ASSIGNMENT-6

R.YAMINI

1 QUESTION No-2.82 (QUADRATIC FORMS)

Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $\begin{pmatrix} 4 & -2 \end{pmatrix} x + 5 = 0$.

2 SOLUTION

Given equation

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -3 & 0 \end{pmatrix} \mathbf{x} + 2 = 0 \tag{2.0.1}$$

we have

$$\therefore \mathbf{u} = \begin{pmatrix} \frac{-3}{2} \\ 0 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.3}$$

$$\implies |\mathbf{V}| = 0 \tag{2.0.4}$$

Thus the curve is a parabola. Now we find the eigen values corresponding to $\mathbf{V}$

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \tag{2.0.5}$$

$$\begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \tag{2.0.6}$$

$$\implies \lambda = 0, 1 \tag{2.0.7}$$

Now we find the eigen vectors corresponding to $\lambda = 0, 1$ respectively.

$$\mathbf{V}\mathbf{x} = \lambda \mathbf{x} \tag{2.0.8}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 0 \implies \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.9}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \mathbf{x} \implies \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.10}$$

Now by eigen decomposition on $\mathbf{V}$

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{2.0.11}$$

where

$$\mathbf{P} = (\mathbf{p}_1 \mathbf{p}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.12}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.13}$$

Now (2.0.11) becomes

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.14}$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.15}$$

Given parallel line equation

$$\begin{pmatrix} 4 & -2 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{2.0.16}$$

Now the tangent to the parabola is parallel to the line equation (2.0.16), The normal vector $\mathbf{n}$ and the direction vector $\mathbf{m}$ is given by

$$\mathbf{n} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{m} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \tag{2.0.18}$$

Now the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u} + \kappa \mathbf{n}^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -\mathbf{f} \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \tag{2.0.19}$$

$$\implies \begin{pmatrix} -3 & \frac{3}{4} \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -2 \\ 0 \\ \frac{3}{4} \end{pmatrix} \tag{2.0.20}$$

Now solving for $\mathbf{q}$ by removing the zero row and representing as augmented matrix and then converting it to echelon form.

$$\begin{pmatrix} -3 & \frac{3}{4} & -2 \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \xleftarrow{R_1 \to (\frac{-1}{3})R_1} \begin{pmatrix} 1 & \frac{-1}{4} & \frac{2}{3} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \tag{2.0.21}$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & \frac{2}{3} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \xleftarrow{R_1 \to R_1 + \frac{1}{4}R_2} \begin{pmatrix} 1 & 0 & \frac{41}{48} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \tag{2.0.22}$$

Hence from the (2.0.22) we get the point of contact to be

$$\mathbf{q} = \begin{pmatrix} \frac{41}{48} \\ \frac{3}{4} \end{pmatrix} \qquad (2.0.23)$$

Now $\mathbf{q}$ is the point on the tangent. Hence the equation of the line can be expressed as

$$\mathbf{n}^\top(\mathbf{x}^\top - \mathbf{q}) = 0 \qquad (2.0.24)$$

where,

$$c = \mathbf{n}^\top\mathbf{q} \qquad (2.0.25)$$

$$= \begin{pmatrix} 4 & -2 \end{pmatrix}\begin{pmatrix} \frac{41}{48} \\ \frac{3}{4} \end{pmatrix} \qquad (2.0.26)$$

$$= \frac{23}{12} \qquad (2.0.27)$$

Hence the equation of the tangent to the curve (2.0.1) parallel to the line (2.0.16) is given by substituting the value of c and $\mathbf{n}$ from (2.0.27) and (2.0.17) respectively to the equation (2.0.24).

$$\begin{pmatrix} 4 & -2 \end{pmatrix}\mathbf{x} = \frac{23}{12} \qquad (2.0.28)$$

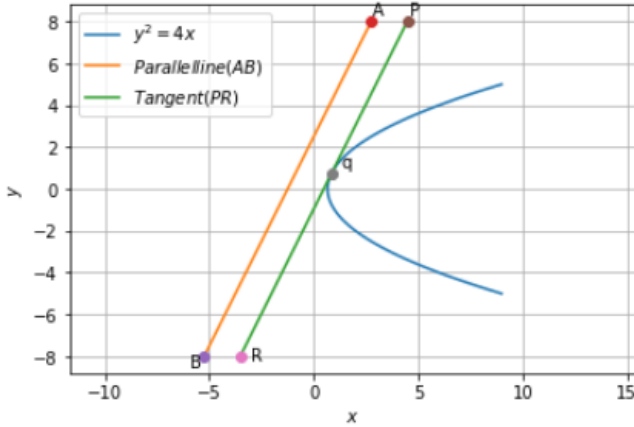Hence (2.0.28) is the equation of the tangent. The plot of the figure is given below

Fig. 2.1: Plot of curve and the lines