7th we have better ways of defining these intervals.

Prime numbers 3 and 5 – The fifth and the third

In rule 2 we found that by dividing the ratio by 2 we could lower the note by an octave. We can apply rule 2 to other whole numbers. We have established that the ratio 3/1 is one octave + a 5th higher than the fundamental. Likewise the ratio 1/3 is an interval one octave + a fifth below the fundamental. To adjust this ratio to our scale we need to raise it by two octaves. Mathematically we multiply 1/3 with 2x2 giving the ratio of 4/3. This is a perfect 4th.

On our guitar we can tune the A string's 2nd harmonic to our fundamental E string. This tunes the A string an octave + a 5th below the fundamental. Playing on the 3rd harmonic yields a perfect 4th to E.

We can now introduce the concept of complementary intervals. Two intervals when added to each other equal an octave are said to be complementary. To mathematically add two intervals we need to multiply their ratios. Multiplying two complementary ratios will always yield 2/1, or the octave. 4/3 x 3/2 = 2/1. I shall use as a basis for for defining my scale the third rule that all intervals has its complementary interval. (rule 3)

Now lets define the complement of the major 3rd 5/4. When we lower the fundamental by a major 3rd we find a minor 6th. Mathematically we multiply the ratio 1/5 with three octaves, 2 x 2 x 2 = 8. 1/5 x 8 = 8/5.

On the guitar we tune the lower E string so that the 4th harmonic is tuned to our fundamental E string. This drops the lowest string to C. Playing the 7th harmonic produces a C that is a minor 6th above the fundamental.

Terce mood

We have just found that adding a major and minor interval yields a perfect octave. (octaves and primes are perfect as 4ths and 5ths are) I shall propose a fourth rule, the adding of what I shall call terce-moods of intervals. (from Latin for third, explanation will come later) Perfect intervals are terce-mood neutral. Major intervals are terce-mood +1 and minors are terce-mood -1. When adding a major and a minor interval, their respective terce-moods will cancel each other out, adding together to be a perfect interval. A deeper explanation of this will be given later. Two perfect intervals will yield a perfect interval. (rule 4)

Lets see how far we have come with our scale. I shall now on visualize the scale as a scale of C.

C	D	E	F	G	Ab	B	C
1/1	9/8	5/4	4/3	3/2	8/5	15/8	2/1

We quickly see that there is at least two important intervals missing, the minor 3rd Eb and the major 6th A. Before we look into these intervals I wish to introduce a new concept.

Vector notation

Joseph Monzo has describes in his “Ratios notated as a Prime series” [http://tonalsoft.com/monzo/article/article.htm#notation] a method of notating ratios using the exponents of prime factors as vectors.

All whole numbers can be expressed as the product of powers of prime numbers. And since the fraction of a ratio is equivalent to the negative exponent, any just interval is expressible as the product of powers of prime numbers. [Douglas Keislar in above]

The major third 5/4, can be expressed as $\frac{5}{2\cdot 2}$ or $2^{-2}\cdot 3^{0}\cdot 5^{1}$, or using just the exponents as vectors [-2,0,1]

The perfect fourth 4/3, can be expressed as $\frac{2 \cdot 2}{3}$ or $2^2 \cdot 3^{-1} \cdot 5^0$ or as vectors [2,-1,0].

Analyzing intervals using primes

There are two reasons why I want to use this method. Firstly it gives us an easy way to add and subtract intervals. (will be shown later).

Secondly (but primarily) it visualizes the concept that I want to introduce in this paper. The prime number 2 is always described in literature as the "octave prime". As we saw multiplying or dividing a ratio by 2 yields notes either an octave above or below. The following, though it might be implied, I have not seen described as a method for constructing and analyzing notes and intervals.

I feel I have already shown that the prime number 3 is the mathematical expression for the perfect fifth and the prime number 5 is the expression for the major third. Multiplying or dividing a ratio by 3 will heighten or lower the interval by a perfect 5^{th}. Doing the same with 5 will change the interval by a major 3^{rd}. I would like to show how we can use this mathematical language to define and analyze intervals, and through this understanding see the function an interval has compared to other intervals.

For example we can analyze the ratio 15/8. Using vector notation [-3,1,1] we clearly see that this interval is a combination of a 5^{th} and and a 3^{rd} above the fundamental, and lowered by three octaves. This is in consequence with our understanding that 15/8 is the ratio for the major 7^{th}, of B in the scale of C. B is the major 3^{rd} to G, which in turn is the perfect 5^{th} to C. We can show this by vector addition.

perfect fifth	[-1, 1, 0]
major third	+ [-2, 0, 1]
major seventh	= [-3, 1, 1]

The complementary interval to 15/8 is 16/15. I have shuddered many times when reading that this is a C#. Lets analyze. Vector notation of 16/15 is [4,-1,-1] telling us that our interval is a 5^{th} and a 3^{rd} below C, giving us a major 7^{th} below, and raised four octaves to give us a minor 2^{nd} or Db. This will also be the consequence of rule 3, that the complement of a major 7^{th} is a minor 2^{nd}.

Now we can use this method to construct a minor 3^{rd} and its complement a major 6^{th}. The minor 3^{rd} does not exist naturally in the harmonic series. It is understood that the minor 3^{rd} is the difference between the 5^{th} and a major 3^{rd}. In effect we find the minor 3^{rd} by going up a 5^{th} and down a major 3^{rd}.

perfect fifth	[-1, 1, 0]
major third	- [-2, 0, 1]
minor third	= [1, 1,-1]

Written as a ratio we get $\frac{2^1 \cdot 3^1}{5^1}$ = 6/5.

We can use vector addition to find the complement, the major 6^{th}.

Octave		[1, 0, 0]
minor third	-	[1, 1,-1]
major sixth	=	[0,-1, 1]

Written as a ratio we find that the major 6^{th} is 5/3. Reading the vector we can see that the major 6^{th} is constructed by going down a fifth to F and up a third to A.

We have now constructed the following scale of C.

C	Db	D	Eb	E	F	G	Ab	A	B	C
1/1	16/15	9/8	6/5	5/4	4/3	3/2	8/5	5/3	15/8	2/1

We are missing one important interval, the minor 7^{th}. Before we look into this matter I would like to take a closer look at our major 2^{nd}.

The “second” problem

9/8 is quite normally used as the ratio for a major 2^{nd}. There are especially two reasons for this. 9/8 is found directly on the harmonic series and it corresponds very closely to the tempered 2^{nd} only 4 cents off. There is however a problem related to this.

First let us vectorize this ratio. 9/8 = [-3,2,0]. Reading the vector tells us that our interval is produced by going first up 1 fifth to G and then another fifth to D. All well so far. The problem is that the fifth is a perfect interval. According to rule 4, adding two perfect intervals yields a perfect interval. So the evidence tells us that 9/8 is a perfect 2^{nd}, or more accurately a perfect 9^{th}. Consider adding two fifths

perfect fifth		[-1, 1, 0]
perfect fifth	+	[-1, 1, 0]
perfect ninth	=	[-2, 2, 0]

Written as a ratio this is equivalent to 9/4. To bring it into scale we have to lower it one more octave.

The terce mood - again

The concept of a perfect 2^{nd} as opposed to a “moody” interval might be hard for some to swallow. Being a moody interval I refer to rule 4 where I introduced the concept of terce-moods. With the term terce-mood I mean that the interval is major or minor. The reason for choosing this term is the fact that “terce” is derived from the Latin word for “third”. Looking at our other ratios we quickly see that the other moody intervals contain 5^1 either above or below the line. All major intervals have 5^1 above and all minors have 5^1 below.

Majors: 5/4, 5/3 and 15/8.
Minors: 16/15, 6/5 and 8/5.

All major intervals have a vector signature [#,#,1] and all minor have [#,#,-1] were # = any number. (rule 5) This is the reason I have introduced the concept that majors are terce-mood 1 and minors are terce-mood -1. We will later see that there are other terce-moods as well.

To find our major 2nd and minor 7th we have to find a ratio that complies to rule 5.
To construct a minor 7th let us consider the following. Let us go up two perfect fifths to D then down a major 3rd to Bb. As vector addition this will look like:

perfect fifth		[-1, 1, 0]
perfect fifth	+	[-1, 1, 0]
major third	-	[-2, 0, 1]
minor seventh	=	[0, 2,-1]

This gives for the minor 7th the ratio of 9/5, and this complies to rule 5.
The complement to the minor 7th is a major 2nd with the ratio of 10/9. Vectorized [1,-2,1] we see that this too compiles to rule 5 as being a major interval.

We can now present our revised edition of the scale in C

C	Db	D	Eb	E	F	G	Ab	A	Bb	B	C
1/1	16/15	10/9	6/5	5/4	4/3	3/2	8/5	5/3	9/5	15/8	2/1

Why is it then that 9/8 is used in most Just tuning systems? A tuning system will always be a compromise. The whole idea of tempered tuning is to find a compromise so all keys can be played equally out of tune. Even a tuning system based on Just intonation must be a compromise. 10/9 is not a perfect fifth to 3/2. One of the most important chords in a scale is the dominant, and when constructing a tuning system it is natural to use the 9/8 variant of the 2nd. But I would like to point out that the dominant chord's 5th is in its function a 9th and not a 2nd, though they can be considered as being enharmonic, like the C# and Db are. Also I see that most scale constructors have a tendency to choose a ratio that comes as close to the Equal tempered scale as possible.

Do we need a perfect 2nd in this scale system. The answer is yes. In our scale we find two places where two major intervals are a 2nd apart, D – E and A – B. Considering rule 4, adding two major 2nd will result in an augmented 3rd. If we add a perfect 2nd to a major interval we will get the result of a major interval, as rule 4 states.

Let us look at the at the difference between these intervals.

E		[-2, 0, 1]	B		[-3, 1, 1]
D	-	[1,-2, 1]	A	-	[0,-1, 1]
9/8	=	[-3, 2, 0]	9/8	=	[-3, 2, 0]

Now we see the significance of the perfect 2nd as the difference of two major intervals or in other words the difference between equal moody intervals. Just consider the difference between a perfect 5th and 4th.

G		[-1, 1, 0]
F	-	[2,-1, 0]
9/8	=	[-3, 2, 0]

My perspective towards scales and intervals is not to define a tuning system. I am trying to understand the basic anatomy of the harmonic series as a natural law of physics and how and why we choose to construct the scales and harmonies the way we do.

I acknowledge both intervals. One of them is for constructing a dominant chord and other issues like the difference between equal moody intervals and the other for minor/major melodic and harmony purposes. I believe that the mathematical basis for each interval tells us what function it has in the harmonic series and the principles of just intonation.

(sharps)

Now we come to the question of sharps, and of coarse the Gb. Since I am using the C scale as an example, sharps in this context are the augmented intervals. Let us first look at the difference between our major and minor intervals.

D	[1,-2, 1]	E	[-2, 0, 1]	A	[0,-1, 1]	B	[-3, 1, 1]
Db -	[4,-1,-1]	Eb -	[1, 1,-1]	Ab -	[-3, 0,-1]	Bb -	[0, 2,-1]
25/24 =	[-3,-1, 2]	25/24 =	[-3,-1, 2]	25/24 =	[-3,-1, 2]	25/24 =	[-3,-1, 2]

It is satisfying to see the conformity that all differences are equal. This wouldn't be the case if we used 9/8 as the major 2^{nd}. Let us analyze this new interval and see if we can find any significant information of its function.

We have two factors of 5 meaning that we must raise our note by 2 major 3^{rd}s, C to E, then E to G#. The negative exponent for the 3 factor states that we lower the note by a perfect 5^{th}, G# to C#.

25/24 is called an augmented unison. Besides being the ratio for the scale note C#, it is also the ratio used to raise an interval from minor to major or perfect to augmented. Because of this it is called the chromatic semitone. This means that all augmented perfect intervals will have an exponent of 5 equal to 2, or in another term have a vector signature of [#,#,2]. this is what I call terce-mood 2. Many will already see that an augmented major will have a terce-mood 3, [#,#,3]. It can be easily shown by adding our augmented unison to, say, a major 6^{th}.

A	[0,-1, 1]
25/24 +	[-3,-1, 2]
A# =	[-3,-2, 3]

Analyzing this ratio 125/72, yields an interval two 5^{th}s down (C-F, F-Bb) and three 3^{rd}s up, (Bb – D, D – F# , F# - A#)

This is the real reason for the need to have a terce-mood. In traditional notation augmenting a perfect and major interval is the same thing. In my system the mathematics show that this is two different entities, though the process of adding the 25/24 ratio is the same.

Now we will be able to construct the diminished 5^{th}. We can take the ratio for a perfect 5^{th} and subtract the augmented unison

G	[-1, 1, 0]
25/24 -	[-3,-1, 2]
Gb =	[2, 2,-2]

This gives a ratio of 36/25. The diminished 5^{th} should be complementary to the augmented 4^{th}.

F [2,-1, 0]
25/24 + [-3,-1, 2]
F# = [-1,-2, 2]

Which gives the ratio 25/18, and is complementary to 36/25.
Another way to define a diminished fifth is by adding two minor 3^{rd}s.

mi 3rd [1, 1,-1]
mi 3rd + [1, 1,-1]
D 5th = [2, 2,-2]

In just intonation there are several candidates that are used as an augmented 4th/diminished 5^{th}. 45/32 and 64/45 respectively are often defined thus.

We can analyze 45/32. The vectors are [-5,2,1]. Reading the vectors tell us to go up two 5^{th}s (C – G, G – D) and up one 3^{rd} (D – F#). Mind you that the D is perfect and adding a major 3^{rd} yields a major 4^{th} or more accurately a major 11^{th}. The vector signature tells us also that this interval is terce-mood 1.

We could do the same with all of the scale notes to find all augmented notes, but I will refrain from doing that here. At the moment I am looking at the scale and augmented (and diminished) are getting out of scale.

Harmonic octaves

Another interesting interval is the syntonic comma, 81/80. Looking at the vectors [-4,4,-1] we can analyze the interval moving up four 5^{th}s to E and down a major 3^{rd} to C. This interval is clearly a unison and since its terce-moode is -1 we can state that this interval is a minor unison. Again we find an interval with an unusual definition. Where do we find a practical application for the syntonic comma?

First we can look at the difference between the perfect 2^{nd} and the major 2^{nd} .

9/8 [-3, 2, 0]
10/9 - [1,-2, 1]
81/80 = [-4, 4,-1]

From this we can deduct that the syntonic comma is the difference between a perfect interval and its major variant. But wait one minute, this implies that the major 2^{nd} is lesser then the perfect 2^{nd}. (9/8 = 1.25 and 10/9 = 1,11) The terce-moode idea gives us the impression that the major is greater than the perfect. This is not the reality. The truth is

dim. minor < minor < major < aug. major

dim. perfect < perfect < aug. perfect

although

minor < dim perfect

major < perfect

The minor/major sequence is not part of the perfect sequence, the two sequences are parallel. It is easier to visualize this using a modified version of Monzo's 5-limit lattice diagram. (Fig. 1) Intervals above each other are a 5th apart, intervals diagonally up to the right are major 3rds apart and diagonally up to the left are minor 3rds apart. Intervals that are aligned vertically have the same terce-mood. Intervals aligned horizontally are 25/24, a accidental semitone apart.

On the lower half of the diagram we find the D minor/major sequence. 7 lines above we can see the D perfect sequence. We could say they are an "octave" apart. Actually they are two octaves apart. Consider the fact that we went up four fifths and down a major 3rd. This gives a ratio of 81/20. (3/2 x 3/2 x 3/2 x 3/2 : 5/4) We divided by two octaves to bring the interval into scale. The ratio 81/80 is the "octave" distance on the lattice diagram.

In a 5-limit system we build our intervals using octaves, 3rds and 5ths. The 5-limit lattice helps us visualize the distance between intervals.

figure 1

Eb 243/200		E 81/64		E# 675/512
	C 81/80		C# 135/128	
Ab 81/50		A 27/16		A# 225/128
	F 27/20		F# 45/32	
Db 27/25		D 9/8		D# 75/64
	Bb 9/5		B 15/8	
Gb 36/25		G 3/2		G# 25/16
	Eb 6/5		E 5/4	
Cb 48/25		C 1/1		C# 25/24
	Ab 8/5		A 5/3	
Fb 32/25		F 4/3		F# 25/18
	Db 16/15		D 10/9	
Bbb 128/75		Bb 16/9		B 50/27
	Gb 64/45		G 40/27	
Ebb 256/225		Eb 32/27		E 100/81
	Cb 256/135		C 160/81	
Abb 1024/675		Ab 128/81		A 400/243

The anatomy of an interval

We can break down the smallest intervals of the scale and see how they are built up. Let's start by looking at the interval C-D. This interval is a whole tone. The whole tone is built up of 2 semitones. The first type of semitone we have is the 25/24, the chromatic semitone. Adding the chromatic semitone to C then we get C#.

C	1/1		[0, 0, 0]
	25/24	+	[-3,-1, 2]
C#	25/24	=	[-3,-1, 2]

The second semitone is the diatonic semitone. This is the same interval as between E-F and B-C.

F	4/3		[2,-1, 0]
E	5/4	-	[-2, 0, 1]
	16/15	=	[4,-1,-1]

If we add 16/15 to C# we get D

C#	25/24		[-3,-1, 2]
	16/15	+	[4,-1,-1]
D	10/9	=	[1,-2, 1]

And of coarse we can turn it around and add the diatonic semitone to C to get Db and then add the chromatic semitone to get D.

C	1/1		[0, 0, 0]
	16/15	+	[4,-1,-1]
Db	16/15	=	[4,-1,-1]
	25/24	+	[-3,-1, 2]
D	10/9	=	[1,-2, 1]

Before we look at the next interval, I would like to look at figure 1. We can clearly see that going from C-Db-D, we are moving downwards on the lattice. It would be natural to continue down to get to our next scale note E. But now we are getting out of scale, far away from our fundamental scale note of C. We have to adjust to get up to the E that is in-scale. Lets start by finding the difference between Eb and D.

Eb	6/5		[1, 1,-1]
D	10/9	-	[1,-2, 1]
	27/25	=	[0, 3,-2]

If we analyze the vectors we will find that this interval is a diminished second. Now this interval is replacing the diatonic semitone (16/15). We can find the difference between these.

27/25		[0, 3,-2]
16/15	-	[4,-1,-1]
81/80	=	[-4, 4,-1]

We have already discussed that the syntonic comma is used to adjust a note to the next “octave level”.

It seems clear that there is a conformity in the construction of the whole-tone interval. It consists of one chromatic semitone and one diatonic semitone. Where we have to adjust to keep in scale, we have to add a syntonic comma. Adding the syntonic comma to the diatonic semitone results in an interval I would like to call the syntonic semitone. (27/25)

Here is the scale with the respective intervals

	16/15		25/24		27/24		25/24		16/15		9/8		16/15		25/24		27/24		25/24		16/15	
C		Db		D		Eb		E		F		G		Ab		A		Bb		B		C

figure 2

I have chosen to let the interval between F and G be the perfect second. There are 2 reasons for this. The first reason is that it graphically shows the beautiful symmetry that natures harmony has. Secondly I only want to include perfect and major/minor intervals in the basic scale. Augmented and diminished intervals start to be off-scale. This said, it is natural, in a scale built up of semitones, that we have a semitone between F and G. All we have to do is decide if the chromatic semitone or the syntonic semitone will be used first. If we use the chromatic semitone first, then we get F – F# (25/18) – G. I personally prefer the F – Gb (36/25) – G because it is consistent with using the flat accidental.

The major triad

I would like to take a closer look at chords, more specifically triads. If we take the first three designated notes of the harmonic series and play them simultaneously we experience a very fulfilling harmony. This is known as a major triad. The major triad is composed of the unison (1/1), the major 3rd (5/4) and the perfect 5th (3/2). There is a relation between all major triads. Lets look at our C major chord.

The relation can be written $\frac{1}{1}:\frac{5}{4}:\frac{3}{2}$.

Finding the common divisor will yield $\frac{4}{4}:\frac{5}{4}:\frac{6}{4}$

or we could write $\frac{4}{1}:\frac{5}{1}:\frac{6}{1}$ which is actually telling us to play the 3rd, 4th and 5th harmonic in concision.

We could just simply write the relation 4:5:6 which means that the three notes in all major triads will relate to each other in this way

3rd to unison	5:4	major 3rd
5th to unison	6:4	perfect 5th (3/2)
5th to 3rd	6:5	minor 3rd

To find other major triads, all we have to do is adjust the unison to the new fundamental of the chord we want to construct. If we want to construct a F major chord (in the key of C) we take our major relation 4:5:6 an divide it by 3, a fifth down from C.

$\frac{4}{3}:\frac{5}{3}:\frac{6}{3}$ which gives us F (4/3), A (5/3) and C (6/3 reduced to 2/1 is the octave).

To find the G major chord we multiply with 3 to find the fifth above.
$\frac{4\cdot3}{1}:\frac{5\cdot3}{1}:\frac{6\cdot3}{1}$ when reduced gives $\frac{3}{2}:\frac{15}{8}:\frac{9}{8}$ and which gives us G, B and a perfect D. It would be more correct to use 9/4 as the 5^{th} to G is a 9^{th} and not a 2^{nd}.

The A major triad will look like this.
$\frac{4\cdot5}{3}:\frac{5\cdot5}{3}:\frac{6\cdot5}{3}$ when reduced gives $\frac{5}{3}:\frac{25}{24}:\frac{5}{4}$ translating to A C# E.

Another look at some intervals

I would like to backtrack and take another look at some of our intervals. The first one I want to look at is the 16/9 interval that many call a minor 7^{th}. Vector analyzing tells us that this is a perfect 7^{th}. So where do we find this interval. In the key of C we have 3 major chords, Tonic (C), Subdominant (F) and the Dominant (G). It is natural to associate the dominant with a 7^{th} in this case the F in the key of C. The difference between F (an octave above) and G is 16/9. It is usually called a minor 7^{th} but since it was analyzed as a perfect 7^{th} I would prefer to specify it as the Dominant 7^{th}.

Another interval worth looking closer at is the 64/45. We have already found that this is not a diminished fifth, but it is rather a minor 5^{th}. Again if we look at the Dominant 7^{th} chord we will find this interval. The difference between the F (octave above) and B is 64/45.

I will now go over to a personal interpretation. I feel that the main effect of the dominant 7^{th} chord is the interval between the dominants 3^{rd} and 7^{th}. They kind of pull towards each other and resolves to the major 3^{rd} of the tonic. Both of these movements from B – C and F – E are chromatic semitones or 16/15. And in regard to the problem of the 2^{nd} I feel it doesn't really matter what type of 3^{rd} ones chooses below the two notes of the minor 5^{th}. In a G7 as a dominant 7^{th} we can swap the D for a D# or Db or even swap the G with a G#. I feel we still get the dominant feeling. I don't think it is necessarily important that the dominant 5^{th} is perfect. Maybe it is of more importance the relationship between the 7^{th} and the 5^{th}.

The minor triad

A minor triad is a chord where the 3^{rd} is a minor interval. If we consider the major triad as being built from a major 3^{rd} and adding a minor 3^{rd} on top of that again, the minor is the opposite where we start with a minor 3^{rd} and add a major 3^{rd} on top.

We can apply the above to the minor triad and we will get the relation 10:12:15. Now this doesn't look quit as nifty. If we instead find a common dividend we will find the normal ratio for a minor triad which is $\frac{1}{6}:\frac{1}{5}:\frac{1}{4}$. We can see that this ratio is some kind of opposite or inverted variant of the major ratio. Lets analyze it and find out what this means.
1/6 says “from the fundamental find a note that is 2 octaves and a 5^{th} below (F).
1/5 says “from the fundamental find a note that is 2 octaves and a major 3^{rd} below (Ab).
1/4 says “from the fundamental find a note that is 2 octaves below (C).

We have described a F minor triad.

But wait one minute!!! We have described a triad that we recognize as a F minor, but constructed it from the fundamental of C.

If we want to construct a C minor we have to raise the fundamental up a 5th to G.
$\frac{3}{6}:\frac{3}{5}:\frac{3}{4}$ reduced to $\frac{1}{1}:\frac{6}{5}:\frac{3}{2}$ giving a C minor.

What does this tell us about the minor triad. Mathematically speaking a minor triad is the opposite of the major. While the major is constructed by adding a major 3rd and a 5th upwards, the minor triad is constructed by adding a major 3rd and a 5th **downwards** from the fundamental. It is almost like the minor chord is a mirror image of the major. The problem with using a system that builds harmonies downwards is that we tend to experience the lowest notes as the basis for the harmony. It seems difficult to learn to use the highest notes as the base (I've tried). It could be that it is the physical property of the standing wave, that the frequency goes higher that sets the standard. But the math tells us that a minor constructed from the fundamental C is a chord we understand as a F minor.

The only reference I know of considering this perspective is Dr. Hugo Riemann, 1849-1919, a German music theorist, who in his book Harmony simplified : The theory of the tonal functions of chords (1896) (http://www.archive.org/details/cu31924022305357) bases his harmony theory on the concept of overclangs and underclangs. He describes the triad A-C-E as the underclang of E. He considered the minor mode as the inverse of the major.

The minor scale

The minor scale would also be interesting to look at in this perspective. If we take our major scale

$$\frac{1}{1}:\frac{10}{9}:\frac{5}{4}:\frac{4}{3}:\frac{3}{2}:\frac{5}{3}:\frac{15}{8}:\frac{2}{1}$$

and invert it we get the following relations

$$\frac{1}{1}:\frac{16}{15}:\frac{6}{5}:\frac{4}{3}:\frac{3}{2}:\frac{8}{5}:\frac{9}{5}:\frac{2}{1}$$

Firstly we notice that like the triad the minor scale is the mirror image of the major scale. The first half step in the minor scale equals the last half step of the major scale and so on.

Secondly we see that the minor scale consists of all of the minor intervals. This seems to be a natural consequence of the idea of a minor scale, play all the minor intervals.

This minor scale doesn't match our normal C minor scale. Again it matches the F minor scale, like the minor triad matched the F minor chord. Actually I don't see a problem with this. I am trying to explain “minor-ness” as the mirror image of the major scale. That we understand the mirror image of the C scale as the F minor scale is because we can't mirror-image the base note of the scale.

End note

This brings me to the end. Not that I feel I am finished but I hope that any input will give me inspiration further.

I really do believe that a theory on harmony shows itself in a symmetrical and harmonious fashion as I feel this does. Thank you for reading

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