# Finding Group Steiner Trees in Graphs with both Vertex and Edge Weights: Some Supplemental Materials

**Road map.** In Section 1, we prove the transformation from group Steiner trees to Steiner trees. In Section 2, we prove the approximation guarantee of LANCET. In Section 3, we show the details of the time complexity of LANCET. In Section 4, we show the details of the time complexity of exENSteiner. In Section 5, we prove the approximation guarantee of exIhlerA. In Section 6, we prove the approximation guarantee of FastAPP. In Section 7, we prove the approximation guarantee of ImprovAPP. In Section 8, we prove the approximation guarantee of PartialOPT. In Section 9, we show the details of the time complexity of PartialOPT. In Section 10, we show that PrunedDP and PrunedDP++ in [2] rely on techniques that do not hold in graphs with vertex weights. In Section 11, we show the memory consumption results. In Section 12, we refine the solutions of exENSteiner, exIhlerA and FastAPP.

## 1 THE TRANSFORMATION

Theorem 1. *Let $G(V, E, w, c)$ be a connected undirected graph, and $\Gamma$ be a set of vertex groups. Let $G_t(V_t, E_t, w_t, c_t)$ be a connected undirected graph, and $T_t \subseteq V_t$ be a set of compulsory vertices. Based on $G$ and $\Gamma$, we construct $G_t$ and $T_t$ in the following way:*

*(1) Initialize $V_t = V$, $E_t = E$, $T_t = \emptyset$, $w_t = (1-\lambda)w$, and $c_t = \lambda c$.*

*(2) For each vertex group $g \in \Gamma$, (i) add a dummy vertex $v_g$ into $T_t$ and $V_t$, such that $w_t(v_g) = 0$, and (ii) add dummy edges $(v_g, j)$ for all $j \in g$ into $E_t$, such that $c_t(v_g, j) = M$, and $M$ is a constant satisfying*

$$M > (1-\lambda)\sum_{v \in V} w(v) + \lambda \sum_{e \in E_{MST}} c(e), \tag{1}$$

*and $E_{MST}$ is the set of edges in a Minimum Spanning Tree of $G$.*

*Let $\Theta_{G_t}$ be an optimal solution to the vertex- and edge-weighted Steiner tree problem in $G_t$, and $\Theta^{non}_{G_t}$ be the non-dummy part of $\Theta_{G_t}$. Then, there is an optimal solution to the vertex- and edge-weighted group Steiner tree problem in $G$, namely, $\Theta_G$, that has the same sets of vertices and edges with $\Theta^{non}_{G_t}$.*

Proof. Since dummy vertices only connect non-dummy vertices, there are at least $|\Gamma|$ dummy edges in $\Theta_{G_t}$. If $c_\lambda(\Theta_G) < c(\Theta^{non}_{G_t})$, then there is a feasible solution to the vertex- and edge-weighted Steiner tree problem in $G_t$: $\Theta'_{G_t}$ such that

$$c(\Theta'_{G_t}) = c_\lambda(\Theta_G) + M|\Gamma| < c(\Theta_{G_t}), \tag{2}$$

which is not possible. Thus, we have $c_\lambda(\Theta_G) \geq c(\Theta^{non}_{G_t})$. Let $\Theta''_{G_t}$ be a tree in $G_t$ such that (i) every dummy vertex $v_g$ is a leaf of $\Theta''_{G_t}$; and (ii) the non-dummy part of $\Theta''_{G_t}$, namely, $\Theta^{non''}_{G_t}$, is in a Minimum Spanning Tree of $G$. Suppose that there is a dummy vertex $v_g$ in $\Theta_{G_t}$ that is not a leaf. Since $c(\Theta^{non''}_{G_t}) < M$, we have

$$c(\Theta_{G_t}) \geq c(\Theta^{non}_{G_t}) + M(|\Gamma|+1) > c(\Theta''_{G_t}) = c(\Theta^{non''}_{G_t}) + M|\Gamma|, \tag{3}$$

which is not possible. Thus, every dummy compulsory vertex $v_g$ is a leaf of $\Theta_{G_t}$. As a result, $\Theta^{non}_{G_t}$ is connected and shares the same sets of vertices and edges with a feasible solution to the vertex- and edge-weighted group Steiner tree problem in $G$, which means that $c_\lambda(\Theta_G) \leq c(\Theta^{non}_{G_t})$. Therefore, $c_\lambda(\Theta_G) = c(\Theta^{non}_{G_t})$. Hence, this theorem holds. □

## 2 THE APPROXIMATION GUARANTEE OF LANCET

LANCET can be regarded as the vertex- and edge-weighted version of the algorithm in [3], which achieves an approximation guarantee of $2(1-1/|T_t|)$ for solving the vertex-unweighted Steiner tree problem. This approximation guarantee relies on the following deduction (*i.e.,* Lemma 1 in [3]): since a pre-order traversal of a tree traverses every edge in this tree exactly twice (see Figure 1 in [3]), in a graph with only edge weights, if we perform a pre-order traversal of an optimal solution tree and sum up every weight that we encounter (including duplicates), then the result is exactly twice the weight of an optimal solution tree. However, in a graph with both vertex and edge weights, summing up the weights that we encounter during this traversal does not always result in twice the weight of an optimal solution tree, since (i) an optimal solution tree may contain non-compulsory vertices with positive weights; and (ii) a pre-order traversal of an optimal solution tree may visit such a vertex more than twice (specifically, the number of times that a pre-order traversal of an optimal solution tree visits such a vertex equals the degree of this vertex in this optimal solution tree). Thus, the above approximation guarantee of $2(1-1/|T_t|)$ does not hold for LANCET. In what follows, we establish the approximation guarantee of LANCET.

Theorem 2. LANCET *has a sharp approximation guarantee of $|T_t|-1$ for solving the vertex- and edge-weighted Steiner tree problem.*

Proof. LANCET merges $|T_t|-1$ LWPs to connect all compulsory vertices together. Suppose that the highest-weight one of these LWPs is $LWP'$, and $\Theta_{opt}$ is an optimal solution. Since $c(LWP')$ is smaller than or equal to the weight of the LWP between a pair of compulsory

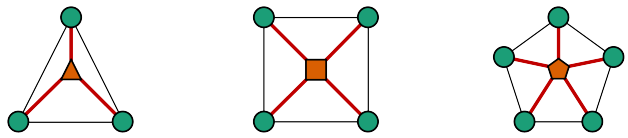


**Figure 1: Touching the approximation guarantee of $|T_t| - 1$.**

vertices, we have

$$c(\Theta_{opt}) \geq c(LWP'). \tag{4}$$

Since there are $|T_t| - 1$ LWPs that have been merged, we have

$$(|T_t| - 1)c(\Theta_{opt}) \geq (|T_t| - 1)c(LWP') \geq c(\Theta). \tag{5}$$

Therefore, LANCET has an approximation guarantee of $|T_t| - 1$. We further show that $|T_t| - 1$ is the sharp approximation guarantee of LANCET. Consider a regular polygon composed of $|T_t|$ compulsory vertices, and a non-compulsory vertex that is in the middle of this polygon and connects $|T_t|$ compulsory vertices (see Figure 1). The weight of each edge between compulsory vertices is $x$, the weight of each edge between a compulsory vertex and the middle non-compulsory vertex is 0, the weight of each compulsory vertex is 0, and the weight of the middle non-compulsory vertex is $z$. Suppose that $x = z - \delta$, where $\delta$ is a tiny positive value; and $z < (|T_t| - 1)x$. Since $x < z$, $\Theta$ contains $|T_t| - 1$ edges between compulsory vertices, and $c(\Theta) = (|T_t| - 1)x$. Since $z < (|T_t| - 1)x$, $\Theta_{opt}$ contains all the edges between compulsory vertices and the middle non-compulsory vertex, and $c(\Theta_{opt}) = z$. We have

$$\lim_{\delta \to 0} \frac{c(\Theta)}{c(\Theta_{opt})} = \frac{(|T_t| - 1)(z - \delta)}{z} = |T_t| - 1. \tag{6}$$

Hence, $|T_t| - 1$ is the sharp approximation guarantee of LANCET. □

## 3 THE TIME COMPLEXITY OF LANCET

**Time complexity of LANCET:**

$$O\Big(|T_t| \cdot (|E_t| + |V_t| \log |V_t|)\Big).$$

The details are as follows. The overhead of the initialization (Lines 1-2) is $O(|T_t|)$. The LWPs from a vertex to the other vertices can be found using Dijkstra's algorithm in $O(|E_t| + |V_t| \log |V_t|)$ time. Thus, the cost of finding the LWPs from every vertex in $V_2$ to the other vertices (Line 3) is $O(|T_t|(|E_t| + |V_t| \log |V_t|))$. It takes $O(|T_t|)$ time to push the LWPs from every vertex in $V_2$ to $i_{rand}$ into $Q$ (Line 4). It concatenates the LWPs between unconnected compulsory vertices and connected vertices using a while loop with $O(|T_t|)$ iterations as follows. Since popping out the top element in the Fibonacci heap takes $O(\log |T_t|)$ time, it pops out $LWP_{min}(V_{min}, E_{min})$ in $O(|T_t| \log |T_t|)$ time throughout the loop (Line 6). We use adjacency lists based on hashes to store graphs. Since adding a vertex or an edge into such an adjacency list takes $O(1)$ time, LANCET merges $LWP_{min}(V_{min}, E_{min})$ into $\Theta$ (Line 7) in $O(|V_t|)$ time in all iterations combined. We use hashes to store $V_1$ and $V_2$. As a result, updating $V_1$ and $V_2$ (Lines 8-9) takes $O(|V_t|)$ time throughout the loop. Since the time complexity of decreasing the key of an element in a min Fibonacci heap is $O(1)$; and LANCET checks and updates the minimum-weight LWPs from every vertex in $V_2$ to $V_1$ only when a new vertex is added into $V_1$, updating the LWPs (Line 10) takes $O(|T_t||V_t|)$ time throughout the loop.

## 4 THE TIME COMPLEXITY OF exENSteiner

**Time complexity of exENSteiner:**

$$O\Big(|\Gamma| \cdot (|E| + |V| \log |V| + |\Gamma| \log |\Gamma| + |\Gamma||V|)\Big).$$

The details are as follows. exENSteiner first transforms the input graph in $O(|E| + |\Gamma||V|)$ time (Line 1). This is because (i) this transformation requires examining all vertices and edges to decide the value of $M$, which takes $O(|V|+|E|)$ time, and (ii) it adds $|\Gamma|$ dummy vertices and $\sum_{g \in \Gamma} |g|$ dummy edges, which incurs $O(|\Gamma||V|)$ overhead. Then, it employs LANCET to find a Steiner tree in $O(|\Gamma|(|E|+|V| \log |V|+|\Gamma| \log |\Gamma|+|\Gamma||V|))$ time (Line 2), since LANCET takes $O(|T_t|(|E_t| + |V_t| \log |V_t|))$ time; and $|T_t| = |\Gamma|$, $|V_t| = |V| + |\Gamma|$ and $|E_t| = |E| + \sum_{g \in \Gamma} |g|$. After that, it removes $|\Gamma|$ dummy vertices and $|\Gamma|$ dummy edges (Line 3), which takes $O(|\Gamma|)$ time. It identifies the MST (Line 4) in $O(|E| + |V| \log |V|)$ time. Notably, since $|\Gamma|$ is often limited in practice, the cost of exENSteiner is close to $O(|\Gamma| \cdot (|E| + |V| \log |V|))$ in practice.

## 5 THE APPROXIMATION GUARANTEE OF exIhlerA

Theorem 3. exIhlerA *has a sharp approximation guarantee of* $|\Gamma| - 1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Suppose that $\Theta_{OPT}(V_{OPT}, E_{OPT})$ is an optimal solution. Let $\Gamma = \{g_1, \ldots, g_{|\Gamma|}\}$. There is a tuple $(v_1, \ldots, v_{|\Gamma|})$ such that $v_i \in V_{OPT} \cap g_i$ for all $i \in \{1, \ldots, |\Gamma|\}$. Without loss of generality, assume that $g_{min} = g_1$. For every $i \in \{2, \ldots, |\Gamma|\}$, there is exactly one simple path between $v_1$ and $v_i$ in $\Theta_{OPT}$, which we refer to as $P_{v_1 v_i}$. We have

$$c_\lambda(P_{v_1 v_i}) \leq c_\lambda(\Theta_{OPT}), \tag{7}$$

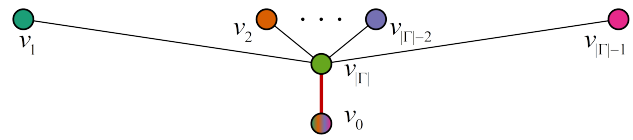



**Figure 2: Touching the approximation guarantee of $|\Gamma|-1$.**

$$\sum_{g\in\Gamma\setminus g_1} c_\lambda(LWP_{\lambda v_1 g}) \le \sum_{i\in\{2,\ldots,|\Gamma|\}} c_\lambda(P_{v_1 v_i}). \tag{8}$$

Thus,

$$c_\lambda(\Theta) \le c_\lambda(G_{v_1}) \le \sum_{g\in\Gamma\setminus g_1} c_\lambda(LWP_{\lambda v_1 g}) \le \sum_{i\in\{2,\ldots,|\Gamma|\}} c_\lambda(P_{v_1 v_i}) \le (|\Gamma|-1)c_\lambda(\Theta_{OPT}). \tag{9}$$

Hence, exIhlerA has an approximation guarantee of $|\Gamma|-1$. We note that this guarantee is sharp. To explain, consider the graph $G(V,E,w,c)$ in Figure 2, where $V=\{v_0,v_1,\ldots,v_{|\Gamma|}\}$, $E=\{(v_{|\Gamma|},v_0),(v_{|\Gamma|},v_1),\ldots,(v_{|\Gamma|},v_{|\Gamma|-1})\}$, $w(i)=0$ for all $i\in V$, $c(v_{|\Gamma|},v_1)=\ldots=c(v_{|\Gamma|},v_{|\Gamma|-1})=1$, and $c(v_{|\Gamma|},v_0)=1+\delta$, where $\delta$ is a tiny positive value. In addition, $\Gamma=\{v_0,v_1\}\cup\ldots\cup\{v_0,v_{|\Gamma|-1}\}\cup\{v_{|\Gamma|}\}$. Let $\lambda=1$. Since $g_{min}=\{v_{|\Gamma|}\}$, exIhlerA produces the solution $\Theta=\{(v_{|\Gamma|},v_1),\ldots,(v_{|\Gamma|},v_{|\Gamma|-1})\}$, and $c_\lambda(\Theta)=|\Gamma|-1$. When $|\Gamma|=2$, $\Theta$ is the optimal solution, *i.e.*, the approximation ratio is $|\Gamma|-1=1$. When $|\Gamma|>2$, we have $\Theta_{OPT}=\{(v_{|\Gamma|},v_0)\}$, and

$$\lim_{\delta\to 0}\frac{c_\lambda(\Theta)}{c_\lambda(\Theta_{OPT})}=\frac{|\Gamma|-1}{1+\delta}=|\Gamma|-1. \tag{10}$$

Hence, the best possible approximation guarantee of exIhlerA is $|\Gamma|-1$. □

## 6 THE APPROXIMATION GUARANTEE OF FastAPP

Theorem 4. FastAPP *has a sharp approximation guarantee of* $|\Gamma|-1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Let $\Theta_{OPT}(V_{OPT},E_{OPT})$ be an optimal solution, and $\Gamma=\{g_1,\ldots,g_{|\Gamma|}\}$. There is a tuple $(v_1,\ldots,v_{|\Gamma|})$ such that $v_i\in V_{OPT}\cap g_i$ for all $i\in\{1,\ldots,|\Gamma|\}$. Without loss of generality, suppose that $g_{min}=g_1$. Let $g_x\in\Gamma\setminus g_1$ be such a vertex group that

$$c_\lambda(LWP_{\lambda v_1 g_x})=\max\{c_\lambda(LWP_{\lambda v_1 g})\mid\forall g\in\Gamma\setminus g_1\}. \tag{11}$$

Since $LWP_{\lambda v_1 g_x}$ links fewer groups to $v_1$ than $\Theta_{OPT}$, we have

$$c_\lambda(LWP_{\lambda v_1 g_x})\le c_\lambda(\Theta_{OPT}). \tag{12}$$

Lines 5-8 in FastAPP guarantee that

$$\max\{c_\lambda(LWP_{\lambda i_{min} g})\mid\forall g\in\Gamma\setminus g_1\}\le c_\lambda(LWP_{\lambda v_1 g_x}). \tag{13}$$

By Lines 10-11 in FastAPP, we have

$$c_\lambda(\Theta)\le(|\Gamma|-1)\cdot\max\{c_\lambda(LWP_{\lambda i_{min} g})\mid\forall g\in\Gamma\setminus g_1\}. \tag{14}$$

Thus,

$$c_\lambda(\Theta)\le(|\Gamma|-1)c_\lambda(LWP_{\lambda v_1 g_x})\le(|\Gamma|-1)c_\lambda(\Theta_{OPT}). \tag{15}$$

Hence, FastAPP has an approximation guarantee of $|\Gamma|-1$. The sharpness of this guarantee can be seen from the example in Section 5, *i.e.*, Figure 2. Hence, this theorem holds. □

## 7 THE APPROXIMATION GUARANTEE OF ImprovAPP

Theorem 5. ImprovAPP *has a sharp approximation guarantee of* $|\Gamma|-1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Let $\Theta_{OPT}(V_{OPT},E_{OPT})$ be an optimal solution. Let $\Gamma=\{g_1,\ldots,g_{|\Gamma|}\}$. There is a tuple $(v_1,\ldots,v_{|\Gamma|})$ such that $v_i\in V_{OPT}\cap g_i$ for all $i\in\{1,\ldots,|\Gamma|\}$. Without loss of generality, suppose that $g_{min}=g_1$. When ImprovAPP processes $v_1$ in the for loop (Lines 5-14), it concatenates $|\Gamma|-1$ lowest weight paths that link $\{g_2,\ldots,g_{|\Gamma|}\}$, respectively, to build $\Theta_{v_1}$. Let $LWP_{\lambda v_x g_y}$ be one of these paths that has the largest regulated weight, and links $g_y\in\{g_2,\ldots,g_{|\Gamma|}\}$. Then,

$$c_\lambda(\Theta_{v_1})\le(|\Gamma|-1)c_\lambda(LWP_{\lambda v_x g_y}). \tag{16}$$

Let $LWP_{\lambda v_1 g_y}$ be the lowest weight path between $v_1$ and $g_y$. Since $LWP_{\lambda v_1 g_y}$ has been pushed into $Q$ initially (Line 6) and has (possibly) been updated to $LWP_{\lambda v_x g_y}$ (Line 12), we have

$$c_\lambda(LWP_{\lambda v_x g_y})\le c_\lambda(LWP_{\lambda v_1 g_y}). \tag{17}$$

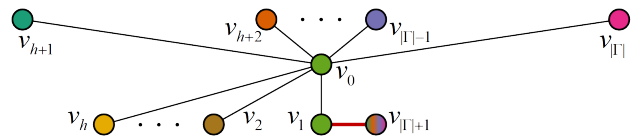



**Figure 3: Touching the approximation guarantee of $|\Gamma| - h + 1$.**

Since $LWP_{\lambda v_1 g_y}$ links fewer groups to $v_1$ than $\Theta_{OPT}$, we have

$$c_\lambda(LWP_{\lambda v_1 g_y}) \leq c_\lambda(\Theta_{OPT}). \tag{18}$$

Thus,

$$c_\lambda(\Theta) \leq c_\lambda(\Theta_{v_1}) \leq (|\Gamma| - 1)c_\lambda(LWP_{\lambda v_x g_y}) \leq (|\Gamma| - 1)c_\lambda(LWP_{\lambda v_1 g_y}) \leq (|\Gamma| - 1)c_\lambda(\Theta_{OPT}). \tag{19}$$

Therefore, ImprovAPP has an approximation guarantee of $|\Gamma| - 1$. The sharpness of this guarantee can be seen from the example in Section 5, *i.e.,* Figure 2. Thus, this theorem holds. □

## 8 THE APPROXIMATION GUARANTEE OF PartialOPT

Theorem 6. PartialOPT *has a sharp approximation guarantee of* $|\Gamma| - h + 1$ *for solving the vertex- and edge-weighted group Steiner tree problem.*

Proof. Suppose that $\Theta_{OPT}(V_{OPT}, E_{OPT})$ is an optimal solution, and $\Gamma = \{g_1, \ldots, g_{|\Gamma|}\}$. There is a tuple $(v_1, \ldots, v_{|\Gamma|})$ such that $v_i \in V_{OPT} \cap g_i$ for all $i \in \{1, \ldots, |\Gamma|\}$. Without loss of generality, let $g_{min} = g_1$. For $v_1 \in g_{min}$, $\Theta^h_{v_1}$ is optimal for $\Gamma_1 = \{\{v_1\}, g_2, \ldots, g_h\}$. Since $\Theta^h_{v_1}$ connects fewer vertex groups to $v_1$ than $\Theta_{OPT}$, we have

$$c_\lambda(\Theta^h_{v_1}) \leq c_\lambda(\Theta_{OPT}). \tag{20}$$

If $\Gamma_2 = \{\{v_1\}\}$ (*i.e.,* $h = |\Gamma|$), then $\Theta^{|\Gamma|}_{v_1} = \{v_1\}$, and $c_\lambda(\Theta) = c_\lambda(\Theta_{OPT})$. Otherwise (*i.e.,* $h < |\Gamma|$), we implement exIhlerA to produce $\Theta^{|\Gamma|}_{v_1}$ for $\Gamma_2 = \{\{v_1\}, g_{h+1}, \ldots, g_{|\Gamma|}\}$. Suppose that $\Theta^{|\Gamma|}_{OPT}$ is an optimal solution for $\Gamma_2$. The proof of Theorem 3 shows that

$$c_\lambda(\Theta^{|\Gamma|}_{v_1}) \leq (|\Gamma| - h)c_\lambda(\Theta^{|\Gamma|}_{OPT}). \tag{21}$$

Since $\Theta^{|\Gamma|}_{OPT}$ connects fewer vertex groups to $v_1$ than $\Theta_{OPT}$, we have

$$c_\lambda(\Theta^{|\Gamma|}_{OPT}) \leq c_\lambda(\Theta_{OPT}). \tag{22}$$

Thus,

$$c_\lambda(\Theta) \leq c_\lambda(G_{v_1}) = c_\lambda(\Theta^h_{v_1} \cup \Theta^{|\Gamma|}_{v_1}) \leq c_\lambda(\Theta^h_{v_1}) + c_\lambda(\Theta^{|\Gamma|}_{v_1}) \leq (|\Gamma| - h + 1)c_\lambda(\Theta_{OPT}). \tag{23}$$

Therefore, PartialOPT has an approximation guarantee of $|\Gamma| - h + 1$. We show that this guarantee is sharp. Consider the graph $G(V, E, w, c)$ in Figure 3, where $V = \{v_0, v_1, \ldots, v_{|\Gamma|+1}\}$, $E = \{(v_0, v_1), (v_0, v_2), \ldots, (v_0, v_{|\Gamma|}), (v_1, v_{|\Gamma|+1})\}$, $w(i) = 0$ for every $i \in \{v_0, \ldots, v_{h-1}\}$, $w(i) = 1$ for every $i \in \{v_h, \ldots, v_{|\Gamma|+1}\}$, $c(v_0, v_1) = \delta_1$, $c(v_1, v_{|\Gamma|+1}) = \delta_2$, where $\delta_1$ and $\delta_2$ are two tiny positive values, and $\delta_1 < \delta_2$, and all the other edge weights are zero. In addition, $\Gamma = \{g_1, \ldots, g_{|\Gamma|}\} = \{v_0, v_1\} \cup \{v_2, v_{|\Gamma|+1}\} \cup \ldots \cup \{v_{|\Gamma|}, v_{|\Gamma|+1}\}$. Let $\lambda = 0.5$, *i.e.,* vertex and edge weights are regulated equally. PartialOPT enumerates two vertices in $g_{min}$: $v_0$ and $v_1$. For $v_0$, PartialOPT produces $\Theta^h_{v_0} = \{(v_0, v_2), \ldots, (v_0, v_h)\}$, and $\Theta^{|\Gamma|}_{v_0} = \{(v_0, v_{h+1}), \ldots, (v_0, v_{|\Gamma|})\}$. Thus, $\Theta_{v_0} = \{(v_0, v_2), \ldots, (v_0, v_{|\Gamma|})\}$. Similarly, for $v_1$, since $\delta_1 < \delta_2$, PartialOPT produces $\Theta_{v_1} = \{(v_0, v_1), \ldots, (v_0, v_{|\Gamma|})\}$. We have $\Theta = \Theta_{v_0}$. When $|\Gamma| = h$, $\Theta$ is the optimal solution, *i.e.,* the approximation ratio is $|\Gamma| - h + 1 = 1$. When $|\Gamma| > h$, we have $\Theta_{OPT} = \{(v_1, v_{|\Gamma|+1})\}$, and

$$\lim_{\delta_2 \to 0} \frac{c_\lambda(\Theta)}{c_\lambda(\Theta_{OPT})} = \frac{|\Gamma| - h + 1}{1 + \delta_2} = |\Gamma| - h + 1. \tag{24}$$

Hence, $|\Gamma| - h + 1$ is the best possible approximation guarantee of PartialOPT. This theorem holds. □

## 9 THE TIME COMPLEXITY OF PartialOPT

**Time complexity of** PartialOPT**:**

$$O\Big(|g_{min}| \cdot \Big(|\Gamma||V| + 3^h|V| + 2^h(|E| + h|V| + |V|\log|V|)\Big)\Big).$$

It initializes $\Theta$ and finds $g_{min}$ in $O(|\Gamma|)$ time. For each vertex in $g_{min}$, it builds $\Gamma_1$ and $\Gamma_2$ in $O(|\Gamma|)$ time (Line 4). It applies DPBF to produce $\Theta^h_i$ in $O(3^h|V| + 2^h(|E| + h|V| + |V|\log|V|))$ time (Line 5; details in [1]), and may invoke exIhlerA to produce $\Theta^{|\Gamma|}_i$ in $O((|\Gamma| - h)|V| + |E| + |V|\log|V|)$ time (Line 9; here, $|g_{min}|$ in the time complexity of exIhlerA is 1, since $\Gamma_2$ contains $\{i\}$). After that, PartialOPT merges $\Theta^h_i$ and $\Theta^{|\Gamma|}_i$ (Line 11)

in $O(|V|)$ time, and computes an MST as $\Theta_i$ (Line 12) in $O(|E| + |V| \log |V|)$ time. It refines $\Theta_i$ in $O(|\Gamma||V| + |V| \log |V|)$ time (Line 13), and updates $\Theta$ in $O(|V|)$ time (Line 14).

## 10 THE RECENT WORK ON ENHANCING DPBF

The PrunedDP and PrunedDP++ algorithms in [2] enhance the DPBF algorithm in [1] for finding optimal vertex-unweighted group Steiner trees. The main idea of this enhancement is to incorporate pruning techniques into the process of DPBF. In this section, we show that PrunedDP and PrunedDP++ rely on pruning techniques that do not hold in graphs with vertex weights. For the sake of simplicity, we do not use $\lambda$ to regulate vertex and edge weights in the examples in this section, *i.e.,* we sum vertex and edge weights directly when calculating the weight of a tree. We use $T(v, \Gamma)$ to signify the minimum-weight tree that roots at vertex $v$ and covers all vertex groups in $\Gamma$.

**Theorem 2 in [2] does not hold in graphs with vertex weights.** Theorem 2 in [2] is the core pruning technique in PrunedDP, and is also an important pruning technique in PrunedDP++. This theorem does not hold in graphs with vertex weights. To explain, we first briefly describe the dynamic programming process of DPBF through an example in Figure 4. Understanding this process is necessary for understanding the reason why Theorem 2 in [2] does not hold in graphs with vertex weights.

In Figure 4, there are three vertex groups $g_1 = \{v_1\}$, $g_2 = \{v_2\}$ and $g_3 = \{v_3\}$. The weight of $u$ is 1, and each of the other vertex and edge weights is $\delta$, and $\delta$ is a tiny positive value. The optimal solution tree is the whole graph, and the weight of this tree is $1 + 6\delta$ (*i.e.,* the sum of vertex and edge weights). To find this tree, DPBF first initializes $T(v_1, \{g_1\})$ as vertex $v_1$; $T(v_2, \{g_2\})$ as vertex $v_2$; and $T(v_3, \{g_3\})$ as vertex $v_3$. Then, DPBF grows $T(v_1, \{g_1\})$, $T(v_2, \{g_2\})$ and $T(v_3, \{g_3\})$ to vertex $u$, and produces $T(u, \{g_1\})$ as edge $(u, v_1)$; $T(u, \{g_2\})$ as edge $(u, v_2)$; and $T(u, \{g_3\})$ as edge $(u, v_3)$. Subsequently, it merges $T(u, \{g_1\})$ and $T(u, \{g_2\})$ as $T(u, \{g_1, g_2\}) = \{(u, v_1), (u, v_2)\}$ (similarly, it also merges $T(u, \{g_1\})$ and $T(u, \{g_3\})$ as $T(u, \{g_1, g_3\})$, and merges $T(u, \{g_2\})$ and $T(u, \{g_3\})$ as $T(u, \{g_2, g_3\})$). At last, it produces the optimal solution tree by merging $T(u, \{g_3\})$ and $T(u, \{g_1, g_2\})$ (or $T(u, \{g_1\})$ and $T(u, \{g_2, g_3\})$, or $T(u, \{g_2\})$ and $T(u, \{g_1, g_3\})$).

Theorem 2 in [2] is that: in DPBF, we can merge two subtrees $T(u, \Gamma')$ and $T(u, \Gamma'')$ for $\Gamma'' \subset \Gamma \setminus \Gamma'$ only when the total weight of these two subtrees is not larger than $\frac{2}{3}$ of the weight of an optimal solution tree. This theorem is true when vertex weights are omitted. For example, if vertex weights are omitted in the above instance, then the weight of the optimal solution tree is $3\delta$. When we merge $T(u, \{g_1\})$ and $T(u, \{g_2\})$ as $T(u, \{g_1, g_2\})$ in the above process, the total weight of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is $2\delta$, which is not larger than $\frac{2}{3}$ of the weight of an optimal solution tree. By Theorem 2 in [2], merging these two subtrees may help produce the optimal solution tree. If the total weight of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is larger than $\frac{2}{3}$ of the weight of an optimal solution tree, then merging these two subtrees does not help produce the optimal solution tree, and thus this merge can be avoided. However, this is not true when vertex weights are considered. For example, if we consider the vertex weights in the above instance, then the total weight of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is $2 + 4\delta$ (since the weight of each of these two trees is $1 + 2\delta$), which is larger than $\frac{2}{3}$ of the weight of an optimal solution tree: $1 + 6\delta$ (notably, even the weight of $T(u, \{g_1, g_2\}) = \{(u, v_1), (u, v_2)\}$, which is $1 + 4\delta$, is larger than $\frac{2}{3}$ of the weight of an optimal solution tree). As a result, if we use Theorem 2 in [2] in the above instance with vertex weights, then the merge of $T(u, \{g_1\})$ and $T(u, \{g_2\})$ is not carried out (similarly, the merge of $T(u, \{g_1\})$ and $T(u, \{g_3\})$, or the merge of $T(u, \{g_2\})$ and $T(u, \{g_3\})$, is not carried out). Consequently, the optimal solution tree will never be found. That is to say, Theorem 2 in [2] does not hold in graphs with vertex weights.

We point out the specific statement in the proof of Theorem 2 in [2] that does not hold in graphs with vertex weights as follows. In the beginning of the proof of Theorem 2 in [2], an optimal solution is assumed to be a tree rooted at vertex $u$ with $k$ subtrees, $T_1, \ldots, T_k$. Each subtree $T_i$ roots at vertex $v_i$, and the weight of each subtree is smaller than half of the weight of an optimal solution tree (*e.g.,* in Figure 4, $T_i$ is the single vertex $v_i$). Let $\overline{T_i}$ be the the edge-grown subtree that is grown by $T_i$ with an edge $(v_i, u)$ (*e.g.,* in Figure 4, $\overline{T_i}$ is the edge $(v_i, u)$). The proof of Theorem 2 in [2] states that there are three different cases: (1) the weight of each $\overline{T_i}$ is smaller than half of the weight of an optimal solution tree; (2) there is only one edge-grown subtree $\overline{T_i}$ that has a weight no smaller than half of the weight of an optimal solution tree; and (3) there are two edge-grown subtrees and the weight of each one is half of the weight of an optimal solution tree. This statement is not true in vertex-weighted scenarios, where there is a fourth case: there are more than two edge-grown subtrees such that the weight of each one is large than half of the weight of an optimal solution tree. For example, in Figure 4, if we consider vertex weights, then the weight of $\overline{T_1}$, $\overline{T_2}$ or $\overline{T_3}$ is $1 + 2\delta$, which is larger than half of the weight of an optimal solution tree.

**Lemmas 2 and 3 in [2] do not hold in graphs with vertex weights.** Except Theorem 2 in [2], another important pruning technique in PrunedDP++ is the tour-based lower bounds construction method for $A^*$-search. There are two types of tour-based lower bounds, which are based on Lemmas 2 and 3 in [2], respectively. We show that these two lemmas do not hold in vertex-weighted scenarios as follows.

First, we introduce the label-enhanced graph in [2], which is constructed by adding dummy vertices and edges into the graph as follows. For each group $g_i \in \Gamma$, we add a dummy vertex $\widetilde{v_i}$, and also add a dummy edge $(\widetilde{v_i}, u)$ with zero weight for every $u \in g_i$. For example, in Figure 5, the graph contains two vertices $v$ and $u$, and one edge $(v, u)$, and there are three vertex groups $g_1 = g_2 = g_3 = \{u\}$. We add dummy vertices $\widetilde{v_1}, \ldots, \widetilde{v_3}$ and dummy edges $(\widetilde{v_1}, u), \ldots, (\widetilde{v_3}, u)$ for creating the label-enhanced graph. [2] uses $W(\widetilde{v_i}, \widetilde{v_j}, \Gamma')$ to refer to the weight of the minimum-weight route that starts from $\widetilde{v_i}$, ends at $\widetilde{v_j}$, and passes through all dummy vertices that correspond to vertex groups in $\Gamma'$. Moreover, [2] uses $d(v, \widetilde{v_i})$ to refer to the weight of the minimum-weight path between non-dummy vertex $v$ and dummy vertex $\widetilde{v_i}$.

Lemma 2 in [2] is that: for every pair of vertex $v \in V$ and a subset of vertex groups $\Gamma' \subseteq \Gamma$, the weight of $T(v, \Gamma')$ is larger than or equal to $lb_1 = \frac{\min_{g_i, g_j \in \Gamma'} \{d(v, \widetilde{v_i}) + W(\widetilde{v_i}, \widetilde{v_j}, \Gamma') + d(\widetilde{v_j}, v)\}}{2}$. This lemma is true when all vertex weights are zero. For example, in Figure 5, let the weight of edge $(v, u)$ be $\delta$, which is a tiny positive value, and all vertex weights be zero, and $\Gamma' = \{g_1, g_2\}$. Then, $d(v, \widetilde{v_1}) = d(\widetilde{v_2}, v) = \delta$, $W(\widetilde{v_1}, \widetilde{v_2}, \Gamma') = 0$,

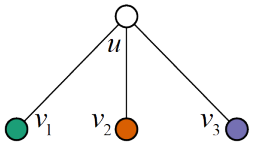



**Figure 4: An example for showing Theorem 2 in [2].**

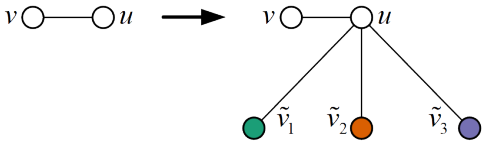



**Figure 5: An example for showing Lemmas 2 and 3 in [2].**

and $T(v, \Gamma')$ is the edge $(v, u)$. As a result, $lb_1 = \delta$, which equals the weight of $T(v, \Gamma')$. Thus, Lemma 2 in [2] holds. This lemma is proven in [2] by first doubling every edge in the label-enhanced $T(v, \Gamma')$ to obtain an Euler tour that starts from $v$, ends at $v$, and passes through all dummy vertices that correspond to vertex groups in $\Gamma'$; and then employing the fact that the total edge weight (including duplicates) that we encounter in this Euler tour is twice the total edge weight in $T(v, \Gamma')$. Nevertheless, Lemma 2 in [2] does not hold in vertex-weighted scenarios. For example, in the above instance, let the weights of $v$ and $u$ be 0 and 1, respectively. Then, $d(v, \widetilde{v_1}) = d(\widetilde{v_2}, v) = 1 + \delta$, and $W(\widetilde{v_1}, \widetilde{v_2}, \Gamma') = 1$. As a result, $lb_1 = \frac{3+2\delta}{2}$, which is larger than the weight of $T(v, \Gamma')$: $1 + \delta$. Thus, Lemma 2 in [2] does not hold any more. The reason is that the above Euler tour encounters $u$ three times, and as a result the weight of $u$ is counted three times in $lb_1$. Generally speaking, the total vertex and edge weight that we encounter in the Euler tour in the proof of Lemma 2 in [2] may be more than twice the weight of $T(v, \Gamma')$, since this tour may visit a vertex in $T(v, \Gamma')$ more than twice. As shown in Section 2 in this supplement, for a similar reason, LANCET does not have an approximation guarantee of 2.

Also for a similar reason, Lemma 3 in [2] does not hold in graphs with vertex weights. The details are as follows. [2] uses $W(\widetilde{v_i}, \Gamma')$ to refer to the weight of the minimum-weight route that starts from $\widetilde{v_i}$, and passes through all dummy vertices that correspond to vertex groups in $\Gamma'$. Lemma 3 in [2] is that: the weight of $T(v, \Gamma')$ is larger than or equal to $lb_2 = \frac{\max_{g_i \in \Gamma'}\{d(v,\widetilde{v_i}) + W(\widetilde{v_i}, \Gamma') + \min_{g_j \in \Gamma'}\{d(\widetilde{v_j}, v)\}\}}{2}$. This lemma is true when all vertex weights are zero. Consider the above instance. If all vertex weights are zero, then $d(v, \widetilde{v_1}) = d(v, \widetilde{v_2}) = \delta$, $W(\widetilde{v_1}, \Gamma') = W(\widetilde{v_2}, \Gamma') = 0$, $\min_{g_j \in \Gamma'}\{d(\widetilde{v_j}, v)\} = \delta$, and the weight of $T(v, \Gamma')$ is $\delta$. Consequently, $lb_2 = \delta$, which equals the weight of $T(v, \Gamma')$. Thus, Lemma 3 in [2] holds. Like Lemma 2, Lemma 3 is proven in [2] by doubling every edge in the label-enhanced $T(v, \Gamma')$ to obtain an Euler tour. Also like Lemma 2, since this tour may visit a vertex in $T(v, \Gamma')$ more than twice, Lemma 3 in [2] does not hold in graphs with vertex weights. For example, suppose that, in Figure 5, the weights of $v$ and $u$ are 0 and 1, respectively. Then, $d(v, \widetilde{v_1}) = d(v, \widetilde{v_2}) = 1 + \delta$, $W(\widetilde{v_1}, \Gamma') = W(\widetilde{v_2}, \Gamma') = 1$, $\min_{g_j \in \Gamma'}\{d(\widetilde{v_j}, v)\} = 1 + \delta$, and the weight of $T(v, \Gamma')$ is $1 + \delta$. As a result, $lb_2 = \frac{3+2\delta}{2}$, which is larger than the weight of $T(v, \Gamma')$. Thus, Lemma 3 in [2] does not hold any more.

Recall that (i) Theorem 2 in [2] is the core pruning technique in PrunedDP, and is also an important pruning technique in PrunedDP++; and (ii) another important pruning technique in PrunedDP++ is the tour-based lower bounds construction method for $A^*$-search, and there are two types of tour-based lower bounds, which are based on Lemmas 2 and 3 in [2], respectively. Since Theorem 2 and Lemmas 2 and 3 in [2] do not hold in graphs with vertex weights, we do not implement PrunedDP and PrunedDP++ in our paper.

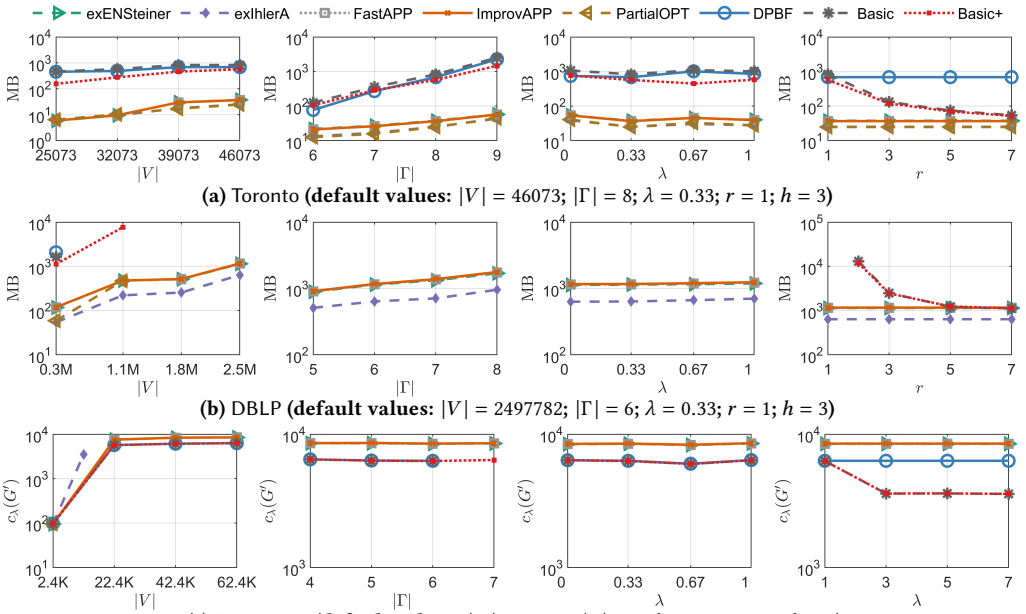
## 11 MEMORY CONSUMPTION RESULTS

Here, we evaluate the memory consumption of algorithms. The reported memory consumption of each algorithm contains the memory consumed by each input of this algorithm (*e.g.,* $G$ and $\Gamma$) as well as any other memory consumed in the process of this algorithm. We use adjacency lists based on hashes to store graphs. Hashes consume more memories than arrays. Our purpose of using adjacency lists based on hashes is to fully optimize the time complexities of algorithms.

**The main experiments.** We report the memory consumption results in the main experiments in our paper in Figures 6 and 7. We observe that DPBF, Basic and Basic+ often consume more memory than the other non-exact algorithms (*e.g.,* when varying $|V|$ in Figure 6a), and the memory consumption of DPBF, Basic and Basic+ often increases quickly with $|\Gamma|$ (*e.g.,* when varying $|\Gamma|$ in Figure 6a). The reason is that, except the $O(|V| + |E|)$ memory consumed by the input graph, DPBF, Basic and Basic+ additionally consume $O(2^{|\Gamma|}|V|)$ memory in the dynamic programming process, while the other non-exact algorithms do not consume such an exponential amount of memory. Notably, the memory consumption of DPBF, Basic and Basic+ does not grow much with $|\Gamma|$ for MovieLens (*e.g.,* when varying $|\Gamma|$ in Figure 6c). The reason is that the MovieLens graph is dense, and as a result the memory consumed by the MovieLens graph dominates the increase of the $O(2^{|\Gamma|}|V|)$ memory. Further note that, the non-exact algorithms may consume more memory than DPBF, Basic and Basic+ in some cases (*e.g.,* Figure 6c). The reason is that these non-exact algorithms build an additional graph that has (i) the same set of edges with the input graph $G$ and (ii) newly defined edge weights for finding lowest weight paths in $G$ (see Section 3.2 in our paper on how to find lowest weight paths), while DPBF, Basic and Basic+ do not build such an additional graph.

Since the dynamic programming process of Basic and Basic+ terminates earlier when $r$ is larger, the memory consumption of Basic and Basic+ decreases with $r$ (*e.g.,* when varying $r$ in Figure 6a). Notably, by incorporating pruning techniques, Basic and Basic+ enumerate fewer trees than DPBF in the dynamic programming process. As a result, Basic and Basic+ may consume a smaller amount of memory than DPBF (*e.g.,* when $|V| = 0.3M$ in Figure 6b). However, DPBF may consume a smaller amount of memory than Basic and Basic+ in some cases (*e.g.,* when $|\Gamma| = 6$ in Figure 6a). There are two reasons. First, Basic and Basic+ store the lowest weight paths between vertices and vertex groups, while DPBF does not store these paths. Second, all these three algorithms iteratively pop trees out of a min priority queue. Basic and Basic+ record trees that have been popped out (details in [2]), while DPBF does not record these trees.

**Our extensions.** We compare the memory consumption of ENSteiner, IhlerA, exENSteiner and exIhlerA in Figure 8, where vertex groups are selected via the uniform approach, and the parameter settings are: for Toronto, $|V| = 46073$, $|\Gamma| = 8$, $\lambda = 0.33$; for DBLP, $|V| = 2497782$,

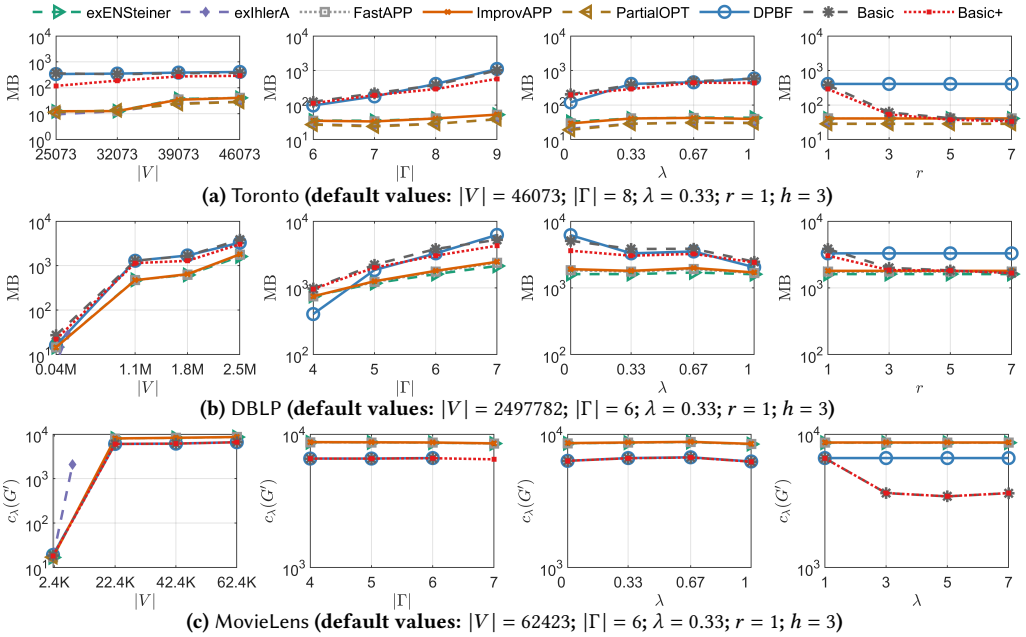



**(a)** Toronto **(default values:** $|V| = 46073$; $|\Gamma| = 8$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**

**(b)** DBLP **(default values:** $|V| = 2497782$; $|\Gamma| = 6$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**

**(c)** MovieLens **(default values:** $|V| = 62423$; $|\Gamma| = 6$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**

**Figure 6: The memory consumption in the main experiments where vertex groups are selected uniformly.**




**(a)** Toronto **(default values:** $|V| = 46073$; $|\Gamma| = 8$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**

**(b)** DBLP **(default values:** $|V| = 2497782$; $|\Gamma| = 6$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**

**(c)** MovieLens **(default values:** $|V| = 62423$; $|\Gamma| = 6$; $\lambda = 0.33$; $r = 1$; $h = 3$**)**

**Figure 7: The memory consumption in the main experiments where vertex groups are selected non-uniformly.**

$|\Gamma| = 6$, $\lambda = 0.33$; for MovieLens, $|V| = 2423$, $|\Gamma| = 6$, $\lambda = 0.33$ (this corresponds to the experiments in Figure 3 in our paper). We observe that exENSteiner and exIhlerA often consume slightly more memory than ENSteiner and IhlerA, respectively. The reason is that exENSteiner and exIhlerA build an additional graph for finding lowest weight paths (as discussed above). In comparison, ENSteiner and IhlerA do not build such an additional graph for finding shortest paths.

**Varying $h$ in** PartialOPT**.** We report the memory consumption of PartialOPT with respect to $h$ in Figure 9, where the Toronto data is used, vertex groups are selected via the uniform approach, $|V| = 46073$, $|\Gamma| = 6$, $\lambda = 0.33$ (this corresponds to the experiments in Figure 7 in our paper). We observe that the memory consumed by PartialOPT grows quickly with $h$. The reason is that PartialOPT employs DPBF to connect $h$ vertex groups optimally, and the space complexity of this process is $O(2^h|V|)$.

**Comparing** LANCET **with** GKA**.** We compare the memory consumption of LANCET and GKA in Figure 10, where the Toronto data is used, vertex groups are selected via the uniform approach, $\lambda = 0.33$, $|\Gamma| = |T_t| = 6$ (this corresponds to the experiments in Figure 8 in our

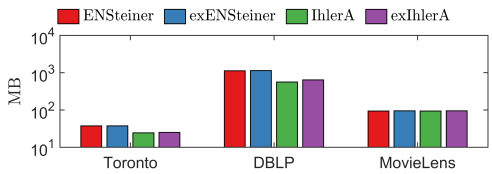



**Figure 8: Our extensions.**

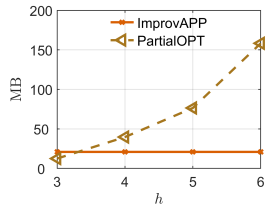



**Figure 9: Varying $h$.**

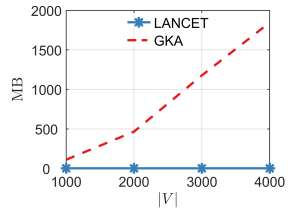


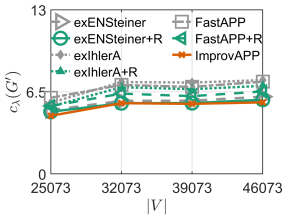

**Figure 10:** LANCET **versus** GKA.

paper). We observe that the memory consumption of GKA increases quickly with $|V|$. The reason is that GKA stores the lowest weight paths between all pairs of vertices, which has a space complexity of $O(|V_t|^2)$, where $|V_t| = |V| + |\Gamma|$. In comparison, LANCET only stores the lowest weight paths from compulsory vertices to the other vertices, which has a space complexity of $O(|T_t||V_t|)$ (see Line 3 of LANCET).

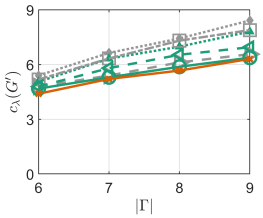
## 12 REFINING THE SOLUTIONS OF exENSteiner, exIhlerA AND FastAPP

There is a solution refinement process in ImprovAPP, *i.e.,* Lines 17-29 in ImprovAPP. This process refines a sub-optimal solution by removing non-unique-group leaves from this solution. Here, we use this process to refine the solutions of exENSteiner, exIhlerA and FastAPP. Notably, this process has already been incorporated into PartialOPT (*i.e.,* Line 13 in PartialOPT). Thus, we do not refine the solutions of PartialOPT here. We report the refinement results in Figures 11 and 12, where exENSteiner+R, exIhlerA+R and FastAPP+R are the refinements of exENSteiner, exIhlerA and FastAPP, respectively.

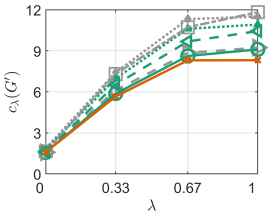
Suppose that there is a feasible solution tree $\Theta(V_\Theta, E_\Theta)$. Then, the time complexity of refining this solution is $O(|\Gamma||V_\Theta| + |V_\Theta| \log |V_\Theta|)$ (details in Section 4.3 in our paper). Since we generally have $|V_\Theta| \ll |V|$ in practice, the running times of refinement are negligible when comparing to the running times of our algorithms. For example, each of our algorithms takes around 100s to produce a feasible solution in the full DBLP graph, while it only takes around 2ms to refine this solution. Thus, we only evaluate the solution qualities in Figures 11 and 12, and do not evaluate the running times of refinement. We observe that ImprovAPP dominates exENSteiner+R, exIhlerA+R and FastAPP+R on solution qualities. We also observe that the refinement is often more effective when vertex groups are selected non-uniformly. For example, the refinement is more effective in Figure 12d than in Figure 11d. The reason is that, when vertex groups are selected non-uniformly, the sizes of the selected vertex groups are often larger, and as a result the leaves in the feasible solutions produced by exENSteiner, exIhlerA and FastAPP are more likely to be non-unique-group leaves.



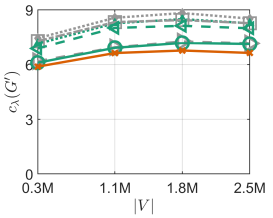

**(a)** Toronto ($|\Gamma| = 8$; $\lambda = 0.33$; $h = 3$)



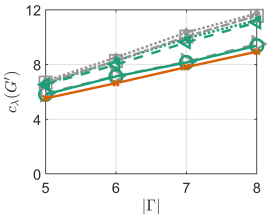

**(b)** Toronto ($|V| = 46073$; $\lambda = 0.33$; $h = 3$)



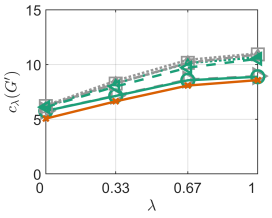

**(c)** Toronto ($|V| = 46073$; $|\Gamma| = 8$; $h = 3$)



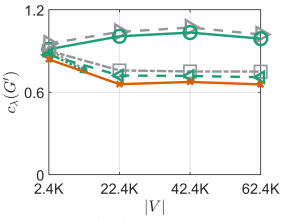

**(d)** DBLP ($|\Gamma| = 6$; $\lambda = 0.33$; $h = 3$)



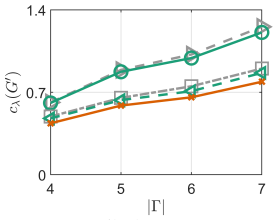

**(e)** DBLP ($|V| = 2497782$; $\lambda = 0.33$)



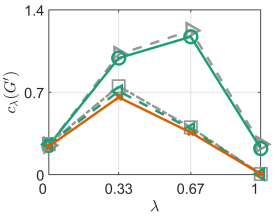

**(f)** DBLP ($|V| = 2497782$; $|\Gamma| = 6$)




**(g)** MovieLens ($|\Gamma| = 6$; $\lambda = 0.33$; $h = 3$)

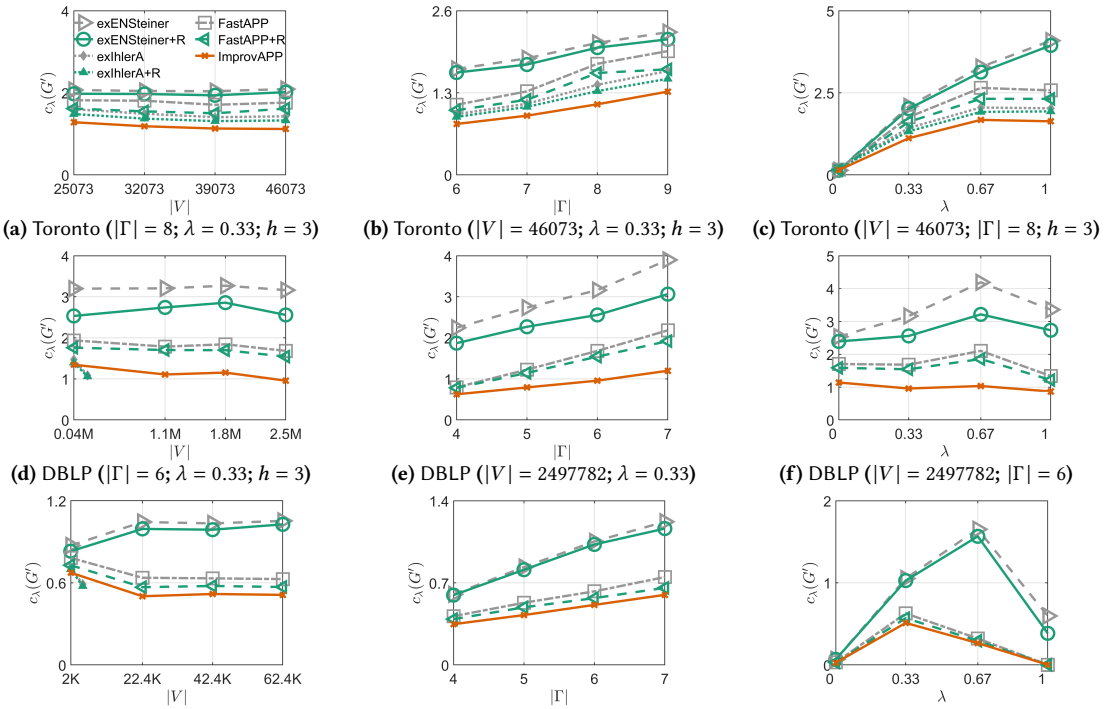



**(h)** MovieLens ($|V| = 62423$; $\lambda = 0.33$)




**(i)** MovieLens ($|V| = 62423$; $|\Gamma| = 6$)

**Figure 11: The refinement results in the main experiments where vertex groups are selected uniformly.**




**(a)** Toronto ($|\Gamma| = 8$; $\lambda = 0.33$; $h = 3$) **(b)** Toronto ($|V| = 46073$; $\lambda = 0.33$; $h = 3$) **(c)** Toronto ($|V| = 46073$; $|\Gamma| = 8$; $h = 3$)

**(d)** DBLP ($|\Gamma| = 6$; $\lambda = 0.33$; $h = 3$) **(e)** DBLP ($|V| = 2497782$; $\lambda = 0.33$) **(f)** DBLP ($|V| = 2497782$; $|\Gamma| = 6$)

**(g)** MovieLens ($|\Gamma| = 6$; $\lambda = 0.33$; $h = 3$) **(h)** MovieLens ($|V| = 62423$; $\lambda = 0.33$) **(i)** MovieLens ($|V| = 62423$; $|\Gamma| = 6$)

**Figure 12: The refinement results in the main experiments where vertex groups are selected non-uniformly.**

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