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EE 5340

HW2

Problem 1:

- Detail the design of CCNOT-gate based oracle for an instance of the search problem with N=4, M=1:

$$N = 4 = 2^2$$

Therefore, we need n = 2 qubits to represent indexing ranged from [0, 3] referred to it as the following:

$$|x> = |q_0 q_1>$$

Number of workspaces will be 1 qubit that is prepared to be in the following state to monitor the change in the sign for a potential solution:

$$|q_2> = H(|1>) = |-> = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \frac{|0> - |1>}{\sqrt{2}}$$

Assuming there is one solution at **index = 3** ($|x_{sol}> = |3>$)which makes f(x) = 1 and 0 otherwise. The function f(x) increases the probability of measuring the solution state.

$$|x>|q_2> \rightarrow |x>|q_2 \oplus f(x)>$$

$$with\ f(x) = q_0 * q_1$$

$$|x>(-1)^{f(x)=q_0*q_1}[\frac{|0> - |1>}{\sqrt{2}}]$$

This transformation can be represented as the following matrix

$$U_f = \begin{bmatrix}\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1}\\0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0\end{bmatrix}$$

$$|x>(-1)^{f(x)=q_0*q_1}\left[\frac{|0> - |1>}{\sqrt{2}}\right] = \left(|x>\left[\frac{|0> - |1>}{\sqrt{2}}\right]\right) U_f$$

The oracle will only flip the sign when $f(x) = q_0 * q_1$ which is $|q_0 q_1> = |11>$. The following shows all possible inputs and its represented output:

$$|000> \rightarrow |000>$$
$$|001> \rightarrow |001>$$

$$|010> \rightarrow |010>$$
$$|011> \rightarrow |011>$$
$$|100> \rightarrow |100>$$
$$|101> \rightarrow |101>$$
$$|\mathbf{110}> \rightarrow |\mathbf{111}>$$
$$|\mathbf{111}> \rightarrow |\mathbf{110}>$$

Problem 2:

i-

$$X \oplus X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$I \oplus H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$CX = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$(X \oplus X)\,(I \oplus H)\,CX\,(I \oplus H)\,(X \oplus X) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -1 * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} == 2|00 >< 00| - I$$

ii-

$$H \,*\, (2\,|0 >< 0| - I) \,*\, H$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & -\cos\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & -\cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\begin{bmatrix} -\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) & -2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ -2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) & -\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right) \end{bmatrix} = \begin{bmatrix} -\cos(\theta) & -sin(\theta) \\ -sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$-1 * \begin{bmatrix} \cos(\theta) & sin(\theta) \\ sin(\theta) & -\cos(\theta) \end{bmatrix} = 2\,|\Psi >< \Psi| - I$$

$$H \,*\, (2\,|0 >< 0| - I) \,*\, H = 2\,|\Psi >< \Psi| - I$$

iii-

$$P = 2\,|\Psi><\Psi| - I = \begin{bmatrix} 2*\cos^2\left(\frac{\theta}{2}\right) - 1 & 2*sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) \\ 2*cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & 2*\sin^2\left(\frac{\theta}{2}\right) - 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & sin(\theta) \\ sin(\theta) & -\cos(\theta) \end{bmatrix}$$

$$O = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$G = PO = \begin{bmatrix} \cos(\theta) & sin(\theta) \\ sin(\theta) & -\cos(\theta) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -sin(\theta) \\ sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$G = \begin{bmatrix} \cos(\theta) & -sin(\theta) \\ sin(\theta) & \cos(\theta) \end{bmatrix}$$

Problem 3:

- More G iterations are needed for M > N/2 because θ will be greater than π/4. Meaning that the least amount of G iterations, which is 1 iteration, will rotate the state vector further from |β> solutions axis by having θ exceeds π/2. Therefore, **adding additional qubit** will solve the problem making M < (2N)/2 always a true statement. The problem will have M solutions out of 2N entries.
- By doubling the number of items (2N), the complexity observations for M > N/2 case **will stay the same**. On the other hand, the search algorithm will change for the number of oracle rotations. The following shows the difference:

1. **M < N/2 case:**

$$oracle\ rotations = R = \frac{\pi}{4}\sqrt{\frac{N}{M}}$$

$$O(\frac{\pi}{4}\sqrt{\frac{N}{M}})$$

2. **M > N/2 case:**

$$oracle\ rotations = R = \frac{\pi}{4}\sqrt{\frac{2 * N}{M}}$$

$$O(\frac{\pi}{4}\sqrt{\frac{N}{M}})$$

Problem 4:

- **Analyze the time complexity of Quantum Fourier Transform:**
  Number of gates for:

$$first\ qubit = n$$
$$second\ qubit = n - 1$$
$$their\ qubit = n - 3$$
$$....$$
$$last\ qubit = 1\ H_{gate} = 1$$

$$gates\ required = \sum_{i=0}^{n-1}(n - i) = \frac{(n^2 + n)}{2}$$
$$Swaps\ gates = \frac{n}{2} * 3\ CNOT_{gates} = \frac{3}{2}n$$

$$Total\ number\ of\ gates = \frac{(n^2 + n)}{2} + \frac{3}{2}n = \frac{(n^2 + 4n)}{2}$$

$$Time\ complexity\ of\ QFT = O(n^2)$$

- **How many swaps are necessary?**
  $Swaps\ gates\ needed = \left\lfloor \frac{n}{2} \right\rfloor$

- **Where should the swap "gate(s)" be placed?**
  At the end of conditional rotations and after all qubits are done.

- **What is the impact on complexity?**
  The impact of swap operations on the total time complexity is not significant. Duo to the conditional rotations dominates total time complexity because it has $O(n^2)$ which is higher than the time complexity of swap operations $O(n)$.

Problem 5:

- According to the paper "*An improved quantum Fourier transform algorithm and applications*" [1] The best time complexity that can be accomplished for classical discrete Fourier transform is

$$O(N\ log\ N) = O(n\ 2^n)$$

The algorithm that achieves this time complexity is referred to it as Fast Fourier transform (FFT) algorithm. FFT manages to reduce the time complexity of discrete Fourier transform from $O(N^2)$

- The time complexity for Quantum Fourier transform (QFT) is

$$O((log\ N)^2) = O(n^2)$$

Making QFT exponentially faster than FFT. The following table explain time complexity difference:

n	FT $O(n\ 2^n)$	QFT $O(n^2)$	computation advantage (FFT/QFT)
2	8	4	2.00
3	24	9	2.67
4	64	16	4.00
5	160	25	6.40
6	384	36	10.67
7	896	49	18.29
8	2048	64	32.00
9	4608	81	56.89
10	10240	100	102.40

- The other major difference aside from the speed between **FFT** and **QFT** is the way data is represented and stored. For instance, the output obtained after FFT algorithm completes is easily accessible, read-out, and manipulate. On the other hand, the output in obtained from QFT is repatriated as a quantum state, which each qubit collapses into a single state. In addition, QFT output would be maximized to collapses into the corresponding FT results but not granted due to injected noise.

References

[1] Hales, Lisa, and Sean Hallgren. "An improved quantum Fourier transform algorithm and applications." *Proceedings 41st Annual Symposium on Foundations of Computer Science*. IEEE, 2000.

i-

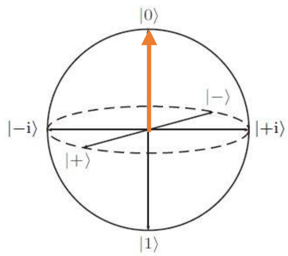
Qubit/bit	Classic Computing	Quantum Computing
1	1	2^1 = 2
2	2	2^2 = 4
3	3	2^3 = 8
4	4	2^4 = 16
5	5	2^5 = 32
n	n	2^n

ii-

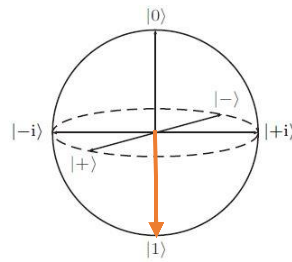
- Yes.
- For an arbitrary number of qubits, there are an infinite number of quantum gates exists. Therefore, any set of qubits state can be represented through a finite number of quantum gates.

iii-

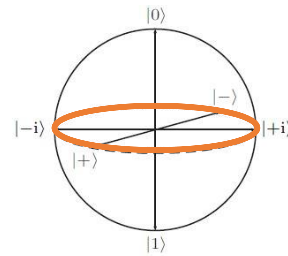
$$(|0> \ |1>) * H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



Input: |0> = $\begin{bmatrix}1\\0\end{bmatrix}$



Input: |1> = $\begin{bmatrix}0\\1\end{bmatrix}$



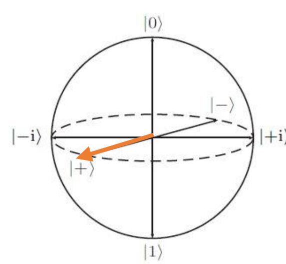
output: $\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$

iv-

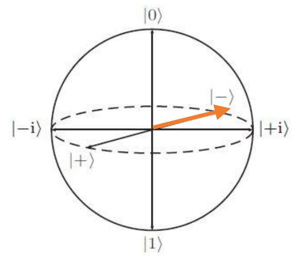
- Degrees of freedom Bloch sphere representation have = 4
- Degrees of freedom in $|\Psi > = \ \alpha\,|0 > \ + \beta\,|1 >$ have = 3
- Mismatch indicates that the quantum state $|\Psi >$ is restricted its two vectors to be normalized to 1 which lower the degree of freedom of Bloch sphere by 1 making it 3.

v-

$$|+> = \frac{1}{\sqrt{2}}\ |0 > \ + \frac{1}{\sqrt{2}}\ |1 > = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

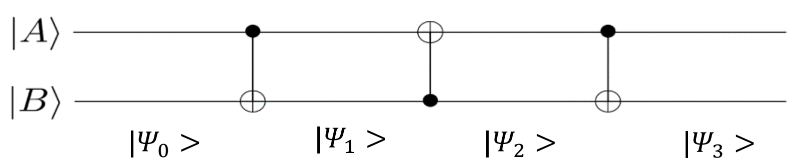


$$|-> = \frac{1}{\sqrt{2}}\ |0 > \ - \frac{1}{\sqrt{2}}\ |1 > = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$



Problem 2:

i-



$|\boldsymbol{\Psi_0}> = \ |A, B>$

$|\boldsymbol{\Psi_1}> = \ |A, A \oplus B>$

$|\boldsymbol{\Psi_2}> = \ |A \oplus (A \oplus B), A \oplus B> = \ |B, A \oplus B>$

$|\boldsymbol{\Psi_3}> = \ |B, B \oplus (A \oplus B)> = \ |B, A>$

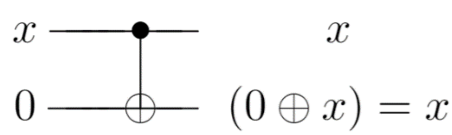
ii-

$$U^*U = I$$
$$U^*_{CNOT} * U_{CNOT} = I$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

iii-

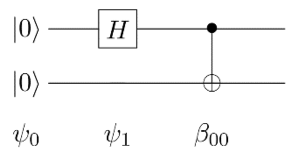
- Quantum circuit which can copy a classic bit:



- We cannot copy the state of a qubit because it violates no-cloning theorem.

iv-

***β_{00} state*:**

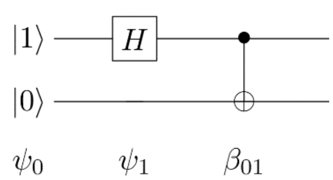


$$\Psi_0 = |00>$$

$$\Psi_1 = \frac{1}{\sqrt{2}}|00> + \frac{1}{\sqrt{2}}|10>$$

$$\beta_{00} = \frac{1}{\sqrt{2}}|00> + \frac{1}{\sqrt{2}}|11>$$

***β_{01} state*:**



$$\Psi_0 = |01>$$

$$\Psi_1 = \frac{1}{\sqrt{2}}|01> + \frac{1}{\sqrt{2}}|11>$$

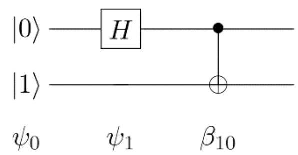
$$\beta_{01} = \frac{1}{\sqrt{2}}|01> + \frac{1}{\sqrt{2}}|10>$$

***β_{10} state*:**

$$\Psi_0 = |10>$$

$$\Psi_1 = \frac{1}{\sqrt{2}}|00> - \frac{1}{\sqrt{2}}|10>$$

$$\beta_{10} = \frac{1}{\sqrt{2}}|00> - \frac{1}{\sqrt{2}}|11>$$

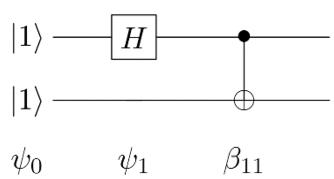


$\boldsymbol{\beta_{11}}$ ***state***:

$$\Psi_0 = |11>$$

$$\Psi_1 = \alpha\,|01> - \beta\,|11>$$

$$\beta_{11} = \alpha|01> - \beta|10>$$

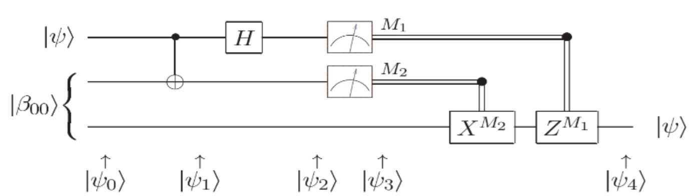


v-

- Yes, quantum information can be sent over a classic communication medium.
- It can be sent by applying the appropriate quantum gates to the entangled qubit. Those quantum gate transformation is controlled classically.
- No, no-cloning theorem persist because the quantum information exists only once in any state. For example, if we are to teleport a state of a qubit, the original qubit will end up measured to 0 or 1 and only the target qubit will carry on the qubit state.

Problem 3:

i- In order to communicate a qubit state, the following quantum circuit is used to teleport the qubit state from one side to the other:



- **Initial state:**

$$|\Psi>= \alpha\,|0> + \beta\,|1>$$

$$\beta_{00} = \frac{1}{\sqrt{2}}|00> + \frac{1}{\sqrt{2}}|11$$

$$|\boldsymbol{\Psi_0}>=|\Psi>\oplus\beta_{00}=\frac{1}{\sqrt{2}}[\alpha\,|0>(|00>+|11>)+\beta|1>(|00>+|11>)]$$

- **CNOT-gate:**

$$|\boldsymbol{\Psi_1}>=\frac{1}{\sqrt{2}}[\alpha\,|0>(|00>+|11>)+\beta|1>(|10>+|01>)]$$

- **H-gate:**

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}[\alpha\,(|0>+|1>)(|00>+|11>)+\beta(|0>-|1>)(|10>+|01>)]$$

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}\,[|00>(\alpha|0>+\beta|1>)+|01>(\alpha|1>+\beta|0>)$$
$$+|10>(\alpha|0>-\beta|1>)+|11>(\alpha|1>-\beta|0>)]$$

- **Measurement:**
  Upon measurement at $|\boldsymbol{\Psi_2}>$ the qubit state of qubit will fall into one of the four states that are equally probable (25%). The measurement value of the first two qubits will determine the phase of the third qubit that is being shared between the two parties. Knowing so will allows was to retrieve the original qubit's state by applying X-gate and Z-gate when needed:

  $|\boldsymbol{\Psi_3}(00)>\rightarrow[\alpha|0>+\beta|1>]$
  $|\boldsymbol{\Psi_3}(01)>\rightarrow[\alpha|1>+\beta|0>]$
  $|\boldsymbol{\Psi_3}(10)>\rightarrow[\alpha|0>-\beta|1>]$
  $|\boldsymbol{\Psi_3}(11)>\rightarrow[\alpha|1>-\beta|0>]$

  At the end, the third qubit will always retrieve the state $|\Psi>$ that is intended to be communicated between the two parties. No-cloning theorem will still stand as the state of qubit that is being teleported only has one copy in any stages.

ii-

$\boldsymbol{\beta_{01}}$ ***state***:

$$|\boldsymbol{\Psi_0}>=|\Psi>\oplus\beta_{01}=\frac{1}{\sqrt{2}}[\alpha\,|0>(|01>+|10>)+\beta|1>(|01>+|10>)]$$

$$|\boldsymbol{\Psi_1}>=\frac{1}{\sqrt{2}}[\alpha\,|0>(|01>+|10>)+\beta|1>(|11>+|00>)]$$

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}[\alpha\,(|0>+|1>)(|01>+|10>)+\beta(|0>-|1>)(|11>)+|00>]$$

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}\,[|00>(\alpha|1>+\beta|0>)+|01>(\alpha|0>+\beta|1>)$$

$$+\,|10>(\alpha|1>-\beta|0>)+|11>(\alpha|0>-\beta|1>)]$$

$$|\boldsymbol{\Psi_3}(00)>\rightarrow[\alpha|1>+\beta|0>]$$
$$|\boldsymbol{\Psi_3}(01)>\rightarrow[\alpha|0>+\beta|1>]$$
$$|\boldsymbol{\Psi_3}(10)>\rightarrow[\alpha|1>-\beta|0>]$$
$$|\boldsymbol{\Psi_3}(11)>\rightarrow[\alpha|0>-\beta|1>]$$

BELL(01) state will teleported the qubit Ψ after applying X-gate transformation

$\boldsymbol{\beta_{10}}$ ***state***:

$$|\boldsymbol{\Psi_0}>=|\Psi>\oplus\beta_{10}=\frac{1}{\sqrt{2}}[\alpha\,|0>(|00>-|11>)+\beta|1>(|00>-|11>)]$$

$$|\boldsymbol{\Psi_1}>=\frac{1}{\sqrt{2}}[\alpha\,|0>(|00>-|11>)+\beta|1>(|10>-|01>)]$$

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}[\alpha\,(|0>+|1>)(|00>-|11>)+\beta(|0>-|1>)(|10>-|01>)]$$

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}\,[|00>(\alpha|0>-\beta|1>)+|01>(-\alpha|1>+\beta|0>)$$

$$+\,|10>(\alpha|0>+\beta|1>)+|11>(-\alpha|1>-\beta|0>)]$$

$$|\boldsymbol{\Psi_3}(00)>\rightarrow[\alpha|0>-\beta|1>]$$
$$|\boldsymbol{\Psi_3}(01)>\rightarrow[-\alpha|1>+\beta|0>]$$
$$|\boldsymbol{\Psi_3}(10)>\rightarrow[\alpha|0>+\beta|1>]$$
$$|\boldsymbol{\Psi_3}(11)>\rightarrow[-\alpha|1>-\beta|0>]$$

- BELL(10) state will teleported the qubit Ψ after applying Z-gate transformation

$\boldsymbol{\beta_{11}}$ ***state***:

$$|\boldsymbol{\Psi_0}>=|\Psi>\oplus\beta_{11}=\frac{1}{\sqrt{2}}[\alpha\,|0>(|01>-|10>)+\beta|1>(|01>-|10>)]$$

$$|\boldsymbol{\Psi_1}>=\frac{1}{\sqrt{2}}[\alpha\,|0>(|01>-|10>)+\beta|1>(|11>-|00>)]$$

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}[\alpha\,(|0>+|1>)(|01>-|10>)+\beta(|0>-|1>)(|11>-|00>)]$$

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}\,[|00>(\alpha|1>-\beta|0>)+|01>(-\alpha|0>+\beta|1>)$$
$$+|10>(\alpha|1>+\beta|0>)+|11>(-\alpha|0>-\beta|1>)]$$

$$|\boldsymbol{\Psi_3}(00)>\rightarrow[\alpha|1>-\beta|0>]$$
$$|\boldsymbol{\Psi_3}(01)>\rightarrow[-\alpha|0>+\beta|1>]$$
$$|\boldsymbol{\Psi_3}(10)>\rightarrow[\alpha|1>+\beta|0>]$$
$$|\boldsymbol{\Psi_3}(11)>\rightarrow[-\alpha|0>-\beta|1>]$$

- BELL(11) state will teleported the qubit Ψ after applying X-gate and then Z-gate transformation.

Problem 4:

i- Observables have eigenvectors that span in Hilbert space and therefore it can be expressed as the following:

$$\Psi=\sum_{1}^{N}c_i*\lambda_i$$

The summation probability of eigenvectors is:

$$\sum_{1}^{N}|c_i|^2=1$$

Which indicate non-degenerate. Therefore, observables correspond to Hermitian operators.

ii-

$$\sigma_x|r>=\,|r>\qquad\qquad \boldsymbol{and}\qquad \sigma_x|l>=\,-|l>$$

$$\begin{bmatrix}\sigma_{x11} & \sigma_{x12}\\ \sigma_{x21} & \sigma_{x22}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \quad \boldsymbol{and} \quad \begin{bmatrix}\sigma_{x11} & \sigma_{x12}\\ \sigma_{x21} & \sigma_{x22}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}=-\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x11}}+\frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x12}}=\frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x21}}+\frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x22}}=\frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x11}}+\frac{-1}{\sqrt{2}}\boldsymbol{\sigma_{x12}}=\frac{-1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x21}}+\frac{-1}{\sqrt{2}}\boldsymbol{\sigma_{x22}}=\frac{1}{\sqrt{2}}$$

Solve four equations and four unknowns to get:

$$\sigma_x=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

iii-

$$\sigma_n=\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ \sin(\theta) & -\cos(\theta)\end{bmatrix},\ |\lambda_1>=\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\end{bmatrix},\ |\lambda_2>=\begin{bmatrix}-\sin(\frac{\theta}{2})\\ \cos(\frac{\theta}{2})\end{bmatrix}$$

$$|\lambda_1><|\lambda_1|-|\lambda_2><|\lambda_2|=\begin{bmatrix}\cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right) & 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\\ 2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & sin^2\left(\frac{\theta}{2}\right)-\cos^2\left(\frac{\theta}{2}\right)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ \sin(\theta) & -\cos(\theta)\end{bmatrix}=\sigma_n$$

iv-

$$\sigma_z\,|l>=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}=|r>$$

$$|r>=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}+\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}-\ upon\ measurement=50\%\rightarrow\ 1\ \boldsymbol{and}\ 50\%\rightarrow\ 0$$

$$|l>=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}-\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}-\ upon\ measurement=50\%\rightarrow 1\ \boldsymbol{and}\ 50\%\rightarrow\ 0$$

- The outcome state would be flipped by applying σ_z to $|l>$ making it transform to $|r>$
- Even though, the state after transformation has changed to the other side of the Bloch sphere, it is still on the same plate. Meaning, upon measurement $|l>$ and $|r>$ would have the same exact portability (50%) of collapsing into 1 or 0.

Problem 5:

i-

$$< o|u >< u|o > = [\gamma \quad \delta] \begin{bmatrix}1\\0\end{bmatrix} * [1 \quad 0] \begin{bmatrix}\gamma\\ \delta\end{bmatrix} = \gamma^* \gamma = \frac{1}{2}$$

$$< o|d >< d|o > = [\gamma \quad \delta] \begin{bmatrix}0\\1\end{bmatrix} * [0 \quad 1] \begin{bmatrix}\gamma\\ \delta\end{bmatrix} = \delta^* \delta = \frac{1}{2}$$

$$< i|u >< i|u > = [\alpha \quad \beta] \begin{bmatrix}1\\0\end{bmatrix} * [1 \quad 0] \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \alpha^* \alpha = \frac{1}{2}$$

$$< i|d >< d|o > = [\alpha \quad \beta] \begin{bmatrix}0\\1\end{bmatrix} * [0 \quad 1] \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \beta^* \beta = \frac{1}{2}$$

$$\gamma^* \gamma = \delta^* \delta = \alpha^* \alpha = \beta^* \beta = \frac{1}{2}$$

ii-

$$< o|l >< l|o > = [\gamma \quad \delta]\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \left(\frac{\gamma}{\sqrt{2}} - \frac{\delta}{\sqrt{2}}\right)\left(\frac{\gamma}{\sqrt{2}} - \frac{\delta}{\sqrt{2}}\right) = \frac{1}{2}$$

$$< o|r >< r|o > = [\gamma \quad \delta]\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \left(\frac{\gamma}{\sqrt{2}} + \frac{\delta}{\sqrt{2}}\right)\left(\frac{\gamma}{\sqrt{2}} + \frac{\delta}{\sqrt{2}}\right) = \frac{1}{2}$$

$$\left(\frac{\gamma}{\sqrt{2}} \pm \frac{\delta}{\sqrt{2}}\right) * \left(\frac{\gamma}{\sqrt{2}} \pm \frac{\delta}{\sqrt{2}}\right) = \frac{1}{2}(\gamma * \gamma \pm \gamma * \delta \pm \gamma * \delta + \delta * \delta) = \frac{1}{2}$$

$$\frac{1}{2}\left(\frac{1}{2} \pm \gamma * \delta \pm \gamma * \delta + \frac{1}{2}\right) = \frac{1}{2}, \qquad since\ \ \gamma^* \gamma = \ \delta^* \delta = \frac{1}{2}$$

$$(\gamma * \delta + \delta * \gamma) = 0$$

$$< i|r >< r|i > = [\alpha \quad \beta]\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \left(\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}}\right)\left(\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}}\right) = \frac{1}{2}$$

$$< i|l >< l|i > = [\alpha \quad \beta]\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \left(\frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{2}}\right)\left(\frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{2}}\right) = \frac{1}{2}$$

$$\left(\frac{\alpha}{\sqrt{2}} \pm \frac{\beta}{\sqrt{2}}\right) * \left(\frac{\alpha}{\sqrt{2}} \pm \frac{\beta}{\sqrt{2}}\right) = \frac{1}{2}(\alpha * \alpha \pm \alpha * \beta \pm \alpha * \beta + \beta * \beta) = \frac{1}{2}$$

$$\frac{1}{2}\left(\frac{1}{2} \pm \alpha * \beta \pm \alpha * \beta + \frac{1}{2}\right) = \frac{1}{2}, \qquad since\ \ \alpha * \alpha = \ \beta^* \beta = \frac{1}{2}$$

$$(\alpha * \beta + \beta * \alpha) = 0$$

$$\alpha * \beta + \beta * \alpha = \gamma * \delta + \delta * \gamma = 0$$

iii-

- Prove α ∗β and γ ∗ δ must be pure imaginary:
  Given the following:

$$\alpha = x_1 + i * y_1$$

$$\beta = x_2 + i * y_2$$

$$\gamma = x_3 + i * y_3$$

$$\delta = x_4 + i * y_4$$

$\alpha * \beta$

$$\alpha * \beta + \beta * \alpha = 0$$

$$\alpha * \beta + \beta * \alpha = (x_1 + i * y_1) * (x_2 + i * y_2) + (x_2 + i * y_2) * (x_1 + i * y_1) = 0$$
$$\alpha * \beta + \beta * \alpha = x_1 x_2 + i * y_2 x_1 + i * y_1 x_2 - y_1 y_2 + x_1 x_2 + i * y_1 x_2 + i * y_2 x_1 - y_1 y_2 = 0$$
$$\boldsymbol{(x_1 x_2 - y_1 y_2) = 0}, \quad \rightarrow \ \boldsymbol{x_1 x_2 = y_1 y_2}$$

$\alpha * \beta = x_1 x_2 + i * y_2 x_1 + i * y_1 x_2 - y_1 y_2$

$$\boldsymbol{x_1 x_2 = y_1 y_2}$$

$\alpha * \beta = i * y_2 x_1 + i * y_1 x_2$

$\gamma * \delta$

$\gamma * \delta = x_3 x_4 + i * y_4 x_3 + i * y_3 x_4 - y_3 y_4$

$$\boldsymbol{x_3 x_4 = y_3 y_4}$$

$\gamma * \delta = i * y_4 x_3 + i * y_3 x_4$

- $\alpha * \beta$ *and* $\gamma * \delta$ must be pure imaginary
- α *and* β cannot be both real because otherwise α*β+β*α=0 won't hold. The same can be said to γ *and* δ as they cannot be both real for the same reason.

Problem 6:

i-

$$\boldsymbol{I} \oplus \boldsymbol{\sigma_x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, |uu> = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, |ud> = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, |du> = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, |dd> = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(\boldsymbol{I} \oplus \boldsymbol{\sigma_x})\,|uu> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |u> \oplus |d> = |ud>$$

$$(\boldsymbol{I} \oplus \boldsymbol{\sigma_x})\,|ud> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |u> \oplus |u> = |uu>

$$(\boldsymbol{I} \oplus \boldsymbol{\sigma_x})\,|du> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |d> \oplus |d> = |dd>$$

$$(\boldsymbol{I} \oplus \boldsymbol{\sigma_x})\,|dd> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |d> \oplus |u> = |du>$$

- The first qubit remains unchanged in all the cases.

ii-

$$\boldsymbol{\sigma_z} \oplus \boldsymbol{\sigma_x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, |uu> = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, |ud> = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, |du> = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, |dd> = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(\boldsymbol{\sigma_z} \oplus \boldsymbol{\sigma_x})\,|uu> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |u> \oplus |d> = |ud>$$

$$(\boldsymbol{\sigma_z} \oplus \boldsymbol{\sigma_x})\,|ud> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |u> \oplus |u> = |uu>$$

$$(\boldsymbol{\sigma_z} \oplus \boldsymbol{\sigma_x})\,|du> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = -|d> \oplus |d> = -|dd>$$

$$(\boldsymbol{\sigma_z} \oplus \boldsymbol{\sigma_x})\,|dd> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = -|d> \oplus |u> = -|du>$$

The information will result into:

$$|uu> = |u> \oplus |u> \quad \rightarrow \quad |ud> = |u> \oplus |d>$$

$$|ud> = |u> \oplus |d> \quad \rightarrow \quad |uu> = |u> \oplus |u>$$

$$|du> = |d> \oplus |u> \quad \rightarrow -|dd> = -|d> \oplus |u>$$

$$|dd> = |d> \oplus |d> \quad \rightarrow -|du> = -|d> \oplus |u>$$

-

- Thus, $\boldsymbol{\sigma_z}$ cats on the first qubit and $\boldsymbol{\sigma_x}$ cats on the second qubit.