Postulate 4: Multiple qubit systems

"The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle$."

Example:

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle = \begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix} \quad \text{and} \quad |\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle = \begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}$$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \begin{bmatrix}\alpha_1\begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}\\ \beta_1\begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\alpha_1\alpha_2\\ \alpha_1\beta_2\\ \beta_1\alpha_2\\ \beta_1\beta_2\end{bmatrix} = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

Entanglement: It is a property of multi qubit ~~system~~ state space. We will examine creating and destroying EPR pair of qubits

(Einstein, Podolsky, Rosen)

start with $|\psi_1\rangle$ in a $|0\rangle$ state

recall $H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ $\qquad$ $|0\rangle$ —[H]—

let $|\psi_1'\rangle = H|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

take another ~~vector~~ qubit in $|0\rangle$ state, $|\psi_2\rangle$

the joint state-space probability vector is the tensor product of these two

$$|\psi_1'\rangle \otimes |\psi_2\rangle = |\psi_1'\psi_2\rangle = \frac{1}{\sqrt{2}}|00\rangle + 0|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + 0|11\rangle$$

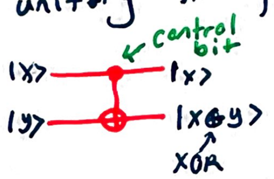
$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$|00\rangle \rightarrow |00\rangle$
$|01\rangle \rightarrow |01\rangle$
$|10\rangle \rightarrow |11\rangle$
$|11\rangle \rightarrow |10\rangle$

let us define a unitary transform

if (•) is zero no change
if (•) is one apply NOT to target ⊕

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



aka EPR pair
aka Bell state

$$|(\psi_1'\psi_2)'\rangle = CNOT|\psi_1'\psi_2\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

* Entangled state space has the property that it can not be decomposed into component spaces. that is for our example, there are no $|\varphi_1\rangle$ and $|\varphi_2\rangle$ such that $|\varphi_1\rangle \otimes |\varphi_2\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

Before CNOT: $|\psi_1'\psi_2\rangle = \frac{1}{\sqrt{2}}|00\rangle + 0\cdot|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + 0|11\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

probability of measuring second bit as zero = 1
" " " " " one = 0

$$M_0^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p^2(0) = \langle\psi_1'\psi_2|\underbrace{M_0^{2\dagger} M_0^2}_{M_0^2}|\psi_1'\psi_2\rangle = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = 1$$

after measmt $\rightarrow \frac{M_0^2|\psi_1'\psi_2\rangle}{\sqrt{p^2(0)}} = \frac{\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}{1}$ → did not change

After CNOT: $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}$

probability of mesuring second qubit as zero is $\frac{1}{2}$
" " " " " " one is $1/2$

$$p^2(0) = \langle\psi|M_0^{2\dagger}M_0^2|\psi\rangle = \langle\psi M_0^2\psi\rangle = \begin{bmatrix}1/\sqrt{2} & 0 & 0 & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}$$
$$= 1/2$$

after measurement: $\dfrac{M_0^2|\psi\rangle}{\sqrt{p^2(0)}} = \dfrac{\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}}{\sqrt{1/2}} = \dfrac{\begin{bmatrix}1/\sqrt{2}\\0\\0\\0\end{bmatrix}}{1/\sqrt{2}} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

$= |00\rangle$

i.e. by measuring one qubit we affect the state of the other entangled qubit. ~~the~~, instantaneously, no matter how far!

How to check two qubits are entangled or not?

~~Example~~: $|\varphi_1\rangle \otimes |\varphi_2\rangle$

$$\begin{bmatrix}\alpha_1\\\beta_1\end{bmatrix}\otimes\begin{bmatrix}\alpha_2\\\beta_2\end{bmatrix} = \begin{bmatrix}\alpha_1\alpha_2\\\alpha_1\beta_2\\\beta_1\alpha_2\\\beta_1\beta_2\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}$$

Example: $\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$

$\alpha_1\alpha_2 = 1/2$
$\alpha_1\beta_2 = 1/2$
$\beta_1\alpha_2 = 1/2$
$\beta_1\beta_2 = 1/2$

$\left.\begin{array}{l}\alpha_2=\beta_2\\ \alpha_1=\beta_1\\ \alpha_2=\alpha_1\end{array}\right\}$ $\alpha_1=\alpha_2=\beta_1=\beta_2=\frac{1}{\sqrt{2}}$

$\therefore = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\otimes\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$\left.\begin{array}{l}\alpha_1\alpha_2 = \frac{1}{\sqrt{2}}\\ \alpha_1\beta_2 = 0\\ \beta_1\alpha_2 = 0\\ \beta_1\beta_2 = \frac{1}{\sqrt{2}}\end{array}\right\}$ 4 eqns, 4 unknowns

~~but~~ since $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$

$\beta_1 = 0$, $\beta_2 = 0$

but $\beta_1\beta_2 = \frac{1}{\sqrt{2}}$

so it is not possible to factorize

$\begin{bmatrix}\alpha_1\\\beta_1\end{bmatrix}\otimes\begin{bmatrix}\alpha_2\\\beta_2\end{bmatrix} \neq \begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}$

therefore entangled.

Kronecker (Tensor) product:

$$A \otimes (B + C) = A \otimes B + A \otimes C$$

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$ (mixed product property)

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

$$(A \otimes B)^{T} = A^{T} \otimes B^{T}$$

$$|A \otimes B| = |A|^{m} |B|^{n}$$ (A is an $n \times n$, B is $m \times m$)

$$A \oplus B = A \otimes I_m + I_n \otimes B$$

ex.

$$\exp(A \oplus B) = \exp(A) \otimes \exp(B)$$

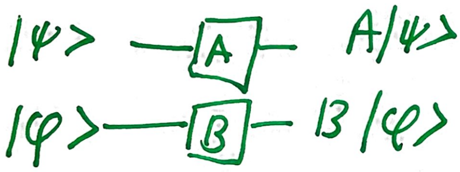
$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$$

Notes about combining quantum gates

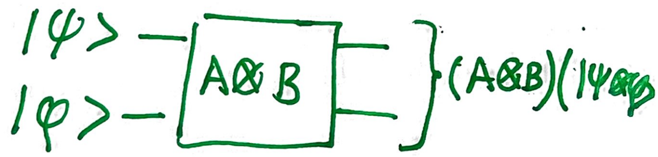
Serial connection:

$\equiv B(A|\psi\rangle)$

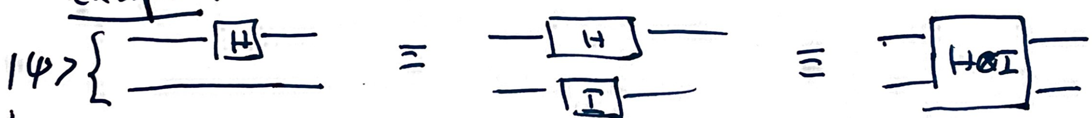
parallel connection



$\equiv$



example:



EPR pair

$$\left.\begin{array}{l} H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}\right\} \quad H\otimes I = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(H\otimes I)(|\psi\rangle) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\equiv \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$