# Modeling Gas Dynamics: Finite Volume Method Approach

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**Abstract**

Gas dynamics have been studied by all methods of science since their derivation in the time of Euler. In this paper, we focus on studying a famous set of coupled non-linear hyperbolic conservation laws, known as the Euler equations, through a computational lens in order to take into account thermal ionization of a weak plasma discharge in the 2D streamer model developed by Liu and Pasko [2004]. The method we choose to use is a finite volume method (FVM), known as Godunov's Method. This method requires the solution to the Riemann Problem which we utilize Roe's Method to solve.

## 1 The Euler Equations

We focus on the incompressible, inviscid, thermally non-conductive Euler Equations:

$$\frac{\partial \rho}{\partial t} + \nabla_{cyl} \cdot (\rho v) = 0 \tag{1}$$

$$\frac{\partial \rho v}{\partial t} + \nabla_{cyl} \cdot (\rho \mathbf{V}^2 + p) = 0 \tag{2}$$

$$\frac{\partial \epsilon}{\partial t} + \nabla_{cyl} \cdot [(\epsilon + p)\mathbf{V}] = Q^T_{eff} \tag{3}$$

$$\epsilon = \frac{5}{2} N k_B T + \frac{1}{2} \rho \mathbf{V}^2 \tag{4}$$

$$p = N k_B T \tag{5}$$

Equation (1) accounts for mass transport (or is otherwise recognized as the mass continuity equation) where $\rho$ represents the mass density determined from all neutral species and $v$ is the bulk velocity. Equation (2) is the momentum density continuity equation accounting for Newton's Second Law by including the necessary contributions of pressure, $p$ as the surface force on the gas. Equation (3) is the translational energy continuity equation which accounts for the balance of its advection, its heat supply, $Q^T_{eff}$ (the effective rate of energy deposition) or Joule heating. Equation (4) is the energy relation. Equation (5) is the equation of state and closes the system.

The equations above form a system of non-linear hyperbolic conservation laws that govern the dynamics of a compressible material, such as gases or liquids at high pressures (**pressure range????**), for which the effects of body forces, viscous forces, and heat flux are neglected.

## 2 Analytical Set-Up

We use the conservative form of the equations which follows the form:

$$U_t + F(U)_x = 0 \tag{6}$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \; F(U) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} \tag{7}$$

where U is a vector of conserved variables and F(U) is a vector of fluxes. To "linearize" the system, we introduce the flux Jacobian matrix,

$$A(U) = \frac{\partial F}{\partial U}. \tag{8}$$

Now, the system is written in 'quasi-linear' form:

$$U_t + \frac{\partial F}{\partial U}\frac{\partial U}{\partial x} = U_t + A(U)U_x = 0. \tag{9}$$

In order to solve this system we must first determine the eigensystem from the Jacobian matrix

$$AK^{(i)} = \lambda_i K^{(i)}. \tag{10}$$

This system is said to be hyperbolic if $\mathbf{A}$ has real eigenvalues, $\lambda_i$, and a corresponding set of linearly independent right eigenvectors $\mathbf{K}^{(i)}$ for $i = 1 : m$, where $m$ is the number of equations. The system is strictly hyperbolic if all the eigenvalues are distinct. Following this procedure allows for the decoupling of the Euler equations, where the dependent variables $U(x,t)$ are transformed into a new set of variables $W(x,t)$.

We then diagonalize $A$,

$$A = K\Lambda K^{-1} \tag{11}$$

where $\Lambda$ is a matrix with the diagonal elements being the eigenvalues. To determine the new dependent variables $W$, we use the transformation

$$W = K^{-1}U \text{ or } U = KW. \tag{12}$$

Now, substituting equations (11) and (12) into equation (9) we get

$$W_t + \Lambda W_x = 0, \tag{13}$$

which is called the characteristic form of the system and be written as

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0,\ i = 1, ..., m. \tag{14}$$

The system is now decoupled with $w_i$ being the single unknown, the characteristic speed is $\lambda_i$, and there are $m$ characteristics curves satisfying $m$ ODEs:

$$\frac{dx}{dt} = \lambda_i, \text{for } i = 1, ..., m. \tag{15}$$

## 2.1 Method of Characteristics

The characteristic curve discussed above is best described in the context of a scalar PDE such as

$$\begin{aligned} \text{PDEs} : u_x + a u_t = 0,\ -\infty < x < \infty,\ t > 0, \\ \text{ICs} : u(x, 0) = u_0(x). \end{aligned} \tag{16}$$

A characteristic curve is defined as $x = x(t)$ in the $x$-$t$ plane along which the PDE becomes an ODE. So $u(x(t), t)$ has a rate of change along $x$ of

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}. \tag{17}$$

If the characteristic curve $x = x(t)$ satisfies the ODE

$$\frac{dx}{dt} = a \tag{18}$$

then the PDE in equation (16) together with equation (17) gives

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0. \tag{19}$$

The time rate of change of $u$ along the characteristic curve $x = x(t)$ satisfying equation (18) is simply zero, implying $u$ is constant along the characteristic curve. The speed $a$ is called the characteristic speed and from equation (18) it is also the slope of the characteristic curve. For instance, if $x(0) = x_0$, then the single characteristic curve that passes through the point $(x_0, t = 0)$ is

$$x = x_0 + at, \tag{20}$$

which is shown in Figure 1. So, if $u(x, 0) = u_0(x)$ at $t = 0$, then along the characteristic curve, equation (20), the solution is

$$u(x, t) = u_0(x_0) = u_0(x - at). \tag{21}$$

Put into words, *given an initial profile $u_0(x)$, the PDE will simply translate this profile with velocity $a$.*

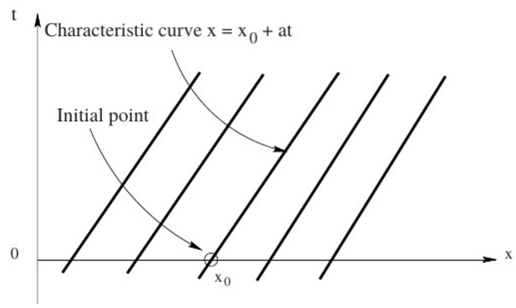


Figure 1:

## 2.2 The General Initial Value Problem

Now, we go back to the general IVP, equation 6, with initial data

$$U^{(0)} = (u_1^{(0)}, u_2^{(0)}, ..., u_m^{(0)})^T. \tag{22}$$

We know from above that upon decoupling the equations, we get a solution like

$$w_i(x,t) = w_i^{(0)}(x - \lambda_i t), \text{for } i = 1, 2, ..., m. \tag{23}$$

The solution of the general IVP in terms of the original variables $\boldsymbol{U}$ is obtained by the transform, as before, $\mathbf{U} = \mathbf{KW}$. This in turn can be written as

$$\mathbf{U}(x,t) = \sum_{i=1}^{m} w_i(x,t)\mathbf{K}^{(i)}. \tag{24}$$

Explicitly, this shows that the function $w_i(x,t)$ is the coefficient of $\mathbf{K}^{(i)}$ in an eigenvector expansion of the vector $\mathbf{U}$. Thus, given a point $(x,t)$ in the $x$-$t$ plane, the solution $\mathbf{U}(x,t)$ at this point depends only on the initial data at the $m$ points $x_0^{(i)} = x - \lambda_i t$. The solution (24) for $\mathbf{U}$ can be seen as the superposition of $m$ waves, each of which is advected independently without change in shape. The $i$-th wave has shape $w_i^{(0)}(x)\mathbf{K}^{(i)}$ and propagates with speed $\lambda_i$.

## 2.3 The Riemann Problem

The Riemann problem (RP) is a special initial value problem where a discontinuity exists between a grid wall as in Figure 2. The problem follows as:

$$\begin{aligned} &\text{PDEs} : U_t + AU_x = 0, \ -\infty < x < \infty, \, t > 0 \\ &\text{ICs} : U(x,0) = U^{(0)}(x) = \begin{cases} u_L & x \leq 0, \\ u_R & x \geq 0. \end{cases} \end{aligned} \tag{25}$$

The solution structure in the $x$-$t$ plane is shown in Figure 3 where there are $m$ waves propagating from the origin, one for each eigenvalue. These waves

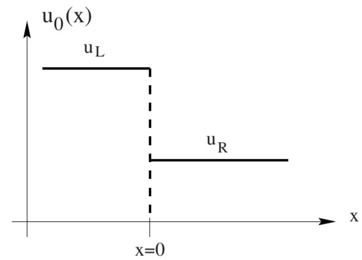


Figure 2:

carry the jump discontinuity in $U$ and propagate with speed $\lambda_i$. It is simple to see that the solution to the left of $\lambda_1$ is $u_L$ and to the right of $\lambda_m$ is $u_R$. The problem is finding the solution in between these waves, a region known as the *Star Region*. Since the eigenvectors are linearly independent, we expand the left and right data as linear combinations of the eigenvectors:

$$u_L = \sum_{i=1}^{m} \alpha_i K^{(i)},\ u_R = \sum_{i=1}^{m} \beta_i K^{(i)}. \tag{26}$$

Where $\alpha$ and $\beta$ are constant coefficients. **Formally, the solution of the IVP is given by (24) in terms of the initial data $w_i^{(0)}(x)$ for the characteristic variables and the right eigenvectors ($K^{(i)}$).** *Note that each of the expansions in (26) is a special case of (24).* Equation (21) along with the initial

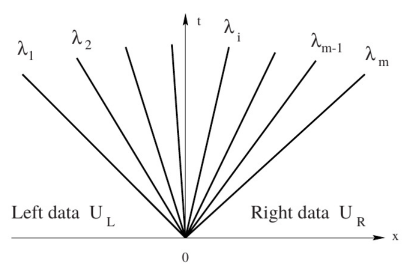


Figure 3:

data

$$w_i^{(0)}(x) = \begin{cases} \alpha_i & \text{if } x \le 0, \\ \beta_i & \text{if } x \ge 0 \end{cases} \tag{27}$$

will have a solution given by

$$w_i(x,t) = w_i^{(0)}(x - \lambda_i t) = \begin{cases} \alpha_i & \text{if } x - \lambda_i t \le 0, \\ \beta_i & \text{if } x - \lambda_i t \ge 0. \end{cases} \tag{28}$$

For a given point $(x,t)$ there is an eigenvalue $\lambda_I$ such that $\lambda_I < \frac{x}{t} < \lambda_{I+1}$, which implies that $x - \lambda_i t > 0\ \forall i \mid i \le I$. So, we can now write the solution to the Riemann Problem in terms of the original variables as

$$U(x,t) = \sum_{i=I+1}^{m} \alpha_i K^{(i)} + \sum_{i=1}^{I} \beta_i K^{(i)}, \tag{29}$$

where the integer $I = I(x,t)$ is the maximum value of the sub-index $i$ for which $x - \lambda_i t > 0$.

## 2.4 Non-Linearities and Shock Waves

For non-linear hyperbolic conservation law systems, such as the case is with the Gas dynamics equations, the outcome has different properties when compared to the linear case. Using the basic initial value problem

$$u_t + f(u)_x, \;\; u(x,0) = u_0(x), \tag{30}$$

we explain these different properties in the following sections.

### 2.4.1 Construction of Solutions on Characteristics

Consider characteristics curves $x = x(t)$ satisfying the ODE,

$$\frac{dx}{dt} = \lambda(u), \; x(0) = x_0, \tag{31}$$

where $\lambda(u) = f'(u)$. By regarding both $u$ and $x$ as functions of $t$, we find the total derivative of $u$ along *x(t)*, namely

$$\frac{du}{dt} = u_t + \lambda(u)u_x = 0. \tag{32}$$

In words, $u$ is constant along the characteristic curves satisfying (31) and therefore the slope $\lambda(u)$ is constant along the characteristic. Hence the characteristic curves are straight lines. The value of $u$ along each curve is the value of $u$ at the initial point $x(0) = x_0$ and so we write

$$u(x,t) = u_0(x). \tag{33}$$

Now the slope $\lambda(u)$ of the characteristic may then be evaluated at $x_0$ so that the solution of the characteristics curves of the ODE equation 31 are

$$x = x_0 + \lambda(u_0(x_0))t. \tag{34}$$

Equations (33) and (34) are the analytic solutions to the ODE. The point $x_0$ depends on the given point $(x,t)$.

### 2.4.2 Wave Steepening and Shock Waves

For non-linear equations, the characteristic speed $\lambda$ becomes $\lambda(u)$, or a function of the solution which creates wave distortion. In Figure 4, we see a smooth initial profile with the initial data $u_0$ along with the initial points $x_0$. If we consider a convex flux function, $\lambda^{'}(u) = f^{''}(u) > 0$, then the characteristic speed is an increasing function of $u$. In the case of non-linear systems of conservation laws, the character of the flux function (convex, concave, or neither) is determined by the Equation of State. The higher values of $u_0$ will therefore travel faster than

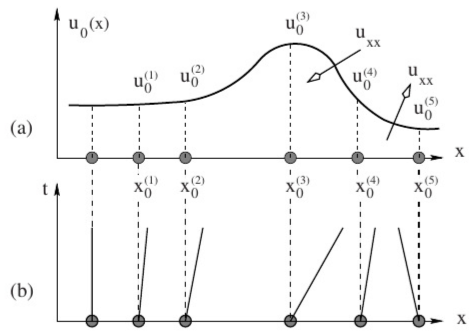


Figure 4:

the lower values. The region to the left of $x_0^{(3)}$ is called the *expansive region* and to the right is called the *compressive region*. In the *expansive region*, the characteristic speed $\lambda_x > 0$ meaning as $x$ increases so to does the wave speed. Whereas in the *compressive region* $\lambda_x < 0$. At a latter time, we would see that the expansive region will have a broader, flatter profile whereas the compressive region will tend to be steeper and narrower. When the characteristics in the compressive region intersect, the wave breaks or becomes a shock wave. A shock wave in air is a small transition layer of very rapid change of physical quantities such as pressure, density, and temperature. Breaking first occurs on the characteristic for which $\lambda_x(x_0)$ is negative and $|\lambda_x(x_0)|$ is a maximum.

To take into account the shock wave, a formula is derived from the integral form of the conservation law system,

$$\frac{d}{dt}\int_{x_L}^{x_R} u(x,t)dx = f(u(x_L,t)) - f(u(x_R,t)), \tag{35}$$

so that the shock wave can be modeled as a mathematical discontinuity (*Shock Wave thickness can be an issue, see Landau and Lifshitz 1959 pp 337-341*). The formula is known as the *Rankine-Hugoniot Condition*,

$$\Delta f = S\Delta u. \tag{36}$$

This equation algebraically relates the jumps between the left and right fluxes, $\Delta f$, the left and right conserved variables, $\Delta u$, and the speed of the discontinuity, $S$.

### 2.4.3 The Entropy Condition

From the initial condition shown in Figure 5(a), we see if the characteristic speed is a function of the data, then a shock wave will form immediately as seen from the characteristic curves shown in Figure 5(b). The discontinuous solution is a shock wave and is compressive in nature and must satisfy the following condition

$$\lambda(u_L) < S < \lambda(u_R), \tag{37}$$

which is known as *the entropy condition*.

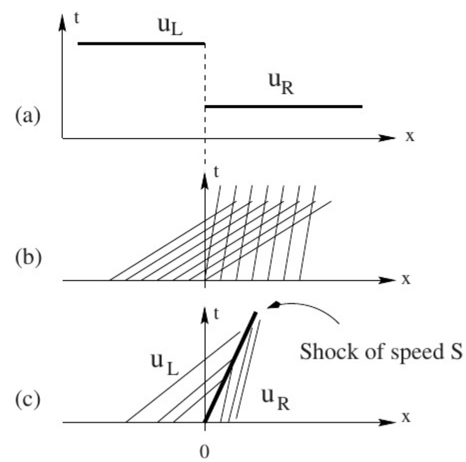


Figure 5:

### 2.4.4 Rarefaction Shock or Entropy-Violating Shock

If the data is flipped so it appears as 6(a), but the flux is still convex, Figure 6(b), a possible solution is the same as above, but will be violating the entropy condition, shown in Figure 6(c).

### 2.4.5 Rarefaction Wave

By replacing the initial data with a linear variation between the discontinuity,

$$u_0(x) = \begin{cases} u_L & \text{if } x \leq x_L, \\ u_L + \frac{(u_R - u_L)}{(x_R - x_L)}(x - x_L) & \text{if } x_L < x < x_R, \\ u_R & \text{if } x \geq x_R, \end{cases} \tag{38}$$

the solution to this problem is found by following characteristics which consists of two constant states, $u_L$ and $u_R$, separated by a region of *smooth transition*. This is called a *rarefaction wave*. The right (*head*) and left (*tail*) edge of the

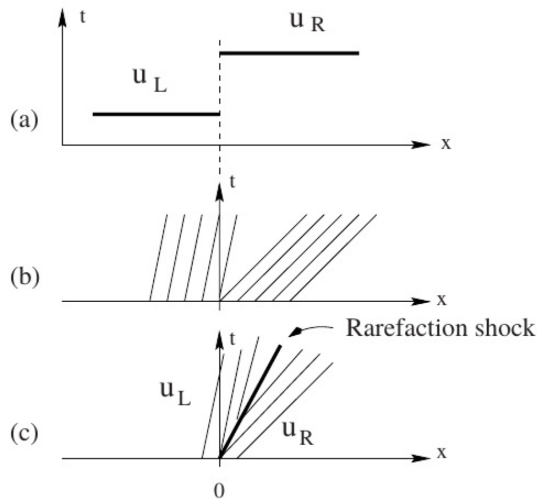


Figure 6:

wave in Figure 7(c) are given by the characteristics emanating from $x_R$ and $x_L$, respectively.

The spreading of waves is a phenomenon not seen in linear hyperbolic systems with constant coefficients. The entire solution is

$$\begin{cases} u(x,t) = u_L & \text{if } \frac{x-x_L}{t} \leq \lambda_L, \\ \lambda(u) = \frac{x-x_L}{t} & \text{if } \lambda_L < \frac{x-x_L}{t} < \lambda_R, \\ u(x,t) = u_R & \text{if } \frac{x-x_R}{t} \geq \lambda_R. \end{cases} \tag{39}$$

Now, there are at least two solutions to the IVP, equation 30, which adds complexity due to the fact that one will be correct and the other spurious. How to distinguish between the two is by analyzing a physical discontinuity using the *Rankine-Hugoniot Condition* and the *entropy condition*.

### 2.4.6 Characteristic Fields

Consider the hyperbolic system of equation 8. The characteristic speed $\lambda_i(\mathbf{U})$ defines a characteristic field, the $\lambda_i$-field or sometimes the $K_i(\mathbf{U})$-field. By defining the gradient of the eigenvalue as

$$\nabla\lambda_i(\mathbf{U}) = (\frac{\partial}{\partial u_1}\lambda_i, \frac{\partial}{\partial u_2}\lambda_i, ..., \frac{\partial}{\partial u_m}\lambda_i)^T, \tag{40}$$

we can determine the type of characteristic field as either *Linearly Degenerate* or *Genuinely Nonlinear*. Both defined as follows

$$\begin{array}{l} \text{Linearly Degenerate} \\ \nabla\lambda_i(\mathbf{U}) \cdot \mathbf{K}^{(i)}(\mathbf{U}) = 0, \, \forall \mathbf{U} \in \Re^m \end{array} \tag{41}$$

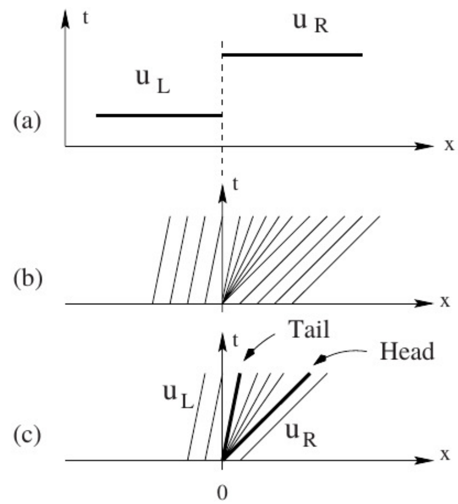


Figure 7:

and

$$\begin{array}{l}\text{Genuinely Nonlinear} \\ \nabla\lambda_i(\mathbf{U})\cdot\mathbf{K}^{(i)}(\mathbf{U}) \neq 0,\, \forall\mathbf{U}\in\Re^m,\end{array} \tag{42}$$

where the $\cdot$ represents the dot product in *phase space*. The *phase space* is space of vectors $\mathbf{U}$; for a 2x2 system it is the $u_1$-$u_2$ phase plane.

#### 2.4.7 Rankine-Hugoniot Condtions

Using the same hyperbolic system as above with a discontinuous wave solution of speed $S_i$ associated with the $\lambda_i$-characteristic field, the *Rankine-Hugoniot Condition* (RHC), depicted in Figure 8, is

$$\Delta F = S_i \Delta U. \tag{43}$$

The *RHC* can be used to determine the solution inside the *Star Region*. For

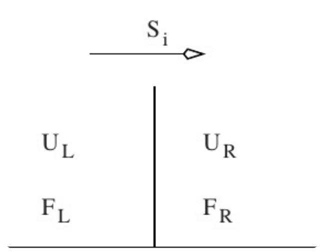


Figure 8:

example, the linearized gas dynamics equations have an eigensystem

$$U = [\rho, u] \ , \ A = \begin{bmatrix} 0 & \rho_0 \\ \frac{a^2}{\rho_0} & 0 \end{bmatrix} \tag{44}$$

and the eigenvalues

$$\lambda_1 = -a, \ \lambda_2 = a. \tag{45}$$

Then by applying the *RHC* across the $\lambda_1$-wave of speed $S_1 = \lambda_1$, we get

$$F(U) = \begin{bmatrix} 0 & \rho_0 \\ \frac{a^2}{\rho_0} & 0 \end{bmatrix} \begin{bmatrix} \rho^* - \rho_L \\ u^* - u_L \end{bmatrix} = -a \begin{bmatrix} \rho^* - \rho_L \\ u^* - u_L \end{bmatrix}. \tag{46}$$

By doing the same for the $\lambda_2$-wave of speed $S_2 = \lambda_2$, then expanding and solving both for $u^*$ we get the exact solution for the *Star Region* unknowns.

### 2.4.8 Generlized Riemann Invariants

Consider the quasi-linear hyperbolic system of equation 9, with

$$\mathbf{W} = [w_1, w_2, ..., w_m]^T. \tag{47}$$

The *Generalized Riemann Invariants* (GRI) are relations that hold true, for certain waves, across the wave structure and lead to the following $m-1$ ODEs

$$\frac{dw_1}{k_1^{(i)}} = \frac{dw_2}{k_2^{(i)}} = \cdots = \frac{dw_m}{k_m^{(i)}}, \tag{48}$$

where $\mathbf{K}^{i)} = [k_1^{(i)}, ..., k_m^{(i)}]$ is the corresponding eigenvector of the wave associated $i$-characteristic field. The ODEs relate ratios of changes due to $dw_s$ of quantities $w_s$ to the respective eigenvector component corresponding to a $\lambda_i$-wave family. With the same linearized gas dynamics equations as above, we use the *GRI* ODEs to find:

$$\begin{array}{l} \text{Across the } \lambda_1\text{-wave} \\ \frac{d\rho}{\rho_0} = \frac{du}{-a} \end{array} \tag{49}$$

and

$$\begin{array}{l} \text{Across the } \lambda_2\text{-wave} \\ \frac{d\rho}{\rho_0} = \frac{du}{a}. \end{array} \tag{50}$$

After integrating these equations and applying them across the left wave, connecting the states $W_L$ and $W^*$, and the right wave, connecting the states $W^*$ and $W_R$, we get respectively

$$u^* + \frac{a}{\rho_0}\rho^* = u_L + \frac{a}{\rho_0}\rho_L \tag{51}$$

$$u^* - \frac{a}{\rho_0}\rho^* = u_R - \frac{a}{\rho_0}\rho_R. \tag{52}$$

Solving both of these for the unknown variables, we get the same solution as above, for the *Star Region*.

## 2.5 Elementary-Wave Solutions of the Riemann Problem

Now putting all of what we learned to use to solve a general $mxm$ non-linear hyperbolic system with initial data

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}) = 0,$$
$$U(x,0) = U^{(0)}(x) = \begin{cases} \mathbf{U}_L & x < 0, \\ \mathbf{U}_R & x > 0. \end{cases} \tag{53}$$

The similarity solution $\mathbf{U}(x/t)$ of this system, consists of *m+1* constant states separated by $m$ waves depicted in Figure 9(A). **A *similarity solution* is the solution to a conservation law that is a function of $x/t$ alone and is self-similar at different times.**

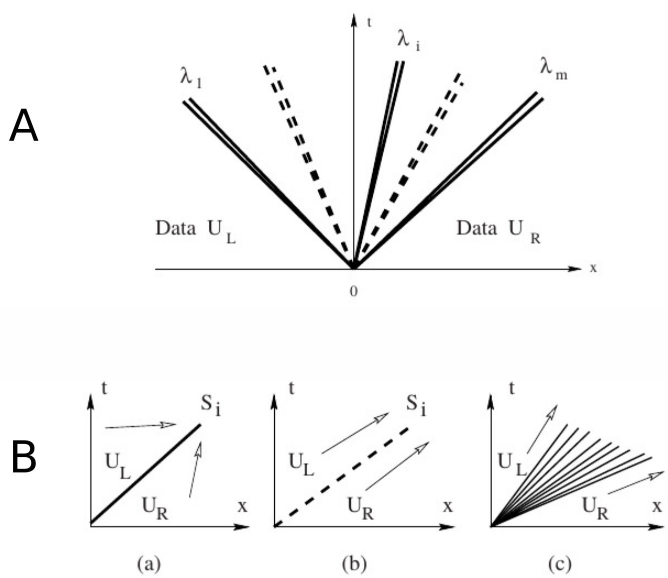


Figure 9:

For each eigenvalue, there is a wave family. If this a linear system with constant coefficients each wave is a discontinuity with speed $S_i = \lambda_i$ and defines a linearly degenerate field.

For non-linear systems, as described above, the waves may be discontinuities such as shock, contact, or rarefaction waves. The possible types of waves in the solution to the Riemann Problem depends critically on the closure conditions, that is to say the equation of state for the Gas Dynamics equations. Figure 9(B) presents the three different wave types for a solution to the Riemann Problem that consists of a single non-trivial wave, where all other waves have zero strength.

### 2.5.1 Shock Waves

Across a *shock wave* $\rho$, $u$, and $p$ change. Central to the analysis of a *shock wave* is the application of the *RHC*, where the shock speed can be determined. The two data states $\mathbf{U}_L$ and $\mathbf{U}_R$ are connected by a single jump discontinuity in a *genuinely nonlinear field* $i$ and the following conditions apply

- Rankine Hugoniot Conditions

$$\Delta\mathbf{F}(\mathbf{U}) = S_i \Delta\mathbf{U} \tag{54}$$

- The Entropy Condition

$$\lambda_i(\mathbf{U}_L) > S_i > \lambda_i(\mathbf{U}_R). \tag{55}$$

This type of wave is shown in Figure 9(B)(a)

### 2.5.2 Contact Waves

A contact wave is a discontinuous wave across which both *pressure* and *velocity* are constant, which is determined via the *GRIs* for the $\mathbf{K}^{(2)}$-wave (relation 56) shown in equation (57), but density jumps discontinuously as do variables that depend on on density, such specific internal energy, temperature, speed of sound, entropy, etc.

$$\mathbf{K}^{(2)} = \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{56}$$

$$\frac{d\rho}{1} = \frac{d(\rho u)}{u} = \frac{dE}{\frac{1}{2}u^2}, \tag{57}$$

where $\alpha_2$ is the associated wave strength. The two data states $\mathbf{U}_L$ and $\mathbf{U}_R$ are connected through a single jump discontinuity of speed $S_i$ in a *linearly degenerate field* $i$ and the following conditions apply

- Rankine Hugoniot Conditions

$$\Delta\mathbf{F}(\mathbf{U}) = S_i \Delta\mathbf{U} \tag{58}$$

- Constancy of the GRIs across the wave, as shown above,

$$\frac{dw_1}{k_1^{(i)}} = \frac{dw_2}{k_2^{(i)}} = \cdots = \frac{dw_m}{k_m^{(i)}}, \tag{59}$$

- The parallel characteristic Condition

$$\lambda_i(\mathbf{U}_L) = S_i = \lambda_i(\mathbf{U}_R). \tag{60}$$

This type of wave is shown in Figure 9(B)(b).

### 2.5.3 Rarefaction Waves

Inspecting the $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(3)}$ eigenvectors, shown in equations (61) and (62), for the primitive-variable formulation shows that $\rho$, $u$, and $p$ change across a rarefaction wave.

$$\mathbf{K}^{(1)} = \alpha_1 \begin{bmatrix} 1 \\ -a/\rho \\ a^2 \end{bmatrix}, \tag{61}$$

$$\mathbf{K}^{(3)} = \alpha_3 \begin{bmatrix} 1 \\ a/\rho \\ a^2 \end{bmatrix}. \tag{62}$$

In summary, a *rarefaction wave* is a smooth wave associated with the 1- and 3-fields across which $\rho$, $u$, and $p$ change. The wave has a fan-type shape and is enclosed by two bounding characteristics. Across the wave the GRIs apply.

The two data states $\mathbf{U}_L$ and $\mathbf{U}_R$ are connected through a *smooth transition region* in a *genuinely nonlinear field* $i$ and the following conditions are met

- Constancy of the GRIs across the wave

$$\frac{dw_1}{k_1^{(i)}} = \frac{dw_2}{k_2^{(i)}} = \cdots = \frac{dw_m}{k_m^{(i)}}, \tag{63}$$

- Divergence of Characteristics

$$\lambda_i(\mathbf{U}_L) < \lambda_i(\mathbf{U}_R). \tag{64}$$

This type of wave is shown in Figure 9(B)(c).

## 2.6 Riemann Problem for Shallow Water Equations

The key issues for designing an **Exact** *Riemann Solver* are

- The variables selected
- The equations used
- The number of equations
- The technique for the **iterative solutions**
- The initial guess
- The way of handling unphysical iterates (i.e. negative pressures)

Before proceeding it is worth mentioning that there does not exist an exact closed-formed solution to the RP for the Euler equations. In order to compute the exact solution to the Riemann Problem with arbitrary left and right states, we do the following:

1. Determine the type of each wave whether it be a shock, rarefaction, or contact (using the abovementioned properties, i.e. entropy condition, Riemann invariants, etc.).
2. Determine the intermediate state, $q_m$.
3. Determine the structure of the solution through any rarefaction waves.

The first and third steps are familiar, but the second step requires some additional work, explained in the following section, and will be helpful in the later *Entropy Fix* section.

We use a slight generalization of the Shock Tube problem (Equation 25) to show this process at work. For the most part, the equation of state (EOS) determines not only the structure of the solution to the RP, but also the mathematical character of the equations of gas dynamics. We use the EOS for ideal gases which obeys the caloric EOS

$$e = \frac{p}{(\gamma - 1)\rho} \tag{65}$$

# 3 Solving the Euler Equations

## 3.1 Conservative versus Non-Conservative

First it is important to note two types of formulations of the Euler Equations. The most well known is the conservative form with

$$U_t + F(U)_x = 0 \tag{66}$$

$$U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \; F(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix}, \tag{67}$$

where $\rho$ is mass density, $u$ is velocity, $E$ is energy density, and $p$ is pressure. The energy density equation is

$$E = \frac{1}{2}\rho(\mathbf{V{=}{<}u{,}0{,}0{>}})^2 + \frac{p}{(\gamma - 1)}. \tag{68}$$

The conservation of mass, momentum, and energy are all well known physical laws.

A non-conservative form most commonly used is with the $primitive-variables$. Which is found by expanding the derivatives in equation (67) which leads to, in quasi-linear form

$$W_t + A(W)W_x = 0 \tag{69}$$

$$W = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}. \tag{70}$$

The conservative form is what we use mainly for the reason that **the solution to the non-conservative form that contains shock waves will give incorrect shock solutions.**

## 3.2 Characteristic Equations

## 3.3 Elementary Wave Solutions of the Riemann Problem

The similarity solution of the Riemann Problem, equation (53), is depicted in Figure 9(A), but in this case $\lambda_i = \lambda_2$ and $\lambda_m = \lambda_3$. The *star-region* is located between $\lambda_1$ and $\lambda_3$. The two rays emanating from the wave solutions indicate that the character is still unknown. These three waves separate four constant states. In order to determine the wave-type, each type allowed in the solution, we first analyze the characteristic fields. We find that the $\mathbf{K}^{(2)}$-field is linearly degenerate, indicating this is a contact wave, and the $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(3)}$ are genuinely non-linear, indicating that they can be a shock or rarefaction wave. One does not inherently know, beforehand, the types of waves present in the solution to the Riemann Problem. The only exception being the middle wave, which is always a contact discontinuity.

### 3.3.1 Contact Wave

### 3.3.2 Rarefaction Wave

### 3.3.3 Shock Wave

## 3.4 2D Euler Equations

Analyzing the 2D Euler Equations by making them quasi-linear and forming the eigensystem, we see that there are now four waves and four associated characteristic speeds. Similar to above, the $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(4)}$ waves are genuinely non-linear and therefore associated with a *shock* or *rarefaction* wave. A difference lies now in the eigenvectors in the *star region*, where $\mathbf{K}^{(2)}$ is a *contact wave* so *pressure* and *velocity* are constant and *density* jumps discontinuously across the wave and $\mathbf{K}^{(3)}$ is a *shear wave* over which the *tangential velocity* component jumps discontinuously across the wave.

# 4 The Method of Godunov for Non-linear Systems

The *REA* Algorithm, below, was originally proposed by Godunov [1959] as a method for solving the nonlinear Euler equations of gas dynamics.

1. **R**econstruct a piecewise polynomial function. A piecewise constant function that takes the value $Q_i^n$ in the ith grid cell.

2. **E**volve the hyperbolic equation exactly or approximately with this data to obtain the solution at a time $t$ later.

3. **A**verage this function over each grid cell to obtain new cell averages.

Application in the context of gas dynamics relies upon whether or not the RP with the piecewise constant data can be solved and that the solution consists of a finite set of waves traveling at constant speeds. The solution is pieced together using the solutions to each Riemann Problem provided that the timestep is short enough to not allow any two waves to interact.

## 4.1 The Wave Propagation Form of Godunov's Method

The most intuitive form of Godunov's method which can be related back to the upwind method is the wave propagation form. In Figure (10), we see two sets of waves emanating from the left, $Q_{i-1/2}$, and right, $Q_{i+1/2}$, grid cell walls of

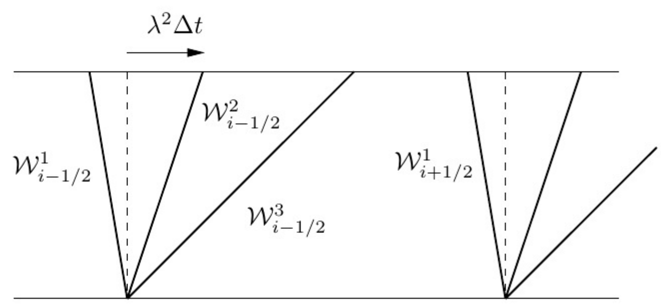


Figure 10: The case of a linear system of three equations.

$Q_i$. By following the above described processes, the three waves are obtained. For a linear system, the solution to the Riemann problem can be expressed by a set of waves,

$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p = \sum_{p=1}^{m} W^p_{i-1/2}. \tag{71}$$

By looking at the $\lambda_2$ wave's effect on the cell average, we note that after a time $\Delta t$ the wave modifies the value of $Q_i$ over a fraction of the grid cell $\lambda_2 \Delta t/\Delta x$. The effect of this wave of the new cell average is

$$-\lambda_2 \frac{\Delta t}{\Delta x} W^2_{i-1/2}. \tag{72}$$

The minus sign arises because the wave value $W^2_{i-1/2}$ measures the jump from left to right. Each of the waves entering the grid cell of $Q_i$ effects the new cell average, which is found by adding up all of the effects and thus

$$Q^n_{i+1} = Q^n_i - \frac{\Delta t}{\Delta x}(\lambda_2 W^2_{i-1/2} + \lambda_3 W^3_{i-1/2} + \lambda_1 W^1_{i+1/2}) \tag{73}$$

To generalize this formula to a system with $m$ equations, we introduce the notation

$$\lambda^+ = max(\lambda, 0),\ \lambda^- = min(\lambda, 0). \tag{74}$$

Supposing that the solution to every Riemann problem consists of $m$ waves $W^p$ traveling at speeds $\lambda_p$, each of which may be negative or positive. The generalized formula is thus

$$Q_{i+1}^n = Q_i^n - \frac{\Delta t}{\Delta x}\Big[\sum_{p=1}^{m}(\lambda_p)^+ W_{i-1/2}^p + \sum_{p=1}^{m}(\lambda_p)^- W_{i+1/2}^p\Big]. \tag{75}$$

The shorthand of this formula is as follows

$$Q_{i+1}^n = Q_i^n - \frac{\Delta t}{\Delta x}\Big[A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\Big]. \tag{76}$$

The shorthand notation symbol $A^{+,-}\Delta Q_{i\mp 1/2}$, or *fluctuation*, should be interpreted as the measure of the net effect of all right(left) going waves from the $x_{i-1/2}$ ($x_{i+1/2}$) interface.

# 5 Approximate Riemann Solvers

The process of solving the RP is often quite expensive due to the necessary use of an iterative process to find solutions. We did not go into depth regarding the process of solving the RP exactly, but what is typically needed is the solution in the star region. The approximate, non-iterative solution provides the necessary items of information for numerical purposes at a fraction of the cost. The two approach types are *approximation to the numerical flux* and *approximation to a state*. The prior is what we use in Roe's Method.

For given data $Q_i$ and $Q_{i-1}$, an approximate Riemann solution will define a function $\hat{Q}_{i-1/2}(x/t)$ that approximates the true similarity solution to the RP. This function will consist of a set of $M_w$ waves $W_{i-1/2}^p$ propagating at speeds $s_{i-1/2}^p$, with

$$Q_i - Q_{i-1/2} = \sum_{p=1}^{M_w} W_{i-1/2}^p. \tag{77}$$

Implemeting the approximate Riemann solution in place of the true Riemann solution in equation 76 and averaging these solutions over the grid cells we get $Q^{n+1}$.

## 5.1 Linearized Riemann Solvers

In similar fashion as linearizing a non-linear system seen in equation 9, only now we implement the approximate form of the linearized equation,

$$\hat{q}_t + \hat{A}_{i-1/2}\hat{q}_x = 0. \tag{78}$$

The matrix $\hat{A}$ is chosen as an approximation to $f^{'}(q)$ valid in the neighborhood of the data $Q_i$ and $Q_{i-1}$ and must satisfy the following:

1. diagonalizable with real eigenvalues so that 78 is hyperbolic

2. $\hat{A}_{i-1/2} \implies f^{'}(\bar{q})$ as $Q_{i-1}, Q_i \implies \bar{q}$

so that the method is consistent with the original conservation law. The approximate solution then consists of $m$ waves proportional to the eigenvectors $\hat{r}_{i-1/2}$ of $\hat{A}_{i-1/2}$, propagating with speeds $s^p_{i-1/2} = \hat{\lambda}^p_{i-1/2}$ given by the eigenvalues.

### 5.1.1 Roe Linearization

We note that near a shock wave, the RPs arising at cell interfaces will typically have a large jump typically in at most one wave family, $W^p$, with $||W^j|| = O(\Delta x)$ for all other waves $j \neq p$.

*If $Q_i$ and $Q_{i-1}$ are connected by a single wave $W^p = Q_i - Q_{i-1}$ in the true Riemann solution, then $W^p$ should also be an eigenvector of $\hat{A}_{i-1/2}$.*

By choosing a specific parameter vector to obtain the approximate Jacobian Roe was able to satisfy

## 5.2 Roe's Method

We begin by presenting Figure 11 in which shows the general IBVP, followed by the solution written in the explicit conservative formula. The definition of the Godunov intercell numerical flux proceeds that and finally the solution to the IBVP is written in the general form. Where $U_{i+1/2}(0)$ is the similarity solution

$$\left.\begin{array}{ll}\text{PDEs} & : \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \, , \\ \text{ICs} & : \mathbf{U}(x,0) = \mathbf{U}^{(0)}(x) \, , \\ \text{BCs} & : \mathbf{U}(0,t) = \mathbf{U}_\mathrm{l}(t) \, , \; \mathbf{U}(L,t) = \mathbf{U}_\mathrm{r}(t) \, , \end{array}\right\}$$

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t}{\Delta x}[\mathbf{F}_{i-\frac{1}{2}} - \mathbf{F}_{i+\frac{1}{2}}] \, ,$$

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}(\mathbf{U}_{i+\frac{1}{2}}(0)) \, .$$

$$\left.\begin{array}{l}\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \, , \\ \mathbf{U}(x,0) = \left\{\begin{array}{l}\mathbf{U}_\mathrm{L} \text{ if } x < 0 \, , \\ \mathbf{U}_\mathrm{R} \text{ if } x > 0 \, , \end{array}\right. \end{array}\right\}$$

Figure 11:

$U_{i+1/2}(x/t)$ of the Riemann problem evaluated at $x/t = 0$. The value $x/t = 0$ for the Godunov Flux corresponds to the $t$-axis.

#### 5.2.1 Failure of Roe's Method

#### 5.2.2 Why impact to thermal ionization is not a highly energetic gas

### 5.3 Transonic Waves

### 5.4 Harten-Hyman Entropy Fix

Suppose there appears to be a transonic rarefaction wave

## 6 end

## References

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