Finding Group Steiner Trees in Graphs with both Vertex and Edge Weights: Some Supplemental Materials

Road map. In Section 1, we prove the transformation from group Steiner trees to Steiner trees. In Section 2, we prove the approximation guarantee of LANCET. In Section 3, we explain the time complexity of LANCET. In Section 4, we explain the time complexity of exENSteiner. In Section 5, we prove the approximation guarantee of exIhlerA. In Section 6, we prove the approximation guarantee of FastAPP. In Section 7, we prove the approximation guarantee of ImprovAPP. In Section 8, we prove the approximation guarantee of PartialOPT. In Section 9, we explain the time complexity of PartialOPT. In Section 10, we show the usefulness of the refinement process (*i.e.,* Lines 16-29) in ImprovAPP. In Section 11, we show that PrunedDP and PrunedDP++ in [2] rely on techniques that do not hold in graphs with vertex weights. In Section 12, we show the memory consumption results. In Section 13, we refine the solutions of exENSteiner, exIhlerA and FastAPP.

1 THE TRANSFORMATION

Theorem 1. *Let $G(V, E, w, c)$ be a connected undirected graph, and Γ be a set of vertex groups. Let $G_t(V_t, E_t, w_t, c_t)$ be a connected undirected graph, and $T_t \subseteq V_t$ be a set of compulsory vertices. Based on G and Γ, we construct G_t and T_t in the following way:*

(1) Initialize $V_t = V$, $E_t = E$, $T_t = \emptyset$, $w_t = (1-\lambda)w$, and $c_t = \lambda c$.

(2) For each vertex group $g \in \Gamma$, (i) add a dummy vertex v_g into T_t and V_t, such that $w_t(v_g) = 0$, and (ii) add dummy edges (v_g, j) for all $j \in g$ into E_t, such that $c_t(v_g, j) = M$, and M is a constant satisfying

$$M > (1-\lambda)\sum_{v \in V} w(v) + \lambda \sum_{e \in E_{MST}} c(e), \tag{1}$$

and E_{MST} is the set of edges in a Minimum Spanning Tree of G.

Let Θ_{G_t} be an optimal solution to the vertex- and edge-weighted Steiner tree problem in G_t, and $\Theta^{non}_{G_t}$ be the non-dummy part of Θ_{G_t}. Then, there is an optimal solution to the vertex- and edge-weighted group Steiner tree problem in G, namely, Θ_G, that has the same sets of vertices and edges with $\Theta^{non}_{G_t}$.

Proof. Since dummy vertices only connect non-dummy vertices, there are at least $|\Gamma|$ dummy edges in Θ_{G_t}. If $c_\lambda(\Theta_G) < c(\Theta^{non}_{G_t})$, then there is a feasible solution to the vertex- and edge-weighted Steiner tree problem in G_t: Θ'_{G_t} such that

$$c(\Theta'_{G_t}) = c_\lambda(\Theta_G) + M|\Gamma| < c(\Theta_{G_t}), \tag{2}$$

which is not possible. Thus, we have $c_\lambda(\Theta_G) \geq c(\Theta^{non}_{G_t})$. Let Θ''_{G_t} be a tree in G_t such that (i) every dummy vertex v_g is a leaf of Θ''_{G_t}; and (ii) the non-dummy part of Θ''_{G_t}, namely, $\Theta^{non''}_{G_t}$, is in a Minimum Spanning Tree of G. Suppose that there is a dummy vertex v_g in Θ_{G_t} that is not a leaf. Since $c(\Theta^{non''}_{G_t}) < M$, we have

$$c(\Theta_{G_t}) \geq c(\Theta^{non}_{G_t}) + M(|\Gamma|+1) > c(\Theta''_{G_t}) = c(\Theta^{non''}_{G_t}) + M|\Gamma|, \tag{3}$$

which is not possible. Thus, every dummy compulsory vertex v_g is a leaf of Θ_{G_t}. As a result, $\Theta^{non}_{G_t}$ is connected and shares the same sets of vertices and edges with a feasible solution to the vertex- and edge-weighted group Steiner tree problem in G, which means that $c_\lambda(\Theta_G) \leq c(\Theta^{non}_{G_t})$. Therefore, $c_\lambda(\Theta_G) = c(\Theta^{non}_{G_t})$. Hence, this theorem holds. □

2 THE APPROXIMATION GUARANTEE OF LANCET

LANCET can be regarded as the vertex- and edge-weighted version of the algorithm in [3], which achieves an approximation guarantee of $2(1 - 1/|T_t|)$ for solving the vertex-unweighted Steiner tree problem. This approximation guarantee relies on the following deduction (*i.e.,* Lemma 1 in [3]): since a pre-order traversal of a tree traverses every edge in this tree exactly twice (see Figure 1 in [3]), in a graph with only edge weights, if we perform a pre-order traversal of an optimal solution tree and sum up every weight that we encounter (including duplicates), then the result is exactly twice the weight of an optimal solution tree. However, in a graph with both vertex and edge weights, summing up the weights that we encounter during this traversal does not always result in twice the weight of an optimal solution tree, since (i) an optimal solution tree may contain non-compulsory vertices with positive weights; and (ii) a pre-order traversal of an optimal solution tree may visit such a vertex more than twice (specifically, the number of times that a pre-order traversal of an optimal solution tree visits such a vertex equals the degree of this vertex in this optimal solution tree). Thus, the above approximation guarantee of $2(1 - 1/|T_t|)$ does not hold for LANCET. In what follows, we establish the approximation guarantee of LANCET.

Theorem 2. LANCET *has a sharp approximation guarantee of $|T_t| - 1$ for solving the vertex- and edge-weighted Steiner tree problem.*

Proof. LANCET merges $|T_t| - 1$ LWPs to connect all compulsory vertices together. Suppose that the highest-weight one of these LWPs is LWP', and Θ_{opt} is an optimal solution. Since $c(LWP')$ is smaller than or equal to the weight of the LWP between a pair of compulsory

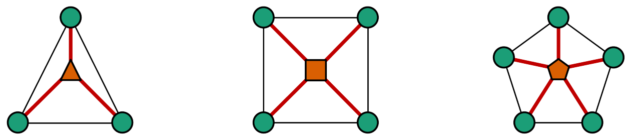


Figure 1: Touching the approximation guarantee of $|T_t| - 1$.

vertices, we have

$$c(\Theta_{opt}) \geq c(LWP'). \tag{4}$$

Since there are $|T_t| - 1$ LWPs that have been merged, we have

$$(|T_t| - 1)c(\Theta_{opt}) \geq (|T_t| - 1)c(LWP') \geq c(\Theta). \tag{5}$$

Therefore, LANCET has an approximation guarantee of $|T_t| - 1$. We further show that $|T_t| - 1$ is the sharp approximation guarantee of LANCET. Consider a regular polygon composed of $|T_t|$ compulsory vertices, and a non-compulsory vertex that is in the middle of this polygon and connects $|T_t|$ compulsory vertices (see Figure 1). The weight of each edge between compulsory vertices is x, the weight of each edge between a compulsory vertex and the middle non-compulsory vertex is 0, the weight of each compulsory vertex is 0, and the weight of the middle non-compulsory vertex is z. Suppose that $x = z - \delta$, where δ is a tiny positive value; and $z < (|T_t| - 1)x$. Since $x < z$, Θ contains $|T_t| - 1$ edges between compulsory vertices, and $c(\Theta) = (|T_t| - 1)x$. Since $z < (|T_t| - 1)x$, Θ_{opt} contains all the edges between compulsory vertices and the middle non-compulsory vertex, and $c(\Theta_{opt}) = z$. We have

$$\lim_{\delta \to 0} \frac{c(\Theta)}{c(\Theta_{opt})} = \frac{(|T_t| - 1)(z - \delta)}{z} = |T_t| - 1. \tag{6}$$

Hence, $|T_t| - 1$ is the sharp approximation guarantee of LANCET. □

3 THE TIME COMPLEXITY OF LANCET

Time complexity of LANCET:

$$O\Big(|T_t| \cdot (|E_t| + |V_t| \log |V_t|)\Big).$$

The details are as follows. The overhead of the initialization (Lines 1-2) is $O(|T_t|)$. The LWPs from a vertex to the other vertices can be found using Dijkstra's algorithm in $O(|E_t| + |V_t| \log |V_t|)$ time. Thus, the cost of finding the LWPs from every vertex in V_2 to the other vertices (Line 3) is $O(|T_t|(|E_t| + |V_t| \log |V_t|))$. It takes $O(|T_t|)$ time to push the LWPs from every vertex in V_2 to i_{rand} into Q (Line 4). It concatenates the LWPs between unconnected compulsory vertices and connected vertices using a while loop with $O(|T_t|)$ iterations as follows. Since popping out the top element in the Fibonacci heap takes $O(\log |T_t|)$ time, it pops out $LWP_{min}(V_{min}, E_{min})$ in $O(|T_t| \log |T_t|)$ time throughout the loop (Line 6). We use adjacency lists based on hashes to store graphs. Since adding a vertex or an edge into such an adjacency list takes $O(1)$ time, LANCET merges $LWP_{min}(V_{min}, E_{min})$ into Θ (Line 7) in $O(|V_t|)$ time in all iterations combined. We use hashes to store V_1 and V_2. As a result, updating V_1 and V_2 (Lines 8-9) takes $O(|V_t|)$ time throughout the loop. Since the time complexity of decreasing the key of an element in a min Fibonacci heap is $O(1)$; and LANCET checks and updates the minimum-weight LWPs from every vertex in V_2 to V_1 only when a new vertex is added into V_1, updating the LWPs (Line 10) takes $O(|T_t||V_t|)$ time throughout the loop.

4 THE TIME COMPLEXITY OF exENSteiner

Time complexity of exENSteiner:

$$O\Big(|\Gamma| \cdot (|E| + |V| \log |V| + |\Gamma| \log |\Gamma| + |\Gamma||V|)\Big).$$

The details are as follows. exENSteiner first transforms the input graph in $O(|E| + |\Gamma||V|)$ time (Line 1). This is because (i) this transformation requires examining all vertices and edges to decide the value of M, which takes $O(|V|+|E|)$ time, and (ii) it adds $|\Gamma|$ dummy vertices and $\sum_{g \in \Gamma} |g|$ dummy edges, which incurs $O(|\Gamma||V|)$ overhead. Then, it employs LANCET to find a Steiner tree in $O(|\Gamma|(|E|+|V| \log |V|+|\Gamma| \log |\Gamma|+|\Gamma||V|))$ time (Line 2), since LANCET takes $O(|T_t|(|E_t| + |V_t| \log |V_t|))$ time; and $|T_t| = |\Gamma|$, $|V_t| = |V| + |\Gamma|$ and $|E_t| = |E| + \sum_{g \in \Gamma} |g|$. After that, it removes $|\Gamma|$ dummy vertices and $|\Gamma|$ dummy edges (Line 3), which takes $O(|\Gamma|)$ time. It identifies the MST (Line 4) in $O(|E| + |V| \log |V|)$ time. Notably, since $|\Gamma|$ is often limited in practice, the cost of exENSteiner is close to $O(|\Gamma| \cdot (|E| + |V| \log |V|))$ in practice.

5 THE APPROXIMATION GUARANTEE OF exIhlerA

THEOREM 3. exIhlerA *has a sharp approximation guarantee of* $|\Gamma| - 1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

PROOF. Suppose that $\Theta_{OPT}(V_{OPT}, E_{OPT})$ is an optimal solution. Let $\Gamma = \{g_1, \ldots, g_{|\Gamma|}\}$. There is a tuple $(v_1, \ldots, v_{|\Gamma|})$ such that $v_i \in V_{OPT} \cap g_i$ for all $i \in \{1, \ldots, |\Gamma|\}$. Without loss of generality, assume that $g_{min} = g_1$. For every $i \in \{2, \ldots, |\Gamma|\}$, there is exactly one simple path between v_1 and v_i in Θ_{OPT}, which we refer to as $P_{v_1 v_i}$. We have

$$c_\lambda(P_{v_1 v_i}) \leq c_\lambda(\Theta_{OPT}), \tag{7}$$

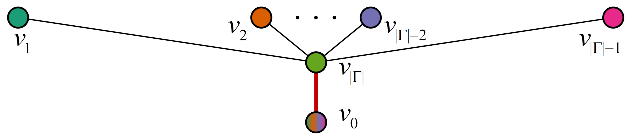


Figure 2: Touching the approximation guarantee of $|\Gamma|-1$.

$$\sum_{g\in\Gamma\setminus g_1} c_\lambda(LWP_{\lambda v_1 g}) \leq \sum_{i\in\{2,\ldots,|\Gamma|\}} c_\lambda(P_{v_1 v_i}). \tag{8}$$

Thus,

$$c_\lambda(\Theta) \leq c_\lambda(G_{v_1}) \leq \sum_{g\in\Gamma\setminus g_1} c_\lambda(LWP_{\lambda v_1 g}) \leq \sum_{i\in\{2,\ldots,|\Gamma|\}} c_\lambda(P_{v_1 v_i}) \leq (|\Gamma|-1)c_\lambda(\Theta_{OPT}). \tag{9}$$

Hence, exIhlerA has an approximation guarantee of $|\Gamma|-1$. We note that this guarantee is sharp. To explain, consider the graph $G(V,E,w,c)$ in Figure 2, where $V=\{v_0,v_1,\ldots,v_{|\Gamma|}\}$, $E=\{(v_{|\Gamma|},v_0),(v_{|\Gamma|},v_1),\ldots,(v_{|\Gamma|},v_{|\Gamma|-1})\}$, $w(i)=0$ for all $i\in V$, $c(v_{|\Gamma|},v_1)=\ldots=c(v_{|\Gamma|},v_{|\Gamma|-1})=1$, and $c(v_{|\Gamma|},v_0)=1+\delta$, where δ is a tiny positive value. In addition, $\Gamma=\{v_0,v_1\}\cup\ldots\cup\{v_0,v_{|\Gamma|-1}\}\cup\{v_{|\Gamma|}\}$. Let $\lambda=1$. Since $g_{min}=\{v_{|\Gamma|}\}$, exIhlerA produces the solution $\Theta=\{(v_{|\Gamma|},v_1),\ldots,(v_{|\Gamma|},v_{|\Gamma|-1})\}$, and $c_\lambda(\Theta)=|\Gamma|-1$. When $|\Gamma|=2$, Θ is the optimal solution, *i.e.*, the approximation ratio is $|\Gamma|-1=1$. When $|\Gamma|>2$, we have $\Theta_{OPT}=\{(v_{|\Gamma|},v_0)\}$, and

$$\lim_{\delta\to 0}\frac{c_\lambda(\Theta)}{c_\lambda(\Theta_{OPT})}=\frac{|\Gamma|-1}{1+\delta}=|\Gamma|-1. \tag{10}$$

Hence, the best possible approximation guarantee of exIhlerA is $|\Gamma|-1$. □

6 THE APPROXIMATION GUARANTEE OF FastAPP

Theorem 4. FastAPP *has a sharp approximation guarantee of* $|\Gamma|-1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Let $\Theta_{OPT}(V_{OPT},E_{OPT})$ be an optimal solution, and $\Gamma=\{g_1,\ldots,g_{|\Gamma|}\}$. There is a tuple $(v_1,\ldots,v_{|\Gamma|})$ such that $v_i\in V_{OPT}\cap g_i$ for all $i\in\{1,\ldots,|\Gamma|\}$. Without loss of generality, suppose that $g_{min}=g_1$. Let $g_x\in\Gamma\setminus g_1$ be such a vertex group that

$$c_\lambda(LWP_{\lambda v_1 g_x})=\max\{c_\lambda(LWP_{\lambda v_1 g})\mid \forall g\in\Gamma\setminus g_1\}. \tag{11}$$

Since $LWP_{\lambda v_1 g_x}$ links fewer groups to v_1 than Θ_{OPT}, we have

$$c_\lambda(LWP_{\lambda v_1 g_x})\leq c_\lambda(\Theta_{OPT}). \tag{12}$$

Lines 5-8 in FastAPP guarantee that

$$\max\{c_\lambda(LWP_{\lambda i_{min} g})\mid \forall g\in\Gamma\setminus g_1\}\leq c_\lambda(LWP_{\lambda v_1 g_x}). \tag{13}$$

By Lines 10-11 in FastAPP, we have

$$c_\lambda(\Theta)\leq(|\Gamma|-1)\cdot\max\{c_\lambda(LWP_{\lambda i_{min} g})\mid \forall g\in\Gamma\setminus g_1\}. \tag{14}$$

Thus,

$$c_\lambda(\Theta)\leq(|\Gamma|-1)c_\lambda(LWP_{\lambda v_1 g_x})\leq(|\Gamma|-1)c_\lambda(\Theta_{OPT}). \tag{15}$$

Hence, FastAPP has an approximation guarantee of $|\Gamma|-1$. The sharpness of this guarantee can be seen from the example in Section 5, *i.e.,* Figure 2. Hence, this theorem holds. □

7 THE APPROXIMATION GUARANTEE OF ImprovAPP

Theorem 5. ImprovAPP *has a sharp approximation guarantee of* $|\Gamma|-1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Let $\Theta_{OPT}(V_{OPT},E_{OPT})$ be an optimal solution. Let $\Gamma=\{g_1,\ldots,g_{|\Gamma|}\}$. There is a tuple $(v_1,\ldots,v_{|\Gamma|})$ such that $v_i\in V_{OPT}\cap g_i$ for all $i\in\{1,\ldots,|\Gamma|\}$. Without loss of generality, suppose that $g_{min}=g_1$. When ImprovAPP processes v_1 in the for loop (Lines 5-14), it concatenates $|\Gamma|-1$ lowest weight paths that link $\{g_2,\ldots,g_{|\Gamma|}\}$, respectively, to build Θ_{v_1}. Let $LWP_{\lambda v_x g_y}$ be one of these paths that has the largest regulated weight, and links $g_y\in\{g_2,\ldots,g_{|\Gamma|}\}$. Then,

$$c_\lambda(\Theta_{v_1})\leq(|\Gamma|-1)c_\lambda(LWP_{\lambda v_x g_y}). \tag{16}$$

Let $LWP_{\lambda v_1 g_y}$ be the lowest weight path between v_1 and g_y. Since $LWP_{\lambda v_1 g_y}$ has been pushed into Q initially (Line 6) and has (possibly) been updated to $LWP_{\lambda v_x g_y}$ (Line 12), we have

$$c_\lambda(LWP_{\lambda v_x g_y})\leq c_\lambda(LWP_{\lambda v_1 g_y}). \tag{17}$$

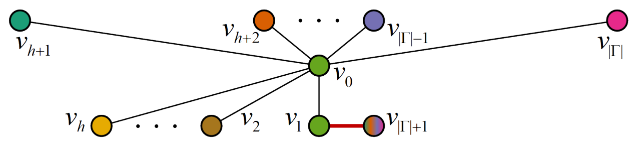


Figure 3: Touching the approximation guarantee of $|\Gamma| - h + 1$.

Since $LWP_{\lambda v_1 g_y}$ links fewer groups to v_1 than Θ_{OPT}, we have

$$c_\lambda(LWP_{\lambda v_1 g_y}) \le c_\lambda(\Theta_{OPT}). \tag{18}$$

Thus,

$$c_\lambda(\Theta) \le c_\lambda(\Theta_{v_1}) \le (|\Gamma|-1)c_\lambda(LWP_{\lambda v_x g_y}) \le (|\Gamma|-1)c_\lambda(LWP_{\lambda v_1 g_y}) \le (|\Gamma|-1)c_\lambda(\Theta_{OPT}). \tag{19}$$

Therefore, ImprovAPP has an approximation guarantee of $|\Gamma| - 1$. The sharpness of this guarantee can be seen from the example in Section 5, *i.e.,* Figure 2. Thus, this theorem holds. □

8 THE APPROXIMATION GUARANTEE OF PartialOPT

Theorem 6. PartialOPT *has a sharp approximation guarantee of* $|\Gamma| - h + 1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Suppose that $\Theta_{OPT}(V_{OPT}, E_{OPT})$ is an optimal solution, and $\Gamma = \{g_1, \dots, g_{|\Gamma|}\}$. There is a tuple $(v_1, \dots, v_{|\Gamma|})$ such that $v_i \in V_{OPT} \cap g_i$ for all $i \in \{1, \dots, |\Gamma|\}$. Without loss of generality, let $g_{min} = g_1$. For $v_1 \in g_{min}$, $\Theta^h_{v_1}$ is optimal for $\Gamma_1 = \{\{v_1\}, g_2, \dots, g_h\}$. Since $\Theta^h_{v_1}$ connects fewer vertex groups to v_1 than Θ_{OPT}, we have

$$c_\lambda(\Theta^h_{v_1}) \le c_\lambda(\Theta_{OPT}). \tag{20}$$

If $\Gamma_2 = \{\{v_1\}\}$ (*i.e.,* $h = |\Gamma|$), then $\Theta^{|\Gamma|}_{v_1} = \{v_1\}$, and $c_\lambda(\Theta) = c_\lambda(\Theta_{OPT})$. Otherwise (*i.e.,* $h < |\Gamma|$), we implement exIhlerA to produce $\Theta^{|\Gamma|}_{v_1}$ for $\Gamma_2 = \{\{v_1\}, g_{h+1}, \dots, g_{|\Gamma|}\}$. Suppose that $\Theta^{|\Gamma|}_{OPT}$ is an optimal solution for Γ_2. The proof of Theorem 3 shows that

$$c_\lambda(\Theta^{|\Gamma|}_{v_1}) \le (|\Gamma| - h)c_\lambda(\Theta^{|\Gamma|}_{OPT}). \tag{21}$$

Since $\Theta^{|\Gamma|}_{OPT}$ connects fewer vertex groups to v_1 than Θ_{OPT}, we have

$$c_\lambda(\Theta^{|\Gamma|}_{OPT}) \le c_\lambda(\Theta_{OPT}). \tag{22}$$

Thus,

$$c_\lambda(\Theta) \le c_\lambda(G_{v_1}) = c_\lambda(\Theta^h_{v_1} \cup \Theta^{|\Gamma|}_{v_1}) \le c_\lambda(\Theta^h_{v_1}) + c_\lambda(\Theta^{|\Gamma|}_{v_1}) \le (|\Gamma| - h + 1)c_\lambda(\Theta_{OPT}). \tag{23}$$

Therefore, PartialOPT has an approximation guarantee of $|\Gamma| - h + 1$. We show that this guarantee is sharp. Consider the graph $G(V, E, w, c)$ in Figure 3, where $V = \{v_0, v_1, \dots, v_{|\Gamma|+1}\}$, $E = \{(v_0, v_1), (v_0, v_2), \dots, (v_0, v_{|\Gamma|}), (v_1, v_{|\Gamma|+1})\}$, $w(i) = 0$ for every $i \in \{v_0, \dots, v_{h-1}\}$, $w(i) = 1$ for every $i \in \{v_h, \dots, v_{|\Gamma|+1}\}$, $c(v_0, v_1) = \delta_1$, $c(v_1, v_{|\Gamma|+1}) = \delta_2$, where δ_1 and δ_2 are two tiny positive values, and $\delta_1 < \delta_2$, and all the other edge weights are zero. In addition, $\Gamma = \{g_1, \dots, g_{|\Gamma|}\} = \{v_0, v_1\} \cup \{v_2, v_{|\Gamma|+1}\} \cup \ldots \cup \{v_{|\Gamma|}, v_{|\Gamma|+1}\}$. Let $\lambda = 0.5$, *i.e.,* vertex and edge weights are regulated equally. PartialOPT enumerates two vertices in g_{min}: v_0 and v_1. For v_0, PartialOPT produces $\Theta^h_{v_0} = \{(v_0, v_2), \dots, (v_0, v_h)\}$, and $\Theta^{|\Gamma|}_{v_0} = \{(v_0, v_{h+1}), \dots, (v_0, v_{|\Gamma|})\}$. Thus, $\Theta_{v_0} = \{(v_0, v_2), \dots, (v_0, v_{|\Gamma|})\}$. Similarly, for v_1, since $\delta_1 < \delta_2$, PartialOPT produces $\Theta_{v_1} = \{(v_0, v_1), \dots, (v_0, v_{|\Gamma|})\}$. We have $\Theta = \Theta_{v_0}$. When $|\Gamma| = h$, Θ is the optimal solution, *i.e.,* the approximation ratio is $|\Gamma| - h + 1 = 1$. When $|\Gamma| > h$, we have $\Theta_{OPT} = \{(v_1, v_{|\Gamma|+1})\}$, and

$$\lim_{\delta_2 \to 0} \frac{c_\lambda(\Theta)}{c_\lambda(\Theta_{OPT})} = \frac{|\Gamma| - h + 1}{1 + \delta_2} = |\Gamma| - h + 1. \tag{24}$$

Hence, $|\Gamma| - h + 1$ is the best possible approximation guarantee of PartialOPT. This theorem holds. □

9 THE TIME COMPLEXITY OF PartialOPT

Time complexity of PartialOPT**:**

$$O\Big(|g_{min}| \cdot \Big(|\Gamma||V| + 3^h|V| + 2^h(|E| + h|V| + |V|\log|V|)\Big)\Big).$$

It initializes Θ and finds g_{min} in $O(|\Gamma|)$ time. For each vertex in g_{min}, it builds Γ_1 and Γ_2 in $O(|\Gamma|)$ time (Line 4). It applies DPBF to produce Θ^h_i in $O(3^h|V| + 2^h(|E| + h|V| + |V|\log|V|))$ time (Line 5; details in [1]), and may invoke exIhlerA to produce $\Theta^{|\Gamma|}_i$ in $O((|\Gamma| - h)|V| + |E| + |V|\log|V|)$ time (Line 9; here, $|g_{min}|$ in the time complexity of exIhlerA is 1, since Γ_2 contains $\{i\}$). After that, PartialOPT merges Θ^h_i and $\Theta^{|\Gamma|}_i$ (Line 11)

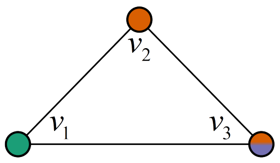


Figure 4: An example for showing the usefulness of the refinement process in ImprovAPP**.**

in $O(|V|)$ time, and computes an MST as Θ_i (Line 12) in $O(|E| + |V| \log |V|)$ time. It refines Θ_i in $O(|\Gamma||V| + |V| \log |V|)$ time (Line 13), and updates Θ in $O(|V|)$ time (Line 14).

10 AN EXAMPLE FOR ImprovAPP

Here, we show the usefulness of the refinement process (*i.e.,* Lines 16-29) in ImprovAPP via a triangular graph in Figure 4. Let this triangular graph be the input graph G. There are three vertex groups: $g_1 = \{v_1\}$, $g_2 = \{v_3\}$ and $g_3 = \{v_2, v_3\}$. Let the vertex weights be $w(v_1) = 2$, $w(v_2) = 6$ and $w(v_3) = 6$. Let the edge weights be $c(v_1, v_2) = 2$, $c(v_1, v_3) = 8$ and $c(v_2, v_3) = 6$. Let $\lambda = 0.5$. Suppose that ImprovAPP selects g_1 as g_{min} at Line 1. When it processes $v_1 \in g_1$ at Line 4, it pushes $LWP_{\lambda v_1 g_2} = \{(v_1, v_3)\}$ and $LWP_{\lambda v_1 g_3} = \{(v_1, v_2)\}$ into Q at Line 6. The regulated weights of these two paths are 8 and 5, respectively. It first merges $LWP_{\lambda v_1 g_3} = \{(v_1, v_2)\}$ into Θ_{v_1} at Line 9. Since the regulated weight of path $\{(v_1, v_3)\}$ is smaller than the regulated weight of path $\{(v_2, v_3)\}$ (*i.e.,* 8 is smaller than 9), it then merges $\{(v_1, v_3)\}$ into Θ_{v_1} at Line 9. Thus, ImprovAPP builds $\Theta_{min} = \Theta_{v_1} = \{(v_1, v_2), (v_1, v_3)\}$. However, since the weight of edge (v_2, v_3) is smaller than the weight of edge (v_1, v_3) (*i.e.,* 6 is smaller than 8), the MST that spans the vertices in Θ_{min} is $\{(v_1, v_2), (v_2, v_3)\}$. Therefore, after the loop at Lines 4-15, Θ_{min} may not be an MST that spans the vertices in Θ_{min}, which means that finding an MST at Line 16 is useful.

If the weight of edge (v_2, v_3) is 9, then ImprovAPP builds $\Theta_{min} = \Theta_{v_1} = \{(v_1, v_2), (v_1, v_3)\}$, and the MST that spans the vertices in Θ_{min} is still Θ_{min}. However, v_2 is not a unique-group leaf of this MST, and can be removed. Thus, after Line 16, it is not guaranteed that all leaves of Θ_{min} are unique-group leaves, which means that implementing Lines 17-29 to remove non-unique-group leaves is also useful.

11 THE RECENT WORK ON ENHANCING DPBF

The PrunedDP and PrunedDP++ algorithms in [2] enhance the DPBF algorithm in [1] for finding optimal vertex-unweighted group Steiner trees. The main idea of this enhancement is to incorporate pruning techniques into the process of DPBF. In this section, we show that PrunedDP and PrunedDP++ rely on pruning techniques that do not hold in graphs with vertex weights. For the sake of simplicity, we do not use λ to regulate vertex and edge weights in the examples in this section, *i.e.,* we sum vertex and edge weights directly when calculating the weight of a tree. We use $T(v, \Gamma)$ to signify the minimum-weight tree that roots at vertex v and covers all vertex groups in Γ.

Theorem 2 in [2] does not hold in graphs with vertex weights. Theorem 2 in [2] is the core pruning technique in PrunedDP, and is also an important pruning technique in PrunedDP++. This theorem does not hold in graphs with vertex weights. To explain, we first briefly describe the dynamic programming process of DPBF through an example in Figure 5. Understanding this process is necessary for understanding the reason why Theorem 2 in [2] does not hold in graphs with vertex weights.

In Figure 5, there are three vertex groups $g_1 = \{v_1\}$, $g_2 = \{v_2\}$ and $g_3 = \{v_3\}$. The weight of u is 1, and each of the other vertex and edge weights is δ, and δ is a tiny positive value. The optimal solution tree is the whole graph, and the weight of this tree is $1 + 6\delta$ (*i.e.,* the sum of vertex and edge weights). To find this tree, DPBF first initializes $T(v_1, \{g_1\})$ as vertex v_1; $T(v_2, \{g_2\})$ as vertex v_2; and $T(v_3, \{g_3\})$ as vertex v_3. Then, DPBF grows $T(v_1, \{g_1\})$, $T(v_2, \{g_2\})$ and $T(v_3, \{g_3\})$ to vertex u, and produces $T(u, \{g_1\})$ as edge (u, v_1); $T(u, \{g_2\})$ as edge (u, v_2); and $T(u, \{g_3\})$ as edge (u, v_3). Subsequently, it merges $T(u, \{g_1\})$ and $T(u, \{g_2\})$ as $T(u, \{g_1, g_2\}) = \{(u, v_1), (u, v_2)\}$ (similarly, it also merges $T(u, \{g_1\})$ and $T(u, \{g_3\})$ as $T(u, \{g_1, g_3\})$, and merges $T(u, \{g_2\})$ and $T(u, \{g_3\})$ as $T(u, \{g_2, g_3\})$). At last, it produces the optimal solution tree by merging $T(u, \{g_3\})$ and $T(u, \{g_1, g_2\})$ (or $T(u, \{g_1\})$ and $T(u, \{g_2, g_3\})$, or $T(u, \{g_2\})$ and $T(u, \{g_1, g_3\})$).

Theorem 2 in [2] is that: in DPBF, we can merge two subtrees $T(u, \Gamma')$ and $T(u, \Gamma'')$ for $\Gamma'' \subset \Gamma \setminus \Gamma'$ only when the total weight of these two subtrees is not larger than $\frac{2}{3}$ of the weight of an optimal solution tree. This theorem is true when vertex weights are omitted. For example, if vertex weights are omitted in the above instance, then the weight of the optimal solution tree is 3δ. When we merge $T(u, \{g_1\})$ and $T(u, \{g_2\})$ as $T(u, \{g_1, g_2\})$ in the above process, the total weight of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is 2δ, which is not larger than $\frac{2}{3}$ of the weight of an optimal solution tree. By Theorem 2 in [2], merging these two subtrees may help produce the optimal solution tree. If the total weight of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is larger than $\frac{2}{3}$ of the weight of an optimal solution tree, then merging these two subtrees does not help produce the optimal solution tree, and thus this merge can be avoided. However, this is not true when vertex weights are considered. For example, if we consider the vertex weights in the above instance, then the total weight of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is $2 + 4\delta$ (since the weight of each of these two trees is $1 + 2\delta$), which is larger than $\frac{2}{3}$ of the weight of an optimal solution tree: $1 + 6\delta$ (notably, even the weight of $T(u, \{g_1, g_2\}) = \{(u, v_1), (u, v_2)\}$, which is $1 + 4\delta$, is larger than $\frac{2}{3}$ of the weight of an optimal solution tree). As a result, if we use Theorem 2 in [2] in the above instance with vertex weights, then the merge of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is not carried out (similarly, the merge of $T(u, \{g_1\})$ and $T(u, \{g_3\})$, or the merge of $T(u, \{g_2\})$ and $T(u, \{g_3\})$, is not carried out). Consequently, the optimal solution tree will never be found. That is to say, Theorem 2 in [2] does not hold in graphs with vertex weights.

We point out the specific statement in the proof of Theorem 2 in [2] that does not hold in graphs with vertex weights as follows. In the beginning of the proof of Theorem 2 in [2], an optimal solution is assumed to be a tree rooted at vertex u with k subtrees, $T_1, \ldots, T_k$. Each

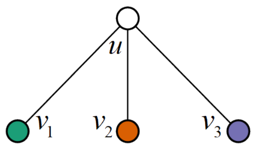


Figure 5: An example for showing Theorem 2 in [2].

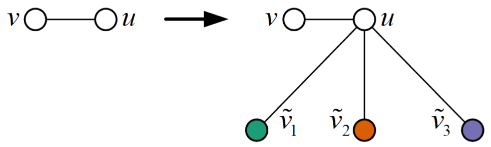


Figure 6: An example for showing Lemmas 2 and 3 in [2].

subtree T_i roots at vertex v_i, and the weight of each subtree is smaller than half of the weight of an optimal solution tree (*e.g.*, in Figure 5, T_i is the single vertex v_i). Let $\overline{T_i}$ be the the edge-grown subtree that is grown by T_i with an edge (v_i, u) (*e.g.*, in Figure 5, $\overline{T_i}$ is the edge (v_i, u)). The proof of Theorem 2 in [2] states that there are three different cases: (1) the weight of each $\overline{T_i}$ is smaller than half of the weight of an optimal solution tree; (2) there is only one edge-grown subtree $\overline{T_i}$ that has a weight no smaller than half of the weight of an optimal solution tree; and (3) there are two edge-grown subtrees and the weight of each one is half of the weight of an optimal solution tree. This statement is not true in vertex-weighted scenarios, where there is a fourth case: there are more than two edge-grown subtrees such that the weight of each one is large than half of the weight of an optimal solution tree. For example, in Figure 5, if we consider vertex weights, then the weight of $\overline{T_1}$, $\overline{T_2}$ or $\overline{T_3}$ is $1 + 2\delta$, which is larger than half of the weight of an optimal solution tree.

Lemmas 2 and 3 in [2] do not hold in graphs with vertex weights. Except Theorem 2 in [2], another important pruning technique in PrunedDP++ is the tour-based lower bounds construction method for A^*-search. There are two types of tour-based lower bounds, which are based on Lemmas 2 and 3 in [2], respectively. We show that these two lemmas do not hold in vertex-weighted scenarios as follows.

First, we introduce the label-enhanced graph in [2], which is constructed by adding dummy vertices and edges into the graph as follows. For each group $g_i \in \Gamma$, we add a dummy vertex $\widetilde{v_i}$, and also add a dummy edge $(\widetilde{v_i}, u)$ with zero weight for every $u \in g_i$. For example, in Figure 6, the graph contains two vertices v and u, and one edge (v, u), and there are three vertex groups $g_1 = g_2 = g_3 = \{u\}$. We add dummy vertices $\widetilde{v_1}, \ldots, \widetilde{v_3}$ and dummy edges $(\widetilde{v_1}, u), \ldots, (\widetilde{v_3}, u)$ for creating the label-enhanced graph. [2] uses $W(\widetilde{v_i}, \widetilde{v_j}, \Gamma')$ to refer to the weight of the minimum-weight route that starts from $\widetilde{v_i}$, ends at $\widetilde{v_j}$, and passes through all dummy vertices that correspond to vertex groups in Γ'. Moreover, [2] uses $d(v, \widetilde{v_i})$ to refer to the weight of the minimum-weight path between non-dummy vertex v and dummy vertex $\widetilde{v_i}$.

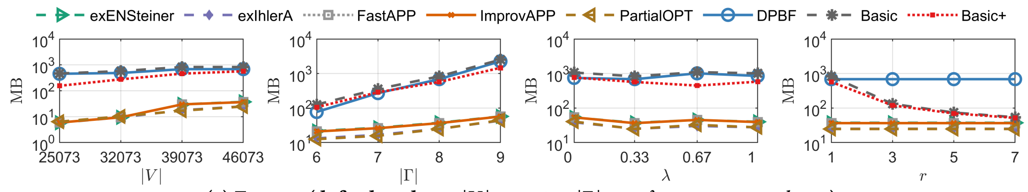
Lemma 2 in [2] is that: for every pair of vertex $v \in V$ and a subset of vertex groups $\Gamma' \subseteq \Gamma$, the weight of $T(v, \Gamma')$ is larger than or equal to $lb_1 = \frac{\min_{g_i, g_j \in \Gamma'} \{d(v,\widetilde{v_i}) + W(\widetilde{v_i}, \widetilde{v_j}, \Gamma') + d(\widetilde{v_j}, v)\}}{2}$. This lemma is true when all vertex weights are zero. For example, in Figure 6, let the weight of edge (v, u) be δ, which is a tiny positive value, and all vertex weights be zero, and $\Gamma' = \{g_1, g_2\}$. Then, $d(v, \widetilde{v_1}) = d(\widetilde{v_2}, v) = \delta$, $W(\widetilde{v_1}, \widetilde{v_2}, \Gamma') = 0$, and $T(v, \Gamma')$ is the edge (v, u). As a result, $lb_1 = \delta$, which equals the weight of $T(v, \Gamma')$. Thus, Lemma 2 in [2] holds. This lemma is proven in [2] by first doubling every edge in the label-enhanced $T(v, \Gamma')$ to obtain an Euler tour that starts from v, ends at v, and passes through all dummy vertices that correspond to vertex groups in Γ'; and then employing the fact that the total edge weight (including duplicates) that we encounter in this Euler tour is twice the total edge weight in $T(v, \Gamma')$. Nevertheless, Lemma 2 in [2] does not hold in vertex-weighted scenarios. For example, in the above instance, let the weights of v and u be 0 and 1, respectively. Then, $d(v, \widetilde{v_1}) = d(\widetilde{v_2}, v) = 1 + \delta$, and $W(\widetilde{v_1}, \widetilde{v_2}, \Gamma') = 1$. As a result, $lb_1 = \frac{3+2\delta}{2}$, which is larger than the weight of $T(v, \Gamma')$: $1 + \delta$. Thus, Lemma 2 in [2] does not hold any more. The reason is that the above Euler tour encounters u three times, and as a result the weight of u is counted three times in lb_1. Generally speaking, the total vertex and edge weight that we encounter in the Euler tour in the proof of Lemma 2 in [2] may be more than twice the weight of $T(v, \Gamma')$, since this tour may visit a vertex in $T(v, \Gamma')$ more than twice. As shown in Section 2 in this supplement, for a similar reason, LANCET does not have an approximation guarantee of 2.

Also for a similar reason, Lemma 3 in [2] does not hold in graphs with vertex weights. The details are as follows. [2] uses $W(\widetilde{v_i}, \Gamma')$ to refer to the weight of the minimum-weight route that starts from $\widetilde{v_i}$, and passes through all dummy vertices that correspond to vertex groups in Γ'. Lemma 3 in [2] is that: the weight of $T(v, \Gamma')$ is larger than or equal to $lb_2 = \frac{\max_{g_i \in \Gamma'} \{d(v,\widetilde{v_i}) + W(\widetilde{v_i}, \Gamma') + \min_{g_j \in \Gamma'} \{d(\widetilde{v_j}, v)\}\}}{2}$. This lemma is true when all vertex weights are zero. Consider the above instance. If all vertex weights are zero, then $d(v, \widetilde{v_1}) = d(v, \widetilde{v_2}) = \delta$, $W(\widetilde{v_1}, \Gamma') = W(\widetilde{v_2}, \Gamma') = 0$, $\min_{g_j \in \Gamma'} \{d(\widetilde{v_j}, v)\} = \delta$, and the weight of $T(v, \Gamma')$ is δ. Consequently, $lb_2 = \delta$, which equals the weight of $T(v, \Gamma')$. Thus, Lemma 3 in [2] holds. Like Lemma 2, Lemma 3 is proven in [2] by doubling every edge in the label-enhanced $T(v, \Gamma')$ to obtain an Euler tour. Also like Lemma 2, since this tour may visit a vertex in $T(v, \Gamma')$ more than twice, Lemma 3 in [2] does not hold in graphs with vertex weights. For example, suppose that, in Figure 6, the weights of v and u are 0 and 1, respectively. Then, $d(v, \widetilde{v_1}) = d(v, \widetilde{v_2}) = 1 + \delta$, $W(\widetilde{v_1}, \Gamma') = W(\widetilde{v_2}, \Gamma') = 1$, $\min_{g_j \in \Gamma'} \{d(\widetilde{v_j}, v)\} = 1 + \delta$, and the weight of $T(v, \Gamma')$ is $1 + \delta$. As a result, $lb_2 = \frac{3+2\delta}{2}$, which is larger than the weight of $T(v, \Gamma')$. Thus, Lemma 3 in [2] does not hold any more.

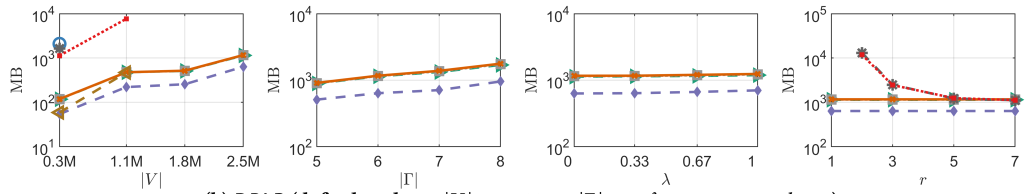
Recall that (i) Theorem 2 in [2] is the core pruning technique in PrunedDP, and is also an important pruning technique in PrunedDP++; and (ii) another important pruning technique in PrunedDP++ is the tour-based lower bounds construction method for A^*-search, and there are two types of tour-based lower bounds, which are based on Lemmas 2 and 3 in [2], respectively. Since Theorem 2 and Lemmas 2 and 3 in [2] do not hold in graphs with vertex weights, we do not implement PrunedDP and PrunedDP++ in our paper.

12 MEMORY CONSUMPTION RESULTS

Here, we evaluate the memory consumption of algorithms. The reported memory consumption of each algorithm contains the memory consumed by each input of this algorithm (*e.g.*, G and Γ) as well as any other memory consumed in the process of this algorithm. We use



(a) Toronto **(default values:** $|V| = 46073$; $|\Gamma| = 8$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**


(b) DBLP **(default values:** $|V| = 2497782$; $|\Gamma| = 6$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**

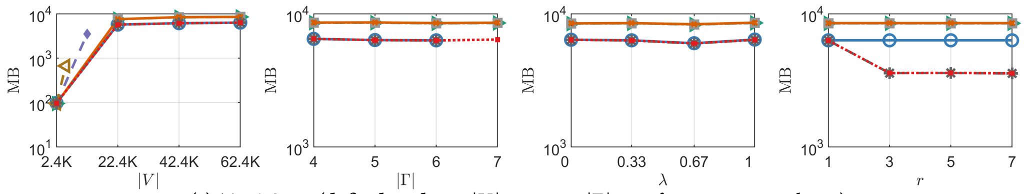
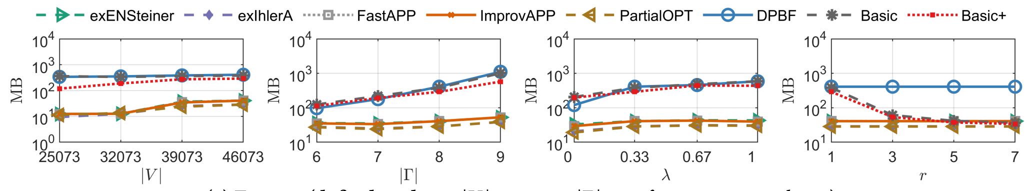
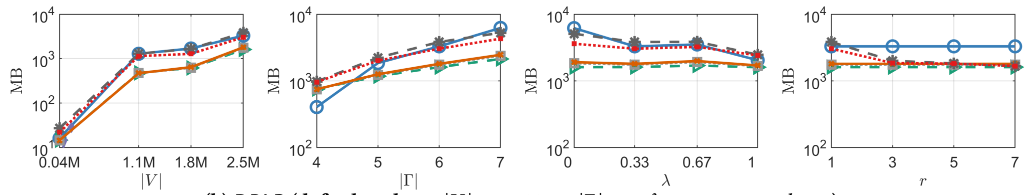

(c) MovieLens **(default values:** $|V| = 62423$; $|\Gamma| = 6$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**

Figure 7: The memory consumption in the main experiments where vertex groups are selected uniformly.



(a) Toronto **(default values:** $|V| = 46073$; $|\Gamma| = 8$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**


(b) DBLP **(default values:** $|V| = 2497782$; $|\Gamma| = 6$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**

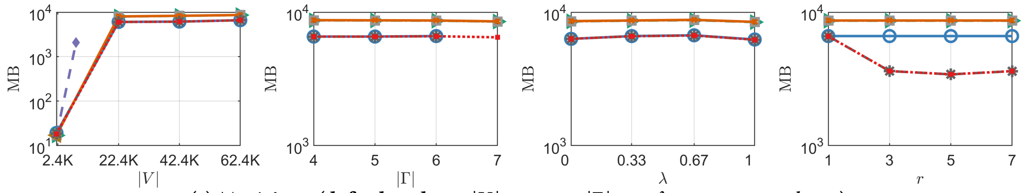

(c) MovieLens **(default values:** $|V| = 62423$; $|\Gamma| = 6$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**

Figure 8: The memory consumption in the main experiments where vertex groups are selected non-uniformly.

adjacency lists based on hashes to store graphs. Hashes consume more memories than arrays. Our purpose of using adjacency lists based on hashes is to fully optimize the time complexities of algorithms.

The main experiments. We report the memory consumption results in the main experiments in our paper in Figures 7 and 8. We observe that DPBF, Basic and Basic+ often consume more memory than the other non-exact algorithms (*e.g.,* when varying $|V|$ in Figure 7a), and the memory consumption of DPBF, Basic and Basic+ often increases quickly with $|\Gamma|$ (*e.g.,* when varying $|\Gamma|$ in Figure 7a). The reason is that, except the $O(|V| + |E|)$ memory consumed by the input graph, DPBF, Basic and Basic+ additionally consume $O(2^{|\Gamma|}|V|)$ memory in the dynamic programming process, while the other non-exact algorithms do not consume such an exponential amount of memory. Notably, the memory consumption of DPBF, Basic and Basic+ does not grow much with $|\Gamma|$ for MovieLens (*e.g.,* when varying $|\Gamma|$ in Figure 7c). The reason is that the MovieLens graph is dense, and as a result the memory consumed by the MovieLens graph dominates the increase of the $O(2^{|\Gamma|}|V|)$ memory. Further note that, the non-exact algorithms may consume more memory than DPBF, Basic and Basic+ in some cases

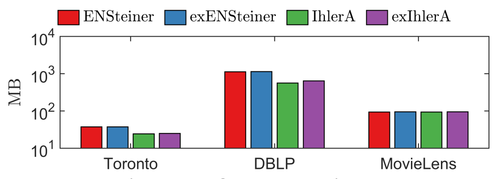


Figure 9: Our extensions.

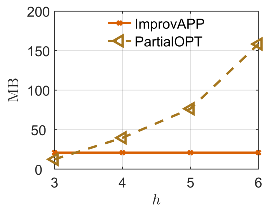


Figure 10: Varying h.

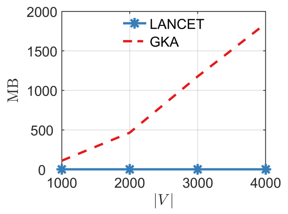


Figure 11: LANCET **versus** GKA**.**

(*e.g.,* Figure 7c). The reason is that these non-exact algorithms build an additional graph that has (i) the same set of edges with the input graph G and (ii) newly defined edge weights for finding lowest weight paths in G (see Section 3.2 in our paper on how to find lowest weight paths), while DPBF, Basic and Basic+ do not build such an additional graph.

Since the dynamic programming process of Basic and Basic+ terminates earlier when r is larger, the memory consumption of Basic and Basic+ decreases with r (*e.g.,* when varying r in Figure 7a). Notably, by incorporating pruning techniques, Basic and Basic+ enumerate fewer trees than DPBF in the dynamic programming process. As a result, Basic and Basic+ may consume a smaller amount of memory than DPBF (*e.g.,* when $|V| = 0.3M$ in Figure 7b). However, DPBF may consume a smaller amount of memory than Basic and Basic+ in some cases (*e.g.,* when $|\Gamma| = 6$ in Figure 7a). There are two reasons. First, Basic and Basic+ store the lowest weight paths between vertices and vertex groups, while DPBF does not store these paths. Second, all these three algorithms iteratively pop trees out of a min priority queue. Basic and Basic+ record trees that have been popped out (details in [2]), while DPBF does not record these trees.

Our extensions. We compare the memory consumption of ENSteiner, IhlerA, exENSteiner and exIhlerA in Figure 9, where vertex groups are selected via the uniform approach, and the parameter settings are: for Toronto, $|V| = 46073$, $|\Gamma| = 8$, $\lambda = 0.33$; for DBLP, $|V| = 2497782$, $|\Gamma| = 6$, $\lambda = 0.33$; for MovieLens, $|V| = 2423$, $|\Gamma| = 6$, $\lambda = 0.33$ (this corresponds to the experiments in Figure 3 in our paper). We observe that exENSteiner and exIhlerA may consume slightly more memory than ENSteiner and IhlerA, respectively. The reason is that exENSteiner and exIhlerA build an additional graph for finding lowest weight paths (as discussed above). In comparison, ENSteiner and IhlerA do not build such an additional graph for finding shortest paths.

Varying h in PartialOPT**.** We report the memory consumption of PartialOPT with respect to h in Figure 10, where the Toronto data is used, vertex groups are selected via the uniform approach, $|V| = 46073$, $|\Gamma| = 6$, $\lambda = 0.33$ (this corresponds to the experiments in Figure 7 in our paper). We observe that the memory consumed by PartialOPT grows quickly with h. The reason is that PartialOPT employs DPBF to connect h vertex groups optimally, and the space complexity of this process is $O(2^h|V|)$.

Comparing LANCET **with** GKA**.** We compare the memory consumption of LANCET and GKA in Figure 11, where the Toronto data is used, vertex groups are selected via the uniform approach, $\lambda = 0.33$, $|\Gamma| = |T_t| = 6$ (this corresponds to the experiments in Figure 8 in our paper). We observe that the memory consumption of GKA increases quickly with $|V|$. The reason is that GKA stores the lowest weight paths between all pairs of vertices, which has a space complexity of $O(|V_t|^2)$, where $|V_t| = |V| + |\Gamma|$. In comparison, LANCET only stores the lowest weight paths from compulsory vertices to the other vertices, which has a space complexity of $O(|T_t||V_t|)$ (see Line 3 of LANCET).

13 REFINING THE SOLUTIONS OF exENSteiner, exIhlerA AND FastAPP

There is a solution refinement process in ImprovAPP, *i.e.,* Lines 17-29 in ImprovAPP. This process refines a sub-optimal solution by removing non-unique-group leaves from this solution. Here, we use this process to refine the solutions of exENSteiner, exIhlerA and FastAPP. Notably, this process has already been incorporated into PartialOPT (*i.e.,* Line 13 in PartialOPT). Thus, we do not refine the solutions of PartialOPT here. We report the refinement results in Figures 12 and 13, where exENSteiner+R, exIhlerA+R and FastAPP+R are the refinements of exENSteiner, exIhlerA and FastAPP, respectively.

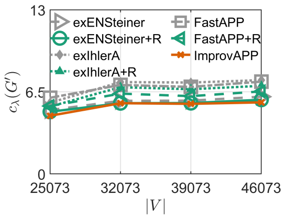
Suppose that there is a feasible solution tree $\Theta(V_\Theta, E_\Theta)$. Then, the time complexity of refining this solution is $O(|\Gamma||V_\Theta| + |V_\Theta| \log |V_\Theta|)$ (details in Section 4.3 in our paper). Since we generally have $|V_\Theta| \ll |V|$ in practice, the running times of refinement are negligible when comparing to the running times of our algorithms. For example, each of our algorithms takes around 100s to produce a feasible solution in the full DBLP graph, while it only takes around 2ms to refine this solution. Thus, we only evaluate the solution qualities in Figures 12 and 13, and do not evaluate the running times of refinement. We observe that ImprovAPP dominates exENSteiner+R, exIhlerA+R and FastAPP+R on solution qualities. We also observe that the refinement is often more effective when vertex groups are selected non-uniformly. For example, the refinement is more effective in Figure 13d than in Figure 12d. The reason is that, when vertex groups are selected non-uniformly, the sizes of the selected vertex groups are often larger, and as a result the leaves in the feasible solutions produced by exENSteiner, exIhlerA and FastAPP are more likely to be non-unique-group leaves.

REFERENCES

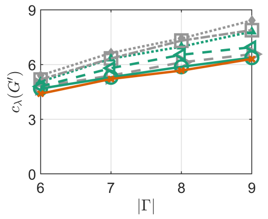
[1] Bolin Ding, Jeffrey Xu Yu, Shan Wang, Lu Qin, Xiao Zhang, and Xuemin Lin. 2007. Finding top-k min-cost connected trees in databases. In *IEEE International Conference on Data Engineering*. IEEE, 836–845.

[2] Rong-Hua Li, Lu Qin, Jeffrey Xu Yu, and Rui Mao. 2016. Efficient and progressive group Steiner tree search. In *Proceedings of the 2016 International Conference on Management of Data*. ACM, 91–106.

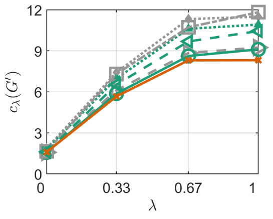
[3] Hiromitsu Takahashi and Akira Matsuyama. 1980. An approximate solution for the Steiner problem in graphs. *Math. Japonica* 24, 6 (1980), 573–577.



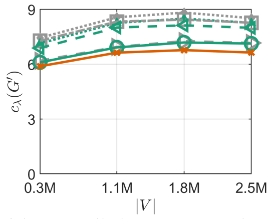
(a) Toronto ($|\Gamma| = 8$; $\lambda = 0.33$)



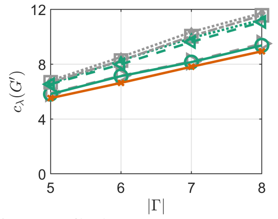
(b) Toronto ($|V| = 46073$; $\lambda = 0.33$)



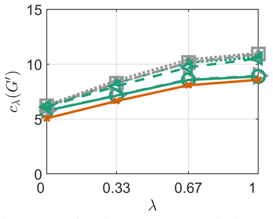
(c) Toronto ($|V| = 46073$; $|\Gamma| = 8$)



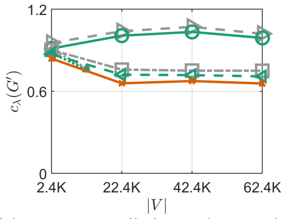
(d) DBLP ($|\Gamma| = 6$; $\lambda = 0.33$)



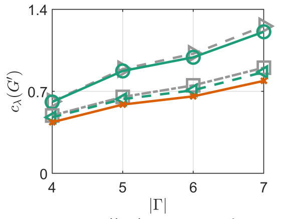
(e) DBLP ($|V| = 2497782$; $\lambda = 0.33$)



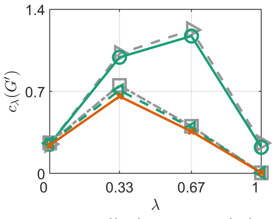
(f) DBLP ($|V| = 2497782$; $|\Gamma| = 6$)



(g) MovieLens ($|\Gamma| = 6$; $\lambda = 0.33$)

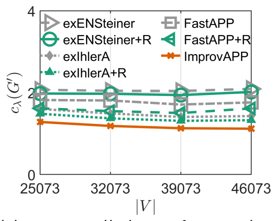


(h) MovieLens ($|V| = 62423$; $\lambda = 0.33$)

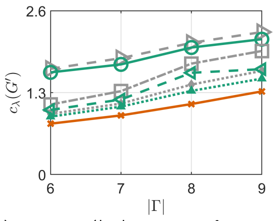


(i) MovieLens ($|V| = 62423$; $|\Gamma| = 6$)

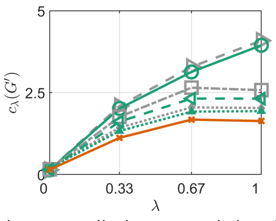
Figure 12: The refinement results in the main experiments where vertex groups are selected uniformly.



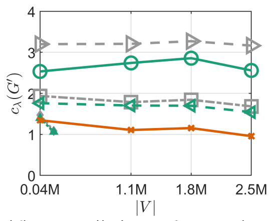
(a) Toronto ($|\Gamma| = 8$; $\lambda = 0.33$)



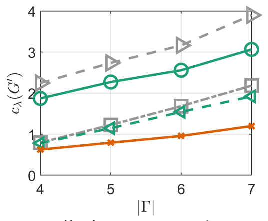
(b) Toronto ($|V| = 46073$; $\lambda = 0.33$)



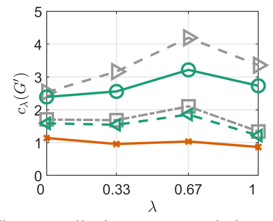
(c) Toronto ($|V| = 46073$; $|\Gamma| = 8$)



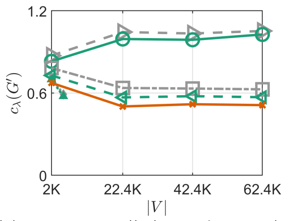
(d) DBLP ($|\Gamma| = 6$; $\lambda = 0.33$)



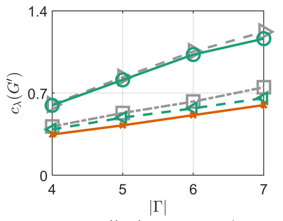
(e) DBLP ($|V| = 2497782$; $\lambda = 0.33$)



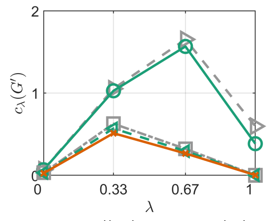
(f) DBLP ($|V| = 2497782$; $|\Gamma| = 6$)



(g) MovieLens ($|\Gamma| = 6$; $\lambda = 0.33$)



(h) MovieLens ($|V| = 62423$; $\lambda = 0.33$)



(i) MovieLens ($|V| = 62423$; $|\Gamma| = 6$)

Figure 13: The refinement results in the main experiments where vertex groups are selected non-uniformly.