# Integral

## Terminology:

- Riemann integral (黎曼积分 / 定积分)
- Improper integral （反常积分 / 不定积分）
- Singularity (奇点)
- Divergence (发散)
- Convergence (收敛)
- Polar coordinate system (极坐标)
- Symmetry (对称性)
- Even Function / odd function (偶函数 / 奇函数)
- Differentiable (可微的)
- Remainder (余项)

## Taylor's formula

If $f(x)$ is k times differentiable at point $x \in (a,b)$, then to any $x_0 \in (a,b)$, we could use the following equation to approximates $f(x)$ :

$$f(x) = \frac{f(x_0)}{0!} + \frac{f\acute{}(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$$

$R_n(x)$ is reminder here, for partial expansion $(x \backsim x_0)$, we use Peano form of the remainder

$$R_n(x) = \circ\left((x - x_0)^n\right)$$

For global expansion, we use Lagrange form of the remainder (k+1 times differentiable)

$$R_n(x) = \frac{f^{(n+1)}(\varrho)}{(n+1)!}(x - x_0)^{n+1} \text{ (ϱ between x and } x_0 \text{ )}$$

e.g. Prove:

$$\sin x = x - \frac{1}{3!}x^3 + \cdots$$

e.g. $f(x)$ is 2 times differentiable when $x \in [0,1]$, $f(0) = 0$ , $f(1) = 1$. $\int_0^1 f(x)dx = 1$. Prove:

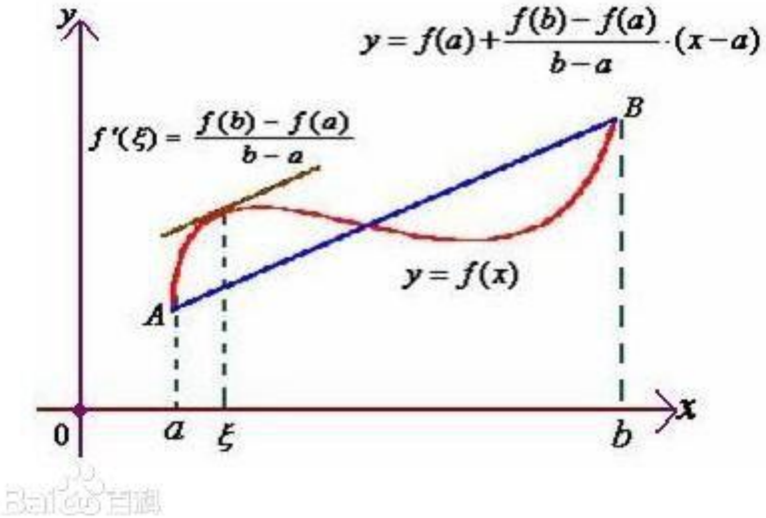
*1.* $\exists\, \varrho \in (\mathbf{0}, \mathbf{1})$ let $\boldsymbol{f\acute{}(\varrho)} = \mathbf{0}$

*2.* $\exists\, \eta \in (\mathbf{0}, \mathbf{1})$, let $\boldsymbol{f''(\eta) < -2}$

**Lagrange Mean Value Theorem**

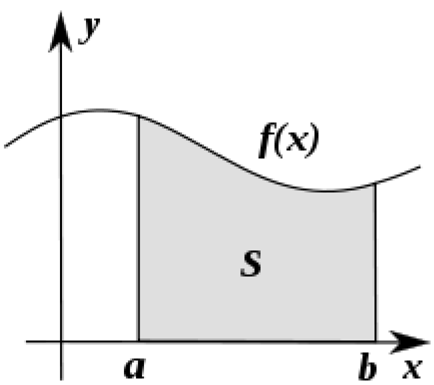
if $f$ is a continuous function on the closed interval $[a, b]$, and differentiable on the open interval $(a, b)$, then there exists a point $\xi$ in $(a, b)$ such that:

$$k = \frac{f(b) - f(a)}{b - a} = f\acute{}(\xi)\ \ (a < \xi < b)$$




e.g. Assume $\{x_n\}$ : $x_1 > 0$ , $x_n e^{x_{n+1}} = e^{x_n} - 1$, n=1,2,3…, Prove $\{x_n\}$ is bounded, and calculate $\lim_{n\to\infty} x_n$

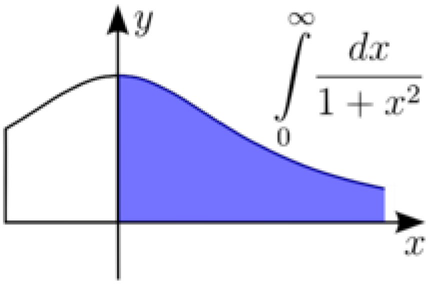
**Riemann integral (definite integral)**



$$S = \sum_{1}^{\infty} f(x)dx = \int_{a}^{b} f(x)dx$$

Has two limits: [a, b] is bounded, f(x) in [a, b.] is bounded

### Improper integral




Break those two limits: [a, b] is bounded, f(x) in [a, b.] is bounded

Principle: we only allow one Singularity exist in one integral, So:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx$$

### Tips:

1. $\frac{dx}{x} = d\ln x$
2.

### Examples

e.g. Assume a, b >0, improper integral $\int_0^{+\infty} \frac{1}{x^a \cdot (2020+x)^b} dx$ is bounded, we could get:

(A) a < 1 and b > 1

(B) a > 1 and b > 1

(C) a < 1 and a + b > 1

(D) a > 1 and a + b < 1

e.g. Assume a > b >0, improper integral $\int_0^{+\infty} \frac{1}{x^a + x^b} dx$ is bounded, we could get:

(A) a > 1 and b > 1

(B) a > 1 and b < 1

(C) a < 1 and a + b > 1 (D) a < 1 and a + b < 1

e.g. Assume a >0, improper integral $\int_1^{+\infty} \frac{\ln(1+\sin\frac{1}{x^a})}{x^b \ln\cos\frac{1}{x}} dx$ is bounded, we could get a + b > 3

Hint: When $\mu \sim 1$, $\ln\mu \sim \mu - 1$, $1 - \cos x \sim \frac{1}{2}x^2$

e.g. Prove:

$$I = \int_2^{+\infty} \frac{1}{x\,(\ln x)^p} dx$$

When p >1, I is Convergence, when p $\leq 1$, I is Divergence

e.g. Calculate the result of (**Symmetry**)

$$\int_0^{10} x(x-1)(x-2)\ldots(x-5)\ldots(x-9)(x-10)dx$$

e.g. Calculate the result of

$$\int_0^4 x\sqrt{4x - x^2}dx$$