Analysis of Texas Hold'em Bonus

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I. Introduction and Rules

Texas Hold 'Em Bonus is a poker-based table game, which is played with a single 52-card deck. Unlike typical Texas Hold'em, Texas Hold 'Em Bonus offers an optional Bonus Net, which pays based on the player's two card hand, and sometimes the dealer's two card hand.

Under Las Vegas Rules, the player firstly makes an Ante wager, plus an optional Bonus bet and two hole cards are dealt face down to the player and dealer. The player will make a decision for the game at each step, i.e.

1. c_1, continue the game (Flop bet) or fold conditional on two hole-cards. The Flop bet must be two times the amount of the ante and three community cards (the Flop) are dealt.

2. c_2, make a Turn bet or check (i.e. do nothing) conditional on two hole-cards and the Flop. The Turn bet must be exactly equal to the ante bet and a fourth community card is dealt (the Turn).

3. c_3, make a River bet or check conditional on two hole-cards, the Flop and the Turn. The River bet must be alsp exactly equal to the ante bet and a fifth community card is dealt (the River).

The player and dealer each make the best five-card hand using any combination of the five community cards and his own two initial hole cards. The higher hand wins. If the dealer has the higher hand the player will lose all wagers, except possibly the Bonus bet. If the player has the higher hand the Flop, Turn, and River bets will pay even money. If the player has a straight or higher the Ante bet will also pay even money, otherwise it will push. If the player and dealer have hands of equal value the Ante, Flop, Turn, and River bets will all push. For the Bonus bet, it will pay based only the player's and dealer's initial cards, according to the pay table posted.

Here our objective is to derive the optimal strategies for the player under standard and non-standard conditions in order to maximize the chance of winning. As the Bonus bet is made at the beginning and pays based on hole-cards, it should not impact the player's strategy. Therefore, if we want to find the optimal strategy for the player, it's equivalent to find how the player should make the decisions $c_i, i = 1,2,3$ at each step during the game. We solve this problem using Dynamic Programming Principle and our analysis is under Las Vegas rules of the game.

II. Analysis under Standard Condition

2.1 Method for deriving optimal strategy

Under standard condition that player knows two own hole-cards at the beginning, we use dynamic programming to derive the optimal strategy for the decisions $c_i, i = 1,2,3$. Dynamic programming is an optimization approach that transforms a complex problem into a sequence of simpler problems. And we can get the globally maximized total expected value earned in the game using Dynamic Programming Principle.

Denote X_1, X_2 as the player's hole cards, X_3, X_4, X_5 as three cards at flop, X_6 as turn card, X_7 as river card, and $Y = (Y_1, Y_2)$ as the dealer's hole cards. From the beginning, for each $X_1 = x_1, X_2 = x_2$, we could have their decision c_1 and subsequent c_2 based on $X_3 = x_3, X_4 = x_4, X_5 = x_5$, c_3 based on $X_6 = x_6$. For the River bet and Turn bet, the player needs to make the bet if the chance of winning the hand is greater than that of losing given the player's two hole-cards and the visible board cards. For the Flop bet, the player should make it if the total expected value of the Ante**,** Flop**,** Turn and River bets with optimal decisions is greater than that of folding (i.e. -1.000000).

The optimal strategy is derived retrospectively as below:

(1) Given x_1, x_2, For each x_{3j}, x_{4j}, x_{5j}, $j = 1, \ldots, \binom{50}{3} = 19600$,

a. Given $x_1, x_2, x_{3j}, x_{4j}, x_{5j}$, for each $x_6 = x_{6i}, i = 1, \ldots, 47$, we calculate $p_{31}^{(i)}$, $p_{32}^{(i)}$, $p_{33}^{(i)}$, the probabilities of winning, winning with a straight or higher and losing given cards $x_1, x_2, x_{3j}, x_{4j}, x_{5j}, x_{6i}$, where

$$p_{31}^{(i)} = p(win|x_1, \dots, x_{6i})$$
$$p_{32}^{(i)} = p(win\ ws\ |x_1, \dots, x_{6i})$$
$$p_{33}^{(i)} = p(lose|x_1, \dots, x_{6i})$$

If $p_{31}^{(i)} > p_{33}^{(i)}$, i.e., $p(win|x_1, \dots, x_{6i}) > p(lose|x_1, \dots, x_{6i})$, the decision for River bet is to make, i.e, $c_{3i} = 1\{p_{31}^{(i)} > p_{33}^{(i)}\}$. And we have the expected value player earn for River round if the River bet is made for each $i = 1, \dots, 47$,

$$E_{2i} = p_{31}^{(i)} - p_{33}^{(i)} (\text{i.e. } p(win|x_1, \dots, x_{6i}) - p(lose|x_1, \dots, x_{6i}))$$

b. After all 47 calculations in the last step, we have the expected value player earn for the River bet with the optimal decision c_3 of it for each $j = 1, \dots, \binom{50}{3} = 19600$, E_{13j}, where

$$E_{13j} = \sum_{x_{6i}} p(x_{6i}|x_1, \dots, x_{5j})\, c_{3i} E_{2i}$$

And we calculate $p_{21}^{(j)}$, $p_{22}^{(j)}$, $p_{23}^{(j)}$, the probabilities of winning, winning with a straight or higher and losing given cards $x_1, x_2, x_{3j}, x_{4j}, x_{5j}$, where

$$p_{21}^{(j)} = \sum_{x_{6i}} p(x_{6i}|x_1, \dots, x_{5j})\, p_{31}^{(i)} = p(win|x_1, \dots, x_{5j})$$

$$p_{22}^{(j)} = \sum_{x_{6i}} p(x_{6i}|x_1, \dots, x_{5j})\, p_{32}^{(i)} = p(win\ ws|x_1, \dots, x_{5j})$$

$$p_{23}^{(j)} = \sum_{x_{6i}} p(x_{6i}|x_1, \dots, x_{5j})\, p_{33}^{(i)} = p(lose|x_1, \dots, x_{5j})$$

If $p_{21}^{(j)} > p_{23}^{(j)}$, i.e. $p(win|x_1, \dots, x_{5j}) > p(lose|x_1, \dots, x_{5j})$, the decision for Turn bet is to make, i.e., $c_{2j} = 1\{p_{21}^{(j)} > p_{23}^{(j)}\}$. And we have the expected value player earn for Turn round if the Turn bet is made for each $j = 1, \dots, \binom{50}{3} = 19600$,

$$E_{12j} = p_{21}^{(j)} - p_{23}^{(j)} = p(win|x_1, \dots, x_{5j}) - p(lose|x_1, \dots, x_{5j})$$

(2) After all 19600 calculations in (1), we have the expected value player earn for the River bet with the optimal decision c_3 for it, E_{13}, where

$$E_{13} = \sum_{x_{3j}, x_{4j}, x_{5j}} p(x_{3j}, x_{4j}, x_{5j}|x_1, x_2) * E_{13j}$$
$$= \sum_{x_{3j}, x_{4j}, x_{5j}} p(x_{3j}, x_{4j}, x_{5j}|x_1, x_2) * \sum_{x_{6i}} p(x_{6i}|x_1, \dots, x_5)\, c_{3i}\big(p(win|x_1, \dots, x_6) - p(lose|x_1, \dots, x_6,)\big)$$

We also have the expected value player earn for the Turn bet with the optimal decision c_2 of it, E_{12},where

$$E_{12} = \sum_{x_{3j}, x_{4j}, x_{5j}} p(x_{3j}, x_{4j}, x_{5j}|x_1, x_2) * c_{2j} E_{12j} = \sum_{x_{3j}, x_{4j}, x_{5j}} p(x_{3j}, x_{4j}, x_{5j}|x_1, x_2) * c_{2j} * \big(p(win|x_1, \dots, x_5) - p(lose|x_1, \dots, x_5)\big)$$

And we calculate p_{11}, p_{12}, p_{13}, the probabilities of winning, winning with a straight or higher and losing given hole-cards x_1, x_2, where

$$p_{11} = \sum_{x_{3j}, x_{4j}, x_{5j}} p(x_{3j}, x_{4j}, x_{5j}|x_1, x_2) * p_{21}^{(j)} = p(win|x_1, x_2)$$

$$p_{12} = \sum_{x_{3j}, x_{4j}, x_{5j}} p(x_{3j}, x_{4j}, x_{5j}|x_1, x_2) * p_{22}^{(j)} = p(win\ ws|x_1, x_2)$$

$$p_{13} = \sum_{x_{3j}, x_{4j}, x_{5j}} p(x_{3j}, x_{4j}, x_{5j}|x_1, x_2) * p_{23}^{(j)} = p(lose|x_1, x_2)$$

So we can calculate the expected value player earn for the Flop bet if it is made, E_{11}, where

$$E_{11} = 2 * (p_{11} - p_{13}) = 2 * \left(p(win|x_1, x_2) - p(lose|x_1, x_2)\right)$$

And for the Ante bet, according to the rules, if the player has the higher hand or the hand of equal value with the dealer's, the Ante bet will push and if the dealer has the higher hand, the player will lose the Ante bet. Moreover, if the player wins with a straight or higher hand, the Ante bet will pay even money. So the expected value player earn for the Ante bet is E_0, where

$$E_0 = p_{12} - p_{13} = p(win\, ws|x_1, x_2) - p(lose|x_1, x_2)$$

By (1) and (2), we can get the total expected value of the Ante**,** Flop**,** Turn and River bets with optimal decisions if the Flop bet is made, E_1, where

$$E_1 = E_{11} + E_{12} + E_{13} + E_0$$

And the player should make the Flop bet if the total expected value of the Ante**,** Flop**,** Turn and River bets with optimal decisions is greater than that of folding (i.e. -1.000000), i.e.,

$$c_1 = 1\{E_1 > -1\}$$

For the Computation, we use methods of Monte Carlo in drawing x_{3j}, x_{4j}, x_{5j} and computing the conditional probabilities in the process of deriving the optimal decisions. Given player's two hole-cards, we can use this method to derive the specific optimal strategy for each bet in the game. The derived strategy is optimal because theoretically it globally maximizes total expected value earned in the game using Dynamic Programming Principle. The theoretical proof for this is in the Appendix.

Moreover, in order to obtain the optimal strategy for the Flop bet, we consider 169 different possible starting hands the player can receive with their own probability of drawing, $p_0^{(k)}, k = 1, \ldots, 169$. There are three kinds of player's two hole-hands,

(1) **Unsuited non-pair:** the player's hand is unsuited non-pair, for example, (2s,3d). And there are $\binom{13}{2} = 78$ possible combinations of different ranks.
(2) **Suited non-pair:** the player's hand is suited non-pair, for example, (2s,3s). There are also 78 different possible combinations in this kind.
(3) **Pair:** the player's hand is a pair, for example, (2s,2d). There are 13 possible starting hand in this kind.

We compute the expected value $E_1^{(k)}, k = 1, \ldots, 169$, for each of them and get the optimal decision for the Flop bet by comparing them with the total expected value of folding (i.e. -1.000000).

2.2 Optimal Strategy

2.2.1 Flop Bet

For each of the 169 different possible player's starting hands, we obtain the total expected values player earn for all bets if making the Flop bet. We use the method above with Monte Carlo simulations, specifically, 10,000 times simulation of three flop cards and 10,000 times simulation of river card and two dealer's hole-cards, which takes time of 76196 seconds. Among the simulated total expected values for 169 different starting hands, there are 5 kinds of hands whose expectation is less than -1, which are shown in the table below.

Player's cards	EV (E_1)
2,3 unsuited	-1.15462
2,4 unsuited	-1.06127
2,5 unsuited	-1.01152
2,6 unsuited	-1.03962
2,7 unsuited	-1.02252

Therefore, at flop stage, the player needs to fold only when the two hole-cards are 2-3 unsuited, 2-4 unsuited, 2-5 unsuited, 2-6 unsuited, 2-7 unsuited, otherwise the player should make the Flop bet.

2.2.2 Turn and River Bets

For the Turn bet and River bet, the player needs to make the bet if the chance of winning the hand is greater than that of losing given the player's two hole-cards and the visible board cards, otherwise the player should check instead. The player can use the method discussed in the last section to derive the optimal decisions for turn and river bets. Specifically, use part (1) in the method to calculate c_2 at turn stage and c_3 at river stage. If we have value c_i equal to 1, we'll bet at the ith stage.

2.3 Expected Value and Variance

After obtaining the total expected values for each of the 169 different player's hole-cards, $E_1^k, k = 1, \dots, 169$, the expected value player earn using the optimal strategy is $E_0 = p_0^{(k)} * \max(-1, E_1^k)$. And the variance could be derived using formula $Var(x) = Var(E(x|y)) + E(var(x|y))$. Using the optimal strategy, under standard condition, the house advantage is **2.0050251%**, which means for every unit ante bet, the player would on average lose approximately 0.02. And using the optimal strategy, the variance of value player earn per initial ante bet is 2.20824. Moreover, we find that if the player plays all hands and never folds at the flop stage, then the house advantage would be 2.2670613%.

And for the Bonus bet, it will pay based only the player's and dealer's initial cards, according to the pay table posted. The expected values and probabilities are shown in the following table.

Player's Hand	Dealer's Hand	Gain	Probability	Expected Value
Ace/Ace	Ace/Ace	1000	0.0000037	0.003694
	Any other	30	0.0045212	0.135636
Ace/King Suited	Any	25	0.0030166	0.075415
Ace/Queen or Ace/Jack Suited	Any	20	0.0060332	0.120664
Ace/King Unsuited	Any	15	0.0090498	0.135747
King/King, Queen/Queen or Jack/Jack	Any	10	0.0135747	0.135747
Ace/Queen or Ace/Jack Unsuited	Any	5	0.0180995	0.090498
10/10 through 2/2	Any	3	0.040724	0.122172
Any Other	Any	-1	0.9049774	-0.904977
TOTAL			1	-0.085406

Therefore, the house advantage of the Bonus bet is 8.5406%, which means for every unit bonus bet, the player would on average lose approximately 0.085406.

III. Analysis under Non-standard Condition: One Dealer Hole-Card

Assume besides the player's two hole-cards, the player can also see one of the dealer's hole-cards when the initial two cards are dealt to the dealer. In this non-standard situation, we analyze the game, find the optimal strategy and corresponding player advantage.

3.1 Method for deriving optimal strategy

We also use dynamic programming to derive the optimal strategy for the decisions $c_i, i = 1,2,3$. The method for deriving the optimal strategy when the player knows one of the dealer's hole-cards has the same process as that under standard condition we discussed before. The only differences are the calculation of the conditional probabilities in each step. For each conditional probability in the process, there is one more known card's value for the given condition, y_1, one of the dealer's hole-cards. For example, $p_{31}^{(i)}$, $p_{32}^{(i)}$, $p_{33}^{(i)}$ here are changed to the probabilities of winning, winning with a straight or higher and losing given cards $y_1, x_1, x_2, x_{3j}, x_{4j}, x_{5j}, x_{6i}$, where

$$p_{31}^{(i)} = p(win|y_1, x_1, \dots, x_{6i})$$
$$p_{32}^{(i)} = p(win\ ws\ |y_1, x_1, \dots, x_{6i})$$
$$p_{33}^{(i)} = p(lose|y_1, x_1, \dots, x_{6i})$$

So if $p_{31}^{(i)} > p_{33}^{(i)}$, i.e., $p(win|y_1, x_1, \dots, x_{6i}) > p(lose|y_1, x_1, \dots, x_{6i})$, the decision for River bet is to make, i.e, $c_{3i} = 1\{p_{31}^{(i)} > p_{33}^{(i)}\}$. Other conditional probabilities are changed similarly in the process of deriving the optimal strategy.

Similarly, we use the method of Monte Carlo in drawing x_{3j}, x_{4j}, x_{5j} and computing the conditional probabilities in the process of deriving the optimal decisions. Without any simplification, there are 66300 different combinations of player hole-hands and one dealer hole-card. In order to obtain the optimal strategy for the Flop bet, we consider 5083 different possible combinations based on the 169 patterns of the player's hand.

(1) If the player's hand is unsuited non-pair: the dealer's hole card can be the first suit, the second suit, or one of the other two suits, for example,

Player Cards	Dealer card, Suit=s	Dealer card, Suit=h	Dealer card, Suit=d(one of other two suits)
2(s),3(h)	12 possible ranks: 3, …, A	12 possible ranks: 2, 4, ..., A	13 possible ranks: 2, 3, …, A

There are total 78*37 possible combinations in this kind.

(2) If the player's hand is suited non-pair: the dealer's card can be the common suit or one of the other three suits, for example,

Player Cards	Dealer card, Suit=s	Dealer card, Suit=d(one of other three suits)
2(s),3(s)	11 possible ranks: 4, 5, …, A	13 possible ranks: 2, 3, …, A

There are total 78*24 possible combinations in this kind.

(3) If the player's hand is a pair: the suit of the dealer's card can be the suits of the cards that make up the pair or one of the other two suits. And if the dealer's card is the same rank as the pair, then the suit of the dealer's card must be one of the other two suits. For example,

Player Cards	Dealer card, Suit=s or h	Dealer card, Suit=d(one of other two suits)
2(s),2(h)	12 possible ranks: 3, …, A	13 possible ranks: 2, 3, …, A

There are total 13*25 possible combinations in this kind.

Therefore, by considering suit-permutations there are all 78*37+78*24+78= 5083 different possible combinations if the player knows one of dealer's hole-cards. With the optimal strategy for turn and river bets, we compute the expected value $E_1^{(k)}, k = 1, \dots, 5083$, for each of them and get the optimal decision for the Flop bet by comparing them with the total expected value of folding (i.e. -1.000000).

3.2 Optimal Strategy

3.2.1 Flop Bet

For each of the 5083 different possible starting hands, we obtain the total expected values player earn for all bets if making the Flop bet. We use the method above with Monte Carlo simulations, specifically, 5,000 times simulation of three flop cards and 1,000 times simulation of river card and another dealer's hole-card, which takes time of 85912 seconds. Among the 5083 simulated total expected values, there are 406 kinds of starting hands whose expectation is less than -1, which means they should be folded before the Flop. And among them, 396 hands correspond to the situation that the player's two hole-cards are unsuited non-pair and 10 hands corresponding to that the player's cards are suited non-pair. All possible starting hands in which the player's cards are a pair have the total expected values E_1 bigger than -1, which means that if player's two hole-cards are a pair, the Flop bet needs to be always made.

Moreover, we find the weakest starting hand which has the lowest total expected value is that the player's cards are 2,8 unsuited and the known dealer's card is 10 with the suit same as 8, for example, (2s,8h) with dealer's card Th. Its expected value is -1.26017. And the edge closest to -1 is that the player's cards are 5,K unsuited and the known dealer's card is Ace with the suit same as K, for example, (5s,Kh) with dealer's card Ah, whose expected value is -1.00012.

Therefore, the optimal strategy for the Flop bet can be described as following.

(1) If the player' s two hole-cards are a pair, player should make the Flop bet.
(2) If the player's cards are suited non-pair, the player needs to fold only when one of the following ten situations occurs,

a. Player's cards are 2,3 suited while the known dealer's card is Ace with the same suit. For example, (2s,3s) with dealer's card As.

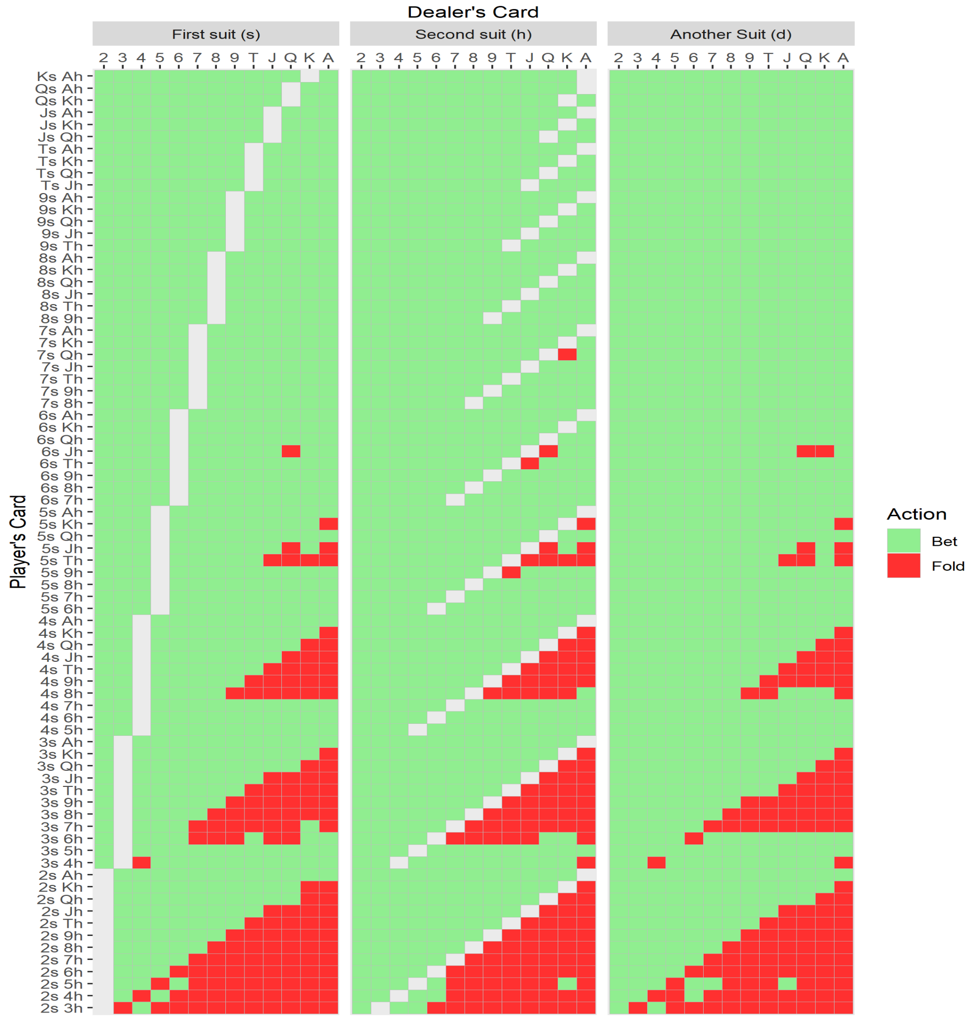
b. Player's cards are 2,7 suited while the known dealer's card is the same suited 9 or 10 or Q or K or A. For example, (2s,7s) with dealer's card 9s or 10s, Qs, Ks, As.

c. Player's cards are 2,8 suited while the known dealer's card is the same suited 10 or J. For example, (2s,8s) with dealer's card 10s or Js.

d. Player's cards are 2,9 suited while the known dealer's card is 10 with the same suit. For example, (2s,9s) with dealer's card 10s.

e. Player's cards are 2,10 suited while the known dealer's card is J with the same suit. For example, (2s,10s) with dealer's card Js.

(3) If the player's cards are unsuited non-pair, the optimal decisions for the Flop bet are summarized in the figure below, in which the squares colored red correspond to starting hands that the player needs to fold and those colored grey correspond to starting hands that are impossible to have. From the figure we can find that if the player's cards are unsuited non-pair with low ranks while the rank of known dealer's card is bigger than the largest rank of the player's cards, the optimal strategy is more likely to be folding regardless of the suit of dealer's card. The complete reference tables of the optimal decision at flop stage for 5083 different starting hand are shown in Appendix.



3.2.2 Turn and River Bets

For the Turn and River bet, the player needs to make the bet if the chance of winning the hand is greater than that of losing given the player's two hole-cards, one dealer's hole card and the visible board cards, otherwise the player should check instead. The player can use the method of deriving optimal strategy to calculate c_2 at turn stage and c_3 at river stage. If we have value c_i equal to 1, we'll bet at the ith stage.

3.3 Expected Value and Variance

After obtaining the total expected values for each of the 5083 different starting hands, $E_1^k, k = 1, \ldots, 5083$, the expected value player earn using the optimal strategy is $E_0 = p_0^{(k)} * \max(-1, E_1^k)$. If the player knows one dealer's hole-card, using the optimal strategy, the player advantage is **7.586434%**, which means for every unit ante bet, the player would on average win approximately 0.07586. And using the optimal strategy, the variance of value player earn per initial ante bet is 3.035162. Moreover, we find that if the player plays all hands and never folds at the flop stage, then the player advantage would be 6.7101647%.

And for the Bonus bet, because this non-standard condition does not affect it, the house advantage of the Bonus bet is still 8.5406%.

IV. Analysis under Non-standard Condition: One Dealer Hole-Card and One River Card

Assume besides the player's two hole-cards, the player can also see one of the dealer's hole-cards and the river card (the last card dealt) at the beginning of game. In this non-standard situation, we analyze the game, find the optimal strategy and corresponding player advantage.

4.1 Method for deriving optimal strategy

Similarly, we use the same process of dynamic programming to derive the optimal strategy for the decisions $c_i, i = 1,2,3$. The only differences are the calculation of the conditional probabilities in each step. For each conditional probability in the process, there is two more known card's values for the given condition, y_1, one of the dealer's hole-cards and x_7, the river card. For example, $p_{31}^{(i)}$, $p_{32}^{(i)}$, $p_{33}^{(i)}$ here are changed to the probabilities of winning, winning with a straight or higher and losing given cards $y_1, x_1, x_2, x_{3j}, x_{4j}, x_{5j}, x_{6i}, x_7$, where

$$p_{31}^{(i)} = p(win|y_1, x_1, \dots, x_{6i}, x_7)$$
$$p_{32}^{(i)} = p(win\ ws\ |y_1, x_1, \dots, x_{6i}, x_7)$$
$$p_{33}^{(i)} = p(lose|y_1, x_1, \dots, x_{6i}, x_7)$$

So if $p_{31}^{(i)} > p_{33}^{(i)}$, i.e., $p(win|y_1, x_1, \dots, x_{6i}, x_7) > p(lose|y_1, x_1, \dots, x_{6i}, x_7)$, the decision for River bet is to make, i.e, $c_{3i} = 1\{p_{31}^{(i)} > p_{33}^{(i)}\}$. Other conditional probabilities are changed similarly in the process of deriving the optimal strategy.

We also use the method of Monte Carlo in drawing x_{3j}, x_{4j}, x_{5j} and computing the conditional probabilities in the process of deriving the optimal decisions. Because the number of possible combinations of the starting hand under this non-standard condition is too huge, i.e., possible combinations of player's two hole-cards, one dealer's hole-card, one river card, here we use Monte Carlo simulations for the starting hand. We simulate 50,000 starting hands and each includes a random dealer's hole-card, a random river card and player's two random hole-cards. Given each of starting hands, with the optimal strategy for turn and river bets, we compute the expected value $E_1^{(k)}, k = 1, \dots, 50000$, for each of them and get the optimal decision for the Flop bet by comparing them with the total expected value of folding (i.e. -1.000000).

4.2 Optimal Strategy

4.2.1 Flop Bet

For each simulated starting hand, we obtain the total expected values player earn for all bets if making the Flop bet. We use the method above with Monte Carlo simulations, specifically, 10,000 times simulation of three flop cards, which takes time of 53802 seconds. Among the 50000 simulated total expected values, there are 8336 starting hands whose expectation is less than -1, which means they should be folded before the Flop. So using the optimal strategy, the player would fold approximately 16.672% of his hands.

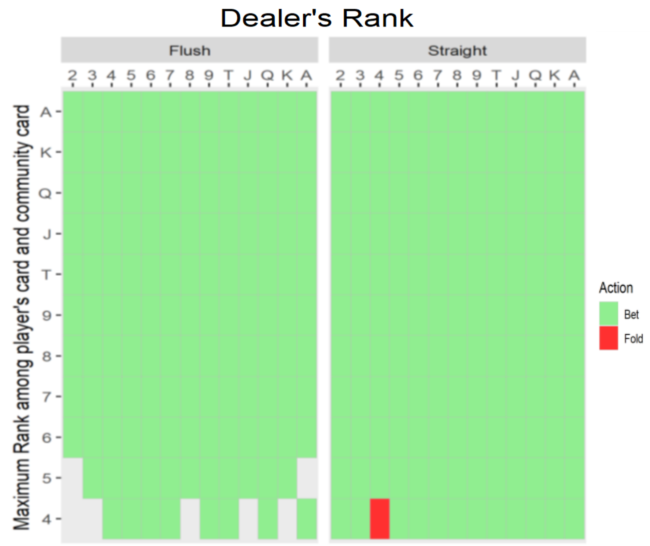
Moreover, we find the weakest starting hand which has the lowest total expected value is that the player's cards are (3d,8h), the river card is Ks and the known dealer's card is Kh. Its expected value is -2.62963. And the edge closest to -1 is that the player's cards are (10h, 3h), the river card is 6d and the known dealer's card is Ks, whose expected value is -1.00028. And the hand with the most positive expected value is that player's cards are (Kc,Kh), the river card is Kd and the known dealer's card is 2s, whose expected value is 4.10368.

In order to get the general optimal strategy for the Flop bet, we consider classifying the 50000 simulated starting hands into five situations, including the player's two hole-cards are a pair, the river card and one of the player's hole-card are a pair, the river card and the player's cards are a straight but not a flush, those three cards are a flush and those three are only a high card. In each situation, we consider different kinds of starting hands with different combinations of the largest rank among the river card and player's two hole-cards and the rank of known dealer's hole-card, and then calculate their average of the expected value. We find that all possible starting hands in which there's a pair among the river card and player's two hole-cards, have the total expected values E_1 bigger than -1, which means the Flop bet needs to be made.

Therefore, the optimal strategy for the Flop bet can be described as following.

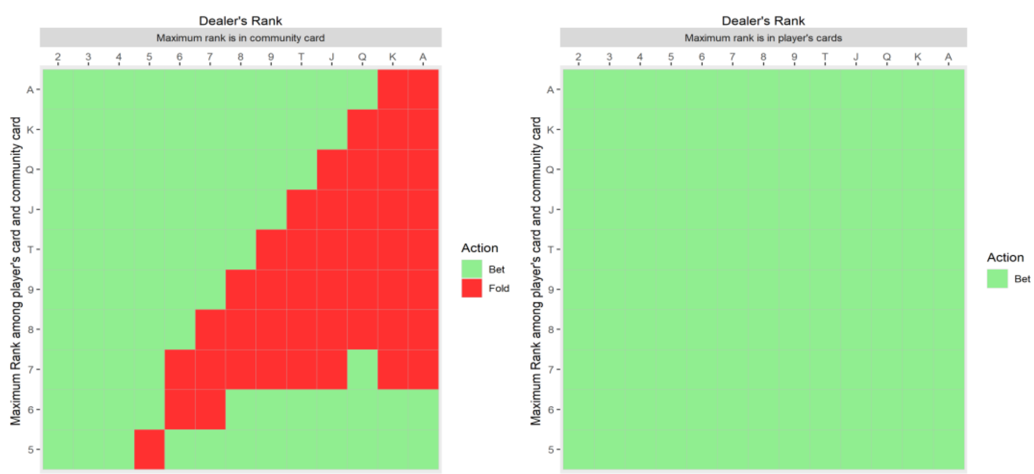
(1) If the player's two hole-cards are a pair, or the river card and one of the player's hole-card are a pair, then player should make the Flop bet whatever the known dealer's card is.

(2) If the player's two hole-cards and the river card is a straight or a flush, the optimal strategies of the Flop bet for each possible kind of starting hand are summarized in the figure below. The squares colored grey corresponding to the kinds of starting hand that are not simulated in those 50,000 starting hands because of their extremely small probabilities.



When the ranks of river card and player's two hole-cards are (2, 3, 4) but they are not a flush and the rank of known dealer's card is 4, the player should fold.

(3) If the river card and player's two hole-cards are only a high card, the optimal strategy depends on the largest rank among those three cards. If one of the player's hole-cards has the largest rank among those three, player should make the Flop bet whatever the dealer's card is. But if the river card has the largest rank among those three cards, player should fold in some situations that the known dealer's card has large rank. The optimal strategies of the Flop bet for the high card situation are summarized in the figure below.



4.2.2 Turn and River Bets

For the Turn and River bet, the player needs to make the bet if the chance of winning the hand is greater than that of losing given the player's two hole-cards, one dealer's hole card, the river card and the visible board cards, otherwise the player should check instead. The player can use the method of deriving optimal strategy to calculate c_2 at turn stage and c_3 at river stage. If we have value c_i equal to 1, we'll bet at the ith stage.

4.3 Expected Value and Variance

After obtaining the total expected values for each of the simulated starting hands, $E_1^k, k = 1, \ldots, 50000$, the expected value player earn using the optimal strategy is $E_0 = 1/50000 * \max(-1, E_1^k)$. If the player knows one dealer's hole-card and the river card, using the optimal strategy, the player advantage is **24.3639%**, which means for every unit ante bet, the player would on average win approximately 0.243639. And using the optimal strategy, the variance of value player earn per initial ante bet is 4.345459. Moreover, we find that if the player plays all hands and never folds at the flop stage, then the player advantage would be 18.8373%. For the Bonus bet, because this non-standard condition does not affect it, the house advantage of the Bonus bet is still 8.5406%.

V. Summary

Under standard condition and non-standard conditions, the number of known cards at each step of the game is summarized in the table below.

# of known cards	**Flop Bet**	**Turn Bet**	**River Bet**
Standard Condition	2	5	6
Know 1-dealer holecard	3	6	7
Know 1-dealer holecard and the river card	4	7	8

Using the optimal strategy under each condition derived by the Dynamic programming, the player's advantages and variances per unit ante bet are summarized in the table below.

	Player's Advantage	**Variance per unit Ante bet**
Standard Condition	**-2.0050251%**	2.20824
Know 1-dealer holecard	**7.586434%**	3.035162
Know 1-dealer holecard and the river card	**24.3639%**	4.345459

And under each condition, the house advantage of the Bonus bet is always 8.5406%. Therefore, we can find that more information player knows, larger the player's advantage as well as the variance.

Reference

[1] https://wizardofodds.com/games/texas-hold-em-bonus/

[2] http://www.reviewpokerrooms.com/casino-games/texas-holdem-bonus/analysis.html

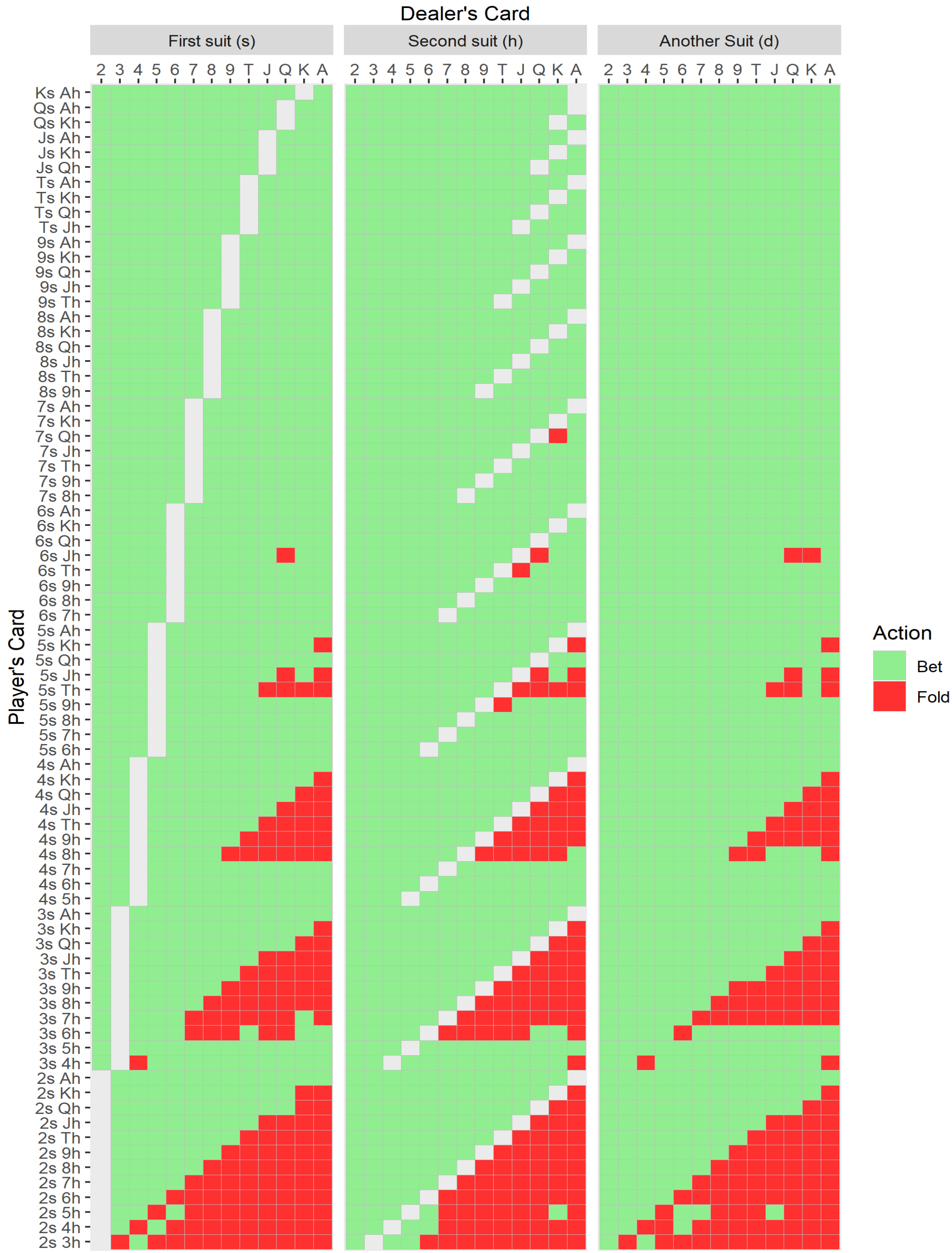
[3] http://www.reviewpokerrooms.com/casino-games/texas-holdem-bonus/bonus-bet-analysis.html

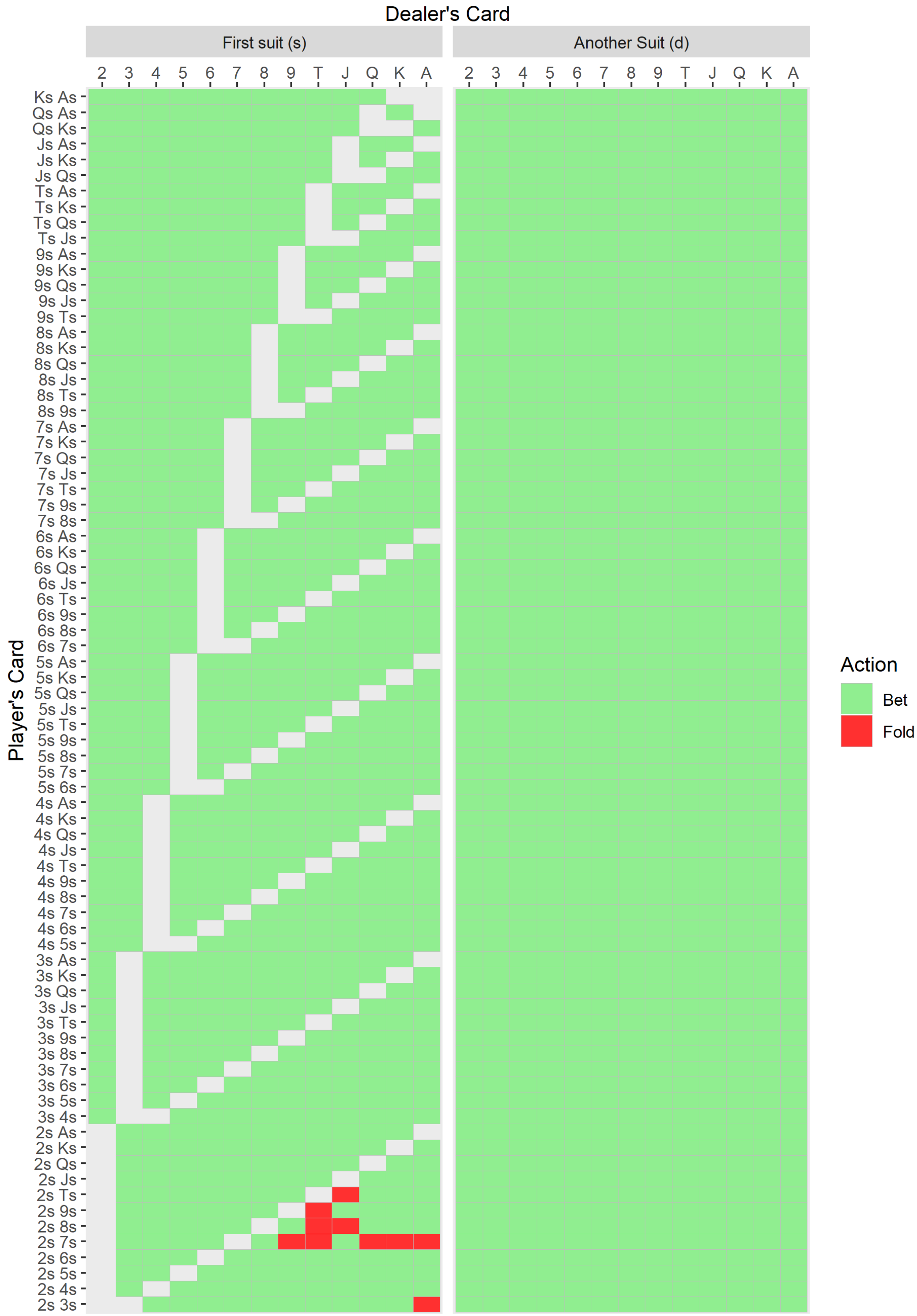
[4]https://www.888casino.com/blog/novelty-games/texas-holdem-bonus-poker-hole-card-play-one-dealer-card

[5]https://www.888casino.com/blog/novelty-games/texas-holdem-bonus-poker-hole-card-play-one-dealer-card-and-one-community-card

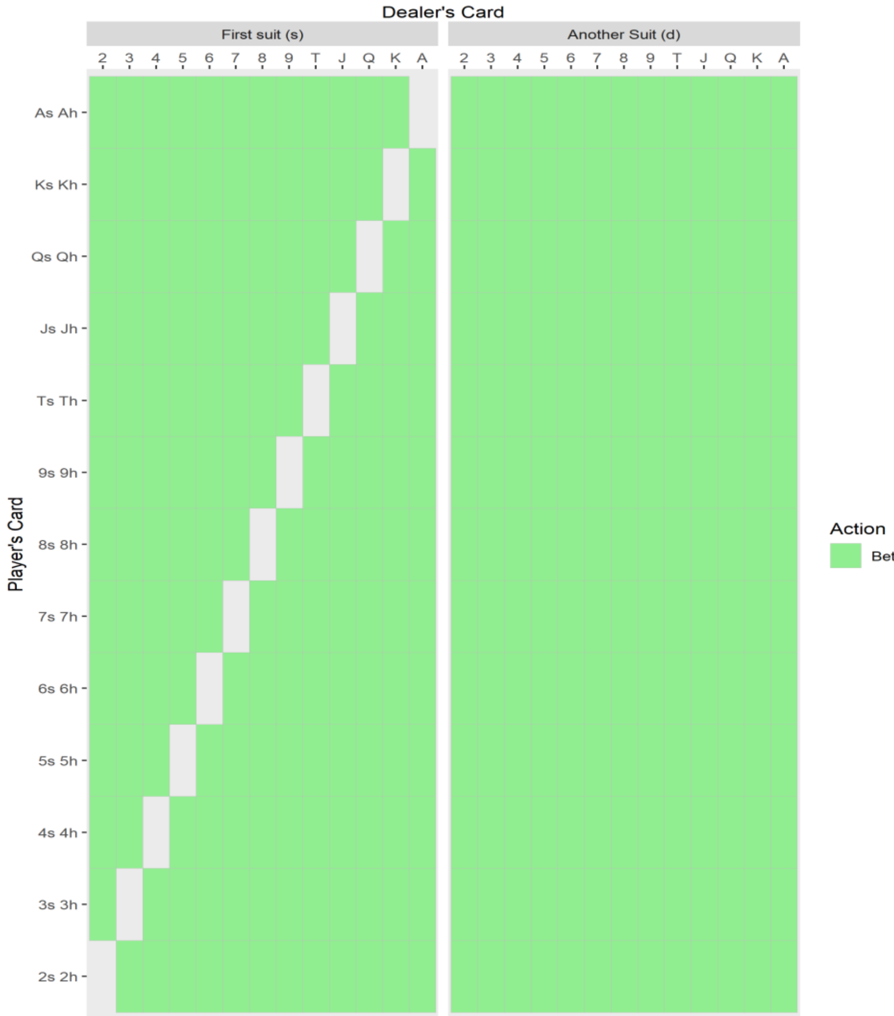
Appendix

A. Optimal deicisions of the Flop bet for 5083 different starting hands if know 1-dealer holecard





Dealer's Card
First suit (s)
Another Suit (d)
2 3 4 5 6 7 8 9 T J Q K A
Player's Card
Ks As
Qs As
Qs Ks
Js As
Js Ks
Js Qs
Ts As
Ts Ks
Ts Qs
Ts Js
9s As
9s Ks
9s Qs
9s Js
9s Ts
8s As
8s Ks
8s Qs
8s Js
8s Ts
8s 9s
7s As
7s Ks
7s Qs
7s Js
7s Ts
7s 9s
7s 8s
6s As
6s Ks
6s Qs
6s Js
6s Ts
6s 9s
6s 8s
6s 7s
5s As
5s Ks
5s Qs
5s Js
5s Ts
5s 9s
5s 8s
5s 7s
5s 6s
4s As
4s Ks
4s Qs
4s Js
4s Ts
4s 9s
4s 8s
4s 7s
4s 6s
4s 5s
3s As
3s Ks
3s Qs
3s Js
3s Ts
3s 9s
3s 8s
3s 7s
3s 6s
3s 5s
3s 4s
2s As
2s Ks
2s Qs
2s Js
2s Ts
2s 9s
2s 8s
2s 7s
2s 6s
2s 5s
2s 4s
2s 3s
Action
Bet
Fold


Dealer's Card
First suit (s)
Another Suit (d)
2 3 4 5 6 7 8 9 T J Q K A
2 3 4 5 6 7 8 9 T J Q K A
Player's Card
As Ah
Ks Kh
Qs Qh
Js Jh
Ts Th
9s 9h
8s 8h
7s 7h
6s 6h
5s 5h
4s 4h
3s 3h
2s 2h
Action
Bet

Appendix B: Proof of Optimality of Strategy for Texas Holdem

May 13, 2020

1 Dynamic Programming Principal

For a discrete deterministic control problem with the control terms denoted as C_1 and C_2, given $\mathcal{X}$ as a fixed set for all different controls, and $f_1(x, C_1)$ and $f_2(x, C_1, C_2)$ as two functions taking elements in $\mathcal{X}$ and the the control terms as input parameters, the problem we would like to optimize $V(C_1, C_2)$ and the optimal control terms $\hat{C}_1, \hat{C}_2$ are:

$$V(C_1, C_2) = \sum_{x \in \mathcal{X}} f_1(x, C_1) + \sum_{x \in \mathcal{X}} f_2(x, C_1, C_2)$$
$$\hat{C}_1, \hat{C}_2 = \arg \max_{C_1, C_2} V(C_1, C_2)$$

If we fix the control term C_1, and let $\hat{V}(C_1) = \max_{C_2} \sum_{x \in \mathcal{X}} f_2(x, C_1, C_2)$, we will have the following theorem:

Theorem 1 *Let $V(C_1, C_2)$ and $\hat{V}(C_1)$ be the same as aforementioned:*

$$\max_{C_1, C_2} V(C_1, C_2) = \max_{C_1} \sum_{x \in \mathcal{X}} [f_1(x, C_1) + \hat{V}(C_1)]$$
$$\hat{C}_2 = \arg \max_{C_2} \sum_{x \in \mathcal{X}} f_2(x, C_1, C_2)$$
$$\hat{C}_1 = \arg \max_{C_1} \sum_{x \in \mathcal{X}} [f_1(x, C_1) + \hat{V}(C_1)]$$

2 Optimal Strategy of Texas Holdem Bonus by Dyanamic Programming

Let the control terms C_1, C_2 and C_3 each be the decisions for the flop round, turn round and the river round. C_1 is 1 if the player decide to play after being dealt two cards and C_1 is 0 if the player decide to fold the cards. C_2 is 1 if the player decide to bet at the turn round and C_2 is 0 if the player decide to check. C_3 is 1 if the player decide to bet at the river round and C_2 is 0 if the player decide to check. Moreover, denote the optimal control terms as C_1^*, C_2^* and C_3^*

2.1 Formulation of the Expectation

Since the decision C_1 is based on the first two cards dealt, we could re-write it as $C_1 = C_1(X_1, X_2)$, similarly, we could re-write $C_2 = C_2(X_1, X_2, X_3, X_4, X_5)$ and $C_3 = C_3(X_1, \cdots, X_6)$, also we denote the event of winning as *win*, the event of losing as *lose* and the event of winning with a straight or higher as *win ws*. Thus the expectation of the game with given controls C_1, C_2 and C_3 and initial ante 1 dollar, denoted by $\mathcal{E}(C_1, C_2, C_3)$, is:

$$
\begin{aligned}
\mathcal{E} = & \sum_{(X_1,X_2)} \sum_{(X_3,X_4,X_5)} \sum_{X_6} C_1(X_1, X_2)\{[2 + C_2(X_1, \cdots, X_5) + C_3(X_1, \cdots, X_6)] \\
& (p(win|X_1, \cdots, X_6) - p(lose|X_1, \cdots, X_6)) + p(win \quad ws|X_1, \cdots, X_6)\}p(X_1, X_2) - 1 \\
& = \sum_{(X_1,X_2)} C_1(X_1, X_2)(p(win|X_1, X_2) - p(lose|X_1, X_2))p(X_1, X_2) \\
& + \sum_{(X_1,X_2)} C_1(X_1, X_2)[\sum_{(X_3,X_4,X_5)} C_2(X_1, \cdots, X_5)(p(win|X_1, \cdots, X_5) - p(lose|X_1, \cdots, X_5)) \\
& p(X_1, \cdots, X_5|X_1, X_2)]p(X_1, X_2) \\
& + \sum_{(X_1,X_2)} \{ \sum_{(X_3,X_4,X_5)} p(X_1, \cdots, X_5|X_1, X_2)[\sum_{X_6} C_3(X_1, \cdots, X_6) \\
& (p(win|X_1, \cdots, X_6) - p(lose|X_1, \cdots, X_6))p(X_1, \cdots, X_6|X_1, \cdots, X_5)]\}p(X_1, X_2) \\
& + \sum_{X_1,X_2} C_1(X_1, X_2)p(win \quad ws|X_1, X_2)p(X_1, X_2) \\
& - \sum_{X_1,X_2} p(X_1, X_2)
\end{aligned}
$$

2.2 Dynamic Programming Principle for Expectation

By dynamic programming principle and denoting $\mathbf{1}$ as the indicator function, it first shall yields that:

$$
\begin{aligned}
C_2^* &= \mathbf{1}\{p(win|X_1, \cdots, X_5) > p(lose|X_1, \cdots, X_5)\} \\
C_3^* &= \mathbf{1}\{p(win|X_1, \cdots, X_6) > p(lose|X_1, \cdots, X_6)\}
\end{aligned}
$$

Namely, if the winning probability is higher than the losing probability given the first five cards, the player shall choose to bet for a turn round; if the winning probability is higher given the first six cards, the player shall choose to bet for the river round.

Before deriving the optimal C_1^*, we first denote the conditioning expectation given the first two cards using C_2^* and C_3^* as decisions for the turn and river rounds as $\mathcal{E}^*(X_1, X_2)$, which is:

$$\mathcal{E}^*(X_1, X_2) = (p(win|X_1, X_2) - p(lose|X_1, X_2))$$
$$+ \sum_{(X_3,X_4,X_5)} C_2^*(p(win|X_1, \cdots, X_5) - p(lose|X_1, \cdots, X_5))p(X_1, \cdots, X_5|X_1, X_2)$$
$$+ \sum_{(X_3,X_4,X_5)} p(X_1, \cdots, X_5|X_1, X_2)[\sum_{X_6} C_3^*(p(win|X_1, \cdots, X_6) - p(lose|X_1, \cdots, X_6))$$
$$p(X_1, \cdots, X_6|X_1, \cdots, X_5)]$$
$$+ p(win \quad ws|X_1, X_2)$$

Thus the optimal C_1^* shall be:

$$C_1^* = \mathbf{1}\{\mathcal{E}^*(X_1, X_2) > 0\}$$

Namely, when the expectation for the turn and river rounds using the optimal C_2^* and C_3^* given the first two cards is greater than 0, the player shall continue to play at the flop round, otherwise, fold the cards.

2.3 Explicit Proof for the Optimality of the Strategy

Proof for Optimality of C_2^* and C_3^*

Let C_2 and C_3 be two arbitrary decisions for the turn and river rounds, and for any arbitrary decision C_1, it shall yields that:

$$\mathcal{E}(C_1, C_2^*, C_3^*) - \mathcal{E}(C_1, C_2, C_3)$$

$$= \sum_{(X_1,X_2)} C_1[\sum_{(X_3,X_4,X_5)} (C_2^* - C_2)(p(win|X_1, \cdots, X_5) - p(lose|X_1, \cdots, X_5))$$
$$p(X_1, \cdots, X_5|X_1, X_2)]p(X_1, X_2)$$
$$+ \sum_{(X_1,X_2)} C_1\{\sum_{(X_3,X_4,X_5)} p(X_1, \cdots, X_5|X_1, X_2)[\sum_{X_6}(C_3^* - C_3)(p(win|X_1, \cdots, X_6) - p(lose|X_1, \cdots, X_6))$$
$$p(X_1, \cdots, X_6|X_1, \cdots, X_5)]\}p(X_1, X_2)$$
$$\geq 0$$

Since:

$$(C_2^* - C_2)(p(win|X_1, \cdots, X_5) - p(lose|X_1, \cdots, X_5)) \geq 0$$
$$(C_3^* - C_3)(p(win|X_1, \cdots, X_6) - p(lose|X_1, \cdots, X_6)) \geq 0$$

Proof for Optimality of C_1^*

Now that we have shown that $\mathcal{E}(C_1, C_2^*, C_3^*) \geq \mathcal{E}(C_1, C_2, C_3)$, we would like to further show that $\mathcal{E}(C_1^*, C_2^*, C_3^*) \geq \mathcal{E}(C_1, C_2, C_3)$ for any arbitrary control C_1, C_2 and C_3

$$\begin{aligned}
\mathcal{E}(C_1^*, C_2^*, C_3^*) - \mathcal{E}(C_1, C_2, C_3) &= [\mathcal{E}(C_1^*, C_2^*, C_3^*) - \mathcal{E}(C_1, C_2^*, C_3^*)] + [\mathcal{E}(C_1, C_2^*, C_3^*) - \mathcal{E}(C_1, C_2, C_3)] \\
&= \sum_{(X_1, X_2)} (C_1^* - C_1)\mathcal{E}_1^*(X_1, X_2)p(X_1, X_2) + [\mathcal{E}(C_1, C_2^*, C_3^*) - \mathcal{E}(C_1, C_2, C_3)] \\
&\geq 0
\end{aligned}$$

Since by definition $(C_1^* - C_1)\mathcal{E}_1^*(X_1, X_2) \geq 0$ and we have already shown that $\mathcal{E}(C_1, C_2^*, C_3^*) \geq \mathcal{E}(C_1, C_2, C_3)$.

Summarizing the previous two parts of proof, we could show that for any other controls: C_1, C_2 and C_3, $\mathcal{E}(C_1^*, C_2^*, C_3^*) \geq \mathcal{E}(C_1, C_2, C_3)$. Therefore, the total expectation of the strategy C_1^*, C_2^* and C_3^* as defined aforementioned, $\mathcal{E}(C_1^*, C_2^*, C_3^*)$ is always higher compared to any other controls. In terms of expectation, the optimal strategy should always be:

$$\begin{gathered}
C_2^* = \mathbf{1}\{p(win|X_1, \cdots, X_5) > p(lose|X_1, \cdots, X_5)\} \\
C_3^* = \mathbf{1}\{p(win|X_1, \cdots, X_6) > p(lose|X_1, \cdots, X_6)\} \\
C_1^* = \mathbf{1}\{\mathcal{E}^*(X_1, X_2) > 0\}
\end{gathered}$$