Postulate 2: Evolution of quantum systems (rotations/reflections)

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"The evolution of a closed quantum system is described by a unitary transformation. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at t_2 by a unitary operator U (norm preserving) which depends only on times t_1 and t_2."

(In $\mathbb{R}^d$ "orthogonal transf. ormation")

i.e. $|\psi'\rangle = U|\psi\rangle$ where if U is unitary then

$$\| U|\psi\rangle \|^2 = \| |\psi\rangle \|^2 \quad \forall |\psi\rangle \in \mathbb{C}^d$$

$$\Rightarrow (U|\psi\rangle)^\dagger (U|\psi\rangle) = \langle\psi|\psi\rangle$$

$$\Rightarrow \langle\psi| U^\dagger U |\psi\rangle = \langle\psi|\psi\rangle$$

$$\Rightarrow U^\dagger U = I$$

then if U is unitary,

$$U^\dagger U = I \iff U U^\dagger = I \iff U^{-1} = U^\dagger \text{ (i.e. invertible)}$$

$$U = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_d \\ | & | & & | \end{pmatrix} \Rightarrow \begin{pmatrix} - & u_1^* & - \\ - & u_2^* & - \\ & \vdots & \\ - & u_d^* & - \end{pmatrix} \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_d \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{pmatrix}$$

Columns (rows) of U are orthonormal.

examples: On qubits, $\mathbb{C}^2$ $\quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

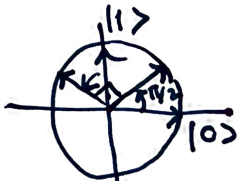
(1) $R_\theta^\dagger R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$$R_{\pi/4} = \begin{pmatrix} 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \cancel{H}$$

Hadamard Gate

$$\cancel{H}|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\cancel{H}|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

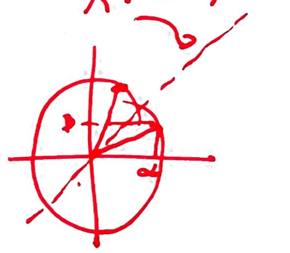


$$\boxed{H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

(2) $\cancel{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

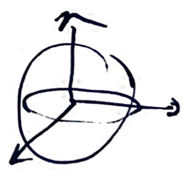
quantum NOT gate $\quad |\psi'\rangle = U|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$

X: a reflection



(3) $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$



(4) $S = \begin{pmatrix} 1 & \\ & i \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & \\ & i \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ i\beta \end{pmatrix}$

Phase shift (rotation by $\pi/2$ in the complex plane)

Postulate 3: Measurement

"Quantum measurements are described by a collection $\{M_m\}$ of measurement ~~operating~~ operators. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

and the state of the system after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

the measurement operators satisfy the completeness equation:

$$\cancel{\sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = I} \quad \sum_m M_m^\dagger M_m = I$$

the completeness equation expresses the fact that probabilities sum to one:

$$1 = \sum_m p(m) = \sum \langle\psi|M_m^\dagger M_m|\psi\rangle$$ "

<u>examples</u>: $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$

$$M_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow M_0 + M_1 = I$, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$p(0) = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = |\alpha|^2$ after measuring $|0\rangle$ the state of the system $\frac{\alpha}{|\alpha|}|0\rangle$

So in summary: if we measure a qubit
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

a) we read $|0\rangle$ with prob. $|\alpha|^2 \xrightarrow{\text{new state}}$ ~~then~~ $|\bar{\psi}\rangle = 1\cdot|0\rangle + 0\cdot|1\rangle$

b) we read $|1\rangle$ with prob. $|\beta|^2 \longrightarrow |\bar{\bar{\psi}}\rangle = 0\cdot|0\rangle + 1|1\rangle$

$+\overline{}$

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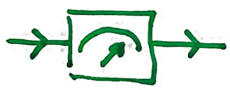
Measuring device:



Elitzur-Vaidman Bomb

11-①

(Lecture notes of Ryan O'Donnel (CMU))

Some experiments with polarizing filter:

Light source

$|0\rangle$

produces heat (i.e. measures $|0\rangle$)
or no heat (i.e. measures $|1\rangle$)

--- No light

100% brightness

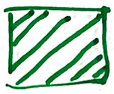
50% brightness

0% brightness

produces heat (i.e. measures $|1\rangle$)
no heat (i.e. measures $|0\rangle$)

all photons here are Horizontally polarized

produces heat (measures $|-\rangle$)
produces no heat (measures $|+\rangle$)

Now let's insert: 

with prob $\frac{1}{2}$ $|0\rangle$ is measured (heat)
with prob 1/2 $|1\rangle$ is measured (pass through)

$|0\rangle$

25% brightness

$|1\rangle$

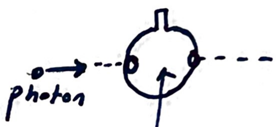
12.5% brightness

50% brightness

$|+\rangle$

with probability $\frac{1}{2}$ measures $|+\rangle$ (passes through)
with prob 1/2 measures $|-\rangle$ (heat)

The Bomb:



inside either there is:
a) nothing ⟶ light passes through
b) horizontal filter as a fuse and attached explosive. ⟶ { if $|0\rangle$ light passes through; if $|1\rangle$ bomb explodes!

* Objective is try to determine if it is a bomb or not (but without explosions if possible!)

① try sending $|0\rangle$ ⟶ nothing happens but also can't be sure if it's a bomb or not.

(better not) try sending $|1\rangle$ ⟶ explodes!

(2) try sending $|+\rangle$ if no explosion measure using (i.e.)

$|+\rangle \longrightarrow \boxed{?} \longrightarrow$ $|+\rangle$ $|-\rangle$

if no bomb we read $|+\rangle$ with 100% probability

if bomb $|+\rangle \to$ ~~change~~ with prob. $1/2$ $|1\rangle$ — explodes

$\to$ with prob $1/2$ $|0\rangle$ passes through

$\quad \underbrace{\text{with prob. } 1/2 \ |+\rangle}_{\text{same as no bomb}}$ $\quad \underbrace{\text{with prob } 1/2 \ |-\rangle}_{\text{we know there is a bomb!}}$

So, if there is a bomb

50% chance of exploding,

25% chance we detect it without exploding!

25% chance inconclusive

it turns out we can make the probability of detection very large, not 100% but very close! (i.e. ~~100~~ 99.9999 %)

1/-3

idea.

if there is a bomb 50% chance of explosion.

instead lets start with a small ε

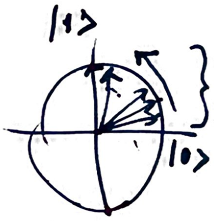
i.e.

$$H_\varepsilon|0\rangle = \cos\varepsilon\,|0\rangle + \sin\varepsilon\,|1\rangle$$

if no bomb qubit exists as it is

if bomb $\Pr[\text{measure } |0\rangle] = (\cos\varepsilon)^2$ and $|0\rangle$ exists

if bomb $\Pr[\text{explode}] = (\sin\varepsilon)^2 \leq \varepsilon^2$



choose a large n $\rightarrow \varepsilon = \frac{\pi}{2n}$ and repeat n times.

if no bomb final state is $|1\rangle$

if bomb, final state is $|0\rangle$ if there were no explosions.

Probability of explosion $\leq \varepsilon^2 n = \frac{\pi}{4n}$

for large n this is very small