ASSIGNMENT-15

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1 QUESTION No-8.6 (GATE Probability)

Suppose X and Y are two random variables such that $aX + bY$ is a normal random variable for all $a, b \in \mathbb{R}$. Consider the following statements P,Q,R and S:
(P): X is a standard normal random variable.
(Q): The conditional distribution of X given Y is normal.
(R): The conditional distribution of X given $X + Y$ is normal.
(S): X - Y has mean 0.
Which of the above statements ALWAYS hold TRUE?

1) both P and Q
2) both Q and R
3) both Q and S
4) both P and S

2 Solution

Definition 1. *Two random variables X and Y are said to be bivariate normal, or jointly normal, if $aX + bY$ has a normal distribution for all $a, b \in \mathbb{R}$.*

(P) X is a standard normal random variable.
By taking $a = 1, b = 0$ we see that X must be normal. Similarly by setting $a = 0, b = 1$ we have Y to be normal. The given information is not sufficient to conclude whether X is a standard normal random variable. Thus this statement does not hold true always.
(Q) The conditional distribution of X given Y is normal.
The pdf for bivariate normal distribution is given as,

$$f_{X,Y(x,y)} = \frac{e^{\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]\right\}}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \tag{2.0.1}$$

where μ_x and σ_x denote the mean and standard deviation of random variable X similarly μ_y and σ_y denote the mean and standard deviation of random variable Y. Thus the conditional distribution of X given Y is given as,

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y(x,y)}}{f_Y(y)} \\ &= \frac{e^{\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]\right\}}\left\{\sigma_y\sqrt{2\pi}\exp^{\left\{\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)\right\}}\right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \\ &= \frac{1}{\sigma_x\sqrt{2\pi}\sqrt{1-\rho^2}}e^{\left\{\frac{-1}{2\sigma_x^2\left(1-\rho^2\right)}\left[(x-\mu_x)-\rho\frac{\sigma_x}{\sigma_y}\left(y-\mu_y\right)\right]^2\right\}} \\ &= \frac{1}{\sigma_x\sqrt{2\pi}\sqrt{1-\rho^2}}e^{\left\{\frac{-1}{2\sigma_x^2\left(1-\rho^2\right)}\left[x-\left(\mu_x+\rho\frac{\sigma_x}{\sigma_y}\left(y-\mu_y\right)\right)^2\right]\right\}} \end{aligned} \tag{2.0.2}$$

So, conditional distribution $X|Y$ follows normal distribution with mean $\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_x)$ and standard deviation $\sigma_x\left(\sqrt{1-\rho^2}\right)$. We can say that $X|Y \sim N\left(\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_x), \sigma_x\left(\sqrt{1-\rho^2}\right)\right)$. Thus the statement holds true always.
(R) The conditional distribution of X given $X + Y$ is normal.
By taking $a = 1, b = 1$ the linear combination $X+Y$ is also normal. Then from the previous statement it directly follows that X given $X+Y$ is normal. Thus the statement holds true always.
(S) X - Y has mean 0.
This is possible only when X and Y are independent standard normal variable.Thus the given information is not sufficient to conclude whether $X-Y$ has mean 0. Thus this statement does not hold true always.