Notes on Kronecker (Tensor) product :

$$A\otimes(B+C) = A\otimes B + A\otimes C$$

$$(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$$ (mixed product property)

$$(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$$

$$(A\otimes B)^{T} = A^{T}\otimes B^{T}$$

$$|A\otimes B| = |A|^{m}|B|^{n}$$ (A is $n\times n$, B is $m\times m$)

Kronecker sum

$$A\oplus B = A\otimes I_m + I_n\otimes B$$

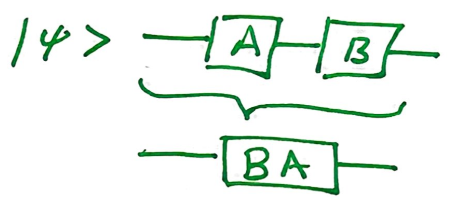
ex.

$$\exp(A\oplus B) = \exp(A)\otimes\exp(B)$$

$$tr(A\otimes B) = tr(A)\,tr(B)$$

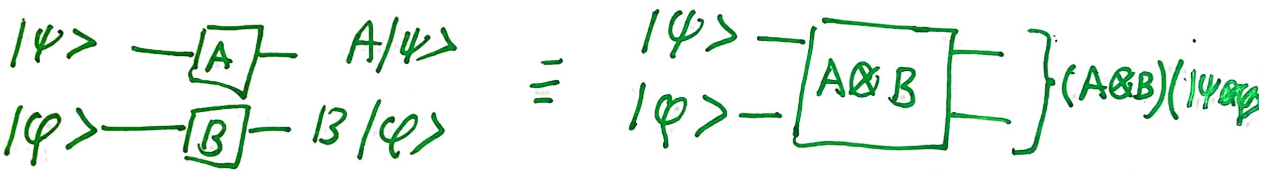
Notes about combining quantum gates

Serial connection:

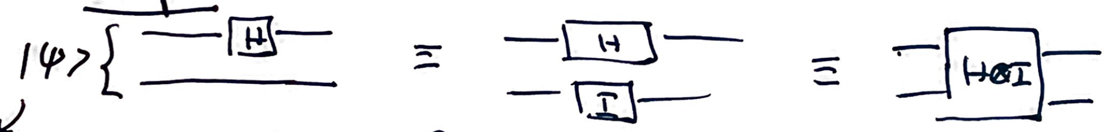


$$\equiv B(A|\psi\rangle)$$

parallel connection



example:



EPR pair

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H \otimes I = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(H \otimes I)(|\psi\rangle) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\equiv \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

Note on the derivation of measurement matrices:
(for multiqubit systems)

$$M_0 = |0\rangle\langle 0| \quad , \quad M_1 = |1\rangle\langle 1|$$

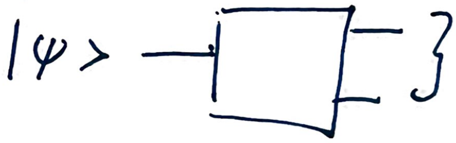
$$M_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M_0^2 = I \otimes M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_0^1 = M_0 \otimes I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_1^2 = I \otimes M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_1^1 = M_1 \otimes I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Non-Cloning theorem:

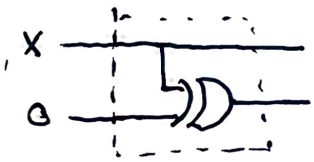
there is no quantum circuit that can do the following:

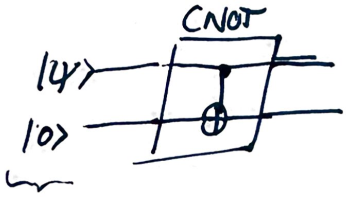


$\Big\}\ |\psi\rangle \otimes |\psi\rangle$ (i.e. two entangled copies of $|\psi\rangle$)

$\forall\ |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Classical: $\longrightarrow$ **Quantum**





$\Big\}\ \alpha|00\rangle + \beta|11\rangle$

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \otimes \begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\alpha\\ 0\\ \beta\\ 0\end{bmatrix}$$

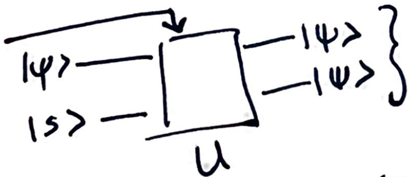
$$\begin{bmatrix}1&&&\\&1&&\\&&0&1\\&&1&0\end{bmatrix}\begin{bmatrix}\alpha\\ 0\\ \beta\\ 0\end{bmatrix} = \begin{bmatrix}\alpha\\ 0\\ 0\\ \beta\end{bmatrix} \quad X \quad \text{since } \alpha\beta \neq 0$$

$$|\psi\rangle \otimes |\psi\rangle = \begin{bmatrix}\alpha^2\\ \alpha\beta\\ \beta\alpha\\ \beta^2\end{bmatrix}$$

$S = |0\rangle$ or $|1\rangle$

proof: (page 532 - Box 12.1)

Assume



$\Big\}\ |\psi\rangle \otimes |\psi\rangle$

$$\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \otimes \begin{bmatrix}s_1\\ s_2\end{bmatrix} = \begin{bmatrix}\psi_1 s_1\\ \psi_1 s_2\\ \psi_2 s_1\\ \psi_2 s_2\end{bmatrix}$$

$$\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} \otimes \begin{bmatrix}s_1\\ s_2\end{bmatrix} = \begin{bmatrix}\varphi_1 s_1\\ \varphi_1 s_2\\ \varphi_2 s_1\\ \varphi_2 s_2\end{bmatrix}$$

$U(|\psi\rangle \otimes |s\rangle) = |\psi\rangle \otimes |\psi\rangle$

$U(|\varphi\rangle \otimes |s\rangle) = |\varphi\rangle \otimes |\varphi\rangle$

$$\begin{bmatrix}\psi_1 s_1\\ \psi_1 s_2\\ \psi_2 s_1\\ \psi_2 s_2\end{bmatrix}^{\dagger} \begin{bmatrix}\psi_1 s_1\\ \psi_1 s_2\\ \psi_2 s_1\\ \psi_2 s_2\end{bmatrix}$$

$\psi_1^* s_1^* \varphi_1 s_1 + \psi_1^* s_2^* \varphi_1 s_2 + \cdots$

$\langle\psi|\varphi\rangle = \langle\psi|\varphi\rangle \otimes \langle s|s\rangle$

$$\langle \psi|\varphi\rangle = \langle \psi|\varphi\rangle \underbrace{\langle s|s\rangle}_{1} = \langle \psi|\varphi\rangle \otimes \langle s|s\rangle$$

$$= (\langle \psi| \otimes \langle s|)(|\varphi\rangle \otimes |s\rangle)$$

$$= \underbrace{(\langle \psi| \otimes \langle s|)U^*}_{(\langle \psi| \otimes \langle \psi|)} \underbrace{U(|\varphi\rangle \otimes |s\rangle)}_{(|\varphi\rangle \otimes |\varphi\rangle)}$$

$$= (\langle \psi|\varphi\rangle)^2$$

$\Rightarrow \langle \psi|\varphi\rangle = 0$ or $\langle \psi|\varphi\rangle = 1$ which contradicts the fact that ψ and φ are arbitrary quantum states.

Quantum teleportation: (Hype!)

* Not the sci-fi ~~the~~ teleportation!

→ We are not teleporting physical objects but information.

The state is transmitted by setting up an entangled state space of 3 qubits and then removing two qubits from the entanglement (via measurement).

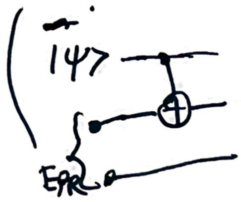
suppose $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and given an EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ the state of the whole system is:

$$\underbrace{\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \otimes \begin{bmatrix}1/\sqrt{2}\\ 0\\ 0\\ 1/\sqrt{2}\end{bmatrix}}_{\frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle+|11\rangle) + \beta|1\rangle(|00\rangle+|11\rangle)\right]} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha\\0\\0\\\alpha\\\beta\\0\\0\\\beta\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle+|11\rangle) + \beta|1\rangle(|00\rangle+|11\rangle)\right] \quad (2)$$

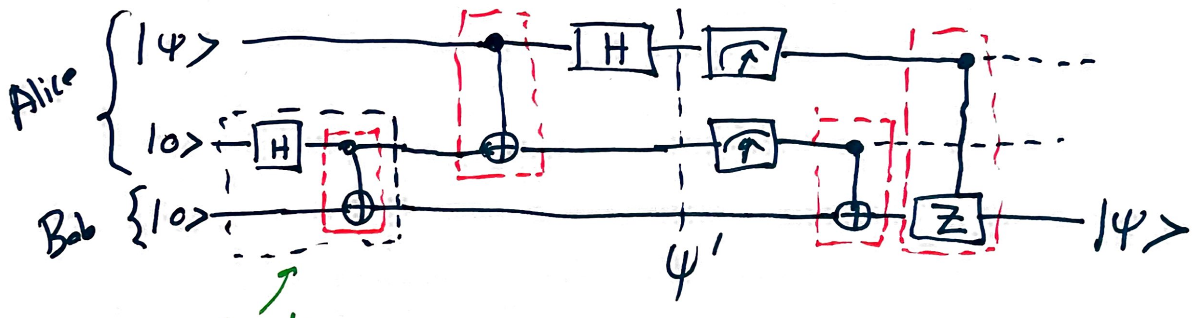
$$= \frac{1}{\sqrt{2}}\alpha|000\rangle + \frac{1}{\sqrt{2}}\alpha|011\rangle + \frac{1}{\sqrt{2}}\beta|100\rangle + \frac{1}{\sqrt{2}}\beta|111\rangle$$

↓ apply CNOT first two bits



$$\underbrace{\frac{1}{\sqrt{2}}\alpha|000\rangle + \frac{1}{\sqrt{2}}\alpha|011\rangle + \frac{1}{\sqrt{2}}\beta|110\rangle + \frac{1}{\sqrt{2}}\beta|101\rangle}_{\frac{1}{\sqrt{2}}\begin{bmatrix}\alpha\\0\\0\\\alpha\\0\\\beta\\\beta\\0\end{bmatrix}}$$

$$= \frac{1}{\sqrt{2}}\left(\alpha|0\rangle(|00\rangle+|11\rangle) + \beta|1\rangle(|10\rangle+|01\rangle)\right) \leftarrow (1)$$



Alice measures the second qubit.

$\frac{1}{2}$ chance it is zero $\rightarrow$ $\alpha|00\rangle + \beta|11\rangle$

$\frac{1}{2}$ chance it is one $\rightarrow$ $\alpha|01\rangle + \beta|10\rangle$

send first bit of (1) through Hadamard gate:

$$\psi' = \frac{1}{2}\left[\alpha(|0\rangle+|1\rangle)(|00\rangle+|11\rangle) + \beta(|0\rangle-|1\rangle)(|10\rangle+|01\rangle)\right)$$

yönü takip et

↓ rewriting this state

$$\frac{1}{2}\Big(|00\rangle(\alpha|0\rangle+\beta|1\rangle) + |01\rangle(\alpha|1\rangle+\beta|0\rangle) + |10\rangle(\alpha|0\rangle-\beta|1\rangle) + |11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big)$$