In [1]:
```
# import necessary libraries
import yfinance as yf
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from arch import arch_model
from sklearn.metrics import mean_absolute_error, mean_squared_error
from statsmodels.graphics.tsaplots import plot_acf
from statsmodels.stats.diagnostic import acorr_ljungbox
from statsmodels.tsa.stattools import adfuller
import warnings
warnings.filterwarnings("ignore")
```

In [2]:
```
# Data retrieval
ticker_symbol = '^GSPC'  # S&P 500 Index
start_date = '2016-01-01'
end_date = '2025-05-31' # Or current date if you want the most rece

data = yf.download(ticker_symbol, start=start_date, end=end_date)

# Preview the data
print(data.head())
```

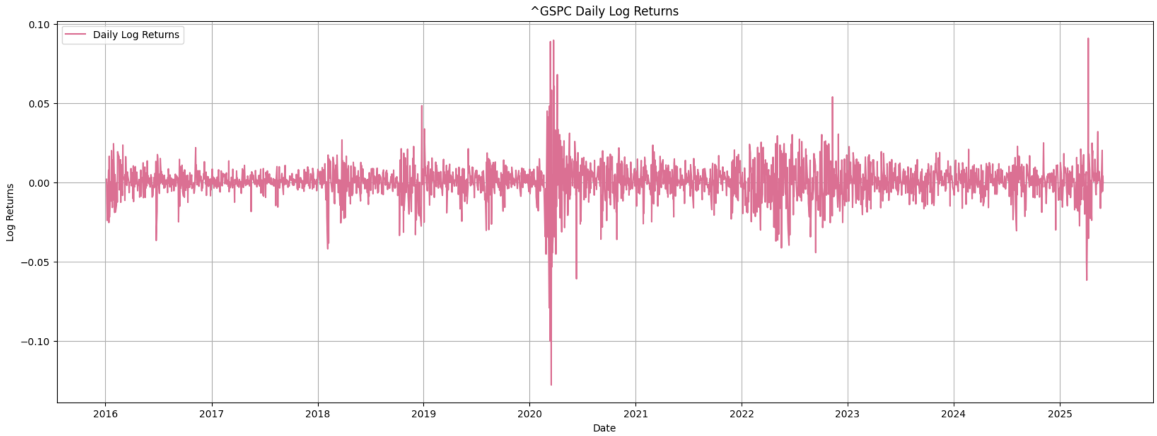
```
[*********************100%***********************]  1 of 1 completed
Price                 Close         High          Low         Open
Volume
Ticker                ^GSPC        ^GSPC        ^GSPC        ^GSPC
^GSPC
Date
2016-01-04  2012.660034  2038.199951  1989.680054  2038.199951  4304
880000
2016-01-05  2016.709961  2021.939941  2004.170044  2013.780029  3706
620000
2016-01-06  1990.260010  2011.709961  1979.050049  2011.709961  4336
660000
2016-01-07  1943.089966  1985.319946  1938.829956  1985.319946  5076
590000
2016-01-08  1922.030029  1960.400024  1918.459961  1945.969971  4664
940000
```

In [3]:
```
# Calculate daily logarithmic returns

# Choose the correct price column
price_col = 'Adj Close' if 'Adj Close' in data.columns else 'Close'

returns = np.log(data[price_col] / data[price_col].shift(1)).dropna

# Display the first few returns
print("\n First few returns:")
print(returns.head())
```

```
 First few returns:
Ticker         ^GSPC
Date
2016-01-05  0.002010
2016-01-06 -0.013202
2016-01-07 -0.023986
2016-01-08 -0.010898
2016-01-11  0.000853
```

In [4]:

```
# Plot the returns to observe volatility clustering
plt.figure(figsize=(20, 7))
plt.plot(returns.index, returns,color = "palevioletred", label='Dai
plt.title(f'{ticker_symbol} Daily Log Returns')
plt.xlabel('Date')
plt.ylabel('Log Returns')
plt.grid(True)
plt.legend()
plt.show()
```



In [5]:

```
# Optional: Check for stationarity (e.g., using Augmented Dickey-Fu

adf_result = adfuller(returns)
print(f'ADF Statistic: {adf_result}')
print(f'P-value: {adf_result[5]}')
# (A low p-value indicates stationarity, which is generally expecte
```

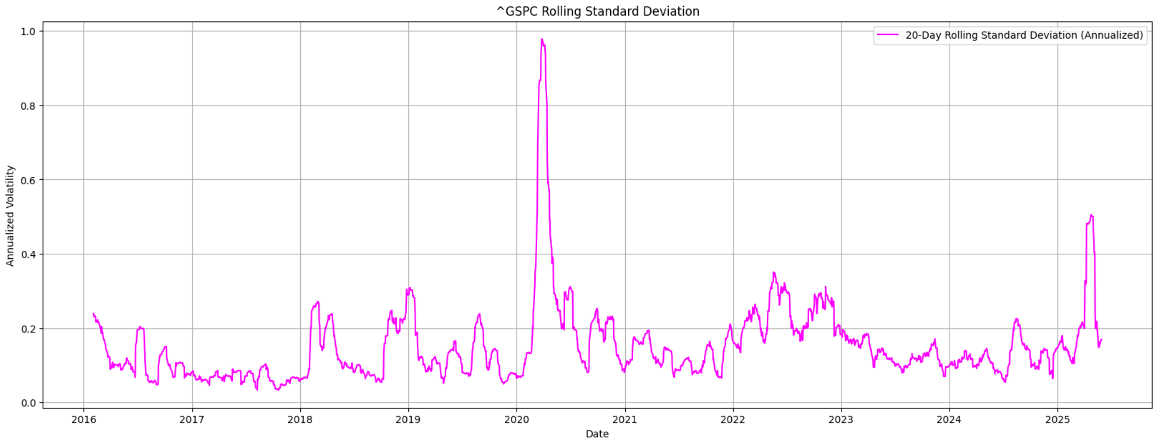
```
ADF Statistic: (np.float64(-15.501836862462824), np.float64(2.410073
622861824e-28), 8, 2356, {'1%': np.float64(-3.433128624404979), '
5%': np.float64(-2.8627675485259805), '10%': np.float64(-2.567423477
1965376)}, np.float64(-14327.675405444457))
P-value: -14327.675405444457
```

In [6]:

```
# Define a rolling window size (e.g., 20 days for approximate month
window_size = 20

# Calculate rolling standard deviation and annualize it
# Annualization factor is sqrt(252) for daily returns
rolling_std = returns.rolling(window=window_size).std() * np.sqrt(2

# Display the first few rolling standard deviations
print("\nFirst few rolling standard deviations (annualized):")
print(rolling_std.head())
```

```
First few rolling standard deviations (annualized):
Ticker      ^GSPC
Date
2016-01-05    NaN
2016-01-06    NaN
2016-01-07    NaN
2016-01-08    NaN
2016-01-11    NaN
```

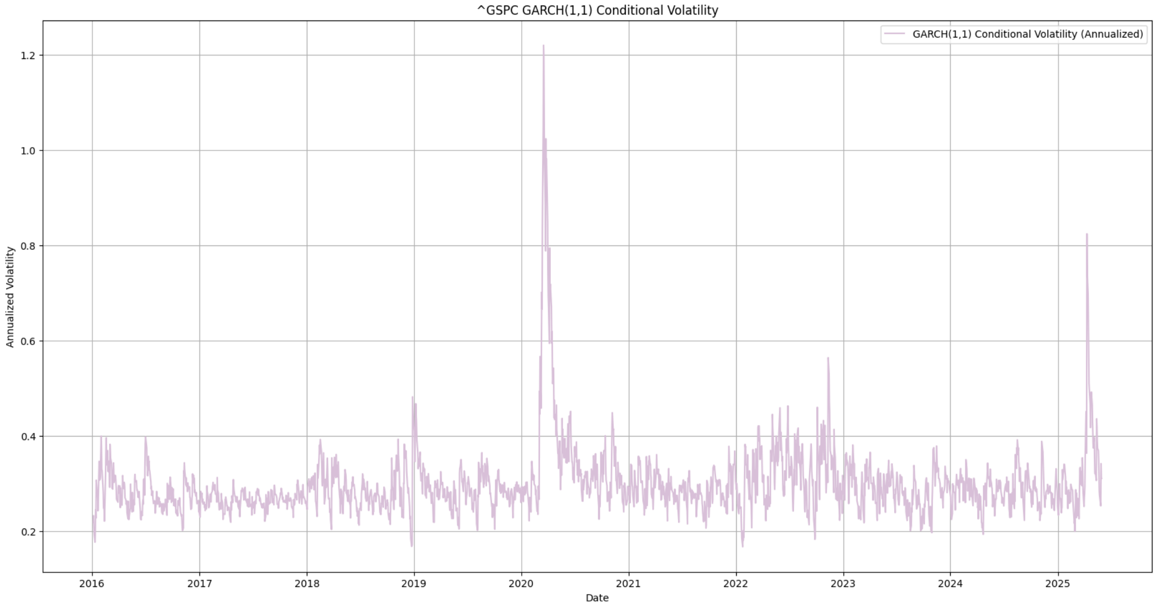
In [7]:
```
# Plot rolling standard deviation
plt.figure(figsize=(20, 7))
plt.plot(rolling_std.index, rolling_std, label=f'{window_size}-Day
plt.title(f'{ticker_symbol} Rolling Standard Deviation')
plt.xlabel('Date')
plt.ylabel('Annualized Volatility')
plt.grid(True)
plt.legend()
plt.show()
```



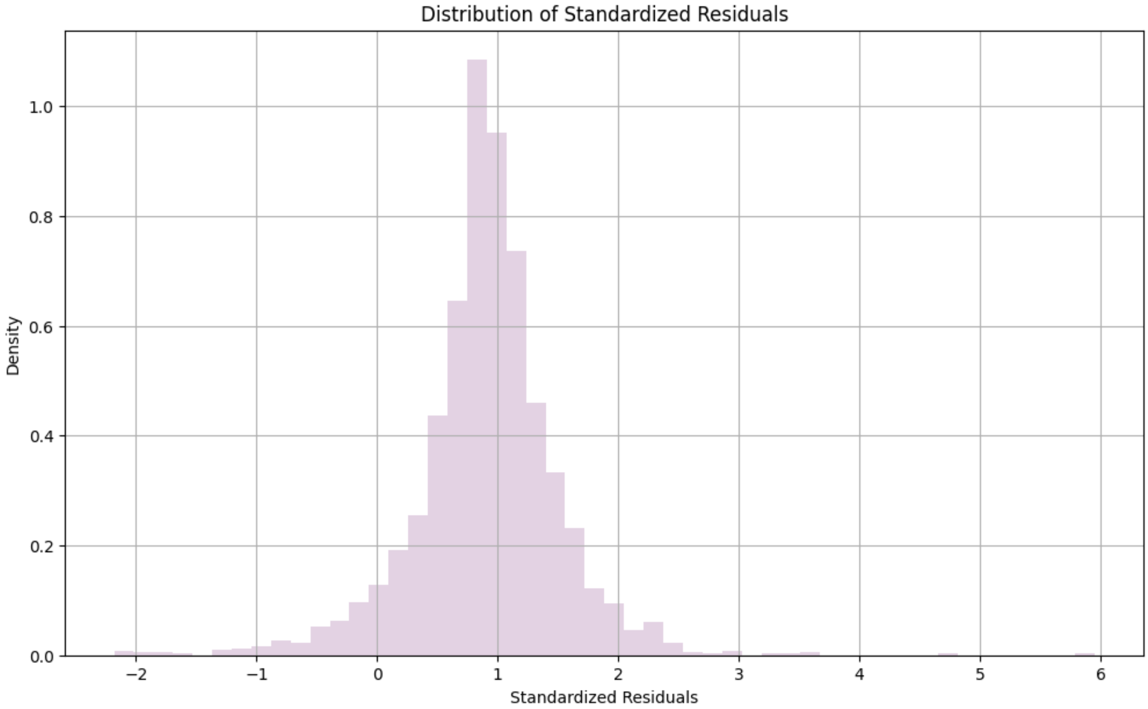
In [8]:
```
'''
The mean return of the S&P 500 is statistically significant but ver
The volatility of the S&P 500 returns is not constant but exhibits
volatility. This is evidenced by the highly significant alpha[1] an
Both past unexpected market movements (alpha[1]) and past volatilit
The sum of alpha[1] and beta[1] being 0.98 indicates very persisten
'''


# Specify the GARCH(1,1) model
# mean='Constant' assumes a constant mean for returns
# vol='Garch' specifies the GARCH volatility model
# p=1, q=1 for GARCH(1,1)
# dist='normal' for Gaussian errors, 'skewt' or 't' for fat tails (
garch_model = arch_model(returns, mean='Constant', vol='Garch', p=1

# Fit the model
# update_freq=10 prints progress every 10 iterations
garch_results = garch_model.fit(disp='off') # disp='off' to suppres

# Print the model summary
print("\nGARCH(1,1) Model Summary:")
print(garch_results.summary())

# Extract conditional volatility from the fitted model
```

```
garch_conditional_volatility = garch_results.conditional_volatility

# Plot conditional volatility
plt.figure(figsize=(20, 10))
plt.plot(garch_conditional_volatility.index, garch_conditional_vola
plt.title(f'{ticker_symbol} GARCH(1,1) Conditional Volatility')
plt.xlabel('Date')
plt.ylabel('Annualized Volatility')
plt.grid(True)
plt.legend()
plt.show()
```

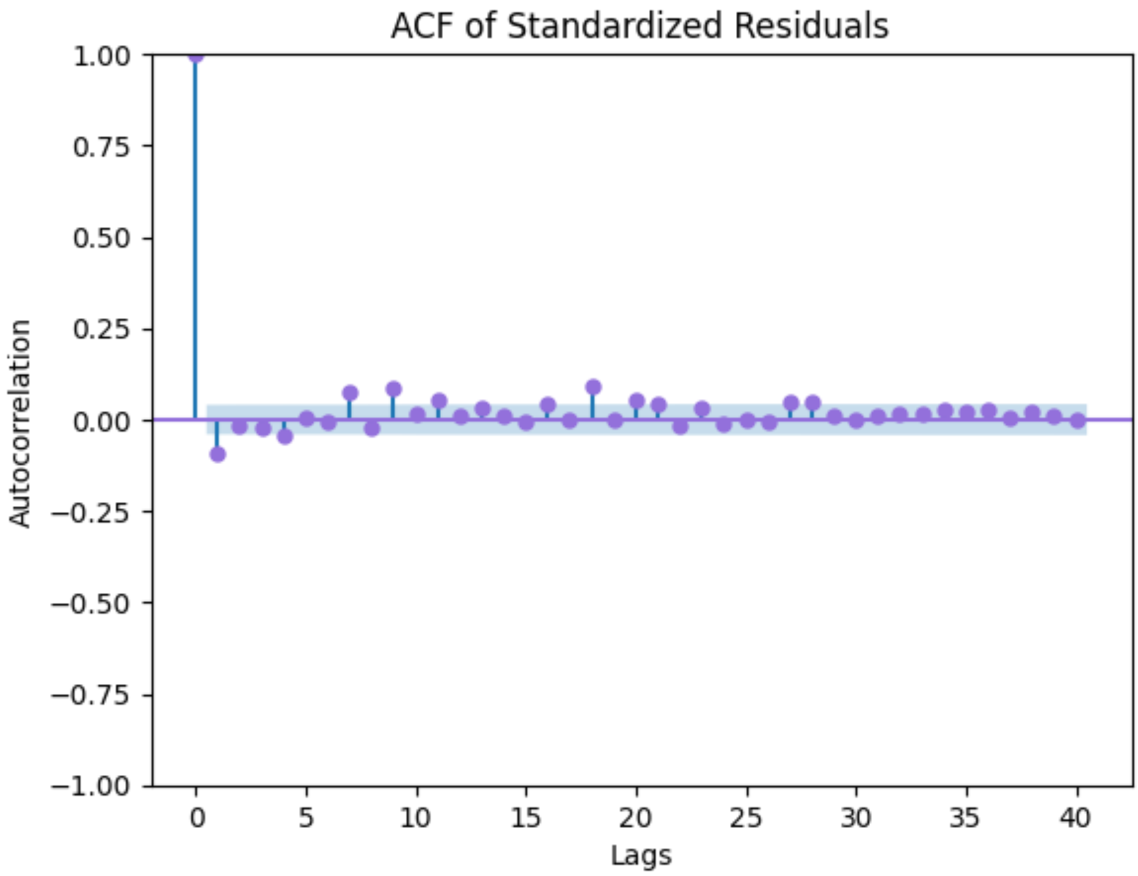
```
GARCH(1,1) Model Summary:
                     Constant Mean - GARCH Model Results
====================================================================
==========
Dep. Variable:                  ^GSPC   R-squared:
0.000
Mean Model:             Constant Mean   Adj. R-squared:
0.000
Vol Model:                      GARCH   Log-Likelihood:
5872.56
Distribution:                  Normal   AIC:
-11737.1
Method:            Maximum Likelihood   BIC:
-11714.0
                                        No. Observations:
2365
Date:                Wed, Jun 25 2025   Df Residuals:
2364
Time:                        09:09:17   Df Model:
1
                                  Mean Model
====================================================================
==========
                 coef    std err          t      P>|t|        95.0%
Conf. Int.
--------------------------------------------------------------------
----------
mu            -0.0160  4.368e-04    -36.599 2.986e-293 [-1.684e-02,-
1.513e-02]
                              Volatility Model
====================================================================
========
                 coef    std err          t      P>|t|      95.0% Co
nf. Int.
--------------------------------------------------------------------
--------
omega      2.7946e-06  2.946e-07      9.487  2.368e-21 [2.217e-06,3.
372e-06]
alpha[1]       0.2001  6.132e-02      3.263  1.103e-03   [7.989e-02,
0.320]
beta[1]        0.7799  7.020e-02     11.110  1.126e-28     [  0.642,
0.918]
====================================================================
========

Covariance estimator: robust
```



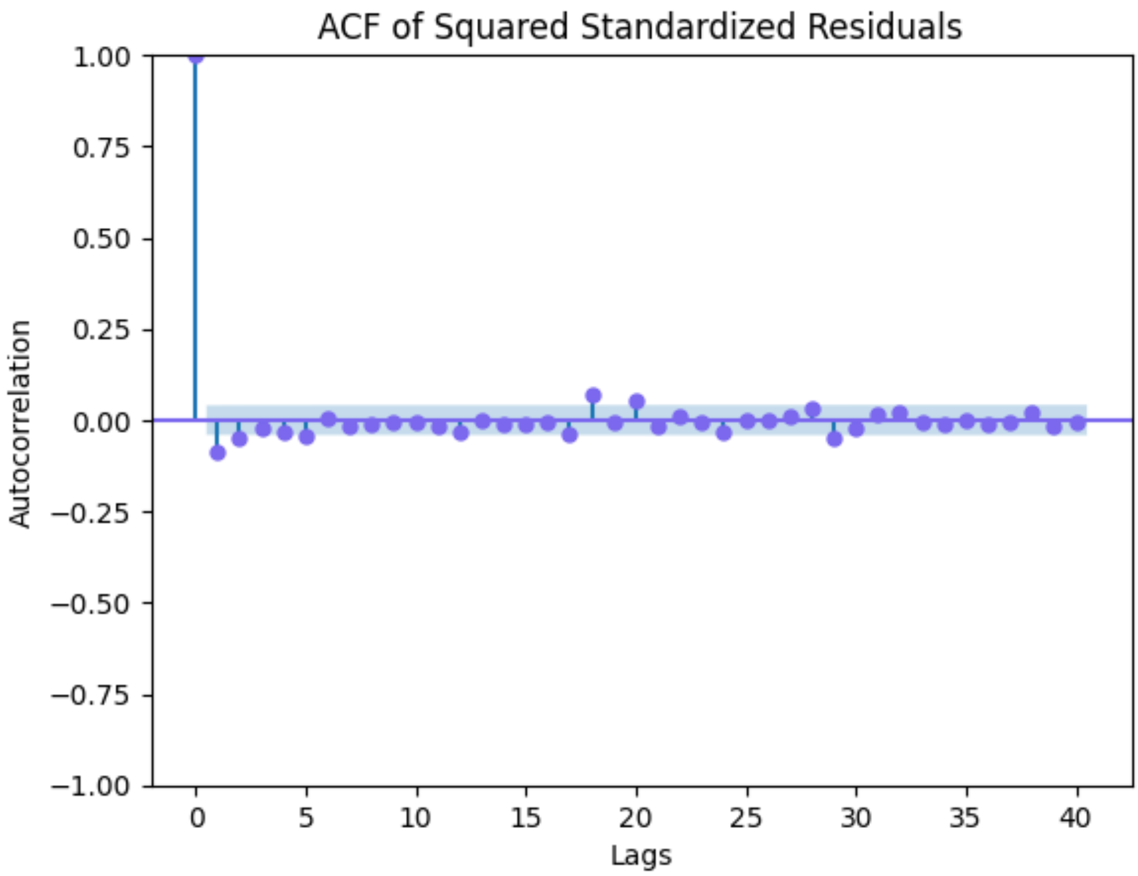
In [14]:
```
# Get standardized residuals
standardized_residuals = garch_results.resid / garch_results.condit

# Plot distribution of standardized residuals
plt.figure(figsize=(12, 7))
plt.hist(standardized_residuals, bins=50, density=True, alpha=0.7,
plt.title('Distribution of Standardized Residuals')
plt.xlabel('Standardized Residuals')
plt.ylabel('Density')
plt.grid(True)
plt.show()

# Plot ACF of standardized residuals
plt.figure(figsize=(20, 10))
plot_acf(standardized_residuals, lags=40, alpha=0.05 , color = 'med
plt.title('ACF of Standardized Residuals')
plt.xlabel('Lags')
plt.ylabel('Autocorrelation')
plt.show()

# Plot ACF of squared standardized residuals (to check for remainin
plt.figure(figsize=(20, 10))
plot_acf(standardized_residuals**2, lags=40, alpha=0.05, color = 'm
plt.title('ACF of Squared Standardized Residuals')
plt.xlabel('Lags')
plt.ylabel('Autocorrelation')
plt.show()
```



<Figure size 2000x1000 with 0 Axes>



<Figure size 2000x1000 with 0 Axes>



In [11]:
```
# Ljung-Box test on standardized residuals
# H0: The data is independently distributed (white noise)
# If p-value < 0.05, reject H0, indicating remaining autocorrelatio

lb_test_residuals = acorr_ljungbox(standardized_residuals, lags=[1,
print("\nLjung-Box Test on Standardized Residuals:")
print(lb_test_residuals)

# Ljung-Box test on squared standardized residuals
lb_test_squared_residuals = acorr_ljungbox(standardized_residuals**
print("\nLjung-Box Test on Squared Standardized Residuals:")
print(lb_test_squared_residuals)

# Interpretation: the given p-values are high (e.g., > 0.05), it su
```

```
Ljung-Box Test on Standardized Residuals:
      lb_stat     lb_pvalue
1   20.602436  5.652415e-06
13  69.076151  1.186958e-09

Ljung-Box Test on Squared Standardized Residuals:
      lb_stat  lb_pvalue
1   17.278624   0.000032
13  36.127976   0.000567
```

In [12]:
```
# Define the size of the training window for rolling forecasts
# A common choice is 250 days (approx. 1 year of trading days)
train_window_size = 250

# Initialize lists to store forecasts and realized volatility
garch_forecasts = []
```

```
realized_volatility_proxy = []
# Using squared returns as a simple proxy for daily realized volati

# Loop through the returns data to perform rolling forecasts
# Start forecasting after the initial training window
for i in range(train_window_size, len(returns)):
    # Define the training data window
    train_data = returns.iloc[i - train_window_size:i]

    # Fit GARCH(1,1) model to the training data
    # disp='off' to suppress output for each fit
    model = arch_model(train_data, mean='Constant', vol='Garch', p=
    res = model.fit(disp='off')

    # Make a one-step-ahead forecast
    # forecast() returns a DataFrame with 'h.1' column for 1-step a
    forecast = res.forecast(horizon=1)
    garch_next_period_variance = forecast.variance.iloc[-1, 0] # Ge

    # Store the forecasted volatility (square root of variance)
    garch_forecasts.append(np.sqrt(garch_next_period_variance) * np

    # Store the actual realized volatility for the next period (squ
    # For simplicity, we use the squared return of the next day as 
    # In practice, realized volatility is often calculated from int
    realized_volatility_proxy.append(np.abs(returns.iloc[i]) * np.s

# Create a DataFrame for forecasts and realized volatility
forecast_index = returns.index[train_window_size:]
garch_forecast_series = pd.Series(garch_forecasts, index=forecast_i
realized_vol_series = pd.Series(realized_volatility_proxy, index=fo

print("\nRolling GARCH Forecasts (first 5):")
print(garch_forecast_series.head())
print("\nRealized Volatility Proxy (first 5):")
print(realized_vol_series.head())
```

```
/opt/anaconda3/envs/yaman/lib/python3.13/site-packages/arch/univaria
te/base.py:768: ConvergenceWarning: The optimizer returned code 4. T
he message is:
Inequality constraints incompatible
See scipy.optimize.fmin_slsqp for code meaning.

  warnings.warn(
/opt/anaconda3/envs/yaman/lib/python3.13/site-packages/arch/univaria
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  warnings.warn(
/opt/anaconda3/envs/yaman/lib/python3.13/site-packages/arch/univaria
te/base.py:768: ConvergenceWarning: The optimizer returned code 4. T
he message is:
Inequality constraints incompatible
See scipy.optimize.fmin_slsqp for code meaning.

  warnings.warn(
```
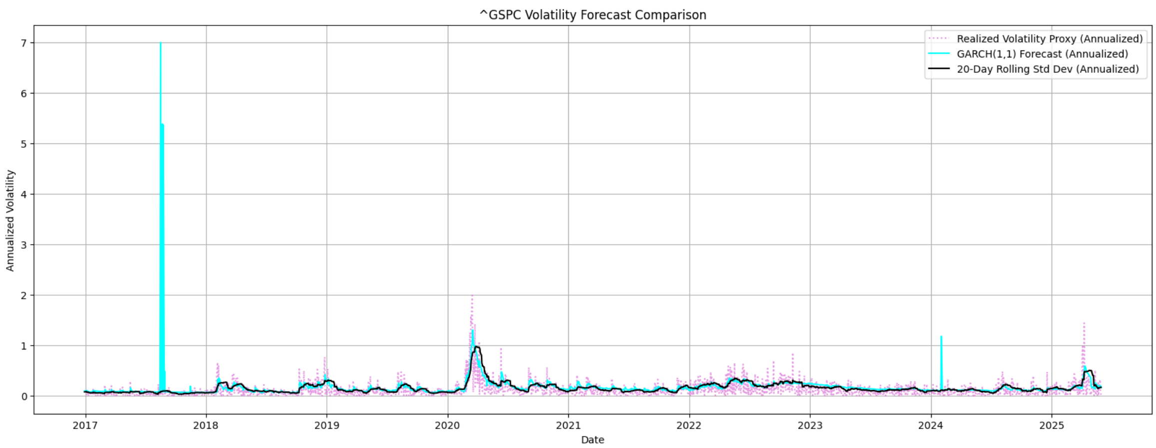

```
Rolling GARCH Forecasts (first 5):
Date
2016-12-30    0.094776
2017-01-03    0.097672
2017-01-04    0.106759
2017-01-05    0.104114
2017-01-06    0.096189
dtype: float64

Realized Volatility Proxy (first 5):
Date
2016-12-30    Ticker
^GSPC    0.073782
Name: 2016-12-30 00:0...
2017-01-03    Ticker
^GSPC    0.134152
Name: 2017-01-03 00:0...
2017-01-04    Ticker
^GSPC    0.090579
Name: 2017-01-04 00:0...
2017-01-05    Ticker
^GSPC    0.012239
Name: 2017-01-05 00:0...
2017-01-06    Ticker
^GSPC    0.055732
Name: 2017-01-06 00:0...
dtype: object
```

In [13]:

```python
# The main issue is that realized_vol_series contains elements that
# so when creating the DataFrame, the 'realized' column becomes a c
# We need to flatten realized_vol_series so each value is a float,

# Convert realized_vol_series to a float Series (extract the value
def flatten_realized_vol(series):
    # If each element is a pandas Series, extract the first value
    return series.apply(lambda x: x.iloc[0] if hasattr(x, 'iloc') e

realized_vol_flat = flatten_realized_vol(realized_vol_series)

# Now align all series to the same index and length
combined_df = pd.DataFrame({
    'realized': realized_vol_flat,
    'garch_forecast': garch_forecast_series,
    'rolling_std': rolling_std.loc[garch_forecast_series.index].squ
}).dropna()

# Extract the aligned series
realized_vol_series_aligned = combined_df['realized']
garch_forecast_series_aligned = combined_df['garch_forecast']
rolling_std_aligned = combined_df['rolling_std']

# Check if the aligned series are empty before proceeding
if realized_vol_series_aligned.empty:
    print("Error: No common data points for comparison after alignm
else:
    # Plotting Forecasted vs. Realized Volatility
```

```
    plt.figure(figsize=(20, 7))
    plt.plot(realized_vol_series_aligned.index, realized_vol_series_
    plt.plot(garch_forecast_series_aligned.index, garch_forecast_se
    plt.plot(rolling_std_aligned.index, rolling_std_aligned, label=
    plt.title(f'{ticker_symbol} Volatility Forecast Comparison')
    plt.xlabel('Date')
    plt.ylabel('Annualized Volatility')
    plt.legend()
    plt.grid(True)
    plt.show()

    # Statistical Metrics for GARCH(1,1)
    garch_mae = mean_absolute_error(realized_vol_series_aligned, ga
    garch_mse = mean_squared_error(realized_vol_series_aligned, gar
    garch_rmse = np.sqrt(garch_mse)
    garch_correlation = garch_forecast_series_aligned.corr(realized_

    print("\n--- GARCH(1,1) Forecast Performance ---")
    print(f"Mean Absolute Error (MAE): {garch_mae:.6f}")
    print(f"Mean Squared Error (MSE): {garch_mse:.6f}")
    print(f"Root Mean Squared Error (RMSE): {garch_rmse:.6f}")
    print(f"Correlation (Forecast vs. Realized): {garch_correlation

    # Statistical Metrics for Rolling Standard Deviation
    rolling_std_mae = mean_absolute_error(realized_vol_series_align
    rolling_std_mse = mean_squared_error(realized_vol_series_aligne
    rolling_std_rmse = np.sqrt(rolling_std_mse)
    rolling_std_correlation = rolling_std_aligned.corr(realized_vol_

    print("\n--- Rolling Standard Deviation Forecast Performance --
    print(f"Mean Absolute Error (MAE): {rolling_std_mae:.6f}")
    print(f"Mean Squared Error (MSE): {rolling_std_mse:.6f}")
    print(f"Root Mean Squared Error (RMSE): {rolling_std_rmse:.6f}"
    print(f"Correlation (Forecast vs. Realized): {rolling_std_corre

    # Summary Comparison Table
    print("\n--- Comparative Performance Summary ---")
    print("| Model | MAE | MSE | RMSE | Correlation |")
    print("|-------------------------|----------|----------|------
    print(f"| GARCH(1,1) | {garch_mae:.6f} | {garch_mse:.6f} | {gar
    print(f"| Rolling Standard Deviation | {rolling_std_mae:.6f} |
```



```
--- GARCH(1,1) Forecast Performance ---
Mean Absolute Error (MAE): 0.103000
Mean Squared Error (MSE): 0.069568
Root Mean Squared Error (RMSE): 0.263758
Correlation (Forecast vs. Realized): 0.2305

--- Rolling Standard Deviation Forecast Performance ---
Mean Absolute Error (MAE): 0.089855
Mean Squared Error (MSE): 0.017252
Root Mean Squared Error (RMSE): 0.131346
Correlation (Forecast vs. Realized): 0.5470

--- Comparative Performance Summary ---
| Model | MAE | MSE | RMSE | Correlation |
|--------------------------|----------|----------|----------|-------------|
| GARCH(1,1) | 0.103000 | 0.069568 | 0.263758 | 0.2305 |
| Rolling Standard Deviation | 0.089855 | 0.017252 | 0.131346 | 0.5470 |
```