ASSIGNMENT-5

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1 QUESTION No-2.98 (Quadratic forms)

Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0$ and inside of the parabola $y^2 = 4x$.

2 Solution

Given equation of the circle

$$\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0. \tag{2.0.1}$$

We know that the general equation of a circle is given by

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

We have $\mathbf{u} = \begin{pmatrix}-4\\0\end{pmatrix}$ and $f = 0$.Thus we have the center and radius as

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix}4\\0\end{pmatrix} \tag{2.0.3}$$

and

$$r = \sqrt{\mathbf{u}^{\mathbf{T}}\mathbf{u} - f} = 4 \tag{2.0.4}$$

respectively. Given equation of the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - 4\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{2.0.5}$$

The plot of the above two curves is

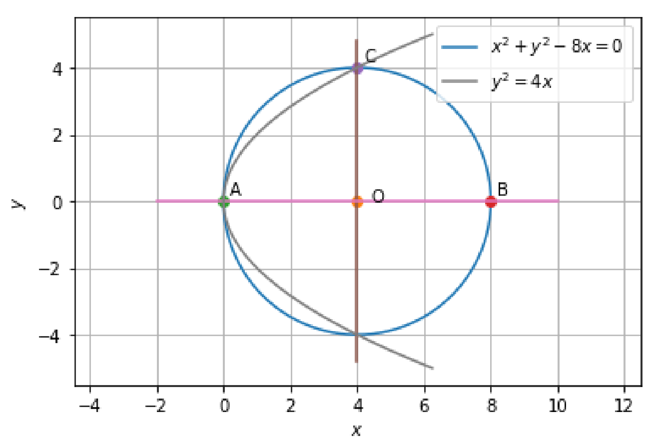

Fig. 2.1: Plot of the curves

We can see that y-axis is the tangent for both the circle and the parabola. Now the parametric equation of the tangent is given by

$$\mathbf{x} = \begin{pmatrix}0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.6}$$

$$= \begin{pmatrix}0\\\lambda\end{pmatrix} \tag{2.0.7}$$

Now substitute (2.0.7) in (3.0.1),

$$\begin{pmatrix}0 & \lambda\end{pmatrix}\begin{pmatrix}0\\\lambda\end{pmatrix} - 8\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\\lambda\end{pmatrix} = 0 \tag{2.0.8}$$

$$\implies \lambda = 0 \tag{2.0.9}$$

Thus the point of contact of the tangent with the circle is $A = \begin{pmatrix}0\\0\end{pmatrix}$. Now to find the point of contact of the tangent with the parabola,substiute (2.0.7) in (3.0.5)

$$\begin{pmatrix}0 & \lambda\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\\lambda\end{pmatrix} - 4\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\\lambda\end{pmatrix} = 0 \tag{2.0.10}$$

$$\implies \lambda = 0 \tag{2.0.11}$$

Thus the point of contact of the parabola with the tangent is $\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}$. Thus $\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}$ is the point of contact for the circle,the parabola and the tangent. Now to find the point of contact of the circle and parabola with the chord. The chord equation in parametric form is given as

$$\mathbf{x} = \begin{pmatrix}4\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.12}$$

$$= \begin{pmatrix}4\\\lambda\end{pmatrix} \tag{2.0.13}$$

Now substitute (2.0.13) in (3.0.1) and (3.0.5)

$$\begin{pmatrix}4 & \lambda\end{pmatrix}\begin{pmatrix}4\\\lambda\end{pmatrix} - 8\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}4\\\lambda\end{pmatrix} = 0 \tag{2.0.14}$$

$$\implies \lambda = \pm 4 \tag{2.0.15}$$

Since the area to be found is above the x-axis we consider $\lambda = +4$ and substituting in (2.0.13) we obtain the point of contact to be $\mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$. Now to find the point of contact of the chord with parabola,sustitute (2.0.13) in (3.0.5),

$$\begin{pmatrix} 4 & \lambda \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ \lambda \end{pmatrix} - 4 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ \lambda \end{pmatrix} = 0 \tag{2.0.16}$$

$$\implies \lambda = \pm 4 \tag{2.0.17}$$

Since the area to be found is above the x-axis we consider $\lambda = +4$ and substituting in (2.0.13) we obtain the point of contact to be $\mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$. Thus $\mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ is the point of contact for the circle,the parabola and the chord.

(**OR**) Now to find the point of contact of the circle and parabola with the chord. Since the chord is parallel to the tangent their equation differ by a constant. So, let the equation of the chord be of the form

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + (\lambda + c) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.18}$$

$$= \begin{pmatrix} 0 \\ (\lambda + c) \end{pmatrix} \tag{2.0.19}$$

Now substitute (2.0.19) in (3.0.5) we have,

Now to find the area bounded above the x-axis,the parabola and the circle. From fig.2.1 the area to be calculated is $AOBCA$.

$$Ar(AOBCA) = Ar(ACOA) + Ar(OCBO) \tag{2.0.20}$$

$$= A_1 + A_2 \tag{2.0.21}$$

To calculate A_1: A_1 is the area enclosed by the parabola $y^2 = 4x$ and the line OC.Thus

$$A_1 = \frac{2}{3}(AO)(OC) \tag{2.0.22}$$

$$= \frac{2}{3}(4)(4) = \frac{32}{3} \tag{2.0.23}$$

To calculate A_2: A_2 is one fourth of the area of the circle.

$$A_2 = \frac{1}{4}\left(\pi r^2\right) \tag{2.0.24}$$

$$= \frac{1}{4}(16\pi) \tag{2.0.25}$$

$$= 4\pi \tag{2.0.26}$$

Now substituting (2.0.5) and (2.0.8) in (2.0.3) we get

$$A_1 + A_2 = \frac{32}{3} + 4\pi \tag{2.0.27}$$

$$= 4\left(\frac{8}{3} + \pi\right) \tag{2.0.28}$$

Thus (2.0.10) is the required area.

3 Solution

Given equation of the circle

$$\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 0. \tag{3.0.1}$$

We know that the general equation of a circle is given by

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.0.2}$$

We have $\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$ and $f = 0$.Thus we have the center and radius as

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{3.0.3}$$

and

$$r = \sqrt{\mathbf{u^T u} - f} = 4 \tag{3.0.4}$$

respectively. Given equation of the parabola

$$\mathbf{x}^T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} - 4\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{3.0.5}$$

The plot of the above two curves is

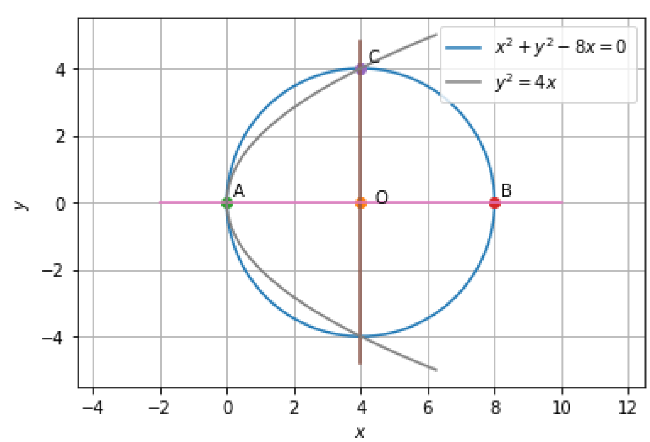

Fig. 3.1: Plot of the curves

We can see that y-axis is the tangent for both the circle and the parabola. Let the parametric equation of the tangent be

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \tag{3.0.6}$$

Substitute (3.0.6) in (3.0.1)

$$(\mathbf{q} + \lambda \mathbf{m})^\top (\mathbf{q} + \lambda \mathbf{m}) - 8 \begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{q} + \lambda \mathbf{m}) = 0 \tag{3.0.7}$$

Now substitute (3.0.6) in (3.0.5)

$$(\mathbf{q} + \lambda \mathbf{m})^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (\mathbf{q} + \lambda \mathbf{m}) - 4 \begin{pmatrix} 0 & 1 \end{pmatrix} (\mathbf{q} + \lambda \mathbf{m}) = 0 \tag{3.0.8}$$

Now subtracting (3.0.8) from (3.0.7),

$$(\mathbf{q} + \lambda \mathbf{m})^\top (\mathbf{q} + \lambda \mathbf{m}) - 8 \begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{q} + \lambda \mathbf{m}) - (\mathbf{q} + \lambda \mathbf{m})^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (\mathbf{q} + \lambda \mathbf{m}) + 4 \begin{pmatrix} 0 & 1 \end{pmatrix} (\mathbf{q} + \lambda \mathbf{m}) = 0 \tag{3.0.9}$$