Penalty Method

Consider a C^1 path $\gamma : [0,1] \to \mathbb{R}^2$ leading from the starting point $\gamma(0) = (0,0)$ to the endpoint $\gamma(1) = (1,1)$. The goal of this study is to minimize

$$H(\gamma) = \frac{1}{2} \int_0 1 \left[x'(t)^2 + y'(t)^2\right] dt$$

under the following constraint: the path must bypass an obstacle assumed to be a disk entirely contained within the unit square of the plane

$$D = \{(x,y) \in \mathbb{R}^2, (x-a)^2 + (y-b)^2 \leq r^2\}.$$

To solve this problem, we will "penalize" the constraint. For $\varepsilon > 0$ and "small", we seek to minimize

$$H_\varepsilon(\gamma) = H(\gamma) + \frac{1}{\varepsilon} R(\gamma)$$

with

$$R(\gamma) = \frac{1}{2} \int_0 1 \max\left(0, r^2 - (x(t)-a)^2 - (y(t)-b)^2\right)^2 dt.$$

Note that the "penalization" term only applies if the point $(x(t), y(t))$ is located within the obstacle. If no point on the path γ is located within the obstacle, then $H_\varepsilon(\gamma) = H(\gamma)$.

Discretization of $H(\gamma)$. For $N \in \mathbb{N}^*$, we set $h = \frac{1}{N}$ and $t_n = nh$ for $n = 0, \ldots, N$. To calculate the integral $H(\gamma)$, we write

$$H(\gamma) = \frac{1}{2} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left[x'(t)^2 + y'(t)^2\right] dt$$

and make the approximation

$$H(\gamma) \approx H(x,y) = \frac{1}{2} \sum_{n=0}^{N-1} \left[\left(\frac{x_{n+1} - x_n}{h}\right)^2 + \left(\frac{y_{n+1} - y_n}{h}\right)^2\right] h,$$

where $x = (x_i)_{i=1}^{N-1} \in \mathbb{R}^{N-1}$, $y = (y_i)_{i=1}^{N-1} \in \mathbb{R}^{N-1}$ and the x_j (resp. y_j) are approximations of $x(t_j)$ (resp. $y(t_j)$) with $(x_0, y_0) = \gamma(0)$, $(x_N, y_N) = \gamma(1)$.

Discretization of $R(\gamma)$. We make the approximation

$$R(\gamma) \approx R(x,y)$$

with

$$R(x,y) = \frac{h}{2}\sum_{n=0}^{N-1} \max\left(0, r^2 - (x_n - a)^2 - (y_n - b)^2\right)^2.$$

We therefore seek to minimize

$$H_\varepsilon(x,y) = H(x,y) + \frac{1}{\varepsilon}R(x,y),$$

for x and y in $\mathbb{R}^{N-1}$.