Chapter 01: Probability Theory (Cont'd)

Section 1.5: Probabilities of Event Intersections

Problem (01):

When a company receives an order, there is a probability of 0.42 that its value is over \$1000. If an order is valued at over \$1000, then there is a probability of 0.63 that the customer will pay with a credit card.

(a) What is the probability that the next three independent orders will each be valued at over \$1000?

(b) What is the probability that the next order will be valued at over \$1000 but will not be paid with a credit card?

(Problem 1.4.2 in textbook)

Solution:

A = the event that the order's value received is over \$1000

B = the event that the customer will pay with a credit card

$P(A) = 0.42$

$P(B \mid A) = 0.63$

$P(B` \mid A) = 1.0 - P(B \mid A) = 1.0 - 0.63 = 0.37$

(a) $P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$

$= 0.42 \times 0.42 \times 0.42$

$= 0.0741$

(b) $P(B` \cap A) = ?!$ (required) :

$$P(B` \mid A) = \frac{P(B` \cap A)}{P(A)}$$

$$0.37 = \frac{P(B` \cap A)}{0.42}$$

$P(B` \cap A) = 0.37 \times 0.42 = 0.1554$

Problem (02):

Show that if the events A and B are independent events, then so are the events:

(a) A and B’ (b) A’ and B (c) A’ and B’

(Problem 1.5.6 in textbook)

Solution:

(a) A and B\` :

We have already proved that (from part a) :

$P(A \cap B) = P(A) \times P(B)$ ……… eqn. 1

Recall that :

$P(B) + P(B`) = 1.0$

$P(B) = 1.0 - P(B`)$

Then :

$P(A) \times P(B) = P(A) \times [\, 1 - P(B`) \,] = P(A) - P(A) \times P(B`)$

And we know that :

$P(A \cap B`) + P(A \cap B) = P(A)$

So that:

$P(A \cap B) = P(A) - P(A \cap B`)$

Now substituting into equation (1) :

$P(A \cap B) = P(A) \times P(B)$

$P(A) - P(A \cap B`) = P(A) - P(A) \times P(B`)$

$- P(A \cap B`) = P(A) - P(A) \times P(B`) - P(A)$

$- P(A \cap B`) = - P(A) \times P(B`)$

$P(A \cap B`) = P(A) \times P(B`)$

The form related to (A and B`) has been finally proved.

(b) A` and B :

We have already proved that (from part a) :

$P(A \cap B) = P(A) \times P(B)$ ……… eqn. 1

Recall that :

$P(A) + P(A`) = 1$

$P(A) = 1 - P(A`)$

Then :

$P(A) \times P(B) = [\, 1 - P(A`)\, ] \times P(B) = P(B) - P(A`) \times P(B)$

And we know that :

$P(A` \cap B) + P(A \cap B) = P(B)$

So that :

$P(A \cap B) = P(B) - P(A` \cap B)$

Now substituting into equation (1) :

$P(A \cap B) = P(A) \times P(B)$

$P(B) - P(A` \cap B) = P(B) - P(A`) \times P(B)$

$- P(A` \cap B) = P(B) - P(A`) \times P(B) - P(B)$

$- P(A` \cap B) = - P(A`) \times P(B)$

$P(A` \cap B) = P(A`) \times P(B)$

The form related to (A` and B) has been finally proved.

(c) A` and B` :

We have already proved that (from part a) :

$P(A \cap B`) = P(A) \times P(B`)$ ……… eqn. 1

Recall that :

$P(A) + P(A`) = 1$

$P(A) = 1 - P(A`)$

Then :

$P(A) \times P(B`) = [\,1 - P(A`)\,] \times P(B`) = P(B`) - P(A`) \times P(B`)$

And we know that :

$P(A \cap B`) + P(A` \cap B`) = P(B`)$

So that :

$P(A \cap B`) = P(B`) - P(A` \cap B`)$

Now substituting into equation (1) :

$P(A \cap B`) = P(A) \times P(B`)$

$P(B`) - P(A` \cap B`) = P(B`) - P(A`) \times P(B`)$

$- P(A` \cap B`) = P(B`) - P(A`) \times P(B`) - P(B`)$

$- P(A` \cap B`) = - P(A`) \times P(B`)$

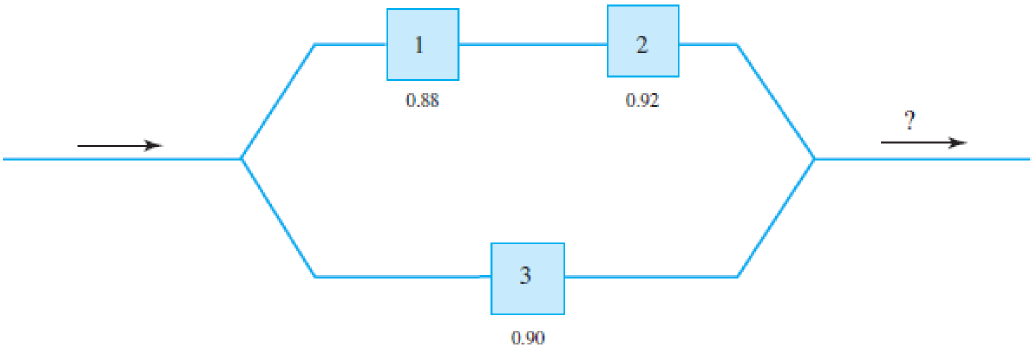
$P(A` \cap B`) = P(A`) \times P(B`)$

The form related to (A` and B`) has been finally proved.

Problem (03):

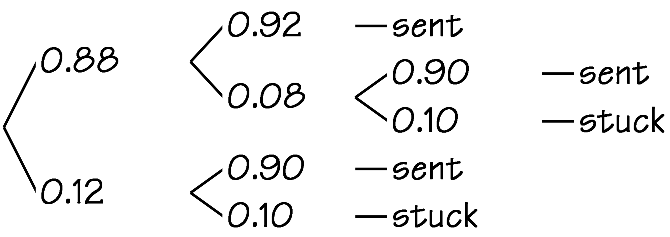
Consider the network given in the Figure below with three switches. Suppose that the switches operate independently of each other and that switch 1 allows a message through with probability 0.88, switch 2 allows a message through with probability 0.92, and switch 3 allows a message through with probability 0.90.

(a) Construct the probability tree for the problem
(b) What is the probability that a message will find a route through the network?



(Problem 1.5.7 in textbook)
Solution:

(a) The probability tree for the problem :



(b) A = the event that a message will be allowed through the network

$$P(A) = (0.88 \times 0.92) + (0.88 \times 0.08 \times 0.90) + (0.12 \times 0.90)$$

$$P(A) = 0.9810$$

Problem (04):

Suppose that 17 lightbulbs in a box of 100 lightbulbs are broken and that 3 are selected at random without replacement.

(a) Construct a probability tree for this problem.
(b) What is the probability that there will be no broken lightbulbs in the sample?

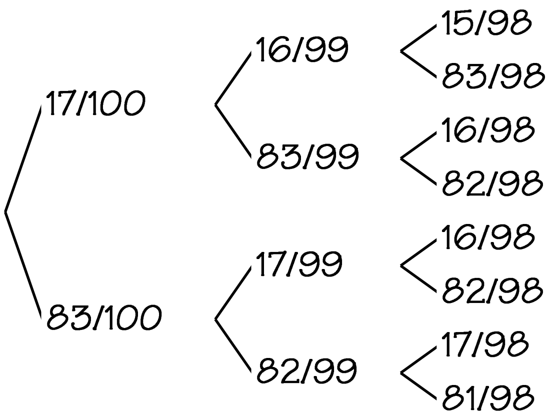
(c) What is the probability that there will be exactly 1 broken lightbulb in the sample?

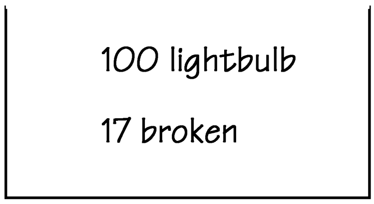
(d) What is the probability that there will be no more than 1 broken lightbulb in the sample?

(Problem 1.5.9 in textbook)

Solution:

(a) The probability tree for the problem :





(b) A = the event that there will be no broken lightbulbs in the sample

$$P(A) = \frac{83}{100} \times \frac{82}{99} \times \frac{81}{98} = 0.5682$$

(c) B = the event that there will be exactly 1 broken lightbulb in the sample

$$P(B) = \left(\frac{17}{100} \times \frac{83}{99} \times \frac{82}{98}\right) + \left(\frac{83}{100} \times \frac{17}{99} \times \frac{82}{98}\right) + \left(\frac{83}{100} \times \frac{82}{99} \times \frac{17}{98}\right)$$
$$= 0.3578$$

(d) C = the event that there will be no more than 1 broken lightbulb in the sample

$$P(C) = 0.5682 + 0.3578 = 0.9260$$

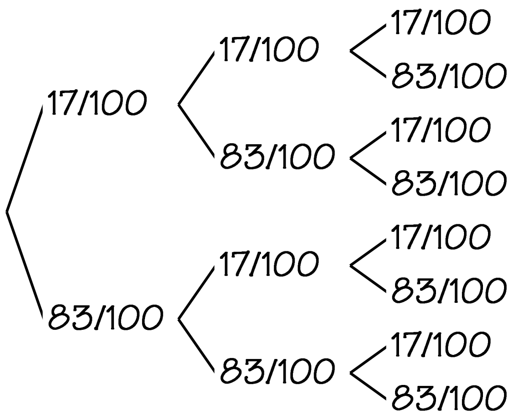
Problem (05):

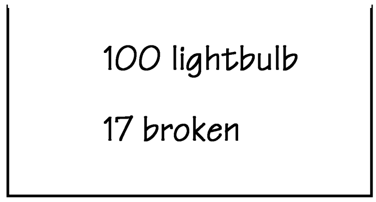
Repeat the previous problem, except that the drawings are made with replacement. Compare your answers with those from the previous problem.

(Problem 1.5.10 in textbook)

Solution:

(a) The probability tree for the problem :





(b) A = the event that there will be no broken lightbulbs in the sample

$$P(A) = \frac{83}{100} \times \frac{83}{100} \times \frac{83}{100} = 0.5718$$

(c) B = the event that there will be exactly 1 broken lightbulb in the sample

$$P(B) = \left(\frac{17}{100} \times \frac{83}{100} \times \frac{83}{100}\right) + \left(\frac{83}{100} \times \frac{17}{100} \times \frac{83}{100}\right) + \left(\frac{83}{100} \times \frac{83}{100} \times \frac{17}{100}\right)$$
$$= 0.3513$$

(d) C = the event that there will be no more than 1 broken lightbulb in the sample

$$P(C) = 0.5718 + 0.3513 = 0.9231$$

Problem (06):

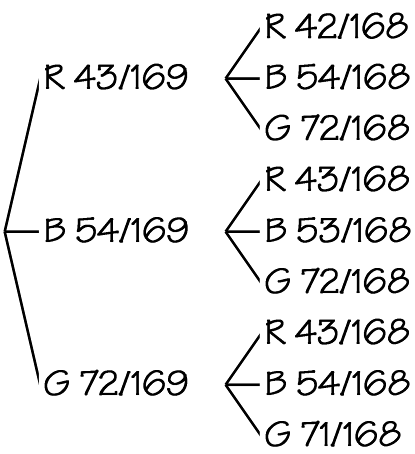
Suppose that a bag contains 43 red balls, 54 blue balls, and 72 green balls, and that 2 balls are chosen at random without replacement.

(a) Construct a probability tree for this problem.
(b) What is the probability that 2 green balls will be chosen?
(c) What is the probability that the 2 balls chosen will have different colours?

(Problem 1.5.11 in textbook)

Solution:

(a) The probability tree for the problem :



(b) A = the event that there 2 green balls will be chosen

$$P(A) = \frac{72}{169} \times \frac{71}{168} = 0.1801$$

(c) B = the event that the 2 balls chosen will have the same colours

C = the event that the 2 balls chosen will have the different colours

$$P(C) = 1.0 - P(B)$$

$$= 1.0 - \left[\left(\frac{43}{169} \times \frac{42}{168}\right) + \left(\frac{54}{169} \times \frac{53}{168}\right) + \left(\frac{72}{169} \times \frac{71}{168}\right)\right]$$

$$= 1.0 - 0.3445 = 0.6556$$

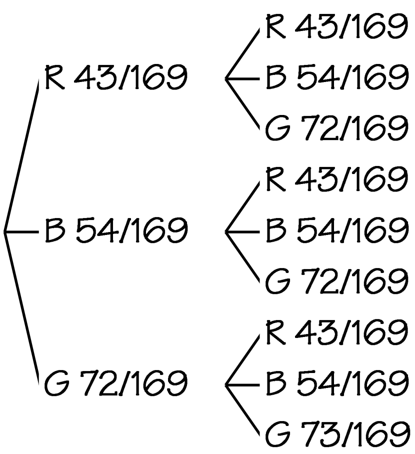
Problem (07):

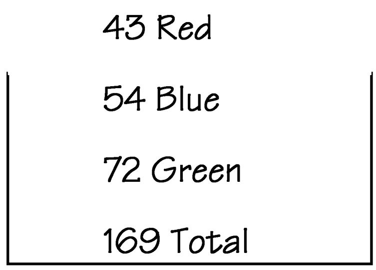
Repeat the previous problem, except that the drawings are made with replacement. Compare your answers with those from the previous problem.

(Problem 1.5.12 in textbook)

Solution:

(a) The probability tree for the problem :





(b) A = the event that there 2 green balls will be chosen

$$P(A) = \frac{72}{169} \times \frac{72}{169} = 0.1815$$

(c) B = the event that the 2 balls chosen will have the same colours

C = the event that the 2 balls chosen will have the different colours

$$P(C) = 1.0 - P(B)$$

$$= 1.0 - \left[\left(\frac{43}{169} \times \frac{43}{169}\right) + \left(\frac{54}{169} \times \frac{54}{169}\right) + \left(\frac{72}{169} \times \frac{72}{169}\right)\right]$$

$$= 1.0 - 0.3483 = 0.6517$$

Problem (08):

A biased coin has a probability p of resulting in a head. If the coin is tossed twice, what value of p minimizes the probability that the same result is obtained on both throws?

(Problem 1.5.13 in textbook)

Solution:

A = the event that the same result obtained on both throws of a coin

P(head) = P

P(tail) = 1 − P

P(A) = P(head) × P(head) + P(tail) × P(tail)

P(A) = P × P + (1 − P) × (1 − P)

$P(A) = P^2 + (1 - P)^2$

$P(A) = P^2 + P^2 - 2P + 1$

$P(A) = 2P^2 - 2P + 1$

$P(A) = 2(P - 0.5)^2 + 0.5$

P(A) is minimized when P = 0.5 (a fair coin)

Problem (09):

(a) If a fair die is rolled six times, what is the probability that each score is obtained exactly once?

(b) If a fair die is rolled seven times, what is the probability that a 6 is not obtained at all?

(Problem 1.5.14 in textbook)

Solution:

(a) A = the event that a 6 is obtained once when a fair die is rolled six times

$$P(A) = \frac{1}{6} \times \frac{2}{6} \times \frac{3}{6} \times \frac{4}{6} \times \frac{5}{6} \times \frac{6}{6} = 0.0154$$

(b) B = the event that a 6 is not obtained when a fair die is rolled seven times

$$P(B) = \left(\frac{5}{6}\right)^7 = 0.2791$$

Problem (10):

Suppose that n components are available, and that each component has a probability of 0.90 of operating correctly, independent of the other components. What value of n is needed so that there is a probability of at least 0.995 that at least one component operates correctly?

(Problem 1.5.16 in textbook)

Solution:

$1.0 - (1.0 - 0.90)^n - 0.995$

is satisfied for $n \geq 3$

Problem (11):

A system has four computers. Computer 1 works with a probability of 0.88; computer 2 works with a probability of 0.78; computer 3 works with a probability of 0.92; computer 4 works with a probability of 0.85. Suppose that the operations of the computers are independent of each other.

(a) Suppose that the system works only when all four computers are working. What is the probability that the system works?

(b) Suppose that the system works only if at least one computer is working. What is the probability that the system works?

(c) Suppose that the system works only if at least three computers are working. What is the probability that the system works?

(Problem 1.5.18 in textbook)

Solution:

(a) When all four computers are working :

$$P(\text{sys works}) = P(\text{all computers working})$$

$$P(\text{sys works}) = 0.88 \times 0.78 \times 0.92 \times 0.85 = 0.537$$

(B) If at least one computer is working :

$$P(\text{sys works}) = P(\text{at least one computer is working})$$

$$P(\text{sys works}) = 1 - P(\text{zero computers working})$$

$$P(\text{sys works}) = 1 - [\,(1 - 0.88) \times (1 - 0.78) \times (1 - 0.92) \times (1 - 0.85)\,]$$

$$P(\text{sys works}) = 0.9997$$

(c) If at least three computers are working :

P(sys works) = P(at least three computer is working)

P(sys works) = P(all computers working)
+ P(computers 1, 2 and 3 working, but 4 not working)
+ P(computers 1, 2 and 4 working, but 3 not working)
+ P(computers 1, 3 and 4 working, but 2 not working)
+ P(computers 2, 3 and 4 working, but 1 not working)

$$
\begin{aligned}
P(\text{sys works}) &= 0.537 + [\,0.88 \times 0.78 \times 0.92 \times (1 - 0.85)\,] \\
&\quad + [\,0.88 \times 0.78 \times (1 - 0.92) \times 0.85\,] \\
&\quad + [\,0.88 \times (1 - 0.78) \times 0.92 \times 0.85\,] \\
&\quad + [\,(1 - 0.88) \times 0.78 \times 0.92 \times 0.85\,] \\
&= 0.903
\end{aligned}
$$

Problem (12):

Suppose that there are two companies such that for each one the revenue is considerably below expectation with probability 0.08, is slightly below expectation with probability 0.19, exactly meets expectation with probability 0.26, is slightly above expectation with probability 0.36, and is considerably above expectation with probability 0.11. Furthermore, suppose that the revenues from both companies are independent. What is the probability that neither company has a revenue below expectation?

(Problem 1.5.19 in textbook)

Solution:

P(considerably below expectation) = 0.08
P(slightly below expectation) = 0.19
P(exactly meets expectation) = 0.26
P(slightly above expectation) = 0.36
P(considerably above expectation) = 0.11

A = the event that the 1st company has not a revenue below expectation
B = the event that the 2nd company has not a revenue below expectation
C = the event that neither company has a revenue below expectation

P(A) = P(exactly meets expectation) + P(slightly above expectation)
+ P(considerably above expectation)

P(A) = 0.26 + 0.36 + 0.11 = 0.73

P(B) = P(exactly meets expectation) + P(slightly above expectation) + P(considerably above expectation)
P(B) = 0.26 + 0.36 + 0.11 = 0.73

P(C) = P(A) × P(B) = 0.73 × 0.73 = 0.5329

Problem (13):

A box contains 10 balls: 4 red and 6 blue. A second box contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each box. The probability that both balls have the same colour is 0.44. Calculate the number of blue balls in the second box.

(Question 1: (8 points) in Midterm Exam 2007)

Solution:

10 balls 4 Red + 6 Blue	? balls 16 Red + ? Blue

A = the event that both balls drawn have the same colour
P(A) = P(Red) x P(Red) + P(Blue) x P(Blue)
Total = X
Blue = Total – Red
Blue = X – 16

$$0.44 = \frac{4}{10} \times \frac{16}{X} + \frac{6}{10} \times \frac{X-16}{X}$$

$$0.44 = \frac{4}{10} \times \frac{16}{X} + \frac{6}{10} \times \frac{X-16}{X}$$

$X = 20$
Total = 20
Blue = Total – Red = 20 – 16 = 4

Problem (14):

Suppose that A and B are mutually exclusive events for which P(A) = 0.3 and P(B) = 0.5. What is the probability that:

(a) (1 point) Either A or B occur.
(b) (1 point) A occurs but B does not.

(c) (1 point) Both A and B occur.
(d) (1 point) Are events A and B independent events? Justify your answer.

(Question 1: (4 points) in Midterm Exam 2009)

Solution:

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 0.3 + 0.5 - 0.0 = 0.8$

(b) $P(A) = 0.3$

(c) $P(A \cap B) = P(\emptyset) = 0.0$

(d) $P(A \cap B) = P(A) \times P(B)$ (check ?!) :

$0.0 = 0.3 \times 0.5$

$0.0 = 0.8$ (Not ok)

i.e. events A and B are not independent.

Problem (15):

Human blood may contain either or both of two antigens, A and B. Blood that contains only the A antigen is called type A, blood that contains only the B antigen is called type B, blood that contains both antigens is called AB, and blood that contains neither A nor B is called type O. At a certain blood bank, 35% of the blood donors have type A, 10% have type B, and 5% have type AB.

(a) (1 point) What is the probability that a randomly chosen blood donor is type O?
(b) (1 point) A recipient with type A blood may safely receive blood from a donor whose blood does not contain the B antigen. What is the probability that a randomly chosen blood donor may donate to a recipient with type A blood?

(c) (2 points) If it is known that Ahmad does not have blood type B, what is the probability that Ahmad has blood type O?

(d) (2 points) What is the probability that two randomly chosen persons have the same blood type?

(Question 2: (4 points) in Midterm Exam 2009)

Solution:

P(A) = 0.35

P(B) = 0.10

P(AB) = 0.05

(a) P(O) = 1.0 – [P(A) + P(B) + P(AB)]

P(O) = 1.0 – [0.35 + 0.10 + 0.05] = 0.50

(b) P(AR) = P(A) + P(O)

= 1.0 – [P(B) + P(AB)]

P(AR) = 1.0 – [0.10 + 0.05] = 0.85

(c) P(O | B`) = ?! (required) :

$$P(O \mid B`) = \frac{P(O \ \cap \ B`)}{P(B`)} = \frac{P(O) \times P(B`)}{P(B`)}$$

$$P(O \mid B`) = \frac{P(O) \times [\, P(A) + P(O) + P(AB) \,]}{P(A) + P(O) + P(AB)}$$

$$P(O \mid B`) = \frac{0.50 \times [\, 0.35 + 0.50 + 0.05 \,]}{0.35 + 0.50 + 0.05} = 0.50$$

Or:

$$P(O \mid B`) = P(O) = 0.50$$

(d) P(same type) = P(A) x P(A) + P(B) x P(B)

+ P(AB) x P(AB) + P(O) x P(O)

$$P(\text{same type}) = 0.35 \times 0.35 + 0.05 \times 0.05 + 0.10 \times 0.10 + 0.50 \times 0.50 = 0.385$$

Problem (16):

A bowl contains seven blue chips and three red chips. Two chips are to be chosen at random and without replacement:

(a) Compute the probability that the 1st draw results in a red chip?

(b) If it was known that the 1st chip is red, what is the probability that the 2nd draw results in a blue chip?

(c) Compute the probability of the red in the 1st draw and blue in the 2nd draw?

(d) Compute the probability of drawing a red chip in each of the two draws?

(Question 1: (4 points) in Midterm Exam 2010)

Solution:

10 Chips

7 Blue + 3 Red

(a) $P(R) = 3/10$

(b) $P(2^{nd}\ B \mid 1^{st}\ R) = 7/9$

(c) $P(1^{st}\ R \text{ and } 2^{nd}\ B) = (3/10) \times (7/9) = 7/30$

(d) $P(1^{st}\ R \text{ and } 2^{nd}\ R) = (3/10) \times (2/9) = 1/15$

Problem (17):

Fifty-two percent of the students at the college of engineering are females. Five percent of the students in this college are majoring in industrial engineering. Two percent of the students are women majoring in industrial engineering.

(a) (2 points) What is the probability that a randomly selected student is majoring in industrial engineering, given that the student is female?
(b) (2 points) What is the probability that a randomly selected student is a male student that is not majoring in industrial engineering?
(c) (2 points) If two students are randomly selected, what is the probability that both students are not majoring in industrial engineering?

(Question 2: (6 paints) in Midterm Exam 2011)

Solution:

A = the event that a randomly selected student is female

B = the event that a randomly selected student is majoring in industrial engineering

P(A) = 0.52

P(A\`) = 0.48

P(B) = 0.05

P(B\`) = 0.95

P(A ∩ B) = 0.02

(a) P(B | A)

$$P(B \mid A) = \frac{P(B \ \cap \ A)}{P(A)} = \frac{0.02}{0.52} = 0.3846$$

(b) P(A\` ∩ B\`) = P(A ∪ B)\` = 1.0 – P(A ∪ B)

= 1.0 – [P(A) + P(B) – P(A ∩ B)]

= 1.0 – [0.52 + 0.05 – 0.02]

= 0.45

(c) P(1stB\` and 2ndB\`) = P(B\`) x P(B\`)

= 0.95 x 0.95

= 0.9025

Problem (18):

Computer keyboard failures are due to faulty electrical connects (12%) or mechanical defects (88%). Mechanical defects are related to loose keys (27%) or improper assembly (73%). Electrical connect defects are caused by defective wires (35%), improper connections (13%), or poorly welded wires (52%).

(a) Find the probability that a failure is due to loose keys.
(b) Find the probability that a failure is due to improperly connected or poorly welded wires.

(Unknown-source problem)

Solution:

(a) P(mechanical defects and loose keys) = 0.88 x 0.27 = 0.2376

(b) P(improper assembly or poorly welded wires) = 0.12 x (0.13 + 0.52)

= 0.078

Problem (19):

A steel plate contains 20 bolts. Assume that 5 bolts are not torqued to the proper limit. Four bolts are selected at random, without replacement, to be checked for torque.

(a) What is the probability that all four of the selected bolts are torqued to the proper limit?
(b) What is the probability that at least one of the selected bolts is not torqued to the proper limit?

(Unknown-source problem)

Solution:

(a) A = the event that the i^{th} bolt selected is not torqued to the proper limit

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_4 \mid A_1 \cap A_2 \cap A_3) \times P(A_1 \cap A_2 \cap A_3)$$

$$= P(A_4 \mid A_1 \cap A_2 \cap A_3) \times P(A_3 \mid A_1 \cap A_2) \times P(A_2 \mid A_1) \times P(A_1)$$

$$= \left(\frac{12}{17}\right)\left(\frac{13}{18}\right)\left(\frac{14}{19}\right)\left(\frac{15}{20}\right) = 0.282$$

(b) B = the event that at least one of the selected bolts is not properly torqued

B` = the event that all bolts are properly torqued

P(B) = 1 – P(B`) = $1 - \left(\frac{15}{20}\right)\left(\frac{14}{19}\right)\left(\frac{13}{18}\right)\left(\frac{12}{17}\right) = 0.718$