Finding Group Steiner Trees in Graphs with both Vertex and Edge Weights: Some Supplemental Materials

Road map. In Section 1, we prove the transformation from group Steiner trees to Steiner trees. In Section 2, we prove the approximation guarantee of LANCET. In Section 3, we prove the approximation guarantee of exIhlerA. In Section 4, we prove the approximation guarantee of FastAPP. In Section 5, we prove the approximation guarantee of ImprovAPP. In Section 6, we prove the approximation guarantee of PartialOPT. In Section 7, we study the recent work in [2]. In Section 8, we show the memory consumption in experiments. In Section 9, we use the refinement process in ImprovAPP to refine the solutions of exENSteiner, exIhlerA and FastAPP.

1 THE TRANSFORMATION

Theorem 1. *Let $G(V, E, w, c)$ be a connected undirected graph, and Γ be a set of vertex groups. Let $G_t(V_t, E_t, w_t, c_t)$ be a connected undirected graph, and $T_t \subseteq V_t$ be a set of compulsory vertices. Based on G and Γ, we construct G_t and T_t in the following way:*

(1) Initialize $V_t = V$, $E_t = E$, $T_t = \emptyset$, $w_t = (1-\lambda)w$, and $c_t = \lambda c$.

(2) For each vertex group $g \in \Gamma$, (i) add a dummy vertex v_g into T_t and V_t, such that $w_t(v_g) = 0$, and (ii) add dummy edges (v_g, j) for all $j \in g$ into E_t, such that $c_t(v_g, j) = M$, and M is a constant satisfying

$$M > (1-\lambda)\sum_{v\in V} w(v) + \lambda \sum_{e\in E_{MST}} c(e), \tag{1}$$

and E_{MST} is the set of edges in a Minimum Spanning Tree of G.

Let Θ_{G_t} be an optimal solution to the vertex- and edge-weighted Steiner tree problem in G_t, and $\Theta^{non}_{G_t}$ be the non-dummy part of Θ_{G_t}. Then, there is an optimal solution to the vertex- and edge-weighted group Steiner tree problem in G, namely, Θ_G, that has the same sets of vertices and edges with $\Theta^{non}_{G_t}$.

Proof. Since dummy vertices only connect non-dummy vertices, there are at least $|\Gamma|$ dummy edges in Θ_{G_t}. If $c_\lambda(\Theta_G) < c(\Theta^{non}_{G_t})$, then there is a feasible solution to the vertex- and edge-weighted Steiner tree problem in G_t: Θ'_{G_t} such that

$$c(\Theta'_{G_t}) = c_\lambda(\Theta_G) + M|\Gamma| < c(\Theta_{G_t}), \tag{2}$$

which is not possible. Thus, we have $c_\lambda(\Theta_G) \geq c(\Theta^{non}_{G_t})$. Let Θ''_{G_t} be a tree in G_t such that (i) every dummy vertex v_g is a leaf of Θ''_{G_t}; and (ii) the non-dummy part of Θ''_{G_t}, namely, $\Theta^{non''}_{G_t}$, is in a Minimum Spanning Tree of G. Suppose that there is a dummy vertex v_g in Θ_{G_t} that is not a leaf. Since $c(\Theta^{non''}_{G_t}) < M$, we have

$$c(\Theta_{G_t}) \geq c(\Theta^{non}_{G_t}) + M(|\Gamma|+1) > c(\Theta''_{G_t}) = c(\Theta^{non''}_{G_t}) + M|\Gamma|, \tag{3}$$

which is not possible. Thus, every dummy compulsory vertex v_g is a leaf of Θ_{G_t}. As a result, $\Theta^{non}_{G_t}$ is connected and shares the same sets of vertices and edges with a feasible solution to the vertex- and edge-weighted group Steiner tree problem in G, which means that $c_\lambda(\Theta_G) \leq c(\Theta^{non}_{G_t})$. Therefore, $c_\lambda(\Theta_G) = c(\Theta^{non}_{G_t})$. Hence, this theorem holds. □

2 THE APPROXIMATION GUARANTEE OF LANCET

LANCET can be regarded as the vertex- and edge-weighted version of the algorithm in [3], which achieves an approximation guarantee of $2(1-1/|T_t|)$ for solving the vertex-unweighted Steiner tree problem. This approximation guarantee relies on the following deduction (*i.e.,* Lemma 1 in [3]): since a pre-order traversal of a tree traverses every edge in this tree exactly twice, in a graph with only edge weights, if we perform a pre-order traversal of an optimal solution tree and sum up every weight that we encounter (including duplicates), then the result is exactly twice the weight of an optimal solution tree. However, in a graph with both vertex and edge weights, summing up the weights that we encounter during this traversal does not always result in twice the weight of an optimal solution tree, since (i) an optimal solution tree may contain non-compulsory vertices with positive weights; and (ii) a pre-order traversal of an optimal solution tree may visit such a vertex more than twice (specifically, the number of times that a pre-order traversal of an optimal solution tree visits such a vertex equals the degree of this vertex in this optimal solution tree). Thus, the above approximation guarantee of $2(1-1/|T_t|)$ does not hold for LANCET. In what follows, we establish the approximation guarantee of LANCET.

Theorem 2. LANCET *has a sharp approximation guarantee of $|T_t|-1$ for solving the vertex- and edge-weighted Steiner tree problem.*

Proof. LANCET merges $|T_t|-1$ LWPs to connect all compulsory vertices together. Suppose that the highest-weight one of these LWPs is LWP', and Θ_{opt} is an optimal solution. Since $c(LWP')$ is smaller than or equal to the weight of the LWP between a pair of compulsory vertices, we have

$$c(\Theta_{opt}) \geq c(LWP'). \tag{4}$$

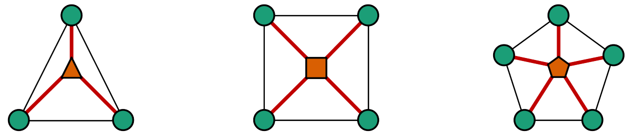


Figure 1: Touching the approximation guarantee of $|T_t| - 1$.

Since there are $|T_t| - 1$ LWPs that have been merged, we have

$$(|T_t| - 1)c(\Theta_{opt}) \geq (|T_t| - 1)c(LWP') \geq c(\Theta). \tag{5}$$

Therefore, LANCET has an approximation guarantee of $|T_t| - 1$. We further show that $|T_t| - 1$ is the sharp approximation guarantee of LANCET. Consider a regular polygon composed of $|T_t|$ compulsory vertices, and a non-compulsory vertex that is in the middle of this polygon and connects $|T_t|$ compulsory vertices. The weight of each edge between compulsory vertices is x, the weight of each edge between a compulsory vertex and the middle non-compulsory vertex is 0, the weight of each compulsory vertex is 0, and the weight of the middle non-compulsory vertex is z. Suppose that $x = z - \delta$, where δ is a tiny positive value; and $z < (|T_t| - 1)x$. Since $x < z$, Θ contains $|T_t| - 1$ edges between compulsory vertices, and $c(\Theta) = (|T_t| - 1)x$. Since $z < (|T_t| - 1)x$, Θ_{opt} contains all the edges between compulsory vertices and the middle non-compulsory vertex, and $c(\Theta_{opt}) = z$. We have

$$\lim_{\delta \to 0} \frac{c(\Theta)}{c(\Theta_{opt})} = \frac{(|T_t| - 1)(z - \delta)}{z} = |T_t| - 1. \tag{6}$$

Hence, $|T_t| - 1$ is the sharp approximation guarantee of LANCET. □

3 THE APPROXIMATION GUARANTEE OF exIhlerA

Theorem 3. exIhlerA *has a sharp approximation guarantee of* $|\Gamma| - 1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Suppose that $\Theta_{OPT}(V_{OPT}, E_{OPT})$ is an optimal solution. Let $\Gamma = \{g_1, \ldots, g_{|\Gamma|}\}$. There is a tuple $(v_1, \ldots, v_{|\Gamma|})$ such that $v_i \in V_{OPT} \cap g_i$ for all $i \in \{1, \ldots, |\Gamma|\}$. Without loss of generality, assume that $g_{min} = g_1$. For every $i \in \{2, \ldots, |\Gamma|\}$, there is exactly one simple path between v_1 and v_i in Θ_{OPT}, which we refer to as $P_{v_1 v_i}$. We have

$$c_\lambda(P_{v_1 v_i}) \leq c_\lambda(\Theta_{OPT}), \tag{7}$$

$$\sum_{g \in \Gamma \setminus g_1} c_\lambda(LWP_{\lambda v_1 g}) \leq \sum_{i \in \{2, \ldots, |\Gamma|\}} c_\lambda(P_{v_1 v_i}). \tag{8}$$

Thus,

$$c_\lambda(\Theta) \leq c_\lambda(G_{v_1}) \leq \sum_{g \in \Gamma \setminus g_1} c_\lambda(LWP_{\lambda v_1 g}) \leq \sum_{i \in \{2, \ldots, |\Gamma|\}} c_\lambda(P_{v_1 v_i}) \leq (|\Gamma| - 1)c_\lambda(\Theta_{OPT}). \tag{9}$$

Hence, exIhlerA has an approximation guarantee of $|\Gamma| - 1$. We note that this guarantee is sharp. To explain, consider the graph $G(V, E, w, c)$ in Figure 2, where $V = \{v_0, v_1, \ldots, v_{|\Gamma|}\}$, $E = \{(v_{|\Gamma|}, v_0), (v_{|\Gamma|}, v_1), \ldots, (v_{|\Gamma|}, v_{|\Gamma|-1})\}$, $w(i) = 0$ for all $i \in V$, $c(v_{|\Gamma|}, v_1) = \ldots = c(v_{|\Gamma|}, v_{|\Gamma|-1}) = 1$, and $c(v_{|\Gamma|}, v_0) = 1 + \delta$, where δ is a tiny positive value. In addition, $\Gamma = \{v_0, v_1\} \cup \ldots \cup \{v_0, v_{|\Gamma|-1}\} \cup \{v_{|\Gamma|}\}$. Let $\lambda = 1$. Since $g_{min} = \{v_{|\Gamma|}\}$, exIhlerA produces the solution $\Theta = \{(v_{|\Gamma|}, v_1), \ldots, (v_{|\Gamma|}, v_{|\Gamma|-1})\}$, and $c_\lambda(\Theta) = |\Gamma| - 1$. When $|\Gamma| = 2$, Θ is the optimal solution, *i.e.*, the approximation ratio is $|\Gamma| - 1 = 1$. When $|\Gamma| > 2$, we have $\Theta_{OPT} = \{(v_{|\Gamma|}, v_0)\}$, and

$$\lim_{\delta \to 0} \frac{c_\lambda(\Theta)}{c_\lambda(\Theta_{OPT})} = \frac{|\Gamma| - 1}{1 + \delta} = |\Gamma| - 1. \tag{10}$$

Hence, the best possible approximation guarantee of exIhlerA is $|\Gamma| - 1$. □

4 THE APPROXIMATION GUARANTEE OF FastAPP

Theorem 4. FastAPP *has a sharp approximation guarantee of* $|\Gamma| - 1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Let $\Theta_{OPT}(V_{OPT}, E_{OPT})$ be an optimal solution, and $\Gamma = \{g_1, \ldots, g_{|\Gamma|}\}$. There is a tuple $(v_1, \ldots, v_{|\Gamma|})$ such that $v_i \in V_{OPT} \cap g_i$ for all $i \in \{1, \ldots, |\Gamma|\}$. Without loss of generality, suppose that $g_{min} = g_1$. Let $g_x \in \Gamma \setminus g_1$ be such a vertex group that

$$c_\lambda(LWP_{\lambda v_1 g_x}) = \max\{c_\lambda(LWP_{\lambda v_1 g}) \mid \forall g \in \Gamma \setminus g_1\}. \tag{11}$$

Since $LWP_{\lambda v_1 g_x}$ links fewer groups to v_1 than Θ_{OPT}, we have

$$c_\lambda(LWP_{\lambda v_1 g_x}) \leq c_\lambda(\Theta_{OPT}). \tag{12}$$

Lines 5-8 in FastAPP guarantee that

$$\max\{c_\lambda(LWP_{\lambda i_{min} g}) \mid \forall g \in \Gamma \setminus g_1\} \leq c_\lambda(LWP_{\lambda v_1 g_x}). \tag{13}$$

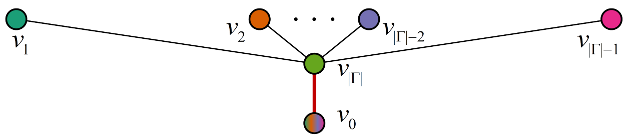


Figure 2: Touching the approximation guarantee of $|\Gamma|-1$.

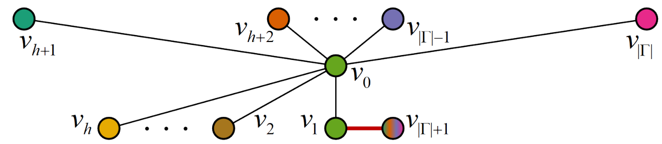


Figure 3: Touching the approximation guarantee of $|\Gamma|-h+1$.

By Lines 10-11 in FastAPP, we have

$$c_\lambda(\Theta) \le (|\Gamma|-1) \cdot \max\{c_\lambda(LWP_{\lambda i_{min} g}) \mid \forall g \in \Gamma \setminus g_1\}. \tag{14}$$

Thus,

$$c_\lambda(\Theta) \le (|\Gamma|-1) c_\lambda(LWP_{\lambda v_1 g_x}) \le (|\Gamma|-1) c_\lambda(\Theta_{OPT}). \tag{15}$$

Hence, FastAPP has an approximation guarantee of $|\Gamma|-1$. The sharpness of this guarantee can be seen from the example in Section 3, *i.e.,* Figure 2. Hence, this theorem holds. □

5 THE APPROXIMATION GUARANTEE OF ImprovAPP

Theorem 5. ImprovAPP *has a sharp approximation guarantee of* $|\Gamma|-1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Let $\Theta_{OPT}(V_{OPT}, E_{OPT})$ be an optimal solution. Let $\Gamma = \{g_1, \dots, g_{|\Gamma|}\}$. There is a tuple $(v_1, \dots, v_{|\Gamma|})$ such that $v_i \in V_{OPT} \cap g_i$ for all $i \in \{1, \dots, |\Gamma|\}$. Without loss of generality, suppose that $g_{min} = g_1$. When ImprovAPP processes v_1 in the for loop (Lines 5-14), it concatenates $|\Gamma|-1$ lowest weight paths that link $\{g_2, \dots, g_{|\Gamma|}\}$, respectively, to build Θ_{v_1}. Let $LWP_{\lambda v_x g_y}$ be one of these paths that has the largest regulated weight, and links $g_y \in \{g_2, \dots, g_{|\Gamma|}\}$. Then,

$$c_\lambda(\Theta_{v_1}) \le (|\Gamma|-1) c_\lambda(LWP_{\lambda v_x g_y}). \tag{16}$$

Let $LWP_{\lambda v_1 g_y}$ be the lowest weight path between v_1 and g_y. Since $LWP_{\lambda v_1 g_y}$ has been pushed into Q initially (Line 6) and has (possibly) been updated to $LWP_{\lambda v_x g_y}$ (Line 12), we have

$$c_\lambda(LWP_{\lambda v_x g_y}) \le c_\lambda(LWP_{\lambda v_1 g_y}). \tag{17}$$

Since $LWP_{\lambda v_1 g_y}$ links fewer groups to v_1 than Θ_{OPT}, we have

$$c_\lambda(LWP_{\lambda v_1 g_y}) \le c_\lambda(\Theta_{OPT}). \tag{18}$$

Thus,

$$c_\lambda(\Theta) \le c_\lambda(\Theta_{v_1}) \le (|\Gamma|-1) c_\lambda(LWP_{\lambda v_x g_y}) \le (|\Gamma|-1) c_\lambda(LWP_{\lambda v_1 g_y}) \le (|\Gamma|-1) c_\lambda(\Theta_{OPT}). \tag{19}$$

Therefore, ImprovAPP has an approximation guarantee of $|\Gamma|-1$. The sharpness of this guarantee can be seen from the example in Section 3, *i.e.,* Figure 2. Thus, this theorem holds. □

6 THE APPROXIMATION GUARANTEE OF PartialOPT

Theorem 6. PartialOPT *has a sharp approximation guarantee of* $|\Gamma|-h+1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Suppose that $\Theta_{OPT}(V_{OPT}, E_{OPT})$ is an optimal solution, and $\Gamma = \{g_1, \dots, g_{|\Gamma|}\}$. There is a tuple $(v_1, \dots, v_{|\Gamma|})$ such that $v_i \in V_{OPT} \cap g_i$ for all $i \in \{1, \dots, |\Gamma|\}$. Without loss of generality, let $g_{min} = g_1$. For $v_1 \in g_{min}$, $\Theta^h_{v_1}$ is optimal for $\Gamma_1 = \{\{v_1\}, g_2, \dots, g_h\}$. Since $\Theta^h_{v_1}$ connects fewer vertex groups to v_1, we have

$$c_\lambda(\Theta^h_{v_1}) \le c_\lambda(\Theta_{OPT}). \tag{20}$$

If $\Gamma_2 = \{v_1\}$, then $\Theta^{|\Gamma|}_{v_1} = \{v_1\}$. Otherwise, we implement exIhlerA to produce $\Theta^{|\Gamma|}_{v_1}$ for $\Gamma_2 = \{\{v_1\}, g_{h+1}, \dots, g_{|\Gamma|}\}$. Suppose that $\Theta^{|\Gamma|}_{OPT}$ is an optimal solution for Γ_2. The proof of Theorem 3 shows that

$$c_\lambda(\Theta^{|\Gamma|}_{v_1}) \le (|\Gamma|-h) c_\lambda(\Theta^{|\Gamma|}_{OPT}). \tag{21}$$

Since $\Theta^{|\Gamma|}_{OPT}$ connects fewer vertex groups to v_1, we have

$$c_\lambda(\Theta^{|\Gamma|}_{OPT}) \le c_\lambda(\Theta_{OPT}). \tag{22}$$

Thus,

$$c_\lambda(\Theta) \le c_\lambda(G_{v_1}) = c_\lambda(\Theta^h_{v_1} \cup \Theta^{|\Gamma|}_{v_1}) \le c_\lambda(\Theta^h_{v_1}) + c_\lambda(\Theta^{|\Gamma|}_{v_1}) \le (|\Gamma|-h+1) c_\lambda(\Theta_{OPT}). \tag{23}$$

Therefore, PartialOPT has an approximation guarantee of $|\Gamma| - h + 1$. We show that this guarantee is sharp. Consider the graph $G(V, E, w, c)$ in Figure 3, where $V = \{v_0, v_1, \ldots, v_{|\Gamma|+1}\}$, $E = \{(v_0, v_1), (v_0, v_2), \ldots, (v_0, v_{|\Gamma|}), (v_1, v_{|\Gamma|+1})\}$, $w(i) = 0$ for every $i \in \{v_0, \ldots, v_{h-1}\}$, $w(i) = 1$ for every $i \in \{v_h, \ldots, v_{|\Gamma|+1}\}$, $c(v_0, v_1) = \delta_1$, $c(v_1, v_{|\Gamma|+1}) = \delta_2$, where δ_1 and δ_2 are two tiny positive values, and $\delta_1 < \delta_2$, and all the other edge weights are zero. In addition, $\Gamma = \{g_1, \ldots, g_{|\Gamma|}\} = \{v_0, v_1\} \cup \{v_2, v_{|\Gamma|+1}\} \cup \ldots \cup \{v_{|\Gamma|}, v_{|\Gamma|+1}\}$. Let $\lambda = 0.5$, *i.e.,* vertex and edge weights are regulated equally. PartialOPT enumerates two vertices in g_{min}: v_0 and v_1. For v_0, PartialOPT produces $\Theta^h_{v_0} = \{(v_0, v_2), \ldots, (v_0, v_h)\}$, and $\Theta^{|\Gamma|}_{v_0} = \{(v_0, v_{h+1}), \ldots, (v_0, v_{|\Gamma|})\}$. Thus, $\Theta_{v_0} = \{(v_0, v_2), \ldots, (v_0, v_{|\Gamma|})\}$. Similarly, for v_1, since $\delta_1 < \delta_2$, PartialOPT produces $\Theta_{v_1} = \{(v_0, v_1), \ldots, (v_0, v_{|\Gamma|})\}$. We have $\Theta = \Theta_{v_0}$. When $|\Gamma| = h$, Θ is the optimal solution, *i.e.*, the approximation ratio is $|\Gamma| - h + 1 = 1$. When $|\Gamma| > h$, we have $\Theta_{OPT} = \{(v_1, v_{|\Gamma|+1})\}$, and

$$\lim_{\delta_2 \to 0} \frac{c_\lambda(\Theta)}{c_\lambda(\Theta_{OPT})} = \frac{|\Gamma| - h + 1}{1 + \delta_2} = |\Gamma| - h + 1. \tag{24}$$

Hence, $|\Gamma| - h + 1$ is the best possible approximation guarantee of PartialOPT. This theorem holds. □

7 THE RECENT WORK ON IMPROVING DPBF

The PrunedDP and PrunedDP++ algorithms in [2] improves DPBF [1] for finding optimal vertex-unweighted group Steiner trees. The main idea of this improvement is to incorporate pruning techniques into the process of DPBF. Here, we show that PrunedDP and PrunedDP++ depend on pruning techniques that do not hold in graph with vertex weights. In the following, we use $T(v, \Gamma)$ to signify the minimum-weight tree that roots at vertex v and covers all vertex groups in Γ.

Theorem 2 in [2] does not hold in graphs with vertex weights. First, note that Theorem 2 in [2] is the core pruning technique in PrunedDP, and is also an important pruning technique in PrunedDP++. This theorem does not hold in graphs with vertex weights. To explain, we first briefly describe the dynamic programming process of DPBF through an example in Figure 4. Understanding this process is necessary for understanding the reason why Theorem 2 in [2] does not hold in graphs with vertex weights.

In Figure 4, there are three vertex groups $g_1 = \{v_1\}$, $g_2 = \{v_2\}$ and $g_3 = \{v_3\}$. The weight of u is 1, and each of the other vertex and edge weights is δ, and δ is a tiny positive value. The optimal solution tree is the whole graph, and the weight of this tree is $1 + 6\delta$ (*i.e.,* the sum of vertex and edge weights). To find this tree, DPBF first initializes $T(v_1, \{g_1\})$ as the single vertex v_1; $T(v_2, \{g_2\})$ as the single vertex v_2; and $T(v_3, \{g_3\})$ as the single vertex v_3. Then, DPBF grows $T(v_1, \{g_1\})$, $T(v_2, \{g_2\})$ and $T(v_3, \{g_3\})$ to vertex u, and produces $T(u, \{g_1\})$ as the edge (u, v_1); $T(u, \{g_2\})$ as the edge (u, v_2); and $T(u, \{g_3\})$ as the edge (u, v_3). Subsequently, it merges $T(u, \{g_1\})$ and $T(u, \{g_2\})$ as $T(u, \{g_1, g_2\}) = \{(u, v_1), (u, v_2)\}$. At last, it merges $T(u, \{g_3\})$ and $T(u, \{g_1, g_2\})$, and produces the optimal solution tree.

Theorem 2 in [2] is that: if all vertex weights are zero, then in DPBF, we can merge two subtrees $T(u, \Gamma')$ and $T(u, \Gamma'')$ for $\Gamma'' \subset \Gamma \setminus \Gamma'$ only when the total weight of these two subtrees is no larger than $\frac{2}{3}$ of the weight of an optimal solution tree. For example, if all vertex weights are zero in the above instance, then the weight of the optimal solution tree is 3δ. When we merge $T(u, \{g_1\})$ and $T(u, \{g_2\})$ as $T(u, \{g_1, g_2\})$ in the above process, the total weight of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is 2δ, which is no larger than $\frac{2}{3}$ of the weight of an optimal solution tree. By Theorem 2 in [2], merging these two subtrees may help produce the optimal solution tree. If the total weight of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is larger than $\frac{2}{3}$ of the weight of an optimal solution tree, then merging these two subtrees does not help produce the optimal solution tree, and thus this merge can be avoided. However, this is true only when all vertex weights are zero. For example, if we consider the vertex weights in the above instance, then the total weight of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is $2 + 4\delta$, which is larger than $\frac{2}{3}$ of the weight of an optimal solution tree: $1 + 6\delta$ (notably, even the weight of $T(u, \{g_1, g_2\}) = \{(u, v_1), (u, v_2)\}$, which is $1 + 4\delta$, is larger than $\frac{2}{3}$ of the weight of an optimal solution tree). As a result, if we use Theorem 2 in [2] in the above instance with vertex weights, then the optimal solution tree will never be found. That is to say, Theorem 2 in [2] does not hold in graphs with vertex weights.

We point out the specific place in the proof of Theorem 2 in [2] that does not hold in graphs with vertex weights as follows. In the beginning of the proof of Theorem 2 in [2], an optimal solution is assumed to be a tree rooted at vertex u with k subtrees, $T_1, \ldots, T_k$. Each subtree T_i roots at v_i, and the weight of each subtree is smaller than half of the weight of an optimal solution tree (*e.g.,* in Figure 4, T_i is the single vertex v_i). Let $\overline{T_i}$ be the the edge-grown subtree that is grown by T_i with an edge (v_i, u) (e.g., in Figure 4, $\overline{T_i}$ is the edge (v_i, u)). The proof of Theorem 2 in [2] then claims that: there are three different cases: (1) the weight of each $\overline{T_i}$ is smaller than half of the weight of an optimal solution tree; (2) there is only one edge-grown subtree $\overline{T_i}$ that has a weight no smaller than half of the weight of an optimal solution tree; and (3) there are two edge-grown subtrees and the weight of each one is half of the weight of an optimal solution tree. This claim is not true in graphs with vertex weights, where there is a fourth case: there are more than two edge-grown subtrees such that the weight of each one is large than half of the weight of an optimal solution tree (*e.g.,* in Figure 4, if we consider vertex weights, then the weight of each $\overline{T_i}$ is $1 + 2\delta$, which is larger than half of the weight of an optimal solution tree).

Since Theorem 2 in [2] is the core pruning technique in PrunedDP and does not hold in graphs with vertex weights, we do not implement PrunedDP in our paper.

Lemma 2 in [2] does not hold in graphs with vertex weights. Theorem 2 in [2] is also an important pruning technique in PrunedDP++. Another important pruning technique in PrunedDP++ is the tour-based lower bounds construction method in [2]. Lemma 2 in [2] is a key of this method. In the following, we show that this lemma does not hold in graphs with vertex weights.

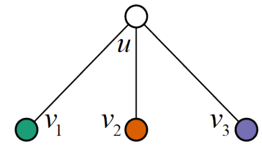


Figure 4: An example for showing Theorem 2 in [2].

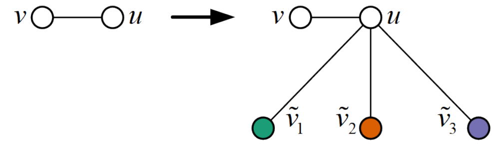


Figure 5: An example for showing Lemma 2 in [2].

First, we briefly introduce the label-enhanced graph in [2], which is constructed by adding dummy vertices and edges into the graph as follows. For each group $g_i \in \Gamma$, we add a dummy vertex $\widetilde{v_i}$, and also add a dummy edge $(\widetilde{v_i}, u)$ with zero weight for every $u \in g_i$. For example, in Figure 5, the graph contains two vertices v and u, and one edge (v, u), and there are three vertex groups $g_1 = g_2 = g_3 = \{u\}$. We add three dummy vertices $\widetilde{v_1}, \ldots, \widetilde{v_3}$ and three dummy edges $(\widetilde{v_1}, u), \ldots, (\widetilde{v_3}, u)$ for creating the label-enhanced graph.

Then, [2] uses $W(\widetilde{v_i}, \widetilde{v_j}, \Gamma')$ to refer to the weight of the minimum-weight route that starts from $\widetilde{v_i}$, ends at $\widetilde{v_j}$, and passes through all dummy vertices that corresponds to vertex groups in Γ'. Moreover, [2] uses $d(v, \widetilde{v_i})$ to refer to the weight of the minimum-weight path between non-dummy vertex v and dummy vertex $\widetilde{v_i}$. Lemma 2 in [2] is that: for any pair of vertex v and a subset of vertex groups $\Gamma' \subseteq \Gamma$, the weight of $T(v, \Gamma')$ is not smaller than $\frac{\min_{g_i, g_j \in \Gamma'} \{d(v,\widetilde{v_i}) + W(\widetilde{v_i}, \widetilde{v_j}, \Gamma') + d(\widetilde{v_j}, v)\}}{2}$.

This lemma is true when all vertex weights are zero. For example, in Figure 5, let the weight of edge (v, u) be δ, which is a tiny positive value, and all vertex weights be zero, and $\Gamma' = \{g_1, g_2\}$. Then, $d(v, \widetilde{v_1}) = d(\widetilde{v_2}, v) = \delta$, and $W(\widetilde{v_1}, \widetilde{v_2}, \Gamma') = 0$. As a result, $\frac{\min_{g_i, g_j \in \Gamma'} \{d(v,\widetilde{v_i}) + W(\widetilde{v_i}, \widetilde{v_j}, \Gamma') + d(\widetilde{v_j}, v)\}}{2} = \delta$, which equals the weight of $T(v, \Gamma')$. Thus, Lemma 2 in [2] holds. Lemma 2 is proven in [2] by first doubling every edge in $T(v, \Gamma')$ to obtain an Euler tour that starts from v and also ends at v, and then employing the fact that the total edge weight (including duplicates) we encounter in this Euler tour is twice the total edge weight in $T(v, \Gamma')$. Nevertheless, this lemma does not hold in graphs with vertex weights. For example, suppose that, in Figure 5, the weights of v and u are 0 and 1, respectively. Still let $\Gamma' = \{g_1, g_2\}$. Then, $d(v, \widetilde{v_1}) = d(\widetilde{v_2}, v) = 1 + \delta$, and $W(\widetilde{v_1}, \widetilde{v_2}, \Gamma') = 1$. As a result, $\frac{\min_{g_i, g_j \in \Gamma'} \{d(v,\widetilde{v_i}) + W(\widetilde{v_i}, \widetilde{v_j}, \Gamma') + d(\widetilde{v_j}, v)\}}{2} = \frac{3+2\delta}{2}$, which is larger than the weight of $T(v, \Gamma')$: $1 + \delta$. Thus, Lemma 2 in [2] does not hold any more. The reason is that the weight of u is counted multiple times, since the tour that starts from v and ends at v encounters u multiple times, *i.e.,* the total vertex and edge weight (including duplicates) we encounter in the Euler tour in the proof of Lemma 2 in [2] may be more than twice the total vertex and edge weight in $T(v, \Gamma')$, since an Euler tour of a tree may visit a vertex in this tree more than twice. Notably, as discussed in Section 2, due to a similar reason, LANCET does not have an approximation guarantee of 2 in graphs with vertex weights.

Both Theorem 2 and Lemma 2 in [2] are core pruning techniques in PrunedDP++. Since both Theorem 2 and Lemma 2 in [2] do not hold in graphs with vertex weights, we do not implement PrunedDP++ in our paper.

8 MEMORY CONSUMPTION IN EXPERIMENTS

Here, we report the memory consumption results in the experiments in our paper. Notably, the reported memory consumption of each algorithm contains the memory consumed by the input graph G and the input set of vertex groups Γ. We use adjacency lists based on hashes to store graphs. Adjacency lists based on hashes consume more memories than adjacency lists based on arrays. Our purpose of using adjacency lists based on hashes is to fully optimize the time complexities of algorithms.

Our extensions. We report the memory consumption of ENSteiner, IhlerA, exENSteiner and exIhlerA in Figure 6, where vertex groups are selected via the uniform approach, and the parameter settings are: for Toronto, $|V| = 46073$, $|\Gamma| = 8$, $\lambda = 0.33$; for DBLP, $|V| = 2497782$, $|\Gamma| = 6$, $\lambda = 0.33$; for MovieLens, $|V| = 2423$, $|\Gamma| = 5$, $\lambda = 0.33$ (this corresponds to the experiments in Figure 2 in our paper). We observe that exENSteiner and exIhlerA consume similar amounts of memory with ENSteiner and IhlerA, respectively.

Comparing DPBF with Basic. We report the memory consumption of DPBF and Basic in Figure 7, where vertex groups are selected via the uniform and non-uniform approaches in the left and right sub-figures, respectively. The parameter settings are: for Toronto, $|V| = 46073$; for DBLP, $|V| = 107782$; for MovieLens, $|V| = 10423$; and for all datasets, $|\Gamma| = 5$, $\lambda = 0.33$ (this corresponds to the experiments in Figure 3 in our paper). We observe that Basic consumes more memory than DPBF. There are two reasons. First, Basic stores the lowest weight paths between vertices and vertex groups, while DPBF does not find or store these paths. Second, recall that both Basic and DPBF enumerate $T(v, \Gamma')$ for every pair of $v \in V$ and $\Gamma' \subseteq \Gamma$, in an increasing order of the weight of $T(v, \Gamma')$, until an optimal solution is found. To enumerate $T(v, \Gamma')$ in an increasing order, both Basic and DPBF iteratively pop $T(v, \Gamma')$ out of a min priority queue. Basic records every $T(v, \Gamma')$ that has been popped out of the queue (details in [2]), while DPBF does not record this. As result, Basic consumes more memory than DPBF.

The main experiments. We report the memory consumption results in the main experiments in our paper in Figures 8 and 9. We observe that the memory consumption of DPBF grows exponentially with $|\Gamma|$ for Toronto and DBLP (*e.g.,* Figures 9b and 9e). The reason is that, except

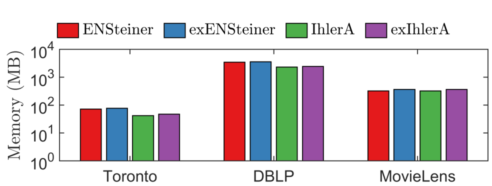


Figure 6: Our extensions.

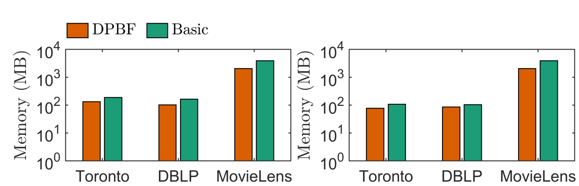
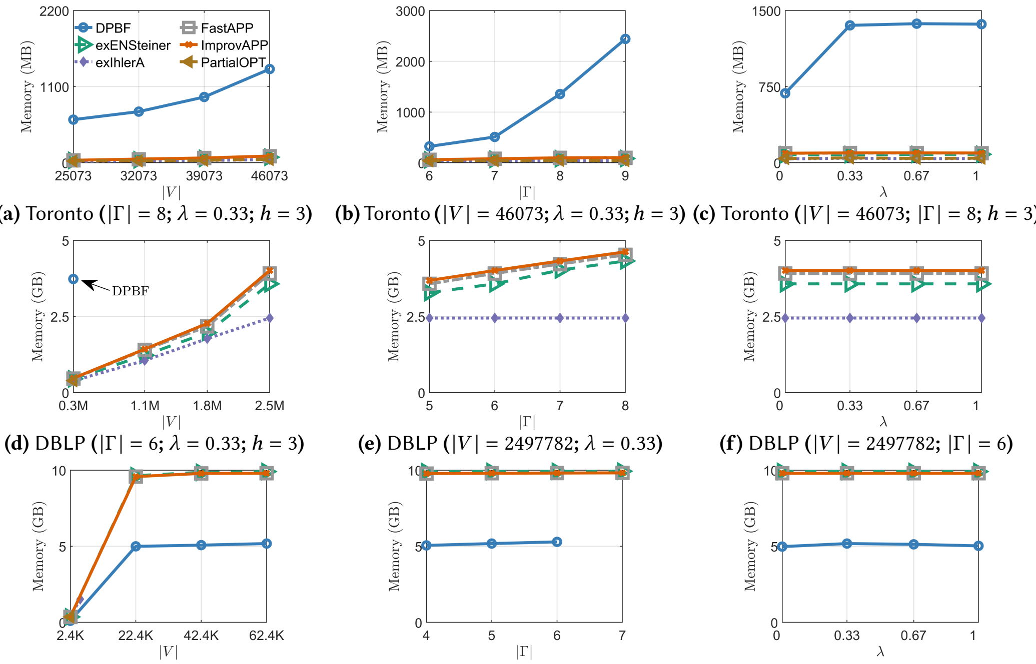


Figure 7: Comparing DPBF with Basic.

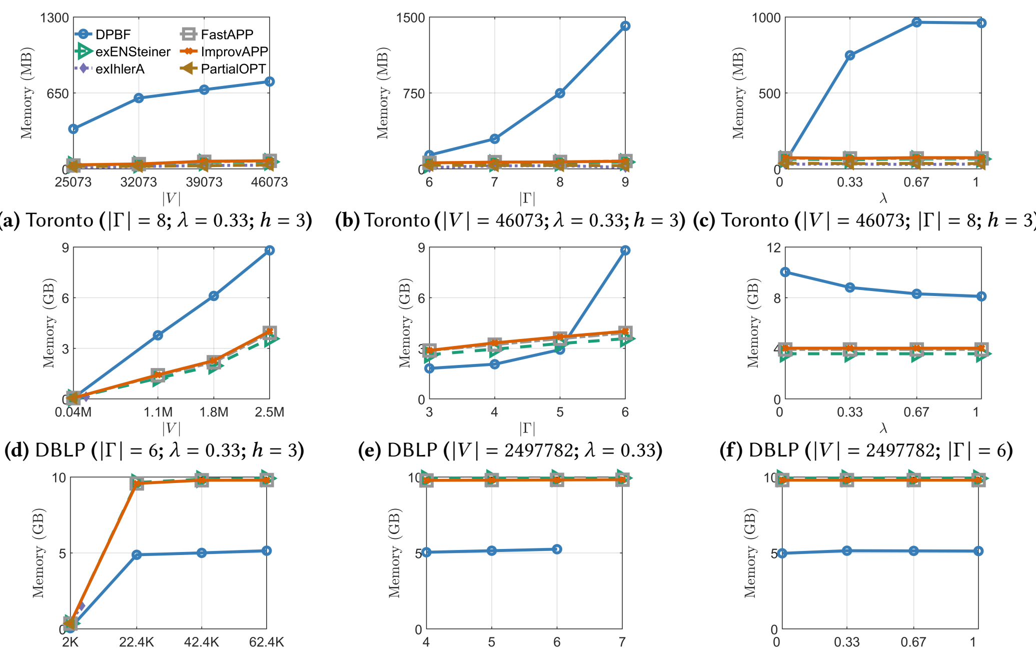


(a) Toronto ($|\Gamma| = 8; \lambda = 0.33; h = 3$) **(b)** Toronto ($|V| = 46073; \lambda = 0.33; h = 3$) **(c)** Toronto ($|V| = 46073; |\Gamma| = 8; h = 3$)

(d) DBLP ($|\Gamma| = 6; \lambda = 0.33; h = 3$) **(e)** DBLP ($|V| = 2497782; \lambda = 0.33$) **(f)** DBLP ($|V| = 2497782; |\Gamma| = 6$)

(g) MovieLens ($|\Gamma| = 5; \lambda = 0.33; h = 3$) **(h)** MovieLens ($|V| = 62423; \lambda = 0.33$) **(i)** MovieLens ($|V| = 62423; |\Gamma| = 5$)

Figure 8: The memory consumption in the main experiment results in which vertex groups are selected uniformly.



(a) Toronto ($|\Gamma| = 8; \lambda = 0.33; h = 3$) **(b)** Toronto ($|V| = 46073; \lambda = 0.33; h = 3$) **(c)** Toronto ($|V| = 46073; |\Gamma| = 8; h = 3$)

(d) DBLP ($|\Gamma| = 6; \lambda = 0.33; h = 3$) **(e)** DBLP ($|V| = 2497782; \lambda = 0.33$) **(f)** DBLP ($|V| = 2497782; |\Gamma| = 6$)

(g) MovieLens ($|\Gamma| = 5; \lambda = 0.33; h = 3$) **(h)** MovieLens ($|V| = 62423; \lambda = 0.33$) **(i)** MovieLens ($|V| = 62423; |\Gamma| = 5$)

Figure 9: The memory consumption in the main experiment results in which vertex groups are selected non-uniformly.

the memory consumed by G and Γ, the memory consumed in the dynamic process in DPBF has a complexity of $O(2^{|\Gamma|}|V|)$. In comparison, the memory consumption of DPBF does not grow much with $|\Gamma|$ for MovieLens (*e.g.,* Figure 8h). The reason is that the MovieLens graph is dense, and as a result the memory consumed by G dominates the $O(2^{|\Gamma|}|V|)$ memory consumed in the dynamic process in DPBF. Similarly, the memories consumed by exENSteiner, FastAPP and ImprovAPP do not increase much with $|\Gamma|$ for MovieLens. Notably, for MovieLens, the memories consumed by exENSteiner, FastAPP and ImprovAPP are roughly twice the memory consumed by DPBF. The reason is that, to find lowest weight paths in G efficiently, exENSteiner, FastAPP and ImprovAPP build a graph that has the same set of edges with G, and the original vertex and edge weights in G are embedded onto edges in this graph. Furthermore, DPBF consumes more memory when vertex groups are selected uniformly. For example, in Figure 8b, DPBF consumes around 2.5GB when $|\Gamma| = 9$, while in Figure 9b, DPBF consumes around 1.5GB when $|\Gamma| = 9$. The reason is that, as we have discussed in our paper, DPBF often enumerates a larger number of trees when vertex groups are selected uniformly.

Varying h in PartialOPT. We report the memory consumed by PartialOPT with different h in Figure 10, where the Toronto data is used, vertex groups are selected via the uniform approach, $|V| = 46073$, $|\Gamma| = 6$, $\lambda = 0.33$ (this corresponds to the experiments in Figure 7 in our paper). We observe that the memory consumed by PartialOPT grows exponentially with h. The reason is that PartialOPT employs DPBF to connect h vertex groups optimally, and the space complexity of this process is $O(2^h|V|)$.

Comparing LANCET with GKA. We report the memory consumed by LANCET and GKA in Figure 11, where the Toronto data is used, vertex groups are selected via the uniform approach, $\lambda = 0.33$, $|\Gamma| = |T_t| = 6$ (this corresponds to the experiments in Figure 8 in our paper). We observe that the memory consumption of GKA increases quickly with $|V|$. The reason is that GKA stores the lowest weight paths between all pairs of vertices, which has a space complexity of $O(|V_t|^2)$, where $|V_t| = |V| + |\Gamma|$. In comparison, LANCET only stores the lowest weight paths from every compulsory vertex to the other vertices, which has a space complexity of $O(|T_t||V_t|)$ (see Line 3 of LANCET).

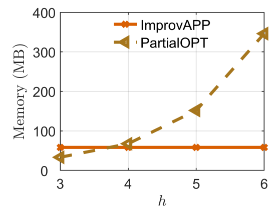


Figure 10: Varying h.

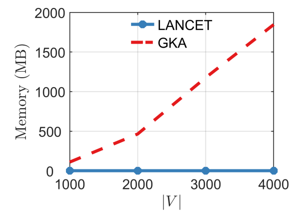
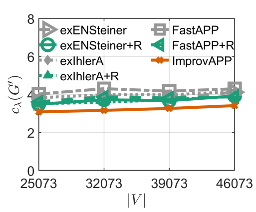


Figure 11: LANCET **versus** GKA.

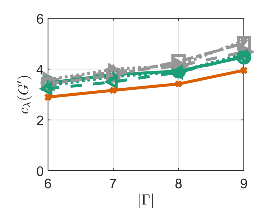
9 REFINING THE SOLUTIONS OF exENSteiner, exIhlerA AND FastAPP

There is a solution refinement process in ImprovAPP, *i.e.,* Lines 17-29 in ImprovAPP. This refinement process refines a sub-optimal solution by removing non-unique-group leaves from this solution. Here, we use this process to refine the solutions of exENSteiner, exIhlerA and FastAPP. Notably, this process has already been incorporated into PartialOPT (*i.e.,* Line 13 in PartialOPT). Thus, we do not refine the solutions of PartialOPT here. We report the refinement results in Figures 12 and 13, where exENSteiner+R, exIhlerA+R and FastAPP+R are the refinements of exENSteiner, exIhlerA and FastAPP, respectively.

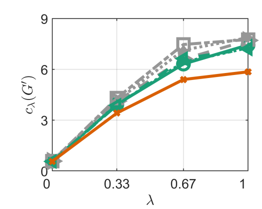
Suppose that there is a feasible solution tree $\Theta(V_\Theta, E_\Theta)$. Then, the time complexity of refining this solution is $O(|\Gamma||V_\Theta| + |V_\Theta| \log |V_\Theta|)$ (details in Section 4.3 in our paper). Since we generally have $|V_\Theta| \ll |V|$ in practice, the running times of refinement are negligible when comparing to the running times of our algorithms. For example, each of our algorithms takes around 100s to produce a feasible solution in



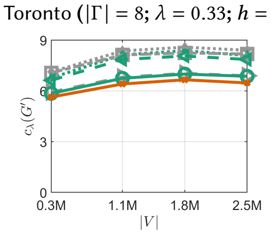
(a) Toronto ($|\Gamma| = 8$; $\lambda = 0.33$; $h = 3$)



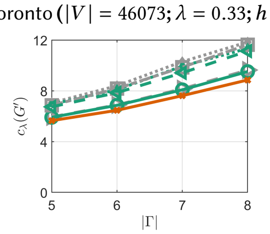
(b) Toronto ($|V| = 46073$; $\lambda = 0.33$; $h = 3$)



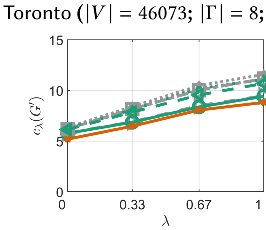
(c) Toronto ($|V| = 46073$; $|\Gamma| = 8$; $h = 3$)



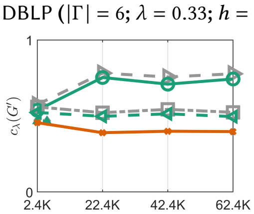
(d) DBLP ($|\Gamma| = 6$; $\lambda = 0.33$; $h = 3$)



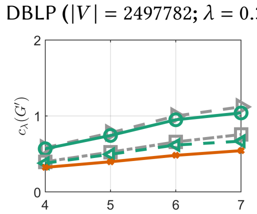
(e) DBLP ($|V| = 2497782$; $\lambda = 0.33$)



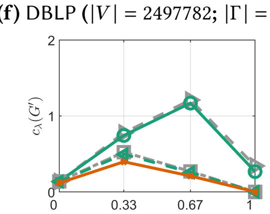
(f) DBLP ($|V| = 2497782$; $|\Gamma| = 6$)



(g) MovieLens ($|\Gamma| = 5$; $\lambda = 0.33$; $h = 3$)

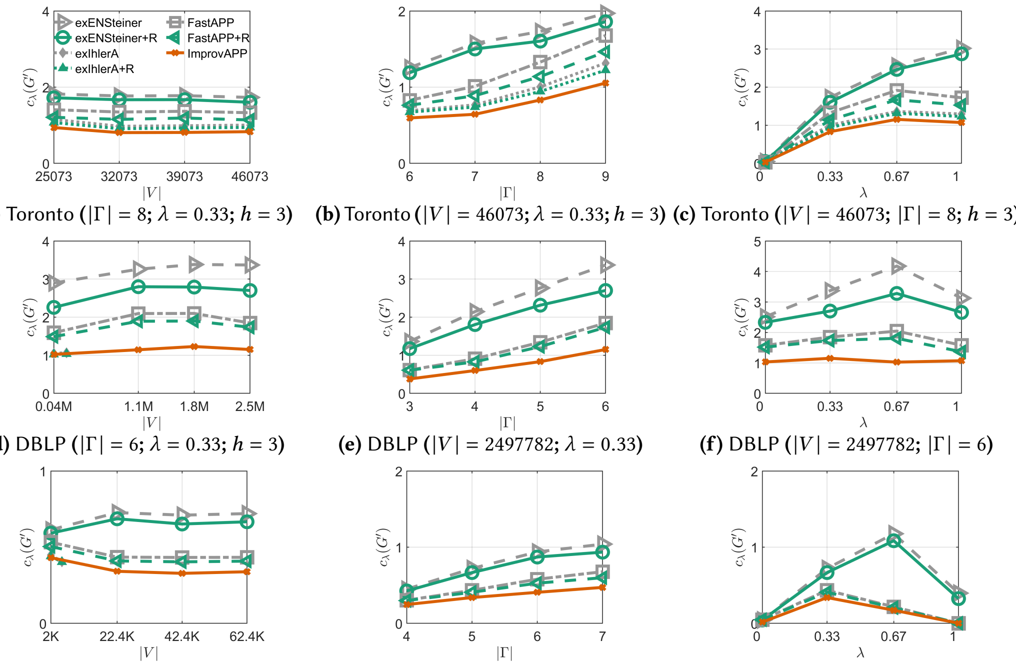


(h) MovieLens ($|V| = 62423$; $\lambda = 0.33$)



(i) MovieLens ($|V| = 62423$; $|\Gamma| = 5$)

Figure 12: The refinement in the main experiment results in which vertex groups are selected uniformly.



(a) Toronto ($|\Gamma| = 8$; $\lambda = 0.33$; $h = 3$) **(b)** Toronto ($|V| = 46073$; $\lambda = 0.33$; $h = 3$) **(c)** Toronto ($|V| = 46073$; $|\Gamma| = 8$; $h = 3$)

(d) DBLP ($|\Gamma| = 6$; $\lambda = 0.33$; $h = 3$) **(e)** DBLP ($|V| = 2497782$; $\lambda = 0.33$) **(f)** DBLP ($|V| = 2497782$; $|\Gamma| = 6$)

(g) MovieLens ($|\Gamma| = 5$; $\lambda = 0.33$; $h = 3$) **(h)** MovieLens ($|V| = 62423$; $\lambda = 0.33$) **(i)** MovieLens ($|V| = 62423$; $|\Gamma| = 5$)

Figure 13: The refinement in the main experiment results in which vertex groups are selected non-uniformly.

the full DBLP graph, while it only takes around 2ms to refine this solution. Thus, we only evaluate the solution qualities in Figures 12 and 13, and do not evaluate the running times of the refinement.

We observe that ImprovAPP dominates exENSteiner+R, exIhlerA+R and FastAPP+R on solution qualities. Thus, a main conclusion in our paper, *i.e.,* ImprovAPP offers a unique option of combining superior efficiency and solution quality when it is too slow to find optimal solutions, still holds here. We also observe that the refinement is often more effective when vertex groups are selected non-uniformly. For example, the refinement is more effective in Figure 13d than in Figure 12d. The reason is that the sizes of vertex groups are often larger when vertex groups are selected non-uniformly, and as a result the leaves in the feasible solutions produced by exENSteiner, exIhlerA and FastAPP are more likely to be non-unique-group leaves. Nevertheless, we consider the refinement process as useful no matter vertex groups are selected uniformly or non-uniformly, given that the running times of the refinement are negligible.

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