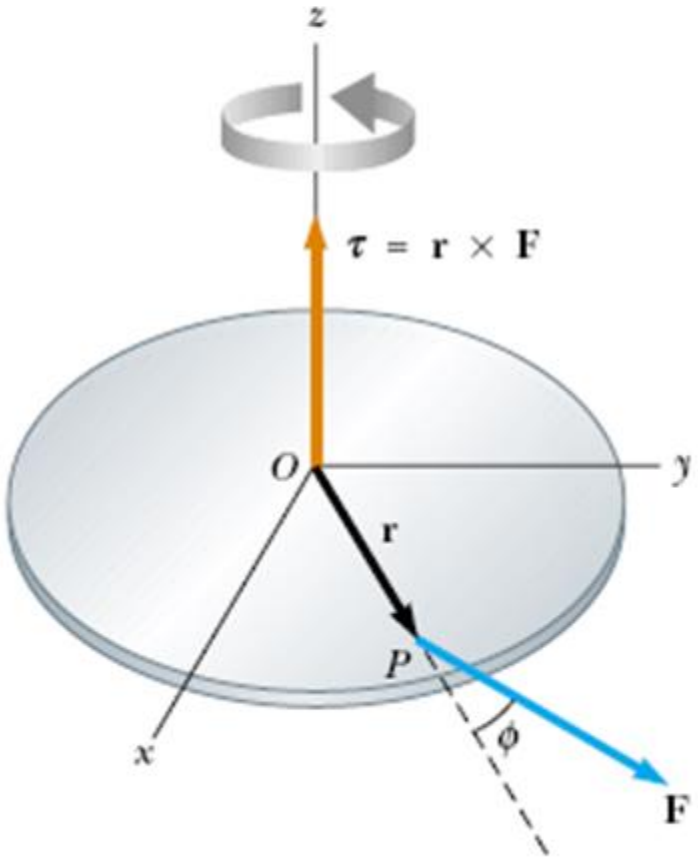
# Bölüm 8 Açısal Momentum

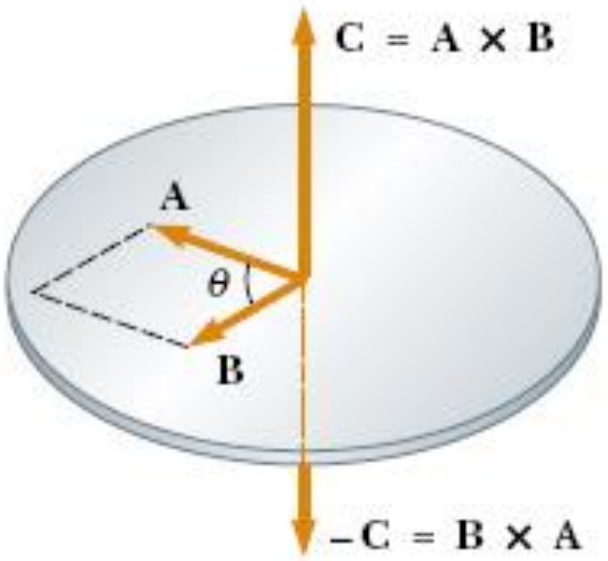
## Vektörel çarpım ve Tork

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$C \equiv AB \sin \theta$$






$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

eğer $\mathbf{A}$ paralel $\mathbf{B}$ $(\theta = 0^\circ$ veya $180^\circ)$ $\mathbf{A} \times \mathbf{B} = 0$

eğer $\mathbf{A}$ dik $\mathbf{B}$, ise $|\mathbf{A} \times \mathbf{B}| = AB$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}$$

**Örnek :** konumu $\mathbf{F} = (2.00\hat{\mathbf{i}} + 3.00\hat{\mathbf{j}})$ N olan bir cisme

$\mathbf{r} = (4.00\hat{\mathbf{i}} + 5.00\hat{\mathbf{j}})$ kuvveti uygulanıyor. Cismin üzerinbdeki tork nedir?

$$\tau = \mathbf{r} \times \mathbf{F} = [(4.00\hat{\mathbf{i}} + 5.00\hat{\mathbf{j}})\ \mathrm{m}] \times [(2.00\hat{\mathbf{i}} + 3.00\hat{\mathbf{j}})\ \mathrm{N}]$$

$$= [(4.00)(2.00)\hat{\mathbf{i}} \times \hat{\mathbf{i}} + (4.00)(3.00)\hat{\mathbf{i}} \times \hat{\mathbf{j}}$$

$$+ (5.00)(2.00)\hat{\mathbf{j}} \times \hat{\mathbf{i}}$$

$$+ (5.00)(3.00)\hat{\mathbf{j}} \times \hat{\mathbf{j}}]\ \mathrm{N \cdot m}$$

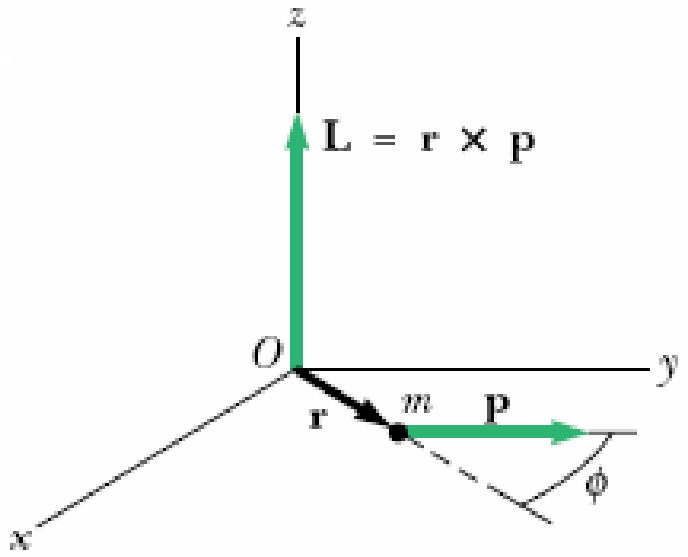
$$= [12.0\hat{\mathbf{i}} \times \hat{\mathbf{j}} + 10.0\hat{\mathbf{j}} \times \hat{\mathbf{i}}]\ \mathrm{N \cdot m} = [12.0\hat{\mathbf{k}} - 10.0\hat{\mathbf{k}})\ \mathrm{N \cdot m} = 2.0\hat{\mathbf{k}}\ \mathrm{N \cdot m}$$

# Açısal momentum

$$\sum \boldsymbol{\tau} = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

eşitliğin sağ tarafama ekleyelim $\frac{d\mathbf{r}}{dt} \times \mathbf{p} = 0$

$$\sum \boldsymbol{\tau} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$




$$\sum \boldsymbol{\tau} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt}$$

$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$ parçacığın açısal momentumu

$$\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

$$L = mvr \sin \phi$$

## Bir parçacık sisteminin açısal momentumu

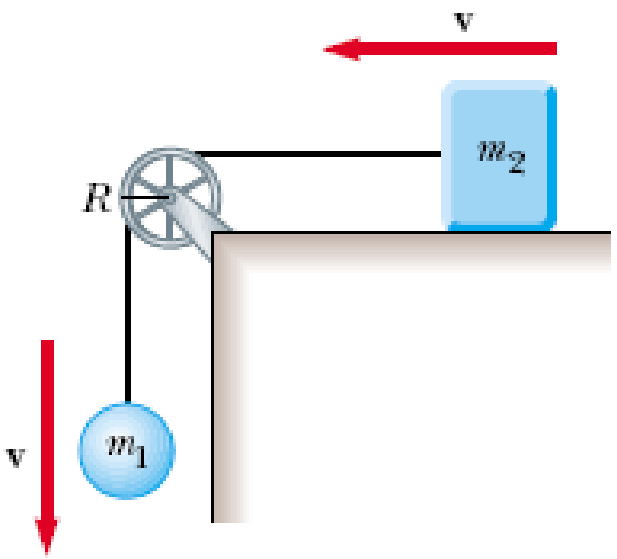
$$\sum \mathbf{F}_{\text{ext}} = \frac{d\mathbf{p}_{top}}{dt}$$

$$\mathbf{L}_{top} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \sum_i \mathbf{L}_i$$

$$\frac{d\mathbf{L}_{top}}{dt} = \sum_i \frac{d\mathbf{L}_i}{dt} = \sum_i \boldsymbol{\tau}_i$$

$$\sum \boldsymbol{\tau}_{dış} = \frac{d\mathbf{L}_{top}}{dt}$$

**Örnek :** Şekildeki sistemin açısal momentumundan hareketle sisitemin ivemsini hesaplayınız




(1) $$L = m_1 vR + m_2 vR + MvR = (m_1 + m_2 + M)vR$$

$$\sum \tau_{dış} = \frac{dL}{dt}$$

$$m_1 gR = \frac{d}{dt}[(m_1 + m_2 + M)vR]$$

(2) $$m_1 gR = (m_1 + m_2 + M)R\,\frac{dv}{dt}$$

$$a = \frac{m_1 g}{m_1 + m_2 + M}$$

## Dönen katı bir cismin açısal momentumu

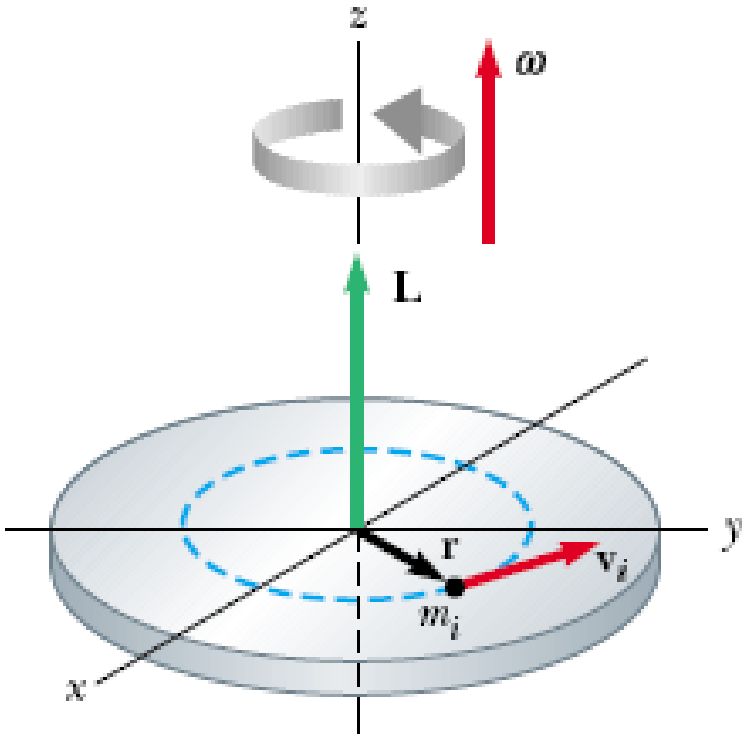
$$L_i = m_i r_i^2 \omega$$

$$L_z = \sum_i L_i = \sum_i m_i r_i^2 \omega = \left( \sum_i m_i r_i^2 \right) \omega$$

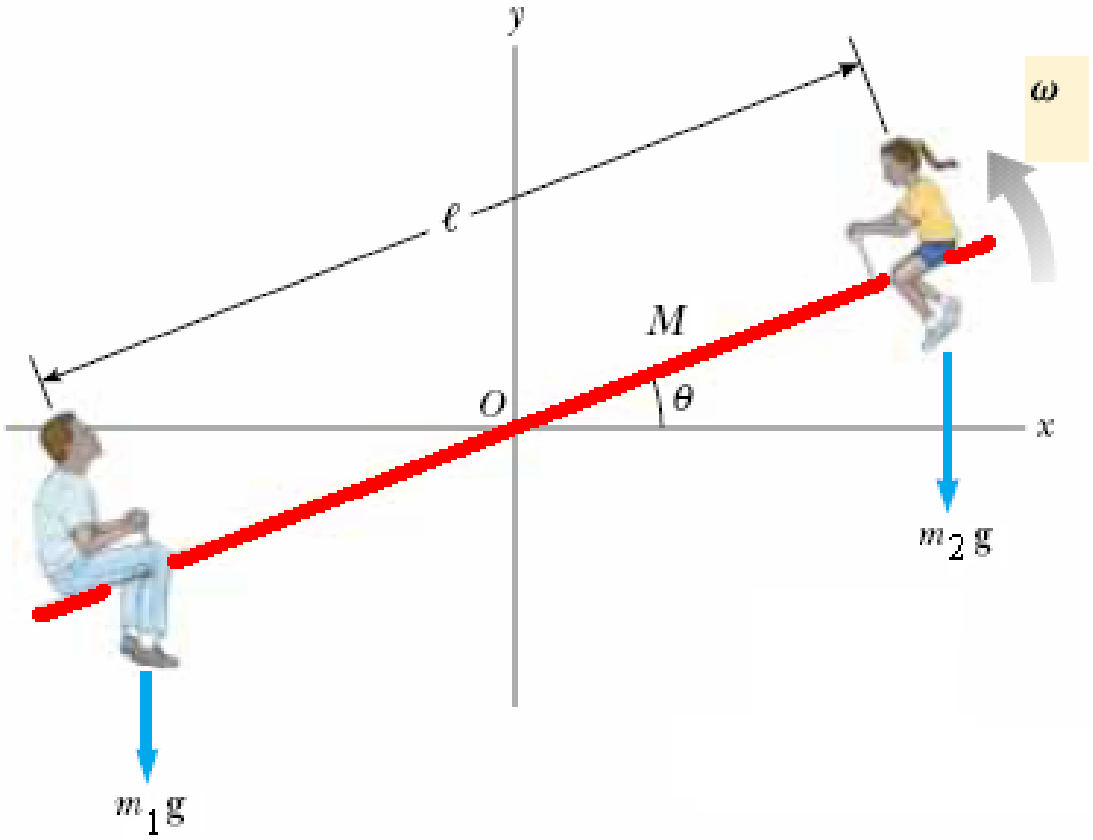
$$L_z = I\omega$$

$$\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha$$

$$\sum \tau_{dış} = I\alpha$$



**Örnek :** Şekildeki sistemin açısal momentumunu hesaplayınız




$$I = \frac{1}{12}M\ell^2 + m_1\left(\frac{\ell}{2}\right)^2 + m_2\left(\frac{\ell}{2}\right)^2 = \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)$$

$$L = I\omega = \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)\omega$$

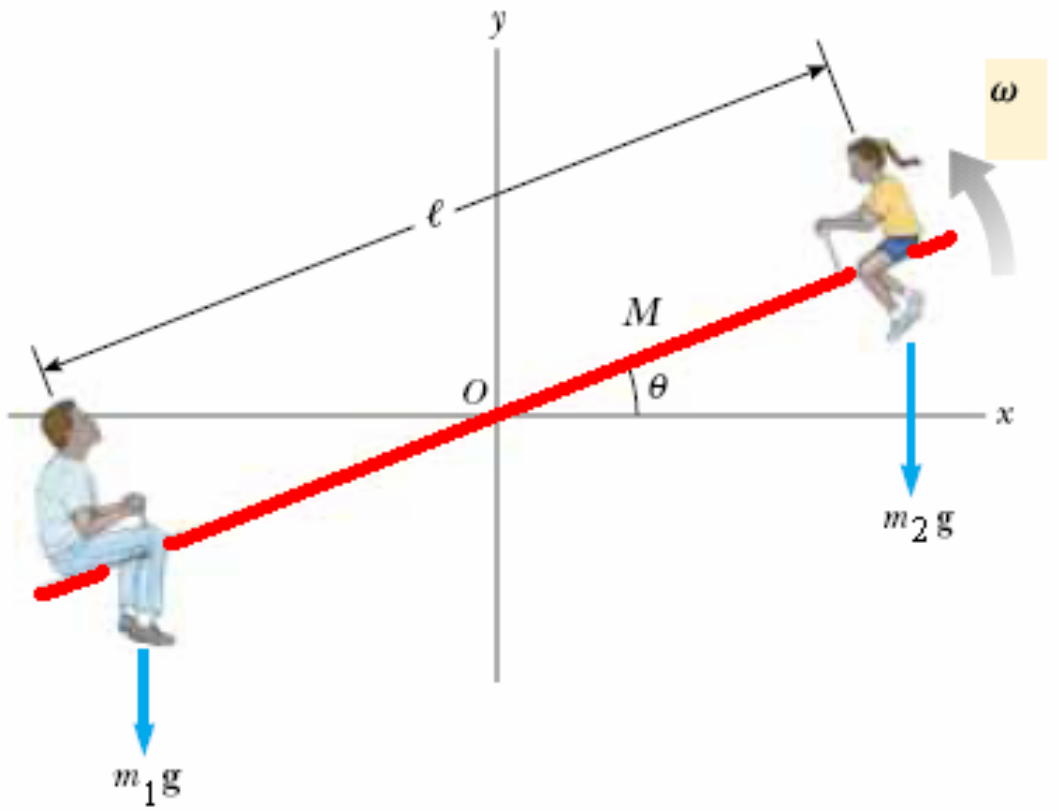
**Örnek :** Sistemin toplam torkunu hesaplayarak buradan hareketle sistemin açısal ivmesini hesaplayınız

$$\tau_1 = m_1 g \frac{\ell}{2} \cos\theta$$

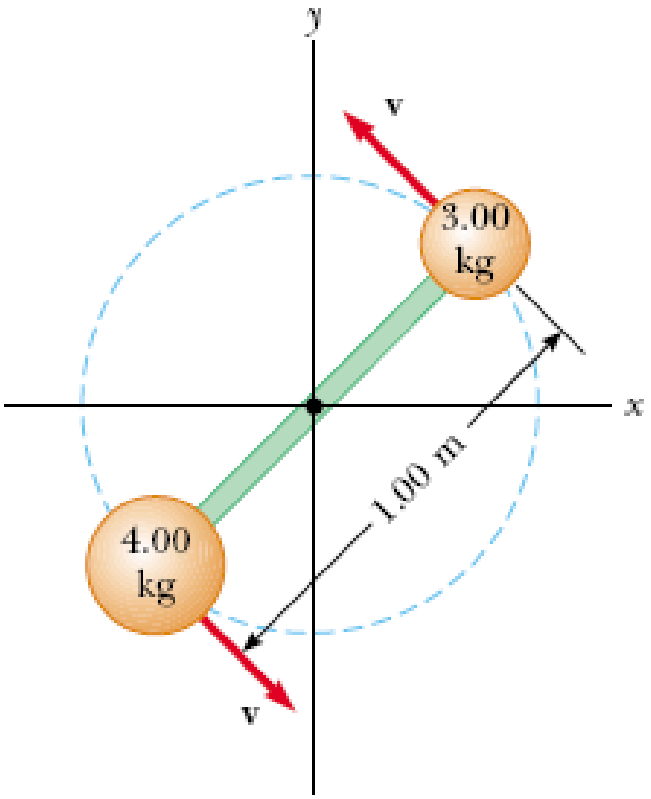
$$\tau_2 = -m_2 g \frac{\ell}{2} \cos\theta$$

$$\sum \tau_{dış} = \tau_1 + \tau_2 = \tfrac{1}{2}(m_1 - m_2) g \ell \cos\theta$$

$$\alpha = \frac{\sum \tau_{dış}}{I} = \frac{2(m_1 - m_2) g \cos\theta}{\ell\left(\frac{M}{3} + m_1 + m_2\right)}$$



**Örnek :** Şekildeki sistem 5 m/sn hızla döndüğünde açısal momentumu ne olur?



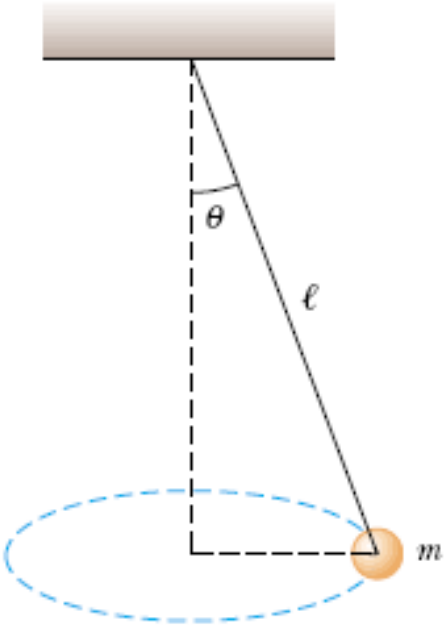

$$L = \sum m_i v_i r_i$$

$$= (4.00\ \text{kg})(5.00\ \text{m/s})(0.500\ \text{m}) + (3.00\ \text{kg})(5.00\ \text{m/s})(0.500\ \text{m})$$

$$L = 17.5\ \text{kg}\cdot\text{m}^2/\text{s}$$

$$\mathbf{L} = \left(17.5\ \text{kg}\cdot\text{m}^2/\text{s}\right)\hat{\mathbf{k}}$$

**Örnek :** Şekildeki gibi hareket eden sistemin açısal momentumunu hesaplayınız.




$$\sum F_x = ma_x \rightarrow T\sin\theta = \frac{mv^2}{r}$$

$$\sum F_y = ma_y \rightarrow T\cos\theta = mg$$

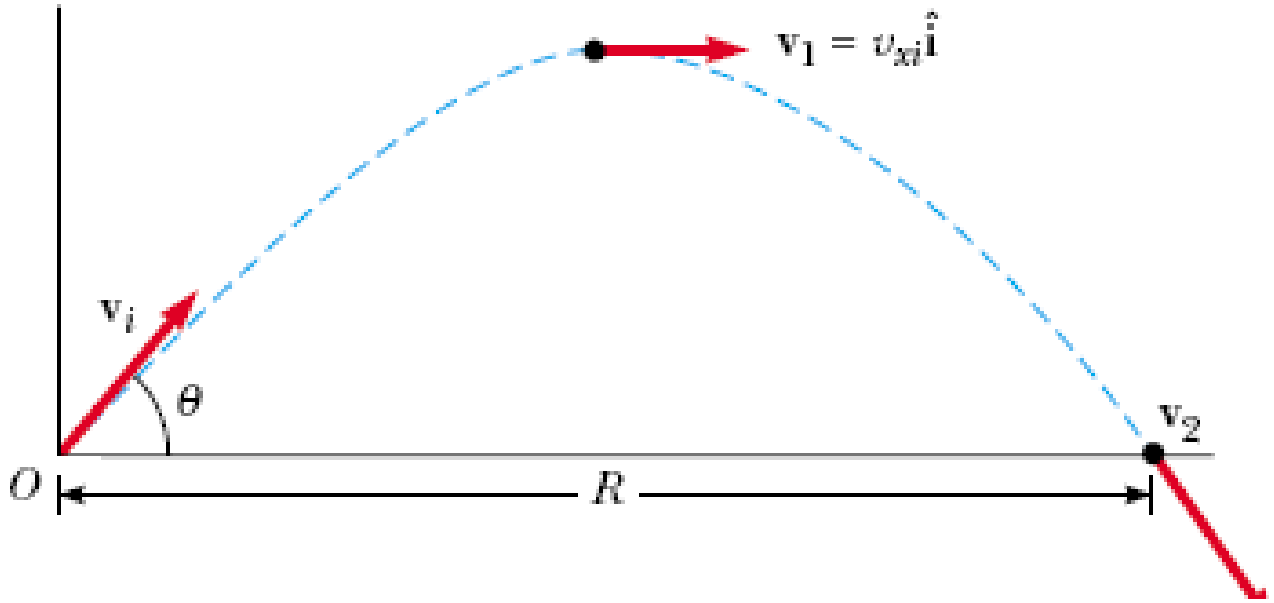
$$\frac{\sin\theta}{\cos\theta} = \frac{v^2}{rg} \rightarrow v = \sqrt{rg\frac{\sin\theta}{\cos\theta}}$$

$$L = rmv\sin 90.0^\circ = rm\sqrt{rg\frac{\sin\theta}{\cos\theta}} = \sqrt{m^2gr^3\frac{\sin\theta}{\cos\theta}}$$

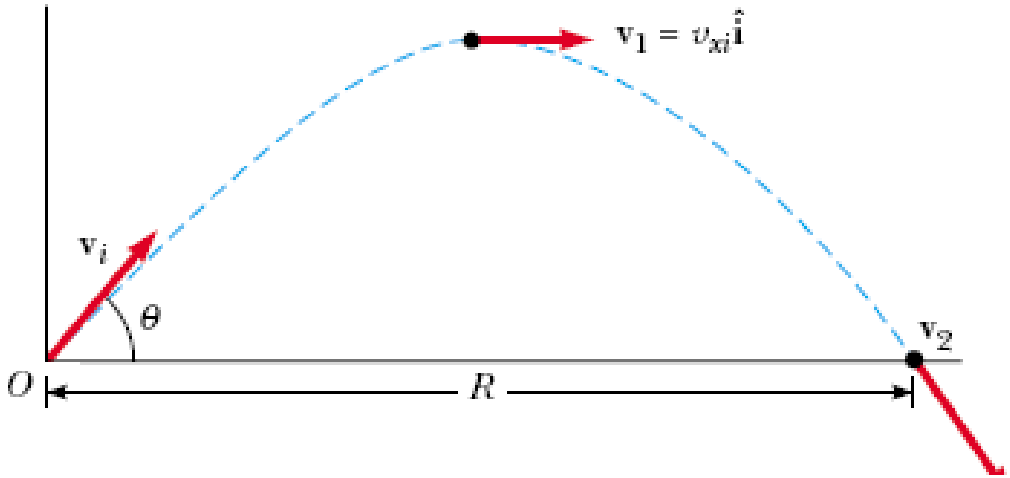
$$r = \ell\sin\theta$$

$$L = \sqrt{m^2g\ell^3\frac{\sin^4\theta}{\cos\theta}}$$

**Örnek :** Şekildeki gibi atılan cismin orijinde, tepe noktasında ve R noktasındaki açısal momentumunu hesaplayınız.




(a) sıfır

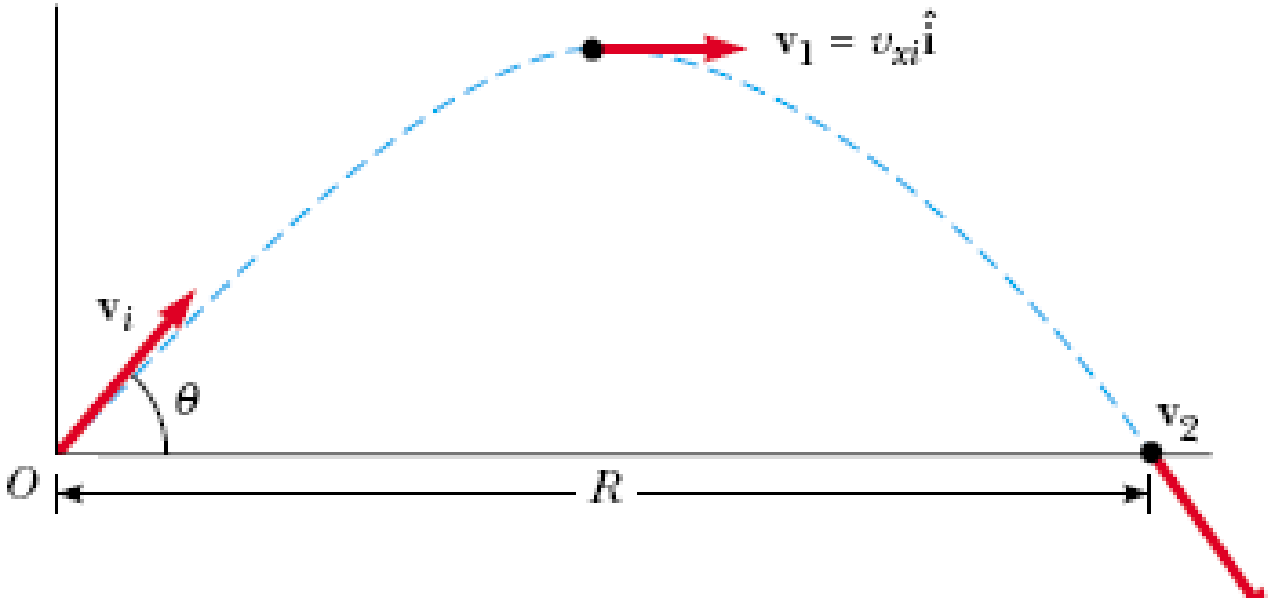



(b) $$x = \frac{1}{2}R = \frac{v_i^2 \sin 2\theta}{2g} \qquad y = h_{\max} = \frac{(v_i \sin\theta)^2}{2g}$$

$$\mathbf{L}_1 = \mathbf{r}_1 \times m\mathbf{v}_1$$

$$= \left[\frac{v_i^2 \sin 2\theta}{2g}\hat{\mathbf{i}} + \frac{(v_i \sin\theta)^2}{2g}\hat{\mathbf{j}}\right] \times m v_{xi}\hat{\mathbf{i}}$$

$$= \boxed{\frac{-m(v_i \sin\theta)^2 v_i \cos\theta}{2g}\hat{\mathbf{k}}}$$

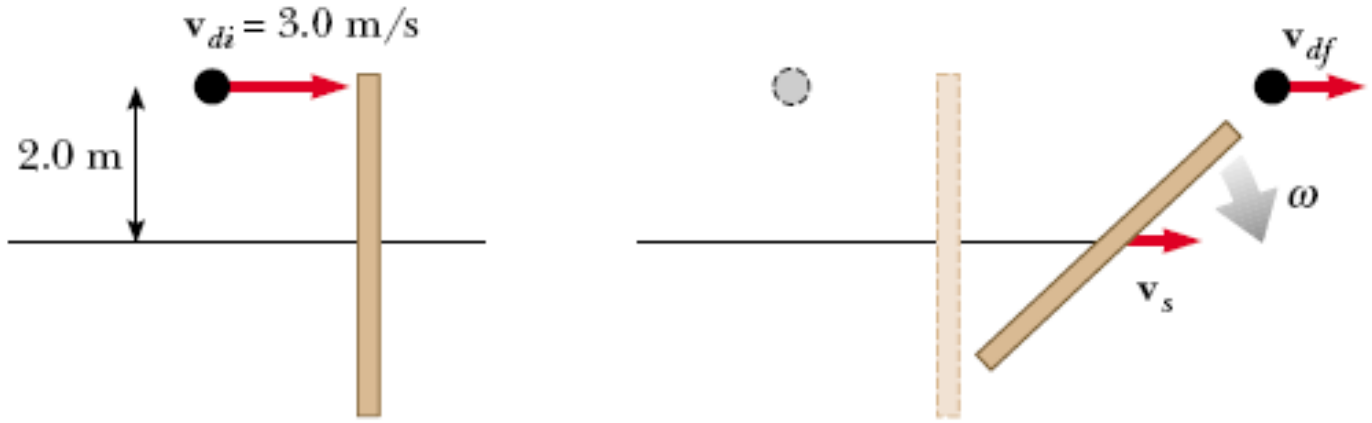



(c) $\mathbf{L}_2 = R\hat{\mathbf{i}} \times m\mathbf{v}_2$ $\qquad R = \dfrac{v_i^2 \sin 2\theta}{g}$

$$= mR\hat{\mathbf{i}} \times \left(v_i \cos\theta\, \hat{\mathbf{i}} - v_i \sin\theta\, \hat{\mathbf{j}}\right)$$

$$= -mRv_i \sin\theta\, \hat{\mathbf{k}} = \frac{-mv_i^3 \sin 2\theta \sin\theta}{g}\hat{\mathbf{k}}$$

**Örnek :** 2 kg kütleli disk 4 m uzunluğunda ve 1 kg kütleli şekildeki çubuğa çarpıyor. Çubuğun eylemsizlik momenti 1.33 kg.m2 olduğuna göre çarpmadan sonra diskin hızını hesaplayınız.

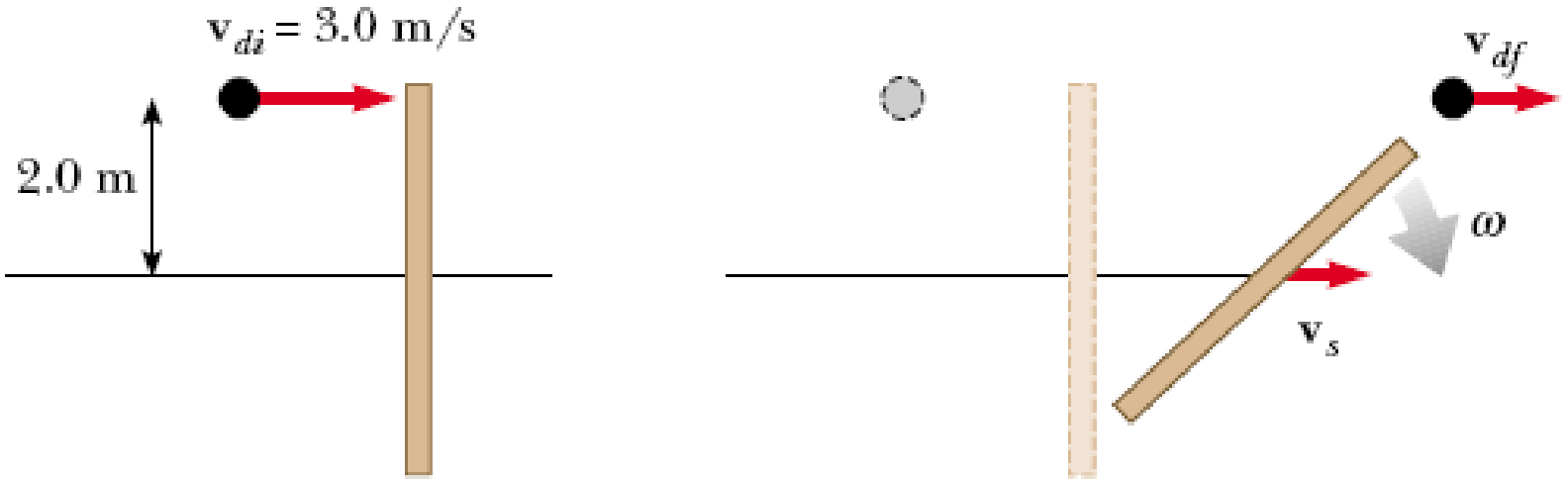



$$p_i = p_f$$

$$m_d v_{di} = m_d v_{df} + m_s v_s$$

$$(2.0\ \text{kg})(3.0\ \text{m/s}) = (2.0\ \text{kg})\, v_{df} + (1.0\ \text{kg})\, v_s$$

$$6.0\ \text{kg}\cdot\text{m/s} - (2.0\ \text{kg})\, v_{df} = (1.0\ \text{kg})\, v_s \qquad (1)$$



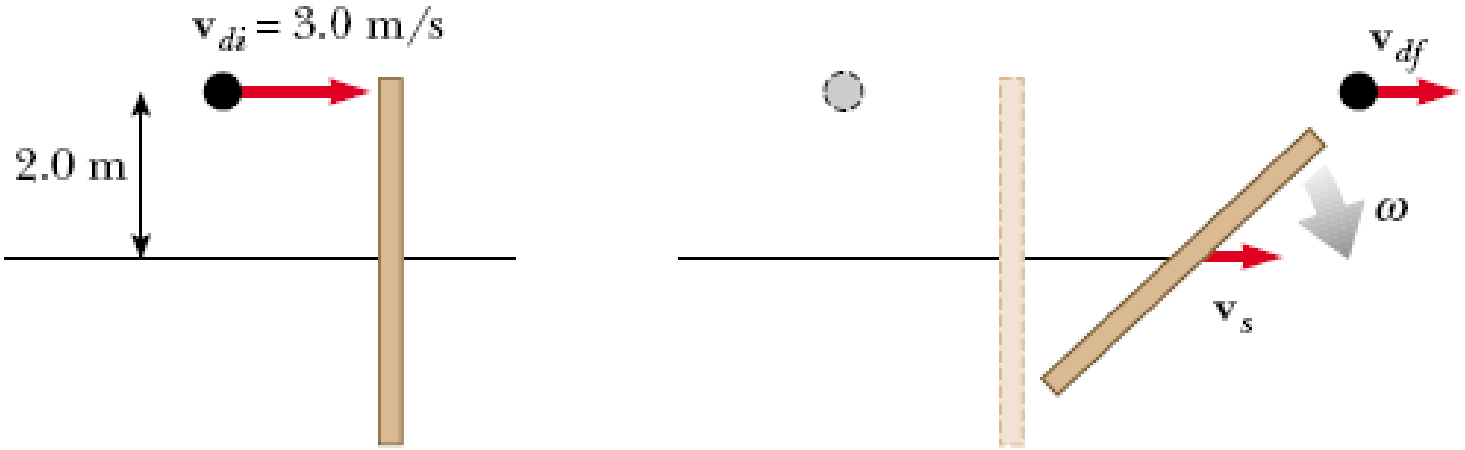

$$L_i = L_f$$

$$-rm_d v_{di} = -rm_d v_{df} + I\omega$$

$$-(2.0\text{ m})(2.0\text{ kg})(3.0\text{ m/s}) = -(2.0\text{ m})(2.0\text{ kg})v_{df} + (1.33\text{ kg}\cdot\text{m}^2)\omega$$

$$-12\text{ kg}\cdot\text{m}^2/\text{s} = -(4.0\text{ kg}\cdot\text{m})v_{df} + (1.33\text{ kg}\cdot\text{m}^2)\omega$$

$$-9.0\text{ rad/s} + (3.0\text{ rad/m})v_{df} = \omega \qquad (2)$$




$$K_i = K_f$$

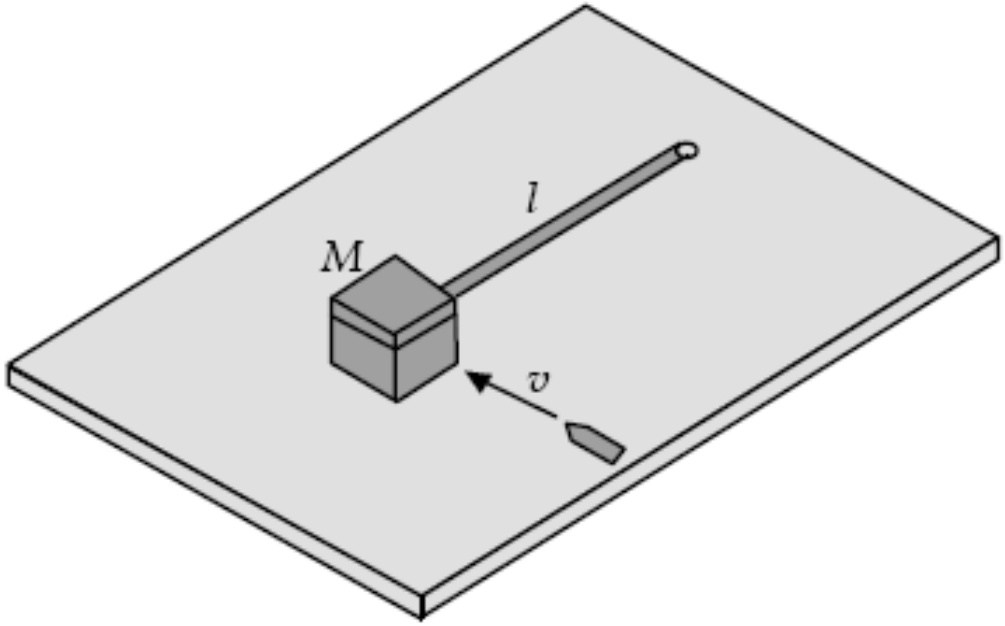
$$\tfrac{1}{2}m_d v_{di}^2 = \tfrac{1}{2}m_d v_{df}^2 + \tfrac{1}{2}m_s v_s^2 + \tfrac{1}{2}I\omega^2$$

$$\tfrac{1}{2}(2.0\text{ kg})(3.0\text{ m/s})^2 = \tfrac{1}{2}(2.0\text{ kg})v_{df}^2 + \tfrac{1}{2}(1.0\text{ kg})v_s^2 + \tfrac{1}{2}(1.33\text{ kg}\cdot\text{m}^2)\omega^2$$

$$18\text{ m}^2/\text{s}^2 = 2.0\,v_{df}^2 + v_s^2 + (1.33\text{ m}^2)\omega^2 \qquad (3)$$

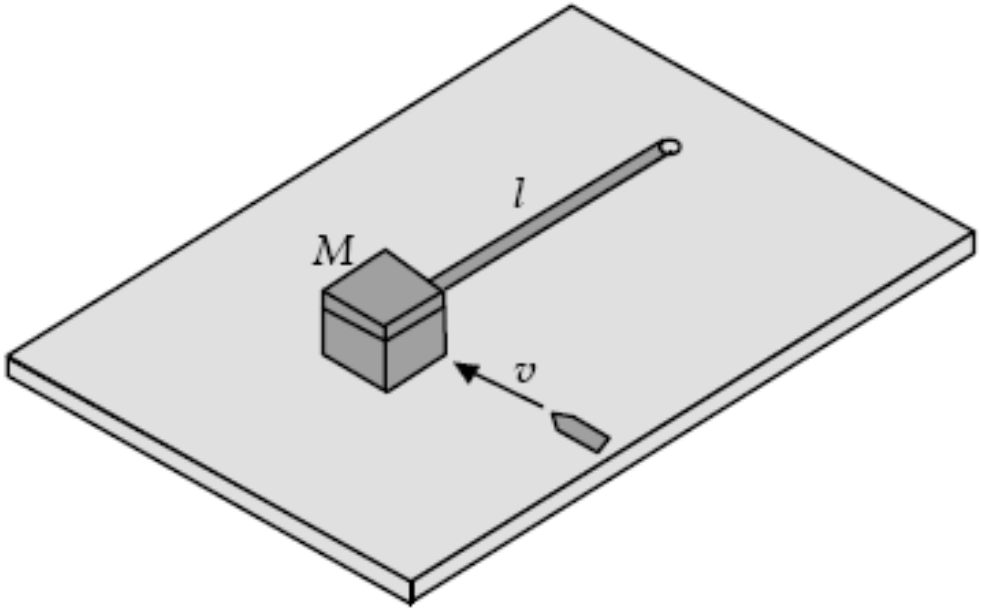
| | | | | | |
|---|---|---|---|---|---|
| $v_{df} =$ | 2.3 m/s | $v_s =$ | 1.3 m/s | $\omega =$ | $-2.0$ rad/s |

**Örnek :** Blok mermi sisteminin açısal momentumu ve kinetik enerjini ne kadarlık oranının sürtünmeye gittiğini hesaplayınız.




(a) $L_i = mv\ell \qquad \sum \tau_{dış} = 0 \qquad L_S = L_i = mv\ell$

$$L_s = (m+M)v_s\ell \qquad v_s = \left(\frac{m}{m+M}\right)v$$

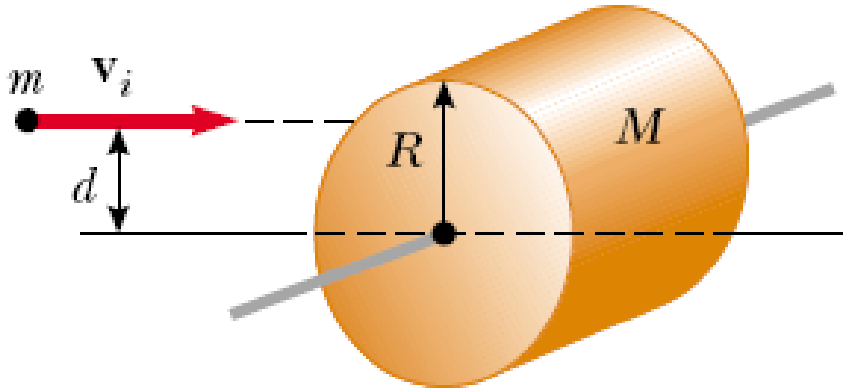



(b) $K_i = \frac{1}{2}mv^2$ $K_s = \frac{1}{2}(M+m)v_s^2$

$$v_s = \left(\frac{m}{M+m}\right)v \Rightarrow$$

$$= \frac{\frac{1}{2}mv^2 - \frac{1}{2}\frac{m^2}{M+m}v^2}{\frac{1}{2}mv^2} = \frac{M}{M+m}$$

**Örnek :** Şekildeki gibi gelen silindirin merkezinden doğruya paralel olarak gelen mermi duran silindire saplanıyor. Silindirin bundan sonraki açısal hızını hesaplayınız.




$$L_i = L_{son}$$

$$mv_i d = I\omega$$

$$mv_i d = \left[\frac{1}{2}MR^2 + mR^2\right]\omega$$

$$\omega = \frac{2mv_i d}{(M+2m)R^2}$$