## Problem 1:

i-

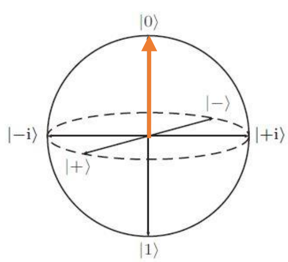
| Qubit/bit | Classic Computing | Quantum Computing |
|---|---|---|
| **1** | 1 | $2^1$ = 2 |
| **2** | 2 | $2^2$ = 4 |
| **3** | 3 | $2^3$ = 8 |
| **4** | 4 | $2^4$ = 16 |
| **5** | 5 | $2^5$ = 32 |
| **n** | $n$ | $2^n$ |

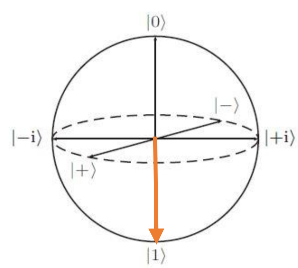
ii-

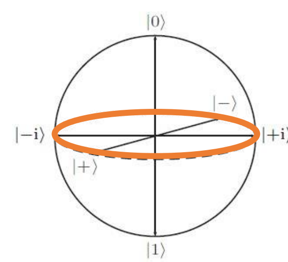
- Yes.
- For an arbitrary number of qubits, there are an infinite number of quantum gates exists. Therefore, any set of qubits state can be represented through a finite number of quantum gates.

iii-

$$(|0> \ |1>) * H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



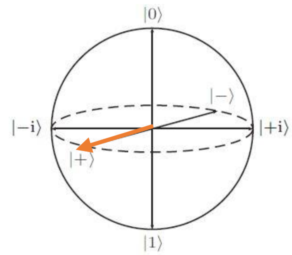

Input: |0> = $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Input: |1> = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ output: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

iv-

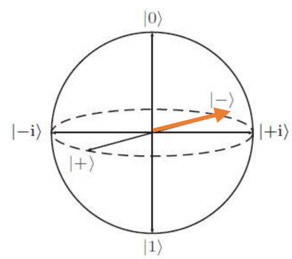
- Degrees of freedom Bloch sphere representation have = 4
- Degrees of freedom in $|\Psi> = \alpha\,|0> + \beta\,|1>$ have = 3
- Mismatch indicates that the quantum state $|\Psi>$ is restricted its two vectors to be normalized to 1 which lower the degree of freedom of Bloch sphere by 1 making it 3.

V-

$$|+> = \frac{1}{\sqrt{2}}\ |0> \ + \frac{1}{\sqrt{2}}\ |1> = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

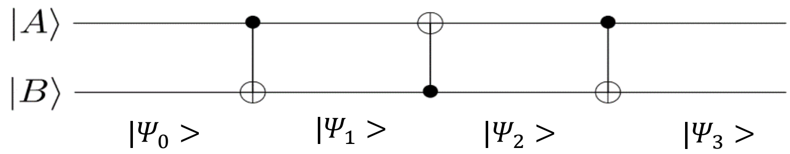



$$|-> = \frac{1}{\sqrt{2}}\ |0> \ - \frac{1}{\sqrt{2}}\ |1> = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$



**Problem 2:**

**i-**




$|\boldsymbol{\Psi_0}> = \ |A, B>$

$|\boldsymbol{\Psi_1}> = \ |A, A \oplus B>$

$|\boldsymbol{\Psi_2}> = \ |A \oplus (A \oplus B), A \oplus B> = \ |B, A \oplus B>$

$|\boldsymbol{\Psi_3}> = \ |B, B \oplus (A \oplus B)> = \ |B, A>$

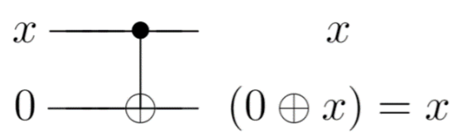
**ii-**

$$U^*U = I$$
$$U^*_{CNOT} * \ U_{CNOT} = I$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**iii-**

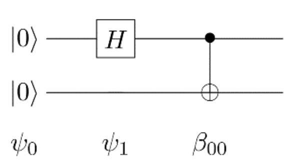
- Quantum circuit which can copy a classic bit:




- We cannot copy the state of a qubit because it violates no-cloning theorem.

**iv-**

***$\beta_{00}$ state*:**

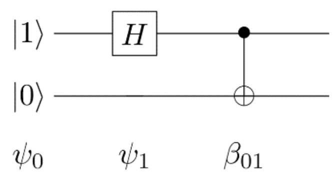



$\Psi_0 = \ |00>$

$\Psi_1 = \ \frac{1}{\sqrt{2}}|00> + \frac{1}{\sqrt{2}}|10>$

$\beta_{00} = \frac{1}{\sqrt{2}} \ |00> + \ \frac{1}{\sqrt{2}} \ |11>$

**$\boldsymbol{\beta_{01}}$ state**:

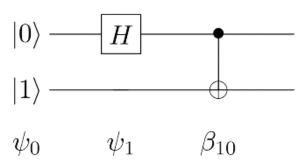



$$\Psi_0 = |01>$$

$$\Psi_1 = \frac{1}{\sqrt{2}}|01> + \frac{1}{\sqrt{2}}|11>$$

$$\beta_{01} = \frac{1}{\sqrt{2}}|01> + \frac{1}{\sqrt{2}}|10>$$

**$\boldsymbol{\beta_{10}}$ state**:

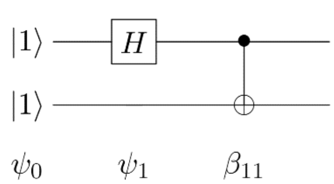



$$\Psi_0 = |10>$$

$$\Psi_1 = \frac{1}{\sqrt{2}}|00> - \frac{1}{\sqrt{2}}|10>$$

$$\beta_{10} = \frac{1}{\sqrt{2}}|00> - \frac{1}{\sqrt{2}}|11>$$

**$\boldsymbol{\beta_{11}}$ state**:




$$\Psi_0 = |11>$$

$$\Psi_1 = \alpha\,|01> - \beta\,|11>$$

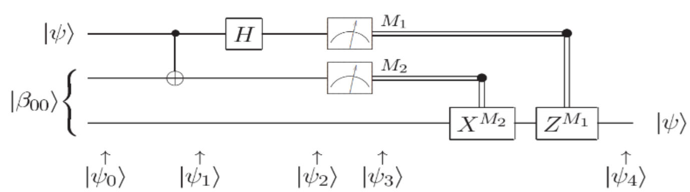
$$\beta_{11} = \alpha|01> - \beta|10>$$

**v-**

- Yes, quantum information can be sent over a classic communication medium.
- It can be sent by applying the appropriate quantum gates to the entangled qubit. Those quantum gate transformation is controlled classically.
- No, no-cloning theorem persist because the quantum information exists only once in any state. For example, if we are to teleport a state of a qubit, the original qubit will end up measured to 0 or 1 and only the target qubit will carry on the qubit state.

**Problem 3:**

**i-** In order to communicate a qubit state, the following quantum circuit is used to teleport the qubit state from one side to the other:




- **Initial state:**

$$|\Psi >= \ \alpha\,|0> \ +\beta\,|1>$$
$$\beta_{00} = \frac{1}{\sqrt{2}}|00> + \frac{1}{\sqrt{2}}|11$$
$$|\boldsymbol{\Psi_0}> = |\Psi> \oplus \beta_{00} = \frac{1}{\sqrt{2}}[\alpha\,|0>(|00>+|11>) + \beta|1>(|00>+|11>)]$$

- **CNOT-gate:**

$$|\boldsymbol{\Psi_1}> = \frac{1}{\sqrt{2}}[\alpha\,|0>(|00>+|11>) + \beta|1>(|10>+|01>)]$$

- **H-gate:**

$$|\boldsymbol{\Psi_2}> = \frac{1}{\sqrt{2}}[\alpha\,(|0>+|1>)(|00>+|11>) + \beta(|0>-|1>)(|10>+|01>)]$$

$$|\boldsymbol{\Psi_2}> = \frac{1}{\sqrt{2}}\,[|00>(\alpha|0>+\beta|1>) + |01>(\alpha|1>+\beta|0>)$$
$$+\,|10>(\alpha|0>-\beta|1>) + |11>(\alpha|1>-\beta|0>)]$$

- **Measurement:**
Upon measurement at $|\boldsymbol{\Psi_2}>$ the qubit state of qubit will fall into one of the four states that are equally probable (25%). The measurement value of the first two qubits will determine the phase of the third qubit that is being shared between the two parties. Knowing so will allows was to retrieve the original qubit's state by applying X-gate and Z-gate when needed:

$$|\boldsymbol{\Psi_3}(00)> \rightarrow [\alpha|0>+\beta|1>]$$
$$|\boldsymbol{\Psi_3}(01)> \rightarrow [\alpha|1>+\beta|0>]$$
$$|\boldsymbol{\Psi_3}(10)> \rightarrow [\alpha|0>-\beta|1>]$$
$$|\boldsymbol{\Psi_3}(11)> \rightarrow [\alpha|1>-\beta|0>]$$

At the end, the third qubit will always retrieve the state $|\Psi>$ that is intended to be communicated between the two parties. No-cloning theorem will still stand as the state of qubit that is being teleported only has one copy in any stages.

**ii-**

$\boldsymbol{\beta_{01}}$ ***state***:

$$|\boldsymbol{\Psi_0}>=|\Psi>\oplus\beta_{01}=\frac{1}{\sqrt{2}}[\alpha\,|0>(|01>+|10>)+\beta|1>(|01>+|10>)]$$

$$|\boldsymbol{\Psi_1}>=\frac{1}{\sqrt{2}}[\alpha\,|0>(|01>+|10>)+\beta|1>(|11>+|00>)]$$

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}[\alpha\,(|0>+|1>)(|01>+|10>)+\beta(|0>-|1>)(|11>)+|00>]$$

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}\,[|00>(\alpha|1>+\beta|0>)+|01>(\alpha|0>+\beta|1>)$$
$$+|10>(\alpha|1>-\beta|0>)+|11>(\alpha|0>-\beta|1>)]$$

$$|\boldsymbol{\Psi_3}(00)>\rightarrow[\alpha|1>+\beta|0>]$$
$$|\boldsymbol{\Psi_3}(01)>\rightarrow[\alpha|0>+\beta|1>]$$
$$|\boldsymbol{\Psi_3}(10)>\rightarrow[\alpha|1>-\beta|0>]$$
$$|\boldsymbol{\Psi_3}(11)>\rightarrow[\alpha|0>-\beta|1>]$$

BELL(01) state will teleported the qubit $\Psi$ after applying X-gate transformation

$\boldsymbol{\beta_{10}}$ ***state***:

$$|\boldsymbol{\Psi_0}>=|\Psi>\oplus\beta_{10}=\frac{1}{\sqrt{2}}[\alpha\,|0>(|00>-|11>)+\beta|1>(|00>-|11>)]$$

$$|\boldsymbol{\Psi_1}>=\frac{1}{\sqrt{2}}[\alpha\,|0>(|00>-|11>)+\beta|1>(|10>-|01>)]$$

$$|\boldsymbol{\Psi_2}>=\frac{1}{\sqrt{2}}[\alpha\,(|0>+|1>)(|00>-|11>)+\beta(|0>-|1>)(|10>-|01>)]$$

$$|\boldsymbol{\Psi_2}> = \frac{1}{\sqrt{2}}\,[|00>(\alpha|0> - \beta|1>) + |01>(-\alpha|1> + \beta|0>)$$
$$+\,|10>(\alpha|0> + \beta|1>) + |11>(-\alpha|1> - \beta|0>)]$$

$$|\boldsymbol{\Psi_3}(00)> \rightarrow [\alpha|0> - \beta|1>]$$
$$|\boldsymbol{\Psi_3}(01)> \rightarrow [-\alpha|1> + \beta|0>]$$
$$|\boldsymbol{\Psi_3}(10)> \rightarrow [\alpha|0> + \beta|1>]$$
$$|\boldsymbol{\Psi_3}(11)> \rightarrow [-\alpha|1> - \beta|0>]$$

- BELL(10) state will teleported the qubit $\Psi$ after applying Z-gate transformation

**$\boldsymbol{\beta_{11}}$ *state*:**

$$|\boldsymbol{\Psi_0}> = |\Psi> \oplus \beta_{11} = \frac{1}{\sqrt{2}}[\alpha\,|0>(|01> - |10>) + \beta|1>(|01> - |10>)]$$

$$|\boldsymbol{\Psi_1}> = \frac{1}{\sqrt{2}}[\alpha\,|0>(|01> - |10>) + \beta|1>(|11> - |00>)]$$

$$|\boldsymbol{\Psi_2}> = \frac{1}{\sqrt{2}}[\alpha\,(|0> + |1>)(|01> - |10>) + \beta(|0> - |1>)(|11> - |00>)]$$

$$|\boldsymbol{\Psi_2}> = \frac{1}{\sqrt{2}}\,[|00>(\alpha|1> - \beta|0>) + |01>(-\alpha|0> + \beta|1>)$$
$$+\,|10>(\alpha|1> + \beta|0>) + |11>(-\alpha|0> - \beta|1>)]$$

$$|\boldsymbol{\Psi_3}(00)> \rightarrow [\alpha|1> - \beta|0>]$$
$$|\boldsymbol{\Psi_3}(01)> \rightarrow [-\,\alpha|0> + \beta|1>]$$
$$|\boldsymbol{\Psi_3}(10)> \rightarrow [\alpha|1> + \beta|0>]$$
$$|\boldsymbol{\Psi_3}(11)> \rightarrow [-\alpha|0> - \beta|1>]$$

- BELL(11) state will teleported the qubit $\Psi$ after applying X-gate and then Z-gate transformation.

**Problem 4:**

**i-** Observables have eigenvectors that span in Hilbert space and therefore it can be expressed as the following:

$$\Psi = \sum_{1}^{N} c_i * \lambda_i$$

The summation probability of eigenvectors is:

$$\sum_{1}^{N} |c_i|^2 = 1$$

Which indicate non-degenerate. Therefore, observables correspond to Hermitian operators.

**ii-**

$$\sigma_x|r> = |r> \qquad \textbf{and} \qquad \sigma_x|l> = -|l>$$

$$\begin{bmatrix}\sigma_{x11} & \sigma_{x12}\\ \sigma_{x21} & \sigma_{x22}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \qquad \textbf{and} \qquad \begin{bmatrix}\sigma_{x11} & \sigma_{x12}\\ \sigma_{x21} & \sigma_{x22}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix} = -\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x11}} + \frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x12}} = \frac{1}{\sqrt{2}}$$
$$\frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x21}} + \frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x22}} = \frac{1}{\sqrt{2}}$$
$$\frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x11}} + \frac{-1}{\sqrt{2}}\boldsymbol{\sigma_{x12}} = \frac{-1}{\sqrt{2}}$$
$$\frac{1}{\sqrt{2}}\boldsymbol{\sigma_{x21}} + \frac{-1}{\sqrt{2}}\boldsymbol{\sigma_{x22}} = \frac{1}{\sqrt{2}}$$

Solve four equations and four unknowns to get:

$$\sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

**iii-**

$$\sigma_n = \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ \sin(\theta) & -\cos(\theta)\end{bmatrix}, \quad |\lambda_1> = \begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\end{bmatrix}, \quad |\lambda_2> = \begin{bmatrix}-\sin(\frac{\theta}{2})\\ \cos(\frac{\theta}{2})\end{bmatrix}$$

$$|\lambda_1><|\lambda_1| - |\lambda_2><|\lambda_2| = \begin{bmatrix}\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) & 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\\ 2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & sin^2\left(\frac{\theta}{2}\right) - \cos^2\left(\frac{\theta}{2}\right)\end{bmatrix}$$
$$= \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ \sin(\theta) & -\cos(\theta)\end{bmatrix} = \sigma_n$$

**iv-**

$$\sigma_z\,|l> = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = |r>$$

$$|r> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} - upon\ measurement = 50\% \rightarrow\ 1\ \boldsymbol{and}\ 50\% \rightarrow\ 0$$

$$|l> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} - upon\ measurement = 50\% \rightarrow 1\ \boldsymbol{and}\ 50\% \rightarrow\ 0$$

- The outcome state would be flipped by applying $\sigma_z$ to $|l>$ making it transform to $|r>$
- Even though, the state after transformation has changed to the other side of the Bloch sphere, it is still on the same plate. Meaning, upon measurement $|l>$ and $|r>$ would have the same exact portability (50%) of collapsing into 1 or 0.

**Problem 5:**

**i-**

$$< o|u >< u|o >= [\gamma \quad \delta]\begin{bmatrix}1\\0\end{bmatrix} * [1 \quad 0]\begin{bmatrix}\gamma\\\delta\end{bmatrix} = \gamma^* \gamma = \frac{1}{2}$$

$$< o|d >< d|o >= [\gamma \quad \delta]\begin{bmatrix}0\\1\end{bmatrix} * [0 \quad 1]\begin{bmatrix}\gamma\\\delta\end{bmatrix} = \delta^* \delta = \frac{1}{2}$$

$$< i|u >< i|u >= [\alpha \quad \beta]\begin{bmatrix}1\\0\end{bmatrix} * [1 \quad 0]\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \alpha^* \alpha = \frac{1}{2}$$

$$< i|d >< d|o >= [\alpha \quad \beta]\begin{bmatrix}0\\1\end{bmatrix} * [0 \quad 1]\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \beta^* \beta = \frac{1}{2}$$

$$\gamma^* \gamma = \delta^* \delta = \alpha^* \alpha = \beta^* \beta = \frac{1}{2}$$

**ii-**

$$< o|l >< l|o > = [\gamma \quad \delta]\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{bmatrix} * \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\gamma\\\delta\end{bmatrix} = \left(\frac{\gamma}{\sqrt{2}} - \frac{\delta}{\sqrt{2}}\right)\left(\frac{\gamma}{\sqrt{2}} - \frac{\delta}{\sqrt{2}}\right) = \frac{1}{2}$$

$$< o|r >< r|o > = [\gamma \quad \delta]\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} * \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\gamma\\\delta\end{bmatrix} = \left(\frac{\gamma}{\sqrt{2}} + \frac{\delta}{\sqrt{2}}\right)\left(\frac{\gamma}{\sqrt{2}} + \frac{\delta}{\sqrt{2}}\right) = \frac{1}{2}$$

$$\left(\frac{\gamma}{\sqrt{2}} \pm \frac{\delta}{\sqrt{2}}\right) * \left(\frac{\gamma}{\sqrt{2}} \pm \frac{\delta}{\sqrt{2}}\right) = \frac{1}{2}(\gamma * \gamma \pm \gamma * \delta \pm \gamma * \delta + \delta * \delta) = \frac{1}{2}$$

$$\frac{1}{2}\left(\frac{1}{2} \pm \gamma * \delta \pm \gamma * \delta + \frac{1}{2}\right) = \frac{1}{2}, \qquad since\ \ \gamma^* \gamma = \delta^* \delta = \frac{1}{2}$$

$$(\gamma * \delta + \delta * \gamma) = 0$$

$$< i|r >< r|i > = [\alpha \quad \beta]\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} * \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \left(\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}}\right)\left(\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}}\right) = \frac{1}{2}$$

$$< i|l >< l|i > = [\alpha \quad \beta]\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{bmatrix} * \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \left(\frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{2}}\right)\left(\frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{2}}\right) = \frac{1}{2}$$

$$\left(\frac{\alpha}{\sqrt{2}} \pm \frac{\beta}{\sqrt{2}}\right) * \left(\frac{\alpha}{\sqrt{2}} \pm \frac{\beta}{\sqrt{2}}\right) = \frac{1}{2}(\alpha * \alpha \pm \alpha * \beta \pm \alpha * \beta + \beta * \beta) = \frac{1}{2}$$

$$\frac{1}{2}\left(\frac{1}{2} \pm \alpha * \beta \pm \alpha * \beta + \frac{1}{2}\right) = \frac{1}{2}, \qquad since\ \ \alpha * \alpha = \beta^* \beta = \frac{1}{2}$$

$$(\alpha * \beta + \beta * \alpha) = 0$$

$$\alpha * \beta + \beta * \alpha = \gamma * \delta + \delta * \gamma = 0$$

**iii-**

- Prove α ∗β and γ ∗ δ must be pure imaginary:
  Given the following:

$$\alpha = x_1 + i * y_1$$

$$\beta = x_2 + i * y_2$$

$$\gamma = x_3 + i * y_3$$

$$\delta = x_4 + i * y_4$$

$\alpha * \beta$

$$\alpha * \beta + \beta * \alpha = 0$$

$$\alpha * \beta + \beta * \alpha = (x_1 + i * y_1) * (x_2 + i * y_2) + (x_2 + i * y_2) * (x_1 + i * y_1) = 0$$

$$\alpha * \beta + \beta * \alpha = x_1x_2 + i * y_2x_1 + i * y_1x_2 - y_1y_2 + x_1x_2 + i * y_1x_2 + i * y_2x_1 - y_1y_2 = 0$$

$$\boldsymbol{(x_1x_2 - y_1y_2) = 0}, \quad \rightarrow \quad \boldsymbol{x_1x_2 = y_1y_2}$$

$$\alpha * \beta = x_1x_2 + i * y_2x_1 + i * y_1x_2 - y_1y_2$$

$$\boldsymbol{x_1x_2 = y_1y_2}$$

$$\alpha * \beta = i * y_2x_1 + i * y_1x_2$$

$\gamma * \delta$

$$\gamma * \delta = x_3x_4 + i * y_4x_3 + i * y_3x_4 - y_3y_4$$

$$\boldsymbol{x_3x_4 = y_3y_4}$$

$$\gamma * \delta = i * y_4x_3 + i * y_3x_4$$

- $\alpha * \beta$ *and* $\gamma * \delta$ must be pure imaginary
- $\alpha$ *and* $\beta$ cannot be both real because otherwise α*β+β*α=0 won't hold. The same can be said to $\gamma$ *and* $\delta$ as they cannot be both real for the same reason.

**Problem 6:**

**i-**

$$I \oplus \sigma_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, |uu> = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, |ud> = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, |du> = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, |dd> = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(I \oplus \sigma_x)\,|uu> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |u> \oplus |d> = |ud>$$

$$(I \oplus \sigma_x)\,|ud> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |u> \oplus |u> = |uu>$$

$$(I \oplus \sigma_x)\,|du> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |d> \oplus |d> = |dd>$$

$$(I \oplus \sigma_x)\,|dd> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |d> \oplus |u> = |du>$$

- The first qubit remains unchanged in all the cases.

**ii-**

$$\sigma_z \oplus \sigma_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, |uu> = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, |ud> = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, |du> = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, |dd> = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(\sigma_z \oplus \sigma_x)\,|uu> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |u> \oplus |d> = |ud>$$

$$(\sigma_z \oplus \sigma_x)\,|ud> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |u> \oplus |u> = |uu>$$

$$(\sigma_z \oplus \sigma_x)\,|du> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = -|d> \oplus |d> = -|dd>$$

$$(\sigma_z \oplus \sigma_x)\,|dd> = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = -|d> \oplus |u> = -|du>$$

The information will result into:

$$|uu> = |u> \oplus |u> \quad \rightarrow \quad |ud> = |u> \oplus |d>$$

$$|ud> = |u> \oplus |d> \quad \rightarrow \quad |uu> = |u> \oplus |u>$$

$$|du> = |d> \oplus |u> \quad \rightarrow -|dd> = -|d> \oplus |u>$$

$$|dd> = |d> \oplus |d> \quad \rightarrow -|du> = -|d> \oplus |u>$$

- Thus, $\boldsymbol{\sigma_z}$ cats on the first qubit and $\boldsymbol{\sigma_x}$ cats on the second qubit.