ANALYSIS ON SIERPINSKI'S TRIANGLE XXX.

BERNARDO AMENEYRO, YAIR A. CASTILLO, AND JAIME D. HERNÁNDEZ

UNIVERSIDAD DE COLIMA

ABSTRACT.We analyze the harmonic function of the fractal to which we refer as Sierpinski's Triangle XXX

Sierpinski's Triangle XXX

Sierpinski's Triangle XXX, which is a variation from the original Sierpinski's Triangle, has as first iteration $\mathcal{V}_0 = \{p_1, p_2, p_3\}$, where $p_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $p_2 = (0,0)$ and $p_3 = (1,0)$, and the next iterations are constructed with the following contractions:

$$\begin{cases} f_1(x) := \frac{x+2p_1}{3} \\ \\ f_2(x) := \frac{x+2p_2}{3} \\ \\ f_3(x) := \frac{x+2p_3}{3} \\ \\ f_4(x) := \frac{x+2p_2}{3} \\ \\ f_5(x) := \frac{x+2p_2}{3} \\ \\ f_6(x) := \frac{x+2p_2}{3} \end{cases}$$

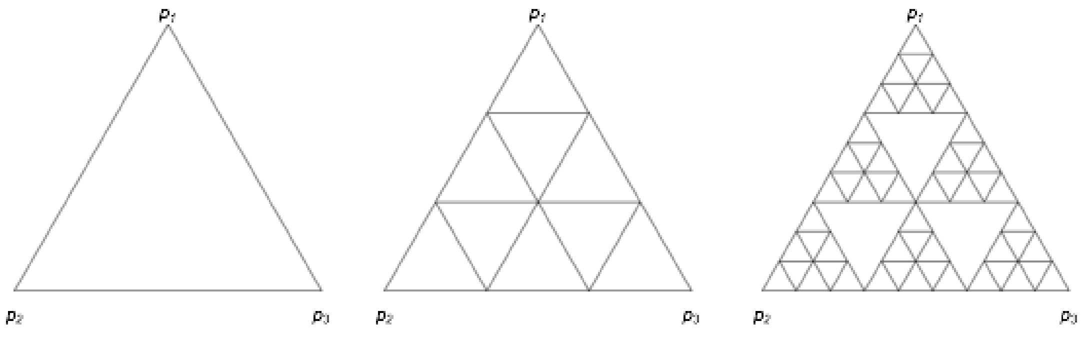


Figure 1: First fourth iterations, V_0, V_1, V_2 and V_3.

Hausdorff Dimension

We say that a compact set $K \subset \mathbb{R}$ is self similar if

$$K = f_1(K) \cup f_2(K) \cup ... \cup f_N(K), \tag{1}$$

where each $f_i : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction, that is,

$$|f(x) - f(y)| \leq \alpha_i |x - y|$$

for all x, $y \in \mathbb{R}$, where $0 \leq \alpha_i < 1$. It is possible to show that K is defined by the contractions f_i, that is, there is a unique compact not empty set that satisfies (1).

Let d be the unique positive number such that

$$\alpha_1^d + \alpha_2^d + ... + \alpha_N^d = 1. \tag{2}$$

We have the next theorem

Theorem. *$dim(K) \leq d$*

To guarantee that $d = dim(K)$ we need a condition that avoids overlaps, like the one given by John E. Hutchinson, called the *open set condition.*

We say that the contractions f_i satisfy the open set condition if there is a open set bounded $U \subset \mathbb{R}^n$ such that:
1.$f_i(U) \subset U$ for each $i = 1, ..., N$; and
2.$f_i(U) \cap f_j(U) = \emptyset$ for each $i, j = 1, ...N, i \neq j$.

Hutchinson Theorem. *Let $f_i : \mathbb{R}^n \to \mathbb{R}^n, i = 1, ..., N$ be contractions such that are self similarities and satisfy the open set condition. If K is the self similar set regarding the f_i, then $dim(K) = d$, where dis the only number that satisfies* (2).

It is clear that our set satisfy the conditions of the open set, therefore, by the Hutchinson theorem its Hausdorff dimension is

$$\mathcal{H}^d = \frac{\log 6}{\log 3} = 1 + \frac{\log 2}{\log 3}$$

since it is the number d that satisfies the equation

$$\left(\frac{1}{3}\right)^d + \left(\frac{1}{3}\right)^d + \left(\frac{1}{3}\right)^d + \left(\frac{1}{3}\right)^d + \left(\frac{1}{3}\right)^d + \left(\frac{1}{3}\right)^d = 1.$$

Self similar Structure

The Critical set of XXX is $\mathcal{C} = \{f_1(p_2) = f_4(p_1), f_2(p_1) = f_4(p_2), f_2(p_3) = f_6(p_2), f_3(p_1) = f_5(p_3), f_1(p_3) = f_5(p_1), f_6(p_1) = f_5(p_2) = f_4(p_3)\}$.

And the Post Critical Set of XXX is $\mathcal{V}_0 = \{p_1, p_2, p_3\}$.

Harmonic Structure

The following matrix is the Laplacian for our fractal, which we call H,

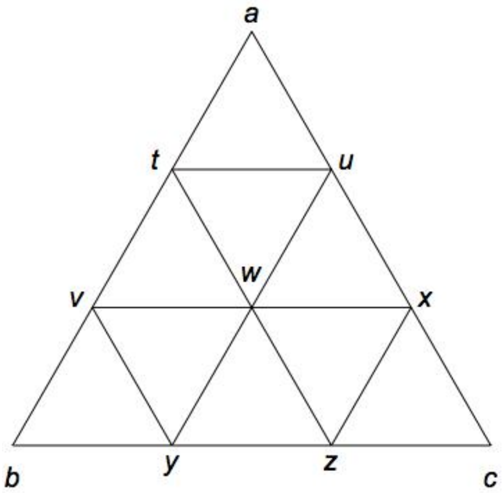


Figure 2: Vertices of $\mathcal{V}_1$ from which we obtain the Laplacian.

$$
H = \begin{array}{c} \\ a \\ b \\ c \\ t \\ v \\ u \\ w \\ x \\ y \\ z \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} a & b & c & t & u & v & w & x & y & z \end{array} \\
\begin{pmatrix}
-2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & -4 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & -4 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & -6 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & -4 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & -4 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -4
\end{pmatrix}
\end{array}
$$

The matrix is a block of matrices, as defined next:

$$
H = \left(\begin{array}{c|c} T & J \\ \hline J^t & X \end{array}\right)
$$

The First one, called T, is the restriction for certain vertices. Since we want to see the restriction for a, b and c, for us,

$$T = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Then

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J^t = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

and,

$$X = \begin{pmatrix} -4 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & -6 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -4 \end{pmatrix}$$

$$X^{-1} = \begin{pmatrix} -\frac{16}{180} & -\frac{29}{180} & -\frac{29}{180} & -\frac{1}{6} & -\frac{19}{180} & -\frac{19}{180} & -\frac{17}{180} \\ -\frac{29}{180} & -\frac{67}{180} & -\frac{19}{180} & -\frac{1}{6} & -\frac{29}{180} & -\frac{17}{180} & -\frac{19}{180} \\ -\frac{29}{180} & -\frac{19}{180} & -\frac{67}{180} & -\frac{1}{6} & -\frac{17}{180} & -\frac{29}{180} & -\frac{19}{180} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{19}{180} & -\frac{29}{180} & -\frac{17}{180} & -\frac{1}{6} & -\frac{67}{180} & -\frac{19}{180} & -\frac{29}{180} \\ -\frac{19}{180} & -\frac{17}{180} & -\frac{29}{180} & -\frac{1}{6} & -\frac{19}{180} & -\frac{67}{180} & -\frac{29}{180} \\ -\frac{17}{180} & -\frac{19}{180} & -\frac{19}{180} & -\frac{1}{6} & -\frac{29}{180} & -\frac{29}{180} & -\frac{67}{180} \end{pmatrix}$$

So as to restrict the energy to the previous iteration, we can use the formula:

$$H_u = T - (J^t)(X^{-1})(J)$$

Thus,

$$H_u = \begin{pmatrix} -\frac{14}{15} & \frac{7}{15} & \frac{7}{15} \\ \frac{7}{15} & -\frac{14}{15} & \frac{7}{15} \\ \frac{7}{15} & \frac{7}{15} & -\frac{14}{15} \end{pmatrix}$$

From H_u, we can notice that,

$$\mathcal{E}_1(v|_{\mathcal{V}_0}) = \frac{7}{15}\mathcal{E}_0(u).$$

We can define $\mathcal{V}_m$ recursively of the form

$$\mathcal{V}_m = f_1(\mathcal{V}_{m-1}) \cup f_2(\mathcal{V}_{m-1}) \cup f_3(\mathcal{V}_{m-1}) \cup f_4(\mathcal{V}_{m-1}) \cup f_5(\mathcal{V}_{m-1}) \cup f_6(\mathcal{V}_{m-1}),$$

we write $x \sim y$ if x and y are adjacent in $\mathcal{V}_m$.

We define the quadratic form $\mathcal{E}_m$ in function in $\mathcal{V}_m$ as

$$\mathcal{E}_m = (\frac{15}{7})^m \sum_{x \sim y} (u(x) - u(y))^2.$$

This implies that

$$min\{\mathcal{E}_{m+1}(v) : v|\mathcal{V}_0 = u\} = \mathcal{E}_m(u),$$

for each $m \geq 0$. Inductively we can verify that

$$min\{\mathcal{E}_{m+1}(v) : v|\mathcal{V}_0 = u\} = \mathcal{E}_0(u),$$

for all $m \geq 0$.

The formula to get the harmonic function, which is the restriction of the other points than a, b and c, is:

$$G_u = -X^{-1}J$$

$$G_u = \begin{pmatrix} \frac{8}{15} & \frac{4}{15} & \frac{1}{5} \\ \frac{8}{15} & \frac{1}{5} & \frac{4}{15} \\ \frac{4}{15} & \frac{8}{15} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{4}{15} & \frac{1}{5} & \frac{8}{15} \\ \frac{1}{5} & \frac{8}{15} & \frac{4}{15} \\ \frac{1}{5} & \frac{4}{15} & \frac{8}{15} \end{pmatrix}$$

Harmonic Function

From its Harmonic Structure, G_u gives us the harmonic function for our fractal,

$$f\left(u(p_w)\right) = \frac{8f\left(u(p_i)\right) + 4f\left(u(p_j)\right) + 3f\left(u(p_k)\right)}{15}$$

and for the central points,

$$f\left(u(p_w)\right) = \frac{f\left(u(p_i)\right) + f\left(u(p_j)\right) + f\left(u(p_k)\right)}{3}$$

where p_i, p_j and p_k are the three points that define the cell of the previous iteration which p_w is in.

On the next graphics we can observe the values for the harmonic function of our fractal with certain values for the points of the first iterations (0, 0 and 1).

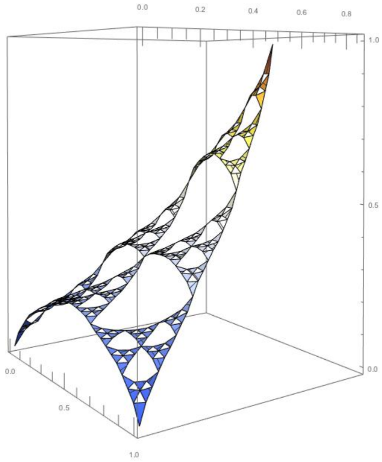


Figure 3: Harmonic Function on Sierpinski's Triangle XXX, with $v(p_1)$=1, $v(p_2)$=0 and $v(p_3)$=0.

Monotony on the edges

Now we focus only on the harmonic function for the interval $[0, 1]$, and will prove that it is monotonous.

Theorem. *Let f be the harmonic function for the Sierpinski's Triangle XXX, with $v(p_1) = \beta$, $a = 0$ and $d = 1$, f is monotonous increasing for $\frac{1}{15} \leq b \leq \frac{2}{3}$ and $\frac{1}{15} \leq c \leq \frac{2}{3}$.*

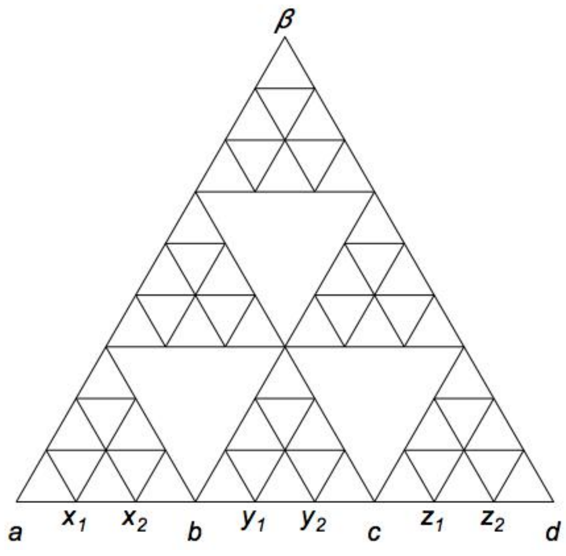


Figure 4: Interval $[0, 1]$ of the Sierpinski's Triangle XXX

Proof. We will suppose that $0 = a < b < c < d = 1$. From the harmonic function we see that,

$$b = \frac{8a + 4d + 3\beta}{15}$$

and

$$c = \frac{15b + 4d + 4a}{15} = b + \frac{4}{15}(d - a)$$

.

We suppose the theorem is true, then, in general,

$$b \in [(1-\theta)a + \theta d, \tau a + (1-\tau)d] \Rightarrow c \in [(1-\tau)a + \tau d, \theta a + (1-\theta)d]$$

with $\theta = \frac{1}{5}$ and $\tau = \frac{1}{3}$.

To prove f is monotonous, we only need to check that

i) $x_1 \in [\frac{14}{15}a + \frac{1}{15}b, \frac{1}{3}a + \frac{1}{3}b]$

ii) $y_1 \in [\frac{14}{15}b + \frac{1}{15}c, \frac{1}{3}b + \frac{1}{3}c]$

iii) $z_1 \in [\frac{14}{15}c + \frac{1}{15}d, \frac{1}{3}c + \frac{1}{3}d]$

i) Writing x_1 in terms of a, b and d, we have

$$\frac{14a + b}{15} \leq \frac{112a + 120b - 7d}{225} \leq \frac{a + 2b}{3}.$$

$$\frac{210a + 15b}{225} \leq \frac{112a + 120b - 7d}{225}$$

$$\frac{98a - 105b}{225} \leq \frac{-7d}{225}$$

$$\frac{14a + d}{155} \leq b,$$

which is true from our hypothesis.

Now the other side of the inequality

$$\frac{112a + 120b - 7d}{225} \leq \frac{75a + 150b}{225}$$

$$\frac{37a - 7d}{225} \leq \frac{30b}{225}$$

$$\frac{-7d}{225} \leq \frac{30b - 37a}{225}$$

$$\frac{-7d}{225} \leq \frac{-7a}{225} \leq \frac{30b - 37a}{225},$$

is true from the hypothesis. Then i) is satisfied.

ii) With y_1 in terms of a, b, c and d, we have

$$\frac{14b+c}{15} \leq \frac{-5a+39b+12c-d}{45} \leq \frac{b+2c}{3}.$$

$$\frac{42b+3c}{15} \leq \frac{-5a+39b+12c-d}{45}$$

$$3b \leq -5a+9c-d$$

$$3b \leq -5a-d+9(b+\frac{4}{15}(d-a))$$

$$-6b \leq \frac{7d-37a}{5}$$

$$\frac{37a-7d}{30} \leq b$$

Since, $\frac{28a+2d}{30} \leq b$,

$$\frac{37a-7d}{30} \leq \frac{28a+2d}{30}$$

$$a \leq d.$$

Now the other side of the inequality

$$\frac{-5a+39b+12c-d}{45} \leq \frac{15b+30c}{45}$$

$$-5a+24b-d \leq 18c = 18(b+\frac{4}{15}(d-a))$$

$$-5a+24b-d \leq 18c = 18(b+\frac{4}{15}(d-a))$$

$$-25a+120b-5d \leq -24a+90b+24d$$

$$b \leq \frac{29d+a}{30}$$

Since, $b \leq \frac{2d+a}{3}$,

$$\frac{20d+10a}{30} \leq \frac{29d+a}{30}$$

$$9a \leq 9d,$$

which is true from the hypothesis. Thus, ii) is satisfied.

iii) Writing z_1 in terms of a, c and d,

$$\frac{14c + d}{15} \leq \frac{-7a + 180c + 52d}{225} \leq \frac{c + 2d}{3}.$$

$$\frac{210c + 15d}{225} \leq \frac{-7a + 180c + 52d}{225}$$

$$\frac{30c - 37d}{225} \leq \frac{-7a}{225}$$

$$\frac{30c - 37d}{225} \leq \frac{-7c}{225} \leq \frac{-7a}{225},$$

that is true from the hypothesis. Now the other side of the inequality.

$$\frac{-7a + 180c + 52d}{225} \leq \frac{75c + 150d}{225}$$

$$105c - 7a \leq 98d$$

Since, $c = b + \frac{4}{15}(d - a)$,

$$-35a + 105b \leq 70d$$

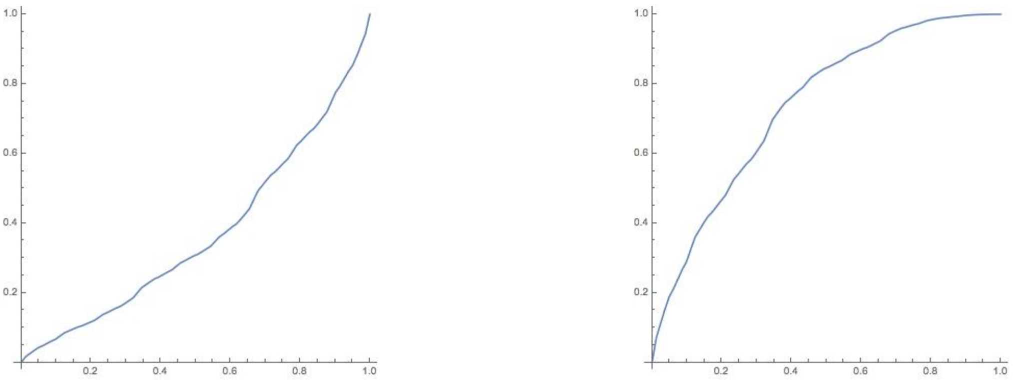
$$b \leq 2d - a,$$

true from the hypothesis. Then, iii) is satisfied.

Therefore, the theorem is true.

$\square$

We can observe the graphics for the Harmonic function on the interval $[0, 1]$ with the critic values stated on the theorem.



(a) Harmonic function on the interval $[0, 1]$, with a=0, b=1/5, d=1.

(b) Harmonic function on the interval $[0, 1]$, with a=0, b=2/3, d=1.

Derivative of the family of harmonic functions

Let $h_0, h_1, h_2 : I \to (0, 1)$ and $r_0, r_1, r_2 : I \to I$ be continuous functions with $h_0(\beta) + h_1(\beta) + h_2(\beta) = 1$.

Proposition. *There exists a family $\{f_\beta\}$ of continuous functions $f_\beta : [0, 1] \to [0, 1]$ such that $f_\beta(0) = 0$, $f_\beta(1) = 1$, and satisfies the functional equations*

$$\begin{cases} f_\beta\left(\dfrac{x}{3}\right) = h_0(\beta) f_{r_0(\beta)}(x) \\ f_\beta\left(\dfrac{x+1}{3}\right) = h_1(\beta) f_{r_1(\beta)}(x) + h_0(\beta) \\ f_\beta\left(\dfrac{x+2}{3}\right) = h_2(\beta) f_{r_2(\beta)}(x) + h_0(\beta) + h_1(\beta) \end{cases}$$

Proof. We can inductively construct the functions f_β on the triadic numbers $\frac{k}{3^m}$, for $m \geq 0$ and $0 \leq k \leq 3^m$, and $k \in \mathbb{Z}$.

$$f\left(\frac{k}{3^m}\right) = \begin{cases} h_0(\beta) f_{r_0(\beta)}\left(\dfrac{k}{3^{m-1}}\right) & k \leq 3^{m-1} \\ h_1(\beta) f_{r_1(\beta)}\left(\dfrac{k - 3^{m-1}}{3^{m-1}}\right) + h_0(\beta) & 3^{m-1} \leq k \leq 2 \cdot 3^{m-1} \\ h_2(\beta) f_{r_2(\beta)}\left(\dfrac{k - 2 \cdot 3^{m-1}}{3^{m-1}}\right) + h_0(\beta) + h_1(\beta) & 2 \cdot 3^{m-1} \leq k \end{cases}$$

Now we have to prove they have continuous extensions to $[0, 1]$:
First case, $k < 3^{m-1}$

$$f_\beta\left(\frac{k+1}{3^m}\right) - f_\beta\left(\frac{k}{3^m}\right) = h_0(\beta)\left(f_{r_0(\beta)}\left(\frac{k+1}{3^{m-1}}\right) - f_{r_0(\beta)}\left(\frac{k}{3^{m-1}}\right)\right)$$

Second case, $3^{m-1} < k < 2 \cdot 3^{m-1}$

$$f_\beta\left(\frac{k+1}{3^m}\right) - f_\beta\left(\frac{k}{3^m}\right) = h_1(\beta)\left(f_{r_1(\beta)}\left(\frac{k+1-3^{m-1}}{3^{m-1}}\right) - f_{r_1(\beta)}\left(\frac{k-3^{m-1}}{3^{m-1}}\right)\right)$$

And the third case, $2 \cdot 3^{m-1} < k$

$$f_\beta\Big(\frac{k+1}{3^m}\Big) - f_\beta\Big(\frac{k}{3^m}\Big) = h_2(\beta)\bigg(f_{r2(\beta)}\Big(\frac{k+1-2\cdot 3^{m-1}}{3^{m-1}}\Big) - f_{r2(\beta)}\Big(\frac{k-2\cdot 3^{m-1}}{3^{m-1}}\Big)\bigg)$$

Let $H = max\{h_o, h_1, h_2\}$, we have $0 < H < 1$ because the image of the $h's$ is $(0, 1)$, and then, by induction:

$$\Big|f_\beta\Big(\frac{k+1}{3^m}\Big) - f_\beta\Big(\frac{k}{3^m}\Big)\Big| \leq H^m.$$

As $H < 1$, we obtain the proposition.

□

And now let us assume that there exists $\epsilon > 0$ so that if, for some $\beta \in I$, $|h_i(\beta) - \frac{1}{3}| < \epsilon$, then $|h_j(r_k(\beta)) - \frac{1}{3}| \geq \epsilon$ for every $i, j, k \in \{0, 1, 2\}$. Then we have the next theorem:

Theorem. *The functions f_β are monotone and, for every x where they are differentiable, $f'(x) = 0$.*

Proof. Monotonicity follows from the construction of the function in the proof of the proposition, since the image of each function h is the interval $(0, 1)$. For the next part we fix β and let x be such that f_β is differentiable at x. We assume $f'(x) \neq 0$ and later we'll arrive at a contradiction.

Consider the trinary expansion of x,

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$$

also written as $.a_1a_2\dots$, and its partial sums $.a_1a_2\dots a_k$, where each $a_i \in \{0, 1, 2\}$. Since

$$\frac{f_\beta(.a_1a_2\dots(a_k+1)) - f_\beta(.a_1a_2\dots a_k)}{\frac{1}{3^k}} \to f'(x) \neq 0$$

where $.a_1a_2\dots(a_k+1)$ means $\Big(\sum_{i=1}^{k}\frac{a_i}{3^i}\Big) + \frac{1}{3^k}$. We have

$$Q_k = \frac{f_\beta(.a_1a_2\dots(a_k+1)) - f_\beta(.a_1a_2\dots a_k)}{f_\beta(.a_1a_2\dots(a_{k-1}+1)) - f_\beta(.a_1a_2\dots a_{k-1})} \to \frac{1}{3}$$

Claim. *$Q_k = h_{a_k}(r_{a_{k-1}} \circ \dots \circ r_{a_2} \circ r_{a_1}(\beta))$ for every β.*

Proof. We prove the claim by induction on the length k of the partial sum of the trinary expansion. For $k = 1$ we have

$$\frac{f_\beta\Big(\frac{a_1+1}{3}\Big) - f_\beta\Big(\frac{a_1}{3}\Big)}{f_\beta(1) - f_\beta(0)} = h_{a_1}(\beta)(f_{r_{a_1}}(1) - f_{r_{a_1}}(0)) = h_{a_1}(\beta)$$

Assume the result is true for $k - 1$. Then

$$
\begin{aligned}
Q_k &= \frac{h_{a_1}(\beta)(f_{r_{a_1}(\beta)}(.a_2 \dots (a_k+1)) - f_{r_{a_1}(\beta)}(.a_2 \dots a_k))}{h_{a_1}(\beta)(f_{r_{a_1}(\beta)}(.a_2 \dots (a_{k-1}+1)) - f_{r_{a_1}(\beta)}(.a_2 \dots a_{k-1}))} \\
&= \frac{f_{r_{a_1}(\beta)}(.a_2 \dots (a_k+1)) - f_{r_{a_1}(\beta)}(.a_2 \dots a_k)}{f_{r_{a_1}(\beta)}(.a_2 \dots (a_{k-1}+1)) - f_{r_{a_1}(\beta)}(.a_2 \dots a_{k-1})} \\
&= h_{a_k}(r_{a_{k-1}} \circ \dots \circ r_{a_2}(r_{a_1}(\beta)))
\end{aligned}
$$

where we have used the induction hypothesis. □

Let $s_k(\beta) = r_{a_{k-1}} \circ \dots \circ r_{a_2} \circ r_{a_1}(\beta)$, with $s_1(\beta) = \beta$, so that we have

$$h_{a_k}(s_k(\beta)) \to \frac{1}{3}$$

Thus there exists K such that, for every $k \geq K$,

$$\left| h_{a_k}(s_k(\beta)) - \frac{1}{3} \right| < \epsilon$$

But by the assumption right before the theorem was stated, this implies that

$$\left| h_{a_{K+1}}(s_{K+1}(\beta)) - \frac{1}{3} \right| = \left| h_{a_{K+1}}(r_{a_K}(s_K(\beta))) - \frac{1}{3} \right| \geq \epsilon$$

a contradiction.
We therefore conclude that $f'(x) = 0$. □

Finally, we will assume the following theorem as true, but you can find the proof in many books of real analysis.

Theorem. *Let f be a monotone continuous function, then f is differentiable at almost every point.*

This means that, for the family of functions f_β that satisfy the conditions of the theorem, $f'(x) = 0$ at almost every point.

Now we wish to prove that the derivative in our variation of Sierpinski's Triangle XXX is zero wherever it is defined. First we have to find the functions h_0, h_1, h_2, r_0, r_1, and r_2 for XXX and the interval I, in which they satisfy the conditions of the theorem.

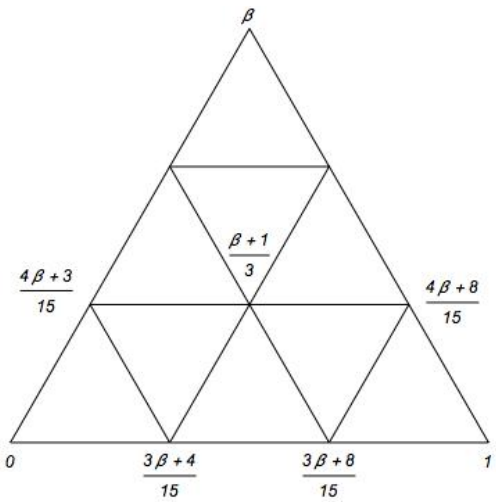


Figure 6: Values for the harmonic function f_β in $\mathcal{V}_1$.

We see from the triangle above that the equations are:

$$\begin{cases} h_0(\beta) = \dfrac{4+3\beta}{15} \\ \\ h_1(\beta) = \dfrac{4}{15} \\ \\ h_2(\beta) = \dfrac{7-3\beta}{15} \end{cases} \qquad \begin{cases} r_0(\beta) = \dfrac{4\beta+3}{4+3\beta} \\ \\ r_1(\beta) = \dfrac{1+2\beta}{4} \\ \\ r_2(\beta) = \dfrac{\beta}{7-3\beta} \end{cases}$$

Now we just have to find the interval I.

Theorem. *The family of harmonic functions f_β in XXX is monotone for $\beta \in [-1, 2]$, and $f'(x) = 0$ at almost every point.*

Proof. First, we will show that $h_i([-1, 2]) \subseteq (0, 1)$ and $r_i([-1, 2]) \subseteq [-1, 2]$ for every $i \in \{0, 1, 2\}$, and then that they don't converge to $\frac{1}{3}$.

Obviously the image of h_1 is contained in $(0, 1)$; and both, h_0 and h_2, are linear, so we just have to evaluate at the extremes:

$$h_0(-1) = \frac{4+3(-1)}{15} = \frac{1}{15} \in (0, 1)$$

$$h_0(2) = \frac{4+3\cdot 2}{15} = \frac{10}{15} \in (0, 1)$$

$$h_2(-1) = \frac{7 - 3(-1)}{15} = \frac{11}{15} \in (0, 1)$$

$$h_2(2) = \frac{7 - 3 \cdot 2}{15} = \frac{1}{15} \in (0, 1)$$

We can see that the image of the h's is indeed contained in $(0, 1)$.

$$\begin{aligned} -1 \leq r_0(\beta) = \frac{4\beta + 3}{3\beta + 4} &\qquad r_0(\beta) = \frac{4\beta + 3}{3\beta + 4} \leq 2 \\ -3\beta - 4 \leq 4\beta + 3 &\qquad 4\beta + 3 \leq 6\beta + 8 \\ -1 \leq \beta &\qquad -\frac{5}{2} \leq \beta \end{aligned}$$

$$\begin{aligned} -1 \leq r_2(\beta) = \frac{\beta}{7 - 3\beta} &\qquad r_2(\beta) = \frac{\beta}{7 - 3\beta} \leq 2 \\ 3\beta - 7 \leq \beta &\qquad \beta \leq 14 - 6\beta \\ \beta \leq \frac{7}{2} &\qquad \beta \leq 2 \end{aligned}$$

And r_1 is linear so we evaluate at the extremes:

$$r_1(-1) = -\frac{1}{4} \in [-1, 2]$$
$$r_1(2) = \frac{5}{4} \in [-1, 2]$$

We have just verified that $r_i([-1, 2]) \subseteq [-1, 2]$ and $h_i([-1, 2]) \subseteq (0, 1)$ for all i, as we wanted. Now we just have to see if h_0 and h_2 stay away from $\frac{1}{3}$.

$$h_0(\beta) = \frac{4 + 3\beta}{15} = \frac{1}{3}$$
$$\Rightarrow \beta_0 = \frac{1}{3}$$

$$h_2(\beta) = \frac{7 - 3\beta}{15} = \frac{1}{3}$$
$$\Rightarrow \beta_2 = \frac{2}{3}$$

In order for the condition of the theorem to be satisfied, we need that $r_i(\beta_j) \neq \beta_k$ for every $i \in \{0,1,2\}$ and $j,k = 0$ or 1, so that in the next iteration, $h_l(r_i(\beta_j)) \neq \frac{1}{3}$.

$$r_0\left(\frac{1}{3}\right) = \frac{\frac{4}{3}+3}{4+1} = \frac{13}{15}$$

$$r_0\left(\frac{2}{3}\right) = \frac{\frac{8}{3}+3}{4+2} = \frac{17}{18}$$

$$r_1\left(\frac{1}{3}\right) = \frac{1+\frac{2}{3}}{4} = \frac{5}{12}$$

$$r_1\left(\frac{2}{3}\right) = \frac{1+\frac{4}{3}}{4} = \frac{7}{12}$$

$$r_2\left(\frac{1}{3}\right) = \frac{\frac{1}{3}}{7-1} = \frac{1}{18}$$

$$r_2\left(\frac{2}{3}\right) = \frac{\frac{2}{3}}{7-2} = \frac{2}{15}$$

Note that neither of this values is either $\frac{1}{3}$ nor $\frac{2}{3}$, therefore the h's don't approach $\frac{1}{3}$ and so the proof is complete. $\square$

This means that the harmonic functions are strictly crescent, yet their derivatives are zero almost everywhere!!!

References

[1] Masaya Yamaguti, Masayoshi Hata, Jun Kigami. *Mathematics of Fractals (Translations of Mathematical Monographs)*. American Mathematical Society, 1998.

[2] Ricardo Sáenz, Alejandra Gaitán. *A Family of Singular Functions on the Interval*. Universidad de Colima, 2013.