Tenney Weighted Integrals of Regular Temperaments

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1 Integral for the Tenney Weighted Invariant

The equation I'm solving today is

$$\int_{\frac{-a}{h_n}}^{\frac{a}{h_n}} \dots \int_{\frac{-a}{h_2}}^{\frac{a}{h_2}} \int_{\frac{-a}{h_1}}^{\frac{a}{h_1}} \boldsymbol{n} \cdot \boldsymbol{x}\, \boldsymbol{x} \, \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \tag{1}$$

where a is an arbitrary real constant, h_i are the logarithms of prime numbers, $\boldsymbol{n}$ is the mapping for an equal temperament of n (notice the difference in boldness) prime dimensions, $\boldsymbol{x}$ is a vector to integrate over, and x_i are the components of $\boldsymbol{x}$. The result is an invariant vector: it describes something like an interval that doesn't change its size when tempered. Whether it's really an interval is up for debate, as it doesn't have integer coefficients. That needn't concern us.

I hope this is some sensible problem to do with the average error of arbitrary intervals in a prime limit bounded by the Tenney harmonic distance. Tenney weighting is implicit in the choice of limits for the integrals. By integrating over a wider range for smaller primes, it means more of those small primes get considered.

The first step to taming this hyperspatial integral is redefining $\boldsymbol{x}$ as $x_i \to \frac{x_i}{h_i}$. That simplifies the integral limits. It also means the mapping $\boldsymbol{n}$ is replaced by the weighted mapping $\boldsymbol{m}$ and because of a stray h_i the resulting vector is an invariant of the weighted mapping rather than the unweighted one. So that gives

$$\int_{-a}^{a} \dots \int_{-a}^{a} \int_{-a}^{a} \boldsymbol{m} \cdot \boldsymbol{x}\, \boldsymbol{x} \, \frac{\mathrm{d}x_1}{h_1} \frac{\mathrm{d}x_2}{h_2} \dots \frac{\mathrm{d}x_n}{h_n} \tag{2}$$

or

$$\frac{1}{\Pi_{i=1}^{n} h_i} \int_{-a}^{a} \dots \int_{-a}^{a} \int_{-a}^{a} \boldsymbol{m} \cdot \boldsymbol{x}\, \boldsymbol{x} \, \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \tag{3}$$

The only point of this vector is that it has the same weighted size before and after tempering. So scaling it by a constant doesn't affect this property and it's simpler as

$$\int_{-a}^{a} \dots \int_{-a}^{a}\int_{-a}^{a} \boldsymbol{m}\cdot\boldsymbol{x}\,\boldsymbol{x}\ \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \tag{4}$$

That's what I'm going to solve.

2 Solving for an Equal Temperament

Equation 4 can easily be split up. All we do is look at each element of $\boldsymbol{x}$ individually. That means solving

$$\int_{-a}^{a} \dots \int_{-a}^{a}\int_{-a}^{a} \sum_{j=1}^{n} m_j x_j x_i\ \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \tag{5}$$

There are two cases to look at: integrating the ith component with respect to x_i, and integrating it with respect to any other element of $\boldsymbol{x}$.

To the first case, then

$$\begin{aligned}
\int_{-a}^{a} \sum_{j=1}^{n} m_j x_j x_i\ \mathrm{d}x_i &= \int_{-a}^{a} \sum_{j=1,j\neq i}^{n} m_j x_j x_i + m_i x_i^2\ \mathrm{d}x_i && (6)\\
&= \left[\frac{1}{2}\sum_{j=1,j\neq i}^{n} m_j x_j x_i^2 + \frac{1}{3} m_i x_i^3\right]_{x_i=-a}^{x_i=a} && (7)\\
&= \left[\frac{1}{2}\sum_{j=1}^{n} m_j x_j x_i^2 - \frac{1}{2} m_i x_i^3 + \frac{1}{3} m_i x_i^3\right]_{x_i=-a}^{x_i=a} && (8)\\
&= \left[\frac{1}{2}\sum_{j=1}^{n} m_j x_j x_i^2 - \frac{1}{6} m_i x_i^3\right]_{x_i=-a}^{x_i=a} && (9)\\
&= -\frac{1}{3} m_i a^3 && (10)
\end{aligned}$$

That's nice and simple because the quadratic term cancels out what with the symmetric limits of the integral. It'll never have to be integrated over x_i again.

Now, here's the integral of the original element with respect to a different element.

$$k \neq i, \int_{-a}^{a} \sum_{j=1}^{n} m_j x_j x_i \,\mathrm{d}x_k = \int_{-a}^{a} \sum_{j=1, j\neq k}^{n} m_j x_j x_i + m_k x_k x_i \,\mathrm{d}x_k \tag{11}$$

$$= \left[\sum_{j=1, j\neq k}^{n} m_j x_j x_i x_k + \frac{1}{2} m_k x_k^2 x_i \right]_{x_k=-a}^{x_k=a} \tag{12}$$

$$= \left[\sum_{j=1}^{n} m_j x_j x_i x_k - m_k x_k^2 x_i + \frac{1}{2} m_k x_k^2 x_i \right]_{x_k=-a}^{x_k=a} \tag{13}$$

$$= 2a \sum_{j=1}^{n} m_j x_j x_i \tag{14}$$

That leaves the form of the term unchanged. So every time you integrate over something other than x_i you add a factor of $2a$. Until you do integrate by x_i, in which case there's a different form to deal with.

$$\int_{-a}^{a} -\frac{1}{3} m_i a^3 \,\mathrm{d}x_k = -\frac{1}{3} m_i a^3 \left[x_k \right]_{x_k=-a}^{x_k=a} \tag{15}$$

$$= -\frac{2}{3} m_i a^4 \tag{16}$$

Like before, the form is unchanged but a factor of $2a$ gets added. So integrating over anything other than x_i always means multiplying by $2a$.

Knowing all this, the general solution for Equation 5 must be

$$\int_{-a}^{a} \ldots \int_{-a}^{a} \int_{-a}^{a} \sum_{j=1}^{n} m_j x_j x_i \,\mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n = -\frac{2^{n-1}}{3} m_i a^{n+2} \tag{17}$$

and Equation 4 is

$$\int_{-a}^{a} \ldots \int_{-a}^{a} \int_{-a}^{a} \boldsymbol{m} \cdot \boldsymbol{x} \, \boldsymbol{x} \,\mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n = -\frac{2^{n-1}}{3} a^{n+2} \boldsymbol{m} \tag{18}$$

That constant isn't interesting, so our invariant vector is simply the weighted mapping of the equal temperament. It's also the same no matter what the size of the hypercube you integrate over. Partly that's because it's an integral over a continuous space, so all intervals are implied by even small chunks. The optimal scale stretch will mean that the weighted mapping gets mapped to itself. And we know what that optimal scale stretch will be — it optimizes the root mean squared error. This follows from the standard equation for a linear least squares problem. For example, using symbols from my Prime Errors and Complexities PDF, it's

$$M^T M G_{\text{opt}} = M^T V \tag{19}$$

The left hand side is the tempered, pre-weighted size of M^T (the weighted mapping treated as an interval) and the right hand side is the equivalent in just intonation (or what would be just intonation if this were an integer matrix).

If you want something like an interval that doesn't get tempered, you have to double-weight the mapping.

3 Arbitrary Regular Temperaments

Generalizing from the optimal scale stretch for an equal temperament to the optimal tuning for any regular temperament is simple. All you do is replace the mapping $\boldsymbol{m}$ with a matrix of weighted mappings M. The integral of any column doesn't affect the others, so the result is the same as the rank 1 case with some substitutions.

$$\int_{-a}^{a} \dots \int_{-a}^{a} \int_{-a}^{a} M\boldsymbol{x}\,\boldsymbol{x}\ \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n = -\frac{2^{n-1}}{3} a^{n+2} M^T \tag{20}$$

That means the invariant weighted vectors are the transposes of the weighted mappings. And, for the same reasons as before, it gives the least squares optimum, or the TOP-RMS tuning.