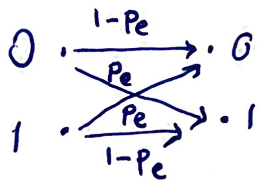
Quantum errors and error correction ①

(Based on the notes of John Preskill + the book)

<u>Classical errors</u> are bit flips; a bit is "flipped" with probability P_e. (If P_e is very large we can still recover the original bit therefore usually we assume $P_e < 0.5$)



<u>Classical error correction</u>: example: 3-bit repetion code, which encodes the bit using 3-bits such as,

$$0 \longrightarrow 000$$

$$1 \longrightarrow 111$$

and decoding is via "majority voting", so for an error to occur either two or three bit should be flipped. Therefore,

$$P_e^{3\text{-bit}} = 3P_e^2(1-P_e) + P_e^3 \sim O(P_e^2)$$

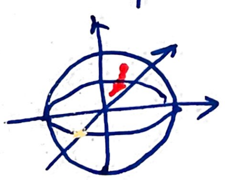
if $P_e < 0.5$ then $\underbrace{P_e^{3\text{-bit}}}_{(\text{i.e. } O(P_e^2))} < P_e$

So, we could try a "quantum" repetition code

$$|\varphi\rangle \longrightarrow |\varphi\rangle \otimes |\varphi\rangle \otimes |\varphi\rangle$$

but this is not possible due to non-cloning theorem.

<u>Furthermore</u>, bit flips are not the only possible errors in quantum computing. There is also a "phase-flip" that could happen.



+ Furthermore, classical errors are discrete in nature and quantum errors are continous.

$$(|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle)$$

+ Furthermore, quantum computers are more prone to errors than classical computers (at least for now), due to.

1) Quantum systems more susceptible to decoherence due to the system interacting with the enviroment (such as heat, mechanical vibrations, cosmic rays etc)
2) Quantum gates can be build upto a certain precision and the measurement devices can be also built upto a certain precision.

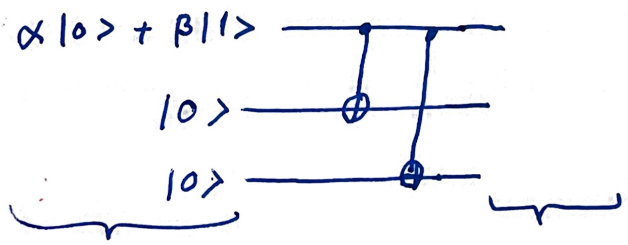
Nevertheless, we can still correct quantum errors.

For example: 3-qubit bit flip code

Rather than cloning, entanglement can play a key role in creating the 3-qubit code. to encode the states as

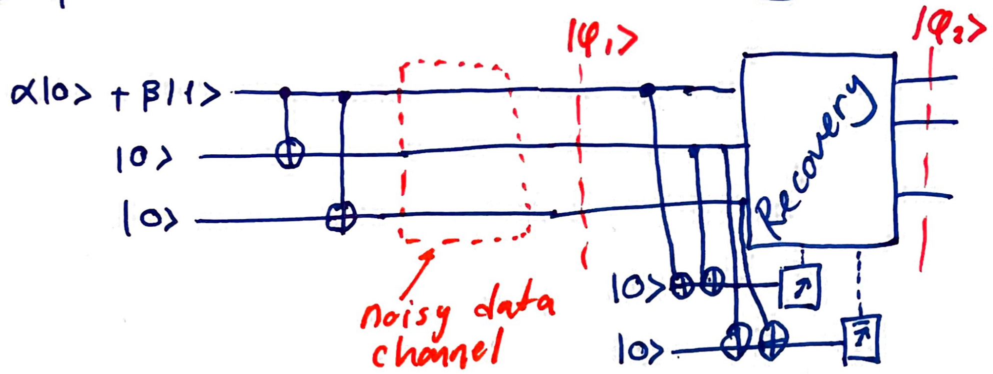
$$|0\rangle \rightarrow |000\rangle$$

$$|1\rangle \rightarrow |111\rangle$$



$$\alpha|000\rangle + \beta|100\rangle \longrightarrow \alpha|000\rangle + \beta|111\rangle$$

3-qubit error detection and recovery



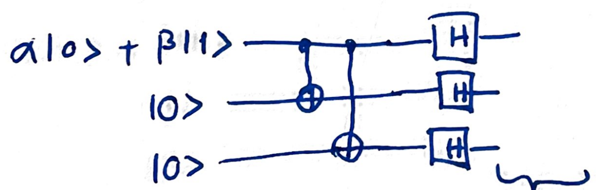
bit flip	$\lvert\varphi_1\rangle$	M_1	M_2	Recovery	$\lvert\varphi_2\rangle$
—	$\alpha\lvert 000\rangle+\beta\lvert 111\rangle$	0	0	$I\otimes I\otimes I$	$\alpha\lvert 000\rangle+\beta\lvert 111\rangle$
1st	$\alpha\lvert 100\rangle+\beta\lvert 011\rangle$	1	0	$X\otimes I\otimes I$	$\alpha\lvert 000\rangle+\beta\lvert 111\rangle$
2nd	$\alpha\lvert 010\rangle+\beta\lvert 101\rangle$	1	1	$I\otimes X\otimes I$	$\alpha\lvert 000\rangle+\beta\lvert 111\rangle$
3rd	$\alpha\lvert 001\rangle+\beta\lvert 110\rangle$	0	1	$I\otimes I\otimes X$	$\alpha\lvert 000\rangle+\beta\lvert 111\rangle$

* the measurements here did not effect the state of $\lvert\varphi_2\rangle$.

→ we can entanglement to create repetition

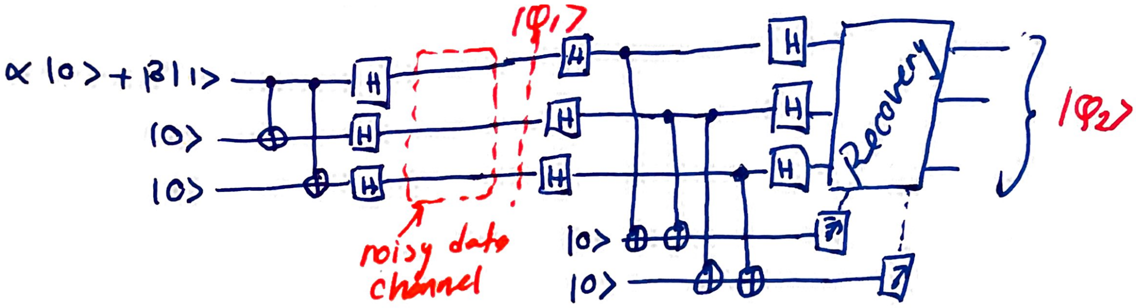
→ we can make measurements that do not destroy information.

example 2: 3-bit phase flip code



$\alpha\lvert 000\rangle+\beta\lvert 100\rangle \longrightarrow \alpha\lvert +++\rangle+\beta\lvert ---\rangle$

* similar to bit-flip code.



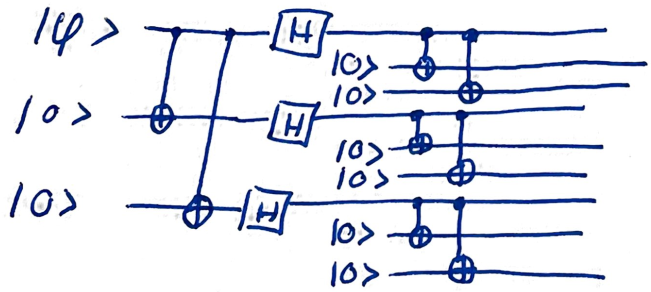
a phase flip does the following:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

phase flip	$\vert\varphi_1\rangle$	M_1	M_2	Recovery	$\vert\varphi_2\rangle$
—	$\alpha\vert{+}{+}{+}\rangle+\beta\vert{-}{-}{-}\rangle$	0	0	$I\otimes I\otimes I$	$\alpha\vert{+}{+}{+}\rangle+\beta\vert{-}{-}{-}\rangle$
1st	$\alpha\vert{-}{+}{+}\rangle+\beta\vert{+}{-}{-}\rangle$	1	0	$Z\otimes I\otimes I$	"
2nd	$\alpha\vert{+}{-}{+}\rangle+\beta\vert{-}{+}{-}\rangle$	1	1	$I\otimes Z\otimes I$	"
3rd	$\alpha\vert{+}{+}{-}\rangle+\beta\vert{-}{-}{+}\rangle$	0	1	$I\otimes I\otimes Z$	"

The Shor Code: a 9-qubit code that combines 3-qubit bit-flip and 3-qubit phase-flip codes as follows;



which encodes the computatitional basis states as

$$|0\rangle \longrightarrow |0_s\rangle = \frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)$$

$$|1\rangle \longrightarrow |1_s\rangle = \frac{1}{2\sqrt{2}}(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$$

Correcting bit flips using Shor code:

suppose we have an arbitrary quantum state $\alpha|0\rangle + \beta|1\rangle$ which is encoded as

$$\frac{1}{2\sqrt{2}}\left(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\right)$$

Suppose 1st qubit is flipped, the new state is

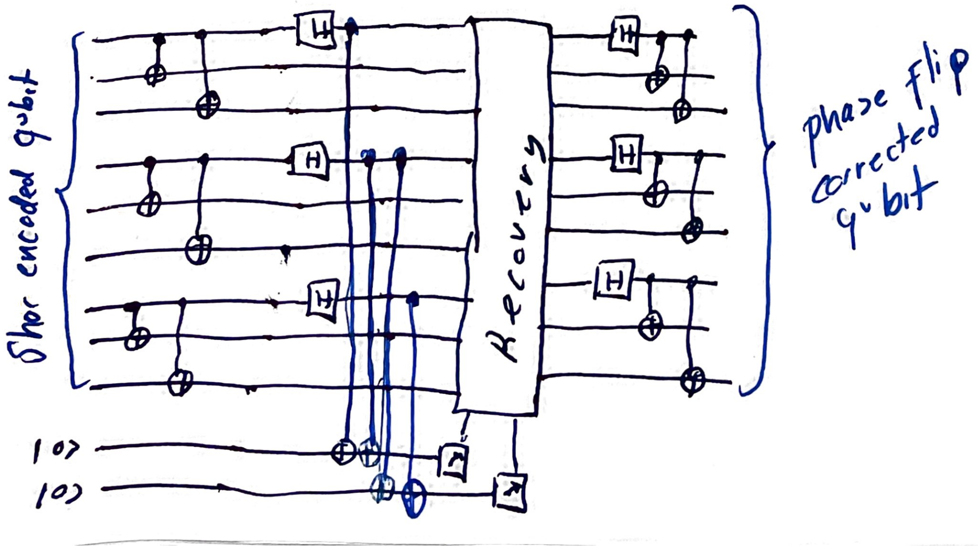
$$\frac{1}{2\sqrt{2}}\left(\alpha(|100\rangle + |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|100\rangle - |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\right)$$

which could be fixed just like in the 3-qubit bit flip code

Correcting phase flips using Shor code:

the Shor code can also correct a phase-flip on any qubit. After a phase flip on the 1st qubit the state becomes

$$\frac{1}{2\sqrt{2}}\left(\alpha(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\right)$$



Correcting any single qubit errors using Shor Code

6

Suppose the 1st qubit encounters an arbitrary error which turns:

$|0\rangle \to a|0\rangle + b|1\rangle$
$|1\rangle \to c|0\rangle + d|1\rangle$, then the state becomes

$$\frac{1}{2\sqrt{2}}\Big(\alpha\big(a|000\rangle + b|100\rangle + c|011\rangle + d|111\rangle\big)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)$$
$$+\beta\big(a|000\rangle + b|100\rangle - c|011\rangle - d|111\rangle\big)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$$

Let $k+m=a$, $k-m=d$, $l+n=b$, $l-n=c$, we get

$$\frac{1}{2\sqrt{2}}\Big(k\big[\alpha(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle) + \beta(|000\rangle-|111\rangle)$$
$$(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\big]^{|00\rangle} + l\big[\alpha(|100\rangle+|011\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)$$
$$+\beta(|100\rangle-|011\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\big]^{|10\rangle} + m\big[\alpha(|000\rangle-|111\rangle)$$
$$(|000\rangle+|111\rangle)(|000\rangle+|111\rangle) + \beta(|000\rangle+|111\rangle)(|000\rangle-|111\rangle)(|000\rangle$$
$$-|111\rangle)\big]^{|00\rangle} + n\big[\alpha(|100\rangle-|011\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)$$
$$+\beta(|100\rangle+|011\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\big]^{|10\rangle}$$

* Now we can do a parity check between 1st and 2nd qubits and 2nd and 3rd qubits, the states of parity checks are appended above in red color.

if the measurement of parity checks are $|00\rangle$ then the state becomes

$$k\big[\alpha(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)+\beta(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$$
$$(|000\rangle-|111\rangle)\big] + m\big[\alpha(|000\rangle-|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)+\beta$$
$$(|000\rangle+|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\big]$$

* in this case there are no bit flips.

if we measure $|10\rangle$ the state becomes

$$l\big[\alpha(|100\rangle+|011\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)+\beta(|100\rangle-|011\rangle)(|000\rangle-|111\rangle)$$
$$(|000\rangle-|111\rangle)\big] + n\big[\alpha(|100\rangle-|011\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)+\beta(|100\rangle$$
$$+|011\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$$

* there is a bit-flip and it can be corrected.

After the parity check and correction for bit flip (7)
we perform a parity check for phase flip. If we measure 0 we get

$$\alpha(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)+\beta(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$$

and if we measure 1,

$$\alpha(|000\rangle-|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)+\beta(|000\rangle+|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$$

Which we can correct, and in all cases we can recover the original state. Therefore the Shor code can not only correct phase and bit flips but also any continous error. by checking and correcting bit and phase flips.

Beyond classical repetition codes there are more sophisticated error correction schemes which are extended to quantum error correction. Such as the Hamming code. The extension of classical error correction codes are called "Calderbank-Shor-Steane" (CSS) codes.
