ASSIGNMENT-6

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1 QUESTION No-2.82 (Quadratic forms)

Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $\begin{pmatrix} 4 & -2 \end{pmatrix}\mathbf{x}+5 = 0$.

2 Solution

Given equation

$$y^2 = 3x - 2 \tag{2.0.1}$$

comparing it with standard equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.2}$$

we have $a = 0, b = 0, c = 1, d = \frac{-3}{2}, e = 0, f = 2$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} \\ 0 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.4}$$

$$\implies |\mathbf{V}| = 0 \tag{2.0.5}$$

Thus the curve is a parabola.Now we find the eigen values corresponding to $\mathbf{V}$

$$|\mathbf{V} - \lambda\mathbf{I}| = 0 \tag{2.0.6}$$

$$\begin{vmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \tag{2.0.7}$$

$$\implies \lambda = 0, 1 \tag{2.0.8}$$

Now we find the eigen vectors corresponding to $\lambda = 0, 1$ respectively.

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \tag{2.0.9}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.10}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.11}$$

Now by eigen decomposition on V

$$\mathbf{V} = \mathbf{PDP^T} \tag{2.0.12}$$

where

$$\mathbf{P} = (\mathbf{p_1}\mathbf{p_2}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.13}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.14}$$

Now (2.0.12) becomes

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.16}$$

Given parallel line equation

$$\begin{pmatrix} 4 & -2 \end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.0.17}$$

$$4x - 2y + 5 = 0 \tag{2.0.18}$$

Now the tangent to the parabola is parallel to the line equation (2.0.18),the general straight line equation is of the form

$$ax + by + c = 0 \tag{2.0.19}$$

The normal vector $\mathbf{n}$ and the direction vector $\mathbf{m}$ is given by

$$\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{m} = \begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \tag{2.0.21}$$

Now the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u} + \kappa\mathbf{n}^\mathbf{T} \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -\mathbf{f} \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{2.0.22}$$

$$\implies \begin{pmatrix} -3 & \frac{3}{4} \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{q} = \begin{pmatrix} -2 \\ 0 \\ \frac{3}{4} \end{pmatrix} \tag{2.0.23}$$

Now solving for $\mathbf{q}$ by removing the zero row and representing as augmented matrix and then converting it to echelon form.

$$\begin{pmatrix} -3 & \frac{3}{4} & -2 \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \xleftrightarrow{R_1 \to \left(\frac{-1}{3}\right) R_1} \begin{pmatrix} 1 & \frac{-1}{4} & \frac{2}{3} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \tag{2.0.24}$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & \frac{2}{3} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \xleftrightarrow{R_1 \to R_1 + \frac{1}{4} R_2} \begin{pmatrix} 1 & 0 & \frac{41}{48} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \tag{2.0.25}$$

Hence from the (2.0.25) we get the point of contact to be

$$\mathbf{q} = \begin{pmatrix} \frac{41}{48} \\ \frac{3}{4} \end{pmatrix} \tag{2.0.26}$$

Now $\mathbf{q}$ is the point on the tangent.Hence the equation of the line can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \tag{2.0.27}$$

where c is,

$$c = \mathbf{n}^T \mathbf{q} \tag{2.0.28}$$

$$= \begin{pmatrix} 4 & -2 \end{pmatrix} \begin{pmatrix} \frac{41}{48} \\ \frac{3}{4} \end{pmatrix} \tag{2.0.29}$$

$$= \frac{23}{12} \tag{2.0.30}$$

Hence the equation of the tangent to the curve (2.0.1) parallel to the line (2.0.18) is given by substituting the value of c and $\mathbf{n}$ from (2.0.30) and (2.0.20) respectively to the equation (2.0.27).

$$\begin{pmatrix} 4 & -2 \end{pmatrix} \mathbf{x} = \frac{23}{12} \tag{2.0.31}$$

$$\begin{pmatrix} 48 & -24 \end{pmatrix} \mathbf{x} = 23 \tag{2.0.32}$$

Hence (2.0.32) is the equation of the tangent. The plot of the figure is given below

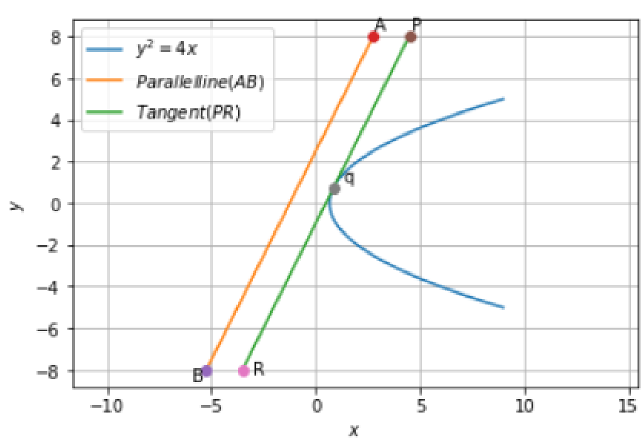

Fig. 2.1: Plot of curve and the lines