Proof of Optimality of Startegy for Texas Holdem

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1 Dynamic Programming Principal

For a discrete deterministic control problem with the control terms denoted as C_1 and C_2, given $\mathcal{X}$ as a fixed set for all different controls, and $f_1(x, C_1)$ and $f_2(x, C_1, C_2)$ as two functions taking elements in $\mathcal{X}$ and the the control terms as input parameters, the problem we would like to optimize $V(C_1, C_2)$ and the optimal control terms $\hat{C}_1, \hat{C}_2$ are:

$$V(C_1, C_2) = \sum_{x \in \mathcal{X}} f_1(x, C_1) + \sum_{x \in \mathcal{X}} f_2(x, C_1, C_2)$$
$$\hat{C}_1, \hat{C}_2 = \arg\max_{C_1, C_2} V(C_1, C_2)$$

If we fix the control term C_1, and let $\hat{V}(C_1) = \max_{C_2} \sum_{x \in \mathcal{X}} f_2(x, C_1, C_2)$, we will have the following theorem:

Theorem 1 *Let $V(C_1, C_2)$ and $\hat{V}(C_1)$ be the same as aforementioned:*

$$\max_{C_1, C_2} V(C_1, C_2) = \max_{C_1} \sum_{x \in \mathcal{X}} [f_1(x, C_1) + \hat{V}(C_1)]$$
$$\hat{C}_2 = \arg\max_{C_2} \sum_{x \in \mathcal{X}} f_2(x, C_1, C_2)$$
$$\hat{C}_1 = \arg\max_{C_1} \sum_{x \in \mathcal{X}} [f_1(x, C_1) + \hat{V}(C_1)]$$

2 Optimal Strategy of Texas Holdem Bonus by Dyanamic Programming

Let the control terms C_1, C_2 and C_3 each be the decisions for the flop round, turn round and the river round. C_1 is 1 if the player decide to play after being dealt two cards and C_1 is 0 if the player decide to fold the cards. C_2 is 1 if the player decide to bet at the turn round and C_2 is 0 if the player decide to check. C_3 is 1 if the player decide to bet at the river round and C_2 is 0 if the player decide to check. Moreover, denote the optimal control terms as C_1^*, C_2^* and C_3^*

2.1 Formulation of the Expectation

Since the decision C_1 is based on the first two cards dealt, we could re-write it as $C_1 = C_1(X_1, X_2)$, similarly, we could re-write $C_2 = C_2(X_1, X_2, X_3, X_4, X_5)$ and $C_3 = C_3(X_1, \cdots, X_6)$, also we denote the event of winning as *win*, the event of losing as *lose* and the event of winning with a straight or higher as *win ws*. Thus the expectation of the game with given controls C_1, C_2 and C_3 and initial ante 1 dollar, denoted by $\mathcal{E}(C_1, C_2, C_3)$, is:

$$
\begin{aligned}
\mathcal{E} = & \sum_{(X_1,X_2)} \sum_{(X_3,X_4,X_5)} \sum_{X_6} C_1(X_1,X_2)\{[2 + C_2(X_1,\cdots,X_5) + C_3(X_1,\cdots,X_6)] \\
& (p(win|X_1,\cdots,X_6) - p(lose|X_1,\cdots,X_6)) + p(win \quad ws|X_1,\cdots,X_6)\}p(X_1,X_2) - 1 \\
= & \sum_{(X_1,X_2)} C_1(X_1,X_2)(p(win|X_1,X_2) - p(lose|X_1,X_2))p(X_1,X_2) \\
& + \sum_{(X_1,X_2)} C_1(X_1,X_2)[\sum_{(X_3,X_4,X_5)} C_2(X_1,\cdots,X_5)(p(win|X_1,\cdots,X_5) - p(lose|X_1,\cdots,X_5)) \\
& p(X_1,\cdots,X_5|X_1,X_2)]p(X_1,X_2) \\
& + \sum_{(X_1,X_2)} \{ \sum_{(X_3,X_4,X_5)} p(X_1,\cdots,X_5|X_1,X_2)[\sum_{X_6} C_3(X_1,\cdots,X_6) \\
& (p(win|X_1,\cdots,X_6) - p(lose|X_1,\cdots,X_6))p(X_1,\cdots,X_6|X_1,\cdots,X_5)]\}p(X_1,X_2) \\
& + \sum_{X_1,X_2} C_1(X_1,X_2)p(win \quad ws|X_1,X_2)p(X_1,X_2) \\
& - \sum_{X_1,X_2} p(X_1,X_2)
\end{aligned}
$$

2.2 Dynamic Programming Principle for Expectation

By dynamic programming principle and denoting $\mathbf{1}$ as the indicator function, it first shall yields that:

$$
\begin{aligned}
C_2^* &= \mathbf{1}\{p(win|X_1,\cdots,X_5) > p(lose|X_1,\cdots,X_5)\} \\
C_3^* &= \mathbf{1}\{p(win|X_1,\cdots,X_6) > p(lose|X_1,\cdots,X_6)\}
\end{aligned}
$$

Namely, if the winning probability is higher than the losing probability given the first five cards, the player shall choose to bet for a turn round; if the winning probability is higher given the first six cards, the player shall choose to bet for the river round.

Before deriving the optimal C_1^*, we first denote the conditioning expectation given the first two cards using C_2^* and C_3^* as decisions for the turn and river rounds as $\mathcal{E}^*(X_1, X_2)$, which is:

$$
\begin{aligned}
\mathcal{E}^*(X_1,X_2) = (&p(win|X_1,X_2) - p(lose|X_1,X_2)) \\
&+ \sum_{(X_3,X_4,X_5)} C_2^*(p(win|X_1,\cdots,X_5) - p(lose|X_1,\cdots,X_5))p(X_1,\cdots,X_5|X_1,X_2) \\
&+ \sum_{(X_3,X_4,X_5)} p(X_1,\cdots,X_5|X_1,X_2)[\sum_{X_6} C_3^*(p(win|X_1,\cdots,X_6) - p(lose|X_1,\cdots,X_6)) \\
&p(X_1,\cdots,X_6|X_1,\cdots,X_5)] \\
&+ p(win \quad ws|X_1,X_2)
\end{aligned}
$$

Thus the optimal C_1^* shall be:

$$C_1^* = \mathbf{1}\{\mathcal{E}^*(X_1,X_2) > 0\}$$

Namely, when the expectation for the turn and river rounds using the optimal C_2^* and C_3^* given the first two cards is greater than 0, the player shall continue to play at the flop round, otherwise, fold the cards.

2.3 Explicit Proof for the Optimality of the Strategy

Proof for Optimality of C_2^* and C_3^*

Let C_2 and C_3 be two arbitrary decisions for the turn and river rounds, and for any arbitrary decision C_1, it shall yields that:

$$\mathcal{E}(C_1, C_2^*, C_3^*) - \mathcal{E}(C_1, C_2, C_3)$$

$$
\begin{aligned}
&= \sum_{(X_1,X_2)} C_1[\sum_{(X_3,X_4,X_5)} (C_2^* - C_2)(p(win|X_1,\cdots,X_5) - p(lose|X_1,\cdots,X_5)) \\
&p(X_1,\cdots,X_5|X_1,X_2)]p(X_1,X_2) \\
&+ \sum_{(X_1,X_2)} C_1\{\sum_{(X_3,X_4,X_5)} p(X_1,\cdots,X_5|X_1,X_2)[\sum_{X_6}(C_3^* - C_3)(p(win|X_1,\cdots,X_6) - p(lose|X_1,\cdots,X_6)) \\
&p(X_1,\cdots,X_6|X_1,\cdots,X_5)]\}p(X_1,X_2) \\
&\geq 0
\end{aligned}
$$

Since:

$$
\begin{aligned}
(C_2^* - C_2)(p(win|X_1,\cdots,X_5) - p(lose|X_1,\cdots,X_5)) \geq 0 \\
(C_3^* - C_3)(p(win|X_1,\cdots,X_6) - p(lose|X_1,\cdots,X_6)) \geq 0
\end{aligned}
$$

Proof for Optimality of C_1^*

Now that we have shown that $\mathcal{E}(C_1, C_2^*, C_3^*) \geq \mathcal{E}(C_1, C_2, C_3)$, we would like to further show that $\mathcal{E}(C_1^*, C_2^*, C_3^*) \geq \mathcal{E}(C_1, C_2, C_3)$ for any arbitrary control C_1, C_2 and C_3

$$\begin{aligned}
\mathcal{E}(C_1^*, C_2^*, C_3^*) - \mathcal{E}(C_1, C_2, C_3) &= [\mathcal{E}(C_1^*, C_2^*, C_3^*) - \mathcal{E}(C_1, C_2^*, C_3^*)] + [\mathcal{E}(C_1, C_2^*, C_3^*) - \mathcal{E}(C_1, C_2, C_3)] \\
&= \sum_{(X_1, X_2)} (C_1^* - C_1)\mathcal{E}_1^*(X_1, X_2) p(X_1, X_2) + [\mathcal{E}(C_1, C_2^*, C_3^*) - \mathcal{E}(C_1, C_2, C_3)] \\
&\geq 0
\end{aligned}$$

Since by definition $(C_1^* - C_1)\mathcal{E}_1^*(X_1, X_2) \geq 0$ and we have already shown that $\mathcal{E}(C_1, C_2^*, C_3^*) \geq \mathcal{E}(C_1, C_2, C_3)$.

Summarizing the previous two parts of proof, we could show that for any other controls: C_1, C_2 and C_3, $\mathcal{E}(C_1^*, C_2^*, C_3^*) \geq \mathcal{E}(C_1, C_2, C_3)$. Therefore, the total expectation of the strategy C_1^*, C_2^* and C_3^* as defined aforementioned, $\mathcal{E}(C_1^*, C_2^*, C_3^*)$ is always higher compared to any other controls. In terms of expectation, the optimal strategy should always be:

$$\begin{gathered}
C_2^* = \mathbf{1}\{p(win|X_1, \cdots, X_5) > p(lose|X_1, \cdots, X_5)\} \\
C_3^* = \mathbf{1}\{p(win|X_1, \cdots, X_6) > p(lose|X_1, \cdots, X_6)\} \\
C_1^* = \mathbf{1}\{\mathcal{E}^*(X_1, X_2) > 0\}
\end{gathered}$$