Lecture Notes of Mark Oskin (Univ. of Washington) (4)

Quantum Mechanics → a mathematical language (like calculus)

Quantum Physics → Explaining nature using quantum mechanics (like explaining nature using calculus in classical physics)

Quantum Computing → Quantum computers are reasoned using quantum mechanics (Like classical computers are reasoned using boolean algebra)

Postulates of Quantum Mechanics:

1. Definition of quantum bits (qubits)
2. How qubits transform (rotate)
3. The effect of measurement
4. How qubits combine into a system of qubits.

Postulate 1:

"Associated to any isolated physical system is a complex vector space with inner product (i.e. Hilbert Space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space."

Definition 1.1: An inner product (also called dot or scalar product) on a complex vector space, $\mathbb{V}$ is a function $\langle \cdot , \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ that satisfies the following for all $v, v_1, v_2 \in \mathbb{V}$ an $c \in \mathbb{C}$

1) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

2) $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$
and $\langle v_1, v_2 + v_3 \rangle = \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$

3) $\langle c v_1, v_2 \rangle = c \langle v_1, v_2 \rangle$ and $\langle v_1, c v_2 \rangle = \bar{c} \langle v_1, v_2 \rangle$

4) $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$

Definition 1.2: A complex inner product space is a complex vector space along with an inner product.

examples:

- $\mathbb{R}^n$: $\langle v_1, v_2 \rangle = v_1^T v_2$
- $\mathbb{C}^n$: $\langle v_1, v_2 \rangle = v_1^+ v_2$
- $\mathbb{R}^{n\times n}$: $\langle A, B \rangle = \text{Trace}(A^T B)$
- $\mathbb{C}^{n\times n}$: $\langle A, B \rangle = \text{Trace}(A^+ B)$

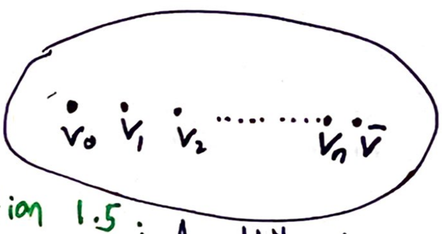
Definition 1.3: Within an inner product space $\mathbb{V}$ a sequence of vectors $v_0, v_1, v_2 \ldots.$ is called the Cauchy sequence if for every $\varepsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that

for all $m, n \geq N_0$ $d(v_m, v_n) \leq \varepsilon$

Definition 1.4: A complex inner product space is called complete if for any cauchy sequence of vectors $v_0, v_1, v_2, \ldots.$ there exists $\bar{v} \in \mathbb{V}$ such that

$$\lim_{n\to\infty} |v_n - \bar{v}| = 0$$

e.g.



Definition 1.5: A Hilbert space is a complex inner product space that is complete.

But don't worry!

Proposition: Every inner product of a finite-dimensional complex vector space is automatically complete. Hence, every finite dimensional complex vector space with an inner product is automatically an Hilbert state.

Now, let's consider a single qubit (a two dimensional state space)

Let $|\phi_0\rangle$ and $|\phi_1\rangle$ be an orthonormal basis for the space then a qubit $|\psi\rangle = \underbrace{\alpha|\phi_0\rangle + \beta|\phi_1\rangle}_{\text{must be a unit vector}}$

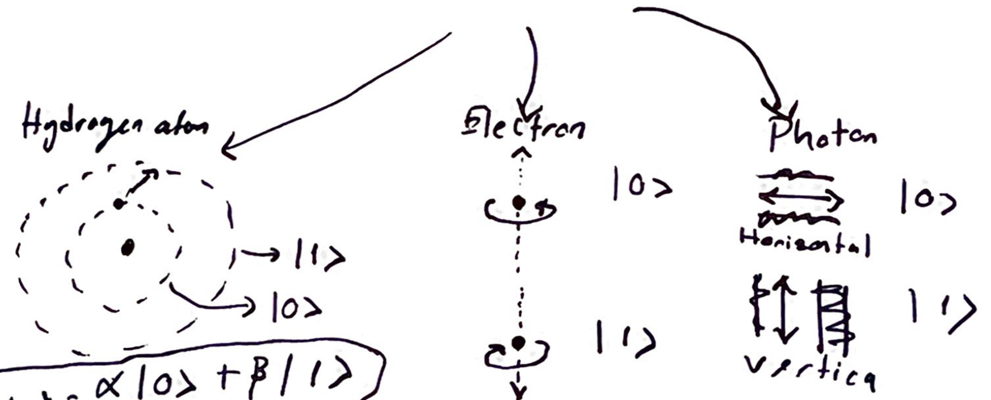
For example: $|\phi_0\rangle = |0\rangle$ and $|\phi_1\rangle = |1\rangle$

an extension of classical computing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \{0, 1\}$

Classical bit | Quantum bit

0 → high voltage | $|0\rangle$

1 → low Voltage | $|1\rangle$

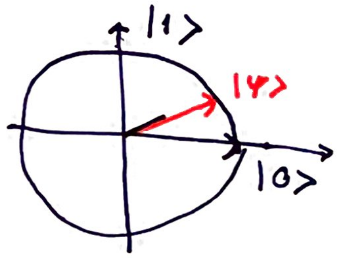


Since $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a unit vector, $\langle\psi|\psi\rangle = 1$ (i.e. $|\alpha|^2 + |\beta|^2 = 1$), in quantum computing $\alpha, \beta \in \mathbb{C}$

then, $|\psi\rangle$ is a qubit.

$$|\psi\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$$

(Bracket) $\begin{bmatrix}1 \\ 0\end{bmatrix}$ $\begin{bmatrix}0 \\ 1\end{bmatrix}$



*Dirac's Bra-ket notation:

$|\cdot\rangle$: a column vector · : "ket"

$\langle\cdot|$: conjugate transpose of $|\cdot\rangle$, row vector · : "bra"

inner product $\langle x, y\rangle = x^{\dagger} y = \langle x|\cdot|y\rangle = \langle x|y\rangle$

outer product $|x\rangle\langle y| = x y^{\dagger}$

examples:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$$

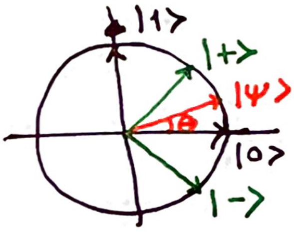
$\nearrow$ $|0\rangle$ with prob. $1/2$

$\searrow$ $|1\rangle$ with prob $1/2$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = |-\rangle$$

" same

$$|\psi\rangle = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$



$|0\rangle$ with prob. $(\cos\theta)^2$

$|1\rangle$ with prob $(\sin\theta)^2$