ASSIGNMENT-16

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1 QUESTION No-8.6 (GATE Probability)

Suppose X and Y are two random variables such that $aX + bY$ is a normal random variable for all $a, b \in \mathbb{R}$. Consider the following statements P,Q,R and S:
(P): X is a standard normal random variable.
(Q): The conditional distribution of X given Y is normal.
(R): The conditional distribution of X given $X + Y$ is normal.
(S): X - Y has mean 0.
Which of the above statements ALWAYS hold TRUE?

1) both P and Q
2) both Q and R
3) both Q and S
4) both P and S

2 Solution

Definition 1. *Two random variables X and Y are said to be bivariate normal, or jointly normal, if $aX + bY$ has a normal distribution for all $a, b \in \mathbb{R}$.*

(P) X is a standard normal random variable.
Let,

$$a = 1, b = 0 \tag{2.0.1}$$
$$\implies X \text{is normal.} \tag{2.0.2}$$

Similarly let

$$a = 0, b = 1 \tag{2.0.3}$$
$$\implies Y \text{is normal.} \tag{2.0.4}$$

The given information is not sufficient to conclude whether X is a standard normal random variable. Thus this statement does not hold true always.
(Q) The conditional distribution of X given Y is normal.

Theorem 2.1. *Let X and Y be two bivariate normal random variables then there exist independent standard normal random variables Z_1 and Z_2 such that*

$$X = \sigma_X(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_X \tag{2.0.5}$$
$$Y = \sigma_Y Z_1 + \mu_Y \tag{2.0.6}$$

By using theorem 2.1 we show that the statement Q is true.Thus given $Y = y$, we have

$$Z_1 = \frac{y-\mu_Y}{\sigma_Y}, \tag{2.0.7}$$
$$\implies X = \sigma_X(\rho \frac{y-\mu_Y}{\sigma_Y} + \sqrt{1-\rho^2} Z_2) + \mu_X. \tag{2.0.8}$$

Since Z_1 and Z_2 are independent we have,

$$E[X|Y=y] = \sigma_X\left(\rho \frac{y-\mu_Y}{\sigma_Y}\right) + \sigma_X\left(\sqrt{1-\rho^2} E[Z_2]\right) + \mu_X \tag{2.0.9}$$

$$= \mu_X + \rho\sigma_X \frac{y-\mu_Y}{\sigma_Y}, \tag{2.0.10}$$
$$Var(X|Y=y) = (1-\rho^2)\sigma_X^2. \tag{2.0.11}$$

where μ_X and σ_X denote the mean and standard deviation of random variable X similarly μ_Y and σ_Y denote the mean and standard deviation of random variable Y and thus

$$X|Y \sim N\left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(Y-\mu_X), \sigma_X\left(\sqrt{1-\rho^2}\right)\right) \tag{2.0.12}$$

Thus the statement holds true always.
(R) The conditional distribution of X given $X + Y$ is normal.
Let,

$$a = 1, b = 1 \tag{2.0.13}$$
$$\implies X + Y \text{is normal} \tag{2.0.14}$$

and let $Z = X + Y$ then $f_{X|Z}$ is also normal which follows from the previous statement. Thus the statement holds true always.
(S) X - Y has mean 0.

In general

$$E[aX + bY] = aE[X] + bE[Y] \tag{2.0.15}$$

$$Var(aX + bY) = a^2Var(X) + 2abCov(X, Y) + b^2Var(Y) \tag{2.0.16}$$

For $a = 1$ and $b = -1$ we have

$$E[X - Y] = E[X] - E[Y] \tag{2.0.17}$$

$$Var(X - Y) = Var(X) - 2Cov(X, Y) + Var(Y) \tag{2.0.18}$$

Mean of $X - Y$ is 0 only when X and Y are independent standard normal variable and the given information is not sufficient to conclude. Thus this statement does not hold true always.