High-dimensional Regression and M-estimator

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1 Preliminaries on random matrix

The root of high-dimensional statistics is dating back to work on random matrix theory and high-dimensional testing problems (Negahban et al. [2012]). To develop theoretical results on high-dimensional regression and M-estimator, we need to introduce some important spectral norm concentration inequalities of random matrix. It's worthy to mention that the "High-dimensional" in this article means that

$$p = n^{\alpha}, \quad \alpha \in (0, 1).$$

For simple normal case, here we states Lemma 9 without proof in Wainwright [2009]:

Lemma 1.1 *For $k \leq n$, let $X \in \mathbb{R}^{n \times k}$ have i.i.d rows $X_i \sim N(0, \Lambda)$ and $\delta(n, k, t) := 2(\sqrt{\frac{k}{n}} + t) + (\sqrt{\frac{k}{n}} + t)^2$*

1. *If the covariance matrix Λ has maximum eigenvalue $C_{max} < \infty$, then for all $t > 0$, we have*
$$\mathbb{P}\left[\left\|\frac{1}{n}X^T X - \Lambda\right\|_2 \geq C_{\max}\delta(n, k, t)\right] \leq 2\exp\left(-nt^2/2\right). \tag{1.1}$$
2. *If the covariance matrix Λ has minimum eigenvalue $C_{min} > 0$, then for all $t > 0$, we have*
$$\mathbb{P}\left[\left\|\left(\frac{X^T X}{n}\right)^{-1} - \Lambda^{-1}\right\|_2 \geq \frac{\delta(n, k, t)}{C_{\min}}\right] \leq 2\exp\left(-nt^2/2\right). \tag{1.2}$$

Next we will generalize the concentration inequality to sub-gaussian case. Recall the operator norm or spectral norm of $m \times n$ matrix A is defined by

$$\|A\|_2 := \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|_2}{\|x\|_2} = \max_{x \in S^{n-1}} \|Ax\|_2,$$

which is the largest singular value of A. For symmetric matrix, the spectral norm is the largest eigenvalue.

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Lemma 1.2 *The covering numbers of the unit Euclidean sphere S^{n-1} satisfy the following for any $\varepsilon > 0$,*

$$\mathcal{N}\left(S^{n-1}, \varepsilon\right) \leq \left(\frac{2}{\varepsilon} + 1\right)^n.$$

Lemma 1.3 *Let A be an $m \times n$ matrix and $\delta > 0$. Suppose that*

$$\left\|A^\top A - I_n\right\| \leq \max\left(\delta, \delta^2\right),$$

then

$$(1-\delta)\|x\|_2 \leq \|Ax\|_2 \leq (1+\delta)\|x\|_2 \quad \textit{for all } x \in \mathbb{R}^n.$$

Proof: W.L.O.G, let $\|x\|_2 = 1$.Using the assumption we have

$$\max\left(\delta, \delta^2\right) \geq \left|\left\langle\left(A^\top A - I_n\right)x, x\right\rangle\right| = \left|\|Ax\|_2^2 - 1\right|.$$

Applying the elementary inequality,

$$\max\left(|z-1|, |z-1|^2\right) \leq \left|z^2 - 1\right|, \quad z \geq 0$$

for $z = \|Ax\|_2$, we concluded that $\|Ax\|_2 - 1| \leq \delta$. ■

Then we introduce the two-sided bounds on the entire spectrum of $m \times n$ matrix A (see Vershynin [2018], page 97).

Theorem 1.4 (Two-sided spectral norm bounds) *Let A be an $m \times n$ matrix whose rows A_i are independent, mean zero, sub-gaussian isotropic random vectors in $\mathbb{R}^n$. Then for any $t > 0$ we have*

$$\sqrt{m} - CK^2(\sqrt{n} + t) \leq s_n(A) \leq s_1(A) \leq \sqrt{m} + CK^2(\sqrt{n} + t) \tag{1.3}$$

with probability at least $1 - 2\exp(-t^2)$. Here $K = \max_i \|A_i\|_{\psi_2}$.

Proof: Using Lemma 1.3, it suffices to show

$$\left\|\frac{1}{m}A^\top A - I_n\right\| \leq K^2 \max\left(\delta, \delta^2\right) \quad \text{where} \quad \delta = C\left(\sqrt{\frac{n}{m}} + \frac{t}{\sqrt{m}}\right).$$

By Lemma 1.2, we can find an $\frac{1}{4}-$net $\mathcal{N}$ of the unit sphere S^{n-1} with cardinality $|\mathcal{N}| \leq 9^n$. Then we can evaluate operator norm on the $\mathcal{N}$,

$$\left\|\frac{1}{m}A^\top A - I_n\right\| \leq 2\max_{x\in\mathcal{N}}\left|\left\langle\left(\frac{1}{m}A^T A - I_n\right)x, x\right\rangle\right| = 2\max_{x\in\mathcal{N}}\left|\frac{1}{m}\|Ax\|_2^2 - 1\right|. \tag{1.4}$$

Let $X_i = x^T A_i$ which is indpendent sub-gaussian random variables, note that

$$\frac{1}{m}\|Ax\|_2^2 - 1 = \frac{1}{m}\sum_{i=1}^m[(x^T A_i)^2 - 1] = \frac{1}{m}\sum_{i=1}^m(X_i^2 - 1),$$

Using the fact that A_i are isotropic and $\|x\|_2 = 1$, $\|X_i\|_{\phi_2} \leq K$. Then $X_i^2 - 1$ is sub-exponential random variables satisfying that $\|X_i^2 - 1\|_{\phi_1} \leq CK$. By Bernstein inequality and we obtain

$$\begin{aligned}
\mathbb{P}\left\{\left|\frac{1}{m}\|Ax\|_2^2 - 1\right| \geq \frac{\varepsilon}{2}\right\} &= \mathbb{P}\left\{\left|\frac{1}{m}\sum_{i=1}^{m} X_i^2 - 1\right| \geq \frac{\varepsilon}{2}\right\} \\
&\leq 2\exp\left[-c_1 \min\left(\frac{\varepsilon^2}{K^4}, \frac{\varepsilon}{K^2}\right) m\right] \\
&= 2\exp\left[-c_1\delta^2 m\right] \\
&\leq 2\exp\left[-c_1 C^2\left(n + t^2\right)\right],
\end{aligned}$$

where the second equality follows that $\frac{\varepsilon}{K^2} = \max\left(\delta, \delta^2\right)$ and the last inequality follows that $(a+b)^2 \geq (a^2 + b^2)$. Using (1.4) we have

$$\begin{aligned}
\mathbb{P}\left(\left\|\frac{1}{m}A^\top A - I_n\right\| \geq K^2 \max\left(\delta, \delta^2\right)\right) &\leq \mathbb{P}\left(2\max_{x\in\mathcal{N}}\left|\frac{1}{m}\|Ax\|_2^2 - 1\right| > K^2\max\left(\delta, \delta^2\right)\right) \\
&\leq 2\cdot 9^n \exp\left[-c_1 C^2\left(n + t^2\right)\right].
\end{aligned}$$

Choose sufficiently large C and the result follows. ■

After proving this conclusion, we can apply this to covariance matrix estimation.

Theorem 1.5 *Let X be a $p-$dimensional multivariate sub-gaussian random variables with covariance matrix Σ and mean $\mathbf{0}$, and there exists $K \geq 1$ such that*

$$\|\langle X, x\rangle\|_{\psi_2} \leq K x^T \Sigma x \text{ for any } x \in \mathbb{R}^p. \tag{1.5}$$

Then for sample covariance matrix $\widehat{\Sigma}_n$ we have

$$\|\Sigma_n - \Sigma\| \leq C\lambda_{max}(\Sigma)K^2\left(\sqrt{\frac{p+t^2}{n}} + \frac{p+t^2}{n}\right) \tag{1.6}$$

holds with probability at least $1 - \exp(-t^2/2)$.

Proof: Let $Z_i = \Sigma^{-1/2}X_i$, then Z_i are independent isotropic sub-gaussian random vector. Using (1.5) we have

$$\|Z_i\|_{\phi_2} = \sup_{x\in S^{p-1}} \|\langle Z_i, x\rangle\|_{\psi_2} \leq K. \tag{1.7}$$

Then note that,

$$\|\Sigma_n - \Sigma\| = \left\|\Sigma^{1/2} R_n \Sigma^{1/2}\right\| \leq \|R_n\|\,\|\Sigma\|,$$

where

$$R_n := \frac{1}{n}\sum_{i=1}^{n} Z_i Z_i^\top - I_p.$$

Let A be the $n \times p$ matrix with rows Z_i, then apply Theorem 1.4 we obtain that

$$\|\Sigma_n - \Sigma\| \leq K^2\|\Sigma\|\max\left(\delta, \delta^2\right)$$

holds with at least probability $1-2\exp(-t^2/2)$. Moreover,

$$\max\left(\delta,\delta^2\right)\le\delta+\delta^2\le C\left(\sqrt{\frac{p+t^2}{n}}+\frac{p+t^2}{n}\right).$$

Thus the proof is completed. ■

Remark. The theorem above implies that for low dimensional setting, i.e., $p<n$

$$\|\Sigma_n-\Sigma\|=O_p\left(\sqrt{\frac{p}{n}}\right). \tag{1.8}$$

Using the fact that

$$\left\|\Sigma_n^{-1}-\Sigma^{-1}\right\|=\Omega_p\left(\|\Sigma_n-\Sigma\|\right),$$

then if $\lambda_{min}(\Sigma)>0$ we have

$$\left\|\Sigma_n^{-1}-\Sigma^{-1}\right\|=O_p\left(\sqrt{\frac{p}{n}}\right). \tag{1.9}$$

2 High dimensional linear regression

Now consider the following linear regression model with random ensembles:

$$y_i=\boldsymbol{X}_i^T\boldsymbol{\beta}^*+e_i,\quad i=1,2,...,n \tag{2.1}$$

where $e_i, i=1,2,...,n$ are independent sub-gaussion random variables with mean 0 and parameter σ and $\boldsymbol{\beta}^*\in\mathbb{R}^p$. We have known that the LSE of $\boldsymbol{\beta}^*$ is

$$\widehat{\boldsymbol{\beta}}=\left(\frac{1}{n}\sum_{i=1}^n\boldsymbol{X}_i\boldsymbol{X}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n y_i\boldsymbol{X}_i\right). \tag{2.2}$$

Theorem 2.1 (Consistence) *For linear regression model (2.1), suppose that X_i are independent sub-gaussion random vectors with same mean $\mathbf{0}$ and covariance matrix Σ and X_i are independent with e_i. Assume that $\lambda_{min}(\Sigma)=\lambda_0>0$ and $\|X_i\|_{\psi_2}\le K$, then*

$$\left\|\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}^*\right\|_2=O_p\left(\sqrt{\frac{p\log p}{n}}\right). \tag{2.3}$$

Proof: By (2.1),

$$\begin{aligned}\left\|\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}^*\right\|_2&=\left\|\left(\frac{1}{n}\sum_{i=1}^n\boldsymbol{X}_i\boldsymbol{X}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n\boldsymbol{X}_ie_i\right)\right\|_2 \qquad (2.4)\\&=\left\|\widehat{\Sigma}_n^{-1}\left(\frac{1}{n}\sum_{i=1}^n\boldsymbol{X}_ie_i\right)\right\|_2\\&\le\|\widehat{\Sigma}_n^{-1}-\Sigma^{-1}\|_2\left\|\left(\frac{1}{n}\sum_{i=1}^n\boldsymbol{X}_ie_i\right)\right\|_2+\|\Sigma^{-1}\|_2\left\|\left(\frac{1}{n}\sum_{i=1}^n\boldsymbol{X}_ie_i\right)\right\|_2.\end{aligned}$$

All we need to do is bounding the term $\left\|\left(\frac{1}{n}\sum_{i=1}^n \boldsymbol{X}_i e_i\right)\right\|_2$, let $Z_{ij} = X_{ij}e_i$. Using the basic inequality $|ab| \le \frac{a^2+b^2}{2}$ and $s^2e^s \le e^{2s}$, for $\eta > 0$ we have

$$\begin{aligned}
\mathbb{E}\left(Z_{ij}^2 e^{\eta|Z_{ij}|}\right) &\le \mathbb{E}\left(\eta^{-2}\exp(2\eta|Z_{ij}|)\right) \\
&\le \eta^2\mathbb{E}\left[\exp\left(2\eta X_{ij}^2\right)\exp\left(2\eta e_i^2\right)\right] \\
&\le \eta^2\sqrt{\mathbb{E}\left[\exp\left(2\eta X_{ij}^2\right)\right]\mathbb{E}\left[\exp\left(2\eta e_i^2\right)\right]}.
\end{aligned}$$

Then by the property of sub-gaussian random variable, there exists some $M > 0$, such that

$$\mathbb{E}\left[\exp\left(2\eta X_{ij}^2\right)\right] \le M, \mathbb{E}\left[\exp\left(2\eta e_i^2\right)\right] \le M.$$

Next use the exponential inequality in Cai et al. [2011], we set $\bar{B}_n^2 = nM\eta^{-2}$

$$\begin{aligned}
\mathbb{P}\left(\max_j^p\left|\frac{1}{n}\sum_{i=1}^n Z_{ij}\right| > C\sqrt{\frac{\log p}{n}}\right) &\le \sum_{j=1}^p \mathrm{P}\left(\left|\sum_{i=1}^n Z_{ij}\right| > C\sqrt{n\log p}\right) \\
&= \sum_{j=1}^p \mathrm{P}\left(\sum_{i=1}^n |Z_{ij}| > C\bar{B}_n M^{-1}\eta\sqrt{\log p}\right) \\
&= p^{-\gamma}.
\end{aligned}$$

And if we choose sufficiently large C, we can obtain that

$$\max_j^p\left|\frac{1}{n}\sum_{i=1}^n Z_{ij}\right| = O_p\left(\sqrt{\frac{\log p}{n}}\right).$$

The proof is completed by (2.4) and Theorem 1.5. ∎

The theorem above implies that if $p\log p = o(n)$, LSE is consistent. Next we will give the central limt theorem for LSE.

Theorem 2.2 (Asymptotic Normality) *Under the condition of Theorem 2.1, and assume that covariates* $\boldsymbol{X}$ *and noise* e *are independent. We have*

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\Sigma^{-1}\right) \tag{2.5}$$

Proof: Note that,

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*\right) = \left(\frac{1}{n}\sum_{i=1}^n \boldsymbol{X}_i\boldsymbol{X}_i^T\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \boldsymbol{X}_i e_i\right). \tag{2.6}$$

By law of large numbers,

$$\frac{1}{n}\sum_{i=1}^n \boldsymbol{X}_i\boldsymbol{X}_i^T \xrightarrow{p} \Sigma.$$

And using the independence, we have $\mathbb{E}\left(\boldsymbol{X}_i e_i\right) = 0$ and

$$\mathbb{E}\left(\boldsymbol{X}_i e_i\right)\left(\boldsymbol{X}_i e_i\right)^T = \sigma^2\Sigma.$$

Thus by multivariate central limt theorem,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \boldsymbol{X}_i e_i \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\Sigma\right)$$

Then the result follows from Slutsky's Lemma. ∎

3 High dimensional M estimator

Given sample $\{X_i, i = 1, 2, ..., n\} \in \mathcal{X}_n$ is drawn independently according to some distribution $\mathbb{P}$. And in the well-specified case the distribution $\mathcal{P}$ is a member of parameterized family $\{\mathbb{P}_\theta, \theta \in \Omega\}$, where $\boldsymbol{\Omega}$ is the parameter space, then the goal is to estimate parameter $\boldsymbol{\theta}^*$. For mis-specified models, in which case the target parameter $\boldsymbol{\theta}^*$ is defined as the minimizer of the population lost function (see Wainwright [2019]).

A function $\mathcal{L}_n : \boldsymbol{\Omega} \times \mathcal{X}_n$ used to measure the goodness of estimation using sample $\boldsymbol{X}_n$, which is called *lost function*. The population lost function is defined as

$$\mathcal{L}(\boldsymbol{\theta}) = \mathbb{E}\left(\mathcal{L}_n\left(\boldsymbol{\theta}, \boldsymbol{X}_n\right)\right), \tag{3.1}$$

where

$$\mathcal{L}_n\left(\boldsymbol{\theta}, \boldsymbol{X}_n\right) = \frac{1}{n}\sum_{i=1}^{n} L(\boldsymbol{\theta}, X_i).$$

Next we define the *target parameter* as the minimum of the population lost function

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}\in\boldsymbol{\Omega}} \mathcal{L}(\boldsymbol{\theta}). \tag{3.2}$$

For example, the negative log-likelihood function is a lost function. Our overall estimator is based on solving the optimization problem

$$\widehat{\theta} \in \arg\min_{\theta\in\Omega} \left\{\mathcal{L}_n\left(\theta; Z_1^n\right) + \lambda_n \Phi(\theta)\right\}, \tag{3.3}$$

where $\lambda_n > 0$ is regularization parameter and $\Phi(\theta) : \boldsymbol{\Omega} \to \mathbb{R}$ is the penalty function. The estimator (3.3) is called **M estimator**, where the "M" stands for minimization (or maximization). We begin with no-penalty problem, and the following assumptions is needed to estabilish theory results, and these assumptions can be found in Zhang et al. [2013] and Jordan et al. [2019].

Assumption 3.1 (Parameter space) *The parameter space Θ is a compact and convex subset of $\mathbb{R}^p$. Moreover, $\theta^* \in \operatorname{int}(\Theta)$ and $R := \sup_{\theta\in\Theta} \|\theta - \theta^*\|_2 > 0$.*

Assumption 3.2 (Local convexity) *The lost function $L(X_i, \boldsymbol{\theta})$ is twice differentiable with respective to $\boldsymbol{\theta}$, and the Hessian matrix $I(\boldsymbol{\theta}) = \nabla^2\mathcal{L}(\boldsymbol{\theta})$ of the population lost function $\mathcal{L}(\boldsymbol{\theta})$ is invertible at $\boldsymbol{\theta}^*$. Moreover, there exists two positive constants $\mu_- < \mu_+$ such that $\mu_- I_d \preceq I(\boldsymbol{\theta}) \preceq \mu_+ I_d$.*

Assumption 3.3 (Smoothness) *There exists some positive constant (G, L) and positive integers (k_0, k_1), such that*

$$\mathbb{E}\left[\|\nabla L(\boldsymbol{\theta}, X)\|_2^{k_0}\right] \leq G^{k_0}, \quad \mathbb{E}\left[\|\nabla^2 L(\boldsymbol{\theta}, X) - \nabla^2\mathcal{L}(\boldsymbol{\theta})\|_2^{k_1}\right] \leq L^{k_1}. \tag{3.4}$$

Moreover, for all $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in U(\boldsymbol{\theta}^*, \rho)$ (a ball around the truth $\boldsymbol{\theta}^*$ with radius $\rho > 0$) there exists some positive constant M and some positive integer k_2 such that

$$\left\|\nabla^2\mathcal{L}(\boldsymbol{\theta}_1, X) - \nabla^2\mathcal{L}(\boldsymbol{\theta}_2, X)\right\|_2 \leq M(X)\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_2, \tag{3.5}$$

and $\mathbb{E}[M(X)^{k_2}] \leq M^{k_2}$.

Before bound the ℓ_2 error between the optimization solution $\widehat{\boldsymbol{\theta}}$ and ture parameter $\boldsymbol{\theta}^*$, we state the following Lemma.

Lemma 3.4 *For convex function $f(x)$, x^* is the global minimizer of $f(x)$. If for any $x \in \{x : |x - \tilde{x}|^2 = a\}$, s.t., $f(x) \geq f(\tilde{x})$, then*

$$|x^* - \tilde{x}| \leq a.$$

Proof: If there exists $x^{'}$ such that $|x^{'} - \tilde{x}|^2 > a$ and $f(x^{'}) \leq f(x^*)$. By the convexity of f, we have

$$f(\alpha x^{'} + (1-\alpha)\tilde{x}) \leq \alpha f(x^{'}) + (1-\alpha) f(\tilde{x}) < f(\tilde{x}),$$

where $0 < \alpha < 1$. Note that

$$|\alpha x^{'} + (1-\alpha)\tilde{x} - \tilde{x}| = \alpha |x^{'} - \tilde{x}|,$$

let $\alpha = |x^{'} - \tilde{x}| / |x^* - \tilde{x}|$, then $|\alpha x^{'} + (1-\alpha)\tilde{x} - \tilde{x}| = a$. But

$$f(\alpha x^{'} + (1-\alpha)\tilde{x}) < f(\tilde{x}),$$

which is a contradiction. ■

Next we state Lemma 7 in Zhang et al. [2013] without proof as following:

Lemma 3.5 *Under Assumption 3.3, there exist some constants C_1 and C_2 (dependent only on the moments k_0 and k_1 respectively) such that*

$$\mathbb{E}\left[\|\nabla \mathcal{L}_n(\boldsymbol{\theta}^*)\|_2^{k_0}\right] \leq C_1 \frac{G^{k_0}}{n^{k_0/2}}, \tag{3.6}$$

$$\mathbb{E}\left[\left\|\nabla^2 \mathcal{L}_n(\boldsymbol{\theta}^*, X) - \nabla^2 \mathcal{L}(\boldsymbol{\theta}^*)\right\|_2^{k_1}\right] \leq C_2 \frac{\log^{k_1/2}(2p) H^{k_1}}{n^{k_1/2}}. \tag{3.7}$$

Theorem 3.6 *Under Assumption 3.2 and Assumption 3.3,*

$$\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\| = O_p\left(\frac{1}{\sqrt{n}}\right). \tag{3.8}$$

Proof: According to Lemma 3.4, it suffices to show that for any $\boldsymbol{\theta}$ satisfying $\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 = O\left(\frac{1}{\sqrt{n}}\right)$ such that

$$\mathcal{L}_n(\boldsymbol{\theta}) \geq \mathcal{L}_n(\boldsymbol{\theta}^*).$$

Taking Taylor expansion for $\mathcal{L}_n(\boldsymbol{\theta})$ at $\boldsymbol{\theta}^*$,

$$\mathcal{L}_n(\boldsymbol{\theta}) = \mathcal{L}_n(\boldsymbol{\theta}^*) + \nabla \mathcal{L}_n(\boldsymbol{\theta}^*)^T (\boldsymbol{\theta} - \boldsymbol{\theta}^*) + \frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{\theta}^*)^T \nabla^2 \mathcal{L}_n(\tilde{\boldsymbol{\theta}}) (\boldsymbol{\theta} - \boldsymbol{\theta}^*), \tag{3.9}$$

where $\tilde{\boldsymbol{\theta}}$ is some point between $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^*$. Define the following three events:

$$\mathcal{E}_0 := \left\{\frac{1}{n}\sum_{i=1}^{n} M(X_i) \leq 2M\right\},$$

$$\mathcal{E}_1 := \left\{\left\|\nabla^2 \mathcal{L}_n(\boldsymbol{\theta}^*, X) - \nabla^2 \mathcal{L}(\boldsymbol{\theta}^*)\right\|_2 \leq \frac{\mu_-}{2}\right\},$$

$$\mathcal{E}_2 := \left\{\|\nabla \mathcal{L}_n(\boldsymbol{\theta}^*)\|_2 \leq \frac{C_0}{\sqrt{n}}\right\}.$$

Using Assumption 3.2, Assumption 3.3 and Markov inequality

$$\mathrm{P}\left(\mathcal{E}_0^c \cup \mathcal{E}_1^c\right) \le \frac{C_3}{n^{k_2/2}} + \frac{C_4 \log^{k_1/2}(2p)}{n^{k_1/2}}.$$

Since $\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 = O\left(\frac{1}{\sqrt{n}}\right)$, there exists some positive constant C such that

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 = \frac{C' \mu_-}{2\sqrt{n}}$$

Under event $\mathcal{E}_0 \cap \mathcal{E}_1$, we can bound $\nabla^2 \mathcal{L}_n(\tilde{\boldsymbol{\theta}})$ by

$$\begin{aligned}
\lambda_{min}\left(\nabla^2 \mathcal{L}_n(\tilde{\boldsymbol{\theta}})\right) &\ge \lambda_{min}\left(I(\boldsymbol{\theta}^*)\right) - \|\nabla^2 \mathcal{L}_n(\boldsymbol{\theta}^*) - I(\boldsymbol{\theta}^*)\|_2 - \|\nabla^2 \mathcal{L}_n(\tilde{\boldsymbol{\theta}}) - \nabla^2 \mathcal{L}_n(\boldsymbol{\theta}^*)\|_2 \\
&\ge \mu_- - \frac{\mu_-}{2} - 2M\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 \\
&= (1 - \frac{2MC'}{\sqrt{n}})\frac{\mu_-}{2}.
\end{aligned}$$

Using (3.6) and Jessen inequlity, we have

$$\begin{aligned}
\mathbb{E}\left[\|\nabla \mathcal{L}_n(\boldsymbol{\theta}^*)\|_2\right] = \mathbb{E}\left[\left(\|\nabla \mathcal{L}_n(\boldsymbol{\theta}^*)\|_2^{k_0}\right)^{1/k_0}\right] &\le \left(\mathbb{E}\left[\|\nabla \mathcal{L}_n(\boldsymbol{\theta}^*)\|_2^{k_0}\right]\right)^{1/k_0} \\
&\le \frac{C_1 G}{\sqrt{n}}.
\end{aligned}$$

Then event $\mathcal{E}_2$ happens with high probability, which follows from $O_p(Y_n) = O(\mathbb{Y}_\ltimes)$. Therefore under event $\mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2$ we have

$$\begin{aligned}
\mathcal{L}_n(\boldsymbol{\theta}) - \mathcal{L}_n(\boldsymbol{\theta}^*) &\ge \nabla \mathcal{L}_n(\boldsymbol{\theta}^*)^T(\boldsymbol{\theta} - \boldsymbol{\theta}^*) + (1 - \frac{2MC'}{\sqrt{n}})\frac{\mu_-}{2}\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2^2 \\
&\ge -\|\nabla \mathcal{L}_n(\boldsymbol{\theta}^*)\|_2 \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 + (1 - \frac{2MC'}{\sqrt{n}})\frac{\mu_-}{2}\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2^2 \\
&\ge -\frac{C'\mu_-}{2\sqrt{n}}\frac{C_0}{\sqrt{n}} + (1 - \frac{2MC'}{\sqrt{n}})\frac{\mu_-}{2}\frac{(C'\mu_-)^2}{4n}.
\end{aligned}$$

If we choose sufficiently large C', $\mathcal{L}_n(\boldsymbol{\theta}) - \mathcal{L}_n(\boldsymbol{\theta}^*) \ge 0$ holds with high probability. ■

The following asymptotic result can help us conduct statistical inference, such as interval estimation and hypothesis testing.

Theorem 3.7 *Under Assumption 3.2 and Assumption 3.3,*

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\right) \xrightarrow{d} \mathcal{N}\left(0, \widetilde{\Sigma}\right), \tag{3.10}$$

where

$$\widetilde{\Sigma} = I(\boldsymbol{\theta}^*)^{-1}\mathbb{E}\left[\nabla L(\boldsymbol{\theta}^*, X)^T \nabla L(\boldsymbol{\theta}^*, X)\right] I(\boldsymbol{\theta}^*)^{-1}.$$

Proof: First we perform Taylor expansion for $\nabla \mathcal{L}_n(\widehat{\boldsymbol{\theta}})$ around $\boldsymbol{\theta}^*$,

$$0 = \nabla \mathcal{L}_n(\widehat{\boldsymbol{\theta}}) = \nabla \mathcal{L}_n(\boldsymbol{\theta}^*) + \nabla^2 \mathcal{L}_n(\boldsymbol{\theta}^*) \left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\right) + uO_p(\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|_2^2),$$

where $u \in \mathbb{R}^p$ is the unit vector. Then taking simple linear algebra we obtain

$$\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^* = -\nabla^2 \mathcal{L}_n(\boldsymbol{\theta}^*)^{-1} \nabla \mathcal{L}_n(\boldsymbol{\theta}^*) + \frac{C}{n} \nabla^2 \mathcal{L}_n(\boldsymbol{\theta}^*)^{-1} u.$$

Using law of large numbers, multivariate central limt theorem and Slutsky's lemma, we have

$$\begin{aligned} \sqrt{n}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\right) &= \left(\frac{1}{n}\sum_{i=1}^n \nabla^2 L(\boldsymbol{\theta}^*, X_i)\right)^{-1} \left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \nabla L(\boldsymbol{\theta}^*, X_i)\right) + \frac{C}{\sqrt{n}} \nabla^2 \mathcal{L}_n(\boldsymbol{\theta}^*)^{-1} u \\ &\xrightarrow{d} \mathcal{N}\left(0, \widetilde{\Sigma}\right). \end{aligned}$$

■

Remark. The following plug-in estimator is a consistent estimator for $\widetilde{\Sigma}$,

$$\left(\frac{1}{n}\sum_{i=1}^n \nabla^2 L(\boldsymbol{\theta}^*, X_i)\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^n \nabla L(\boldsymbol{\theta}^*, X_i) L(\boldsymbol{\theta}^*, X_i)^T\right) \left(\frac{1}{n}\sum_{i=1}^n \nabla^2 L(\boldsymbol{\theta}^*, X_i)\right)^{-1} \tag{3.11}$$

More generally, by Assumption 3.3 we set $\rho \in (0,1)$, then choosing the potentially smaller radius $\delta_\rho = \min\{\rho, \rho\mu_-/4L\}$. We can define the following good events

$$\begin{aligned} \mathcal{E}_0 &:= \left\{\frac{1}{n}\sum_{i=1}^n M\left(X_i\right) \leq 2M\right\}, \\ \mathcal{E}_1 &:= \left\{\left\|\nabla^2 \mathcal{L}_n\left(\boldsymbol{\theta}^*, X\right) - \nabla^2 \mathcal{L}\left(\boldsymbol{\theta}^*\right)\right\|_2 \leq \frac{\rho\mu_-}{2}\right\}, \\ \mathcal{E}_2 &:= \left\{\|\nabla \mathcal{L}_n\left(\boldsymbol{\theta}^*\right)\|_2 \leq \frac{(1-\rho)\mu_-\delta_\rho}{2}\right\}. \end{aligned}$$

The following lemma is Lemma 6 in Zhang et al. [2013].

Lemma 3.8 *Under the events $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$, we have*

$$\|\theta_1 - \theta^*\|_2 \leq \frac{2\left\|\nabla F_1\left(\theta^*\right)\right\|_2}{(1-\rho)\mu_-}, \quad \text{and} \quad \nabla^2 F_1(\theta) \succeq (1-\rho)\mu_- I_{p\times p}. \tag{3.12}$$

We can assume that $\|\hat{\theta} - \theta^*\|_2 \leq R$, then make decomposition as

$$\begin{aligned} \mathbb{E}\left[\left\|\hat{\theta} - \theta^*\right\|_2^k\right] &= \mathbb{E}\left[1_{(\mathcal{E})}\left\|\hat{\theta} - \theta^*\right\|_2^k\right] + \mathbb{E}\left[1_{(\mathcal{E}^c)}\left\|\hat{\theta} - \theta^*\right\|_2^k\right] \\ &\leq \frac{2^k \mathbb{E}\left[1_{(\mathcal{E})}\left\|\nabla \mathcal{L}_n\left(\theta^*\right)\right\|_2^k\right]}{(1-\rho)^k \lambda^k} + \mathbb{P}\left(\mathcal{E}^c\right) R^k \\ &\leq \frac{2^k \mathbb{E}\left[\left\|\nabla \mathcal{L}_n\left(\theta^*\right)\right\|_2^k\right]}{(1-\rho)^k \lambda^k} + \mathbb{P}\left(\mathcal{E}^c\right) R^k. \end{aligned}$$

Using Assumption 3.2, Assumption 3.3 and Lemma 3.4, we can prove

$$\mathbb{P}\left(\mathcal{E}^c\right) \leq C_2 \frac{1}{n^{k_2/2}} + C_1 \frac{\log^{k_1/2}(2d) H^{k_1}}{n^{k_1/2}} + C_0 \frac{G^{k_0}}{n^{k_0/2}},$$

for some universal constants $C_0,\ C_1,\ C_2$. Therefore for any $k \in \mathbb{N}$ with $k \leq \min\{k_0, k_1, k_2\}$ we have

$$\mathbb{E}\left[\|\theta_1 - \theta^*\|_2^k\right] = \mathcal{O}\left(n^{-k/2} \cdot \frac{G^k}{(1-\rho)^k \lambda^k} + n^{-k_0/2} + n^{-k_1/2} + n^{-k_2/2}\right) = \mathcal{O}\left(n^{-k/2}\right). \quad (3.13)$$

We can also obtain the ℓ_2 error bound $\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|_2 = O_p\left(\frac{1}{\sqrt{n}}\right)$ form (3.13).

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