Hypothesis Testing for the Homogeneity of Two Zero-and-one-inflated Poisson Populations

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Abstract. Melkersson and Olsson (1999) proposed *zero-and-one-inflated Poisson* (ZOIP) as an extension of the *zero-inflated Poisson* (ZIP) distribution in order to deal with count data with excess zeros and ones simultaneously. Zhang *et al.* (2016) explored the distributional theory and corresponding properties of the ZOIP, and developed likelihood-based methods for parameters of interest. However, most of the works about ZOIP distribution were conducted in one ZOIP population case but relatively few works are discussing comparison of two ZOIP models. In this paper, we consider the problem of comparing the parameters of two ZOIP populations. With statistical inferences of ZOIP populations by the *expectation-maximization* (EM) algorithm and proper transformations, we derive likelihood ratio tests, score tests and Wald tests for homogeneity of two ZOIP populations under five different hypotheses: (1) there's no difference between two populations in the group of zeros; (2) there's no difference between two populations in the group of ones; (3) there's no difference between two populations in the group of non-zero-and-ones; (4) there's no difference between two populations in both groups of zeros and ones; (5) there's no difference between two distributions in groups of zeros, ones and non-zero-and-ones. Simulation studies are conducted to compare the proposed tests by assessing error rate and power for different sample sizes. Finally, three real data sets are used to illustrate the proposed methods of testing the homogeneity of two ZOIP populations.

Keywords: Zero-and-one-inflated Poisson distribution; Hypothesis testing; EM algorithm; LRT; Score test; Wald test.

1. Introduction

Traditional Poisson model performs not well in dealing with count data with excess zeros which arise frequently in different fields such as patent applications (Crepon & Duguet, 1997), species abundance (Welsh *et al.* 1996; Faddy, 1998), road safety (Miaou, 1994), etc. *Zero-inflated Poisson* (ZIP) model proposed by Lambert plays a key role to address this issue. A review of useful methods to handle count data with excess zeros was reported by Ridout *et al.* (1998) later. Tse *et al.* (2009) derived likelihood ratio tests (LRT) for testing treatment difference in the occurrence of a safety parameter in a randomized parallel-group comparative clinical trial under the assumption that the number of occurrence follows a ZIP distribution. Also approximate formulas for sample size calculation to achieve a desired power for detecting a clinically meaningful difference are obtained. However, in many cases, there's a high proportion of both zeros and ones in the count data. For example, the visits to a dentist in a year, count data from Swedish Level of Living Survey from 1968 and 1981 reported by Eriksson and Åberg (1987), contain excess zeros and ones simultaneously. To fit the dentist visiting data, Melkersson and Olsson (1999) proposed *zero-and-one-inflated Poisson* (ZOIP) as an extension of the ZIP. A Bayesian analysis of the same dentist visiting data were developed by Saito and Radrigues (2005) subsequently without considering any covariates. Zhang *et al.* (2016) explored the distributional theory and corresponding properties of the ZOIP and developed likelihood-based methods for parameters of interest. Five equivalent stochastic representations for the ZOIP random variable were constructed and maximum likelihood estimate of parameters were obtained by both the Fisher scoring and expectation-maximization algorithms. Bootstrap confidence intervals for parameters and testing hypotheses under large sample sizes were also provided by them. Later, Liu *et al.* (2018) investigated the maximum likelihood estimation (MLE) and Bayesian estimation for the ZOIP and derived the reference prior and the Jeffreys prior. They also discussed about the ZOIP regression model.

Most of the works cited above were conducted in one ZOIP population case but relatively few works are discussing comparison of two ZOIP models. Given two ZOIP models, there are five possible scenarios of common interest: (1) there's no difference between two populations in the group of zeros; (2) there's no difference between two populations in the group of ones; (3) there's no difference between two populations in the group of non-zero-and-ones; (4) there's no difference between two populations in both groups of zeros and ones; (5) there's no difference between two distributions in groups of zeros, ones and non-zero-and-ones. In this article, the main objective is to derive likelihood ratio tests, score tests and Wald tests for homogeneity of two ZOIP populations under the null hypotheses specified in each of the five scenarios after making the likelihood-based inferences.

The remainder of the paper is organized as follows. Section 2 makes the likelihood-based inferences about combined data of two ZOIP populations by the *expectation-maximization* (EM) algorithm, considering transformation. In Section 3, likelihood ratio tests, score tests and Wald tests for homogeneity of two ZOIP populations are derived under the five hypotheses using proper transformations. Simulation studies to compare likelihood ratio tests, score tests and Wald tests are conducted in Section 4. In Section 5, three real data sets are used to illustrate the proposed tests for homogeneity of two ZOIP populations. A discussion is given in Section 6.

2. Likelihood-based Inferences

Assume that $Y_{11}, \ldots, Y_{1n_1} \stackrel{\text{iid}}{\sim} \text{ZOIP}(\phi_{10}, \phi_{11}; \lambda_1)$ and $y_{11}, \ldots, y_{1n_1}$ denote their realizations($\phi_{10} \leq 0,\ \phi_{11} \leq 0,\ \lambda_1 \leq 0$). Also $Y_{21}, \ldots, Y_{2n_2} \stackrel{\text{iid}}{\sim} \text{ZOIP}(\phi_{20}, \phi_{21}; \lambda_2)$ and $y_{21}, \ldots, y_{2n_2}$ denote their realizations Let $Y_{\text{obs}} = \{y_{ij}\}_{j=1}^{n_i} (i = 1, 2)$ denote the observed data. Moreover, let $\mathbb{Q}_{10} \hat{=} \{j | y_{1j} = 0,\ 1 \leqslant j \leqslant n_1\}$ and $m_{10} = \sum_{j=1}^{n} I(y_{1j} = 0)$ denote the number of elements in $\mathbb{Q}_{10}$; Similarly, let $\mathbb{Q}_{20} \hat{=} \{j | y_{2j} = 0,\ 1 \leqslant j \leqslant n_2\}$ and $m_{20} = \sum_{j=1}^{n} I(y_{2j} = 0)$ denote the number of elements in $\mathbb{Q}_{20}$. $\mathbb{Q}_{11} \hat{=} \{j | y_{1j} = 1,\ 1 \leqslant j \leqslant n_2\}$ and $m_{11} = \sum_{i=1}^{n} I(y_{1j} = 1)$ denote the number of elements in $\mathbb{Q}_{11}$. Also $\mathbb{Q}_{21} \hat{=} \{j | y_{2j} = 1,\ 1 \leqslant j \leqslant n_2\}$ and $m_{21} = \sum_{i=1}^{n} I(y_{2j} = 1)$ denote the number of elements in $\mathbb{Q}_{21}$. The observed-data likelihood function for $(\phi_{10}, \phi_{11}, \lambda_1, \phi_{20}, \phi_{21}, \lambda_2)$ is then given by

$$\begin{aligned} L(\phi_{10}, \phi_{11}, \lambda_1, \phi_{20}, \phi_{21}, la_2 | Y_{\text{obs}}) \ = \ & (\phi_{10} + \phi_{12} \mathrm{e}^{-\lambda_1})^{m_{10}} \times (\phi_{11} + \phi_{12} \lambda_1 \, \mathrm{e}^{-\lambda_1})^{m_{11}} \times \phi_{12}^{n_1 - m_{10} - m_{11}} \\ & \times \prod_{j \notin \mathbb{Q}_{10} \cup \mathbb{Q}_{11}} \frac{\lambda^{y_{1j}} \mathrm{e}^{-\lambda_1}}{y_{1j}!} \times (\phi_{20} + \phi_{22} \mathrm{e}^{-\lambda_2})^{m_{20}} \times (\phi_{21} + \phi_{22} \lambda_2 \, \mathrm{e}^{-\lambda_2})^{m_{21}} \\ & \times \phi_{22}^{n_2 - m_{20} - m_{21}} \prod_{j \notin \mathbb{Q}_{20} \cup \mathbb{Q}_{21}} \frac{\lambda^{y_{2j}} \mathrm{e}^{-\lambda_2}}{y_{2j}!}, \end{aligned} \tag{2.1}$$

so that the log-likelihood function is

$$
\begin{aligned}
\ell \;\hat{=}\; & \ell(\phi_{10}, \phi_{20}, \phi_{11}, \phi_{21}, \lambda_1, \lambda_2 | Y_{\rm obs}) = m_{10} \log(\phi_{10} + \phi_{12} \mathrm{e}^{-\lambda_1}) + m_{11} \log(\phi_{11} + \phi_{12}\lambda_1\, \mathrm{e}^{-\lambda_1}) \\
& + m_{20} \log(\phi_{20} + \phi_{22} \mathrm{e}^{-\lambda_2}) + m_{21} + \log(\phi_{11} + \phi_{12}\lambda_1\, \mathrm{e}^{-\lambda_1}) + N_1 \log \lambda_1 + N_2 \log \lambda_2 \\
& + (n_1 - m_{10} - m_{11})(\log \phi_{12} - \lambda_1) + (n_2 - m_{20} - m_{21})(\log \phi_{22} - \lambda_2),
\end{aligned}
$$

where $\phi_{12} = 1 - \phi_{10} - \phi_{11}$ and $N_1 = \sum_{j \notin \mathbb{Q}_{10} \cup \mathbb{Q}_{11}} y_{1j}$, and $\phi_{22} = 1 - \phi_{20} - \phi_{21}$ and $N_2 = \sum_{j \notin \mathbb{Q}_{20} \cup \mathbb{Q}_{21}} y_{2j}$. To calculate the Fisher information matrix, we need the following results. From Zhang *et al.* (2016), we get following theorem:

Theorem 1 (Expectations). The expectations of m_{10}, m_{11}, m_{20}, m_{21}, N_1 and N_2 are given by

$$
\begin{aligned}
E(m_{10}) &= n_1(\phi_{10} + \phi_{12}\mathrm{e}^{-\lambda_1}), \quad E(m_{11}) = n_1(\phi_{11} + \phi_{12}\lambda_1\, \mathrm{e}^{-\lambda_1}) \\
E(m_{20}) &= n_2(\phi_{20} + \phi_{22}\mathrm{e}^{-\lambda_2}), \quad E(m_{21}) = n_2(\phi_{21} + \phi_{22}\lambda_2\, \mathrm{e}^{-\lambda_2}) \\
E(N_1) &= n_1\phi_{12}\lambda_1(1 - \mathrm{e}^{-\lambda_1}), \quad E(N_2) = n_2\phi_{22}\lambda_2(1 - \mathrm{e}^{-\lambda_2})
\end{aligned} \tag{2.2}
$$

Proof. The expressions of $E(m_{10})$, $E(m_{11})$, $E(m_{20})$, and $E(m_{21})$ are easy to show. To verify the $E(N_1)$ and $E(N_2)$, we noted that

$$
N_1 = \sum_{i \notin \mathbb{Q}_{10} \cup \mathbb{Q}_{11}} y_{1i} = \sum_{i=1}^{n_1} y_{1i} - m_{11}. \tag{2.3}
$$

Thus, $E(N_1) = n_1 E(Y_1) - E(m_1) \overset{(2.3)}{=} n_1(\phi_{11} + \phi_{12}\lambda_1) - n_1(\phi_{11} + \phi_{12}\lambda_1\, \mathrm{e}^{-\lambda_1}) = n_1\phi_{12}\lambda_1(1 - \mathrm{e}^{-\lambda_1})$. Similarly, by the same process, we can proof $E(N_2)$. □

2.1 MLEs via the EM algorithm

The *extra zeros* from the degenerate distribution at zero because of population variability at the point zero and the *structural zeros* from an ordinary Poisson distribution formed the *zero* observations from a ZOIP distribution. Similarly, the *one* observations can be categorized into the *extra ones* resulted from the degenerate distribution at one because of population variability at the point one and *structural ones* came from an ordinary Poisson distribution. Thus, we partition

$$\begin{aligned} \mathbb{Q}_{10} &= \mathbb{Q}_{10}^{\text{extra}} \cup \mathbb{Q}_{10}^{\text{structural}}, \quad \mathbb{Q}_{11} = \mathbb{Q}_{11}^{\text{extra}} \cup \mathbb{Q}_{11}^{\text{structural}}, \\ \mathbb{Q}_{20} &= \mathbb{Q}_{20}^{\text{extra}} \cup \mathbb{Q}_{20}^{\text{structural}} \quad \text{and} \quad \mathbb{Q}_{21} = \mathbb{Q}_{21}^{\text{extra}} \cup \mathbb{Q}_{21}^{\text{structural}} \end{aligned}$$

Note that a major obstacle from obtaining explicit solutions of MLEs of parameters from (2.1) is the first, second, fifth, and sixth terms in the right-hand-side of (2.1). To overcome this difficulty, we augment Y_{obs} with four latent variables Z_{10}, Z_{11}, Z_{20}, and Z_{21}, where Z_{10} denotes the number of $\mathbb{Q}_{10}^{\text{extra}}$ to split m_{10} into Z_{10} and $m_{10} - Z_{10}$, Z_{20} denotes the number of $\mathbb{Q}_{20}^{\text{extra}}$ to split m_{20} into Z_{20} and $m_{20} - Z_{20}$. Similarly, Z_{11} denotes the number of $\mathbb{Q}_{11}^{\text{extra}}$ to split m_{11} into Z_{11} and $m_{11} - Z_{11}$. Z_{21} denotes the number of $\mathbb{Q}_{21}^{\text{extra}}$ to split m_{21} into Z_{21} and $m_{21} - Z_{21}$. Thus, the resulting conditional predictive distributions of Z_{10}, Z_{11}, Z_{20} , and Z_{21} are given by

$$Z_{10}|(Y_{\text{obs}}, \phi_{10}, \phi_{11}, \lambda_1) \sim \text{Binomial}\left(m_{10}, \frac{\phi_{10}}{\phi_{10} + \phi_{12}\mathrm{e}^{-\lambda_1}}\right) \tag{2.4}$$

$$Z_{11}|(Y_{\text{obs}}, \phi_{10}, \phi_{11}, \lambda_1) \sim \text{Binomial}\left(m_{11}, \frac{\phi_{11}}{\phi_{11} + \phi_{12}\lambda_1\,\mathrm{e}^{-\lambda_1}}\right) \tag{2.5}$$

$$Z_{20}|(Y_{\text{obs}}, \phi_{20}, \phi_{21}, \lambda_2) \sim \text{Binomial}\left(m_{20}, \frac{\phi_{20}}{\phi_{20} + \phi_{22}\mathrm{e}^{-\lambda_2}}\right) \tag{2.6}$$

$$Z_{21}|(Y_{\text{obs}}, \phi_{20}, \phi_{21}, \lambda_2) \sim \text{Binomial}\left(m_{21}, \frac{\phi_{21}}{\phi_{21} + \phi_{22}\lambda_2\,\mathrm{e}^{-\lambda_2}}\right) \tag{2.7}$$

respectively. On the other hand, the complete-data likelihood function is proportional to

$$
\begin{aligned}
L(\phi_{10}, \phi_{11}, \lambda_1, \phi_{20}, \phi_{21}, la_2|Y_{\rm com}) &\propto \phi_{10}^{z_{10}}[\phi_{12}\mathrm{e}^{-\lambda_1}]^{m_{10}-z_{10}} \times \phi_{11}^{z_{11}}[\phi_{12}\lambda_1\,\mathrm{e}^{-\lambda_1}]^{m_{11}-z_{11}} \\
&\quad \times \phi_{20}^{z_{20}}[\phi_{22}\mathrm{e}^{-\lambda_2}]^{m_{20}-z_{20}} \times \phi_{21}^{z_{21}}\phi_{22}\lambda_2\,\mathrm{e}^{-\lambda_2}]^{m_{21}-z_{21}} \\
&\quad \times \phi_{12}^{n_1-m_{10}-m_{11}}\mathrm{e}^{-(n_1-m_{10}-m_{11})\lambda_1}\lambda_1^{N_1} \\
&\quad \times \phi_{22})^{n_2-m_{20}-m_{21}}\mathrm{e}^{-(n_2-m_{20}-m_{21})\lambda_2}\lambda_2^{N_2} \\
&= \phi_{10}^{z_{10}}\phi_{11}^{z_{11}}(1-\phi_{10}-\phi_{11})^{n_1-z_{10}-z_{11}}\mathrm{e}^{-(n_1-z_{10}-z_{11})\lambda_1}\lambda_1^{m_{11}-z_{11}+N_1} \\
&\quad \times\phi_{20}^{z_{20}}\phi_{21}^{z_{21}}(1-\phi_{20}-\phi_{21})^{n_2-z_{20}-z_{21}} \times \mathrm{e}^{-(n_2-z_{20}-z_{21})\hat{\lambda}_2} \\
&\quad \times\lambda_2^{m_{21}-z_{21}+N_2}. \qquad (2.8)
\end{aligned}
$$

Thus the M-step is to find the complete-data MLEs

$$
\begin{aligned}
&\hat{\phi}_{10} = \frac{z_{10}}{n_1}, \hat{\phi}_{11} = \frac{z_{11}}{n_1}, \hat{\phi}_{20} = \frac{z_{20}}{n_2}, \hat{\phi}_{21} = \frac{z_{21}}{n_2}, \\
&\hat{\lambda}_1 = \frac{N_1 + m_{11} - z_{11}}{n_1 - z_{10} - z_{11}} = \frac{N_1 + m_{11} - n_1\hat{\phi}_{11}}{n_1(1-\hat{\phi}_{10}-\hat{\phi}_{11})}, \quad \text{and} \\
&\hat{\lambda}_2 = \frac{N_2 + z_{21} - z_{21}}{n_2 - z_{20} - z_{21}} = \frac{N_2 + m_{21} - n_2\hat{\phi}_{21}}{n_2(1-\hat{\phi}_{20}-\hat{\phi}_{21})}. \qquad (2.9)
\end{aligned}
$$

The E-step is to replace z_{10}, z_{11}, z_{20}, and z_{21} in (2.8) by their conditional expectations:

$$
\begin{aligned}
E(Z_{10}|Y_{\rm obs}, \phi_{10}, \phi_{11}, \lambda_1) &= \frac{m_{10}\phi_{10}}{\phi_{10} + (1-\phi_{10}-\phi_{11})\mathrm{e}^{-\lambda_1}} \\
E(Z_{20}|Y_{\rm obs}, \phi_{20}, \phi_{21}, \lambda_2) &= \frac{m_{20}\phi_{20}}{\phi_{20} + (1-\phi_{20}-\phi_{21})\mathrm{e}^{-\lambda_2}} \\
E(Z_{11}|Y_{\rm obs}, \phi_{10}, \phi_{11}, \lambda_1) &= \frac{m_{11}\phi_{11}}{\phi_{11} + (1-\phi_{10}-\phi_{11})\mathrm{e}^{-\lambda_1}\lambda_1} \\
E(Z_{21}|Y_{\rm obs}, \phi_{20}, \phi_{21}, \lambda_2) &= \frac{m_{21}\phi_{21}}{\phi_{21} + (1-\phi_{20}-\phi_{21})\mathrm{e}^{-\lambda_2}\lambda_2}.
\end{aligned}
\qquad (2.10)
$$

3. Testing Homogeneity of Two ZOIP Populations with Large Sample Sizes

In this section, we utilize LRT, score test, and Wald test for testing several hypotheses of $(i)H_{01} : (\lambda_1, \phi_{10}, \phi_{11}) = (\lambda_2, \phi_{20}, \phi_{21})$; $(ii)H_{02} : (\phi_{10}, \phi_{11}) = (\phi_{20}, \phi_{21})$; $(iii)H_{03} : \lambda_1 = \lambda_2$; $(iv)H_{04} : \phi_{10} = \phi_{20}$; $(v)H_{05} : \phi_{11} = \phi_{21}$. Also, in order to derive tests, we need to make a transformation:

$$\begin{aligned} \alpha_1 &= \lambda_1 - \lambda_2,\ \beta_1 = \phi_{10} - \phi_{20},\ \gamma_1 = \phi_{11} - \phi_{21}, \\ \alpha_2 &= \lambda_2,\ \beta_2 = \phi_{20},\ \gamma_2 = \phi_{21}. \end{aligned} \tag{3.1}$$

Since the parameter values are limited in the bounded parameter space, we conduct logistic transformation to parameters above:

$$\begin{aligned} \theta_{10} &= \log \frac{\beta_1 + \beta_2}{1 - (\beta_1 + \beta_2)}, \quad \theta_{20} = \log \frac{\beta_2}{1 - \beta_2}, \\ \theta_{11} &= \log \frac{\gamma_1 + \gamma_2}{1 - (\gamma_1 + \gamma_2)}, \quad \theta_{21} = \log \frac{\gamma_2}{1 - \gamma_2}. \end{aligned} \tag{3.2}$$

Then, it can be easily proved that testing several hypotheses specified before is equivalent to testing $(i)H_{01}^* : (\lambda_1, \theta_{10}, \theta_{11}) = (\lambda_2, \theta_{20}, \theta_{21})$; $(ii)H_{02}^* : (\theta_{10}, \theta_{11}) = (\theta_{20}, \theta_{21})$; $(iii)H_{03}^* : \lambda_1 = \lambda 2$; $(iv)H_{04}^* : \theta_{10} = \phi_{20}$; $(v)H_{05}^* : \theta_{11} = \theta_{21}$.

3.1 Tests for equality of zero-and-one inflation parameters and non-zero-and-one means between two populations

In this section, we will develop likelihood ratio test, score test and Wald test to examine equality of zero-and-one inflation parameters and non-zero-and-one means between two ZOIP populations. The null and alternative hypotheses are:

$$H_{01}\colon\ (\lambda_1, \phi_{10}, \phi_{11}) = (\lambda_2, \phi_{20}, \phi_{21}) \quad \text{against} \quad H_{11}\colon\ (\lambda_1, \phi_{10}, \phi_{11}) \neq (\lambda_2, \phi_{20}, \phi_{21}). \tag{3.3}$$

3.1.1 Likelihood ratio test

By transformation, the null hypothesis and alternative hypotheses specified by (3.3) are equivalent to:

$$H_{01,L} : (\alpha_1, \beta_1, \gamma_1) = (0, 0, 0) \quad \text{against} \quad H_{11,L} : (\alpha_1, \beta_1, \gamma_1) \neq (0, 0, 0) \tag{3.4}$$

Under $H_{01,L}$, the LRT test statistic is given by

$$T_1 = -2 \left\{ \ell(0, 0, 0, \hat{\alpha}_{2,H_{01}}, \hat{\beta}_{2,H_{01}}, \hat{\gamma}_{2,H_{01}}) - \ell(\hat{\alpha}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2) \right\} \tag{3.5}$$

where $(\hat{\alpha}_{2,H_{01}}, \hat{\beta}_{2,H_{01}}, \hat{\gamma}_{2,H_{01}})$ denotes the constrained MLEs of $(\alpha_2, \beta_2, \gamma_2)$ under $H_{01,L}$ and $(\hat{\alpha}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2)$ denotes the unconstrained MLEs of $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$, which are obtained by EM algorithm from (2.4)-(2.10). Note that under H_0, the MLEs of $(\alpha_2, \beta_2, \gamma_2)$ can be calculated by the following EM iteration

$$\begin{aligned}
\alpha_{2,H_{01}}^{(t+1)} &= \frac{N_1 + N_2 + m_{11} + m_{21} - (n_1 + n_2)\gamma_{2,H_{01}}^{(t)}}{((n1 + n2)(1 - \beta_{2,H_{01}}^{(t)} - \gamma_{2,H_{01}}^{(t)}))} \\
\beta_{2,H_{01}}^{(t+1)} &= \frac{(m_{10} + m_{20})\beta_{2,H_{01}}^{(t)}}{(n_1 + n_2)\left[\beta_{2,H_{01}}^{(t)} + (1 - \beta_{2,H_{01}}^{(t)} - \gamma_{2,H_{01}}^{(t)})e^{-\alpha_{2,H_{01}}^{(t)}}\right]} \\
\gamma_{2,H_{01}}^{(t+1)} &= \frac{(m_{11} + m_{21})\gamma_{2,H_{01}}^{(t)}}{(n_1 + n_2)\left[\gamma_{2,H_{01}}^{(t)} + (1 - \beta_{2,H_{01}}^{(t)} - \gamma_{2,H_{01}}^{(t)})e^{-\alpha_{2,H_{01}}^{(t)}}\alpha_{2,H_{01}}^{(t)}\right]}
\end{aligned} \tag{3.6}$$

Standard large-sample theory suggests that the asymptotic null distribution of T_1 is $\chi^2(3)$. Hence the corresponding p-value is

$$p_{v1} = Pr(T_1 > t_1 | H_{01,L}) = Pr(\chi^2(3) > t_1). \tag{3.7}$$

3.1.2 Score test

Next, we develop a score test to examine equality of zero-and-one inflation parameters and non-zero-and-one means. By transformation, four new parameters are introduced as follows:

$$
\begin{aligned}
\nu_1 &= \theta_{10} - \theta_{20} = \log \frac{\beta_1 + \beta_2}{1 - (\beta_1 + \beta_2)} - \log \frac{\beta_2}{1 - \beta_2} = \log \frac{\phi_{10}}{1 - \phi_{10}} - \log \frac{\phi_{20}}{1 - \phi_{20}}, \\
\omega_1 &= \theta_{11} - \theta_{21} = \log \frac{\gamma_1 + \gamma_2}{1 - (\gamma_1 + \gamma_2)} - \log \frac{\gamma_2}{1 - \gamma_2} = \log \frac{\phi_{11}}{1 - \phi_{11}} - \log \frac{\phi_{21}}{1 - \phi_{21}}, \\
\nu_2 &= \theta_{20} = \log \frac{\beta_2}{1 - \beta_2} = \log \frac{\phi_{20}}{1 - \phi_{20}}, \\
\omega_2 &= \theta_{21} = \log \frac{\gamma_2}{1 - \gamma_2} = \log \frac{\phi_{21}}{1 - \phi_{21}}.
\end{aligned}
\tag{3.8}
$$

Then, testing H_{01} against H_{11} specified by (3.3) is equivalent to testing

$$
H_{01,S} : (\alpha_1, \nu_1, \omega_1) = (0, 0, 0) \qquad \textit{against} \qquad H_{11,S} : (\alpha_1, \nu_1, \omega_1) \neq (0, 0, 0). \tag{3.9}
$$

And $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are defined by (3.1). The observed-data log-likelihood function now becomes

$$
\begin{aligned}
\ell_2 \;\hat{=}\; & \ell_2(\alpha_1, \nu_1, \omega_1, \alpha_2, \nu_2, \omega_2) = m_{10} \log[\frac{e^{\nu_1+\nu_2}}{1 + e^{\nu_1+\nu_2}} + (1 - \frac{e^{\nu_1+\nu_2}}{1 + e^{\nu_1+\nu_2}} - \frac{e^{\omega_1+\omega_2}}{1 + e^{\omega_1+\omega_2}})e^{-(\alpha_1+\alpha_2)}] \\
& + m_{20} \log[\frac{e^{\nu_2}}{1 + e^{\nu_2}} + (1 - \frac{e^{\nu_2}}{1 + e^{\nu_2}} - \frac{e^{\omega_2}}{1 + e^{\omega_2}})e^{-\alpha_2}] \\
& + m_{11} \log[\frac{e^{\omega_1+\omega_2}}{1 + e^{\omega_1+\omega_2}} + (1 - \frac{e^{\nu_1+\nu_2}}{1 + e^{\nu_1+\nu_2}} - \frac{e^{\omega_1+\omega_2}}{1 + e^{\omega_1+\omega_2}})(\alpha_1 + \alpha_2)e^{-(\alpha_1+\alpha_2)}] \\
& + m_{21} \log[\frac{e^{\omega_2}}{1 + e^{\omega_2}} + (1 - \frac{e^{\nu_2}}{1 + e^{\nu_2}} - \frac{e^{\omega_2}}{1 + e^{\omega_2}})\alpha_2 e^{-\alpha_2}] \\
& + (n_1 - m_{10} - m_{11})[\log(1 - \frac{e^{\nu_1+\nu_2}}{1 + e^{\nu_1+\nu_2}} - \frac{e^{\omega_1+\omega_2}}{1 + e^{\omega_1+\omega_2}}) - (\alpha_1 + \alpha_2)] \\
& + (n_2 - m_{20} - m_{21})[\log(1 - \frac{e^{\nu_2}}{1 + e^{\nu_2}} - \frac{e^{\omega_2}}{1 + e^{\omega_2}}) - \alpha_2] \\
& + N_1 \log(\alpha_1 + \alpha_2) + N_2 \log \alpha_2
\end{aligned}
$$

In order to show equations briefly, some symbols are used to denote common parts in equations,

$$
\begin{aligned}
t_1 &\hat{=} \frac{\mathrm{e}^{\nu_2}}{1+\mathrm{e}^{\nu_2}} \hat{=} \frac{\mathrm{e}^{\theta_{20}}}{1+\mathrm{e}^{\theta_{20}}},\ t_2 \hat{=} \frac{\mathrm{e}^{\omega_2}}{1+\mathrm{e}^{\omega_2}} \hat{=} \frac{\mathrm{e}^{\theta_{21}}}{1+\mathrm{e}^{\theta_{21}}},\\
t_3 &\hat{=} \frac{\mathrm{e}^{\nu_1+\nu_2}}{1+\mathrm{e}^{\nu_1+\nu_2}} \hat{=} \frac{\mathrm{e}^{\theta_{10}}}{1+\mathrm{e}^{\theta_{10}}},\ t_4 \hat{=} \frac{\mathrm{e}^{\omega_1+\omega_2}}{1+\mathrm{e}^{\omega_1+\omega_2}} \hat{=} \frac{\mathrm{e}^{\theta_{11}}}{1+\mathrm{e}^{\theta_{11}}},\\
r_1 &\hat{=} t_1+t_2-1 \hat{=} \frac{\mathrm{e}^{\nu_2}}{1+\mathrm{e}^{\nu_2}} + \frac{\mathrm{e}^{\omega_2}}{1+\mathrm{e}^{\omega_2}} - 1 \hat{=} \frac{\mathrm{e}^{\theta_{20}}}{1+\mathrm{e}^{\theta_{20}}} + \frac{\mathrm{e}^{\theta_{21}}}{1+\mathrm{e}^{\theta_{21}}} - 1,\\
r_2 &\hat{=} t_3+t_4-1 \hat{=} \frac{\mathrm{e}^{\nu_1+\nu_2}}{1+\mathrm{e}^{\nu_1+\nu_2}} + \frac{\mathrm{e}^{\omega_1+\omega_2}}{1+\mathrm{e}^{\omega_1+\omega_2}} - 1 \hat{=} \frac{\mathrm{e}^{\theta_{10}}}{1+\mathrm{e}^{\theta_{10}}} + \frac{\mathrm{e}^{\theta_{11}}}{1+\mathrm{e}^{\theta_{11}}} - 1,\\
p_1 &\hat{=} t_1^2-t_1,\ p_2 \hat{=} t_3-t_3^2,\ p_3 \hat{=} t_4^2-t_4,\ p_4 \hat{=} t_2^2-t_2\\
q_1 &\hat{=} 2t_3^3-3t_3^2+t_3,\ q_2 \hat{=} 2t_1^3-3t_1^2+t_1,\ q_3 \hat{=} 2t_4^3-3t_4^2+t_4,\ q_4 \hat{=} 2t_2^3-3t_2^2+t_2\\
j_1 &\hat{=} \mathrm{e}^{-\lambda_1}\lambda_1 r_2 \hat{=} \mathrm{e}^{-\lambda_1}\lambda_1(\frac{\mathrm{e}^{\nu_1+\nu_2}}{1+\mathrm{e}^{\nu_1+\nu_2}} + \frac{\mathrm{e}^{\omega_1+\omega_2}}{1+\mathrm{e}^{\omega_1+\omega_2}} - 1) \hat{=} \mathrm{e}^{-\lambda_1}\lambda_1(\frac{\mathrm{e}^{\theta_{10}}}{1+\mathrm{e}^{\theta_{10}}} + \frac{\mathrm{e}^{\theta_{11}}}{1+\mathrm{e}^{\theta_{11}}} - 1),\\
j_2 &\hat{=} \mathrm{e}^{-\lambda_2} r_1 \hat{=} \mathrm{e}^{-\lambda_2}(\frac{\mathrm{e}^{\nu_2}}{1+\mathrm{e}^{\nu_2}} + \frac{\mathrm{e}^{\omega_2}}{1+\mathrm{e}^{\omega_2}} - 1) \hat{=} \mathrm{e}^{-\lambda_2}(\frac{\mathrm{e}^{\theta_{20}}}{1+\mathrm{e}^{\theta_{20}}} + \frac{\mathrm{e}^{\theta_{21}}}{1+\mathrm{e}^{\theta_{21}}} - 1).
\end{aligned}
$$

The score vector is now denoted by

$$
U(\alpha_1,\nu_1,\omega_1,\alpha_2,\nu_2,\omega_2) = \left(\frac{\partial\ell_2}{\partial\alpha_1},\frac{\partial\ell_2}{\partial\nu_1},\frac{\partial\ell_2}{\partial\omega_1},\frac{\partial\ell_2}{\partial\alpha_2},\frac{\partial\ell_2}{\partial\nu_2},\frac{\partial\ell_2}{\partial\omega_2}\right)^\top,
$$

where

$$
\begin{aligned}
\frac{\partial\ell_2}{\partial\alpha_1} &= m_{10}\left(1+\frac{r_2e^{-t_1}}{-r_2e^{-t_1}+t_4}\right) + m_{11}\left(1-\frac{-j_1+r_2e^{-t_1}}{-j_1+t_5}\right) + \frac{N_1}{t_1} - n_1\\
\frac{\partial\ell_2}{\partial\nu_1} &= -\frac{m_{10}p_2\left(e^{-t_1}-1\right)}{-r_2e^{-t_1}+t_4} - \frac{m_{11}t_1p_2e^{-t_1}}{-j_1+t_5} - \frac{p_2\left(m_{10}+m_{11}-n_1\right)}{r_2}\\
\frac{\partial\ell_2}{\partial\omega_1} &= \frac{m_{10}p_3e^{-t_1}}{-r_2e^{-t_1}+t_4} + \frac{m_{11}p_3\left(t_1e^{-t_1}-1\right)}{-j_1+t_5} + \frac{p_3\left(m_{10}+m_{11}-n_1\right)}{r_2}\\
\frac{\partial\ell_2}{\partial\alpha_2} &= m_{10}\left(1+\frac{r_2e^{-t_1}}{-r_2e^{-t_1}+t_4}\right) + m_{11}\left(1-\frac{-j_1+r_2e^{-t_1}}{-j_1+t_5}\right) + m_{20}\left(1-\frac{j_2}{j_2-t_2}\right)\\
&\quad + m_{21}\left(1-\frac{-\alpha_2j_2+j_2}{-\alpha_2j_2+t_3}\right) - n_2 + \frac{N_1}{t_1} + \frac{N_2}{\alpha_2} - n_1
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell_2}{\partial \nu_2} &= -\frac{m_{10}p_2\left(e^{-t_1}-1\right)}{-r_2e^{-t_1}+t_4}-\frac{m_{11}t_1p_2e^{-t_1}}{-j_1+t_5}-\frac{m_{20}p_1\left(e^{-\alpha_2}-1\right)}{j_2-t_2}+\frac{m_{21}\alpha_2p_1e^{-\alpha_2}}{-\alpha_2j_2+t_3} \\
&\quad +\frac{p_1\left(m_{20}+m_{21}-n_2\right)}{r_1}-\frac{p_2\left(m_{10}+m_{11}-n_1\right)}{r_2} \\
\frac{\partial \ell_2}{\partial \omega_2} &= \frac{m_{10}p_3e^{-t_1}}{-r_2e^{-t_1}+t_4}+\frac{m_{11}p_3\left(t_1e^{-t_1}-1\right)}{-j_1+t_5}-\frac{m_{20}p_4e^{-\alpha_2}}{j_2-t_2}+\frac{m_{21}p_4\left(\alpha_2e^{-\alpha_2}-1\right)}{-\alpha_2j_2+t_3} \\
&\quad +\frac{p_4\left(m_{20}+m_{21}-n_2\right)}{r_1}+\frac{p_3\left(m_{10}+m_{11}-n_1\right)}{r_2}
\end{aligned}
$$

The second derivatives are derived from the score vector and given in Appendix. Since we know that

$$
\begin{aligned}
E[m_{10}] &= n_1[\frac{\mathrm{e}^{\nu_1+\nu_2}}{1+\mathrm{e}^{\nu_1+\nu_2}}+(1-\frac{\mathrm{e}^{\nu_1+\nu_2}}{1+\mathrm{e}^{\nu_1+\nu_2}}-\frac{\mathrm{e}^{\omega_1+\omega_2}}{1+\mathrm{e}^{\omega_1+\omega_2}})\mathrm{e}^{-(\alpha_1+\alpha_2)}] \\
&\hat{=} n_1[t_3-r_2\mathrm{e}^{-\lambda_1}], \\
E[m_{11}] &= n_1[\frac{\mathrm{e}^{\omega_1+\omega_2}}{1+\mathrm{e}^{\omega_1+\omega_2}}+(1-\frac{\mathrm{e}^{\nu_1+\nu_2}}{1+\mathrm{e}^{\nu_1+\nu_2}}-\frac{\mathrm{e}^{\omega_1+\omega_2}}{1+\mathrm{e}^{\omega_1+\omega_2}})(\alpha_1+\alpha_2)\mathrm{e}^{-(\alpha_1+\alpha_2)}] \\
&\hat{=} n_1[t_4-r_2\lambda_1\mathrm{e}^{-\lambda_1}], \\
E[m_{20}] &= n_2[\frac{\mathrm{e}^{\nu_2}}{1+\mathrm{e}^{\nu_2}}+(1-\frac{\mathrm{e}^{\nu_2}}{1+\mathrm{e}^{\nu_2}}-\frac{\mathrm{e}^{\omega_2}}{1+\mathrm{e}^{\omega_2}})\mathrm{e}^{-\alpha_2}] \\
&\hat{=} n_2[t_1-r_1\mathrm{e}^{-\lambda_2}], \\
E[m_{21}] &= n_2[\frac{\mathrm{e}^{\omega_2}}{1+\mathrm{e}^{\omega_2}}+(1-\frac{\mathrm{e}^{\nu_2}}{1+\mathrm{e}^{\nu_2}}-\frac{\mathrm{e}^{\omega_2}}{1+\mathrm{e}^{\omega_2}})\alpha_2\mathrm{e}^{-\alpha_2}] \\
&\hat{=} n_2[t_2-r_1\lambda_2\mathrm{e}^{-\lambda_2}], \\
E[N_1] &= n_1(1-\frac{\mathrm{e}^{\nu_1+\nu_2}}{1+\mathrm{e}^{\nu_1+\nu_2}}-\frac{\mathrm{e}^{\omega_1+\omega_2}}{1+\mathrm{e}^{\omega_1+\omega_2}})(\alpha_1+\alpha_2)(1-\mathrm{e}^{-(\alpha_1+\alpha_2)}) \\
&\hat{=} -n_1r_2\lambda_1(1-\mathrm{e}^{-\lambda_1}), \\
E[N_2] &= n_2(1-\frac{\mathrm{e}^{\nu_2}}{1+\mathrm{e}^{\nu_2}}-\frac{\mathrm{e}^{\omega_2}}{1+\mathrm{e}^{\omega_2}})\alpha_2(1-\mathrm{e}^{-\alpha_2}) \\
&\hat{=} -n_2r_1\lambda_2(1-\mathrm{e}^{-\lambda_2}),
\end{aligned}
$$

the Fisher information matrix can be calculated as follows:

$$
J(\alpha_1,\nu_1,\omega_1,\alpha_2,\nu_2,\omega_2)=(J_{jj'})=-E\left[\nabla^2\ell_2(\alpha_1,\nu_1,\omega_1,\alpha_2,\nu_2,\omega_2)\right]
$$

Under $H_{01,S}$, the score test statistic is

$$\begin{aligned} S_1 &= U^\top(0,0,0,\hat{\alpha}_{2,H_{01}},\hat{\nu}_{2,H_{01}},\hat{\omega}_{2,H_{01}})\, J^{-1}(0,0,0,\hat{\alpha}_{2,H_{01}},\hat{\nu}_{2,H_{01}},\hat{\omega}_{2,H_{01}})\, U(0,0,0,\hat{\alpha}_{2,H_{01}},\hat{\nu}_{2,H_{01}},\hat{\omega}_{2,H_{01}}) \\ &\dot{\sim}\ \chi^2(3), \end{aligned} \tag{3.10}$$

where $\hat{\alpha}_{2,H_{01}}$, $\hat{\nu}_{2,H_{01}} = \log \frac{\hat{\beta}_{2,H_{01}}}{1-\hat{\beta}_{2,H_{01}}}$ and $\hat{\omega}_{2,H_{01}} = \log \frac{\hat{\gamma}_{2,H_{01}}}{1-\hat{\gamma}_{2,H_{01}}}$ denote the constrained MLEs of α_2, ν_2 and ω_2 under H_{01}, and $\hat{\alpha}_{2,H_{01}}, \hat{\beta}_{2,H_{01}}, \hat{\gamma}_{2,H_{01}}$ are determined by (3.6). The corresponding p-value is

$$p_{\mathrm{s}_1} = \Pr(S_1 > s_1 | H_{01}) = \Pr(\chi^2(3) > s_1). \tag{3.11}$$

3.1.3 Wald test

To compute the Wald statistic, we use the logistic transformation introduced before. Let $\hat{\boldsymbol{\theta}} = (\hat{\lambda}_1, \hat{\theta}_{10}, \hat{\theta}_{11}, \hat{\lambda}_2, \hat{\theta}_{20}, \hat{\theta}_{21})^\intercal$ be estimator of $\boldsymbol{\theta} = (\lambda_1, \theta_{10}, \theta_{11}, \lambda_2, \theta_{20}, \theta_{21})^\intercal$ where $\theta_{10}, \theta_{11}, \theta_{20}, \theta_{21}$ are defined by (3.2). It can be approximated by normal distribution with covariance matrix $\boldsymbol{V}$, $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \to \mathbb{N}(\mathbf{0}, \boldsymbol{V})$. The test of three hypotheses on the six parameters can be expressed with a 3x6 matrix $\boldsymbol{R_1}$.

$$\boldsymbol{R_1} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

Then the null and alternative hypotheses specified by (3.3) are equivalent to:

$$H_{01,W} : \boldsymbol{R_1}\boldsymbol{\theta} = \mathbf{0} \qquad \textit{against} \qquad H_{11,W} : \boldsymbol{R_1}\boldsymbol{\theta} \neq \mathbf{0} \tag{3.12}$$

The observed-data log-likelihood function now becomes

$$
\begin{aligned}
\ell_3 \quad \hat{=} \quad & \ell_3(\lambda_1, \theta_{10}, \theta_{11}, \lambda_2, \theta_{20}, \theta_{21}) = m_{10} \log[\frac{\mathrm{e}^{\theta_{10}}}{1+\mathrm{e}^{\theta_{10}}} + (1 - \frac{\mathrm{e}^{\theta_{10}}}{1+\mathrm{e}^{\theta_{10}}} - \frac{\mathrm{e}^{\theta_{11}}}{1+\mathrm{e}^{\theta_{11}}})\mathrm{e}^{-\lambda_1}] \\
& + m_{20} \log[\frac{\mathrm{e}^{\theta_{20}}}{1+\mathrm{e}^{\theta_{20}}} + (1 - \frac{\mathrm{e}^{\theta_{20}}}{1+\mathrm{e}^{\theta_{20}}} - \frac{\mathrm{e}^{\theta_{21}}}{1+\mathrm{e}^{\theta_{21}}})\mathrm{e}^{-\lambda_2}] \\
& + m_{11} \log[\frac{\mathrm{e}^{\theta_{11}}}{1+\mathrm{e}^{\theta_{11}}} + (1 - \frac{\mathrm{e}^{\theta_{10}}}{1+\mathrm{e}^{\theta_{10}}} - \frac{\mathrm{e}^{\theta_{11}}}{1+\mathrm{e}^{\theta_{11}}})\lambda_1\mathrm{e}^{-\lambda_1}] \\
& + m_{21} \log[\frac{\mathrm{e}^{\theta_{21}}}{1+\mathrm{e}^{\theta_{21}}} + (1 - \frac{\mathrm{e}^{\theta_{20}}}{1+\mathrm{e}^{\theta_{20}}} - \frac{\mathrm{e}^{\theta_{21}}}{1+\mathrm{e}^{\theta_{21}}})\lambda_2\mathrm{e}^{-\lambda_2}] \\
& + (n_1 - m_{10} - m_{11})[\log(1 - \frac{\mathrm{e}^{\theta_{10}}}{1+\mathrm{e}^{\theta_{10}}} - \frac{\mathrm{e}^{\theta_{11}}}{1+\mathrm{e}^{\theta_{11}}}) - \lambda_1] \\
& + (n_2 - m_{20} - m_{21})[\log(1 - \frac{\mathrm{e}^{\theta_{20}}}{1+\mathrm{e}^{\theta_{20}}} - \frac{\mathrm{e}^{\theta_{21}}}{1+\mathrm{e}^{\theta_{21}}}) - \lambda_2] \\
& + N_1 \log(\lambda_1) + N_2 \log \lambda_2
\end{aligned}
$$

In terms of the symbols specified before, the second derivatives are derived from score vector and given in Appendix. The Fisher information matrix can be calculated as follows:

$$
\boldsymbol{J_w}(\lambda_1, \theta_{10}, \theta_{11}, \lambda_2, \theta_{20}, \theta_{21}) = -E\left[\nabla^2 \ell_3(\lambda_1, \theta_{10}, \theta_{11}, \lambda_2, \theta_{20}, \theta_{21})\right]
$$

The Wald test statistic is

$$
W_1 = (\boldsymbol{R_1}\hat{\boldsymbol{\theta}})^\intercal (\boldsymbol{R_1}\hat{\boldsymbol{V}}\boldsymbol{R_1}^\intercal)^{-1}(\boldsymbol{R_1}\hat{\boldsymbol{\theta}}) \dot{\sim} \chi^2(3) \tag{3.13}
$$

where $\boldsymbol{R_1}\hat{\boldsymbol{\theta}}$ equals to $(\hat{\lambda}_1 - \hat{\lambda}_2, \hat{\theta}_{10} - \hat{\theta}_{20}, \hat{\theta}_{11} - \hat{\theta}_{21})^\intercal$, and $(\hat{\lambda}_1, \hat{\theta}_{10}, \hat{\theta}_{11}, \hat{\lambda}_2, \hat{\theta}_{20}, \hat{\theta}_{21})$ denotes the unconstrained MLEs of $(\lambda_1, \theta_{10}, \theta_{11}, \lambda_2, \theta_{20}, \theta_{21})$, which are obtained by the EM algorithm. We use inverse Fisher information matrix $\boldsymbol{J_w^{-1}}$ as the estimator of the covariance matrix $\boldsymbol{V}$. Under H_0^*, W_1 is supposed to follow asymptotically $\chi^2(3)$. And the corresponding p-value is

$$
p_{\mathrm{w}_1} = \Pr(W_1 > w_1 | H_{01,W}) = \Pr(\chi^2(3) > w_1). \tag{3.14}
$$

3.2 Tests for equality of zero-and-one inflation parameters between two populations

In this section, we would like to test the equality of ZOIP parameters between two populations. The null and alternative hypotheses are as follows:

$$H_{02} : (\phi_{10}, \phi_{11}) = (\phi_{20}, \phi_{21}) \quad \text{against} \quad H_{12} : (\phi_{10}, \phi_{11}) \neq (\phi_{20}, \phi_{21}). \tag{3.15}$$

3.2.1 Likelihood ratio test

To test whether two samples have similar inflated zeros and ones, the null hypothesis and alternative hypotheses specified by (3.15) are equivalent to:

$$H_{02,L} : (\beta_1, \gamma_1) = (0, 0) \quad \text{against} \quad H_{12,L} : (\beta_1, \gamma_1) \neq (0, 0) \tag{3.16}$$

Under $H_{02,L}$, the LRT statistic is given by

$$T_{02,L} = -2 \left\{ \ell(\hat{\alpha}_{1,H_{02}}, 0, 0, \hat{\alpha}_{2,H_{02}}, \hat{\beta}_{2,H_{02}}, \hat{\gamma}_{2,H_{02}}) - \ell(\hat{\alpha}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2) \right\} \tag{3.17}$$

where $(\hat{\beta}_{2,H_{02}}, \hat{\gamma}_{2,H_{02}})$ denotes the constrained MLEs of (β_2, γ_2) under $H_{02,L}$ and $(\hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2)$ denotes the unconstrained MLEs of $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$, which are obtained by EM algorithm (2.4)-(2.10). Note that under H_0, the MLEs of (β_2, γ_2) can be calculated by the following EM iteration:

$$\begin{aligned}
\alpha_{1,H_{02}}^{(t+1)} &= \frac{m_{11} - z_{11} + N_1}{n_1 - z_{10} - z_{11}} - \frac{m_{21} - z_{21} + N_2}{n_2 - z_{20} - z_{21}}, \\
\alpha_{2,H_{02}}^{(t+1)} &= \frac{m_{21} - z_{21} + N_2}{n_2 - z_{20} - z_{21}}, \\
\beta_{2,H_{02}}^{(t+1)} &= \frac{z_{10} + z_{20}}{n_1 + n_2}, \\
\gamma_{2,H_{02}}^{(t+1)} &= \frac{z_{11} + z_{21}}{n_1 + n_2},
\end{aligned} \tag{3.18}$$

where

$$
\begin{aligned}
z_{10} &= \frac{m_{10}\beta_{2,H_{02}}^{(t)}}{\beta_{2,H_{02}}^{(t)} + (1-\beta_{2,H_{02}}^{(t)} - \gamma_{2,H_{02}}^{(t)})e^{-(\alpha_{1,H_{02}}^{(t)}+\alpha_{2,H_{02}}^{(t)})}}, \\
z_{20} &= \frac{m_{20}\beta_{2,H_{02}}^{(t)}}{\beta_{2,H_{02}}^{(t)} + (1-\beta_{2,H_{02}}^{(t)} - \gamma_{2,H_{02}}^{(t)})e^{-\alpha_{2,H_{02}}^{(t)}}}, \\
z_{11} &= \frac{m_{11}\gamma_{2,H_{02}}^{(t)}}{\gamma_{2,H_{02}}^{(t)} + (1-\beta_{2,H_{02}}^{(t)} - \gamma_{2,H_{02}}^{(t)})(\alpha_{1,H_{02}}^{(t)}+\alpha_{2,H_{02}}^{(t)})e^{-(\alpha_{1,H_{02}}^{(t)}+\alpha_{2,H_{02}}^{(t)})}}, \quad \text{and} \\
z_{21} &= \frac{m_{21}\gamma_{2,H_{02}}^{(t)}}{\gamma_{2,H_{02}}^{(t)} + (1-\beta_{2,H_{02}}^{(t)} - \gamma_{2,H_{02}}^{(t)})\alpha_{2,H_{02}}^{(t)}e^{-\alpha_{2,H_{02}}^{(t)}}}.
\end{aligned}
$$

Standard large-sample theory suggests that the asymptotic null distribution of T_2 is $\chi^2(2)$. Hence the corresponding p-value is

$$p_{v2} = Pr(T_2 > t_2 | H_{02,L}) = Pr(\chi^2(2) > t_1). \tag{3.19}$$

3.2.2 Score test

Alternatively, the score test can be used for testing H_{02} against H_{12} specified in (3.15), which is equivalent to testing

$$H_{02,S} : (\nu_1, \omega_1) = (0, 0) \qquad \textit{against} \qquad H_{12,S} : (\nu_1, \omega_1) \neq (0, 0). \tag{3.20}$$

Let$(\alpha_1, \nu_1, \omega_1, \alpha_2, \nu_2, \omega_2)$ be defined by (3.8). Under $H_{02,S}$, the score test statistic is

$$
\begin{aligned}
S_2 &= U^{\top}(\hat{\alpha}_{1,H_{02}}, 0, 0, \hat{\alpha}_{2,H_{02}}, \hat{\nu}_{2,H_{02}}, \hat{\omega}_{2,H_{02}})\, J^{-1}(\hat{\alpha}_{1,H_{02}}, 0, 0, \hat{\alpha}_{2,H_{02}}, \hat{\nu}_{2,H_{02}}, \hat{\omega}_{2,H_{02}}) \\
&\quad U(\hat{\alpha}_{1,H_{02}}, 0, 0, \hat{\alpha}_{2,H_{02}}, \hat{\nu}_{2,H_{02}}, \hat{\omega}_{2,H_{02}}) \\
&\dot{\sim} \ \chi^2(2), \qquad (3.21)
\end{aligned}
$$

where the score vector function $U(\alpha_1, \nu_1, \omega_1, \alpha_2, \nu_2, \omega_2)$ and the Fisher information matrix function $J(\alpha_1, \nu_1, \omega_1, \alpha_2, \nu_2, \omega_2)$ have been calculated in Part 3.1.2. And $\hat{\alpha}_{1,H_{02}}$, $\hat{\alpha}_{2,H_{02}}$, $\hat{\nu}_{2,H_{02}} = \log \frac{\hat{\beta}_{2,H_{02}}}{1-\hat{\beta}_{2,H_{02}}}$ and $\hat{\omega}_{2,H_{02}} = \log \frac{\hat{\gamma}_{2,H_{02}}}{1-\hat{\gamma}_{2,H_{02}}}$ denote the constrained MLEs of α_1, α_2, ν_2

and ω_2 under H_{02}, and $\hat{\alpha}_{1,H_{02}}, \hat{\alpha}_{2,H_{02}}, \hat{\beta}_{2,H_{02}}, \hat{\gamma}_{2,H_{02}}$ are determined by (3.18). The corresponding p-value is

$$p_{s_2} = \Pr(S_2 > s_2 | H_{02,S}) = \Pr(\chi^2(2) > s_2). \tag{3.22}$$

3.2.3 Wald test

Let $\hat{\boldsymbol{\theta}} = (\hat{\lambda}_1, \hat{\theta}_{10}, \hat{\theta}_{11}, \hat{\lambda}_2, \hat{\theta}_{20}, \hat{\theta}_{21})^\intercal$ be estimator of $\boldsymbol{\theta} = (\lambda_1, \theta_{10}, \theta_{11}, \lambda_2, \theta_{20}, \theta_{21})^\intercal$. In this case, we construct a new 3x6 matrix $\boldsymbol{R_2}$ to express the test of two hypotheses on the six parameters.

$$\boldsymbol{R_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

Then the null and alternative hypotheses specified in (3.15) are equivalent to:

$$H_{02,W} : \boldsymbol{R_2\theta} = \boldsymbol{0} \qquad against \qquad H_{12,W} : \boldsymbol{R_2\theta} \neq \boldsymbol{0} \tag{3.23}$$

The Wald test statistic is

$$W_2 = (\boldsymbol{R_2}\hat{\boldsymbol{\theta}})^\intercal (\boldsymbol{R_2}\hat{\boldsymbol{V}}\boldsymbol{R_2}^\intercal)^{-1} (\boldsymbol{R_2}\hat{\boldsymbol{\theta}}) \dot{\sim} \chi^2(2) \tag{3.24}$$

where $\boldsymbol{R_2}\hat{\boldsymbol{\theta}}$ equals to $(\hat{\theta}_{10} - \hat{\theta}_{20}, \hat{\theta}_{11} - \hat{\theta}_{21})^\intercal$, and $(\hat{\theta}_{10}, \hat{\theta}_{11}, \hat{\theta}_{20}, \hat{\theta}_{21})$ denotes the unconstrained MLEs of $(\theta_{10}, \theta_{11}, \theta_{20}, \theta_{21})$, which are obtained by the EM algorithm. We use inverse Fisher information matrix $\boldsymbol{J_w^{-1}}$ as the estimator of the covariance matrix $\boldsymbol{V}$.
The corresponding p-value is

$$p_{w_2} = \Pr(W_2 > w_2 | H_{02,W}) = \Pr(\chi^2(2) > w_2). \tag{3.25}$$

3.3 Tests for equality of non-zero-and-one means between two populations

In this section, we would like to test the equality of non-zero-and-one means between two populations. The null and alternative hypotheses are as follows:

$$H_{03} : \lambda_1 = \lambda_2 \quad \text{against} \quad H_{13} : \lambda_1 \neq (\lambda_2). \tag{3.26}$$

3.3.1 Likelihood ratio test

To test equality of non-zero-and-one means between two populations, the null hypothesis and alternative hypotheses specified by (3.26) are equivalent to:

$$H_{03,L} : \alpha_1 = 0 \quad \text{against} \quad H_{13,L} : \alpha_1 \neq 0 \tag{3.27}$$

Under H_{03}, the LRT statistic is given by

$$T_3 = -2 \left\{ \ell(0, \hat{\beta}_{1,H_{03}}, \hat{\gamma}_{1,H_{03}}, \hat{\alpha}_{2,H_{03}}, \hat{\beta}_{2,H_{03}}, \hat{\gamma}_{2,H_{03}}) - \ell(\hat{\alpha}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2) \right\} \tag{3.28}$$

where $(\hat{\beta}_{1,H_{03}}, \hat{\gamma}_{1,H_{03}}, \hat{\alpha}_{2,H_{03}}, \hat{\beta}_{2,H_{03}}, \hat{\gamma}_{2,H_{03}})$ denotes the constrained MLEs of $(\beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$ under $H_{03,L}$ and $(\hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2)$ denote the unconstrained MLEs of $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$, which are obtained by EM algorithm (2.4)-(2.10). Note that under $H_{03,L}$, the MLEs of $(\beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$ can be calculated by the following EM iteration:

$$\begin{aligned}
\beta_{1,H_{03}}^{(t+1)} &= \frac{z_{10}}{n_1} - \frac{z_{20}}{n_2}, \\
\beta_{2,H_{03}}^{(t+1)} &= \frac{z_{20}}{n_2}, \\
\gamma_{1,H_{03}}^{(t+1)} &= \frac{z_{11}}{n_1} - \frac{z_{21}}{n_2}, \\
\gamma_{2,H_{03}}^{(t+1)} &= \frac{z_{21}}{n_2}, \\
\lambda_{2,H_{03}}^{(t+1)} &= \frac{N_1 + N_2 + m_{21} + m_{11} - z_{21} - z_{11}}{n_1 + n_2 - z_{10} - z_{20} - z_{11} - z_{21}},
\end{aligned} \tag{3.29}$$

where

$$\begin{aligned}
z_{10} &= \frac{m_{10}(\beta_{1,H_{03}}^{(t)} + \beta_{2,H_{03}}^{(t)})}{(\beta_{1,H_{03}}^{(t)} + \beta_{2,H_{03}}^{(t)}) + \left[1 - (\beta_{1,H_{03}}^{(t)} + \beta_{2,H_{03}}^{(t)}) - (\gamma_{1,H_{03}}^{(t)} + \gamma_{2,H_{03}}^{(t)})\right] e^{-\alpha_{2,H_{03}}^{(t)}}}, \\
z_{20} &= \frac{m_{20}(\beta_{1,H_{03}}^{(t)} + \beta_{2,H_{03}}^{(t)})}{(\beta_{1,H_{03}}^{(t)} + \beta_{2,H_{03}}^{(t)}) + \left[1 - (\beta_{1,H_{03}}^{(t)} + \beta_{2,H_{03}}^{(t)}) - (\gamma_{1,H_{03}}^{(t)} + \gamma_{2,H_{03}}^{(t)})\right] e^{-\alpha_{2,H_{03}}^{(t)}}}, \\
z_{11} &= \frac{m_{11}(\gamma_{1,H_3}^{(t)} + \gamma_{2,H_3}^{(t)})}{(\gamma_{1,H_{03}}^{(t)} + \gamma_{2,H_{03}}^{(t)}) + \left[1 - (\beta_{1,03}^{(t)} + \beta_{2,H_{03}}^{(t)}) - (\gamma_{1,H_{03}}^{(t)} + \gamma_{2,H_{03}}^{(t)})\right] \alpha_{2,H_{03}}^{(t)} e^{-\alpha_{2,H_{03}}^{(t)}}}, \\
z_{21} &= \frac{m_{21}(\gamma_{1,H_{03}}^{(t)} + \gamma_{2,H_{03}}^{(t)})}{(\gamma_{1,H_{03}}^{(t)} + \gamma_{2,H_{03}}^{(t)}) + \left[1 - (\beta_{1,H_{03}}^{(t)} + \beta_{2,H_{03}}^{(t)}) - (\gamma_{1,H_{03}}^{(t)} + \gamma_{2,H_{03}}^{(t)})\right] \alpha_{2,H_{03}}^{(t)} e^{-\alpha_{2,H_{03}}^{(t)}}}.
\end{aligned}$$

Standard large-sample theory suggests that the asymptotic null distribution of T_3 is $\chi^2(1)$. Hence the corresponding p-value is

$$p_{v3} = Pr(T_3 > t_3|H_{03,L}) = Pr(\chi^2(1) > t_1). \tag{3.30}$$

3.3.2 Score test

Let $(\alpha_1, \nu_1, \omega_1, \alpha_2, \nu_2, \omega_2)$ be defined by (3.8). The score test for testing H_{03} against H_{13} specified in (3.26) is equivalent to that for testing

$$H_{03,S} : \alpha_1 = 0 \qquad against \qquad H_{13,S} : \alpha_1 \neq 0. \tag{3.31}$$

Under $H_{03,S}$, the score test statistic is

$$\begin{aligned} S_3 &= U^{\top}(0, \hat{\nu}_{1,H_{03}}, \hat{\omega}_{1,H_{03}}, \hat{\alpha}_{2,H_{03}}, \hat{\nu}_{2,H_{03}}, \hat{\omega}_{2,H_{03}})\, J^{-1}(0, \hat{\nu}_{1,H_{03}}, \hat{\omega}_{1,H_{03}}, \hat{\alpha}_{2,H_{03}}, \hat{\nu}_{2,H_{03}}, \hat{\omega}_{2,H_{03}}) \\ & \quad U(0, \hat{\nu}_{1,H_{03}}, \hat{\omega}_{1,H_{03}}, \hat{\alpha}_{2,H_{03}}, \hat{\nu}_{2,H_{03}}, \hat{\omega}_{2,H_{03}}) \\ &\dot{\sim} \; \chi^2(1), \end{aligned} \tag{3.32}$$

where the score vector function $U(\alpha_1, \nu_1, \omega_1, \alpha_2, \nu_2, \omega_2)$ and the Fisher information matrix function $J(\alpha_1, \nu_1, \omega_1, \alpha_2, \nu_2, \omega_2)$ have been calculated in Part 3.1.2. And $\hat{\nu}_{1,H_{03}} = \log \frac{\hat{\beta}_{1,H_{03}}+\hat{\beta}_{2,H_{03}}}{1-(\hat{\beta}_{1,H_{03}}+\hat{\beta}_{2,H_{03}})} - \log \frac{\hat{\beta}_{2,H_{03}}}{1-\hat{\beta}_{2,H_{03}}}$, $\hat{\omega}_{1,H_{03}} = \log \frac{\hat{\gamma}_{1,H_{03}}+\hat{\gamma}_{2,H_{03}}}{1-(\hat{\gamma}_{1,H_{03}}+\hat{\gamma}_{2,H_{03}})} - \log \frac{\hat{\gamma}_{2,H_{03}}}{1-\hat{\gamma}_{2,H_{03}}}$, $\hat{\alpha}_{2,H_{03}}$, $\hat{\nu}_{2,H_{03}} = \log \frac{\hat{\beta}_{2,H_{03}}}{1-\hat{\beta}_{2,H_{03}}}$ and $\hat{\omega}_{2,H_{03}} = \log \frac{\hat{\gamma}_{2,H_{03}}}{1-\hat{\gamma}_{2,H_{03}}}$ denote the constrained MLEs of ν_1, ω_1, α_2, ν_2 and ω_2 under H_{03}, and $\hat{\beta}_{1,H_{03}}, \hat{\gamma}_{1,H_{03}}, \hat{\alpha}_{2,H_{03}}, \hat{\beta}_{2,H_{03}}, \hat{\gamma}_{2,H_{03}}$ are determined by (3.29). The corresponding p-value is

$$p_{s_3} = \Pr(S_3 > s_3|H_{03,S}) = \Pr(\chi^2(1) > s_3). \tag{3.33}$$

3.3.3 Wald test

Let $\hat{\boldsymbol{\theta}} = (\hat{\lambda}_1, \hat{\theta}_{10}, \hat{\theta}_{11}, \hat{\lambda}_2, \hat{\theta}_{20}, \hat{\theta}_{21})^{\intercal}$ be estimator of $\boldsymbol{\theta} = (\lambda_1, \theta_{10}, \theta_{11}, \lambda_2, \theta_{20}, \theta_{21})^{\intercal}$. In this case, we construct a new 1x6 vector $\boldsymbol{R_3}$ to express the test of one hypotheses on the six parameters.

$$\boldsymbol{R_3} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the null and alternative hypotheses specified in (3.26) are equivalent to:

$$H_{03,W} : \boldsymbol{R_3\theta} = 0 \qquad against \qquad H_{13,W} : \boldsymbol{R_3\theta} \neq 0 \tag{3.34}$$

The Wald test statistic is

$$W_3 = (\boldsymbol{R_3\hat{\theta}})^\intercal(\boldsymbol{R_3\hat{V}R_3}^\intercal)^{-1}(\boldsymbol{R_3\hat{\theta}}) \dot\sim \chi^2(1) \tag{3.35}$$

where $\boldsymbol{R_3\hat{\theta}}$ equals to $(\hat{\lambda}_1 - \hat{\lambda}_2)^\intercal$, and $(\hat{\lambda}_1, \hat{\lambda}_2)$ denotes the unconstrained MLES of (λ_1, λ_2), which are obtained by the EM algorithm. We use inverse Fisher information matrix $\boldsymbol{J_w^{-1}}$ as the estimator of the covariance matrix $\boldsymbol{V}$.
The corresponding p-value is

$$p_{\mathrm{w}_3} = \Pr(W_3 > w_3|H_{03,W}) = \Pr(\chi^2(3) > w_3). \tag{3.36}$$

3.4 Tests for equality of zero inflation parameters between two populations

In this section, we would like to test the equality of zero inflation parameters between two populations. The null and alternative hypotheses are as follows:

$$H_{04} : \phi_{10} = \phi_{20} \quad \text{against} \quad H_{14} : \phi_{10} \neq \phi_{20}. \tag{3.37}$$

3.4.1 Likelihood ratio test

To test equality of zero inflation parameters between two populations, the null hypothesis and alternative hypotheses specified by (3.37) are equivalent to:

$$H_{04,L} : \beta_1 = 0 \quad \text{against} \quad H_{14,L} : \beta_1 \neq 0 \tag{3.38}$$

Under $H_{04,L}$, the LRT statistic is given by

$$T_4 = -2\left\{\ell(\hat{\alpha}_{1,H_{04}}, 0, \hat{\gamma}_{1,H_{04}}, \hat{\alpha}_{2,H_{04}}, \hat{\beta}_{2,H_{04}}, \hat{\gamma}_{2,H_{04}}) - \ell(\hat{\alpha}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2)\right\} \tag{3.39}$$

where $(\hat{\alpha}_{1,H_{04}}, \hat{\gamma}_{1,H_{04}}, \hat{\alpha}_{2,H_{04}}, \hat{\beta}_{2,H_{04}}, \hat{\gamma}_{2,H_{04}})$ denotes the constrained MLEs of $(\alpha_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$ under $H_{04,L}$ and $(\hat{\alpha}_1\hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2)$ denotes the unconstrained MLEs of $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$, which are obtained by EM algorithm from (2.4)-(2.10). Note that under $H_{04,L}$, the MLEs of $(\alpha_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$ can be calculated by the following EM iteration:

$$\begin{aligned}
\alpha_{1,H_{04}}^{(t+1)} &= \frac{m_{11} - z_{11} + N_1}{n_1 - z_{10} - z_{11}} - \frac{m_{21} - z_{21} + N_2}{n_2 - z_{20} - z_{21}}, \\
\alpha_{2,H_{04}}^{(t+1)} &= (m_{21} - z_{21} + N_2)/(n_2 - z_{20} - z_{21}), \\
\beta_{2,H_{04}}^{(t+1)} &= \frac{z_{10} + z_{20}}{n_1 + n_2}, \\
\gamma_{1,H_{04}}^{(t+1)} &= \frac{z_{11}(n_1 + n_2 - z_{10} - z_{20})}{(n_1 + n_2)(n_1 - z_{10})} - \frac{z_{21}(n_1 + n_2 - z_{10} - z_{20})}{(n_1 + n_2)(n_2 - z_{20})}, \\
\gamma_{2,H_{04}}^{(t+1)} &= \frac{z_{21}(n_1 + n_2 - z_{10} - z_{20})}{(n_1 + n_2)(n_2 - z_{20})},
\end{aligned} \tag{3.40}$$

where

$$\begin{aligned}
z_{10} &= \frac{m_{10}(\beta_{2,H_{04}}^{(t)})}{\beta_{2,H_{04}}^{(t)} + \left[1 - \beta_{2,H_{04}}^{(t)} - (\gamma_{1,H_{04}}^{(t)} + \gamma_{2,H_{04}}^{(t)})\right] e^{-(\alpha_{1,H_{04}}^{(t)} + \alpha_{2,H_{04}}^{(t)})}}, \\
z_{20} &= \frac{m_{20}\beta_{2,H_{04}}^{(t)}}{\beta_{2,H_{04}}^{(t)} + \left[1 - \beta_{2,H_{04}}^{(t)} - (\gamma_{1,H_{04}}^{(t)} + \gamma_{2,H_{04}}^{(t)})\right] e^{-(\alpha_{1,H_{04}}^{(t)} + \alpha_{2,H_{04}}^{(t)})}}, \\
z_{11} &= \frac{m_{11}(\gamma_{1,H_{04}}^{(t)} + \gamma_{2,H_{04}}^{(t)})}{(\gamma_{1,H_{04}}^{(t)} + \gamma_{2,H_{04}}^{(t)}) + \left[1 - \beta_{2,H_{04}}^{(t)} - (\gamma_{1,H_{04}}^{(t)} + \gamma_{2,H_{04}}^{(t)})\right] (\alpha_{1,H_{04}}^{(t)} + \alpha_{2,H_{04}}^{(t)}) e^{-(\alpha_{1,H_{04}}^{(t)} + \alpha_{2,04}^{(t)})}}, \\
z_{21} &= \frac{m_{21}(\gamma_{1,H_{04}}^{(t)} + \gamma_{2,H_{04}}^{(t)})}{\left[1 - \beta_{2,H_{04}}^{(t)} - (\gamma_{1,H_{04}}^{(t)} + \gamma_{2,H_{04}}^{(t)})\right] (\alpha_{1,H_{04}}^{(t)} + \alpha_{2,H_{04}}^{(t)}) e^{-(\alpha_{1,H_{04}}^{(t)} + \alpha_{2,H_{04}}^{(t)})}}.
\end{aligned}$$

Standard large-sample theory suggests that the asymptotic null distribution of T_4 is $\chi^2(1)$. Hence the corresponding p-value is

$$p_{v4} = Pr(T_4 > t_4 | H_{04,L}) = Pr(\chi^2(1) > t_4). \tag{3.41}$$

3.4.2 Score test

The score test can be used for testing H_{04} against H_{14} specified in (3.37), which is equivalent to testing

$$H_{04,S} : \nu_1 = 0 \qquad \textit{against} \qquad H_{14,S} : \nu_1 \neq 0. \tag{3.42}$$

Let $(\alpha_1, \nu_1, \omega_1, \alpha_2, \nu_2, \omega_2)$ be defined by (3.8). Under $H_{04,S}$, the score test statistic is

$$\begin{aligned} S_4 &= U^{\top}(\hat{\alpha}_{1,H_{04}}, 0, \hat{\omega}_{1,H_{04}}, \hat{\alpha}_{2,H_{04}}, \hat{\nu}_{2,H_{04}}, \hat{\omega}_{2,H_{04}})\, J^{-1}(\hat{\alpha}_{1,H_{04}}, 0, \hat{\omega}_{1,H_{04}}, \hat{\alpha}_{2,H_{04}}, \hat{\nu}_{2,H_{04}}, \hat{\omega}_{2,H_{04}}) \\ & \quad U(\hat{\alpha}_{1,H_{04}}, 0, \hat{\omega}_{1,H_{04}}, \hat{\alpha}_{2,H_{04}}, \hat{\nu}_{2,H_{04}}, \hat{\omega}_{2,H_{04}}) \\ &\dot{\sim} \ \chi^2(1), \end{aligned} \tag{3.43}$$

where $\hat{\alpha}_{1,H_{04}}$, $\hat{\omega}_{1,H_{04}} = \log \frac{\hat{\gamma}_{1,H_{04}}+\hat{\gamma}_{2,H_{04}}}{1-(\hat{\gamma}_{1,H_{04}}+\hat{\gamma}_{2,H_{04}})} - \log \frac{\hat{\gamma}_{2,H_{04}}}{1-\hat{\gamma}_{2,H_{04}}}$, $\hat{\alpha}_{2,H_{04}}$, $\hat{\nu}_{2,H_{04}} = \log \frac{\hat{\beta}_{2,H_{04}}}{1-\hat{\beta}_{2,H_{04}}}$ and $\hat{\omega}_{2,H_{04}} = \log \frac{\hat{\gamma}_{2,H_{04}}}{1-\hat{\gamma}_{2,H_{04}}}$ denote the constrained MLEs of α_1, ω_1, α_2, ν_2 and ω_2 under H_{04}, and $\hat{\alpha}_{1,H_{04}}, \hat{\gamma}_{1,H_{04}}, \hat{\alpha}_{2,H_{04}}, \hat{\beta}_{2,H_{04}}, \hat{\gamma}_{2,H_{04}}$ are determined by (3.40). The corresponding p-value is

$$p_{s_4} = \Pr(S_4 > s_4 | H_{04,S}) = \Pr(\chi^2(1) > s_4). \tag{3.44}$$

3.4.3 Wald test

Let $\hat{\boldsymbol{\theta}} = (\hat{\lambda}_1, \hat{\theta}_{10}, \hat{\theta}_{11}, \hat{\lambda}_2, \hat{\theta}_{20}, \hat{\theta}_{21})^{\intercal}$ be estimator of $\boldsymbol{\theta} = (\lambda_1, \theta_{10}, \theta_{11}, \lambda_2, \theta_{20}, \theta_{21})^{\intercal}$. Similarly, we construct a new 3x6 vector $\boldsymbol{R_4}$ to express the test of one hypotheses on the six parameters.

$$\boldsymbol{R_4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the null and alternative hypothesis specified in (3.37) are equivalent to:

$$H_C : \boldsymbol{R_4}\boldsymbol{\theta} = 0 \qquad \textit{against} \qquad H_{14,W} : \boldsymbol{R_4}\boldsymbol{\theta} \neq 0 \tag{3.45}$$

The Wald test statistic is

$$W_4 = (\boldsymbol{R_4}\hat{\boldsymbol{\theta}})^{\intercal}(\boldsymbol{R_4}\hat{\boldsymbol{V}}\boldsymbol{R_4}^{\intercal})^{-1}(\boldsymbol{R_4}\hat{\boldsymbol{\theta}}) \mathrel{\dot{\sim}} \chi^2(1) \tag{3.46}$$

where$\boldsymbol{R_4}\hat{\boldsymbol{\theta}}$ equals to $(\hat{\theta}_{10} - \hat{\theta}_{20})^{\intercal}$, and $(\hat{\theta}_{10}, \hat{\theta}_{20})$ denotes the unconstrained MLES of $(\theta_{10}, \theta_{20})$, which are obtained by the EM algorithm. We use inverse Fisher information matrix $\boldsymbol{J_w^{-1}}$ as the estimator of the covariance matrix $\boldsymbol{V}$.

The corresponding p-value is

$$p_{\mathrm{w}_4} = \Pr(W_4 > w_4 | H_{04,W}) = \Pr(\chi^2(1) > w_4). \tag{3.47}$$

3.5 Tests for equality of one inflation parameters between two populations

To test equality of one inflation parameters between two populations, we consider the following null hypothesis and alternative hypotheses

$$H_{05} : \phi_{11} = \phi_{21} \quad \text{against} \quad H_{15} : \phi_{11} \neq \phi_{21}. \tag{3.48}$$

3.5.1 Likelihood ratio test

To test equality of one inflation parameters between two populations, the null and alternative hypotheses specified by (3.48) are equivalent to:

$$H_{05,L} : \gamma_1 = 0 \quad \text{against} \quad H_{15,L} : \gamma_1 \neq 0 \tag{3.49}$$

Under $H_{05,L}$, the LRT statistic is given by

$$T_5 = -2\left\{\ell(\hat{\alpha}_{1,H_{05}}, \hat{\beta}_{1,H_{05}}, 0, \hat{\alpha}_{2,H_{05}}, \hat{\beta}_{2,H_{05}}, \hat{\gamma}_{2,H_{05}}) - \ell(\hat{\alpha}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2)\right\} \tag{3.50}$$

where $(\hat{\alpha}_{1,H_{05}}, \hat{\beta}_{1,H_{05}}, \hat{\alpha}_{2,H_{05}}, \hat{\beta}_{2,H_{05}}, \hat{\gamma}_{2,H_{05}})$ denotes the constrained MLEs of $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_2)$ under $H_{05,L}$ and $(\hat{\alpha}_1 \hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2)$ denote the unconstrained MLEs of $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$, which are obtained by EM algorithm from (2.4)-(2.10). Note that under H_0, the MLEs of

$(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_2)$ can be calculated by the following EM iteration:

$$\begin{aligned}
\alpha_{1,H_{05}}^{(t+1)} &= \frac{m_{11} - z_{11} + N_1}{n_1 - z_{10} - z_{11}} - \frac{m_{21} - z_{21} + N_2}{n_2 - z_{20} - z_{21}}, \\
\alpha_{2,H_{05}}^{(t+1)} &= \frac{m_{21} - z_{21} + N_2}{n_2 - z_{20} - z_{21}}, \\
\beta_{1,H_{05}}^{(t+1)} &= \frac{z_{10}(n_1 + n_2 - z_{11} - z_{21})}{(n_1 + n_2)(n_1 - z_{11})} - \frac{z_{20}(n_1 + n_2 - z_{11} - z_{21})}{(n_1 + n_2)(n_2 - z_{21})}, \\
\beta_{2,H_{05}}^{(t+1)} &= \frac{z_{20}(n_1 + n_2 - z_{11} - z_{21})}{(n_1 + n_2)(n_2 - z_{21})}, \\
\gamma_{2,H_{05}}^{(t+1)} &= \frac{z_{11} + z_{21}}{n_1 + n_2}, \qquad (3.51)
\end{aligned}$$

where

$$\begin{aligned}
z_{10} &= \frac{m_{10}(\beta_{1,H_{05}}^{(t)} + \beta_{2,H_{05}}^{(t)})}{(\beta_{1,H_{05}}^{(t)} + \beta_{2,H_{05}}^{(t)}) + \left[1 - (\beta_{1,H_{05}}^{(t)} + \beta_{2,H_{05}}^{(t)}) - \gamma_{2,H_{05}}^{(t)}\right] e^{-(\alpha_{1,H_{05}}^{(t)} + \alpha_{2,H_{05}}^{(t)})}}, \\
z_{20} &= \frac{m_{20}(\beta_{1,H_{05}}^{(t)} + \beta_{2,H_{05}}^{(t)})}{(\beta_{1,H_{05}}^{(t)} + \beta_{2,H_{05}}^{(t)}) + \left[1 - (\beta_{1,H_{05}}^{(t)} + \beta_{2,H_{05}}^{(t)}) - \gamma_{2,H_{05}}^{(t)}\right] e^{-(\alpha_{1,H_{05}}^{(t)} + \alpha_{2,H_{05}}^{(t)})}}, \\
z_{11} &= \frac{m_{11}\gamma_{2,H_{05}}^{(t)}}{\gamma_{2,H_{05}}^{(t)} + \left[1 - (\beta_{1,H_{05}}^{(t)} + \beta_{2,H_{05}}^{(t)}) - \gamma_{2,H_{05}}^{(t)}\right] (\alpha_{1,H_{05}}^{(t)} + \alpha_{2,H_{05}}^{(t)}) e^{-(\alpha_{1,H_{05}}^{(t)} + \alpha_{2,H_{05}}^{(t)})}}, \\
z_{21} &= \frac{m_{21}\gamma_{2,H_{05}}^{(t)}}{\gamma_{2,H_{05}}^{(t)} + \left[1 - (\beta_{1,H_{05}}^{(t)} + \beta_{2,H_{05}}^{(t)}) - \gamma_{2,H_{05}}^{(t)}\right] \alpha_{2,H_{05}}^{(t)} e^{-\alpha_{2,H_{05}}^{(t)}}}.
\end{aligned}$$

Standard large-sample theory suggests that the asymptotic null distribution of T_5 is $\chi^2(1)$. Hence the corresponding p-value is

$$p_{v5} = Pr(T_5 > t_5 | H_{05,L}) = Pr(\chi^2(1) > t_5). \qquad (3.52)$$

3.5.2 Score test

Let $(\alpha_1, \nu_1, \omega_1, \alpha_2, \nu_2, \omega_2)$ be defined by (3.8), testing H_{05} against H_{15} specified in (3.48) is equivalent to testing

$$H_{05,S} : \omega_1 = 0 \qquad \textit{against} \qquad H_{15,S} : \omega_1 \neq 0. \qquad (3.53)$$

Under $H_{05,S}$, the score test statistic is

$$\begin{aligned} S_5 &= U^{\top}(\hat{\alpha}_{1,H_{05}}, \hat{\nu}_{1,H_{05}}, 0, \hat{\alpha}_{2,H_{05}}, \hat{\nu}_{2,H_{05}}, \hat{\omega}_{2,H_{05}})\, J^{-1}(\hat{\alpha}_{2,H_{05}}, \hat{\nu}_{1,H_{05}}, 0, \hat{\alpha}_{2,H_{05}}, \hat{\nu}_{2,H_{05}}, \hat{\omega}_{2,H_{05}}) \\ & \quad U(\hat{\alpha}_{1,H_{05}}, \hat{\nu}_{1,H_{05}}, 0, \hat{\alpha}_{2,H_{05}}, \hat{\nu}_{2,H_{05}}, \hat{\omega}_{2,H_{05}}) \\ &\dot{\sim}\ \chi^2(1), \end{aligned} \tag{3.54}$$

where $\hat{\alpha}_{1,H_{05}},\ \hat{\nu}_{1,H_{05}} = \log \frac{\hat{\beta}_{1,H_{05}}+\hat{\beta}_{2,H_{05}}}{1-(\hat{\beta}_{1,H_{05}}+\hat{\beta}_{2,H_{05}})} - \log \frac{\hat{\beta}_{2,H_{05}}}{1-\hat{\beta}_{2,H_{05}}},\ \hat{\alpha}_{2,H_{05}},\ \hat{\nu}_{2,H_{05}} = \log \frac{\hat{\beta}_{2,H_{05}}}{1-\hat{\beta}_{2,H_{05}}}$ and $\hat{\omega}_{2,H_{05}=\log \frac{\hat{\gamma}_{2,H_{05}}}{1-\hat{\gamma}_{2,H_{05}}}}$ denote the constrained MLEs of α_1, ν_1, α_2, ν_2 and ω_2 under H_{05}, and $\hat{\alpha}_{1,H_{05}}, \hat{\beta}_{1,H_{05}}, \hat{\alpha}_{2,H_{05}}, \hat{\beta}_{2,H_{05}}, \hat{\gamma}_{2,H_{05}}$ are determined by (3.51). The corresponding p-value is

$$p_{s_5} = \Pr(S_5 > s_5 | H_{05,S}) = \Pr(\chi^2(1) > s_5). \tag{3.55}$$

3.5.3 Wald test

Let $\hat{\boldsymbol{\theta}} = (\hat{\lambda}_1, \hat{\theta}_{10}, \hat{\theta}_{11}, \hat{\lambda}_2, \hat{\theta}_{20}, \hat{\theta}_{21})^{\intercal}$ be estimator of $\boldsymbol{\theta} = (\lambda_1, \theta_{10}, \theta_{11}, \lambda_2, \theta_{20}, \theta_{21})^{\intercal}$. Similarly, we construct a new 3x6 vector $\boldsymbol{R_5}$ to express the test of one hypotheses on the six parameters.

$$\boldsymbol{R_5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

Then the null and alternative hypotheses specified in (3.48) are equivalent to:

$$H_{05,W} : \boldsymbol{R_5\theta} = 0 \qquad \textit{against} \qquad H_{15,W} : \boldsymbol{R_5\theta} \neq 0 \tag{3.56}$$

The Wald test statistic is

$$W_5 = (\boldsymbol{R_5}\hat{\boldsymbol{\theta}})^{\intercal}(\boldsymbol{R_5}\hat{\boldsymbol{V}}\boldsymbol{R_5}^{\intercal})^{-1}(\boldsymbol{R_5}\hat{\boldsymbol{\theta}})\ \dot{\sim}\ \chi^2(1) \tag{3.57}$$

where $\boldsymbol{R_5}\hat{\boldsymbol{\theta}}$ equals to $(\hat{\theta}_{11} - \hat{\theta}_{21})^{\intercal}$, and $(\hat{\theta}_{11}, \hat{\theta}_{21})$ denotes the unconstrained MLES of $(\theta_{11}, \theta_{21})$, which are obtained by the EM algorithm. We use inverse Fisher information matrix $\boldsymbol{J_w}^{-1}$ as the estimator of the covariance matrix $\boldsymbol{V}$.

The corresponding p-value is

$$p_{w_5} = \Pr(W_5 > w_5 | H_{05,W}) = \Pr(\chi^2(1) > w_5). \tag{3.58}$$

4. Simulation Studies

To investigate the performance of the *likelihood ratio test* (LRT), the score test and the Wald test, we compare the type I error rate and the power of the three tests for five hypotheses (1) $(\alpha_1, \beta_1, \gamma_1) = (0, 0, 0)$, (2) $(\beta_1, \gamma_1) = (0, 0)$, (3) $\alpha_1 = 0$, (4)$\beta_1 = 0$, (5)$\gamma_1 = 0$. The sample sizes are set to be $n = 100(50)500$.

4.1 Tests for equality of zero-and-one inflations and non-zero-and-one means

In this subsection, we compare the type I error rates(with H_0: $(\alpha_1, \beta_1, \gamma_1) = (0, 0, 0)$) and powers (with H_1: $(\alpha_1, \beta_1, \gamma_1) \neq (0, 0, 0)$) among the LRT, score test and Wald test for various sample sizes via simulations. Without loss of generality, the values of α_1 in H_1 are chosen to be $0, 1, 2$ and the values of β_1 and γ_1 are chosen to be $0, 0.05, 0.1, 0.15$. For a given combination of $(n, \lambda_1, \phi_{10}, \phi_{11}, \lambda_2 = 2, \phi_{20} = 0.1, \phi_{21} = 0.1)$, we first independently draw $\mathbf{z}_{j1}^{(l)}, \ldots, \mathbf{z}_{jn}^{(l)} \stackrel{\text{iid}}{\sim} \text{Multinomial}\,(1; \phi_{j0}, \phi_{j1}, \phi_{j2})$ for $l = 1, \ldots, L$, where $\mathbf{z}_{ji}^{(l)} = (Z_{ji0}^{(l)}, Z_{ji1}^{(l)}, Z_{ji2}^{(l)})^{\top}$, $j = 1, 2$, $i = 1, \ldots, n$. And then we independently generate $X_{j1}^{(l)}, \ldots, X_{jn}^{(l)} \stackrel{\text{iid}}{\sim} \text{Poisson}(\lambda_j)$. Finally, we set

$$Y_{ji}^{(l)} = Z_{ji1}^{(l)} + Z_{ji2}^{(l)} \cdot X_{ji}^{(l)}, \quad j = 1, 2; \quad i = 1, \ldots, n; \quad l = 1, \ldots, L\,(L = 10^6). \tag{4.1}$$

All hypothesis testings are conducted at the significant level $\alpha = 0.05$. Let s_k denote the number of rejecting the null hypothesis H_0: $(\alpha_1, \beta_1, \gamma_1) = (0, 0, 0)$ by the test statistics T_k $(k = 1, 2, 3)$ given by (3.5), (3.10), and (3.13) respectively. Thus we can estimate the actual significance level by s_k/L with $(\alpha_1, \beta_1, \gamma_1) = (0, 0, 0)$ and estimated the power of the test statistic T_k by s_k/L with $(\alpha_1, \beta_1, \gamma_1) \neq (0, 0, 0)$.

Figure 1 shows that the comparison of type I error rates among three general tests: likelihood ratio test, score test, and Wald test for various sample sizes, and the 95% CIs for the empirical significance level connected with three tests. In this case, the Wald test has a better performance in controlling its type I error rates around the pre-chosen nominal level than the LRT. We can find that the score test, LRT and Wald test have the correct

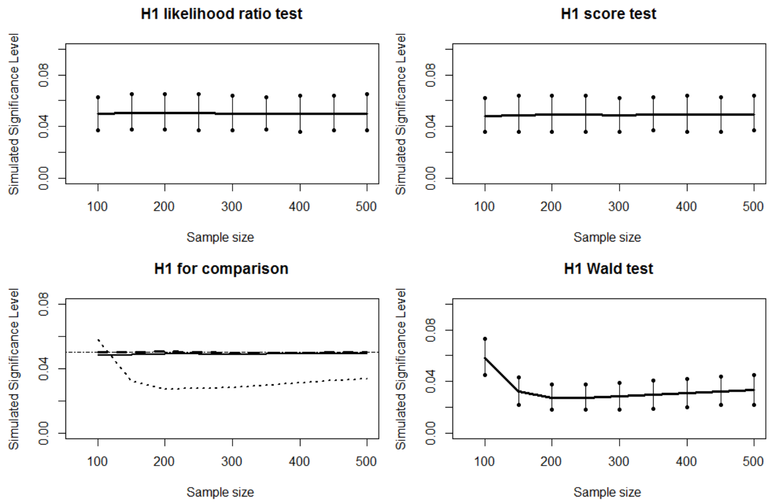

Figure 1: Comparison of type I error rates of three tests for H_{01}. For the plot that contains three different types of lines:the dashed line represents the power of the LRT, the dotted line represents the power of the Wald test, and the solid line represents the power of the score test.

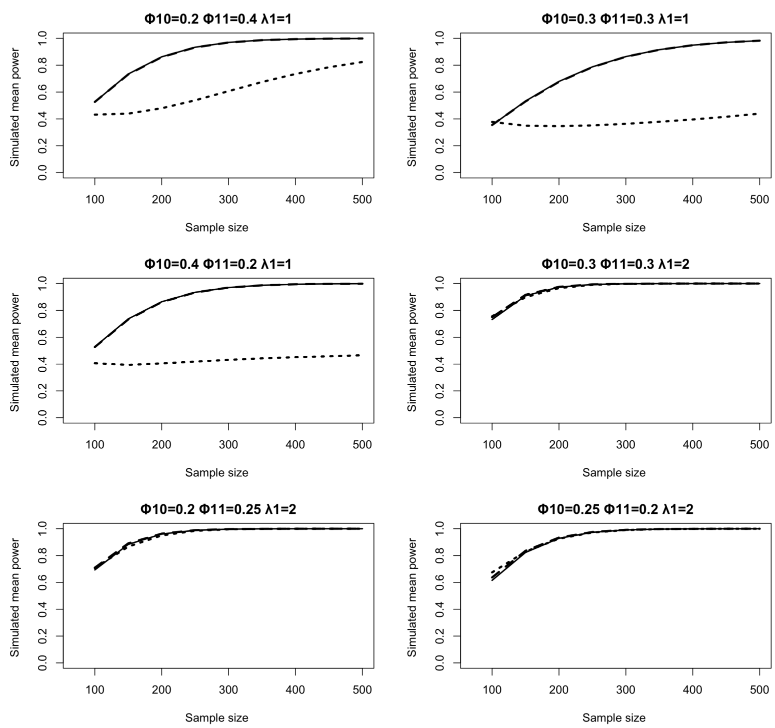

Figure 2: Simulated mean power of three tests for H_{01}. Also the dashed line represents the power of the LRT, the dotted line represents the power of the Wald test, and the solid line represents the power of the score test.

size around 0.05. As is known to all, the lower the type I error rate is, the better the test's performance is. Thus, Wald test has the best performance in this case.

Figure 2 gives the comparison of powers among three tests: the LRT, the score test, and the Wald test for different values of ϕ_{10}, ϕ_{11} and λ_1. We set ($\phi_{10} =$, ϕ_{11}, λ_1) as $(0.2, 0.15, 1)$, $(0.25, 0.15, 1)$, $(0.2, 0.25, 1)$, $(0.25, 0.25, 1)$, $(0.2, 0.15, 2)$, and $(0.25, 0.15, 2)$. It is easy to find that the Wald test is less powerful than other two tests. The empirical levels/ powers of the LRT statistic T_1, the score test statistic S_1, and the Wald test statistic W_1 are summarized in Tables 1, 2 and 3, respectively, for different scenarios.

Table 1 Empirical levels/powers of the LRT statistic T_1 be set on 10^6 replications

Sample Size (n_1&n_2)	Empirical Level	Empirical Power (ϕ_{10},ϕ_{11},λ_1) (0.2,0.15,1)	(0.25,0.15,1)	(0.2,0.25,1)	(0.25,0.25,1)	(0.2,0.15,2)	(0.25,0.15,2)
100	0.050	0.527	0.354	0.527	0.749	0.710	0.637
150	0.050	0.732	0.530	0.733	0.916	0.891	0.835
200	0.050	0.860	0.676	0.862	0.975	0.963	0.934
250	0.050	0.933	0.786	0.934	0.994	0.989	0.976
300	0.050	0.969	0.863	0.970	0.998	0.997	0.992
350	0.050	0.987	0.915	0.987	1.000	0.999	0.997
400	0.050	0.994	0.949	0.995	1.000	1.000	0.999
450	0.050	0.998	0.970	0.998	1.000	1.000	1.000
500	0.050	0.999	0.983	0.999	1.000	1.000	1.000

Table 2 Empirical levels/powers of the score statistic S_1 based on 10^6 replications

Sample Size (n_1&n_2)	Empirical Level	Empirical Power (ϕ_{10},ϕ_{11},λ_1) (0.2,0.15,1)	(0.25,0.15,1)	(0.2,0.25,1)	(0.25,0.25,1)	(0.2,0.15,2)	(0.25,0.15,2)
100	0.048	0.526	0.353	0.529	0.732	0.693	0.615
150	0.049	0.734	0.533	0.737	0.909	0.883	0.823
200	0.049	0.863	0.679	0.864	0.973	0.960	0.928
250	0.049	0.934	0.788	0.935	0.993	0.988	0.973
300	0.049	0.970	0.864	0.970	0.998	0.997	0.991
350	0.049	0.987	0.916	0.987	1.000	0.999	0.997
400	0.049	0.995	0.949	0.995	1.000	1.000	0.999
450	0.049	0.998	0.970	0.998	1.000	1.000	1.000
500	0.049	0.999	0.983	0.999	1.000	1.000	1.000

Table 3 Empirical levels/powers of the Wald statistic W_1 based on 10^6 replications

Sample Size (n_1&n_2)	Empirical Level	Empirical Power (ϕ_{10}, ϕ_{11},λ_1)					
		(0.2,0.15,1)	(0.25,0.15,1)	(0.2,0.25,1)	(0.25,0.25,1)	(0.2,0.15,2)	(0.25,0.15,2)
100	0.058	0.432	0.377	0.406	0.756	0.707	0.674
150	0.032	0.439	0.349	0.394	0.899	0.866	0.829
200	0.027	0.479	0.346	0.405	0.966	0.950	0.926
250	0.028	0.539	0.352	0.418	0.991	0.984	0.972
300	0.028	0.607	0.363	0.431	0.997	0.995	0.990
350	0.030	0.675	0.378	0.443	0.999	0.999	0.997
400	0.031	0.735	0.396	0.451	1.000	1.000	0.999
450	0.033	0.785	0.416	0.458	1.000	1.000	1.000
500	0.033	0.824	0.439	0.466	1.000	1.000	1.000

4.2 Tests for equality of zero-and-one inflations

In this subsection, we compare the type I error rates (with H_{20}: $(\beta_1, \gamma_1) = (0, 0)$) and powers (with H_{21}: $(\beta_1, \gamma_1) \neq (0, 0)$) between the LRT and the score test for various sample sizes. Firstly we set one sample parameter's value $\alpha_2 = 1,\ \beta_2 = 0.1,\ \gamma_2 = 0.1$. Then for a given value of $\alpha_1 = 0,\ 1,\ 2$, we set combinations as follows: $(\beta_1, \gamma_1) = (0, 0)$, $(0.05, 0)$, $(0.1, 0)$, $(0.15, 0.0)$, $(0, 0.05)$, $(0.5, 0.05)$, $(0.1, 0.05)$, $(0.15, 0.05)$, $(0, 0.1)$, $(0.05, 0.1)$, $(0.1, 0.1)$, $(0.15, 0.1)$, $(0, 0.15)$, $(0.05, 0.15)$, $(0.1, 0.15)$, $(0.15, 0.15)$. Then we generate

$$Y_{11}^{(l)}, \ldots, Y_{1n}^{(l)} \overset{\text{iid}}{\sim} \text{ZOIP}(\beta_1, \gamma_1; \beta_2, \gamma_2, \alpha_1, \alpha_2)$$

with $L = 10^6$ replications. All hypothesis testings are conducted at the significant level $\alpha = 0.05$.

Let r_k denote the number of rejecting the null hypothesis H_{20}: $(\beta_1, \gamma_1) = (0, 0)$ by the test statistics T_k $(k = 4,\ 5,\ 6)$ given by (3.17), (3.21), and (3.24) respectively. Hence, the actual significance level can be estimated by r_k/L with $(\beta_1, \gamma_1) = (0, 0)$ and the power of the test statistic T_k can be estimated by r_k/L with $(\beta_1, \gamma_1) \neq (0, 0)$.

Figure 3 shows that the comparison of type I error rates between the LRT, the score test and Wald test. In general, the score test is slightly better than the other two tests. And when the sample size is small, the Wald test is not good as the other two tests. But as

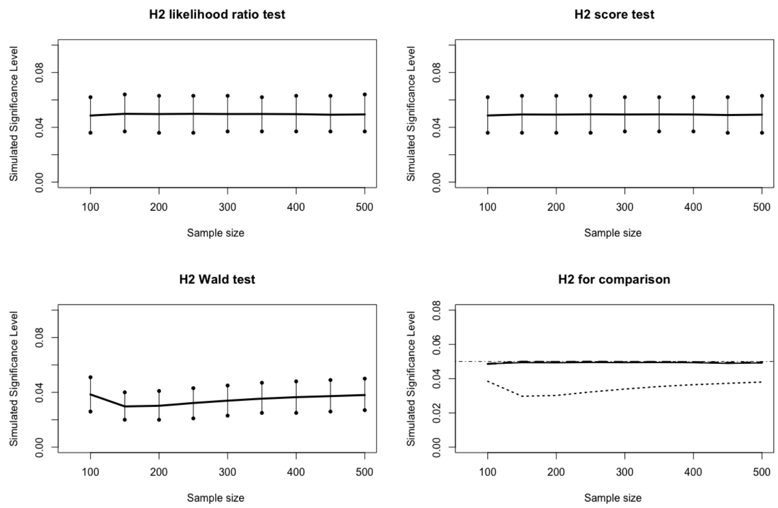

Figure 3: Comparison of type I error rates of three tests for H_{02}. For the plot that contains three different types of lines:the dashed line represents the power of the LRT, the dotted line represents the power of the Wald test, and the solid line represents the power of the score test.

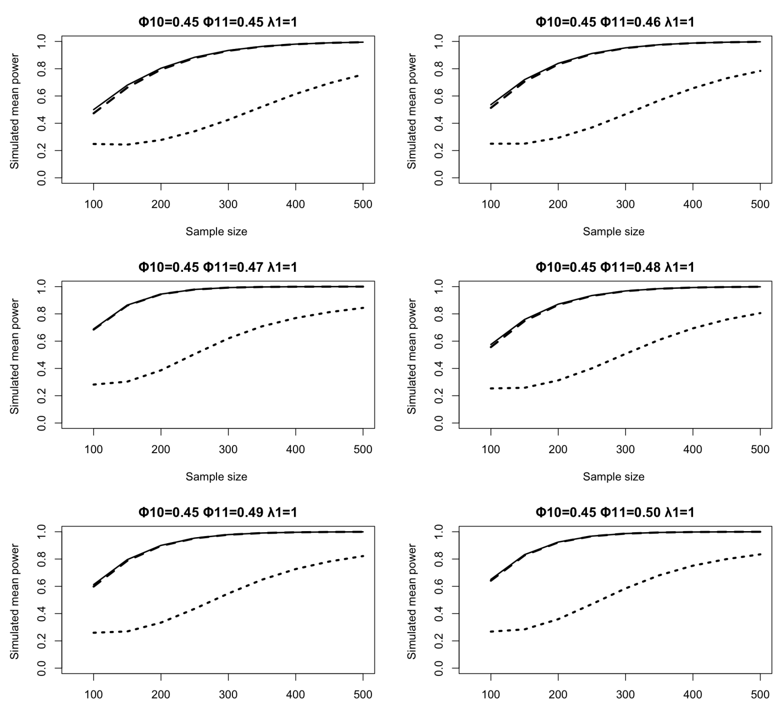

Figure 4: Simulated mean power of three tests for H_{02}. The dashed line represents the power of the LRT, the dotted line represents the power of the Wald test, and the solid line represents the power of the score test.

the sample size grows larger, the performance of the Wald test becomes much better than before.

Figure 4 gives the comparison of powers between the LRT and the score test for different combination values of (α_1, β_1). From the figure, we can find that score test and likelihood ratio test have little difference between each other. The curves of them are almost overlapped. And the Wald test is always worse than score and LRT test in any different scenarios.

The empirical levels/powers of the LRT statistic T_2, the score test statistic S_2, and the Wald test statistic W_2 are summarized in Tables 4, 5 and 6, respectively, for different scenarios.

Table 4 Empirical levels/powers of the LRT statistic T_2 based on 10^6 replications

Sample Size (n_1&n_2)	Empirical Level	Empirical Power (ϕ_{10},ϕ_{11},λ_1) (0.2,0.15,1)	(0.25,0.15,1)	(0.2,0.25,1)	(0.25,0.25,1)	(0.2,0.15,2)	(0.25,0.15,2)
100	0.050	0.473	0.513	0.685	0.556	0.598	0.641
150	0.050	0.662	0.705	0.862	0.747	0.788	0.827
200	0.050	0.794	0.832	0.944	0.866	0.897	0.922
250	0.050	0.879	0.908	0.979	0.932	0.952	0.967
300	0.049	0.932	0.951	0.993	0.967	0.979	0.987
350	0.050	0.962	0.975	0.997	0.984	0.991	0.995
400	0.049	0.980	0.988	0.999	0.993	0.996	0.998
450	0.049	0.989	0.994	1.000	0.997	0.999	0.999
500	0.049	0.995	0.997	1.000	0.999	0.999	1.000

Table 5 Empirical levels/powers of the score statistic S_2 based on 10^6 replications

Sample Size (n1&n2)	Empirical Level	Empirical Power (ϕ_{10},ϕ_{11},λ_1) (0.2,0.15,1)	(0.25,0.15,1)	(0.2,0.25,1)	(0.25,0.25,1)	(0.2,0.15,2)	(0.25,0.15,2)
100	0.049	0.500	0.536	0.690	0.575	0.613	0.651
150	0.049	0.680	0.720	0.864	0.759	0.796	0.832
200	0.049	0.804	0.839	0.945	0.871	0.900	0.925
250	0.049	0.884	0.912	0.979	0.935	0.953	0.968
300	0.049	0.934	0.953	0.993	0.968	0.979	0.987
350	0.049	0.963	0.976	0.997	0.985	0.991	0.995
400	0.049	0.980	0.988	0.999	0.993	0.996	0.998
450	0.049	0.990	0.994	1.000	0.997	0.999	0.999
500	0.049	0.995	0.997	1.000	0.999	0.999	1.000

Table 6 Empirical levels/powers of the Wald statistic W_2 based on 10^6 replications

Sample Size (n_1&n_2)	Empirical Level	Empirical Power (ϕ_{10}, ϕ_{11},λ_1)					
		(0.2,0.15,1)	(0.25,0.15,1)	(0.2,0.25,1)	(0.25,0.25,1)	(0.2,0.15,2)	(0.25,0.15,2)
100	0.038	0.248	0.250	0.282	0.253	0.260	0.268
150	0.030	0.244	0.250	0.303	0.258	0.269	0.284
200	0.030	0.277	0.293	0.386	0.312	0.334	0.359
250	0.032	0.341	0.369	0.506	0.401	0.435	0.470
300	0.034	0.425	0.466	0.620	0.507	0.547	0.585
350	0.035	0.522	0.568	0.709	0.610	0.649	0.681
400	0.036	0.615	0.657	0.770	0.694	0.727	0.751
450	0.037	0.694	0.730	0.813	0.759	0.782	0.800
500	0.038	0.757	0.784	0.844	0.806	0.822	0.834

4.3 Tests for equality of non-zero-and-one means

In this subsection, we compare the type I error rates (with H_0: $\alpha_1 = 0$) and powers (with H_1: $\lambda \neq \alpha_1$) between the LRT,the score test, and Wald test for various sample sizes and different values of α_1 via simulations, where the values of α_1 are chosen to be $1, 2, 3, 4, 5, 6, 7$. For a given combination of $(n_1, \beta_1 = 0, \gamma_1 = 0, \alpha_2 = 1, \beta_2 = 0.1, \gamma_2 = 0.1)$. We generate

$$Y_1^{(l)}, \ldots, Y_n^{(l)} \overset{\text{iid}}{\sim} \text{ZOIP}(\beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2; \alpha_1)$$

with $L = 10^6$ replications. All hypothesis testings are conducted at the significant level $\alpha = 0.05$. Let r_k denote the number of rejecting the null hypothesis $H_0 : \alpha_1 = 0$ by the test statistics T_k ($k = 7, 8, 9$) given by (3.28), (3.32), and (3.35) respectively. Hence, the actual significance level can be estimated by r_k/L with $\alpha_1 = 0$ and the power of the test statistic T_k can be estimated by r_k/L with $\alpha_1 \neq 0$.

Figure 5 shows that some comparison of type I error rates among three tests: the LRT test, the score test and the Wald test. When $\alpha_1 = 0$, the Wald test's type I rate can not be controlled at the significance level $\alpha = 0.05$. But as the sample size goes larger, type I error rate can be controlled. And in large sample size, the three tests can control its type I error rate respectively. In general, there is no significant difference among performance of type I error rate of three tests.

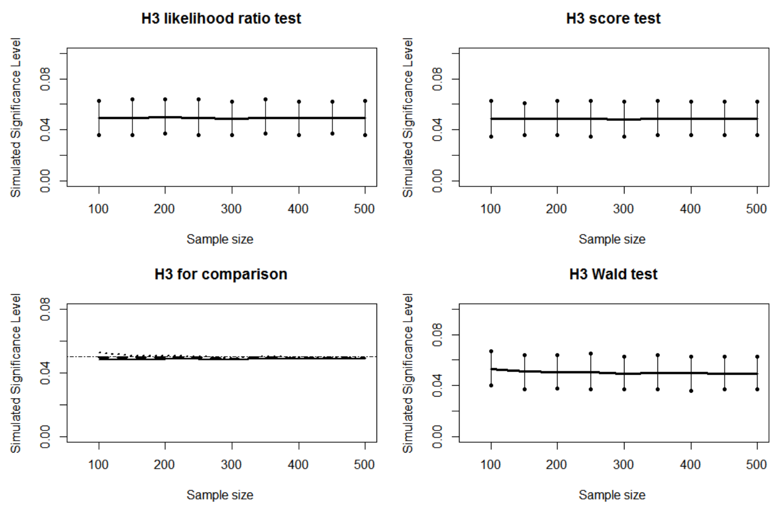

Figure 5: Comparison of type I error rates of three tests for H_{03}. For the plot that contains three different types of lines:the dashed line represents the power of the LRT, the dotted line represents the power of the Wald test, and the solid line represents the power of the score test.

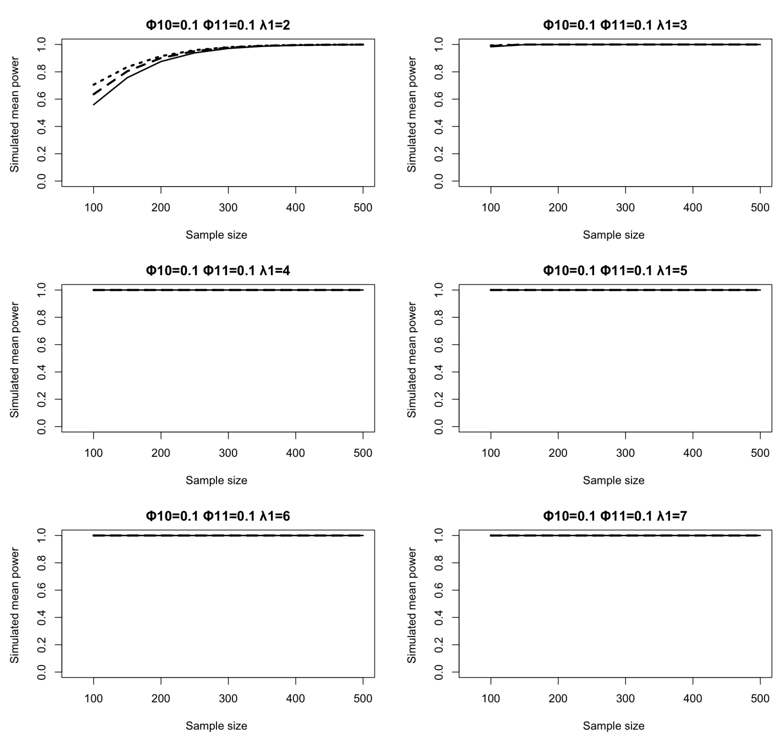

Figure 6: The simulated mean power of three tests for H_{03}. The dashed line represents the power of the LRT, the dotted line represents the power of the Wald test, and the solid line represents the power of the score test.

Figure 6 shows that some comparison of simulated mean power among three tests: the LRT, the score test and the Wald test. When $\alpha_1 = 1$, the Wald test is more powerful than the other tests. When $\lambda \geqslant 1$, the three performance of three tests are almost the same. In general, there is no significant difference between the three tests' powers.

The empirical levels/powers of the LRT statistic T_3, the score test statistic S_3, and the Wald test statistics W_3 are summarized in Tables 7, 8, and 9, respectively.

Table 7 Empirical levels/powers of the LRT statistic T_3 based on 10^6 replications

Sample Size	Empirical	Empirical Power					
		λ_1					
(n_1&n_2)	Level	2	3	4	5	6	7
100	0.050	0.636	0.990	1.000	1.000	1.000	1.000
150	0.050	0.804	0.999	1.000	1.000	1.000	1.000
200	0.050	0.901	1.000	1.000	1.000	1.000	1.000
250	0.050	0.952	1.000	1.000	1.000	1.000	1.000
300	0.049	0.978	1.000	1.000	1.000	1.000	1.000
350	0.050	0.990	1.000	1.000	1.000	1.000	1.000
400	0.049	0.996	1.000	1.000	1.000	1.000	1.000
450	0.049	0.998	1.000	1.000	1.000	1.000	1.000
500	0.049	0.999	1.000	1.000	1.000	1.000	1.000

Table 8 Empirical levels/powers of the score statistic S_3 based on 10^6 replications

Sample Size	Empirical	Empirical Power					
		λ_1					
(n_1&n_2)	Level	2	3	4	5	6	7
100	0.048	0.559	0.983	1.000	1.000	1.000	1.000
150	0.048	0.757	0.999	1.000	1.000	1.000	1.000
200	0.049	0.875	1.000	1.000	1.000	1.000	1.000
250	0.049	0.939	1.000	1.000	1.000	1.000	1.000
300	0.048	0.971	1.000	1.000	1.000	1.000	1.000
350	0.049	0.987	1.000	1.000	1.000	1.000	1.000
400	0.049	0.994	1.000	1.000	1.000	1.000	1.000
450	0.049	0.998	1.000	1.000	1.000	1.000	1.000
500	0.049	0.999	1.000	1.000	1.000	1.000	1.000

Table 9 Empirical levels/powers of the Wald statistic W_3 based on 10^6 replications

Sample Size (n_1&n_2)	Empirical Level	Empirical Power λ_1					
		2	3	4	5	6	7
100	0.033	0.706	0.993	1.000	1.000	1.000	1.000
150	0.026	0.835	1.000	1.000	1.000	1.000	1.000
200	0.024	0.915	1.000	1.000	1.000	1.000	1.000
250	0.009	0.958	1.000	1.000	1.000	1.000	1.000
300	0.021	0.981	1.000	1.000	1.000	1.000	1.000
350	0.028	0.991	1.000	1.000	1.000	1.000	1.000
400	0.027	0.996	1.000	1.000	1.000	1.000	1.000
450	0.02	0.998	1.000	1.000	1.000	1.000	1.000
500	0.015	0.999	1.000	1.000	1.000	1.000	1.000

4.4 Tests for equality of zero inflation

In this subsection, we compare the type I error rates (with $H_0 : \beta_1 = 0$) and powers (with $H_1 : \beta_1 \neq \beta_1$) between the LRT, the score test, and Wald test for various sample sizes and different values of β_1 via simulations, where the values of β_1 are chosen to be $0.15, 0.16, 0.17, 0.18, 0.19, 0.2$. For a given combination of $(n_1, \alpha_1 = 0, \gamma_1 = 0, \alpha_2 = 1, \beta_2 = 0.1, \gamma_2 = 0.1)$, we generate $Y_1^{(l)}, \ldots, Y_n^{(l)} \stackrel{\text{iid}}{\sim} \text{ZOIP}(\alpha_1, \gamma_1, \alpha_2, \beta_2, \gamma_2; \beta_1)$ with $L = 10^6$ replications. All hypothesis testings are conducted at the significant level $\alpha = 0.05$. Let r_k denote the number of rejecting the null hypothesis H_0: $\beta_1 = 0$ by the test statistics T_k ($k = 10, 11, 12$) given by (3.39), (3.4.2), and (3.46), respectively. Hence, the actual significance level can be estimated by r_k/L with $\beta_1 = 0$ and the power of the test statistic T_k can be estimated by r_k/L with $\beta_1 \neq 0$.

Figure 7 shows that some comparison of type I error rates among three tests: the LRT test, the score test and the Wald test. When $\alpha_1 = 0$, the Wald test's type I rate can not be controlled at the significance level $\alpha = 0.05$.But as the sample size goes larger, type I error rate can be controlled.And in large sample size, the three tests can control its type I error rate respectively. In general, there is no significant difference among performance of type I error rate of three tests. Figure 8 shows that some comparison of simulated mean power

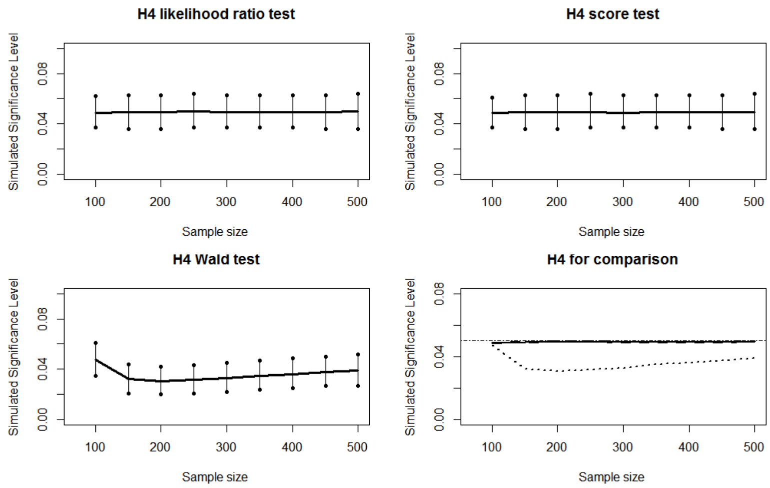

Figure 7: Comparison of type I error rates of three tests for H_{04}. For the plot that contains three different types of lines:the dashed line represents the power of the LRT, the dotted line represents the power of the Wald test, and the solid line represents the power of the score test.

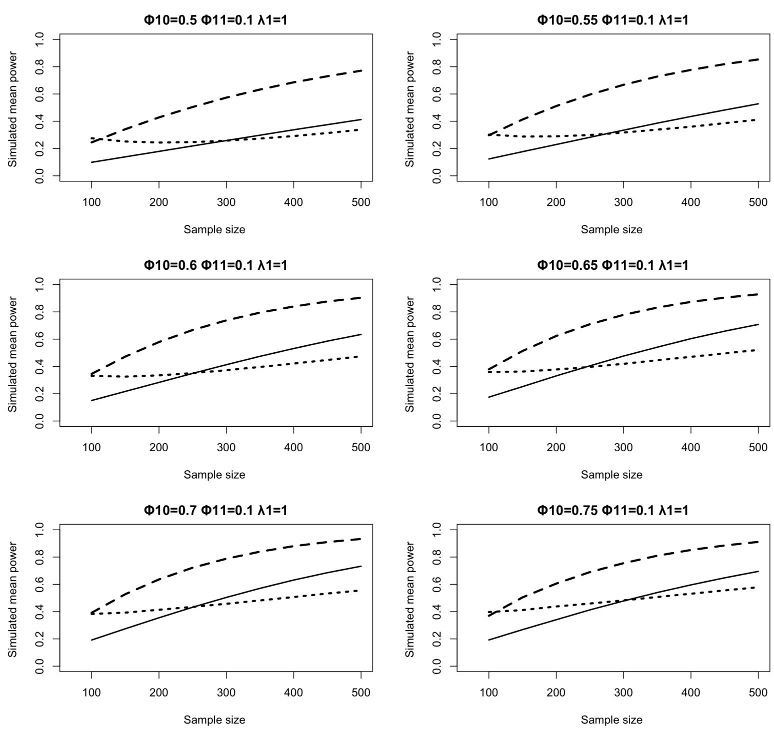

Figure 8: Simulated mean power of three tests for H_{04}. The dashed line represents the power of the LRT, the dotted line represents the power of the Wald test, and the solid line represents the power of the score test.

among three tests: the LRT, the score test and the Wald test. When $\beta_1 \geqslant 0$, LRT is the most powerful of three tests. And in general Wald test is better than the score test. But in small size, the score test and Wald test have a slight difference.

The empirical levels/powers of the LRT statistic T_4, the score test statistic S_4, and Wald statistic W_4 are summarized in Tables 10, 11, and 12, respectively.

Table 10 Empirical levels/powers of the LRT statistic T_4 based on 10^6 replications

Sample	Empirical	Empirical Power					
Size		ϕ_{10}					
$(n_1 \& n_2)$	Level	0.25	0.26	0.27	0.28	0.29	0.3
100	0.049	0.245	0.297	0.345	0.379	0.392	0.370
150	0.049	0.342	0.413	0.472	0.513	0.528	0.504
200	0.049	0.428	0.512	0.579	0.623	0.636	0.605
250	0.050	0.505	0.596	0.667	0.710	0.722	0.690
300	0.049	0.573	0.667	0.738	0.778	0.788	0.755
350	0.049	0.633	0.728	0.795	0.831	0.839	0.810
400	0.049	0.686	0.777	0.840	0.873	0.880	0.852
450	0.049	0.730	0.819	0.876	0.904	0.910	0.885
500	0.050	0.771	0.853	0.904	0.929	0.933	0.911

Table 11 Empirical levels/powers of the score statistic S_4 based on 10^6 replications

Sample	Empirical	Empirical Power					
Size		ϕ_{10}					
$(n_1 \& n_2)$	Level	0.25	0.26	0.27	0.28	0.29	0.3
100	0.049	0.099	0.124	0.150	0.175	0.192	0.192
150	0.049	0.139	0.177	0.217	0.252	0.274	0.268
200	0.049	0.179	0.230	0.283	0.331	0.355	0.340
250	0.049	0.219	0.283	0.349	0.405	0.432	0.413
300	0.049	0.259	0.335	0.413	0.476	0.504	0.478
350	0.049	0.298	0.386	0.474	0.541	0.571	0.540
400	0.049	0.338	0.435	0.531	0.604	0.631	0.596
450	0.049	0.375	0.482	0.586	0.659	0.685	0.648
500	0.050	0.413	0.528	0.634	0.708	0.733	0.695

Table 12 Empirical levels/powers of the Wald statistic W_4 based on 10^6 replications

Sample Size (n_1&n_2)	Empirical Level	Empirical Power ϕ_{10}					
		0.25	0.26	0.27	0.28	0.29	0.3
100	0.047	0.275	0.302	0.331	0.359	0.383	0.397
150	0.032	0.252	0.288	0.325	0.362	0.392	0.411
200	0.031	0.244	0.290	0.335	0.378	0.414	0.438
250	0.031	0.247	0.299	0.351	0.396	0.434	0.459
300	0.033	0.258	0.317	0.372	0.419	0.457	0.482
350	0.035	0.273	0.339	0.396	0.446	0.482	0.507
400	0.036	0.292	0.360	0.421	0.470	0.507	0.531
450	0.037	0.314	0.386	0.448	0.496	0.533	0.555
500	0.039	0.339	0.412	0.474	0.521	0.555	0.579

4.5 Tests for equality of one inflation

In this subsection, we compare the type I error rates (with H_0: $\gamma_1 = 0$) and powers (with H_1: $\gamma_1 \neq 0$) between the LRT, the score test, and Wald test for various sample sizes and different values of γ_1 via simulations, where the values of γ_1 are chosen to be $0.25, 0.26, 0.27, 0.28.29, 0.3$. For a given combination of $(n_1, \alpha_1 = 0, \beta 1 = 0, \alpha_2 = 1, \beta_2 = 0.1, \gamma_2 = 0.1)$, we generate $Y_1^{(l)}, \ldots, Y_n^{(l)} \overset{\text{iid}}{\sim} \text{ZOIP}(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_2; \beta_1)$ with $L = 10^6$ replications. All hypothesis testings are conducted at the significant level $\alpha = 0.05$. Let r_k denote the number of rejecting the null hypothesis H_0: $\gamma_1 = 0$ by the test statistics T_k $(k = 7, 8)$ given by (3.50), (3.5.2), and (3.57) respectively. Hence, the actual significance level can be estimated by r_k/L with $\gamma_1 = 0$ and the power of the test statistic T_k can be estimated by r_k/L with $\gamma_1 \neq 0$.

Figure 9 shows that some comparison of type I error rates among three tests: the LRT test,the score test and the Wald test. When $\gamma_1 = 0$, the Wald test's type I rate can not be controlled at the significance level $\alpha = 0.05$. But as the sample size goes larger, type I error rate can be controlled. And in large sample size, the three tests can control its type I error rate respectively. In general, there is no significant difference among performance of type I error rate of three tests.

Figure 10 shows that some comparison of type I error rates between the LRT and the score test. When $\lambda = 1$, the score test has a better performance in controlling its type I error

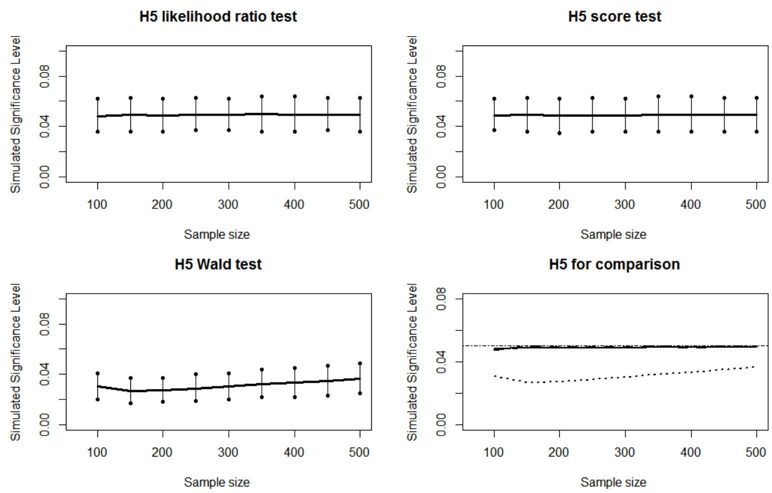

Figure 9: Comparison of type I error rates of three tests for H_{05}. For the plot that contains three different types of lines:the dashed line represents the power of the LRT, the dotted line represents the power of the Wald test, and the solid line represents the power of the score test.

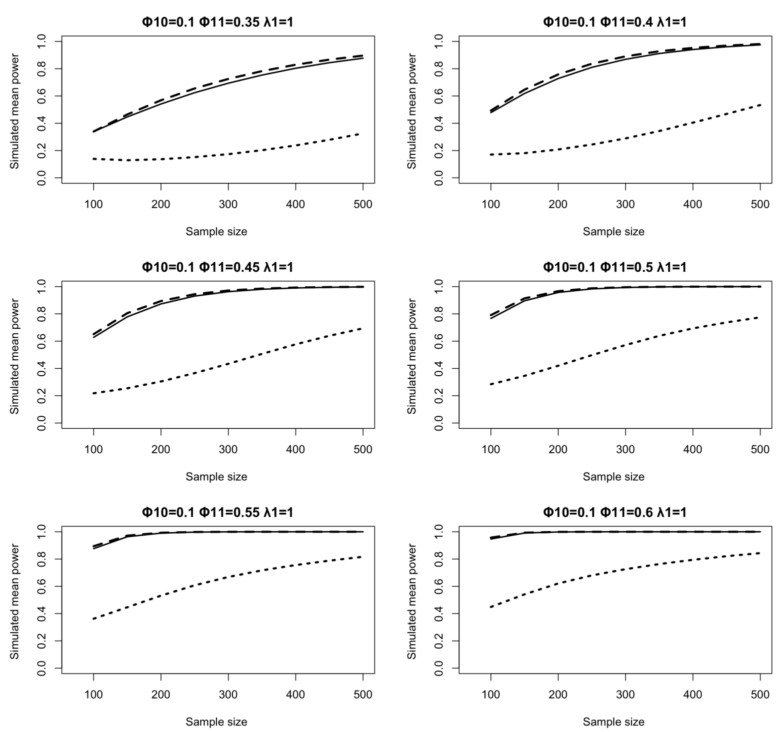

Figure 10: Simulated mean power of three tests for H_{05}. The dashed line represents the power of the LRT, the dotted line represents the power of the Wald test, and the solid line represents the power of the score test.

rates around the pre-chosen nominal level than the LRT. When $\lambda \geqslant 3$, it can be found that the two lines fluctuate with several points and sections, so we cannot say which one is more powerful. In general, there is no significant difference between the two tests' performance in controlling their type I error rates around the pre-chosen nominal level.

The empirical levels/powers of the LRT statistic T_5, the score test statistic S_5, and the Wald statistic W_5 are summarized in Tables 13, 14 and 15, respectively.

Table 13 Empirical levels/powers of the LRT statistic T_5 based on 10^6 replications

Sample Size	Empirical	Empirical Power					
		ϕ_{11}					
(n_1&n_2)	Level	0.35	0.36	0.37	0.38	0.39	0.4
100	0.048	0.339	0.493	0.651	0.791	0.894	0.957
150	0.049	0.465	0.647	0.805	0.914	0.972	0.993
200	0.049	0.569	0.758	0.894	0.967	0.993	0.999
250	0.049	0.655	0.837	0.944	0.988	0.998	1.000
300	0.049	0.725	0.891	0.971	0.996	1.000	1.000
350	0.050	0.782	0.928	0.986	0.998	1.000	1.000
400	0.049	0.829	0.953	0.993	0.999	1.000	1.000
450	0.049	0.867	0.970	0.996	1.000	1.000	1.000
500	0.050	0.896	0.981	0.998	1.000	1.000	1.000

Table 14 Empirical levels/powers of the score statistic S_5 based on 10^6 replications

Sample Size	Empirical	Empirical Power					
		ϕ_{11}					
(n_1&n_2)	Level	0.35	0.36	0.37	0.38	0.39	0.4
100	0.048	0.192	0.339	0.478	0.628	0.767	0.876
150	0.049	0.268	0.447	0.619	0.778	0.896	0.964
200	0.049	0.340	0.541	0.728	0.873	0.957	0.990
250	0.049	0.413	0.624	0.810	0.931	0.983	0.998
300	0.049	0.478	0.694	0.869	0.963	0.994	0.999
350	0.050	0.540	0.753	0.912	0.981	0.998	1.000
400	0.049	0.596	0.803	0.941	0.990	0.999	1.000
450	0.049	0.648	0.844	0.961	0.995	1.000	1.000
500	0.049	0.695	0.877	0.975	0.998	1.000	1.000

Table 15 Empirical levels/powers of the Wald statistic W_5 based on 10^6 replications

Sample Size (n_1&n_2)	Empirical Level	Empirical Power ϕ_{11} 0.35	0.36	0.37	0.38	0.39	0.4
100	0.031	0.139	0.170	0.218	0.284	0.362	0.449
150	0.027	0.129	0.181	0.254	0.346	0.447	0.541
200	0.027	0.136	0.208	0.304	0.420	0.532	0.621
250	0.029	0.152	0.244	0.366	0.498	0.608	0.680
300	0.030	0.173	0.289	0.434	0.572	0.668	0.726
350	0.032	0.202	0.343	0.507	0.639	0.717	0.763
400	0.033	0.237	0.404	0.577	0.694	0.756	0.794
450	0.035	0.279	0.469	0.640	0.738	0.788	0.821
500	0.037	0.325	0.534	0.694	0.775	0.817	0.844

5. Applications

In this section, three real data sets are used to illustrate the proposed methods, where the Newton–Raphson algorithm for finding the MLEs of parameters is not available in all examples because the corresponding observed information matrices are nearly singular, while the EM algorithm works well in all examples.

5.1 Stressful Trajectories with or without Health Terms on Mortality Data

The VA Normative Aging Study(NAS) screened 1443 mens from 1985 to 2002(18 years long) about stressful life event trajectories. To measure the stressful life event, the Elders Life Stress Inventory(ELSI) is introduced to this survey, which includes thirty items that assesses events likely to occur in middle-aged and older adults during the past year. We scored ELSI in two ways: a total sum which includes all 30 items and one which omits the two items which tap health items, therefore we get two samples: one sample is the number of events with health items for a man in 18-year-lifespan and the other sample is the number of events without health items. And we supposed that they are independent. However, because approximately 30% of NAS men did not report stressful life events, it was necessary to use a zero-inflated Poisson (ZIP) model to account for the fact that there are more zeros than would be expect under the Poisson assumption. Moreover, we observe that this two-sample data is appropriate for Zero-and-One Inflated Poisson(ZOIP) model. Thus we use ZOIP model to fit the data.

Table 16 The dentist visiting data from Swedish Level of Living Surveys (Eriksson and Åberg, 1987; Melkersson and Olsson, 1999)

Count	0	1	2	3	4	5	6	7	8	9	10	11++
SEW/OHI^{*}	29.4	21.8	17.3	11.6	7.32	4.57	2.95	1.95	1.21	0.69	0.45	0.76
$SEWHI^{**}$	37.8	23.1	15.6	9.77	5.27	3.68	1.79	1.22	0.73	0.34	0.25	0.38

** SEW/OHI=Stressful events without health items ** SEWHI=Stressful events with health items*

Table 17 AIC and BIC

SEWH/OI	Poisson	ZIP	ZOIP
AIC	431.3566	403.8649	307.3065
BIC	433.9618	409.0752	405.1220
SEWHI	Poisson	ZIP	ZOIP
AIC	388.0451	360.8633	355.6591
BIC	390.6496	366.0723	363.4726

** SEW/OHI=Stressful events without health items * SEWHI=Stressful events with health items*

The hypotheses of interest is $H_0 : (\lambda_1, \phi_{10}, \phi_{11}) = (\lambda_2, \phi_{20}, \phi_{21})$ or equivalently $H_0 : (\alpha_1, \beta_1, \gamma_1) = (0, 0, 0)$ against H_1 : not all are equal. Based on the data, the EM algorithm gives the unconstrained MLEs of the parameters:$(\hat{\alpha_1}, \hat{\beta_1}, \hat{\gamma_1}) = (0.37411, -0.07854, -0.0005)$. The test statistic $\chi^2_{LRT} < \chi^2(3)$, $\chi^2_{ST} < \chi^2(3)$ and $\chi^2_{WT} < \chi^2(3)$. The p-values are $p_{LRT} = 2.047143e - 06$, $p_{ST} = 2.251822e - 06$ and $p_{WT} = 2.06252e - 06$, which suggests that there is every reason to believe that we should reject H_0. Thus there's a big difference between the distribution of the OPP and OFNP.

5.2 OPP and OFNP data

Provided by National Medical ExpenditureSurvey(NMES), the data was conducted in 1987 and 1988 to research how Americans use and pay for health services. Here we only consider two circumstances: visits to a physician in a hospital outpatient setting (OPP) and visits to a non-physician in an office setting (OFNP). Table 18 shows the propotion of OPP and OFNP. Considering the relatively higher propotions of zero and one counts, we apply the ZOIP distibution to fit the data. What's more, the AIC and BIC listed in Table 19 also support the assumption. Our aim is to find whether there's a difference between the distribution of the OPP and OFNP.

Table 18 OPP and OFNP data (NMES, 1987-1988)

Count	0	1	2	3	4	5	6++
OPP	77.1	11.9	4.6	1.7	1.2	0.8	2.6
OFNP	68.2	12.9	4.7	2.8	2.4	1.6	7.5

Table 19 AIC and BIC

OPP	Poisson	ZIP	ZOIP
AIC	234.0064	187.5795	182.3755
BIC	236.6101	192.7879	190.1881
OFNP	Poisson	ZIP	ZOIP
AIC	347.5401	257.5805	245.5464
BIC	350.1462	262.7968	253.3650

The hypotheses of interest is $H_0 : (\lambda_1, \phi_{10}, \phi_{11}) = (\lambda_2, \phi_{20}, \phi_{21})$ or equivalently $H_0 : (\alpha_1, \beta_1, \gamma_1) = (0, 0, 0)$ against H_1 : not all are equal. Based on the data, the EM algorithm gives the unconstrained MLEs of the parameters:$(\hat{\alpha_1}, \hat{\beta_1}, \hat{\gamma_1}) = (-0.8798, 0.0888, -0.0129)$. The test statistic $\chi^2_{LRT} < \chi^2(3)$, $\chi^2_{ST} < \chi^2(3)$ and $\chi^2_{WT} < \chi^2(3)$. The p-values are $p_{LRT} = 0.259$, $p_{ST} = 0.276$ and $p_{WT} = 0.251$, which suggests that we have no evidence to reject H_0. Thus there's no difference between the distribution of the OPP and OFNP.

5.3 Poliomyelitis data

Scott L.Zeger (1988) analyzed the data of the monthly number of cases of poliomyelitis reported by the U.S.Centers for Disease Control for the years 1970 to 1983. We devide the data into two parts and recalculate. Table 20 lists the number of cases reported for the years 1970 to 1976 and the years 1977 to 1983. AIC and BIC shown in Table 21 suggest that it is reasonable to fit the data with ZOIP model. We want to test equality of the parameters:$(\lambda, \phi_0, \phi_1)$.

Table 20 Poliomyelitis data(U.S.Centers, 1970-1983)

Count	0	1	2	3	4	5	6++
1970-1976	30	30	9	7	1	3	3
1977-1983	34	25	13	5	4	0	3

Table 21 AIC and BIC

1970-1976	Poisson	ZIP	ZOIP
AIC	274.2781	270.8456	259.6154
BIC	276.6818	275.6137	266.5707
1977-1983	Poisson	ZIP	ZOIP
AIC	267.9687	261.6684	259.3580
BIC	270.3995	266.5300	266.6504

Similarly, the hypotheses of interest in equal to $H_0 : (\alpha_1, \beta_1, \gamma_1) = (0, 0, 0)$ against H_1 : not all are equal. The unconstrained MLEs given by EM algorithm of the parameters are $(\hat{\alpha_1}, \hat{\beta_1}, \hat{\gamma_1}) = (0.4782122, -0.020563, 0.1046636)$. The test statistic $\chi^2_{LRT} < \chi^2(3)$, $\chi^2_{ST} < \chi^2(3)$ and $\chi^2_{WT} < \chi^2(3)$. The p-values are $p_{LRT} = 0.684$, $p_{ST} = 0.684$ and $p_{WT} = 0.702$. It clearly shows that we have no evidence to reject H_0. Thus there's no difference between the distribution of the number of cases reported for the years 1970 to 1976 and the years 1977 to 1983.

6. Discussion

In this paper, the problem of comparing the parameters of two ZOIP populations is considered. Using proper transformations, we derive likelihood ratio tests, score tests and Wald tests for homogeneity of two ZOIP populations under five different hypotheses, with statistical inferences of ZOIP populations by the *expectation-maximization* (EM) algorithm. By the simulation studies to compare likelihood ratio tests, score tests and Wald tests, we noted that for different hypotheses, the optimal tests are different. Moreover, theses tests perform well when the sample sizes are large enough and the difference between two ZOIP populations is relatively significant. Also, the performance of those tests is related to the accuracy of statistical inferences of ZOIP populations and in this paper, we derive *expectation-maximization* (EM) algorithm. In this paper, we only discussed the homogeneity of two ZOIP populations and considered five possible scenarios of common interest to compare two ZOIP populations. In the future, the homogeneity of multiple ZOIP populations is worthy of consideration. And the generalization of the ZOIP distribution to the bivariate or multivariate version is our future research interest.

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A Second derivatives of the observed-data log-likelihood functions

The second derivatives of the observed-data log-likelihood function ℓ_2 are given by

$$
\begin{aligned}
\frac{\partial^2 \ell_2}{\partial \alpha_1^2} &= -\frac{m_{10} r_2 t_3 e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}+\frac{m_{11}}{\left(t_4-j_1\right)^2}[\left(t_4-j_1\right)\left(2 r_2 e^{-\lambda_1}-j_1\right)-\left(r_2 e^{-\lambda_1}-j_1\right)^2]-\frac{N_1}{\lambda_1^2}, \\
\frac{\partial^2 \ell_2}{\partial \alpha_1 \partial \nu_1} &= \frac{m_{10} p_2\left(-r_2+t_3\right) e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}-\frac{m_{11} p_2 e^{-\lambda_1}}{\left(t_4-j_1\right)^2}\left(\lambda_1 r_2 e^{-\lambda_1}-\lambda_1 t_4+t_4-j_1\right), \\
\frac{\partial^2 \ell_2}{\partial \alpha_1 \partial \omega_1} &= -\frac{m_{10} t_3 p_3 e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}+\frac{m_{11} p_3}{\left(t_4-j_1\right)^2}[\left(-\lambda_1+1\right)\left(t_4-j_1\right) e^{-\lambda_1}+\left(\lambda_1 e^{-\lambda_1}-1\right)\left(r_2 e^{-\lambda_1}-j_1\right)], \\
\frac{\partial^2 \ell_2}{\partial \alpha_1 \partial \alpha_2} &= \frac{\partial^2 \ell_2}{\partial \alpha_1^2}, \\
\frac{\partial^2 \ell_2}{\partial \alpha_1 \partial \nu_2} &= \frac{\partial^2 \ell_2}{\partial \alpha_1 \partial \nu_1}, \\
\frac{\partial^2 \ell_2}{\partial \alpha_1 \partial \omega_2} &= \frac{\partial^2 \ell_2}{\partial \alpha_1 \partial \omega_1}, \\
\frac{\partial^2 \ell_2}{\partial \nu_1^2} &= -\frac{m_{10}\left(1-e^{-\lambda_1}\right)}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}[p_2^2\left(1-e^{-\lambda_1}\right)-q_1\left(-r_2 e^{-\lambda_1}+t_3\right)]-\frac{m_{11} \lambda_1 e^{-\lambda_1}}{\left(t_4-j_1\right)^2}[\lambda_1 p_2^2 e^{-\lambda_1}+q_1\left(t_4-j_1\right)] \\
&\quad -\frac{\left(p_2^2-r_2 q_1\right)\left(m_{11}+m_{10}-n_1\right)}{r_2^2}, \\
\frac{\partial^2 \ell_2}{\partial \nu_1 \partial \omega_1} &= \frac{m_{10} p_2 p_3\left(-1+e^{-\lambda_1}\right) e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}+\frac{m_{11} \lambda_1 p_2 p_3 e^{-\lambda_1}}{\left(t_4-j_1\right)^2}\left(\lambda_1 e^{-\lambda_1}-1\right)-\frac{p_2 p_3\left(m_{11}+m_{10}-n_1\right)}{r_2^2}, \\
\frac{\partial^2 \ell_2}{\partial \nu_1 \partial \alpha_2} &= \frac{\partial^2 \ell_2}{\partial \alpha_1 \partial \nu_1}, \\
\frac{\partial^2 \ell_2}{\partial \nu_1 \partial \nu_2} &= \frac{\partial^2 \ell_2}{\partial \nu_1^2}, \\
\frac{\partial^2 \ell_2}{\partial \nu_1 \partial \omega_2} &= \frac{\partial^2 \ell_2}{\partial \nu_1 \partial \omega_1}, \\
\frac{\partial^2 \ell_2}{\partial \omega_1^2} &= -\frac{m_{10} e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}[q_3\left(-r_2 e^{-\lambda_1}+t_3\right)+p_3^2 e^{-\lambda_1}] \\
&\quad -\frac{m_{11}\left(\lambda_1 e^{-\lambda_1}-1\right)}{\left(t_4-j_1\right)^2}[q_3\left(t_4-j_1\right)+p_3^2\left(\lambda_1 e^{-\lambda_1}-1\right)]+\frac{\left(-r_2 q_3+p_3^2\right)\left(m_{11}+m_{10}-n_1\right)}{r_2^2}, \\
\frac{\partial^2 \ell_2}{\partial \omega_1 \partial \alpha_2} &= \frac{\partial^2 \ell_2}{\partial \alpha_1 \partial \omega_1},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell_2}{\partial \omega_1 \partial \nu_2} &= \frac{\partial^2 \ell_2}{\partial \nu_1 \partial \omega_1}, \\
\frac{\partial^2 \ell_2}{\partial \omega_1 \partial \omega_2} &= \frac{\partial^2 \ell_2}{\partial \omega_1^2}, \\
\frac{\partial^2 \ell_2}{\partial \alpha_2^2} &= -\frac{m_{10} r_2 t_3 e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1} + t_3\right)^2} + \frac{m_{11}}{\left(t_4 - j_1\right)^2}[\left(t_4 - j_1\right)\left(2 r_2 e^{-\lambda_1} - j_1\right) - \left(r_2 e^{-\lambda_1} - j_1\right)^2] - \frac{N_1}{\lambda_1^2} \\
&\quad - \frac{m_{20} j_2 t_1}{\left(j_2 - t_1\right)^2} + \frac{m_{21} j_2}{\left(t_2 - j_2 y\right)^2}[-j_2\left(-y+1\right)^2 + \left(t_2 - j_2 y\right)\left(-y+2\right)] - \frac{N_2}{y^2}, \\
\frac{\partial^2 \ell_2}{\partial \alpha_2 \partial \nu_2} &= \frac{m_{10} p_2 \left(-r_2 + t_3\right) e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1} + t_3\right)^2} - \frac{m_{11} p_2 e^{-\lambda_1}}{\left(t_4 - j_1\right)^2}[\lambda_1\left(r_2 e^{-\lambda_1} - j_1\right) + \left(-\lambda_1 + 1\right)\left(t_4 - j_1\right)] \\
&\quad + \frac{m_{20} p_1 \left(j_2 - t_1 e^{-y}\right)}{\left(j_2 - t_1\right)^2} + \frac{m_{21} t_2 p_1 e^{-y}\left(-y+1\right)}{\left(t_2 - j_2 y\right)^2}, \\
\frac{\partial^2 \ell_2}{\partial \alpha_2 \partial \omega_2} &= -\frac{m_{10} t_3 p_3 e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1} + t_3\right)^2} + \frac{m_{11} p_3}{\left(t_4 - j_1\right)^2}[\left(-\lambda_1 + 1\right)\left(t_4 - j_1\right) e^{-\lambda_1} + \left(\lambda_1 e^{-\lambda_1} - 1\right)\left(r_2 e^{-\lambda_1} - j_1\right)] \\
&\quad - \frac{m_{20} p_4 t_1 e^{-y}}{\left(j_2 - t_1\right)^2} + \frac{m_{21} p_4 \left(-y+1\right)\left(t_2 e^{-y} - j_2\right)}{\left(t_2 - j_2 y\right)^2}, \\
\frac{\partial^2 \ell_2}{\partial \nu_2^2} &= \frac{m_{10}\left(1 - e^{-\lambda_1}\right)}{\left(-r_2 e^{-\lambda_1} + t_3\right)^2}[-p_2^2\left(1 - e^{-\lambda_1}\right) + q_1\left(-r_2 e^{-\lambda_1} + t_3\right)] - \frac{m_{11} \lambda_1 e^{-\lambda_1}}{\left(t_4 - j_1\right)^2}[\lambda_1 p_2^2 e^{-\lambda_1} + q_1\left(t_4 - j_1\right)] \\
&\quad + \frac{m_{20}\left(-1 + e^{-y}\right)}{\left(j_2 - t_1\right)^2}[q_2\left(j_2 - t_1\right) - p_1^2\left(-1 + e^{-y}\right)] - \frac{m_{21} y e^{-y}}{\left(t_2 - j_2 y\right)^2}[q_2\left(t_2 - j_2 y\right) + p_1^2 y e^{-y}] \\
&\quad + \frac{\left(p_2^2 - r_2 q_1\right)\left(m_{11} + m_{10} - n_1\right)}{r_2^2} + \frac{\left(-r_1 q_2 + p_1^2\right)\left(m_{21} + m_{20} - n_2\right)}{r_1^2}, \\
\frac{\partial^2 \ell_2}{\partial \nu_2 \partial \omega_2} &= \frac{m_{10} p_2 p_3 \left(-1 + e^{-\lambda_1}\right) e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1} + t_3\right)^2} + \frac{m_{11} \lambda_1 p_2 p_3 e^{-\lambda_1}}{\left(t_4 - j_1\right)^2}\left(\lambda_1 e^{-\lambda_1} - 1\right) - \frac{m_{20} p_4 p_1 e^{-y}\left(-1 + e^{-y}\right)}{\left(j_2 - t_1\right)^2} \\
&\quad - \frac{m_{21} p_4 p_1 y e^{-y}\left(y e^{-y} - 1\right)}{\left(t_2 - j_2 y\right)^2} - \frac{p_2 p_3\left(m_{11} + m_{10} - n_1\right)}{r_2^2} + \frac{p_4 p_1\left(m_{21} + m_{20} - n_2\right)}{r_1^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell_2}{\partial \omega_2^2} &= -\frac{m_{10}e^{-\lambda_1}}{\left(-r_2e^{-\lambda_1}+t_3\right)^2}[q_3\left(-r_2e^{-\lambda_1}+t_3\right)+p_3^2e^{-\lambda_1}] \\
&\quad -\frac{m_{11}\left(\lambda_1e^{-\lambda_1}-1\right)}{\left(t_4-j_1\right)^2}[q_3\left(t_4-j_1\right)+p_3^2\left(\lambda_1e^{-\lambda_1}-1\right)]+\frac{m_{20}e^{-y}}{\left(j_2-t_1\right)^2}[q_4\left(j_2-t_1\right)-p_4^2e^{-y}] \\
&\quad -\frac{m_{21}\left(ye^{-y}-1\right)}{\left(t_2-j_2y\right)^2}[q_4\left(t_2-j_2y\right)+p_4^2ye^{-y}-p_4^2]+\frac{\left(-r_2q_3+p_3^2\right)\left(m_{11}+m_{10}-n_1\right)}{r_2^2} \\
&\quad +\frac{\left(-r_1q_4+p_4^2\right)\left(m_{21}+m_{20}-n_2\right)}{r_1^2},
\end{aligned}
$$

The second derivatives of the observed-data log-likelihood function ℓ_3 are given by

$$\begin{aligned}
\frac{\partial^2 \ell_3}{\partial \lambda_1^2} &= -\frac{m_{10} r_2 t_3 e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}+\frac{m_{11}}{\left(t_4-j_1\right)^2}[\left(t_4-j_1\right)\left(2 r_2 e^{-\lambda_1}-j_1\right)-\left(r_2 e^{-\lambda_1}-j_1\right)^2]-\frac{N_1}{\lambda_1^2}, \\
\frac{\partial^2 \ell_3}{\partial \lambda_1 \partial \theta_{10}} &= \frac{m_{10} p_2\left(-r_2+t_3\right) e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}-\frac{m_{11} p_2 e^{-\lambda_1}}{\left(t_4-j_1\right)^2}\left(\lambda_1 r_2 e^{-\lambda_1}-\lambda_1 t_4+t_4-j_1\right), \\
\frac{\partial^2 \ell_3}{\partial \lambda_1 \partial \theta_{11}} &= -\frac{m_{10} t_3 p_3 e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}+\frac{m_{11} p_3}{\left(t_4-j_1\right)^2}[\left(-\lambda_1+1\right)\left(t_4-j_1\right) e^{-\lambda_1}+\left(\lambda_1 e^{-\lambda_1}-1\right)\left(r_2 e^{-\lambda_1}-j_1\right)], \\
\frac{\partial^2 \ell_3}{\partial \lambda_1 \partial \lambda_2} &= 0, \\
\frac{\partial^2 \ell_3}{\partial \lambda_1 \partial \theta_{20}} &= 0, \\
\frac{\partial^2 \ell_3}{\partial \lambda_1 \partial \theta_{21}} &= 0, \\
\frac{\partial^2 \ell_3}{\partial \theta_{10}^2} &= -\frac{m_{10}\left(1-e^{-\lambda_1}\right)}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}[p_2^2\left(1-e^{-\lambda_1}\right)-q_1\left(-r_2 e^{-\lambda_1}+t_3\right)]-\frac{m_{11} \lambda_1 e^{-\lambda_1}}{\left(t_4-j_1\right)^2}[\lambda_1 p_2^2 e^{-\lambda_1}+q_1\left(t_4-j_1\right)] \\
&\quad -\frac{\left(p_2^2-r_2 q_1\right)\left(m_{11}+m_{10}-n_1\right)}{r_2^2}, \\
\frac{\partial^2 \ell_3}{\partial \theta_{10} \partial \theta_{11}} &= \frac{m_{10} p_2 p_3\left(-1+e^{-\lambda_1}\right) e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}+\frac{m_{11} \lambda_1 p_2 p_3 e^{-\lambda_1}}{\left(t_4-j_1\right)^2}\left(\lambda_1 e^{-\lambda_1}-1\right)-\frac{p_2 p_3\left(m_{11}+m_{10}-n_1\right)}{r_2^2}, \\
\frac{\partial^2 \ell_3}{\partial \theta_{10} \partial \lambda_2} &= 0, \\
\frac{\partial^2 \ell_3}{\partial \theta_{10} \partial \theta_{20}} &= 0, \\
\frac{\partial^2 \ell_3}{\partial \theta_{10} \partial \theta_{21}} &= 0, \\
\frac{\partial^2 \ell_3}{\partial \theta_{11}^2} &= -\frac{m_{10} e^{-\lambda_1}}{\left(-r_2 e^{-\lambda_1}+t_3\right)^2}[q_3\left(-r_2 e^{-\lambda_1}+t_3\right)+p_3^2 e^{-\lambda_1}] \\
&\quad -\frac{m_{11}\left(\lambda_1 e^{-\lambda_1}-1\right)}{\left(t_4-j_1\right)^2}[q_3\left(t_4-j_1\right)+p_3^2\left(\lambda_1 e^{-\lambda_1}-1\right)]+\frac{\left(-r_2 q_3+p_3^2\right)\left(m_{11}+m_{10}-n_1\right)}{r_2^2}, \\
\frac{\partial^2 \ell_3}{\partial \theta_{11} \partial \lambda_2} &= 0,
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \ell_3}{\partial \theta_{11} \partial \theta_{20}} &= 0, \\
\frac{\partial^2 \ell_3}{\partial \theta_{11} \partial \theta_{21}} &= 0, \\
\frac{\partial^2 \ell_3}{\partial \lambda_2^2} &= -\frac{m_{20} j_2 t_1}{\left(j_2 - t_1\right)^2} + \frac{m_{21} j_2}{\left(t_2 - j_2 y\right)^2}\left[-j_2 \left(-y+1\right)^2 + \left(t_2 - j_2 y\right)\left(-y+2\right)\right] - \frac{N_2}{y^2}, \\
\frac{\partial^2 \ell_3}{\partial \lambda_2 \partial \theta_{20}} &= \frac{m_{20} p_1 \left(j_2 - t_1 e^{-y}\right)}{\left(j_2 - t_1\right)^2} + \frac{m_{21} t_2 p_1 e^{-y} \left(-y+1\right)}{\left(t_2 - j_2 y\right)^2}, \\
\frac{\partial^2 \ell_3}{\partial \lambda_2 \partial \theta_{21}} &= -\frac{m_{20} p_4 t_1 e^{-y}}{\left(j_2 - t_1\right)^2} + \frac{m_{21} p_4 \left(-y+1\right)\left(t_2 e^{-y} - j_2\right)}{\left(t_2 - j_2 y\right)^2}, \\
\frac{\partial^2 \ell_3}{\partial \theta_{20}^2} &= \frac{m_{20}\left(-1 + e^{-y}\right)}{\left(j_2 - t_1\right)^2}\left[q_2\left(j_2 - t_1\right) - p_1^2\left(-1 + e^{-y}\right)\right] - \frac{m_{21} y e^{-y}}{\left(t_2 - j_2 y\right)^2}\left[q_2\left(t_2 - j_2 y\right) + p_1^2 y e^{-y}\right] \\
&\quad + \frac{\left(-r_1 q_2 + p_1^2\right)\left(m_{21} + m_{20} - n_2\right)}{r_1^2}, \\
\frac{\partial^2 \ell_3}{\partial \theta_{20} \partial \theta_{21}} &= -\frac{m_{20} p_4 p_1 e^{-y}\left(-1 + e^{-y}\right)}{\left(j_2 - t_1\right)^2} - \frac{m_{21} p_4 p_1 y e^{-y}\left(y e^{-y} - 1\right)}{\left(t_2 - j_2 y\right)^2} \\
&\quad + \frac{p_4 p_1 \left(m_{21} + m_{20} - n_2\right)}{r_1^2}, \\
\frac{\partial^2 \ell_3}{\partial \theta_{21}^2} &= \frac{m_{20} e^{-y}}{\left(j_2 - t_1\right)^2}\left[q_4\left(j_2 - t_1\right) - p_4^2 e^{-y}\right] - \frac{m_{21}\left(y e^{-y} - 1\right)}{\left(t_2 - j_2 y\right)^2}\left[q_4\left(t_2 - j_2 y\right) + p_4^2 y e^{-y} - p_4^2\right] \\
&\quad + \frac{\left(-r_1 q_4 + p_4^2\right)\left(m_{21} + m_{20} - n_2\right)}{r_1^2}.
\end{aligned}
$$