

# ASSIGNMENT-15

R.YAMINI

## 1 QUESTION No-2.9(OPTIMIZATION)

Let

Find the maximum area of an isosceles triangle inscribed in the ellipse  $\mathbf{x}^\top \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = a^2 b^2$  with its vertex at one end of the major axis.

$$f(x, y) = \frac{1}{2} \mathbf{x}^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}^\top + \begin{pmatrix} a \\ 0 \end{pmatrix}^\top \mathbf{x} \quad (2.0.4)$$

$$g(x, y) = \mathbf{x}^\top \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} - a^2 b^2 \quad (2.0.5)$$

and we solve this using Lagrange multipliers,

$$\nabla f = \lambda (\nabla g) \quad (2.0.6)$$

$$\begin{pmatrix} a + y \\ x \end{pmatrix} = \lambda \begin{pmatrix} 2a^2 x \\ 2b^2 y \end{pmatrix} \quad (2.0.7)$$

we obtain the following from (2.0.7)

$$a + y = 2\lambda a^2 x \quad (2.0.8)$$

$$x = 2\lambda b^2 y \quad (2.0.9)$$

$$\Rightarrow \lambda = \frac{x}{2b^2 y} \quad (2.0.10)$$

substitute (2.0.9) in (2.0.8)

$$a + y = 2\lambda a^2 (2\lambda b^2 y) \quad (2.0.11)$$

now substitute (2.0.10) in (2.0.11)

$$a + y = \frac{x^2 a^2}{b^2 y} \quad (2.0.12)$$

using (2.0.3) in (2.0.12) we have,

$$ab^2 y + 2b^2 y^2 = a^2 b^2 \quad (2.0.13)$$

$$\Rightarrow 2y^2 + ay - a^2 = 0 \quad (2.0.14)$$

$$\Rightarrow y = -a, \frac{1}{2}a. \quad (2.0.15)$$

We see that  $y = -a$  doesn't give any value for  $x$  so substitute  $y = \frac{1}{2}a$  in (2.0.12),

$$a + \left(\frac{1}{2}a\right) = \frac{x^2 a^2}{b^2 \left(\frac{1}{2}a\right)} \quad (2.0.16)$$

$$\Rightarrow x^2 = \frac{3ab^2}{4} \quad (2.0.17)$$

$$\Rightarrow x = \frac{(\sqrt{3}a)b}{2}. \quad (2.0.18)$$

## 2 SOLUTION

Let  $OD = y$  and  $BD = x$ . From the below figure we obtain  $\mathbf{A} = \begin{pmatrix} 0 \\ a \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ ,  $\mathbf{C} = \begin{pmatrix} x \\ -y \end{pmatrix}$  and the area of the  $\triangle ABC$  as,

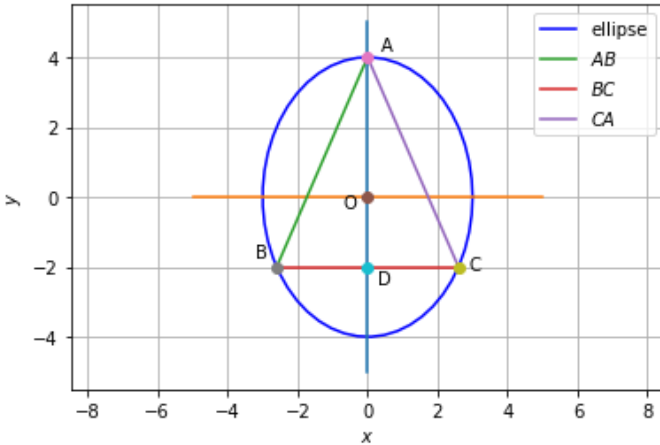


Fig. 2.1: isosceles triangle inscribed in the ellipse

$$\frac{1}{2} |\mathbf{B} - \mathbf{C}| |\mathbf{A} - \mathbf{D}| = \frac{1}{2} \mathbf{x}^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}^\top + \begin{pmatrix} a \\ 0 \end{pmatrix}^\top \mathbf{x} \quad (2.0.1)$$

Now the objective function is given as

$$\max Z = \frac{1}{2} \mathbf{x}^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}^\top + \begin{pmatrix} a \\ 0 \end{pmatrix}^\top \mathbf{x} \quad (2.0.2)$$

subject to constraint

$$\mathbf{x}^\top \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} - a^2 b^2 = 0. \quad (2.0.3)$$

Hence  $\mathbf{x} = \begin{pmatrix} (\frac{\sqrt{3}a)b}{\frac{1}{2}a} \end{pmatrix}$  and thus we obtain

$$Z = \frac{3\sqrt{3}ab}{4}. \quad (2.0.19)$$

Thus the maximum area of an isosceles triangle inscribed in ellipse is  $\frac{3\sqrt{3}ab}{4}$ .