Exercises for the course
**Deep Learning 2**
Summer Semester 2025

Machine Learning Group
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# Exercise Sheet **Equivariant Neural Networks**

## Exercise 1: Rotational equivariance (40 P)

In this excercise, we will derive an equivariant function given a function $f : \mathbb{R}^n \to \mathbb{R}$ that is invariant to the rotation group, i.e.

$$f(\mathbf{x}) = f(g\mathbf{x})$$

for any $g \in \mathrm{SO}(n)$.
*Hint:* Use the fact that $g$ is orthogonal.

(a) Show that $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$ is invariant under SO(n).

$$\begin{aligned} \|g\boldsymbol{x}\| &= \sqrt{\boldsymbol{x}^T g^T g\, \boldsymbol{x}} \\ &= \sqrt{\boldsymbol{x}^T \boldsymbol{x}} \\ &= \|\boldsymbol{x}\| \end{aligned}$$

(b) Show that the derivative of any $f$ is equivariant under SO(n):

$$g\nabla f(\boldsymbol{x}) = \nabla f(g\boldsymbol{x})$$

for any $g \in \mathrm{SO}(n)$.
Take the derivative on both sides of invariance relation:

$$\begin{aligned} \frac{\partial f}{\partial \boldsymbol{x}}(\boldsymbol{x}) &= \nabla f(\boldsymbol{x}) \\ \frac{\partial f \circ g}{\partial \boldsymbol{x}}(\boldsymbol{x}) &= g^T \nabla f(g\boldsymbol{x}) \\ \Rightarrow \nabla f(\boldsymbol{x}) &= g^T \nabla f(g\boldsymbol{x}) \\ \Rightarrow g\nabla f(\boldsymbol{x}) &= g\, g^T \nabla f(g\boldsymbol{x}) = \nabla f(g\boldsymbol{x}) \end{aligned}$$

(c) Calculate the gradient $u(\boldsymbol{x}) = \nabla f(\boldsymbol{x}) = \nabla\|\boldsymbol{x}\|$ to obtain an equivariant function $u : \mathbb{R}^n \to \mathbb{R}^n$.

$$u(\boldsymbol{x}) = \nabla f(\boldsymbol{x}) = \frac{x}{\|x\|}$$

(d) Analog to (c), derive an equivariant function $v : \mathbb{R}^n \to \mathbb{R}^{n \times n}$

$$v(\boldsymbol{x}) = H_f(\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{x}} \frac{x}{\|x\|} = \frac{I\|\boldsymbol{x}\|^2 - xx^T}{\|x\|^3}$$

## Exercise 2: Programming (60 P)

Download the programming files on ISIS and follow the instructions.