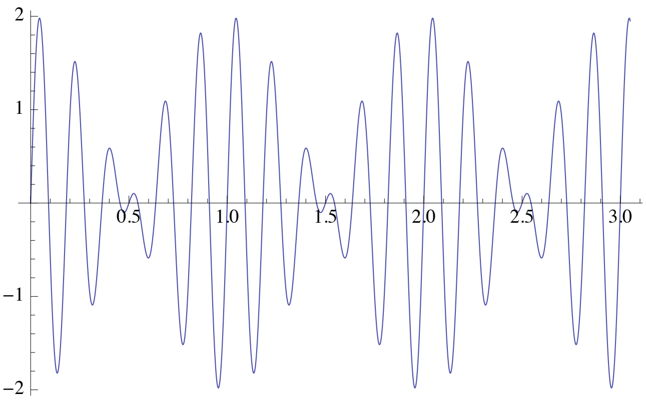
Second Order GCD's:

An Alternative to Fourier Analysis

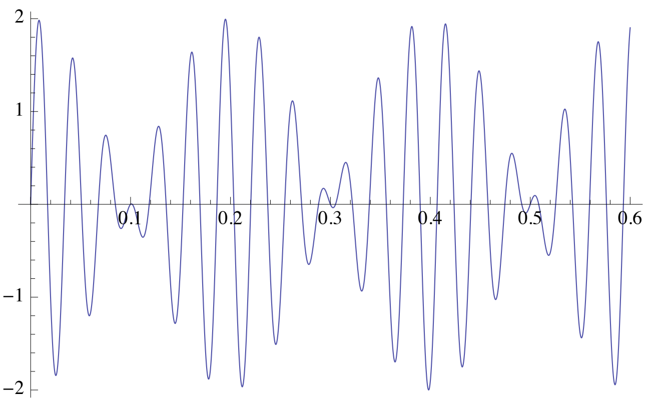
Rick Ballan 2010

By way of introduction, consider a graphic comparison between the wave of the epimoric minor third interval 5:6 and two non-epimoric ones, the Pythagorean 27:32 and the rational tempered 16:19 minor thirds:

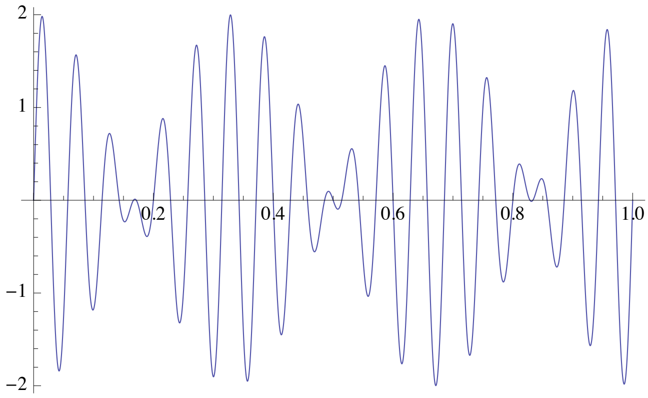
```
Plot[Sin[2 π 5 t] + Sin[2 π 6 t], {t, 0, 3.05}]
```




```
Plot[Sin[2 π 27 t] + Sin[2 π 32 t], {t, 0, 0.6}]
```



Plot[(Sin[2 π 16 t] + Sin[2 π 19 t]), {t, 0, 1}]



The frequency of the first wave, which of course corresponds to the GCD of 1, can be clearly seen graphically in the time between the first two largest wave peaks (maxima). Therefore this minor third interval appears as the 5th and 6th harmonic to the tonic 1. Now, comparingthis to the other two we see that there seems to be a distinctly similar pattern to all three, a certain "minor third-ness" they all have in common. We might guess,for example, that the frequency inverse of the time between the first two largest maxima in the second and third graphs also serves as some type of 'GCD' or tonic to the 27th/16th and 32nd/19th harmonics, respectively. The following article shows that this is indeed the case. When dealing with large-numbered intervals, one of the problems has been in trying to place the role of a GCD that is so far removed from the component frequencies. The minor third 27:32 and 16:19 above are two cases in point. If 16 → A440Hz, for example,then this would place the GCD at 440/16 = 27.5Hz which is approaching the sub-audible cut-off of 20Hz. On the other hand, if 5 → A440Hz then 440/5 = 88Hz.

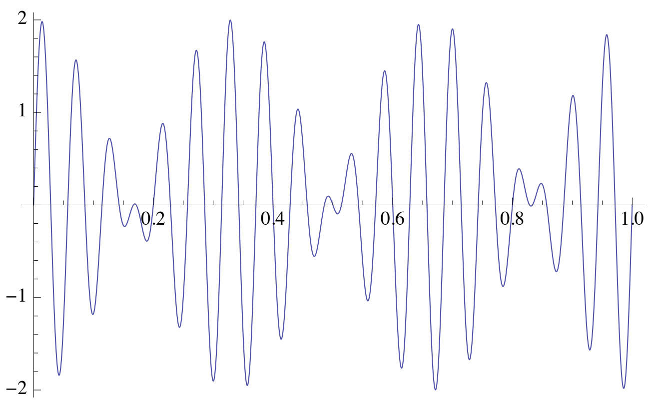
This problem is solved, however, when it is seen that these peaks do in fact correspond to something relatively close to the GCD. Once identified, it is then seen that these '2nd order GCD's'correspond to a *sub-harmonic of the sum (average) frequency*. That is, for two frequencies a:b,the 'GCD'∈ (a+b)/n, n = 1,2,3... What comes as a complete surprise is the fact that this value n will actually equal the sum of the small-numbered (epimoric) intervals to which it approximates. For the example above, this value will equal 440Hz(1 + (19/16))/(5 + 6) = 87.5Hz. As we see,it does indeed approximate 88Hz. And before we proceed, I'd suggest briefly looking at the graph at the bottom of the next chapter to see the degree of concordance we can expect from this method.

Those intervals which have proved useful in traditional tonal tunings will likely possess a sub-harmonic that approximates the GCD of its epimoric counterpart. OTOH, those that give a bad match to any GCD are of little use for tonality. Of particular interest is the fact thatthese frequencies, being semi-periodic, possess no Fourier component and yet seem as 'real' as any wave can be. Flying under the radar of Fourier analysis, it is little wonder then that this important class of frequencies seems to have passed by completely undetected. The following is only a brief introduction.

Mathematical Derivation

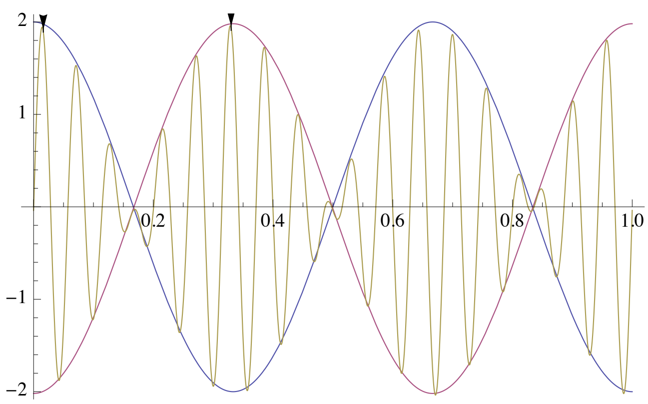
Here I will begin by finding the times at which the firsttwo maxima appear for 16:19. The results are then generalised to a universal formula. Using a standard trig ID, the 16:19 sine wave above can be rewritten as an averaged wave multiplied by a modulated amplitude:

```
Plot[(2 Sin[π (16 + 19) t] Cos[π (19 - 16) t]), {t, 0, 1}]
```



Plotting this with the modulated amplitude (envelope) gives:

```
Plot[{(2 Sin[π (16 + 19) t] Cos[π (19 - 16) t]),
  2 Cos[π (19 - 16) t], -2 Cos[π (19 - 16) t]}, {t, 0, 1}]
```



Next, we wish to find maximas where the original wave and the envelope meet. The simultaneous Equation for the modulated amplitude and the original wave is

2 Sin[π(16 + 19)t] Cos[π(19 - 16)t] =
± 2Cos[π(19 - 16)t] or

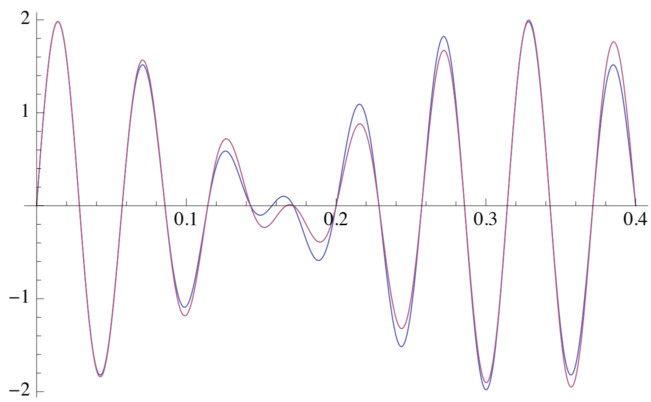
Sin[π (16 + 19) t] = ±1.

Solving for time gives

t = (2n - 1)/(2*35), n = 1, 2, 3, ...

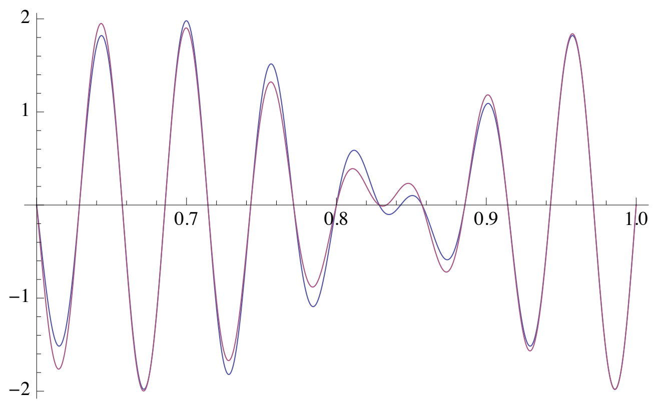
Now, the first two *largest* maxima that fit the above cosine curve (marked by arrows) occur at n = 1and n = 12 giving the times t = 1/70 and t = 23/70. Taking the difference between these times and inverting gives a frequency of 35/11. Multiplying by 5 and 6 gives 15.90909 ...and 19.0909 ... implying that the original frequencies are sufficiently close to the 5th and 6th harmonics of 'GCD' tonic 35/11 to be regarded as slightly detuned harmonics to this fundamental. Below is an overlay of the two waves which we see is a very good match.

```
Plot[{Sin[2 π 5 * (35 / 11) t] + Sin[2 π 6 * (35 / 11) t], Sin[2 π 16 t] + Sin[2 π 19 t]}, {t, 0, 0.4}]
```



Although the two waves become more out-of-phase as time progresses, it must be kept in mind that this match exists within each cycle of the original wave. The point is to demonstrate that the time between these wave peaks, which is only part of the cycle of 16:19, corresponds closely to one full cycle of the 5:6, and that this pattern is repeated for every cycle of the 16:19. The implication is that we can sometimes perceive this ~ GCD as a tonic. Observe for instance that, since we are dealing with sine waves, then the function is odd and the latter part of the cycle will be a reversed reflection of the first part:

```
Plot[{-Sin[2 π 5 * (35 / 11) (1 - t)] + -Sin[2 π 6 * (35 / 11) (1 - t)],
  Sin[2 π 16 t] + Sin[2 π 19 t]}, {t, 0.6, 1}]
```



And just to reiterate, note also that the value 11 here is just the sum of 5 and 6, the epimoric interval to which we are approximating. Investigating further it is seen that this does indeed apply to all cases, which leads us to the following definition:

General Formula

As one would expect, the 'GCD' is much easier to deduce if we already know the low-numbered (epimoric) ratio to which it approximates. And if we don' t know this interval, then it can easily be deduced via the usual method of rounding - off or continued fractions. In any case, once this lower ratio is known, if we take a/b as the higher interval, p/q as the lower, then the second order GCD is defined as

~GCD = (a + b)/(p + q),

a ≃ p[(a + b)/(p + q)] and b ≃ q[(a + b)/(p + q)].

It is finally seen that this value now represents the *ratio of the sum frequencies*.

[NOTE: Another, perhaps more rigorous, method for obtaining thedesired maxima involves finding the peak which lies closest to that of the envelope. Substituting t = (2n - 1)/(2*35) into the envelope wave and applying an approximation argument gives:

± Cos(π3(2n - 1)/2*35)≃ 1

or

3(2n - 1)/2*35)≃ 2N, N = 0,1,2,... for + cos,
3(2n - 1)/2*35)≃ (2N - 1), for - cos.

In general (a - b)(2n - 1)/2(a + b)≃ N.

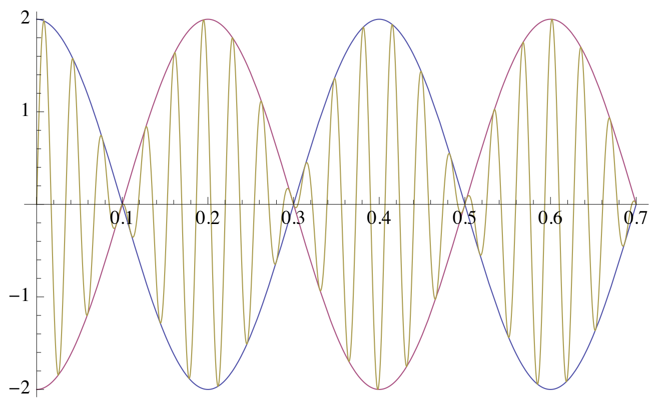
Since our peak occurs for - cos around π, then N = 1, from which we obtain n = 12, the expected result.

Corollary: Observe that this breaks downfor epimoric harmonics. Since (a - b) = 1 and 2(a + b) is even, then the closest values will be the two odds at either side of 2(a + b). Then (2n - 1) = 2(a + b) + 1, gives n = (a + b) + 1 and (a + b)/(n - 1) = 1 i.e. itself. This leaves (2n - 1) = 2(a + b) - 1 and (a + b)/((a + b) - 1)) as our only remaining possibility. But the only number which has a GCD of itself minus 1 is the number 2, in which case a = b = 1 and (a - b)= 0, *contra hyp*. IOW there exist no smaller numbers that approximate consecutive harmonics, which of course we 'knew' already.]

The Pythagorean Minor Third

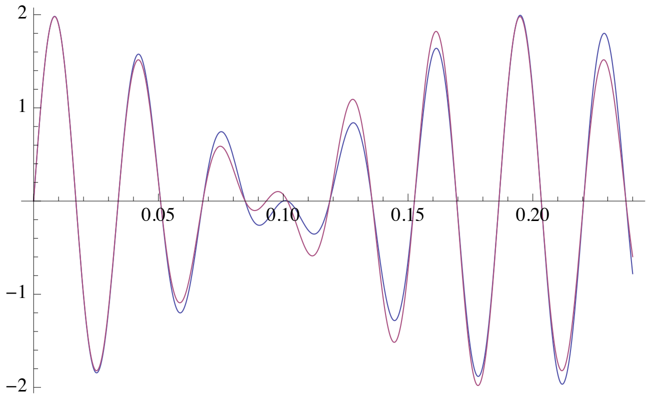
Let us now apply this process to the Pythagorean Minor Third 27 : 32.

```
Plot[{2 Cos[π (32 - 27) t], -2 Cos[π (32 - 27) t],
  (2 Sin[π (27 + 32) t] Cos[π (32 - 27) t])}, {t, 0, 0.7}]
```



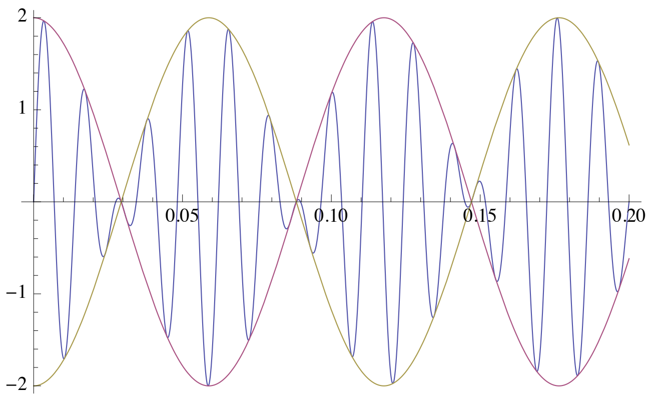
5(2n - 1)/2*59) ≃ 1, for - cos. Therefore n = 12, 'GCD' = 59/11 and once again we see that our initial pair correspond approximately to the 5th and 6th harmonic of this tonic:

```
Plot[{Sin[2 π 27 t] + Sin[2 π 32 t], Sin[2 π 5 * (59 / 11) t] + Sin[2 π 6 * (59 / 11) t]}, {t, 0, 0.24}]
```



The Pythagorean Major Third 64 : 81

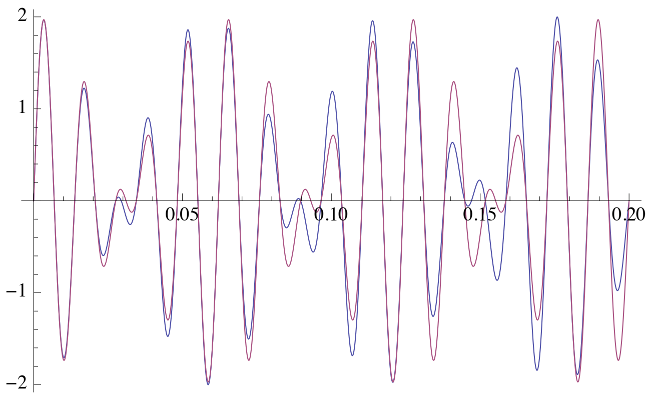
```
Plot[{ (2 Sin[π (64 + 81) t] Cos[π (81 - 64) t]),
  2 Cos[π (81 - 64) t], -2 Cos[π (81 - 64) t]}, {t, 0, 0.2}]
```



Before I discovered the general formula above I had to deduce the correct value geometrically. Sometimes mistakes can form a good basis for comparison so here is an example.

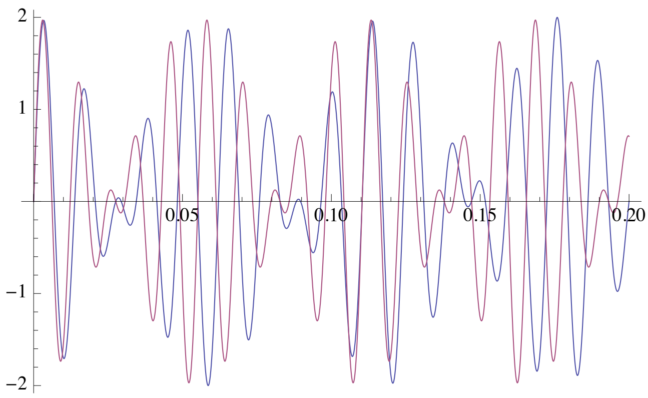
17(2n - 1)/2*145) ≃ 1. Therefore 290/17 ≃ 17 and n = 9. However, 'GCD' = 145/8 is not the correct result. 'Eyeballing' the wave we see that the crest we need has slightly overstepped the midway-point. Therefore we choose instead n = 10, which still gives a result approximate to 1, and 'GCD' = 145/9. A plot of the two waves confirms this:

```
Plot[{(2 Sin[π (64 + 81) t] Cos[π (81 - 64) t]),
  Sin[2 π 4 * (145 / 9) t] + Sin[2 π 5 * (145 / 9) t]}, {t, 0, 0.2}]
```



As we see below, the first choice is an ideal example of a bad match:

```
Plot[{(2 Sin[π (64 + 81) t] Cos[π (81 - 64) t]),
  Sin[2 π 4 * (145 / 8) t] + Sin[2 π 5 * (145 / 8) t]}, {t, 0, 0.2}]
```



Interval Matching

Here is an example of how the technique may help to find the correct interval match/classification. Take for eg 33/27 which is between the minor third as 32/29 and major third as 34/27. Using the techniques I've given already we have ~ GCD = (a + b)/(p + q) where a/b is the larger interval and p/q the smaller, then (33 + 27)/(4 + 5) = 60/9 for the major third and (33 + 27)/(5 + 6) = 60/11 for the minor. This gives 4*(60/9) = 26.6666 and 5*(60/9) = 33.333 as our approx 27 and 33 for the major, 5*(60/11) = 27.2727... and 6*(60/11) = 32.7272.-..for the minor. We see that the minor is the closer one.

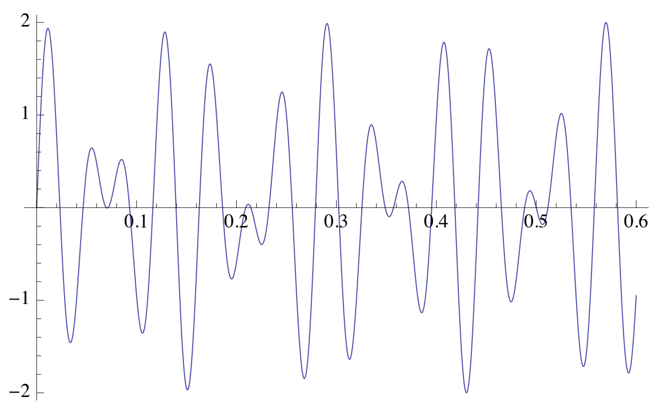
Notice that in both cases we have a whole number + remainder i.e. 27.2727...= 27 + 3/11. Obviously the p,q pair that give the smallest remainder will be the closer match. For the remainder R, the general formula is:

p[(a + b)/(p + q)] = a - R,
q[(a + b)/(p + q)] = b + R, where
R = (aq - pb)/(p + q).

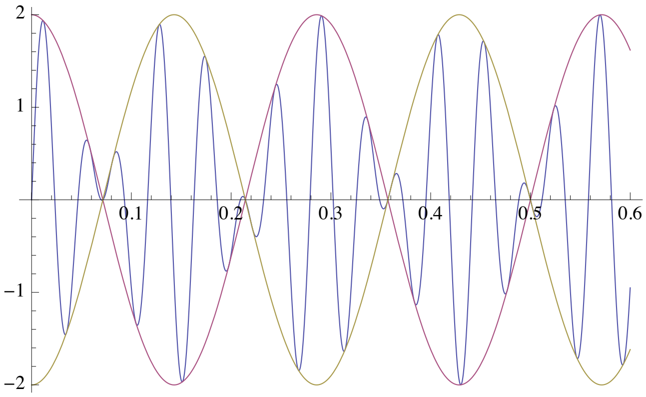
Therefore we need only to plug our values in this equation for R to see which pair p and q gives the smallest value. However, having now established that 33/27 is to correspond to the minor third and not the major, we are now also in a position to see why history has chosen 32/27 as our minor third in this vicinity of numbers and not 33/27. From the formula above we see that for 32/27, R = -2/11. Since |-2/11| < |3/11| then it is the greater approximation.

Other Random Examples

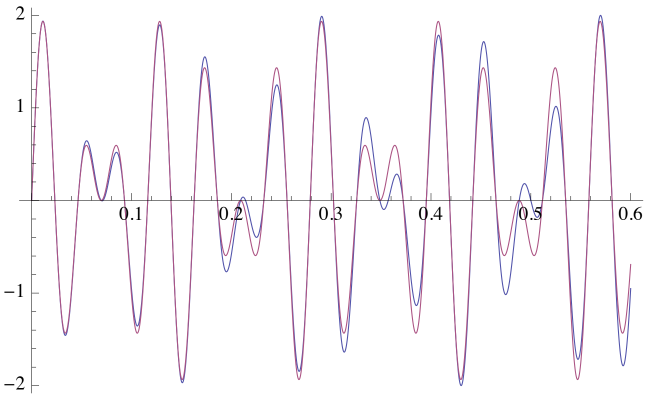
```
Plot[Sin[2 π 18 t] + Sin[2 π 25 t], {t, 0, 0.6}]
```



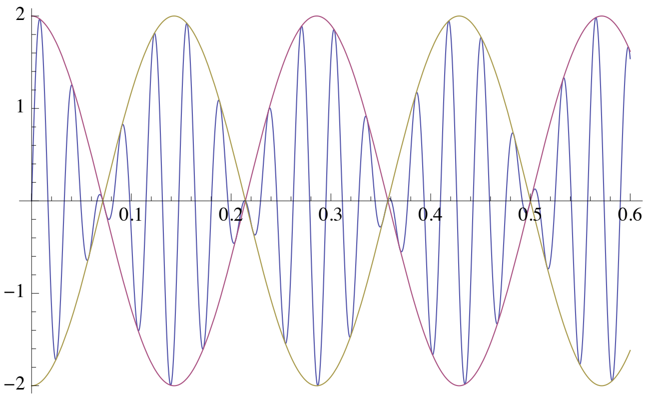
```
Plot[{Sin[2 π 18 t] + Sin[2 π 25 t], 2 Cos[π (25 - 18) t], -2 Cos[π (25 - 18) t]}, {t, 0, 0.6}]
```



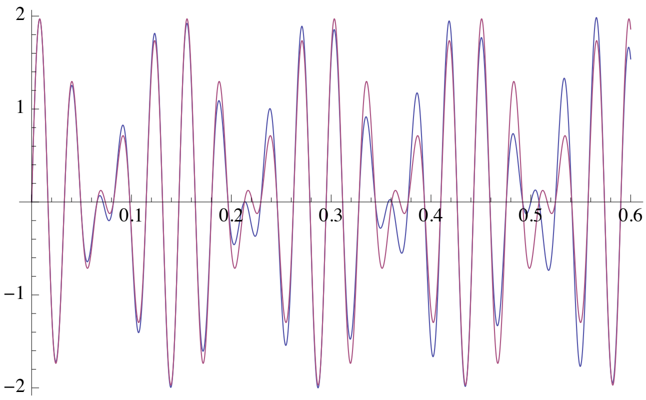
```
Plot[{Sin[2 π 18 t] + Sin[2 π 25 t], Sin[2 π 5 * (43 / 12) t] + Sin[2 π 7 * (43 / 12) t]}, {t, 0, 0.6}]
```



```
Plot[{Sin[2 π 27 t] + Sin[2 π 34 t], 2 Cos[π (34 - 27) t], -2 Cos[π (34 - 27) t]}, {t, 0, 0.6}]
```

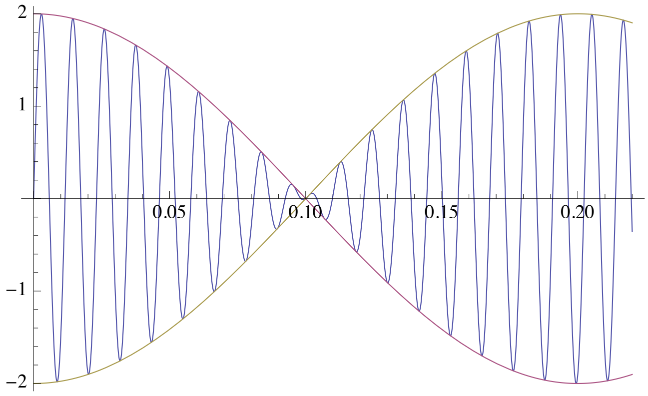


```
Plot[{Sin[2 π 27 t] + Sin[2 π 34 t], Sin[2 π 4 * (61 / 9) t] + Sin[2 π 5 * (61 / 9) t]}, {t, 0, 0.6}]
```

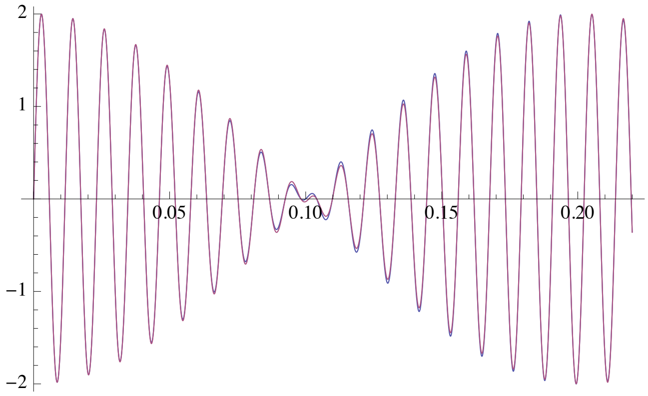


Rational Approximation to Tempered Semitone: 89/84 = 1.059523...

```
Plot[{Sin[2 π 84 t] + Sin[2 π 89 t], 2 Cos[π 5 t], -2 Cos[π 5 t]}, {t, 0, 0.22}]
```

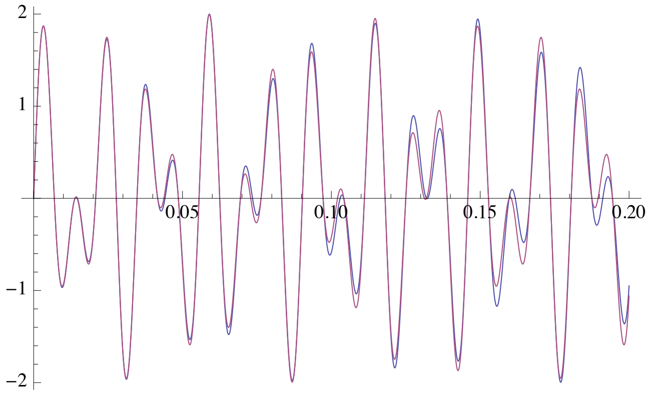



```
Plot[{Sin[2 π 84 t] + Sin[2 π 89 t],
  Sin[2 π 17 * (173 / 35) t] + Sin[2 π 18 * (173 / 35) t]}, {t, 0, 0.22}]
```



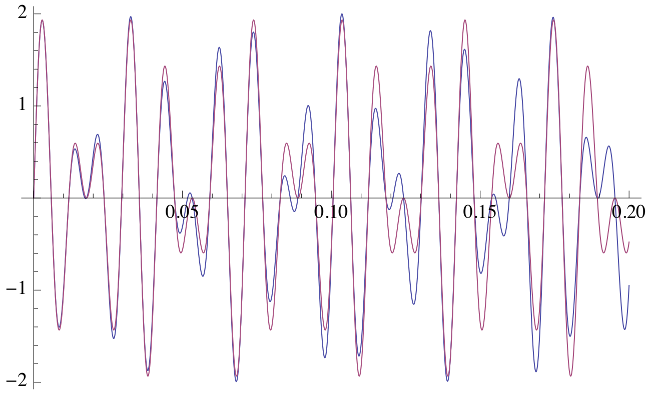
Golden Ratio. Observe that 55:89 predicts a lower interval from the Fibbinachi Series, 8:13.

```
Plot[{Sin[2 π 55 t] + Sin[2 π 89 t],
  Sin[2 π 8 * (144 / 21) t] + Sin[2 π 13 * (144 / 21) t]}, {t, 0, 0.2}]
```

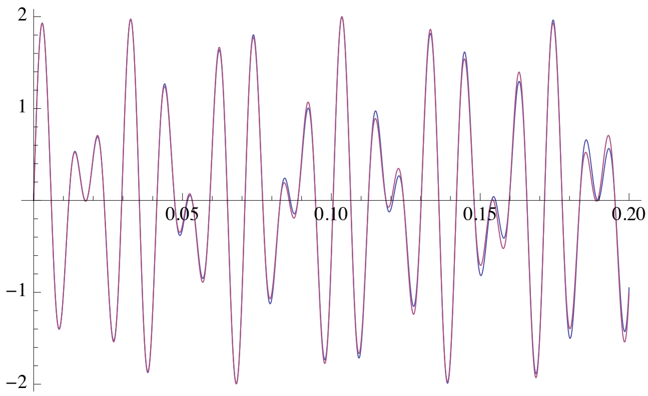


Square Root of 2. Notice that 12:17 is a much better fit than 5:7.

```
Plot[{Sin[2 π 70 t] + Sin[2 π 99 t],
  Sin[2 π 5 * (169 / 12) t] + Sin[2 π 7 * (169 / 12) t]}, {t, 0, 0.2}]
```

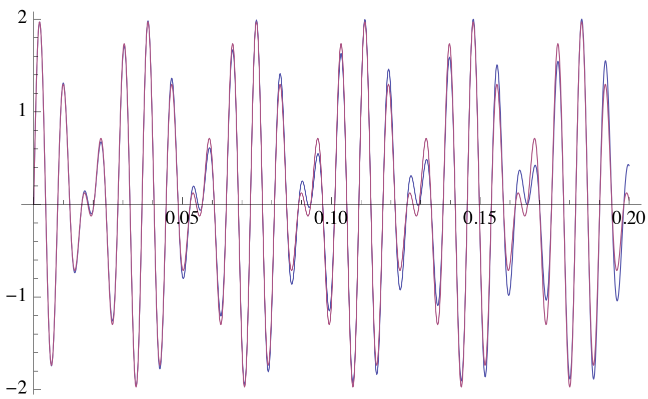



```
Plot[{Sin[2 π 70 t] + Sin[2 π 99 t],
  Sin[2 π 12 * (169 / 29) t] + Sin[2 π 17 * (169 / 29) t]}, {t, 0, 0.2}]
```



Lucy Tuned major third.

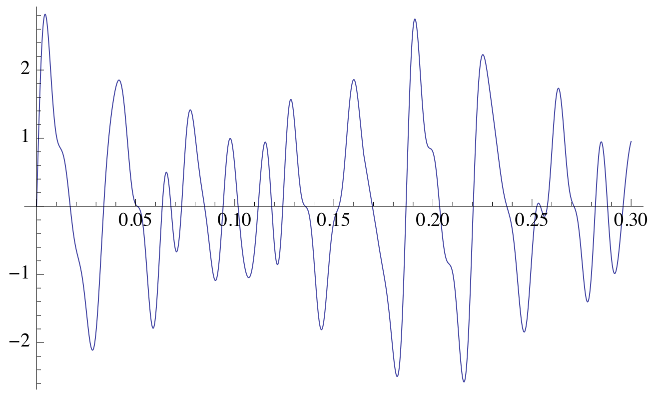
```
Plot[{Sin[2 π 110 t] + Sin[2 π 137.156 t],
  Sin[2 π 4 * (247.156 / 9) t] + Sin[2 π 5 * (247.156 / 9) t]}, {t, 0, 0.2}]
```



Upper Harmonics

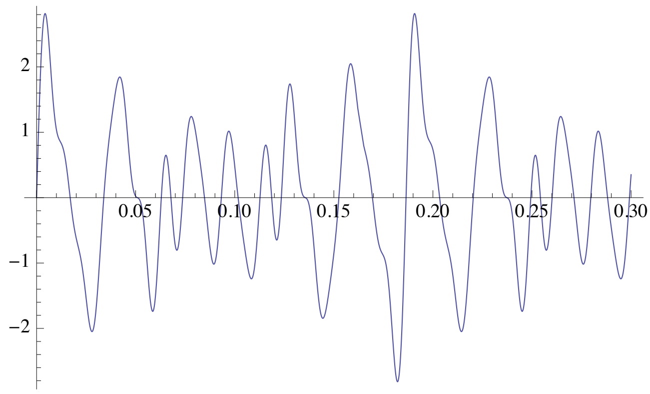
The interval matching also holds for the addition of upper harmonics. The first three harmonics of 27:32 gives:

```
Plot[Sin[2 π 27 t] + 0.5 Sin[2 π 54 t] + 0.3 Sin[2 π 81 t] +
  Sin[2 π 32 t] + 0.5 Sin[2 π 64 t] + 0.3 Sin[2 π 96 t], {t, 0, 0.3}]
```



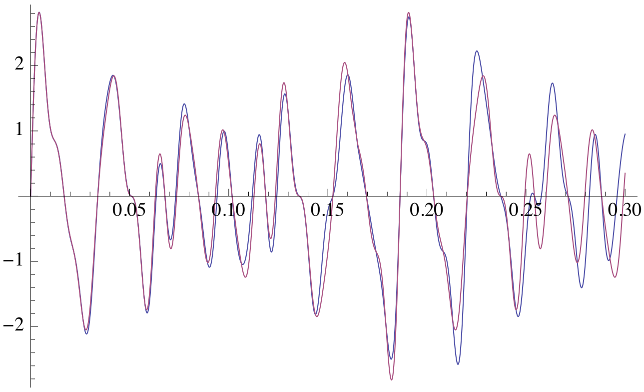
The first three harmonics of 5(59/11):6(59/11) gives:

```
Plot[Sin[2 π 5 * (59 / 11) t] + 0.5 Sin[2 π 10 * (59 / 11) t] + 0.3 Sin[2 π 15 * (59 / 11) t] +
  Sin[2 π 6 * (59 / 11) t] + 0.5 Sin[2 π 12 * (59 / 11) t] + 0.3 Sin[2 π 18 * (59 / 11) t], {t, 0, 0.3}]
```



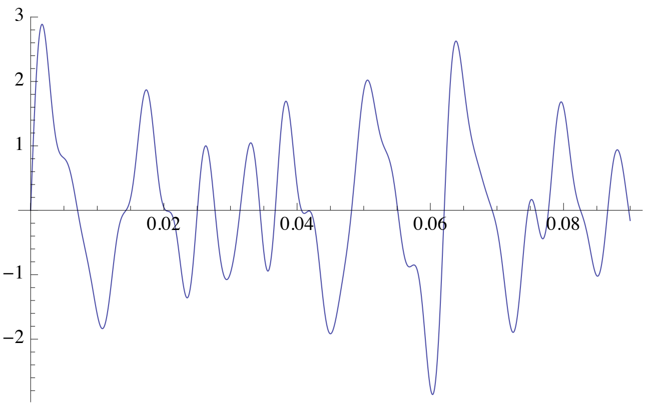
The graph below shows that the match holds:

```
Plot[{Sin[2 π 27 t] + 0.5 Sin[2 π 54 t] + 0.3 Sin[2 π 81 t] +
    Sin[2 π 32 t] + 0.5 Sin[2 π 64 t] + 0.3 Sin[2 π 96 t], Sin[2 π 5 * (59 / 11) t] +
    0.5 Sin[2 π 10 * (59 / 11) t] + 0.3 Sin[2 π 15 * (59 / 11) t] + Sin[2 π 6 * (59 / 11) t] +
    0.5 Sin[2 π 12 * (59 / 11) t] + 0.3 Sin[2 π 18 * (59 / 11) t]}, {t, 0, 0.3}]
```

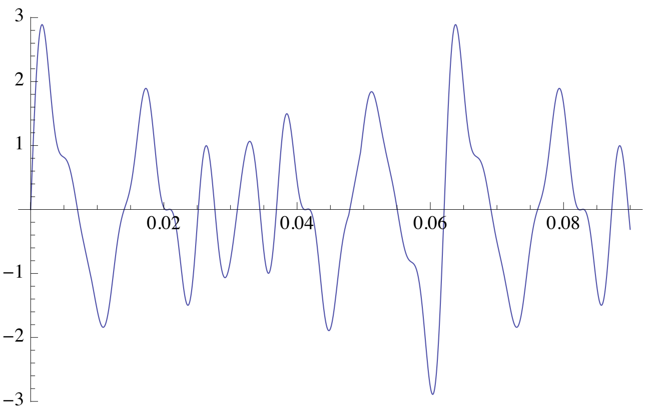


Same for the Pythagorean major third:

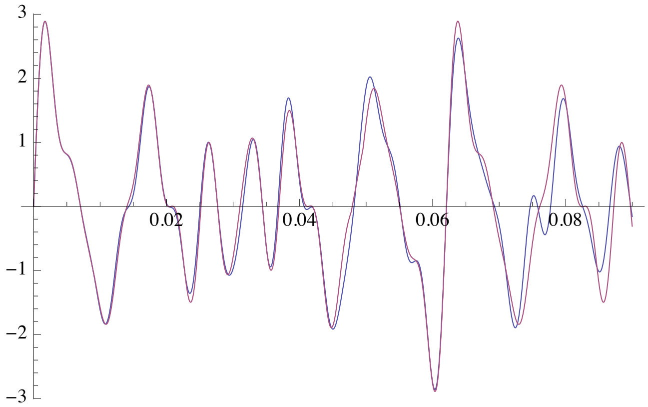
```
Plot[Sin[2 π 64 t] + 0.5 Sin[2 π 2 * 64 t] + 0.4 Sin[2 π 3 * 64 t] +
  Sin[2 π 81 t] + 0.5 Sin[2 π 2 * 81 t] + 0.3 Sin[2 π 3 * 81 t], {t, 0, 0.09}]
```




```
Plot[Sin[2 π 4 * (145 / 9) t] + 0.5 Sin[2 π 8 * (145 / 9) t] + 0.4 Sin[2 π 12 * (145 / 9) t] +
  Sin[2 π 5 * (145 / 9) t] + 0.5 Sin[2 π 10 * (145 / 9) t] + 0.3 Sin[2 π 15 * (145 / 9) t], {t, 0, 0.09}]
```



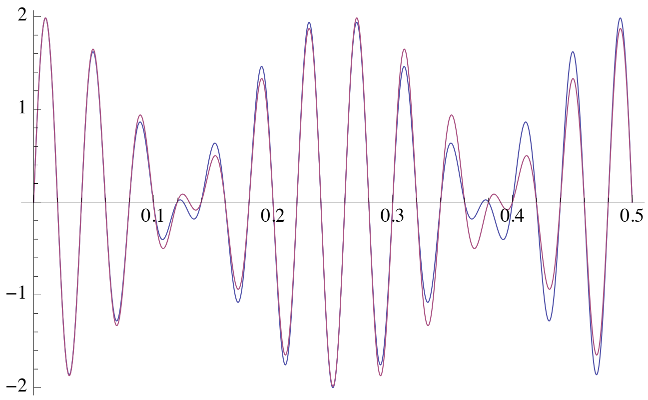
```
Plot[{Sin[2 π 64 t] + 0.5 Sin[2 π 2 * 64 t] + 0.4 Sin[2 π 3 * 64 t] +
   Sin[2 π 81 t] + 0.5 Sin[2 π 2 * 81 t] + 0.3 Sin[2 π 3 * 81 t], Sin[2 π 4 * (145 / 9) t] +
   0.5 Sin[2 π 8 * (145 / 9) t] + 0.4 Sin[2 π 12 * (145 / 9) t] + Sin[2 π 5 * (145 / 9) t] +
   0.5 Sin[2 π 10 * (145 / 9) t] + 0.3 Sin[2 π 15 * (145 / 9) t]}, {t, 0, 0.09}]
```



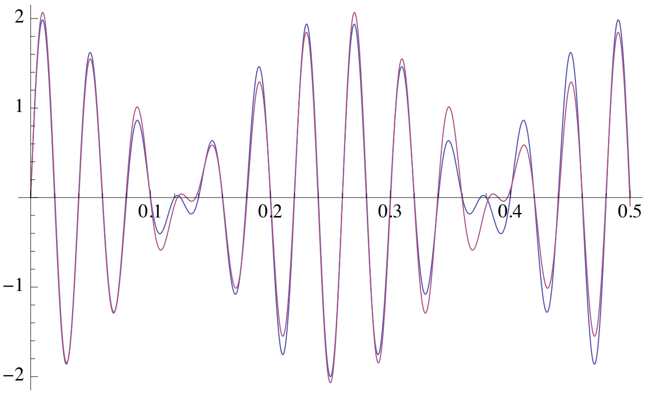
Octave Equivalence

Most importantly it is seen that the addition of octave equivalentsof the ~GCD, that is 2^-N, N =1,2,3,..., merely changes the amplitude while the characteristic wave shape remains unaffected. Since waves which have the same shape essentially producethe same sound, then this property seems to reflect why the addition of 8ve's is generally regarded as not altering the harmonic content. It might also explain why we can sometimes hear pitches that are 8ve equivalent to the GCD or ~GCD. Here is an example of the flattened minor third as 23:27. Below this graph are some others where lower 8ves with small amplitude have been added:

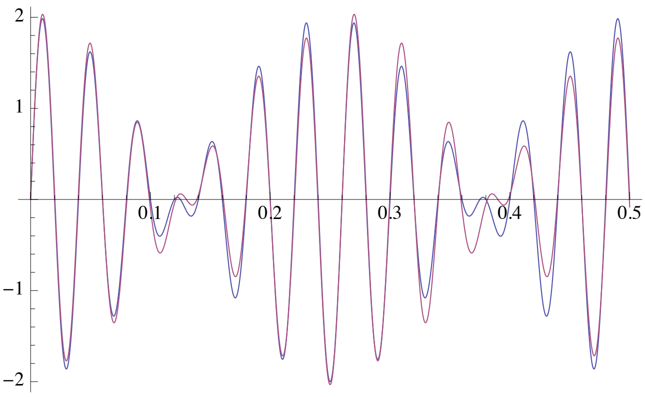
```
Plot[{Sin[2 π 23 t] + Sin[2 π 27 t], Sin[2 π 6 * (50 / 13) t] + Sin[2 π 7 * (50 / 13) t]}, {t, 0, 0.5}]
```



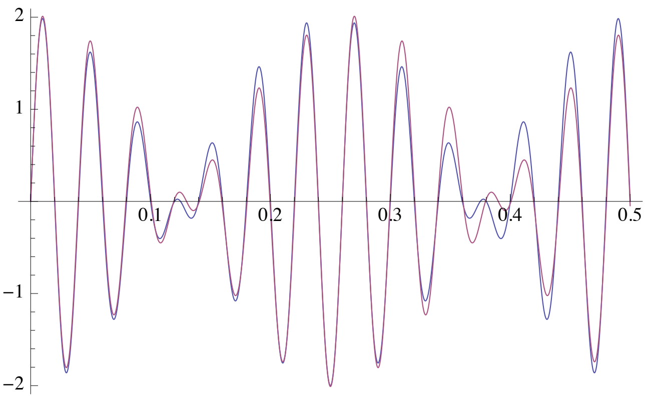
```
Plot[{Sin[2 π 23 t] + Sin[2 π 27 t],
  0.1 Sin[2 π 4 * (50 / 13) t] + Sin[2 π 6 * (50 / 13) t] + Sin[2 π 7 * (50 / 13) t]}, {t, 0, 0.5}]
```



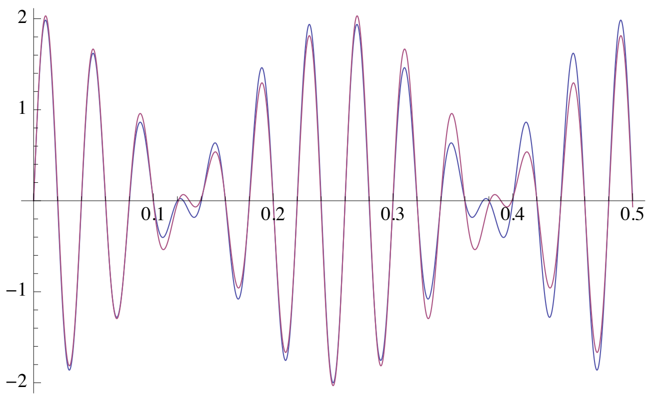

```
Plot[{Sin[2 π 23 t] + Sin[2 π 27 t],
  0.1 Sin[2 π 2 * (50 / 13) t] + Sin[2 π 6 * (50 / 13) t] + Sin[2 π 7 * (50 / 13) t]}, {t, 0, 0.5}]
```




```
Plot[{Sin[2 π 23 t] + Sin[2 π 27 t],
  0.1 Sin[2 π (50 / 13) t] + Sin[2 π 6 * (50 / 13) t] + Sin[2 π 7 * (50 / 13) t]}, {t, 0, 0.5}]
```



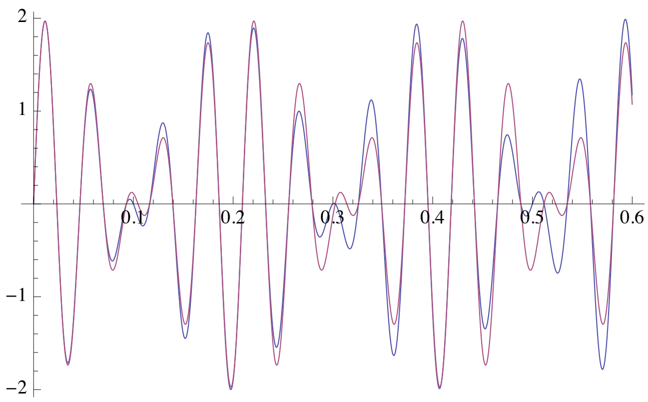

```
Plot[{Sin[2 π 23 t] + Sin[2 π 27 t], 0.03 Sin[2 π (50 / 13) t] + 0.03 Sin[2 π 2 * (50 / 13) t] +
   0.03 Sin[2 π 4 * (50 / 13) t] + Sin[2 π 6 * (50 / 13) t] + Sin[2 π 7 * (50 / 13) t]}, {t, 0, 0.5}]
```



Ratios Between Approx. GCD's

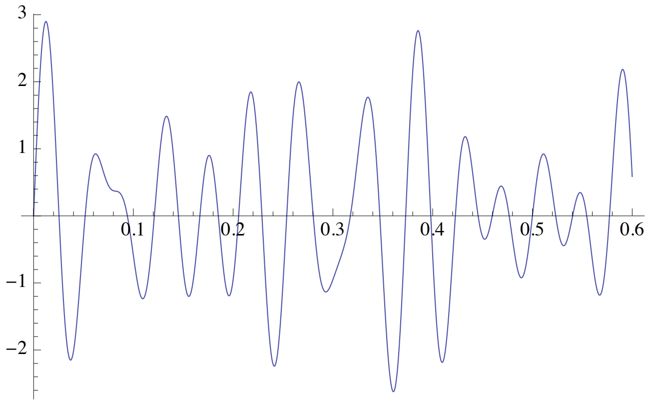
For chords which contain more than one interval it is necessary to investigate which ones will best correspond to the wave in question. For example, given the minor triad as 16:19:24, since 24/16 = 3/2 is excluded, then we need to decide whether the 16:19 or 19:24 intervals make the best match, or both. Do we take our 'GCD' from the 16:19 and apply 5:6:(7.5)? Or do we instead take the 'GCD' from the major third and our triad as 3.33..:4:5? In fact the graphs below show that any of these choices will hold:

```
Plot[{(Sin[2 π 19 t] + Sin[2 π 24 t]),
  (Sin[2 π 4 * ((19 + 24) / 9) t] + Sin[2 π 5 * ((19 + 24) / 9) t])}, {t, 0, 0.6}]
```

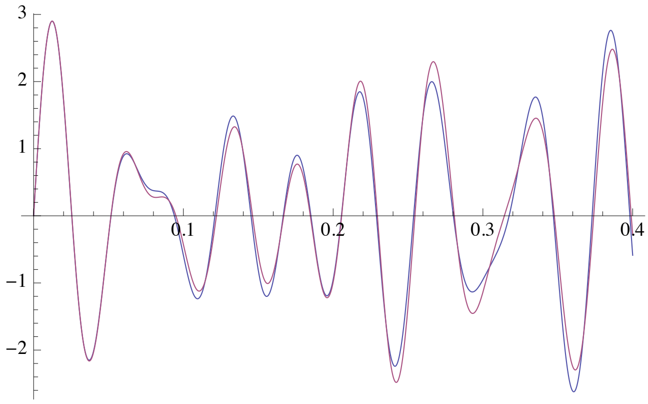


The minor triad as 16:19:24 gives:

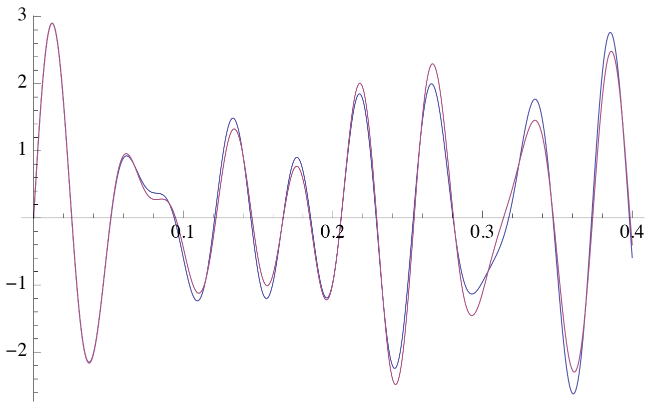
```
Plot[(Sin[2 π 16 t] + Sin[2 π 19 t] + Sin[2 π 24 t]), {t, 0, 0.6}]
```



```
Plot[{(Sin[2 π 16 t] + Sin[2 π 19 t] + Sin[2 π 24 t]),
  (Sin[2 π 5 * ((16 + 19) / 11) t] Sin[2 π 6 * ((16 + 19) / 11) t] +
    Sin[2 π 7.5 * ((16 + 19) / 11) t])}, {t, 0, 0.4}]
```

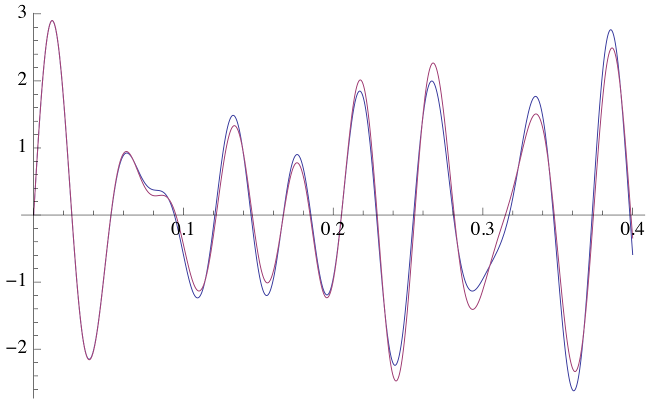


```
Plot[{(Sin[2 π 16 t] + Sin[2 π 19 t] + Sin[2 π 24 t]),
  (Sin[2 π 3.3333 * ((19 + 24) / 9) t] Sin[2 π 4 * ((19 + 24) / 9) t] +
    Sin[2 π 5 * ((19 + 24) / 9) t])}, {t, 0, 0.4}]
```

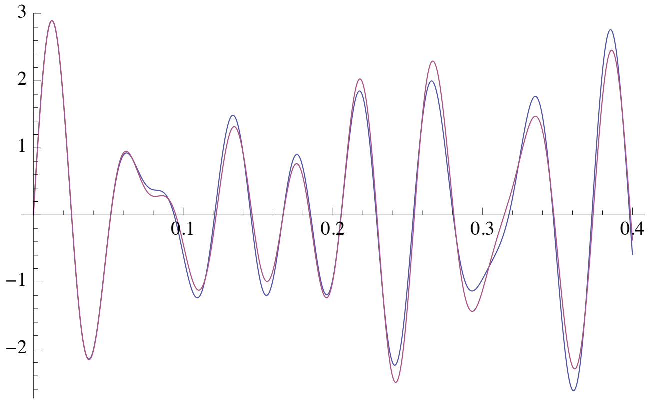


Both of these mixed options below also appear to be good choices (observe that (16 + 24)/(2 + 3) = 8, the second and third harmonics of which are 16 and 24. It is therefore excluded):

```
Plot[{(Sin[2 π 16 t] + Sin[2 π 19 t] + Sin[2 π 24 t]), (Sin[2 π 5 * ((16 + 19) / 11) t] +
    Sin[2 π 6 * ((16 + 19) / 11) t] + Sin[2 π 5 * ((19 + 24) / 9) t])}, {t, 0, 0.4}]
```



```
Plot[{(Sin[2 π 16 t] + Sin[2 π 19 t] + Sin[2 π 24 t]),
  (Sin[2 π 5 * ((16 + 19) / 11) t] + Sin[2 π 4 * ((19 + 24) / 9) t] + Sin[2 π 5 * ((19 + 24) / 9) t])}, {t,
  0, 0.4}]
```



Conclusion

It is well known that adding or subtacting harmonic sine wave components to or from a periodic wave alters the tone while leaving the pitch unaffected. Graphically speaking, this can be seen as a change in wave shape without a change in GCD period. Yet it has become common practice to regard the fundamental component as an exception to this general rule, even to the point of establishing a whole separate category for the case when it is not present (See for eg, Terhardt's distinction between "spectral"and "virtual" pitch, for the cases when the fundamental is present and absent, respectively). Yet another name for this is the "missing fundamental". Now in all fairness this statement should come with a caveat; for when we consider that the exact GCD is often far removed from the audible range of the component frequencies and yet something close to those frequencies could still sometimes be heard, then psychoacoustics reached the only viable conclusion remaining, that the brain itself somehow has the innate ability to sub-harmonically match overtones to the fundamental of a series. (Once we have eliminated every other valid possibility then the one that is remaining is likely the correct one, however unbelievable or improbable). Furthermore, a similar oversight occurs in wave theory itself. While it is well-known that a wave subjected to boundary conditions will produce ratios between whole-numbered frequencies, what has not been sufficiently considered is the fact that these ratios also occur *between* boundary conditions. In other words,we must have known the concept of the GCD in advance of Fourier Analysis, and therefore all that is required for periodic waves to exist is that their frequencies bear a rational relation and occupy the same space and time simultaneously.

However, the discovery of these 'near GCD's' places us in a unique position to re-evaluate the case where precise GCD's exist and are sufficiently close to the component frequencies to be heard, safe in the knowledge that we are no longer restricted to ratios between small-whole numbers.

First of all, it is well known that the frequency of the resulting wave will correspond to the GCD. That is, according to both definitions of frequency as 1). the number of cycles occurring per unit time, or 2). the inverse of period. The fact that we can hear this is merely proof that our previous identification of 'pitch = that of the fundamental sine wave' was insufficient. Secondly, from a mathematical standpoint, a frequency corresponds to the *class* of all waves that share a common period, which means that they share the same GCD. The cases when the first harmonic *happens* to be present are but one among infinitely many. Further, the notion that all periodic waves can be Fourier Analysed into elementary sine wave components is itself also only a point of view. If two or more sine frequencies create a GCD, then adding overtones to each component produces a GCD harmonic series with the GCD as'fundamental'. For example, 6 and 9 produce the GCD of 3. The GCD between the 2nd harmonics of 6 and 9, i.e. between 12 and 18, is 6 which is the 2nd harmonic of 3. 18 and

27 make up the 3rd harmonics which have a GCD of
9, the 3rd harmonic of 3, and so on:

[sin(2π6t) + sin(2π12t) + sin(2π18t) +...] +

[sin(2π9t) + sin(2π18t) + sin(2π27t) +...] =

[sin(2π6t) + sin(2π9t)] + [sin(2π2*6t) + sin(2π2*9t)] +

[sin(2π3*6t) + sin(2π3*9t)] +...

IOW it is just as valid to regard the GCD as the 'elementary component'. (Of course we can always *say* that these too can be further analysed into sine waves, but that is missing the point, which is that according to class or set theory GCD's are the general rule, not sine waves. And this is much closer to how music behaves, that we can play the *same* melody on a *variety* of instruments).

Finally, by now it should be clear that everything that was just said also applies to the approximate GCD's. The fact that they are close enough to the sine wave frequencies now makes the entire prospect viable that we can sometimes hear the GCD directly. IOW, since these frequencies exist in the wave then the "missing fundamental" does not seem to be missing at all! Isn't it more accurate to say that to perceive or extract *any* pitch from frequency *always* requires an element of interpretation or 'virtuality' which is ever-present in the listener? If so, then why should GCD's be considered any more 'virtual' than any other pitch? After all, they are certainly no less real than the sine waves that went into producing them. Isn't it in fact more accurate to say that it is our perception of *tone* that is truly virtual, not pitch, and that inconsideration of this distinction led to the false designation? Reviewing the evidence, all of the usual experiments that are cited as proof of virtual-pitch-as-tonic - the fact that there is no resonance at that frequency or that it disappears when passed through a high-pass filter - no longer seem convincing. If we screen out the sine waves of the upperharmonics that went into making the wave then of course it is going to disappear. It is like taking apart the Eiffel Tower to 'prove' that its shape is an illusion. And tuning forks are designed to produce sine waves. When all is said and done, we could always cause tuning forks to resonate at the upper harmonics. But then their composite wave will be just recreating the initial conditions of our original experiment. And as far as binaural superposition goes (the argument that we can hear the tonic when the components are separated into each ear), I have personally never been able to hear this myself. I suspect that it is a bias inherited from Newtonian mechanics which would have music classified as some type of therapeutic illusion. But again, even if we do grant this to be true, then this 'virtuality' would still have to apply to our perception of pitch across the board; for it is sheer madness to assume that pitch can exist without *some* form of objective stimuli. Since these approximate GCD's are now a feature of superposition itself, no less so than Fourier Analysis, then the brain ought to have a method of processing them as well. Perhaps they would apply as much to the electromagnetic waves of our neural networks as to the sound waves that trigger them? At any rate, at least we now have something capable of being *tested*.