Chapter 5: The Normal Distribution

Section 5.1: Probability Calculations Using the Normal Distribution

Problem (01):

Suppose that $Z \sim N(0, 1)$. Find:

(a) $P(Z \leq 1.34)$

(b) $P(Z \geq -0.22)$

(c) $P(-2.19 \leq Z \leq 0.43)$

(d) $P(0.09 \leq Z \leq 1.76)$

(e) $P(|Z| \leq 0.38)$

(f) The value of x for which $P(Z \leq x) = 0.55$

(g) The value of x for which $P(Z \geq x) = 0.72$

(h) The value of x for which $P(|Z| \leq x) = 0.31$

(Problem 5.1.1 in textbook)

Solution:

(a) $P(Z \leq 1.34) = \Phi(1.34) = 0.9099$

(b) $P(Z \geq -0.22) = 1 - P(Z \leq -0.22) = 1 - \Phi(-0.22) = 0.5871$

(c) $P(-2.19 \leq Z \leq 0.43) = \Phi(0.43) - \Phi(-2.19) = 0.6521$

(d) $P(0.09 \leq Z \leq 1.76) = \Phi(1.76) - \Phi(0.09) = 0.4249$

(e) $P(|Z| \leq 0.38) = P(-0.38 \leq Z \leq 0.38)$

$= \Phi(0.38) - \Phi(-0.38)$

$= 0.2960$

(f) $P(Z \le x) = 0.55$

$$P\left(\frac{x-\mu}{\sigma}\right) = 0.55$$

$$\Phi\left(\frac{x-0.0}{1.0}\right) = 0.55$$

$$\frac{x-0.0}{1.0} = 0.1257$$

$x = 0.1257$

(g) $\Phi(Z \ge x) = 1.0 - \Phi(Z \le x) = 0.72$

$$1.0 - P\left(\frac{x-0.0}{1.0}\right) = 0.72$$

$$\Phi\left(\frac{x-0.0}{1.0}\right) = 1.0 - 0.72$$

$$\Phi\left(\frac{x-0.0}{1.0}\right) = 0.28$$

$$\frac{x-0.0}{1.0} = -0.5828$$

$x = -0.5828$

(h) $P(|Z| \le x) = P(-x \le Z \le x) = \Phi(x) - \Phi(-x) = 0.31$

$\Phi(x) - [1.0 - \Phi(x)] = 0.31$

$2 \times \Phi(x) = 1.0 - 0.31$

$2 \times \Phi(x) = 1.31$

$\Phi(x) = 1.31 / 2$

$\Phi(x) = 0.655$

$x = 0.3989$

Problem (02):

Suppose that $X \sim N(10, 2)$. Find:

(a) $P(X \leq 10.34)$

(b) $P(X \geq 11.98)$

(c) $P(7.67 \leq X \leq 9.90)$

(d) $P(10.88 \leq X \leq 13.22)$

(e) $P(|X - 10| \leq 3)$

(f) The value of x for which $P(X \leq x) = 0.81$

(g) The value of x for which $P(X \geq x) = 0.04$

(h) The value of x for which $P(|X - 10| \geq x) = 0.63$

(Problem 5.1.3 in textbook)

Solution:

(a) $P(X \leq 10.34) = \Phi\left(\frac{10.34-10}{\sqrt{2}}\right) = \Phi(0.2404) = 0.5950$

(b) $P(X \geq 11.98) = 1.0 - \Phi\left(\frac{11.98-10}{\sqrt{2}}\right)$

$$= 1.0 - \Phi(1.4001)$$

$$= 1.0 - 0.9193$$

$$= 0.0807$$

(c) $P(7.67 \leq X \leq 9.90) = \Phi\left(\frac{9.90-10}{\sqrt{2}}\right) - \Phi\left(\frac{7.67-10}{\sqrt{2}}\right)$

$$= \Phi(-0.0707) - \Phi(-1.6475)$$

$$= 0.4221$$

(d) $P(10.88 \leq X \leq 13.22) = \Phi\left(\frac{13.22-10}{\sqrt{2}}\right) - \Phi\left(\frac{10.88-10}{\sqrt{2}}\right)$

$$= \Phi(2.2769) - \Phi(0.6223)$$

$$= 0.2555$$

(e) $P(|X-10| \leq 3) = P(7 \leq X \leq 13)$

$= \Phi\left(\frac{13-10}{\sqrt{2}}\right) - \Phi\left(\frac{7-10}{\sqrt{2}}\right)$

$= \Phi(\, 2.1213) - \Phi(\, -2.1213)$

$= 0.9662$

(f) $P(X \leq x) = \Phi\left(\frac{x-10}{\sqrt{2}}\right) = 0.81$

$$\frac{x-10}{\sqrt{2}} = 0.8779$$

$x = 11.2415$

(g) $P(X \geq x) = \Phi\left(\frac{x-10}{\sqrt{2}}\right) = 0.04$

$$\frac{x-10}{\sqrt{2}} = 1.7507$$

$x = 12.4758$

(h) $P(|X-10| \geq x) = 0.63$

$$P(X \geq 10+x) + P(X \leq 10-\ x) = 0.63$$

$$\Phi(X \geq 10+x) + \Phi(X \leq 10-\ x) = 0.63$$

$$\Phi\left(\frac{(10+x)-10}{\sqrt{2}}\right) + \Phi\left(\frac{(10-\ x)-10}{\sqrt{2}}\right) = 0.63$$

$$\Phi\left(\frac{x}{\sqrt{2}}\right) + \Phi\left(\frac{-x}{\sqrt{2}}\right) = 0.63 \quad \ldots\ \ldots\ \ldots\ eqn.1$$

Since : $\Phi(\mathrm{x}) + \Phi(-\mathrm{x}) = 1.0$

$$\Phi\left(\frac{x}{\sqrt{2}}\right) + \Phi\left(\frac{-x}{\sqrt{2}}\right) = 1.0$$

$$\Phi\left(\frac{-x}{\sqrt{2}}\right) = 1.0 - \Phi\left(\frac{x}{\sqrt{2}}\right) \ldots\ldots\ldots eqn.2$$

Therefore, substituting from equation (2) into equation (1) results in :

$$\Phi\left(\frac{x}{\sqrt{2}}\right) + \left[1.0 - \Phi\left(\frac{x}{\sqrt{2}}\right)\right] = 0.63$$

$$2\Phi\left(\frac{x}{\sqrt{2}}\right) = 1.63$$

$$\Phi\left(\frac{x}{\sqrt{2}}\right) = \frac{1.63}{2}$$

$$\Phi\left(\frac{x}{\sqrt{2}}\right) = 0.815$$

$$\frac{x}{\sqrt{2}} = 0.4817$$

x = 0.6812

Problem (03):

Suppose that X ~ N(μ, σ^2) and that: P(X ≤ 5) = 0.8 and P(X ≥ 0) = 0.6
What are the values of μ and σ^2?
(Problem 5.1.5 in textbook)

Solution:

$P(X \leq 5) = \Phi\left(\frac{5.0-\mu}{\sigma}\right) = 0.80$

$$\frac{5-\mu}{\sigma} = 0.845 \qquad \dots \dots \dots\ eqn.1$$

$P(X \geq 0) = \Phi\left(\frac{0.0-\mu}{\sigma}\right) = 0.60$

$$\frac{0.0-\mu}{\sigma} = -0.255 \qquad \dots \dots \dots\ eqn.2$$

Solving equations (1) and (2) results in :

μ = 1.1569

σ = 4.5663

Problem (04):

(a) What are the upper and lower quartiles of a N(0, 1) distribution?
(b) What is the interquartile range?
(c) What is the interquartile range of a $N(\mu, \sigma^2)$ distribution?

(Problem 5.1.8 in textbook)

Solution:

(a) Upper and lower quartiles :

Upper quartile :

$\Phi(x) = 0.75$

$x = 0.6745$

Lower quartile :

$\Phi(x) = 0.25$

$x = -0.6745$

(b) Interquartile range :

Interquartile range (IQR) = Upper quartile (UQ) − Lower quartile (LQ)

$= 0.6745 - (-0.6745)$

$= 1.3490$

(c) Interquartile range of a $N(\mu , \sigma^2)$ distribution :

Interquartile range (IQR) $= 1.3490 \times \sigma$

Problem (05):

The thicknesses of glass sheets produced by a certain process are normally distributed with a mean of $\mu = 3.00$ mm and a standard deviation of $\sigma = 0.12$ mm.

(a) What is the probability that a glass sheet is thicker than 3.2 mm?
(b) What is the probability that a glass sheet is thinner than 2.7 mm?

(c) What is the value of c for which there is a 99% probability that a glass sheet has a thickness within the interval [3.00 − c, 3.00 + c]?

(Problem 5.1.9 in textbook)

Solution:

(a) $P(X \geq 3.2)$:

$$P(X \geq 3.2) = 1.0 - \Phi\left(\frac{3.2 - 3.0}{0.12}\right)$$

$$= 1.0 - \Phi(1.6667)$$

$$= 1.0 - 0.9522$$

$$= 0.0478$$

(b) $P(X \leq 2.7)$:

$$P(X \geq 2.7) = 1.0 - \Phi\left(\frac{2.7 - 3.0}{0.12}\right)$$

$$= 1.0 - \Phi(-2.5000)$$

$$= 1.0 - 0.9938$$

$$= 0.0062$$

(c) $P(3.0 - C \leq X \leq 3.0 + C) = 0.99$

$C = \sigma \times Z_{0.005} = 0.12 \times 2.5758 = 0.3091$

Problem (06):

The amount of sugar contained in 1 kg packets is actually normally distributed with a mean of $\mu = 1.03$ kg and a standard deviation of $\sigma = 0.014$ kg.

(a) What proportion of sugar packets are underweight?
(b) If an alternative package-filling machine is used for which the weights of the packets are normally distributed with a mean of $\mu = 1.05$ kg and a standard deviation of $\sigma = 0.016$ kg, does this result in an increase or a decrease in the proportion of underweight packets?

(c) In each case, what is the expected value of the excess package weight above the advertised level of 1 kg?

(Problem 5.1.10 in textbook)

Solution:

(a) $N(1.03, 0.014^2)$

$P(X \leq 1) = \Phi\left(\frac{1.0-1.03}{0.014}\right) = \Phi(-2.1429) = 0.0161$

(b) $N(1.05, 0.016^2)$

$P(X \leq 1) = \Phi\left(\frac{1.0-1.05}{0.016}\right) = \Phi(-3.125) = 0.0009$

There is a decrease in the proportion of underweight packets

(c) The expected excess weight is $\mu - 1$ which is 0.03 and 0.05

Problem (07):

The thicknesses of metal plates made by a particular machine are normally distributed with a mean of 4.3 mm and a standard deviation of 0.12 mm.

(a) What is the interquartile range of the metal plate thicknesses?

(b) What is the value of c for which there is 80%probability that a metal plate has a thickness within the interval [4.3 – c , 4.3 + c]?

(Problem 5.1.11 in textbook)

Solution:

$X \sim N(4.3, 0.12^2)$

(a) Interquartile range :

Upper quartile :

$P(X \leq x) = 0.75$

$$P\left(Z \leq \frac{x-4.3}{0.12}\right) = 0.75$$

$$\Phi\left(\frac{x-4.3}{0.12}\right) = 0.75$$

$$\frac{x-4.3}{0.12}=0.675$$

x = 4.3809

Lower quartile :

P(X ≤ x) = 0.25

$$P\left(Z \le \frac{x-4.3}{0.12}\right)=0.25$$

$$\Phi\left(\frac{x-4.3}{0.12}\right)=0.25$$

$$\frac{x-4.3}{0.12}=-0.675$$

x = 4.2191

Interquartile range (IQR) = Upper quartile (UQ) – Lower quartile (LQ)

= 4.3809 – 4.2191

= 0.1618

(b) The value of c :

P(4.3 – c ≤ X ≤ 4.3 + c) = 0.80

$$P\left(\frac{(4.3-c)-4.3}{0.12} \le Z \le \frac{(4.3+c)-4.3}{0.12}\right)=0.8$$

$$P(-8.33c \le Z \le 8.33c)=0.8$$

$$\Phi(8.33c)-\Phi(-8.33c)=0.8 \ldots\ldots\ldots eqn.1$$

Since : $\Phi(x)+\Phi(-x)=1.0$

$$\Phi(8.33c)+\Phi(-8.33c)=1.0$$

$$\Phi(-8.33c)=1.0-\Phi(8.33c) \ldots\ldots\ldots eqn.2$$

Therefore, substituting from equation (2) into equation (1) results in :

$$\Phi(8.33c) - [1.0 - \Phi(8.33c)] = 0.8$$

$$2\Phi(8.33c) = 1.8$$

$$\Phi(8.33c) = \frac{1.8}{2}$$

$$\Phi(8.33c) = 0.9$$

$$8.33c = 1.285$$

$$c = 0.1538$$

Alternative solution :

$P(4.3 - C \leq X \leq 4.3 + C) = 0.80$

$P(4.3 - \sigma \times Z_\alpha \leq X \leq 4.3 + \sigma \times Z_\alpha) = 0.80$

$C = \sigma \times Z_\alpha = 0.12 \times Z_{0.10} = 0.12 \times 1.282 = 0.1538$

Problem (08):

The resistance in milliohms of 1 meter of copper cable at a certain temperature is normally distributed with a mean of $\mu = 23.8$ and a variance of $\sigma^2 = 1.28$.

(a) What is the probability that a 1 meter segment of copper cable has a resistance less than 23.0?
(b) What is the probability that a 1 meter segment of copper cable has a resistance greater than 24.0?
(c) What is the probability that a 1 meter segment of copper cable has a resistance between 24.2 and 24.5?
(d) What is the upper quartile of the resistance level?
(e) What is the 95th percentile of the resistance level?

(Problem 5.1.13 in textbook)

Solution:

(a) $P(X \leq 23.0) = \mathrm{P}\left(Z \leq \frac{23.0-23.8}{1.28}\right)$

$$= \Phi(-0.625)$$

$$= 0.2398$$

(b) $P(X \geq 24.0) = 1.0 - P(X \leq 24.0)$

$$= 1.0 - P\left(Z \leq \frac{24.0 - 23.8}{1.28}\right)$$

$$= \Phi(0.15625)$$

$$= 0.4298$$

(c) $P(24.2 \leq X \leq 24.5) = P\left(\frac{24.5-23.8}{1.28} \leq X \leq \frac{24.5-23.8}{1.28}\right)$

$$= P\left(Z \leq \frac{24.5 - 23.8}{1.28}\right) - P\left(Z \leq \frac{24.2 - 23.8}{1.28}\right)$$

$$= \Phi(0.5469) - \Phi(0.3125)$$

$$= 0.0937$$

(d) $P(X \leq x) = 0.75$

$$P\left(Z \leq \frac{x - 23.8}{1.28}\right) = 0.75$$

$$\Phi\left(\frac{x - 23.8}{1.28}\right) = 0.75$$

$$\frac{x - 23.8}{1.28} = 0.675$$

$x = 24.56$

(e) $P(X \leq x) = 0.95$

$$P\left(Z \leq \frac{x - 23.8}{1.28}\right) = 0.95$$

$$\Phi\left(\frac{x - 23.8}{1.28}\right) = 0.95$$

$$\frac{x - 23.8}{1.28} = 1.64$$

x = 25.66

Problem (09):

The weights of bags filled by a machine are normally distributed with a standard deviation of 0.05 kg and a mean that can be set by the operator. At what level should the mean be set if it is required that only 1% of the bags weigh less than 10 kg?

(Problem 5.1.14 in textbook)

Solution:

$X \sim N(\mu, 0.05^2)$

$P(X \leq 10) = 0.01$

$$P\left(Z \leq \frac{10-\mu}{0.05}\right) = 0.01$$

$$\frac{10-\mu}{0.05} = 2.3263$$

$\mu = 10.1163$

Alternative solution :

$\mu_x = \mu + \sigma \times Z_{0.01} = 10 + 0.05 \times 2.3263 = 10.1163$

Problem (10):

Suppose a certain mechanical component produced by a company has a width that is normally distributed with a mean of $\mu = 2600$ and a standard deviation of $\sigma = 0.6$.

(a) What proportion of the components have a width outside the range 2599 to 2601?

(b) If the company needs to be able to guarantee to its purchaser that no more than 1 in 1000 of the components have a width outside the range 2599 to2601, by how much does the value of σ need to be reduced?

(Problem 5.1.15 in textbook)

Solution:

(a) Proportion of the components have a width outside the range :

Range = 2599 to 2601

The proportion of components have a width inside the range :

$$
\begin{aligned}
\mathrm{P}(2599 \le \mathrm{X} \le 2601) &= \mathrm{P}\left(\frac{2599-2600}{0.6} \le \mathrm{Z} \le \frac{2601-2600}{0.6}\right) \\
&= \mathrm{P}(-1.67 \le \mathrm{Z} \le 1.67) \\
&= \Phi(1.67) - \Phi(-1.67) \\
&= 0.9525 - 0.0475 \\
&= 0.905
\end{aligned}
$$

Therefore, the proportion of components which have a width outside the range = 1.0 – 0.905 = 0.095

(b) The value of σ need to be reduced :

P(outside) = 1 / 1000 = 0.001

P(inside) = 1 – 0.001 = 0.999

$$\mathrm{P}(2599 \le \mathrm{X} \le 2601) = 0.999$$

$$\mathrm{P}\left(\frac{2599-2600}{\sigma} \le \mathrm{Z} \le \frac{2601-2600}{\sigma}\right) = 0.999$$

$$\mathrm{P}\left(-\frac{1}{\sigma} \le \mathrm{Z} \le \frac{1}{\sigma}\right) = 0.999$$

$$\Phi\left(\frac{1}{\sigma}\right) - \Phi\left(-\frac{1}{\sigma}\right) = 0.999 \quad \dots\dots\dots eqn.1$$

$$\Phi\left(\frac{1}{\sigma}\right) + \Phi\left(-\frac{1}{\sigma}\right) = 1.0$$

$$\Phi\left(-\frac{1}{\sigma}\right) = 1.0 - \Phi\left(\frac{1}{\sigma}\right) \qquad \dots\dots\dots eqn.2$$

Therefore, substituting from equation (2) into equation (1) results in :

$$\Phi\left(\frac{1}{\sigma}\right) - \left[1.0 - \Phi\left(\frac{1}{\sigma}\right)\right] = 0.999$$

$$\Phi\left(\frac{1}{\sigma}\right) - 1.0 + \Phi\left(\frac{1}{\sigma}\right) = 0.999$$

$$2\Phi\left(\frac{1}{\sigma}\right) = 1.999$$

$$\Phi\left(\frac{1}{\sigma}\right) = 0.9995$$

$$\frac{1}{\sigma} = 3.29$$

$$\sigma = 0.303$$

Problem (11):

Manufactured items have a strength that has a normal distribution with a standard deviation of 4.2. The mean strength can be altered by the operator. At what value should the mean strength be set so that exactly 95% of the items have a strength less than 100?

(Problem 5.1.17 in textbook)

Solution:

$X \sim N(\mu, 4.2^2)$

$P(X \leq 100) = 0.95$

$$P\left(Z \leq \frac{100-\mu}{4.2}\right) = 0.95$$

$$\frac{100-\mu}{4.2} = Z_{0.05}$$

$$\frac{100-\mu}{4.2} = 1.645$$

$\mu = 93.09$

Section 5.2: Linear Combinations of Normal Random Variables

Problem (01):

Suppose that $X \sim N(3.2, 6.5)$, $Y \sim N(-2.1, 3.5)$, and $Z \sim N(12.0, 7.5)$ are independent random variables. Find the probability that:

(a) $X + Y \geq 0$

(b) $X + Y - 2Z \leq -20$

(c) $3X + 5Y \geq 1$

(d) $4X - 4Y + 2Z \leq 25$

(e) $|X + 6Y + Z| \geq 2$

(f) $|2X - Y - 6| \leq 1$

(Problem 5.2.1 in textbook)

Solution:

(a) $X + Y \geq 0.0$

$\mu = 3.2 + (-2.1) = 1.1$

$\sigma^2 = 6.5 + 3.5 = 10$

$$P(N(1.1\,,10) \geq 0.0) = P\left(Z \geq \frac{0.0 - 1.1}{\sqrt{10}}\right)$$

$$= 1 - P\left(Z \leq \frac{0.0 - 1.1}{\sqrt{10}}\right)$$

$$= 1 - \Phi(-0.3162)$$

$$= 0.6360$$

(b) $X + Y - 2Z \leq -20$

$\mu = 3.2 + (-2.1) - (2 \times 12.0) = -22.9$

$\sigma^2 = 6.5 + 3.5 + (2^2 \times 7.5) = 40$

$$P(N(-22.9\,,40) \geq -20) = P\left(Z \geq \frac{-20 - (-22.9)}{\sqrt{40}}\right)$$

$$= \Phi(0.4585)$$

$$= 0.6767$$

(c) $3X + 5Y \geq 1$

$\mu = (3 \times 3.2) + (5 \times (-2.1)) = -0.9$

$\sigma^2 = (3^2 \times 6.5) + (5^2 \times 3.5) = 146$

$$\mathrm{P}(\mathrm{N}(-0.9\,,146) \geq 1) = \mathrm{P}\left(\mathrm{Z} \geq \frac{1-(-0.9)}{\sqrt{146}}\right)$$

$$= 1 - \mathrm{P}\left(\mathrm{Z} \leq \frac{1-(-0.9)}{\sqrt{146}}\right)$$

$$= 1 - \Phi(0.1211)$$

$$= 0.4375$$

(d) $4X - 4Y + 2Z \leq 25$

$\mu = (4 \times 3.2) - (4 \times (-2.1)) + (2 \times 12.0) = 45.2$

$\sigma^2 = (4^2 \times 6.5) + (4^2 \times 3.5) + (2^2 \times 7.5) = 190$

$$\mathrm{P}(\mathrm{N}(45.2\,,190) \geq 25) = \mathrm{P}\left(\mathrm{Z} \leq \frac{25-45.2}{\sqrt{190}}\right)$$

$$= \Phi(0.1211)$$

$$= 0.0714$$

(f) $|\,X + 6Y + Z\,| \geq 2$

$\mu = 3.2 + (6 \times (-2.1)) + 12.0 = 2.6$

$\sigma^2 = 6.5 + (6^2 \times 3.5) + 7.5 = 140$

$$\mathrm{P}(\mathrm{N}(2.6\,,140) \geq 2) = \mathrm{P}\left(\mathrm{Z} \geq \frac{2-2.6}{\sqrt{140}}\right)$$

$$= 1 - \mathrm{P}\left(\mathrm{Z} \leq \frac{2-2.6}{\sqrt{140}}\right)$$

$$= 1 - \Phi(0.1211)$$

$$= 0.8689$$

(f) $| 2X - Y - 6 | \leq 1$

$\mu = (2 \times 3.2) - (-2.1) - 6 = 2.5$

$\sigma^2 = (2^2 \times 6.5) + 3.5 = 29.5$

$$P(N(2.5\,, 29.5) \leq 2) = P\left(Z \leq \frac{2-2.5}{\sqrt{29.5}}\right)$$

$$= 1 - \Phi(0.1211)$$

$$= 0.8689$$

Problem (02):

Consider a sequence of independent random variables Xi, each with a standard normal distribution.

(a) What is $P(|Xi| \leq 0.5)$?

(b) If $\bar{X}$ is the average of eight of these random variables, what is $P(|\bar{X}| \leq 0.5)$?

(c) In general, if $\bar{X}$ is the average of n of these random variables, what is the smallest value of n for which $P(|\bar{X}| \leq 0.5) \geq 0.99$?

(Problem 5.2.3 in textbook)

Solution:

(a) $P(| Xi | \leq 0.5) = P(0.5 \leq Xi \leq 0.5)$

$$= \Phi(0.5) - \Phi(-0.5)$$

$$= \Phi(0.5) - \Phi(-0.5)$$

$$= 0.6915 - 0.3085$$

$$= 0.3830$$

Problem (03):

A machine part is assembled by fastening two components of type A and three components of type B, end to end. The lengths of components of type A in mm are independent N(37.0, 0.49) random variables, and the lengths of components of type B in mm are independent N(24.0, 0.09) random

variables. What is the probability that a machine part has a length between 144 and 147 mm?
(Problem 5.2.5 in textbook)

Solution:

X1 = type A random variable
X2 = type B random variable

X1 ~ N(37.0, 0.49)
X2 ~ N(24.0, 0.09)

Y = machine random variable
Y = X1 x X1 + X2 + X2 + X2

$\mu_Y = \mu_{X1} + \mu_{X1} + \mu_{X2} + \mu_{X2} + \mu_{X2} = 37 + 37 + 24 + 24 + 24 = 146$

$\sigma_Y^2 = \sigma_{X1}^2 + \sigma_{X1}^2 + \sigma_{X2}^2 + \sigma_{X2}^2 + \sigma_{X2}^2 = 0.49 + 0.49 + 0.09 + 0.09 + 0.09 = 1.25$

Y ~ N(146, 1.25)

$$\begin{aligned} P(144 \le Y \le 147) &= P\left(\frac{144-146}{\sqrt{1.25}} \le Z \le \frac{147-146}{\sqrt{1.25}}\right) \\ &= P(-1.78 \le Z \le 0.89) \\ &= \Phi(0.89) - \Phi(-1.78) \\ &= 0.8133 - 0.0375 \\ &= 0.7758 \end{aligned}$$

Problem (04):

The thicknesses of glass sheets produced by a certain process are normally distributed with a mean of μ = 3.00 mm and a standard deviation of σ = 0.12 mm.

(a) What is the probability that three glass sheets placed one on top of another have a total thickness greater than 9.50 mm?
(b) What is the probability that seven glass sheets have an average thickness less than 3.10 mm?

(Problem 5.2.8 in textbook)

Solution:

(a) Three glass sheets placed one on top of another have a total thickness greater than 9.50 mm :

$Y = X1 + X2 + X3$

$\mu_Y = a_1 \times \mu_{X1} + a_2 \times \mu_{X2} + a_2 \times \mu_{X2} = 1 \times 3 + 1 \times 3 + 1 \times 3 = 9$

$\sigma_Y^2 = a_1^2 \times \sigma_{X1}^2 + a_2^2 \times \sigma_{X2}^2 + a_3^2 \times \sigma_{X3}^2 = 1 \times 0.12^2 + 1 \times 0.12^2 + 1 \times 0.12^2$
$= 0.0432$

$Y \sim N(9, 0.0432)$

$$
\begin{aligned}
P(Y \geq 9.5) &= 1 - P(Y \leq 9.5) \\
&= 1 - P\left(Z \leq \frac{9.5 - 9}{\sqrt{0.0432}}\right) \\
&= 1 - P(Z \leq 2.4) \\
&= 1 - \Phi(2.4) \\
&= 1 - 0.9918 \\
&= 0.0082
\end{aligned}
$$

(b) Seven glass sheets have an average thickness less than 3.10 mm :

$$
\begin{aligned}
&\mu = 3.0 \\
&\sigma^2 = \frac{0.12^2}{7} = 0.00205 \\
&P(X \leq 3.1) = P\left(Z \leq \frac{3.1 - 3}{\sqrt{0.00205}}\right) \\
&\qquad = \Phi(2.2) \\
&\qquad = 0.9861
\end{aligned}
$$

Problem (05):

Sugar packets have weights with N(1.03, 0.0142) distributions. A box contains 22 sugar packets.

(a) What is the distribution of the total weight of sugar in a box?
(b) What are the upper and lower quartiles of the total weight of sugar in a box?

(Problem 5.2.9 in textbook)

Solution:

(a) The distribution of the total weight of sugar in a box :

$\mu = \mu_1 + \mu_2 + \dots + \mu_{22} = 22 \times 1.03 = 22.6$

$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_{X3}^2 = 22 \times 0.014^2 = 4.312 \times 10^{-3}$

$X \sim N(22.66, 4.312 \times 10^{-3})$

(b) The upper and lower quartiles of the total weight of sugar in a box :

Upper quartile :

$\Phi(x) = 0.75$

$$P\left(Z \leq \frac{x - 22.66}{\sqrt{4.312 \times 10^{-3}}}\right) = 0.75$$

$x = 22.704$

Lower quartile :

$\Phi(x) = 0.25$

$$P\left(Z \leq \frac{x - 22.66}{\sqrt{4.312 \times 10^{-3}}}\right) = 0.25$$

$x = 22.616$

Problem (06):

The amount of timber available from a certain type of fully grown tree has a mean of 63400 with a standard deviation of 2500.

(a) What are the mean and the standard deviation of the total amount of timber available from 20 trees?

(b) What are the mean and the standard deviation of the average amount of timber available from 30 trees?

(Problem 5.2.17 in textbook)

Solution:

(a) The mean and the standard deviation of the total amount of timber available from 20 trees :

$E(X) = 20\mu = 20 \times 63400 = 1268000$

$$\sigma^2 = 20 \times \sigma^2 = 20 \times (2500)^2 = 125000000$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{125000000} = 11180.3399$$

(b) The mean and the standard deviation of the average amount of timber available from 30 trees :

$E(X) = \mu = 63400$

$$\sigma^2 = \frac{\sigma^2}{n} = \frac{(2500)^2}{30} = 208333.3333$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{208333.3333} = 456.4351$$

Problem (07):

A chemist can set the target value for the elasticity of a polymer compound. The resulting elasticity is normally distributed with a mean equal to the target value and a standard deviation of 47.

(a) What target value should be set if it is required that there is only a 10% probability that the elasticity is less than 800?

(b) Suppose that a target value of 850 is used. What is the probability that the average elasticity of ten samples is smaller than 875?

(Problem 5.2.18 in textbook)

Solution:

(a) $P(X < 800) = 0.10$

$$P(X < 800) = \Phi\left(\frac{800 - \mu}{47}\right) = 0.10$$

$$\frac{800 - \mu}{47} = -Z_{0.10}$$

$$\frac{800 - \mu}{47} = -1.282$$

$$\mu = 860.3$$

(b) The probability that the average elasticity of 10 samples is smaller than 875 :

$E(X) = \mu = 850$

$$\sigma^2 = \frac{\sigma^2}{n} = \frac{(47)^2}{10} = 220.9$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{220.9} = 14.8324$$

$$P(Y < 875) = \Phi\left(\frac{875 - 850}{14.8324}\right) = \Phi(1.6855) = 0.954$$