ASSIGNMENT-15

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1 QUESTION No-2.9(Optimization)

Find the maximum area of an isosceles triangle inscribed in the ellipse $\mathbf{x}^\top \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = a^2b^2$ with its vertex at one end of the major axis.

2 Solution

Let $OD = y$ and $BD = x$. From the below figure we obtain $\mathbf{A} = \begin{pmatrix} 0 \\ a \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -x \\ -y \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} x \\ -y \end{pmatrix}$ and the area of the $\triangle ABC$ as,

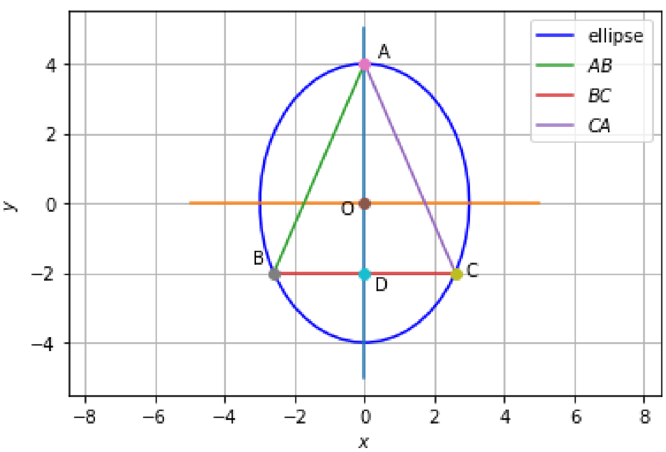

Fig. 2.1: isosceles triangle inscribed in the ellipse

$$\frac{1}{2}\left|\mathbf{B}-\mathbf{C}\right|\left|\mathbf{A}-\mathbf{D}\right| = \frac{1}{2}\mathbf{x}^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} a \\ 0 \end{pmatrix}^\top \mathbf{x} \tag{2.0.1}$$

Now the objective function is given as

$$\max Z = \frac{1}{2}\mathbf{x}^\top \mathbf{P}\mathbf{x} + \mathbf{q}^\top \mathbf{x} \tag{2.0.2}$$

where $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and subject to constraint

$$\mathbf{x}^\top \mathbf{D}\mathbf{x} - a^2b^2 = 0 \tag{2.0.3}$$

where $\mathbf{D} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$. Let

$$\mathbf{f}(\mathbf{x}) = \frac{1}{2}\mathbf{x}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x}^\top + \begin{pmatrix} a \\ 0 \end{pmatrix}^\top \mathbf{x} \tag{2.0.4}$$

$$\mathbf{g}(\mathbf{x}) = \mathbf{x}^\top \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\mathbf{x} - a^2b^2 \tag{2.0.5}$$

and we solve this using Lagrange multipliers,

$$\nabla \mathbf{f} = \lambda\left(\nabla \mathbf{g}\right) \tag{2.0.6}$$

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} + a \\ \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} \end{pmatrix} = \lambda\begin{pmatrix} \begin{pmatrix} 2a^2 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 2b^2 \end{pmatrix} \end{pmatrix}\mathbf{x} \tag{2.0.7}$$

we obtain the following from (2.0.7)

$$\begin{pmatrix} -2\lambda a^2 & 1 \\ 1 & -2\lambda b^2 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -a \\ 0 \end{pmatrix} \tag{2.0.8}$$

$$\implies x = 2\lambda b^2 y \tag{2.0.9}$$

$$\implies \lambda = \frac{x}{2b^2 y} \tag{2.0.10}$$

From (2.0.8) we have

$$a + y = 2\lambda a^2 x \tag{2.0.11}$$

substitute (2.0.9) in (2.0.11)

$$a + y = 2\lambda a^2\left(2\lambda b^2 y\right) \tag{2.0.12}$$

now substitute (2.0.10) in (2.0.12)

$$a + y = \frac{x^2a^2}{b^2y} \tag{2.0.13}$$

using (2.0.3) in (2.0.13) we have,

$$ab^2y + 2b^2y^2 = a^2b^2 \tag{2.0.14}$$

$$\implies 2y^2 + ay - a^2 = 0 \tag{2.0.15}$$

$$\implies y = -a, \frac{1}{2}a. \tag{2.0.16}$$

We see that $y = -a$ doesn't give any value for x so substitute $y = \frac{1}{2}a$ in (2.0.13),

$$a + \left(\frac{1}{2}a\right) = \frac{x^2a^2}{b^2\left(\frac{1}{2}a\right)} \tag{2.0.17}$$

$$\implies x^2 = \frac{3ab^2}{4} \tag{2.0.18}$$

$$\implies x = \frac{(\sqrt{3a})b}{2}. \tag{2.0.19}$$

Hence $\mathbf{x} = \begin{pmatrix} \frac{(\sqrt{3a})b}{2} \\ \frac{1}{2}a \end{pmatrix}$ and thus we obtain

$$Z = \frac{3\sqrt{3}ab}{4}. \tag{2.0.20}$$

Thus the maximum area of an isosceles triangle inscribed in ellipse is $\frac{3\sqrt{3}ab}{4}$.