Deutsch-Jozsa Algorithm:

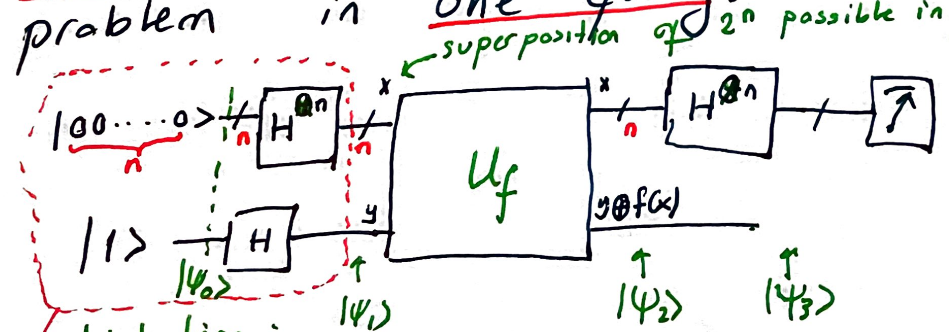
(A generalization of Deutsch's algorithm)

Instead of considering $f(x): \{0,1\} \to \{0,1\}$, let us consider $f(x)\ \{0,1\}^n \longrightarrow \{0,1\}$; i.e. f accepts a string of 0's and 1's of size n, and returns 0 or 1. (We can also consider this input as a ~~natural~~ whole number between 0 and 2^n-1)

The function (f) is balanced if half of the inputs give 0 and the other half 1; and if is constant if it always returns 0 for all inputs and returns 1 for all inputs. We know f is either constant or balanced. How can we find out if it is constant/balanced?

The classical solution: Try different inputs. if two inputs produce different output we can say it is balanced. In the worst case we need to make just one more query than the half of the possible inputs. ($\frac{2^n}{2}+1$ queries), then we can be sure if it is constant or balanced.

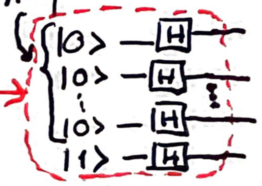
The Quantum Solution: Deutsch-Jozsa algorithm solves this problem in one query!



New Notation:

$\underbrace{|00\cdots0\rangle}_{n\text{-qubits}} = |0\rangle^{\otimes n} \rightarrow |\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle$ and $H^{\otimes n} = \underbrace{H\otimes H\otimes\cdots\otimes H}_{n\text{-of them}}$

i.e.



$|\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle$

$\langle\cdot,\cdot\rangle : \{0,1\}^n \times \{0,1\}^n \longrightarrow \{0,1\}$

$\langle x,y\rangle = (x_0 y_0) \oplus (x_1 y_1) \cdots \oplus (x_{n-1} y_{n-1})$

$\langle x_1 \oplus x_2, y\rangle = \langle x_1, y\rangle \oplus \langle x_2, y\rangle$

$\langle x, y_1 \oplus y_2\rangle = \langle x, y_1\rangle \oplus \langle x, y_2\rangle$

$$|\psi_1\rangle = \sum_{\underbrace{x\in\{0,1\}^n}_{\text{all possible bit strings of size } n}} \frac{|x\rangle}{\sqrt{2^n}}\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

(for example if $n=2$: 00, 01, 10, 11)

since $U_f : |x, y\rangle \longrightarrow |x, y\oplus f(x)\rangle$

$$|\psi_2\rangle = \sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}}\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

consider $x=0$ and $x=1$ for a single qubit:

$$H|x\rangle = \sum_{z\in\{0,1\}} (-1)^{\langle x,z\rangle}|z\rangle/\sqrt{2}$$, thus

$$H^{\otimes n}|x_1 x_2 \ldots x_n\rangle = \frac{\sum_{z_1, z_2 \ldots z_n} (-1)^{x_1 z_1 \oplus x_2 z_2 \oplus \ldots \oplus x_n z_n} |z_1 z_2 \ldots z_n\rangle}{\sqrt{2^n}}$$

$$\rightarrow H^{\otimes n}|x\rangle = \frac{\sum_z (-1)^{\langle x,z\rangle}|z\rangle}{\sqrt{2^n}}$$

$$\rightarrow |\psi_3\rangle = \sum_z \sum_x \frac{(-1)^{\langle x,z\rangle + f(x)}|z\rangle}{2^n}\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Q: what is the probability of measuring top qubits of $|\psi_3\rangle$ as $|0\rangle$?

A: set $z = |0\rangle$, then $\langle x, 0\rangle = 0$ for all x, then

$$|\psi_3\rangle = \sum_x \underbrace{\frac{(-1)^{f(x)}}{2^n}}_{\text{amplitude of state } |0\rangle^{\otimes n}}|0\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

if $f(x)$ is constant 1 then the top qubits become $-\frac{2^n}{2^n}|0\rangle = -|0\rangle$

if $f(x)$ is constant 0 then $+|0\rangle$

if $f(x)$ is balanced $\underline{0}|0\rangle \rightarrow$ some qubits are not zero

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