Griffith Theory Revisited

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1 The Formulation

Throughout the section, Ω denotes a bounded connected open domain of $\mathbb{R}^N$, $1 \leq N \leq 3$, with smooth boundary $\partial\Omega$ the surface measure of which is finite and such that Ω is the interior of $\overline{\Omega}$. As such, Ω represents the crack-free reference configuration of an elastic body.

1.1 The Ingredients

The family of possible cracks and their surface energy. In classical fracture mechanics, the family of possible cracked configurations are few, restricted in general to a few number of smooth surfaces; by contrast, our family is rather extended, since it is made up of all closed subdomains of $\overline{\Omega}$, independently of their shape, subject only to the condition that their dimension is not greater than $N-1$ so that they do not consume too much surface energy. Indeed, an energy is assigned to each member of the family. In the spirit of Griffith, if $k(x)$ represents the energy required to create an "infinitesimal" crack at the point x of $\overline{\Omega}$, then the surface energy associated to the crack $\Gamma \subset \overline{\Omega}$ is given by

$$E_s(\Gamma) = \int_\Gamma k(x) d\mathcal{H}^{N-1}(x), \tag{1}$$

where $\mathcal{H}^{N-1}$ denotes the $N-1$-dimensional Hausdorff measure (a "surface" measure which amounts to the usual surface measure for smooth enough hypersurfaces).

Remark 1. *The energy density $k(x)$, the toughness of the given material, is assumed to be strictly positive (***and even bounded away from zero***), so that energy is created by introducing additional cracks. Furthermore $k(x)$ can take the value ∞, which corresponds to a material that would not be breakable at the given point x. This will prove useful when analyzing debonding in a composite material, see Section 3. We could also introduce a possible dependence of k upon the normal vector to the crack. This is necessary for anisotropic materials.*

The type of loading. For reasons that will be clear in Section 5 we consider only the cases where the displacements are prescribed (at a value U) on a part $\partial_d\Omega$ of the boundary, while the remaining part of the boundary and the lips of the cracks are free (of forces), and the body forces vanish.

Remark 2. *A no-contact condition on the "lips" of the crack is thus assumed. Moreover, the crack can be partially located at the boundary where the displacements are prescribed. In such a case the "inner trace" v^- of an admissible displacement v does not satisfy the boundary condition.*

In conclusion, the kinematically admissible set of displacement fields is a function of both the loading U and the crack Γ :

$$\mathcal{C}(\Gamma, U) = \{v \in W^{1,2}(\Omega \setminus \Gamma; \mathbb{R}^N) \ : \ v = U \text{ on } \partial_d\Omega \setminus \Gamma\}. \tag{2}$$

The material behavior. In all that follows the material is assumed to behave elastically and to only undergo infinitesimal transformations. If $\varepsilon(v)$ denotes the symmetric gradient of v, an elastic energy density $W(x, \varepsilon(v)(x))$ is thus given at each point x of $\Omega \setminus \Gamma$ by

$$W(x, \xi) = \frac{1}{2} A(x)\xi \cdot \xi, \tag{3}$$

where $A(x)$ denotes the elasticity tensor at x. $A(x)$ is symmetric and definite positive, *i.e.* such that $0 < \alpha I \leq A(x) \leq \beta I$ for a.e. $x \in \Omega$.

We accordingly define the bulk energy as

$$E_d(\Gamma, U) = \inf_{v \in \mathcal{C}(\Gamma, U)} \int_{\Omega \setminus \Gamma} W(x, \varepsilon(v)(x)) dx. \tag{4}$$

The total energy of the body for a given crack Γ and a given loading U is then given by

$$E(\Gamma, U) = E_d(\Gamma, U) + E_s(\Gamma). \tag{5}$$

1.2 The Evolution Law

We now propose to follow the response of the body to a time-dependent loading. To this effect a time-parameterized loading $U(t)$ is applied to $\partial_d \Omega$. Assume that a (maybe empty) initial crack Γ_0 is present in the body at the onset of the loading process. Our goal is to determine the evolution of the crack(s) during the loading, *i.e.*, to obtain the time-parameterized mapping $t \mapsto \Gamma(t)$. (In the sequel $\Gamma(t)$ will be called the *cracked state* at time t.) The basic idea is as follows. At a given time t, and for the corresponding loading $U(t)$ the crack state $\Gamma(t)$ will be the closed subset of $\overline{\Omega}$ which minimizes $E(\Gamma, U(t))$ among all cracks Γ which contain all previous $\Gamma(s),\ s < t$.

There are two important features in this formulation. On the one hand, the driving principle is global energy minimization. On the other hand, the geometry and size of the crack is limited by its predecessors; this is an attempt at expressing the irreversibility of the cracking process, and the absence of healing.

If the conceptual principle is fairly clear, its precise formulation is not so easily laid out. We proceed in two steps, the first of which addresses time-discretized evolutions, while the second pertains to time-continuous evolutions, albeit for a specific subclass of all possible loadings.

Law 1. Discrete evolutions:

Assume that Γ_0 is given. Let U_i, $1 \leq i \leq p$ be a sequence of loadings. Then, the corresponding cracked states Γ_i have to satisfy

$$\Gamma_i \supset \Gamma_{i-1}, \qquad E(\Gamma_i, U_i) \leq E(\Gamma, U_i) \quad \textit{for every } \Gamma \supset \Gamma_{i-1}. \tag{6}$$

As such, the evolution is dicretization-dependent. The real evolution should be constructed as a limit of the discrete evolution as the time-step tends to 0. It has been recently shown in Francfort and Larsen (2002) that a precise law can then be derived for all loading scenarios. In the present study we will only investigate *monotonically increasing loadings*. Assume therefore that the loading is of the form

$$U(x,t) = \begin{cases} tU_0(x), & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{7}$$

for all x's on $\partial_d \Omega$.

Law 2. Continuous monotone evolutions:

The cracked state $\Gamma(t)$ have to satisfy

(i) $\Gamma(t) = \Gamma_0$ *for* $t < 0$, *and* $\Gamma(t)$ *does not decrease with* t,

(ii) $E(\Gamma(t), U(t)) \leq E(\Gamma, U(t))$, *for all* $\Gamma \supset \Gamma^-(t) \equiv \cup_{s<t}\Gamma(s)$,

(iii) $E(\Gamma(t), U(t)) \leq E(\Gamma, U(t))$, *for all* $\Gamma \subset \Gamma(t)$.

The above law immediately calls for few comments. The first condition is merely the requirement that the crack increases with time from its original state Γ_0. (We show in the next paragraph that $\Gamma(0) = \Gamma_0$.) That the energy should be minimum among all possible cracks at a given time is expressed in the second condition which can be formally derived from Law 1 upon replacing t_i by t and Γ_{i-1} by $\Gamma^-(t)$. The third condition is a comparison with the energy that the body would have if it were left in a smaller cracked state but submitted to the present load. It is a condition stronger than that introduced in Francfort and Marigo (1998).

Remark 3. *Noting that there does not exist a total order relation between all possible cracked states, Law 2 simply requires that the crack grows and gives to the body the smallest energy among the subfamily of greater or smaller cracked configurations.*

1.3 Some Elementary Properties

The energies E_d and E_s enjoy the following elementary but very useful properties, direct consequences of their definition :

P 1. E_s *is positive and strictly monotonically increasing in* Γ *: If* $\Gamma_1 \subset \Gamma_2$, *then* $0 \leq E_s(\Gamma_1) \leq E_s(\Gamma_2)$. *Moreover if* $E_s(\Gamma_1) = E_s(\Gamma_2) < +\infty$, *then* $\Gamma_1 = \Gamma_2$ *(up to a set of zero* $\mathcal{H}^{N-1}$*-measure).*

Proof. It is a direct consequence of the assumed strict positivity of the toughness. □

P 2. E_d *is positive and monotonically decreasing in* Γ *for any fixed* U *:*

$$E_d(\Gamma_1, U) \geq E_d(\Gamma_2, U) \geq 0 \quad if \quad \Gamma_1 \subset \Gamma_2.$$

Proof. If $\Gamma_1 \subset \Gamma_2$ then $\mathcal{C}(\Gamma_1) \subset \mathcal{C}(\Gamma_2)$, and since the bulk energy $\mathcal{E}_d(\Gamma, U)$ is the infimum on $\mathcal{C}(\Gamma)$, the result follows. □

P 3. E_d *is homogeneous of degree 2 in* U *:*

$$E_d(\Gamma, tU) = t^2 E_d(\Gamma, U).$$

Proof. It is a direct consequence of the infinitesimal linear elasticity assumption. □

We can immediately deduce that $\mathit{\Gamma}(0) = \mathit{\Gamma}_0$. Indeed, by P3, and *(i)* and *(iii)* of Law 2 we have $E(\mathit{\Gamma}(0), U(0)) = E_s(\mathit{\Gamma}(0)) \leq E_s(\mathit{\Gamma}_0)$. But, by *(i)* of Law 2 we have also $\mathit{\Gamma}_0 \subset \mathit{\Gamma}(0)$ and we can conclude owing to P1.

Another interesting property for the bulk energy is that it is bounded from below as a function of the crack. Let us introduce its greatest lower bound :

$$E_d^{min}(U_0) = \inf_{\mathit{\Gamma}\ :\ E_s(\mathit{\Gamma})<+\infty} E_d(\mathit{\Gamma}, U_0) \geq 0. \tag{8}$$

The value of $E_d^{min}(U_0)$ depends both on the loading, on the geometry and on the material properties. For instance, we have the following

Proposition 1. *If all the materials as well as all the interfaces are breakable,* i.e. *if $k(x) \leq k_M < \infty$ for all $x \in \overline{\mathit{\Omega}}$, then $E_d^{min}(U_0) = 0$.*

Proof. Indeed, take $\mathit{\Gamma} = \partial_d \mathit{\Omega}$ (its surface energy is finite by hypothesis on the smoothness of $\partial\mathit{\Omega}$ and on the finitude of the toughness), then $v = 0$ becomes an admissible displacement field. Therefore we deduce from (4) and P2 that $E_d(\mathit{\Gamma}, U_0) = 0$. □

In other contexts the lowest bulk energy can be strictly positive. It is the case in the pull-out problem treated in Section 3 where the interface alone is breakable, and the fiber and the matrix are clamped at one end. Moreover it could happen that this infimum is reached by no cracked state.

1.4 Elementary Examples

As a first test of the proposed formulation, three examples for which exact solutions can be computed are presented below: that of a one-dimensional bar under tension for which Law 2 is used, that of a three-dimensional cylinder under uniaxial traction (controlled by a hard device) and that of the tearing of a reinforcement in a cylindrical domain for which Law 1 is used.

Tension of a bar. Assume that $\mathit{\Omega} = (0, L)$, that the bar is homogeneous with stiffness ES and fracture toughness kS (S representing the area of the cross-section of the “real” three-dimensional body). Assume also that the bar is initially uncracked, $\mathit{\Gamma}_0 = \emptyset$, fixed at one end and submitted to an increasing displacement at the other end :

$$u(0-) = 0 \qquad u(L+) = t. \tag{9}$$

Accordingly, the set of admissible displacements reads as

$$\mathcal{C}(\mathit{\Gamma}, t) = \{v \in W^{1,2}([0, L] \setminus \mathit{\Gamma})\ :\ v(0-) = 0,\ v(L+) = t\}, \tag{10}$$

while the possible cracked configurations of the bar with finite energy are

$$\mathit{\Gamma} = \emptyset \qquad or \qquad \mathit{\Gamma} = \{x_1, \cdots, x_n\} \quad with \quad n \geq 1, \quad 0 \leq x_1 < \cdots < x_n \leq L. \tag{11}$$

Recalling that the bulk energy is given by

$$E_d(\Gamma, t) = \inf_{v \in \mathcal{C}(\Gamma, t)} \frac{1}{2} \int_0^L ESv'(x)^2 \, dx, \tag{12}$$

an easy computation would show that the displacement field, say $u(\emptyset, t)$, which achieves the infimum for the crack free bar is given by

$$u(\emptyset, t)(x) = t\frac{x}{L} \tag{13}$$

and the corresponding energies are

$$E_s(\emptyset) = 0, \qquad E(\emptyset, t) = E_d(\emptyset, t) = \frac{ESt^2}{2L}. \tag{14}$$

On the other hand, when the crack is $\Gamma = \{x_1, \cdots, x_n\}$ with $n \geq 1$, we easily find that the bulk energy vanishes and then

$$E_d(\Gamma, t) = 0 \qquad E(\Gamma, t) = E_s(\Gamma) = nkS. \tag{15}$$

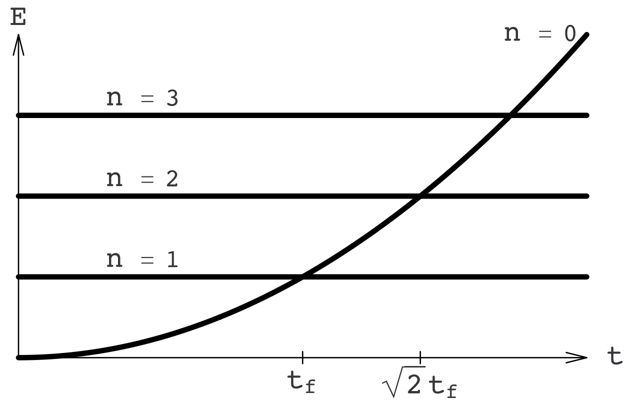


Figure 1. Graphs of the total energy for the different cracked configurations of the bar.

The graphs of the total energy corresponding to these different cracked configurations versus the load parameter are plotted in Figure 1. We deduce then the following result

Proposition 2. *The bar remains crack-free as long as the loading parameter t remains strictly less than $t_f = \sqrt{\frac{2kL}{E}}$. For $t > t_f$, the bar is cut into two pieces at an arbitrary* *point $x_1 \in [0, L]$. In other words, the (family indexed by x_1 of) crack evolution is*

$$\Gamma(t) = \begin{cases} \emptyset, & \textit{when} \quad t < t_f \\ \emptyset \textit{ or } \{x_1\} & \textit{when} \quad t = t_f \\ \{x_1\} & \textit{when} \quad t > t_f \end{cases} \tag{16}$$

Proof. ▷ The proof is divided into 2 steps.

(i) *Existence.* Let us first verify that the proposed crack evolution $t \mapsto \Gamma(t)$ satisfied the three items of Law 2. $\Gamma(0) = \emptyset$ and $\Gamma(t)$ grows with t, hence item *(i)* is satisfied. It can be easily verified on Figure 1 that, for each t, $\Gamma(t)$ is the cracked state giving the smallest energy among the set of all possible cracked states. Therefore items *(ii)* and *(iii)* are satisfied. □

(ii) *Uniqueness.* Let us show now that there does not exist another solution. Let $t \mapsto \Gamma(t)$ be an evolution verifying items *(i)* –*(iii)* and $t \mapsto n(t)$ the corresponding evolution of the number of cuts. When $t < t_f$, by choosing $\Gamma = \emptyset$ in item *(iii)* we obtain that $E(\Gamma(t), t) \leq E(\emptyset, t)$, which is only possible if $\Gamma(t) = \emptyset$. Consider now t such that $t_f < t < \sqrt{2}\, t_f$. By choosing once more $\Gamma = \emptyset$ in item *(iii)* , we deduce from Figure 1 that $0 \leq n(t) \leq 1$. But $n(t) = 0$ is impossible by *(ii)* . Indeed, if $n(t) = 0$ we could choose $\Gamma = \{x_1\}$ in *(ii)* and we should obtain $E(\emptyset, t) \leq kS$, *i.e.* a contradiction. Therefore $n(t) = 1$, the bar is cut at one point. But by virtue of *(i)* this point must be the same at each t. So $\Gamma(t) = \{x_1\}$ when $t_f < t < \sqrt{2}\, t_f$. Consider now $t \geq \sqrt{2}\, t_f$. Item *(i)* requires that $\Gamma(t) \supset \{x_1\}$. By choosing $\Gamma = \{x_1\}$, item *(iii)* requires that $E(\Gamma(t), t) \leq kS$ and therefore that $n(t) = 1$. But then *(i)* allows to conclude that $\Gamma(t) = \{x_1\}$. □ ◁

Uniaxial traction of a three-dimensional cylinder. Assume that $\Omega = S \times (0, L)$, with S smooth open connected domain of $\mathbb{R}^2$, that the material is isotropic and homogeneous with Young's modulus E, Poisson's ratio ν, and fracture toughness $k(x) = k$. Assume also that $\Gamma_0 = \emptyset$, and that $u_3 = 0$ on $S \times \{0\}$ while $u_3 = t$ on $S \times \{L\}$. In other words, the boundary conditions look like

$$\begin{cases} u_3 = 0 \textit{ and } \sigma_{13} = \sigma_{23} = 0 \text{ on } S \times \{0\}, \\ u_3 = t \textit{ and } \sigma_{13} = \sigma_{23} = 0 \text{ on } S \times \{L\}, \\ \qquad\qquad \sigma \cdot n = 0 \qquad \text{on } \partial S \times (0, L). \end{cases} \tag{17}$$

Correspondingly, the set of admissible displacements reads as

$$\mathcal{C}(\Gamma, t) = \{v \in W^{1,2}(\Omega \setminus \Gamma; \mathbb{R}^3) \;:\; v_3 = 0 \text{ in } S \times \{0\} \setminus \Gamma, v_3 = t \text{ in } S \times \{L\} \setminus \Gamma\}. \tag{18}$$

An easy computation would show that the displacement field at equilibrium for the crack free cylinder is given (up to an admissible rigid displacement) by

$$u(\emptyset, t)(x) = -\nu t \frac{x_1}{L} e_1 - \nu t \frac{x_2}{L} e_2 + t \frac{x_3}{L} e_3, \tag{19}$$

the corresponding stress field reads as

$$\sigma(\emptyset, t)(x) = E\frac{t}{L}e_3 \otimes e_3. \tag{20}$$

The corresponding energies are

$$E_s(\emptyset) = 0, \qquad E(\emptyset, t) = E_d(\emptyset, t) = \frac{1}{2}E\frac{t^2}{L}\mathcal{H}^2(S). \tag{21}$$

Remark 4. *In the absence of crack, the stress field is then uniaxial but the longitudinal extension is controlled. This type of control is essential for the questions of energy minimization, see* Charlotte *et al. (2000). We will call such a test a uniaxial traction controlled by a hard device.*

Let Γ be an arbitrary crack, denote by $P(\Gamma)$ its projection onto S and define

$$\Theta(\Gamma) = \frac{\mathcal{H}^2(P(\Gamma))}{\mathcal{H}^2(S)}. \tag{22}$$

The energy of the body in an arbitrary cracked state is not explicitly computable but an estimate from below will prove sufficient for our purpose. This is the object of the following

Proposition 3. $E(\Gamma, t) \geq (1 - \Theta(\Gamma))E(\emptyset, t) + \Theta(\Gamma)k\mathcal{H}^2(S)$. *The equality holds if and only if* $\Gamma = \emptyset$ *or* $\Gamma = S \times \{z\}$ *with* $0 \leq z \leq L$.

Proof. The equality is trivial when $\Gamma = \emptyset$. When $\Gamma = S \times \{z\}$, the equality holds because $E_d(\Gamma, t) = 0$ and $\Theta(\Gamma) = 1$. Let us consider now an arbitrary Γ. Then

$$E_s(\Gamma, t) \geq \Theta(\Gamma)k\mathcal{H}^2(S), \tag{23}$$

the inequality being strict unless $\mathcal{H}^2(P(\Gamma)) = \mathcal{H}^2(\Gamma)$, that is unless Γ is transverse. For the strain energy, classical dual methods in linear elasticity give

$$\begin{aligned} E_d(\Gamma, t) &= \inf_{v \in \mathcal{C}(\Gamma, t)} \frac{1}{2}\int_{\Omega_\Gamma} A\varepsilon(v)\cdot\varepsilon(v)dx \\ &= \inf_{v \in \mathcal{C}(\Gamma, t)} \sup_{\tau \in [L^2(\Omega_\Gamma)]^9_s} \int_{\Omega_\Gamma} (\tau \cdot \varepsilon(v) - \frac{1}{2}A^{-1}\tau\cdot\tau)dx \\ &\geq \sup_{\tau \in [L^2(\Omega_\Gamma)]^9_s} \inf_{v \in \mathcal{C}(\Gamma, t)} \int_{\Omega_\Gamma} (\tau \cdot \varepsilon(v) - \frac{1}{2}A^{-1}\tau\cdot\tau)dx \\ &\geq \inf_{v \in \mathcal{C}(\Gamma, t)} \int_{\Omega_\Gamma} (\sigma(t)\cdot\varepsilon(v) - \frac{1}{2}A^{-1}\sigma(t)\cdot\sigma(t))dx, \end{aligned} \tag{24}$$

where

$$\sigma(t) = \begin{cases} 0 & \textit{in} \quad P(\Gamma) \times (0, L) \\ \sigma(\emptyset, t) & \textit{elsewhere} \end{cases} \tag{25}$$

is an admissible stress field. Now each element of $\mathcal{C}(\Gamma,t)$ has a third component with a well-defined trace on $(S \setminus P(\Gamma)) \times \{0 \text{ (resp.} L)\}$, namely 0 (resp. t). Thus, since the only non-zero component of $\sigma(t)$ is its 3×3 component,

$$\begin{aligned}
\int_{\Omega_\Gamma} \sigma(t) \cdot \varepsilon(v) dx &= \int_{\Omega_\Gamma} \sigma_{33}(t) \cdot \frac{\partial v_3}{\partial x_3} dx \\
&= \int_{(S\setminus P(\Gamma))\times(0,L)} \sigma_{33}(\emptyset, t) \cdot \frac{\partial u_3(\emptyset,t)}{\partial x_3} dx \\
&= \int_{(S\setminus P(\Gamma))\times(0,L)} (\sigma(\emptyset,t) \cdot \varepsilon(u(\emptyset,t)) dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
E_d(\Gamma,t) \geq \int_{S\setminus P(\Gamma)\times(0,L)} &(\sigma(\emptyset,t)\cdot\varepsilon(u(\emptyset,t)) - 1/2A^{-1}\sigma(\emptyset,t)\cdot\sigma(\emptyset,t))dx \\
= (1 - \Theta(\Gamma))E(\emptyset,t), &\qquad (26)
\end{aligned}$$

which, together with (23), proves the inequality of the Proposition.

Assume that the equalities hold in (23) and (24). Then Γ is transverse, $\sigma(t)$ is the true stress field $\sigma(\Gamma,t)$, and $u(\Gamma,t)_{3,3}$ is equal to 0 in $(P(\Gamma)\times(0,L))\setminus\Gamma$ and to t/L in $(S\setminus P(\Gamma))\times(0,L)$. When $t \neq 0$, the continuity of $u(\Gamma,t)_3$ on $\partial P(\Gamma)\times(0,L)$ is then impossible, unless $P(\Gamma)=\emptyset$ or $P(\Gamma)=S$. The former case corresponds to $\Gamma=\emptyset$. In the latter case, we obtain $\sigma(\Gamma,t)=0$ and $\varepsilon(u(\Gamma,t))=0$ in $\overline{\Omega}\setminus\Gamma$. Hence $u(\Gamma,t)$ is a rigid displacement in each connected component of $\overline{\Omega}\setminus\Gamma$. By the boundary conditions there exists at least two connected components, but since Γ is transverse there exists exactly two connected components. Hence Γ is of the form $S\times\{z\}$, $0\leq z\leq L$. $\square$

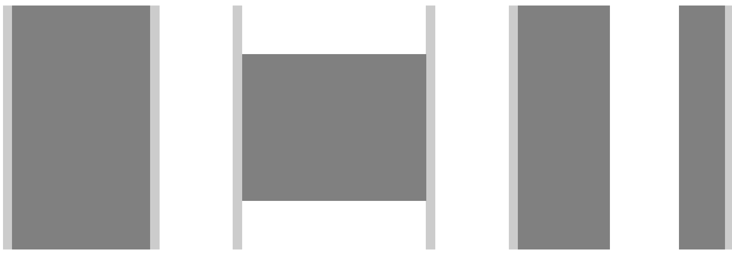


Figure 2. *Failure of the cylinder under uniaxial traction controlled by a hard device.*

We are now in a position to prove the following

Proposition 4. *The cylinder remains crack-free as long as the loading parameter t remains strictly less than $t_f = \sqrt{\frac{2kL}{E}}$. For $t > t_f$, a solution consists in cutting the cylinder into two pieces along an arbitrary transverse section.*

Furthermore these are the unique evolutions which are limits of evolutions satisfying Law 1 when the step of discretization goes to 0.

Proof. Let $z \in [0, L]$ and let $\Gamma_z(t)$ be defined for $t \geq 0$ by

$$\Gamma_z(t) = \begin{cases} \emptyset & \textit{when} \quad t < t_f \\ \emptyset \textit{ or } S \times \{z\} & \textit{when} \quad t = t_f \\ S \times \{z\} & \textit{when} \quad t > t_f \end{cases}. \tag{27}$$

$\Gamma_z(t)$ is a global minimizer of the energy:

$$E(\Gamma_z(t), t) \leq E(\Gamma, t), \qquad \forall \Gamma. \tag{28}$$

Indeed, $E(\Gamma_z(t), t) = E(\emptyset, t) \leq k\mathcal{H}^2(S)$ when $t \leq t_f$ while $E(\Gamma_z(t), t) = k\mathcal{H}^2(S) \leq E(\emptyset, t)$ when $t \geq t_f$. Thus, by virtue of Proposition 3,

$$E(\Gamma, t) \geq (1 - \Theta(\Gamma))E(\emptyset, t) + \Theta(\Gamma)k\mathcal{H}^2(S) \geq \min\{E(\emptyset, t), k\mathcal{H}^2(S)\} = E(\Gamma_z(t), t)$$

and the first inequality is strict unless $\Gamma = \emptyset$ or $\Gamma = S \times \{z'\}$ for some $z' \in [0, L]$.

Let us consider an arbitrary discretization of the loading: $0 = t_0 < t_1 < \cdots < t_n < \cdots$. It is clear that $\{\Gamma_z(t_n)\}_{n\in\mathbb{N}}$ satisfies Law 1 because $\Gamma_z(t_n)$ is increasing with n and an energy minimizer.

Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be another solution. Since $\Gamma_0 = \emptyset$, Law 1 requires that Γ_1 is a global minimizer of the energy at t_1. Hence $\Gamma_1 = \emptyset$ if $t_1 < t_f$. By repeating the same argument, as long as $t_i < t_f$ Law 1 yields $\Gamma_i = \emptyset$. Let n such that $t_{n-1} < t_f$ and $t_n > t_f$. By Law 1 Γ_n must be a global minimizer of the energy at t_n. Hence $\Gamma_n = S \times \{z\}$ for some $z \in [0, L]$. Since $\Gamma_{n+1} \supset \Gamma_n$, since $k\mathcal{H}^2(S) = E(\Gamma_n, t_{n+1})$ and since $k\mathcal{H}^2(S)$ is the minimum of the energy at t_{n+1}, Law 1 requires that $\Gamma_{n+1} = \Gamma_n$. By induction, $\Gamma_i = \Gamma_n$ for all $i \geq n$. We have then proved that $\Gamma_i = \Gamma_z(t_i)$ for all i whatever the load-discretization. (The value of z is of course dependent on the discretization, but that indetermination is by essence due to the non uniqueness of the solution.) $\square$

Tearing of a reinforcement. A three-dimensional cylinder $\Omega = S \times (0, L)$ is investigated; its cross-section S is an annulus of inner (resp. outer) radius R_i (resp. R_o); it is made of a homogeneous, isotropic elastic material with Lamé coefficients λ, μ and toughness k. A rigid, unbreakable reinforcement is bonded to the inner surface $\mathcal{C}_i$ of the cylinder. Assume that the cylinder is initially crack-free ($\Gamma(t) = \emptyset, t < 0$), and that the following load is applied:

$$u_1 = u_2 = \sigma_{33} = 0 \text{ on } S \times \{0\}, S \times \{L\};\ u = 0 \text{ on } \Sigma_o; u = te_3 \text{ on } \mathcal{C}_i, \tag{29}$$

where Σ_o denotes the outer surface of the cylinder. Therefore the set of kinematically admissible displacement fields reads as

$$\begin{aligned} \mathcal{C}(\Gamma, t) = \{\ & v \in W^{1,2}(\Omega \setminus \Gamma; \mathbb{R}^3)\ : \\ & v_1 = v_2 = 0 \text{ on } (S \times \{0, L\}) \setminus \Gamma, \\ & v = 0 \text{ on } \Sigma_o \setminus \Gamma, v = te_3 \text{ on } \mathcal{C}_i \setminus \Gamma\}, \end{aligned} \tag{30}$$

The result is the following

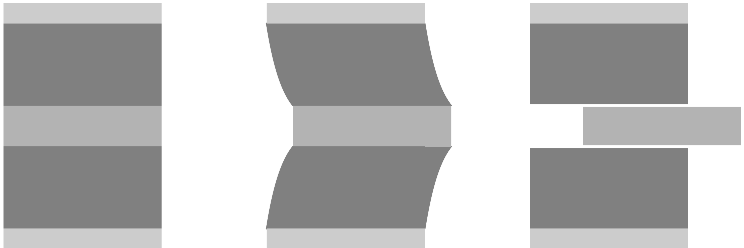


Figure 3. *Tearing of a reinforcement.*

Proposition 5. *If we consider only "axial cracks", that is cracks of the form $\Gamma = \gamma \times (0, L)$, $\gamma \subset \bar{S}$, then the unique evolution which is the limit of an evolution satisfying Law 1 when the step of discretization goes to 0 is the following:*

The cylinder remains crack-free as long as

$$t < t_i = t_f \equiv \sqrt{\frac{2kR_i}{\mu} \log(\frac{R_o}{R_i})}, \tag{31}$$

while there is total debonding of the reinforcement from the inner surface of the cylinder as soon as $t > t_f$.

Proof. The proof is very similar to that of Proposition 4; it is sketched below. We must note that, by virtue of the assumptions on the axial form of the cracks and of the assumed boundary conditions, the problem becomes an antiplane problem.

The crack-free equilibrium fields are

$$u(\emptyset, t) = \frac{\log(r/R_o)}{\log(R_i/R_o)} t e_3, \tag{32}$$

$$\sigma(\emptyset, t) = \frac{t\mu}{2r\log(R_i/R_o)}(e_r \otimes e_3 + e_3 \otimes e_r), \tag{33}$$

with corresponding energies

$$E(\emptyset, t) = E_d(\emptyset, t) = \pi\mu L \frac{t^2}{\log(R_o/R_i)}. \tag{34}$$

Let $\Gamma = \gamma \times (0, L)$ be an arbitrary axial cracked configuration. A lower bound on the energy $E(\Gamma, t)$ is first established by projecting Γ onto the cylinder with radius R_i.

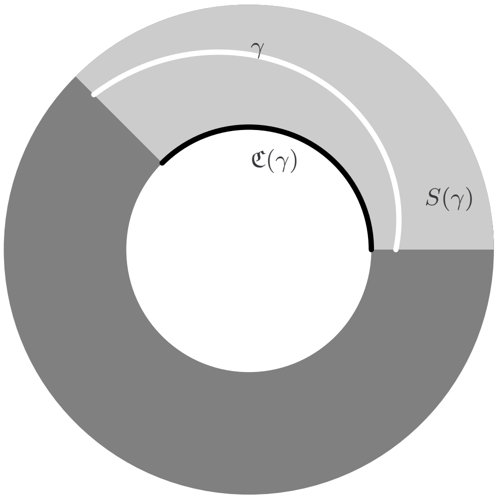


Figure 4. The trace of the axial crack on the cross section and its projection.

Let $\mathfrak{C}(\gamma)$ be the radial projection of γ onto the inner circle $\mathfrak{C}_i = \{[R_i, \theta] \ : \ 0 \leq \theta \leq 2\pi\}$[1], *i.e.*, see Figure 4,

$$\mathfrak{C}(\gamma) = \{[R_i, \theta] \ : \ \exists r \ such\ that\ [r, \theta] \in \gamma\} \subset \mathfrak{C}_i, \tag{35}$$

$\omega(\gamma)$ the length fraction of $\mathfrak{C}(\gamma)$ with respect to $\mathfrak{C}_i$:

$$\omega(\gamma) = \frac{\mathcal{H}^1(\mathfrak{C}(\gamma))}{2\pi \ R_i} \tag{36}$$

and $S(\gamma)$ the angular sector of the cross section which contains the crack, see Figure 4 :

$$S(\gamma) = \left\{[r, \theta] \in \overline{S} \ : \ [R_i, \theta] \in \mathfrak{C}(\gamma)\right\}. \tag{37}$$

We first establish the following inequalities :

$$E_s(\Gamma, t) \geq \omega(\gamma) k 2\pi R_i L, \qquad E_d(\Gamma, t) \geq (1 - \omega(\gamma))\, E_d(\emptyset, t). \tag{38}$$

The first inequality is a direct consequence of the definition of $\mathfrak{C}(\gamma)$, projection of γ onto $\mathfrak{C}_i$. Moreover the equality holds if and only if $\gamma = \mathfrak{C}(\gamma) \subset \mathfrak{C}_i$. To obtain the second inequality, we use, like in the case of the traction test, dual estimates by choosing as the statically stress field τ :

$$\tau(x) = \begin{cases} 0 & if\ x \in S(\gamma) \times (0, L), \\ \sigma(\emptyset, t) & otherwise. \end{cases} \tag{39}$$

[1] The square brackets denote polar coordinates of a point of the cross section, e.g. $[r, \theta] \in S$.

A simple calculation gives the expected inequality. We have then obtained the following estimate of the energy $E(\Gamma, t)$:

$$E(\Gamma, t) \geq (1 - \omega(\gamma))\, E(\emptyset, t) + \omega(\gamma) k 2\pi R_i L. \tag{40}$$

Furthermore, we deduce that, when $t \neq 0$, the equality holds if and only if $\gamma = \emptyset$ or $\gamma = \mathfrak{C}_i$, *i.e.* if and only if either the cylinder is crack free or the reinforcement is entirely debonded.

The result is then obtained through an argument identical to that developed in the proof of Proposition 4. □

Remark 5.

1. *The solutions given by the least energy principle are (at least qualitatively) in good agreement with the commonly observed modes of failure for these two basic tests.*
2. *In both cases, since the body is initially crack free, Griffith law is unable to assess the onset of cracking.*
3. *By using the three items of the Law 2, the solution is unique for the latter test. (The non uniqueness for the traction test is of course expected.)*
4. *In both cases the cracking is brutal.*

2 General Properties

This section investigates various characteristics of the crack evolution as predicted by the least energy principle. We compare them with those foreseen by Griffith's theory of brittle fracture in which the crack evolution is based on the criterion of the critical energy release rate, see Proposition 8 where a precise statement of Griffith's law is given (in a particular case). Our analysis is, as in the previous section, restricted to the no-contact case between the lips of the cracks, to linear elastic materials and to increasing boundary displacement loadings of the form

$$U(x, t) = \begin{cases} tU_0(x) \ , \, t \geq 0, \\ 0 \qquad\quad , \, t < 0. \end{cases}$$

2.1 Crack Initiation and Failure of the Sample

We propose to demonstrate in this subsection that the model predicts crack initiation in a crack-free environment, in striking contrast with Griffith's theory; we also evidence eventual mechanical failure under monotonic loadings. Let us denote by t_i the initiation load, *i.e.*, the load when the cracked state first evolves (recall that Γ_0 can be empty),

$$t_i = \sup\{t \ : \ \Gamma(t) = \Gamma_0\}. \tag{41}$$

By using the infimum $E_d^{min}(U_0)$ of the bulk energy introduced in (8), we immediately obtain:

Proposition 6.

$$t_i = \infty \text{ if and only if } E_d(\varGamma_0, U_0) = E_d^{min}(U_0).$$

Proof. Since $\varGamma(t) \supset \varGamma_0$ by *(i)* of Law 2, $E_s(\varGamma(t)) \geq E_s(\varGamma_0)$. If $E_d(\varGamma_0, U_0) = E_d^{min}(U_0)$, then, because of the monotonicity and the positivity of the bulk energy, $E_d(\varGamma(t), U_0) = E_d^{min}(U_0)$ for all $t > 0$. According to *(iii)* in Law 2, $E_s(\varGamma(t)) \leq E_s(\varGamma_0)$. Then the surface energies are equal and, recalling P1, $\varGamma(t) = \varGamma_0$.

Conversely, if $t_i = \infty$, then $\varGamma(t) = \varGamma_0, \forall t > 0$, and by virtue of (ii) in Law 2,

$$t^2 E_d(\varGamma_0, U_0) + E_s(\varGamma_0) \leq t^2 E_d(\varGamma, U_0) + E_s(\varGamma), \text{ for all } \varGamma \supset \varGamma_0.$$

Let $\varGamma'_n$ be a minimizing sequence for the bulk energy, *i.e.* a sequence such that

$$\lim_{n \to \infty} E_d(\varGamma'_n, U_0) = E_d^{min}(U_0).$$

Take $\varGamma_n = \varGamma_0 \cup \varGamma'_n$, then $\varGamma_n$ is also a minimizing sequence because $E_d(\varGamma_n, U_0) \leq E_d(\varGamma'_n, U_0)$. We first obtain

$$t^2 E_d(\varGamma_0, U_0) + E_s(\varGamma_0) \leq t^2 E_d(\varGamma_n, U_0) + E_s(\varGamma_n), \qquad \forall t > 0,$$

then dividing by t^2 and passing to the limit when t goes to ∞ give

$$E_d(\varGamma_0, U_0) \leq E_d(\varGamma_n, U_0)$$

which is only possible if $E_d(\varGamma_0, U_0) = E_d^{min}(U_0)$. □

Remark 6. *An immediate consequence of the above Proposition is that: if $\varGamma_0 = \emptyset$ (no preexisting crack), all materials and interfaces are breakable and U_0 is not compatible with a rigid displacement, then* **a crack will always appear at a finite load**. *(Indeed, in such a case $E_d(\emptyset, U_0) > E_d^{min}(U_0) = 0$.)*

Concerning the complete fracture, we have

Proposition 7.

(i) $\lim_{t \to \infty} E_d(\varGamma(t), U_0) = E_d^{min}(U_0)$,
(ii) If there exists a finite load t_r such that $E_d(\varGamma(t_r), U_0) = E_d^{min}(U_0)$, then $\varGamma(t) = \varGamma(t_r), \forall t \geq t_r$.

The greatest lower bound of such loads t_r is referred to as the failure load for the sample and denoted by t_f.

Proof. Let $\varGamma'_n$ be a minimizing sequence for the bulk energy. For a given n, let $\varGamma = \varGamma'_n \cup \varGamma(t)$. Therefore, (ii) in Law 2 implies that

$$\begin{aligned} t^2 E_d(\varGamma(t), U_0) + E_s(\varGamma(t)) &\leq t^2 E_d(\varGamma, U_0) + E_s(\varGamma) \\ &\leq t^2 E_d(\varGamma'_n, U_0) + E_s(\varGamma'_n) + E_s(\varGamma(t)). \end{aligned}$$

Dividing by t^2 in the above inequality and letting t tend to ∞ yields

$$\lim_{t \to \infty} E_d(\Gamma(t), U_0) \leq E_d(\Gamma'_n, U_0),$$

then passing to the limit in n gives *(i)* .

If $E_d(\Gamma(t_r), U_0) = E_d^{min}(U_0)$ for some t_r, then *(iii)* in Law 2 implies that $E_s(\Gamma(t)) \leq E_s(\Gamma(t_r))$ for $t \geq t_r$, and, since $\Gamma(t) \supset \Gamma(t_r)$, (ii) is established. $\square$

Remark 7. *This result demonstrates that, under monotonic loadings, the crack will expand until a configuration of minimum elastic energy. This can be interpreted as a configuration of mechanical failure. This, together with Remark 6, provides a rather complete picture of crack evolution in our main setting, namely that of no-contact between the lips of the cracks and linear elasticity:* **A crack will appear at a finite load and will expand until mechanical failure at a finite or infinite load**.

Remark 8. *Whenever* $0 < t_i = t_f < \infty$ *the crack evolution is as follows:*

$$\Gamma(t) = \begin{cases} \Gamma_0 & if \quad t < t_i = t_f, \\ \Gamma_f & if \quad t > t_i = t_f, \end{cases} \tag{42}$$

where Γ_f *is such that* $E_d(\Gamma_f, U_0) = E_d^{min}(U_0)$. *Such is the case in the three examples developed in Section 1.4.*

2.2 Griffith versus Energy Minimization

This subsection is devoted to a comparison between Griffith's theory and the least energy principle. The framework is that of two-dimensional linear elasticity. The following is assumed throughout this subsection:

Hypothesis (Predefined crack path). *The crack path is a rectifiable curve* Γ *of finite length* L *with origin* x_0 *and end point* x_L. *It is parameterized by its arc-length* $x(s)$ *with* $x(0) = x_0$. *Define*

$$\Gamma(l) \equiv \Gamma_0 \cup \{x(s); 0 \leq s \leq l\}. \tag{43}$$

Then the bulk and surface energies become functions of l, *namely,*

$$\mathcal{E}(l) \equiv E_d(\Gamma(l), U_0), \qquad \mathcal{G}(l) \equiv E_s(\Gamma(l)). \tag{44}$$

These functions are respectively monotonically decreasing and increasing with l; *we also assume that they are as smooth as needed.*

It now remains to investigate the trajectory of the crack along its path, *i.e.*, the function $t \to l(t)$. The first result concerns progressive crack evolution.

Proposition 8. *If* $l(t)$ *is an absolutely continuous function of* t, *then the crack evolution follows Griffith's law. In other words,*

(i) $l(0) = 0$,
(ii) $\dot{l}(t) \geq 0$,
(iii) $-t^2\mathcal{E}'(l(t)) \leq k(x(l(t)))$,
(iv) $\dot{l}(t) = 0$ *if the inequality is strict in (iii)* .

Proof. Items *(i)* and *(ii)* are obvious. Item *(iii)* is obtained by applying *(ii)* of Law 2 at load t, with $\Gamma(l(t)+h), h \geq 0$ as test crack:

$$t^2\mathcal{E}(l(t)) + \int_0^{l(t)} k(x(s))ds \leq t^2\mathcal{E}(l(t)+h) + \int_0^{l(t)+h} k(x(s))ds. \tag{45}$$

Dividing by h and passing to the limit in h give *(iii)* .

Item *(iv)* is obtained by applying *(iii)* of Law 2 at load t with $\Gamma(l(t-h))$ as test crack:

$$t^2\mathcal{E}(l(t)) + \int_0^{l(t)} k(x(s))ds \leq t^2\mathcal{E}(l(t-h)) + \int_0^{l(t-h)} k(x(s))ds. \tag{46}$$

Dividing by h and passing to the limit in h give $\Big(t^2\mathcal{E}'(l(t)) + k\big(x(l(t))\big)\Big)\dot{l}(t) \leq 0$. By using items *(ii)* and *(iii)* of Griffith's law the result follows. □

Remark 9. *Item (iii) in the above Proposition is precisely Griffith's criterion, since $-t^2\mathcal{E}'(l(t))$ represents the energy release rate. Item (iv) is the classical statement that propagation does not take place unless the energy release rate becomes critical.*

If now $l(t)$ experiences a jump at load t_0, the following Proposition holds:

Proposition 9 (The jump condition). *If at load t_0, $l^-(t_0) \neq l^+(t_0)$, then,*

$$-t_0^2\Big(\mathcal{E}(l^+(t_0)) - \mathcal{E}(l^-(t_0))\Big) = \mathcal{G}(l^+(t_0)) - \mathcal{G}(l^-(t_0)). \tag{47}$$

Proof. Apply *(ii)* of Law 2 to load $t_0 - h$ with $\Gamma(l(t_0+h))$ as test crack; then,

$$(t_0-h)^2\mathcal{E}(l(t_0-h)) + \mathcal{G}(l(t_0-h)) \leq (t_0-h)^2\mathcal{E}(l(t_0+h)) + \mathcal{G}(l(t_0+h)), \tag{48}$$

and passing to the limit in h yields one inequality. The reverse inequality is obtained by applying *(iii)* of Law 2 to $t = t_0 + h$, taking $\Gamma(l(t_0-h))$ as test crack, so that

$$(t_0+h)^2\mathcal{E}(l(t_0+h)) + \mathcal{G}(l(t_0+h)) \leq (t_0+h)^2\mathcal{E}(l(t_0-h)) + \mathcal{G}(l(t_0-h)) \tag{49}$$

which yields the result upon passing to the limit in h. □

Remark 10. *The above condition ensures nothing but the continuity of the total energy with respect to the load parameter : even when the crack evolves brutally, the finite release of strain energy is just used to produce the needed finite surface energy. The reader must note that this "energy conservation" during a brutal evolution of a crack is deduced from the least energy principle, it is thus an optimality condition and not an extra condition like in* Hashin *(1996). In a homogeneous medium it can be reinterpreted as a propagation criterion for the mean energy release rate, i.e.,*

$$\frac{-t_0^2}{l^+(t_0) - l^-(t_0)} \int_{l^-(t_0)}^{l^+(t_0)} \frac{d\mathcal{E}}{dl}(l)dl = k. \tag{50}$$

2.3 Progressive Evolution and Brutal Evolution

In this subsection, we investigate the circumstances that preside over the brutal versus progressive character of crack growth. We always adopt the assumption of "predefined crack path", as detailed in the previous subsection, and we reparameterize the crack path in terms of its surface energy by setting

$$\lambda = \mathcal{G}(l), \qquad \hat{\mathcal{E}}(\lambda) = \mathcal{E}(\mathcal{G}^{-1}(\lambda)); \tag{51}$$

such a change is licit since, in view of Remark 1, $\mathcal{G}$ is a continuous strictly increasing function of l. Note that $\hat{\mathcal{E}}$ is a monotonically decreasing function of λ in view of (44).

Remark 11. *In the present context the evolution law described in Law 2 requires that the crack trajectory $t \to \lambda(t)$ must be such that*
(i) $\lambda(t)$ *is a monotonically increasing function of* t*,*
(ii) $t^2\hat{\mathcal{E}}(\lambda(t)) + \lambda(t) \le t^2\hat{\mathcal{E}}(\lambda) + \lambda, \quad \forall \lambda \ge \lambda^-(t)$*,*
(iii) $t^2\hat{\mathcal{E}}(\lambda(t)) + \lambda(t) \le t^2\hat{\mathcal{E}}(\lambda) + \lambda, \quad \forall \lambda \le \lambda(t)$*.*

We immediately obtain the following

Proposition 10. *If $\hat{\mathcal{E}}$ is continuous, then $\lambda(t)$ lives, for each t, among the minimizers of $t^2\hat{\mathcal{E}}(\lambda) + \lambda$ on $[0, \mathcal{G}(L)]$.*

Proof. A minimizer exists for each t because the energy function is continuous on a compact set. Let $\lambda(t)$ be (one of) the minimizer(s) of $t^2\hat{\mathcal{E}}(\lambda) + \lambda$ on $[0, \mathcal{G}(L)]$. Of course, it satisfies *(ii)* and *(iii)* . Moreover by using the optimality of $\lambda(t_1)$ and $\lambda(t_2)$ at t_1 and t_2 respectively, we get

$$t_1^2\hat{\mathcal{E}}(\lambda(t_1)) + \lambda(t_1) \le t_1^2\hat{\mathcal{E}}(\lambda(t_2)) + \lambda(t_2),$$
$$t_2^2\hat{\mathcal{E}}(\lambda(t_2)) + \lambda(t_2) \le t_2^2\hat{\mathcal{E}}(\lambda(t_1)) + \lambda(t_1).$$

Adding the two inequalities yields $(t_1^2 - t_2^2)\big(\hat{\mathcal{E}}(\lambda(t_1)) - \hat{\mathcal{E}}(\lambda(t_2))\big) \le 0$ and the required growth *(i)* of $\lambda(t)$ follows from the monotonicity of $\hat{\mathcal{E}}$.

Conversely, if $\lambda(t)$ verifies *(i)* –*(iii)* , then it is a minimizer (note that $\lambda^-(t) \le \lambda(t)$ by *(i)*). □

The brutal versus progressive character of crack growth is then intimately linked to the convexity properties of $\hat{\mathcal{E}}$. Specifically, we obtain the following

Proposition 11.

(i) If $\hat{\mathcal{E}}$ is $\mathcal{C}^1$ and strictly convex, then $\lambda(t)$ is given by

$$\lambda(t) = \begin{cases} 0 & if\ 0 \le t \le t_i, \\ \hat{\mathcal{E}}'^{-1}(-\frac{1}{t^2}) & if\ t_i \le t \le t_L, \\ \mathcal{G}(L) & if\ t_L \le t, \end{cases}, \tag{52}$$

with

$$\begin{cases} t_i = \sqrt{1/(-\hat{\mathcal{E}}'(0))}, \\ t_L = \sqrt{1/(-\hat{\mathcal{E}}'(\mathcal{G}(L)))}; \end{cases} \tag{53}$$

in other words, the crack growth is progressive,

(ii) If $\hat{\mathcal{E}}$ is concave, then $\lambda(t)$ is given by

$$\lambda(t) = \begin{cases} 0 & if\ 0 \le t < t_i = t_L, \\ \mathcal{G}(L) & if\ t_i = t_L < t, \end{cases} , \tag{54}$$

with

$$t_L \equiv \sqrt{\mathcal{G}(L)/(\hat{\mathcal{E}}(0) - \hat{\mathcal{E}}(\mathcal{G}(L)))}; \tag{55}$$

in other words, the crack growth is brutal.

Proof. If $\hat{\mathcal{E}}$ is convex and $\mathcal{C}^1$, $t^2\hat{\mathcal{E}}(\lambda) + \lambda$ attains its minimum at $\lambda = 0$ if $t^2\hat{\mathcal{E}}'(0) + 1 \ge 0$, at $\mathcal{G}(L)$ if $t^2\hat{\mathcal{E}}'(\mathcal{G}(L)) + 1 \le 0$, and at λ such that $t^2\hat{\mathcal{E}}'(\lambda) + 1 = 0$ otherwise ; application of Proposition 10 then yields *(i)* . If however $\hat{\mathcal{E}}$ is concave, $t^2\hat{\mathcal{E}}(\lambda) + \lambda$ can only attain its minimum at $\lambda = 0$ if $t < t_L$, and at $\lambda = \mathcal{G}(L)$ if $t > t_L$; application of Proposition 10 then yields *(ii)* . □

Remark 12. *Whenever $\hat{\mathcal{E}}$ is not convex, the minima of $t^2\hat{\mathcal{E}}(\lambda) + \lambda$ will be those of $t^2\hat{\mathcal{E}}^{**}(\lambda) + \lambda$, where $\hat{\mathcal{E}}^{**}$ denotes the convexification of $\hat{\mathcal{E}}$. Thus, the crack evolution will be progressive on the strictly convex parts of the graph of $\hat{\mathcal{E}}^{**}$ and brutal on the linear parts of that graph.*

When comparing the above results to those yielded by Griffith's theory, it is easily seen that the results are identical in the strictly convex case while in the concave case Griffith's criterion will actually predict initiation at load $t_G = \sqrt{1/(-\hat{\mathcal{E}}'(0))}$ (cf. (iv) of Proposition 8), but then, the criterion will not be met for $t > t_G$ by any λ. The crack is then baptized "unstable" and henceforth intractable. Note also that the predicted initiation load t_G is greater than that, namely t_L, obtained through our formulation. If, in general, $\hat{\mathcal{E}}'(0) = 0$, Griffith's theory will not permit crack initiation (cf. (iv) of Proposition 8), whereas ours will generate brutal growth, as is immediately deduced from the preceding results.

3 An Application : The Pull-Out Problem

The fiber pull-out problem is a particularly interesting case so as to illustrate the properties presented in the previous section. However, we treat it in a very simplified context by neglecting internal stresses and friction. The reader interested by a more realistic approach should refer to Hutchinson and Jensen (1990) and Kerans and Parthasarathy (1991).

3.1 Statement of the Problem

We consider a straight cylindrical beam of length L with cross circular section of radius R reinforced by a centered fiber with circular cross section of radius R_f (see Figure 5). We denote by $S = \{(x_1, x_2) \ : \ r < R\}$ and $\Omega = S \times (0, L)$ the section and the domain,

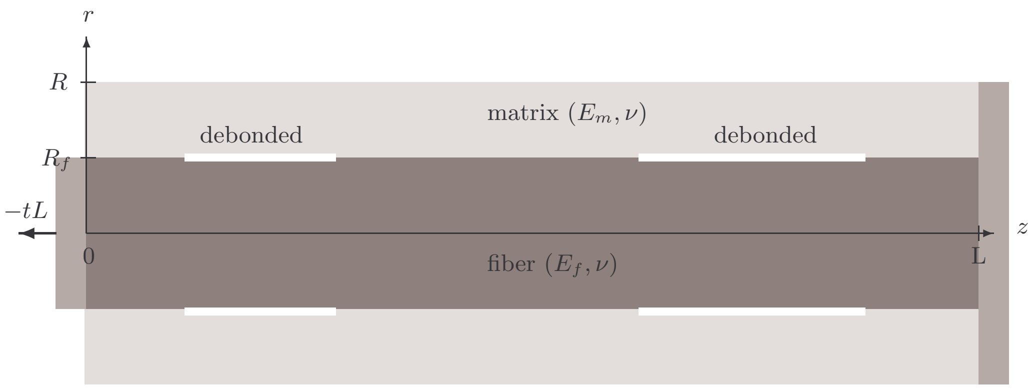


Figure 5. *Geometry and loading of the pull-out problem.*

as well as S_f and $V_f = R_f^2/R^2$ (respectively, S_m et V_m) the corresponding section and the volume fraction of the fiber (respectively, of the matrix). Both fiber and matrix are assumed to be perfectly bonded before loading and made of linearly elastic, unbreakable and isotropic materials with respective Young moduli E_f and E_m, while Poisson ratios are equal $\nu_f = \nu_m = \nu$. Their toughness k_f and k_m is then infinite, only the interface is breakable, its toughness being k. The beam is clamped on its section $S \times \{L\}$ while a longitudinal displacement $-tL$ is prescribed on $S_f \times \{0\}$, t increasing from 0 and playing the role of a loading parameter. The system is free of body forces and the lateral surface and the extremity $S_m \times \{0\}$ are not submitted to surface forces. Moreover we assume that the lips of the debonded surfaces are never in contact. Due to the symmetry of the problem we only consider *axisymmetric* debonding along the interface I. Therefore the debonding state is defined by a characteristic function χ defined on $[0, L]$ which takes the value 0 at sections z where fiber and matrix are bonded and the value 1 elsewhere. We denote by

$$I_\chi = \left\{x = (x_1, x_2, x_3) \ : \ x_1^2 + x_2^2 = R_f^2, \quad \chi(x_3) = 1\right\} \tag{56}$$

the debonded part of the interface I and Ω_χ refers to the beam Ω in “the cracked configuration χ”. The set of the kinematically admissible displacements, denoted $\mathcal{C}(\chi)$, is :

$$\mathcal{C}(\chi) = \left\{\mathbf{v} \in W^{1,2}(\Omega_\chi; \mathbb{R}^3) : \ v_z = -1 \textit{ on } S_f \times \{0\}, \ \mathbf{v} = 0 \textit{ on } S \times \{L\}\right\}. \tag{57}$$

The pull-out problem consists in finding the load-dependent characteristic function $\chi(t)$ of the debonded zone which has to satisfy

Law 3. Debonding evolution:

(i) $\chi(t)$ *is increasing with* t, $\quad \chi(0) = 0$,

(ii) $t^2 E_d(\chi(t)) + E_s(\chi(t)) \le t^2 E_d(\chi) + E_s(\chi), \qquad \forall \chi \ge \chi(t-)$,

(iii) $t^2 E_d(\chi(t)) + E_s(\chi(t)) \le t^2 E_d(\chi) + E_s(\chi), \qquad \forall \chi \le \chi(t)$,

where $E_d(\chi)$ is the elastic energy stored in the χ-damaged beam at equilibrium under a *unit* load and $E_s(\chi)$ is the Griffith surface energy

$$E_d(\chi) = \min_{\mathbf{v} \in \mathcal{C}(\chi)} \int_{\Omega_\chi} \tfrac{1}{2} A\varepsilon(v) \cdot \varepsilon(v) dx, \tag{58}$$

$$E_s(\chi) = \kappa \int_0^L \chi(z) dz, \qquad \kappa = 2\pi R_f k. \tag{59}$$

3.2 First Properties

Since the fiber and the matrix are unbreakable, the greatest crack with finite surface energy corresponds to the entire debonding of the interface, hence the minimum of the bulk energy is reached when $\chi(z) = 1$, $\forall z$. Let us show that the bulk energy of the beam when the interface is perfectly bonded is greater than this lower bound :

$$E_d(0) > E_d(1) = E_d^{min}. \tag{60}$$

The inequality $E_d(0) \ge E_d(1)$ holds because of the monotonicity of E_d. Let $u(0)$ and $u(1)$ be the displacement fields which achieve the minimum in (58) when $\chi = 0$ and $\chi = 1$, respectively. Assume that $E_d(0) = E_d(1)$, then, since $u(0) \in \mathcal{C}(1)$ and since the minimizer of $E_d(1)$ is unique, we should have $u(0) = u(1)$. Let $u_f(1)$ and $u_m(1)$ be the restrictions of $u(1)$ to the fiber and matrix, respectively. Since these two fields are linked by no relation and since $u_m(1)$ is only submitted to homogeneous boundary condition, then $u_m(1) = 0$. On the other hand, $u_m(0)$ and $u_f(0)$ are equal along the interface. Since $u_f(0)_z = -1$ at $z = 0$, $u_m(0)$ cannot be identically zero. It is a contradiction, thus $E_d(0) \ne E_d(1)$.

We deduce from Proposition 6 that debonding will appear at the interface at a finite load t_i. Moreover Proposition 7 says that the debonding will continue until the entire surface is debonded. If the load of total debonding t_f is finite, then the fiber will behave elastically for $t > t_f$ (whereas the matrix is unstrained).

The main issues are to find accurate estimates for t_i, t_f and for the debonding process in the interval (t_i, t_f). The next subsections are devoted to this task.

3.3 A First Approximation for Slender Beams

We now consider slender beams and we introduce the small parameter ϵ :

$$\epsilon = R/L. \tag{61}$$

The asymptotic bulk energy. Let χ be a given debonding state. Following the asymptotic analysis of Geymonat et al. (1987) for slender structures made of an unidirectional fibrous composite, it is shown that the displacement field $\mathbf{u}^\epsilon$ which achieves the minimum in (58) converges (when ϵ goes to 0) to a displacement field $\mathbf{u}^0 = u^0\,\mathbf{e}_3$ which satisfies the classical Navier-Bernoulli's kinematical assumptions ($\varepsilon_{rr}(\mathbf{u}^0) = \varepsilon_{\theta\theta}(\mathbf{u}^0) = \varepsilon_{rz}(\mathbf{u}^0) = 0$) and takes the form

$$u^0(r,z) = \begin{cases} U_m(z) \text{ when } & r > R_f, \\ U_f(z) \text{ when } & r < R_f, \end{cases} \tag{62}$$

U_m and U_f representing the axial displacement of the matrix and the fiber, respectively. Moreover, these displacements have to be equal in the bonded zone and thus they satisfy

$$(1-\chi(z))(U_f(z) - U_m(z)) = 0, \quad \forall z \in [0, L]. \tag{63}$$

Furthermore the elastic energy $E_d^\epsilon(\chi)$ converges to $E_d^0(\chi)$ which is obtained by solving the following minimization problem, the minimizer of which is just (U_f, U_m)

$$E_d^0(\chi) = \min_{(V_f,V_m)\in\mathcal{C}^0(\chi)} \frac{1}{2}\int_0^L \Big(A_f V_f'^2 + A_m V_m'^{\,2}\Big)dz. \tag{64}$$

Here $\mathcal{C}^0(\chi)$ denotes the set of associated admissible Navier-Bernoulli displacements,

$$\mathcal{C}^0(\chi) = \Big\{(V_f, V_m) \in W^{1,2}(0,L)^2 \ : \ V_f(0) = -1, V_f(L) = V_m(L) = 0, \\ (1-\chi(z))(V_f(z) - V_m(z)) = 0 \ a.e.\Big\}, \tag{65}$$

whereas A_f and A_m represent the stiffness of the fiber and matrix cross section, respectively, that is :

$$A_f = \pi R_f^2 E_f, \qquad A_m = \pi(R^2 - R_f^2)E_m. \tag{66}$$

Let us denote by $\ell(\chi)$ the length of the debonded zone : $\ell(\chi) = \int_0^L \chi(z)dz$. So as to solve

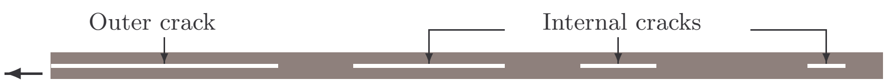


Figure 6. *Outer crack, internal cracks and bonded zone.*

(64) and to obtain the minimizer (U_f, U_m), we divide the interval $(0, L)$ into three parts : $I_d^1(\chi) = (0, \ell_1(\chi))$, $I_d^2(\chi)$ and $I_c(\chi)$. The first corresponds to the debonded zone in the neighborhood of $z = 0$ (it is called *the outer crack*); the second, empty if $\ell_1(\chi) = \ell(\chi)$, corresponds to the remainder of the debonded zone, made of *the internal cracks*; the third,

empty if $\ell(\chi) = L$, corresponds to the bonded zone. The Euler equations corresponding to the equilibrium equations read as :

$$In\ I_c(\chi) : (A_f + A_m)U_f'(z) = F, \quad U_f(z) = U_m(z) \tag{67}$$

$$In\ I_d^1(\chi) : A_f U_f'(z) = F, \quad A_m U_m'(z) = 0 \tag{68}$$

$$In\ I_d^2(\chi) : A_f U_f'(z) = G, \quad A_m U_m'(z) = F - G, \tag{69}$$

where F and G are two constants. From (69) we deduce that $U_f' - U_m'$ is constant on the internal cracks, but, since they are equal at the ends of these inner cracks (by continuity), we get $U_f = U_m$ on $I_d^2(\chi)$, *i.e.* the fiber does not slip along the matrix along the internal cracks. The system becomes

$$A_f U_f'(z) = F, \quad A_m U_m'(z) = 0, \quad \text{when} \quad 0 < z < \ell_1(\chi), \tag{70}$$

$$(A_f + A_m)U_f'(z) = F, \quad U_f(z) = U_m(z), \quad \text{when} \quad \ell_1(\chi) < z < L, \tag{71}$$

with the boundary conditions : $U_f(0) = -1$, $U_m(L) = U_f(L) = 0$, and the continuity at $z = \ell_1(\chi) : U_m(\ell_1(\chi)) = U_f(\ell_1(\chi))$. The determination of F is now straightforward and we finally get

$$E_d^0(\chi) = \frac{F}{2} = \frac{1}{2} \frac{A_f(A_f + A_m)}{A_f L + A_m \ell_1(\chi)}. \tag{72}$$

Remark 13. *The simplicity of the expression of the stiffness coefficients and of the bulk energy is essentially due to the assumption that the Poisson ratio is constant and that the lips of the crack are free. The same coefficients can be obtained in a closed (but intricate) form in a more general context and the interested reader is referred to* Bilteryst *and* Marigo *(2002).*

The asymptotic debonding problem. By approximating the true elastic energy $E_d^\epsilon(\chi)$ of the beam in a debonding state χ by its asymptotic limit $E_d^0(\chi)$, the "limit" debonding evolution $t \mapsto \chi^0(t)$ must satisfy :

Law 4. Asymptotic debonding evolution:

(i) $\chi^0(t)$ *is increasing with* t, $\quad \chi^0(0) = 0$,

(ii) $t^2 E_d^0(\chi^0(t)) + E_s(\chi^0(t)) \leq t^2 E_d^0(\chi) + E_s(\chi), \quad \forall \chi \geq \chi^0(t-)$,

(iii) $t^2 E_d^0(\chi^0(t)) + E_s(\chi^0(t)) \leq t^2 E_d^0(\chi) + E_s(\chi), \quad \forall \chi \leq \chi^0(t)$.

The asymptotic solution. Let $\ell \in [0, L]$ and let χ be a characteristic function such that

$$\int_0^L \chi(z) dz = \ell. \tag{73}$$

The characteristic function of the interval $[0, \ell]$ is noted $\chi_{[0,\ell]}$ ($\chi_{[0,\ell]}(z)$ equals 1 when $0 \leq z \leq \ell$ and 0 elsewhere). Since only the length of the outer crack appears in (72), we have

$$E_d^0(\chi) = E_d^0(\chi_{[0,\ell_1(\chi)]}) \geq E_d^0(\chi_{[0,\ell]}) \equiv \mathcal{E}_0(\ell) \tag{74}$$

and equality holds only if $\chi = \chi_{[0,\ell]}$.

Let us prove that, if there exists a solution, it corresponds to an outer crack, *i.e.*

$$\chi^0(t) = \chi_{[0,\ell^0(t)]}. \tag{75}$$

Indeed, let $\chi^0(t)$ be a solution and let $\ell^0(t) = \ell_1(\chi^0(t))$ the length of its outer crack. Using $\chi_{[0,\ell^0(t)]}$ as a test function in *(iii)* of Law 4 and referring to (74) yields

$$\begin{aligned} t^2 E_d^0(\chi^0(t)) + E_s(\chi^0(t)) &\leq t^2 E_d^0(\chi_{[0,\ell^0(t)]}) + E_s(\chi_{[0,\ell^0(t)]}) \\ &= t^2 E_d^0(\chi^0(t)) + E_s(\chi_{[0,\ell^0(t)]}). \end{aligned} \tag{76}$$

Since, on the other hand, $\ell(\chi^0(t)) \geq \ell^0(t)$, the equality holds and since $\chi^0(t) \geq \chi_{[0,\ell^0(t)]}$, Eq. (75) follows.

Thus we are exactly arrived at the situation of a predefined crack path like in Section 2.3. Since $\mathcal{E}_0$ is a strictly convex function of ℓ, Propositions 10 and 11 yield $\ell^0(t)$ as the unique minimizer of $t^2\mathcal{E}_0(\ell) + \kappa\ell$. The minimum is at 0 as long as $t^2\mathcal{E}_0'(0) + \kappa \geq 0$, at L as soon as $t^2\mathcal{E}_0'(1) + \kappa \leq 0$, and at ℓ such that $t^2\mathcal{E}_0'(\ell) + \kappa = 0$ otherwise. Thus the debonding length is given by :

$$\ell^0(t) = \begin{cases} 0 & if \quad 0 \leq t \leq t_i^0 \\ \dfrac{t - t_i^0}{t_r^0 - t_i^0} & if \quad t_i^0 \leq t \leq t_f^0 \\ 1 & if \quad t \geq t_f^0 \end{cases} \tag{77}$$

where t_i^0 and t_f^0 are given by

$$t_i^0 = \sqrt{\frac{4\pi R_f k L^2 A_f}{(A_m + A_f)A_m}}, \qquad t_f^0 = \sqrt{\frac{4\pi R_f k L^2 (A_f + A_m)}{A_m A_f}}. \tag{78}$$

It remains to verify that this candidate satisfies all the items of Law 4. The growth of $\ell^0(t)$ is verified.

Let us check that *(ii)* of Law 4 is satisfied. Remark that ℓ^0 is a continuous function of t and let $\chi \geq \chi_{[0,\ell^0(t)]}$. Since $\ell^0(t)$ is a minimizer of $t^2\mathcal{E}_0(\ell) + \kappa\ell$, we have

$$\begin{aligned} t^2 E_d^0(\chi_{[0,\ell^0(t)]}) + E_s(\chi_{[0,\ell^0(t)]}) &= t^2\mathcal{E}_0(\ell^0(t)) + \kappa\ell^0(t) \\ &\leq t^2\mathcal{E}_0(\ell_1(\chi)) + \kappa\ell_1(\chi) \\ &= t^2 E_d^0(\chi) + \kappa\ell_1(\chi) \\ &\leq t^2 E_d^0(\chi) + E_s(\chi), \end{aligned}$$

which is the required inequality.

Let us check that *(iii)* of Law 4 is satisfied. Let $\chi \leq \chi_{[0,\ell^0(t)]}$. By using the optimality of $\ell^0(t)$ and (74), we obtain

$$\begin{aligned} t^2 E_d^0(\chi_{[0,\ell^0(t)]}) + E_s(\chi_{[0,\ell^0(t)]}) &= t^2\mathcal{E}_0(\ell^0(t)) + \kappa\ell^0(t) \\ &\leq t^2\mathcal{E}_0(\ell(\chi)) + \kappa\ell(\chi) \\ &= t^2 E_d^0(\chi_{[0,\ell(\chi)]}) + E_s(\chi) \\ &\leq t^2 E_d^0(\chi) + E_s(\chi), \end{aligned}$$

which is the required inequality.

We have established the

Proposition 12 (Asymptotic solution of the debonding). *The asymptotic pull-out problem (see Law 4), admits a unique solution*

$$\chi^0(t) = \chi_{[0,\ell^0(t)]}$$

with $\ell^0(t)$ given by (77) *and* (78) *which is interpreted as follows :*

When the load reaches the critical value t_i^0 defined by $-{t_i^0}^2\mathcal{E}_0'(0) = \kappa$, then Griffith's condition of propagation is satisfied and a crack appears at $z = 0$. Then its length $\ell^0(t)$ grows with the loading t in such a manner that Griffith's condition of propagation is always fulfilled, i.e. *the equality $-t^2\mathcal{E}_0'(\ell^0(t)) = \kappa$ holds. It reaches the end $z = 1$ when the load attains the critical value t_f^0 given by $-{t_f^0}^2\mathcal{E}_0'(1) = \kappa$, after which the fiber behaves elastically since the end $z = 1$ is assumed to be clamped.*

3.4 Possible Scenarios

We return now to the 3D problem and, since we do not know enough about $\mathcal{E}(\chi)$ for an arbitrary characteristic function, we will only consider outer cracks by assuming that $\chi(t) = \chi_{[0,\ell(t)]}$. According to the previous asymptotic result, this is not too restrictive an assumption. The bulk energy of a such debonding configuration is denoted by $\mathcal{E}(\ell)$:

$$\mathcal{E}(\ell) \equiv E_d(\chi_{[0,\ell]}). \tag{79}$$

By virtue of Proposition 10, the problem is then reduced to the determination of $\ell(t)$ such that

$$t^2\mathcal{E}(\ell(t)) + \kappa\ell(t) \leq t^2\mathcal{E}(\ell) + \kappa\ell \quad \forall\ell \in [0, L]. \tag{80}$$

For small values of ℓ, it is shown in Leguillon (1989) that the presence of a singularity at the interface on $S\times\{0\}$ leads to the following expansion of the elastic energy:

$$\mathcal{E}(\ell) = \mathcal{E}(0) - C\ell^{2\alpha} + o(\ell^{2\alpha}), \tag{81}$$

where α is the real part of the power of the singularity of the displacement field and C is a positive factor depending on the applied load and on the shape of the defect, but not on α.

In our case it turns out that $\alpha > 0.5$ when the fiber is stiffer than the matrix ($E_f > E_m$) while $\alpha < 0.5$ when the matrix is stiffer than the fiber ($E_f < E_m$), see Leguillon et al. (2000). Therefore, when the fiber is stiffer than the matrix the energy release rate is of the order of $\ell^{2\alpha-1}$ and tends to 0 with ℓ, whereas when the matrix is stiffer than the fiber the energy release rate tends to $-\infty$ when $\ell \to 0$:

$$\mathcal{E}'(0) = 0 \quad if \quad E_f > E_m, \qquad \mathcal{E}'(0) = -\infty \quad if \quad E_f < E_m. \tag{82}$$

The main features. We recall or establish now a series of properties (several of them are already contained in Section 2) concerning this problem; these do not require a precise expression for $\mathcal{E}(\ell)$. We simply use the fact that $\mathcal{E}$ is a continuously differentiable and (strictly) decreasing function of ℓ.

(i): ***A solution of*** (80) ***exists and*** $\ell(t)$ ***increases with*** t. The existence is ensured because one has to minimize a continuous function on a compact set. However uniqueness is not ensured.

(ii): ***The solution of*** (80) ***is not trivial, a debonded zone will appear at a load*** $t_i, 0 \leq t_i < \infty$.

(iii): ***The debonding process continues until the interface is entirely debonded***.

(iv): ***At each load, Griffith's criterion holds :***$-t^2\mathcal{E}'(\ell(t)) \leq \kappa$. It suffices to apply (80) at t with $\ell = \ell(t) + h$ and $h > 0$, to divide by h, and to pass to the limit when h tends to 0.

(v): ***At each load*** t ***where the debonding evolves continuously, Griffith's condition of propagation holds :*** $-t^2\mathcal{E}'(\ell(t)) = \kappa$. See Proposition 8.

(vi): ***The total energy is a continuous function of*** t. See Proposition 9.

The above properties deserve several remarks. Thus, (i)–(iii) demonstrate the advantage of the minimization principle in investigating the debonding process. However, the principle is intimately related to Griffith's law as it is showed by (iv) and (v). The fundamental difference lies in the ability of the minimization principle to take into account discontinuous debonding for which a new criterion emerges from energy continuity (vi).

(vii): ***If the fiber is stiffer than the matrix, then the onset of debonding starts at*** $t_i > 0$ ***and it is brutal, whereas, if the matrix is stiffer than the fiber, then the debonding starts at*** $t = 0$ ***and grows continuously at first***.

Proof. Assume that $E_f > E_m$. We know by (ii) that a debonding will appear at t_i, but it cannot appear and grow smoothly from $\ell = 0$, because it should satisfy Griffith's condition of propagation by (v), which is impossible since $\mathcal{E}'(0) = 0$. Thus at t_i, a crack of finite length, say ℓ_i, appears. Formally : $\ell(t_i-) = 0$, $\ell(t_i+) = \ell_i > 0$. By (vi), t_i and ℓ_i must satisfy $t_i^2\mathcal{E}(\ell_i) + \kappa\ell_i = t_i^2\mathcal{E}(0)$ which forces $t_i > 0$.

Assume now that $E_m > E_f$. Since $\mathcal{E}'(0) = -\infty$, (iv) yields $t_i = 0$. Moreover, if the onset was brutal at $t = 0$, we should have by (vi) $\ell(0+) = 0$ which is a contradiction. $\square$

Another important property is the stability of the minimization principle with respect to the limit process when ϵ goes to 0. By reintroducing the parameter ϵ, the elastic energy and the debonding length are denoted by $\mathcal{E}_\epsilon$ and ℓ^ϵ. Since $\mathcal{E}_\epsilon$ converges uniformly to $\mathcal{E}_0$, then the following convergence result holds :

(viii): $\ell^\epsilon(t)$ ***converges to*** $\ell^0(t)$ ***when*** $\epsilon \to 0$. For a given t, since $\ell^\epsilon(t)$ is bounded, it converges to a limit, say $\ell^*(t)$ (*a priori* up to the possible extraction of subsequences, but, since the limit is unique, the whole sequence really converges). We can pass to the limit in each term of (80) and immediately see that $\ell^*(t)$ is necessarily the minimizer of $t^2\mathcal{E}_0(\ell) + \kappa\ell$, namely $\ell^0(t)$.

The case of a stiffer fiber. We consider the case when the fiber is stiffer than the matrix. We cannot continue the analysis of the debonding process without additional information concerning the detailed behavior of the elastic energy. Since there are many parameters in the problem, it is not possible to investigate all possibilities. We first remark that $\mathcal{E}_\epsilon$ is necessarily concave in a neighborhood of $\ell = 0$ since it is decreasing and its derivative at 0 vanishes. On the other hand, since $\mathcal{E}_\epsilon$ converges (when ϵ tends to 0) to $\mathcal{E}_0$ which is strictly convex, we can expect that the concavity of $\mathcal{E}_\epsilon$ should change for sufficiently large value of ℓ when ϵ is small enough. As a consequence, we need to consider only two potential situations : the first, where the graph of $\mathcal{E}_\epsilon$ lies always above the chord $A_\epsilon C_\epsilon$ joining the point $A_\epsilon = (0, \mathcal{E}_\epsilon(0))$ and $C_\epsilon = (1, \mathcal{E}_\epsilon(1))$, see Figure 7; the second, where the graph of $\mathcal{E}_\epsilon$ intersects the chord $A_\epsilon C_\epsilon$ and is at first concave, then convex, see Figure 8.

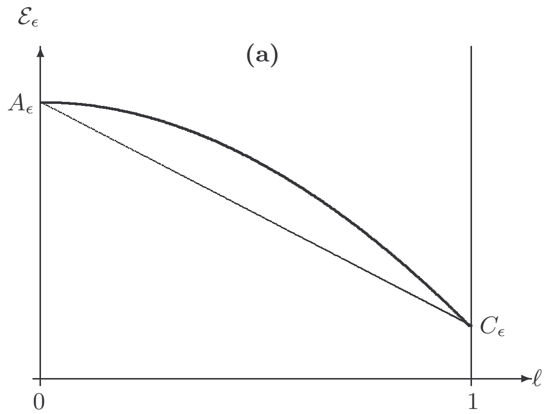


Figure 7. *Graphs of the elastic energy versus the debond length for short beams.*

(a): ***The graph of $\mathcal{E}_\epsilon$ is above its chord***. We can expect that it is the case for short beams, that is, when ϵ is large, but the rigorous proof of such a result requires a careful analysis of the asymptotic behavior of $\mathcal{E}_\epsilon$ when ϵ becomes infinite, which is out of the scope of the present study. In any case, the minimum of $t^2\mathcal{E}_\epsilon(\ell) + \kappa\ell$ is then reached at $\ell = 0$ or at $\ell = 1$. Specifically, we obtain

$$\ell^\epsilon(t) = \begin{cases} 0 & if \quad t < t_i^\epsilon \\ 1 & if \quad t > t_i^\epsilon \end{cases} \quad with \quad t_i^\epsilon = \sqrt{\frac{\kappa}{\mathcal{E}_\epsilon(0) - \mathcal{E}_\epsilon(1)}}. \tag{83}$$

In other words, when the load attains the critical value t_i^ϵ, the fiber is suddenly entirely debonded.

(b): ***The graph of $\mathcal{E}_\epsilon$ intersects its chord, and is at first concave, then strictly convex***. By virtue of (vii) a crack of length ℓ_i^ϵ appears at $t = t_i^\epsilon$ and by virtue of

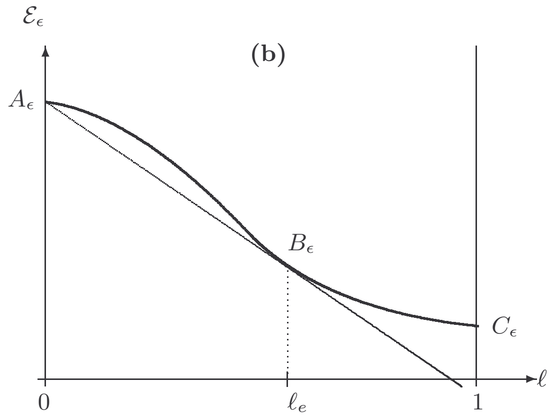


Figure 8. *Graph of the elastic energy versus the debond length for slender beams.*

the continuity property (vi) and of the minimum property (80), ℓ_i^ϵ and t_i^ϵ have to satisfy

$$\mathcal{E}_\epsilon(0) - \mathcal{E}_\epsilon(\ell_i^\epsilon) = \frac{\kappa}{t_i^{\epsilon 2}}\ell_i^\epsilon, \qquad \mathcal{E}_\epsilon(0) - \mathcal{E}_\epsilon(\ell) \leq \frac{\kappa}{t_i^{\epsilon 2}}\ell, \quad \forall \ell \in [0,1]. \tag{84}$$

Hence the graph of $\mathcal{E}_\epsilon$ must be above the chord passing through A_ϵ and $B_\epsilon = (\ell_i^\epsilon, \mathcal{E}_\epsilon(\ell_i^\epsilon))$. Since by hypothesis, the graph of $\mathcal{E}_\epsilon$ intersects the chord $(A_\epsilon C_\epsilon)$, B_ϵ is not at C_ϵ and consequently the chord $A_\epsilon B_\epsilon$ is tangent at B_ϵ to the graph of $\mathcal{E}_\epsilon$, cf Figure 8. This last property gives us a second relation between ℓ_i^ϵ and t_i^ϵ which have thus to satisfy

$$-t_i^{\epsilon 2}\mathcal{E}_\epsilon'(\ell_i^\epsilon) = \kappa, \qquad t_i^{\epsilon 2}\Big(\mathcal{E}_\epsilon(0) - \mathcal{E}_\epsilon(\ell_i^\epsilon)\Big) = \kappa \ell_i^\epsilon. \tag{85}$$

The assumed strict convexity of $\mathcal{E}_\epsilon$ after its inflection point implies the existence of a unique pair $(\ell_i^\epsilon, t_i^\epsilon)$ solution of (85). Moreover, by inserting the expression of t_i^ϵ given by (85a) into (85b) we obtain the following equation for ℓ_i^ϵ

$$\mathcal{E}_\epsilon(0) - \mathcal{E}_\epsilon(\ell_i^\epsilon) + \mathcal{E}_\epsilon'(\ell_i^\epsilon)\ell_i^\epsilon = 0 \tag{86}$$

in which the toughness κ does not appear. The first debonding length is independent of the toughness but depends only on the elastic coefficients and on the geometric parameters. Of course, the critical load t_i^ϵ depends on the toughness.

After this brutal initial debonding, the length of the crack grows continuously with the load until it reaches 1; the evolution is then governed by Griffith law. The crack arrives at 1 when t attains t_f^ϵ :

$$t_f^\epsilon = \sqrt{\frac{\kappa}{-\mathcal{E}_\epsilon'(1)}}. \tag{87}$$

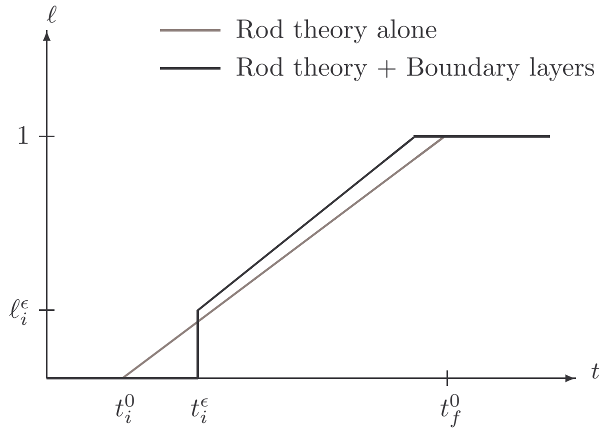


Figure 9. *The debonding process for slender beams and stiffer fiber.*

Remark 14. *In Figure 9 we compare the debonding evolution obtained from the asymptotic expression of the bulk energy with that given by the true 3D problem. The difference essentially lies into the onset of the debonding. In the former case, the onset is smooth and follows Griffith law, while it is brutal in the latter case and cannot be deduced from Griffith's law. There is no contradiction, as is proved in* Bilteryst *and* Marigo *(2002) by using matched asymptotic expansions like in* Abdelmoula *and* Marigo *(1991), because the finite initial debond length ℓ_i^ϵ tends to 0 as $\sqrt{\epsilon}$. On the other hand, it is necessary to use the least energy principle instead of Griffith law in the full 3D context, otherwise debonding never starts.*

4 Numerical Treatment

The numerical implementation of the revisited theory of brittle fracture developed in section 1 is presented. A computational method based on a variational approximation of the original functional is proposed. It is tested on two examples where we consider the discrete evolution Law 1 which consists, given $\varGamma_0$, of finding $\varGamma_i$, for $i \geq 1$ such that

(i) $\varGamma_i \supset \varGamma_{i-1}$,

(ii) $E(\varGamma_i, t_i) \leq E(\varGamma, t_i)$, for all $\varGamma \supset \varGamma_i$.

4.1 Numerical Implementation

Antiplane problems. Our law of brittle fracture is close to a model of image segmentation, namely that obtained through the minimization of the Mumford-Shah functional (Mumford and Shah, 1989). For a given grey level image, (*i.e.*, a real valued function g, defined on a bounded open domain $\varOmega$), the goal is to minimize the following "energy"

$$\int_{\varOmega \setminus \varGamma} |\nabla u|^2 + |u - g|^2 \, dx + \mathcal{H}^{n-1}(\varGamma), \tag{88}$$

over all compact subsets Γ of $\mathbb{R}^n$ (the "edge set" of the image) and all real valued functions u (the "segmented image"), continuous on $\Omega \setminus \Gamma$.

The antiplane isotropic elasticity case with constant elasticity and fracture toughness is the closest in spirit to the Mumford-Shah functional. In the former however, non-homogeneous Dirichlet boundary conditions will replace the $\int_\Omega |u-g|^2\,dx$ term while the set of admissible cracks will be somewhat different from the set of edges. Indeed, while in the image context, edges are to be detected *inside* the domain, cracks are to be accounted for *inside* and *at the boundary* of the domain.

In what follows, Ω denotes a bounded open domain of $\mathbb{R}^2$. The body is homogeneous and isotropic with constant fracture toughness. In the "strong" formulation, the energy of the body is given by the functional

$$\mathcal{E}(u,\Gamma) = \frac{1}{2}\int_{\Omega\setminus\Gamma} \mu\nabla u\cdot\nabla u\,dx + k\mathcal{H}^1(\Gamma), \tag{89}$$

for each admissible crack set $\Gamma \subset \overline{\Omega}$ and each admissible function, $u \in W^{1,2}(\Omega\setminus\Gamma)$ such that $u = U$ on $\partial_d\Omega$. The complementary boundary part $\partial_f\Omega = \partial\Omega \setminus \partial_d\Omega$ will then be traction free. Since jumps of u on $\partial\Omega$ are allowed, the functional $\mathcal{E}$ is redefined on a "large enough" open set $\widetilde{\Omega}$, containing $\overline{\Omega}$ (an estimate of "how large" $\widetilde{\Omega}$ should be will be given later) and the boundary condition is automatically satisfied by imposing $u = U$ in $\widetilde{\Omega}\setminus\Omega$. The "weak" formulation (that is not detailed here, see Bourdin et al. (2000)) consists of introducing a larger space of displacement fields (namely, the space of special functions of bounded variation) and of taking as the cracked state Γ the set of points S_u where u is discontinuous, see Ambrosio and Tortorelli (1990) or Braides (1998).

Approximation by an elliptic functional. From a numerical standpoint, the main difficulty in solving the minimization problem in its original form is that we have to investigate a family of free discontinuity surfaces. An alternative solution is to use a regularized formulation where the displacements are not discontinuous but can undergo large gradients. It is the method used in Ambrosio and Tortorelli (1990) for the Mumford-Shah functional; we will adapt it in our context. The main idea in this kind of approximation is to introduce an auxiliary variable (subsequently denoted by v) taking its values in the interval [0,1] and approaching (in some sense) the crack. We introduce the following regularized functional, for each $u \in W^{1,2}(\widetilde{\Omega})$ and $v \in W^{1,2}(\widetilde{\Omega};[0,1])$,

$$\mathcal{E}_c(u,v) = \frac{1}{2}\int_{\widetilde{\Omega}} (v^2+k_c)\mu\nabla u\cdot\nabla u\,dx + \int_{\widetilde{\Omega}} k\Big(c|\nabla v|^2 + \frac{(1-v)^2}{4c}\Big)\,dx, \tag{90}$$

where c and k_c are positive constants. Then, if $k_c \ll c$ when $c \to 0$, it can be proved by using $\Gamma-$*convergence* arguments that the minimum of $\mathcal{E}_c$ converges to the minimum of $\mathcal{E}$, the sequence of the (smooth) minimizers u_c converge to the non smooth field u, the set where u is discontinuous, S_u playing the role of the crack in the "weak" formulation. (The interested reader is referred to Ambrosio and Tortorelli (1990) or Bourdin et al. (2000) for precise statements and proofs.) Thus, in the limit, the minimization of $\mathcal{E}$ and that of $\mathcal{E}_c$ are equivalent. The proofs of the Γ–limit estimates further demonstrate that the first part of the regularized energy $\int_{\widetilde{\Omega}}(v_c^2+k_c)|\nabla u_c|^2\,dx$ converges to the bulk energy $\int_{\widetilde{\Omega}}|\nabla u|^2\,dx$ while the second one converges to the surface energy $\mathcal{H}^1(S_u)$. Finally the auxiliary function v in $\mathcal{E}_c$ converges pointwise to 1 on $\widetilde{\Omega}\setminus S_u$ and to 0 on S_u.

The same Γ–convergence result holds for the discrete functional $\mathcal{E}_{c,h}$, defined by the projection of $\mathcal{E}_c$ over a piecewise affine finite element space, provided that the characteristic length of the mesh h tends to zero faster than the infinitesimal k_c and c, see Bellettini and Coscia (1994). In the following subheading, we discuss the numerical solving of the discrete minimization problem.

Numerical solving of the discrete problem. We propose to minimize the regularized functional for small c's. At this stage, quite a few technical issues are to be addressed. Firstly, because of the presence of the term $v^2|\nabla u|^2$, the functional $\mathcal{E}_{c,h}$ is not convex in (u,v). However, it is convex and coercive in *each* variable, so that, once one of the fields is fixed, the minimization with respect to the other variable is easy. The idea is then to iterate minimizations in each variable until the successive minimizers are close enough to one another. We are unfortunately unable to prove the convergence of this algorithm; the sequence of optimal energies does converge and, up to a subsequence, the alternate minimizers converge to some critical point of $\mathcal{E}_{c,h}$. Both minimization problems are implemented by a standard finite element method, with triangular first order Lagrange elements. To enforce irreversibility on the crack set evolution, one should disconnect, at the end of each loading step, the nodes where a crack is detected, or, equivalently, add restrictions on the admissible set for the crack field by imposing some homogeneous Dirichlet conditions on the detected crack field v. The second solution is that which has been implemented.

As far as the choice of the different numerical parameters is concerned, we simply make the following comments (the interested reader is referred to Bourdin et al. (2000) for a more detailed discussion):

– c must be chosen small enough to prevent a softening effect that causes the bulk energy to be underestimated and large enough, compared to the discretization step h, close to the cracks to avoid overestimating the surface energy.
– k_c must be large enough to prevent the divergence of the numerical scheme but small enough so that no rigidity remains on the cracks.
– The logical domain $\widetilde{\Omega}$ must be chosen according to c so that the energy of cracks near the boundary of the physical domain Ω is well represented.

Remarks on the plane elasticity case. The study of the plane elasticity problem is still in its infancy. The equivalence between strong and weak formulation is not established at this time, nor is the weak formulation so clear. Furthermore, non-interpenetration of the crack lips should be imposed . The numerical adaptation of the algorithm to a linear isotropic elasticity problem in the absence of unilateral conditions is similar to that of the antiplane case. The regularized functional that has to be minimized is

$$\int_{\widetilde{\Omega}}(v^2+k_c)W(\varepsilon(u))\,dx+\int_{\widetilde{\Omega}}k\Big(c|\nabla u|^2+\frac{(1-v)^2}{4c}\Big)\,dx, \tag{91}$$

where $2W(\xi) = \lambda tr(\xi)^2 + 2\mu \, \xi \cdot \xi$. This is the method we used for the computations of the traction on a reinforced matrix, Section 4.2. Implementing unilateral conditions is however an open problem as of yet, for want of the proper regularized functional in place of (91).

4.2 Numerical experiments

In this section two numerical experiments, the first in antiplane elasticity, the second in plane elasticity, are described and compared with the theoretical predictions in Sections 1 and 2. The figures presented below represent the "damage" v-field. The crack site is included within the set of points where v is near zero (black or red color-coded in the figures).

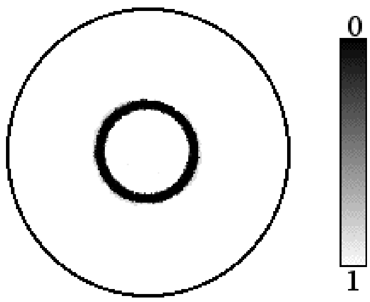


Figure 10. The computed damage field at $t = 1.5$.

Tearing of a reinforcement. The tearing of a rigid fiber out of a three-dimensional axisymmetric cylinder is solved in a closed form in Section 1.4. It was obtained there that the response is elastic without any crack as long as the axial displacement t prescribed to the fiber is smaller than a critical value t_f given by (31) while there is total debonding of the reinforcement from the inner surface of the cylinder (where the fiber is initially glued) as soon as $t > t_f$. Here, we take the following values of the material and geometrical parameters :

$$\mu = 1, \quad k = 1, \quad R_i = 1, \quad R_o = 3.$$

The failure load t_f, the bulk energy and the surface energy are given by

$$t_f = \sqrt{2 \ln 3} \sim 1.48, \tag{92}$$

$$E_d(t) \sim \begin{cases} 2.86\, t^2 & if \ t < 1.48 \\ 0 & if \ t > 1.48 \end{cases}, \quad E_s(t) \sim \begin{cases} 0 & if \ t < 1.48 \\ 6.28 & if \ t > 1.48 \end{cases} \tag{93}$$

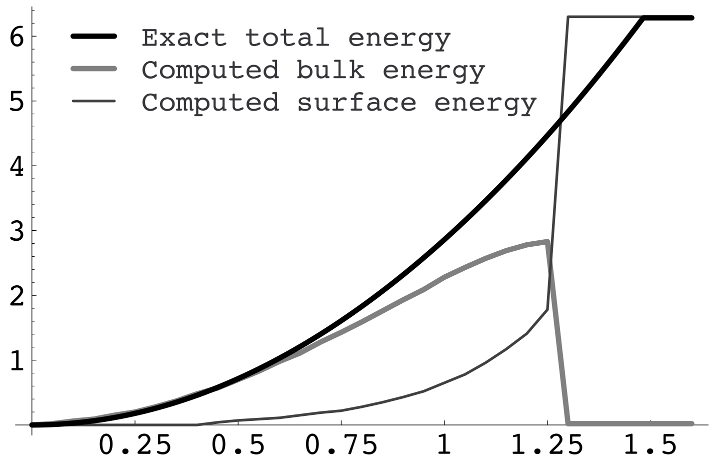


Figure 11. Evolution of the computed or exact energies with t.

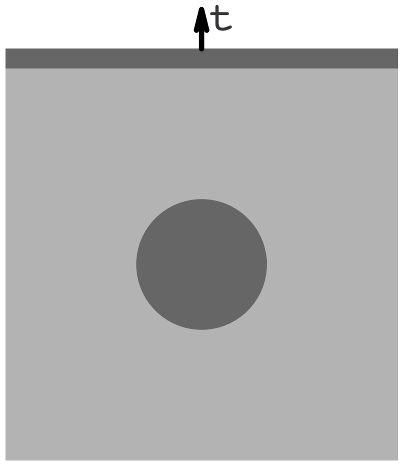
Our goal is to test the numerical scheme on this example. Since the computer code has not been developed in the full $3D$ context, we will test the antiplane version. So we assume that the problem is independent of the axial direction and we try to describe the crack evolution on a cross section. We make no assumption concerning the axisymmetry of the solution. The parameters are $h = 10^{-3}$, $c = 10^{-1}$, $k_c = 10^{-4}$. In Figure 11, we plot the computed bulk energy, the computed surface energy and the exact total energy versus the load t, whereas in Figure 10 we plot the computed "damage" field v at $t = 1.5$.

Figure 11 shows that, at $t \sim 1.30$, brutal cracking appears in the section because the surface energy jumps from 1.78 to 6.30, while, at the same time, the computed bulk energy jumps from 2.83 to nearly 0. Then, the energy of the body does not evolve. Figure 10 indicates what happens at $t \sim 1.30$, namely the total debonding of the fiber. Therefore the computations are qualitatively in perfect agreement with the theoretical predictions. From a quantitative point of view, one sees that the computed critical load is underestimated (1.30 instead of 1.48). Note also that the surface energy is not 0 before the critical load, even if there is no localization of the damage at this stage. The reason is probably due to an insufficiently small value of the parameter c (and of the mesh parameter). On the other hand, the surface energy is correct within 1%, once the fiber is debonded.

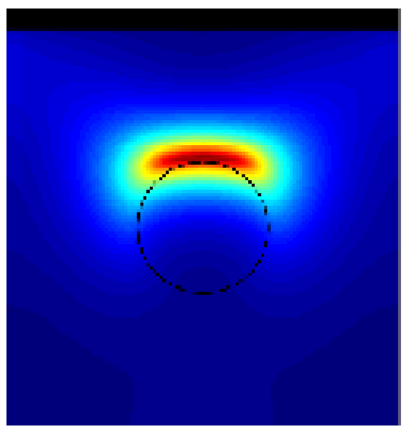
Traction experiment on a fiber reinforced matrix. A square elastic matrix is reinforced by a rigid circular fiber as shown in the figure below. The fiber remains fixed, while a uniform displacement field te_2 is imposed on the upper side of the square; the remaining sides are traction-free.

The computed crack evolution. The following evolution is observed as t grows:

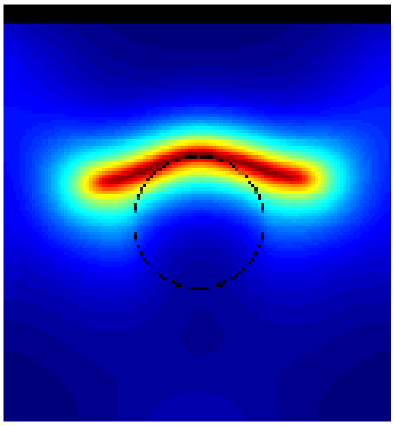
Phase 1 : Elastic response. As long as $t < 0.2$, the matrix remains purely elastic. Note however on Figure 12, that the surface energy is not exactly 0 during this phase, probably because the chosen value of the parameter c is not sufficiently small.



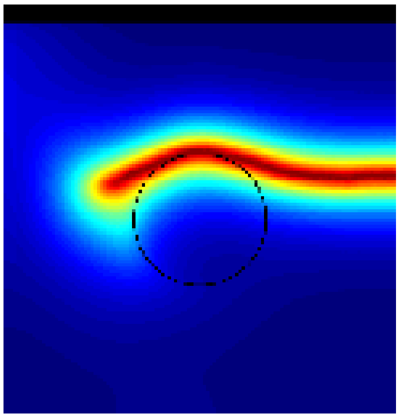
Phase 2 : Brutal onset. At $t \sim 0.2$, a crack of finite length brutally appears near the north pole of the inclusion. Note that this first crack is symmetric with respect to the 2 axis, but that is not straight. (It is not a purely numerical effect, because this path does not follow the mesh.) This brutal onset is confirmed in Figure 12 where, at this load, the surface energy increases while the bulk energy decreases.



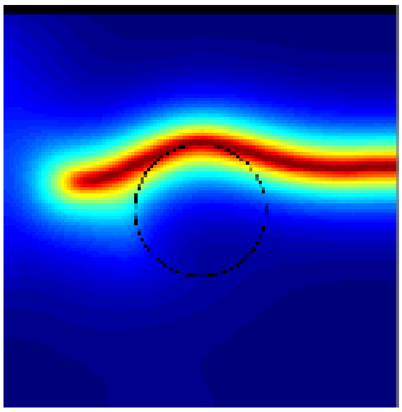
Phase 3 : Progressive and symmetric evolution of the cracking. When t varies between 0.2 and 0.32, the crack progressively grows in the matrix. The evolution is smooth as is shown in Figure 12 where the surface energy increases smoothly, while the bulk energy is nearly constant. Note that the propagation is symmetric but not straight.



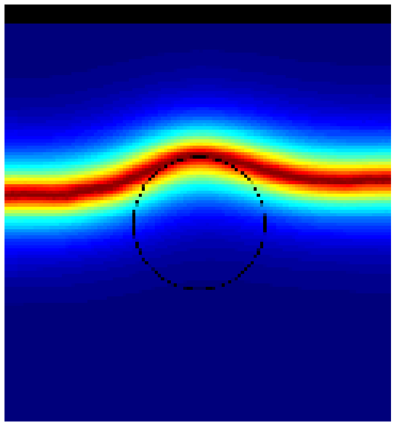
Phase 4 : Rupture of the right ligament. At $t \sim 0.32$, the right hand-side of the matrix is brutally cut. The brutal character of the phenomenon can be verified on Figure 12 with jump discontinuity of both the surface and bulk energies. Note that the cracked state is no longer symmetric.



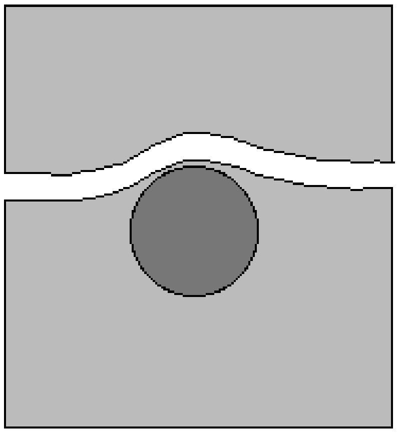
Phase 5 : Progressive propagation of the left branch. When t varies between 0.32 and 0.37, the left part of the crack progressively grows, the surface energy slowly increases, whereas the bulk energy is nearly constant. Note also that the crack is trying to recover symmetry.



Phase 6 : Rupture of the left ligament. At $t \sim 0.37$, the crack brutally severs the remaining filament of uncracked material, see also Figure 12 where the jump of the energies is clearly evidenced. Note also that, in the final stage the right and left parts of the crack are nearly symmetric.



Phase 7 : The sample is broken. For $t > 0.37$, the sample is split into two parts. Theoretically, the lower piece should be now in its natural configuration, while the upper one should be simply translated. Note however in figure 12 that the elastic energy is not exactly equal to 0, because of the presence of the coefficient k_c in the functional.



Comparisons with the theoretical properties. Of course, an exact solution is not available for this example. We can however compare some features of the computations with the theoretical properties established in Section 2 or in Francfort and Marigo (1998).

1. The onset of cracking is brutal. This is in agreement with item 4 of Proposition 4.19 in Francfort and Marigo (1998) and Proposition 4.2 in Section 2. Indeed, if x_0 is a non singular point for the purely elastic solution, the bulk energy release due to a crack of small length ℓ near x_0 is of the order of ℓ^2. Therefore its derivative tends to 0 with ℓ and the bulk energy is a concave function of the crack length near $\ell = 0$.
2. The energy is (nearly) conserved during phases 2, 4, 6 of brutal growth, as theoretically expected.
3. Griffith law is satisfied during the phases 3 and 5 of progressive growth of the crack. Figure 13 is an attempt at checking this. Denoting by $\ell(t)$, $E(t)$ and $E_d(t) = t^2\mathcal{E}_d(\ell(t))$ the length of the crack, the total energy and the bulk energy, respectively, when the

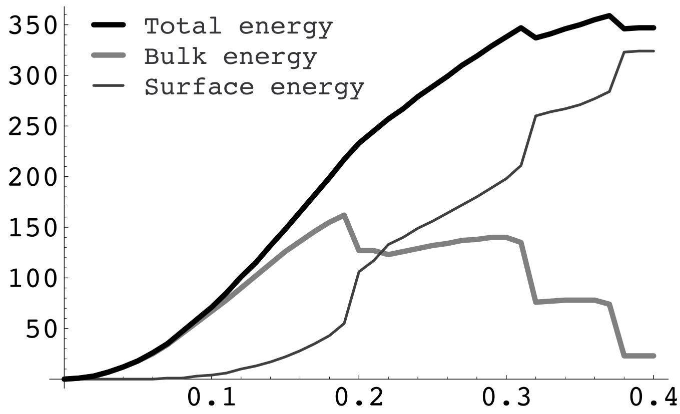


Figure 12. *Evolution of the energies with the load.*

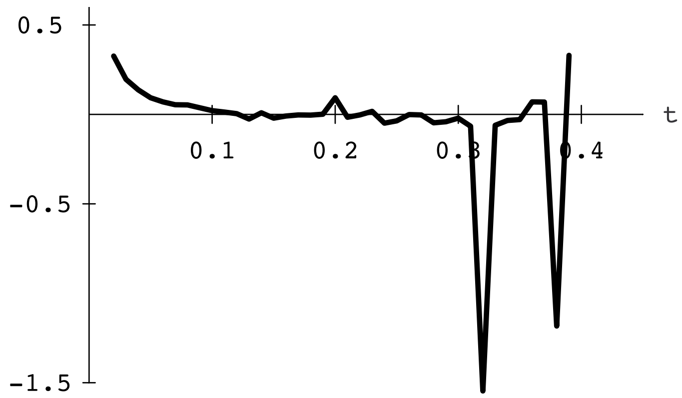


Figure 13. *Verification of the Griffith law.*

load is t we have

$$E(t) = E_d(t) + k\ell(t), \tag{94}$$

while Griffith law reads as

$$t^2 \frac{d\mathcal{E}_d}{d\ell}(\ell(t)) + k = 0. \tag{95}$$

Differentiating (94) with respect to t and using (95) yields

$$E'(t) - \frac{2E_d(t)}{t} = 0.$$

In Figure 13, $E' - 2E_d(t)/t$ is plotted as a function of t, and we see that the criterion is indeed met during the progressive phases of the evolution, while it is not satisfied at $t = 0.32$ and at $t = 0.37$, *i.e.* when the right and left ligaments break. Note however that it is nearly satisfied at $t = 0.2$ which is in agreement with the results obtained in Section 3 for the pull-out test.

Remark 15. *This numerical experiment is exemplary because of the wide range of associated crack behaviors throughout the evolution: brutal versus progressive fracture, edge versus bulk crack, symmetric versus asymmetric path, curvilinear crack, ...*

5 Obstacles and Perspectives

Before illustrate them in the one dimensional context let us briefly present the main limitations of our formulation.

1. As mentioned in Section 1, our formulation is severely impaired when surface or body force-loads are applied to the sample. Indeed, as illustrated in the sequel, the total energy (which includes now the potential of the applied forces) is generally not bounded from below and no **global** minimizer exists.
2. On the other hand, if we investigate **local** minimizers, since the purely elastic response is, for any value of the loading, a local minimizer, it is not possible to predict the onset of cracking in a crack free body.
3. Moreover, Griffith's assumption on the surface energy leads in general to **size effects** which are not conform to experiments.

Accordingly, it is necessary to change the surface energy and to weaken the least energy principle.

5.1 Limits of the Current Formulation.

No global minimum when forces are prescribed. We consider a homogeneous bar of natural length L, with cross-sectional area S, constituted of an elastic breakable material with Young modulus E and toughness k. Before any loading, the bar is assumed to be crack free. The end $x = 0$ is fixed, a tensile force tF $(F > 0)$ is applied at $x = L$ and the bar is submitted to a uniform distribution of tensile body forces tg, $g > 0$, t being a loading parameter increasing from 0. If we adopt Griffith's surface energy, the total energy of the bar is the following functional on admissible displacement fields v :

$$\mathcal{E}_t(v) = \frac{ES}{2}\int_0^L v'(x)^2 dx + kS\, card(\Gamma(v)) - tg\int_0^L v(x)dx - tFv^+(L), \tag{96}$$

where $\Gamma(v)$ denotes the set of points where v is discontinuous and will play the role of the cracked state. The set $\mathcal{C}$ of admissible displacements is the space of functions v of bounded variation such that $v^-(0) = 0$[2] and with positive jumps ($[\![v]\!] = v^+ - v^- \geq 0$) :

$$\mathcal{C} = \Big\{ v \in BV(\mathbb{R}) \ : \ v = 0 \ on \ (-\infty, 0), \ v = const \ on \ (L, \infty), \ [\![v]\!] \geq 0 \ on \ \Gamma(v) \Big\}. \tag{97}$$

[2] Since v can be discontinuous at 0 or at L, v is defined on the whole real line by setting $v(x) = 0$ when $x < 0$ and $v(x) = constant$ when $x > L$.

We immediately obtain the

Proposition 13. *When $t > 0$, the energy $\mathcal{E}_t$ is not bounded from below. Thus a global minimizer does not exist.*

Proof. Let v be the field such that $v(x) = 0$ for $x < L/2$ and $v(x) = U > 0$ for $x > L/2$. This field is admissible, its elastic energy vanishes, its surface energy is equal to kS and then its total energy is $kS - t(F + gL/2)U$. Since U can be taken arbitrarily large, the result follows. $\square$

The elastic response is the unique local minimum. We say that an admissible field u is a *local minimizer* of $\mathcal{E}_t$ if there exists a ball (in the sense of the natural norm of functions of bounded variation, see Braides (1998)) $\mathcal{B}(u, r)$ of center u and of radius $r > 0$ such that

$$\mathcal{E}_t(u) \leq \mathcal{E}_t(v) \quad \forall v \in \mathcal{B}(u, r) \cap \mathcal{C}. \tag{98}$$

Denoting by u_t the purely elastic response of the bar for the loading parameter t, *i.e.*

$$u_t(x) = \frac{tg}{2ES}\, x(2L - x) + \frac{tF}{ES}\, x, \tag{99}$$

the following Proposition holds true :

Proposition 14. *For any value of the loading parameter t, the elastic response u_t is a local minimizer.*

Proof. Let $r > 0$ and $\phi \in \mathcal{C}$ with $\|\phi\| = 1$ and $card(\Gamma(\phi)) \geq 1$. Direct calculations give

$$\begin{aligned}
\mathcal{E}_t(u_t + r\phi) - \mathcal{E}_t(u_t) &= r\left(\int_0^L \Big(ESu_t'(x)\phi'(x) - tg\phi(x)\Big)dx - tF\phi^+(L)\right) \\
&\quad + kS\ card(\Gamma(\phi)) + \frac{r^2}{2}\int_0^L \phi'(x)^2 dx \\
&\geq rt\int_0^L \Big(\big(g(L-x) + F\big)\phi'(x) - g\phi(x)\Big)dx - rtF\phi^+(L) + kS \\
&= -rt\sum_{x\in\Gamma(\phi)} \Big(g(L-x) + F\Big)[\![\phi]\!](x) + kS \\
&\geq -rt(gL + F) + kS,
\end{aligned}$$

where the last inequality follows from $\|\phi\| = 1 \geq \sum_{x\in\Gamma(\phi)} |[\![\phi]\!](x)|$. Thus, if we choose r sufficiently small we obtain

$$\mathcal{E}_t(v) \geq \mathcal{E}_t(u_t), \qquad \forall v \in \mathcal{B}(u_t, r) \cap \{v \in \mathcal{C}\ :\ card(\Gamma(v)) \geq 1\}. \tag{100}$$

If we now consider $v \in \mathcal{C}$ such that $card(\Gamma(v)) = 0$, *i.e.* smooth v, we classically obtain

$$\mathcal{E}_t(v) - \mathcal{E}_t(u_t) = \frac{1}{2}\int_0^L (v'(x) - u_t'(x))^2 dx \geq 0, \tag{101}$$

and the result follows. $\square$

Remark 16. *This result means that the elastic response is never unstable if we consider Griffith's surface energy. It remains true in the general context of* $3D$ *Fracture Mechanics, provided that the elastic field* u_t *does not contain strong singularities, see* Francfort *and* Marigo *(1998).*

It is possible to obtain a stronger result by proving that, in our $1D$ context of prescribed forces, the elastic response $u_t, t > 0$, is the unique local minimizer, see Charlotte et al. (2000). Let us simply verify here that no discontinuous field v is a local minimizer. Let $v \in \mathcal{C}$ with $card(\Gamma(v)) \geq 1$, let $x_1 \in \Gamma(v)$ and let v_n be the following sequence of admissible fields converging to v when n goes to infinity :

$$v_n(x) = \begin{cases} v(x) & if\ x < x_1 \\ v(x) + \dfrac{1}{n} & if\ x > x_1 \end{cases}. \tag{102}$$

Noting that $card(\Gamma(v_n)) = card(\Gamma(v))$, a direct calculation gives then

$$\mathcal{E}_t(v_n) - \mathcal{E}_t(v) = -\frac{t}{n}\Big(g(L - x_1) + F\Big) < 0. \tag{103}$$

Spurious size effects. Let us consider the case of applied displacements with Griffith's model. We have found in Section 4.2 that the cylinder breaks when the prescribed displacement U reaches the critical value

$$U_c = \sqrt{\frac{2kL}{E}}$$

corresponding to the critical strain $\sqrt{\frac{2k}{EL}}$ and to the critical stress $\sqrt{\frac{2kE}{L}}$. Thus, the longer is the bar, the smaller is the rupture stress. Moreover, when the bar length tends to infinity, the rupture stress tends to 0. This size effect is general (see also the tearing of the reinforcement in Section 1.4 and the pull-out test in Section 3) and is due to Griffith assumption on the surface energy which is based on the comparison of surface energy with volume energy and brings necessarily into the theory an internal length scale.

5.2 More Realistic Models.

Barenblatt's model As outlined by Griffith himself, the surface energy is proportional to the area of the surface of discontinuity only when the distance between the lips of the crack are large with respect to the characteristic atomic length. In the spirit of what happens at an atomic scale when atomic bonds break, we could assume, following the idea of Barenblatt (1962), that the surface energy depends on the value of the displacement jump, starting from 0 and progressively growing to its effective Griffith value k when the displacement jump becomes large with respect to the characteristic atomic length. It is then more convenient to assume that the surface energy takes the following form :

$$\mathcal{E}_s(u) = \int_{\Gamma(u)} \kappa(x, \nu(x), [\![u(x)]\!]) d\mathcal{H}^{N-1}(x), \tag{104}$$

where $\nu(x)$ denotes the unit normal vector on $\mathit{\Gamma}(u)$ at x. The surface energy is now a function of the displacement field u.

In the present $1D$ context we will assume that the surface energy reads as

$$\mathcal{E}_s(u) = \sum_{\mathit{\Gamma}(u)} S\kappa([\![u]\!](x)), \tag{105}$$

with

$$\kappa(0) = 0, \quad \kappa \text{ twice differentiable and increasing}, \quad \kappa(+\infty) = k. \tag{106}$$

In the analysis of local minima, the derivative of κ at 0 plays a fundamental role. Noticing that κ has the dimension of an energy per unit surface and that its derivative has the dimension of a stress, we put

$$\sigma_c = \kappa'(0). \tag{107}$$

Remark also that κ can be defined only for positive real number, because of the non-penetration condition.

The presence of a critical stress. A quite general study of this model has been achieved in Choksi et al. (1999), Del Piero (1999) or Charlotte et al. (2000), always in the $1D$ context. We will simply show in the present paragraph how Barenblatt's surface energy destabilizes the elastic response when the loading parameter is sufficiently large. The total energy reading now

$$\mathcal{E}_t(v) = \frac{ES}{2}\int_0^L v'(x)^2 dx + \sum_{x\in\mathit{\Gamma}(v)} \kappa([\![v]\!](x))S - tg\int_0^L v(x)dx - tFv^+(L), \tag{108}$$

we obtain

Proposition 15. *The elastic response u_t is not a local minimum when $t > t_f$, with*

$$t_f = \frac{\sigma_c S}{gL + F}. \tag{109}$$

Proof. Let v_n be the following sequence of admissible fields converging to u_t when n goes to infinity :

$$v_n(x) = \begin{cases} u_t(x) + \dfrac{1}{n} & if\ x > 0 \\ 0 & if\ x < 0 \end{cases}.$$

Let us note that v_n is discontinuous at $x = 0$, $[\![v_n]\!](0) = 1/n$. We get, for sufficiently large n :

$$\begin{aligned} \mathcal{E}_t(v_n) - \mathcal{E}_t(u_t) &= -\frac{t}{n}(gL + F) + \kappa\left(\frac{1}{n}\right)S \\ &= -\frac{t - t_f}{n}(gL + F) + o(1/n) \\ &< 0, \end{aligned}$$

and the result follows. □

Remark 17. *The value $t(gL+F)$ represents the reaction of the wall at which the bar is fixed. It is the maximal tension that the bar must sustain. Hence, the proposition says that the bar is necessarily broken when the tension at this point is greater than $\sigma_c S = \kappa'(0)S$. Thus σ_c plays the role of a critical stress for the material.*

It is possible to prove the converse result, i.e. *the elastic response remains a local minimum as long as the tension at $x = 0$ is less than σ_c, see* Charlotte *et al. (2000).*

Let us finally remark that this energy based analysis suggests that the bar will break at $x = 0$ as soon as $t > t_f$. But, since the bar is then no more at equilibrium, this "post-mortem" analysis requires to consider dynamical effects.

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