ASSIGNMENT-7

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1 QUESTION No-2.72(k) (Quadratic forms)

Find the equation of the ellipse, with major axis on x-axis and passing through the points $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}6\\2\end{pmatrix}$

2 Solution

Given,

$$\mathbf{p} = \begin{pmatrix}4\\3\end{pmatrix}, \mathbf{q} = \begin{pmatrix}6\\2\end{pmatrix} \tag{2.0.1}$$

are the points on the ellipse. The general form of the conic is given by

$$\mathbf{x}^\top \mathbf{D}\mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}, \lambda_1, \lambda_2 > 0 \tag{2.0.2}$$

The points $\mathbf{p}$ and $\mathbf{q}$ satisfy (2.0.2), and thus we have

$$\mathbf{p}^\top \mathbf{D}\mathbf{p} = 1, \tag{2.0.3}$$
$$\mathbf{q}^\top \mathbf{D}\mathbf{q} = 1 \tag{2.0.4}$$

which can be further expressed as,

$$\begin{aligned}\mathbf{p}^\top \mathbf{P}\mathbf{d} &= 1,\\ \mathbf{q}^\top \mathbf{Q}\mathbf{d} &= 1\end{aligned} \tag{2.0.5}$$

where,

$$\mathbf{d} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}, \mathbf{P} = \begin{pmatrix}4 & 0\\0 & 3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}6 & 0\\0 & 2\end{pmatrix}. \tag{2.0.6}$$

(2.0.5) can then be expressed as,

$$\begin{pmatrix}\mathbf{p}^\top \mathbf{P}\\\mathbf{q}^\top \mathbf{Q}\end{pmatrix}\mathbf{d} = \begin{pmatrix}1\\1\end{pmatrix} \tag{2.0.7}$$
$$\begin{pmatrix}16 & 9\\36 & 4\end{pmatrix}\mathbf{d} = \begin{pmatrix}1\\1\end{pmatrix} \tag{2.0.8}$$

The augmented matrix is

$$\begin{pmatrix}16 & 9 & 1\\36 & 4 & 1\end{pmatrix} \tag{2.0.9}$$

and we perform row reduction,

$$\begin{pmatrix}16 & 9 & 1\\1 & 16 & 1\end{pmatrix} \xleftrightarrow{R_1\to\frac{R_1}{16}} \begin{pmatrix}1 & \frac{9}{16} & \frac{1}{16}\\36 & 4 & 1\end{pmatrix} \tag{2.0.10}$$
$$\xleftrightarrow{R_2\to R_2-36R_1} \begin{pmatrix}1 & \frac{9}{16} & \frac{1}{16}\\0 & \frac{-65}{4} & \frac{-5}{4}\end{pmatrix} \tag{2.0.11}$$
$$\xleftrightarrow{R_2\to\frac{-4}{65}R_2} \begin{pmatrix}1 & \frac{9}{16} & \frac{1}{16}\\0 & 1 & \frac{1}{13}\end{pmatrix} \tag{2.0.12}$$
$$\xleftrightarrow{R_1\to R_1-\frac{9}{16}R_2} \begin{pmatrix}1 & 0 & \frac{1}{52}\\0 & 1 & \frac{1}{13}\end{pmatrix} \tag{2.0.13}$$
$$\implies \mathbf{d} = \begin{pmatrix}\frac{1}{52}\\\frac{1}{13}\end{pmatrix}. \tag{2.0.14}$$

Thus we have,

$$\mathbf{D} = \begin{pmatrix}\frac{1}{52} & 0\\0 & \frac{1}{13}\end{pmatrix} \tag{2.0.15}$$

Hence equation of ellipse is given by,

$$\mathbf{x}^\top \begin{pmatrix}\frac{1}{52} & 0\\0 & \frac{1}{13}\end{pmatrix}\mathbf{x} = 1 \tag{2.0.16}$$

The plot of the ellipse is given below

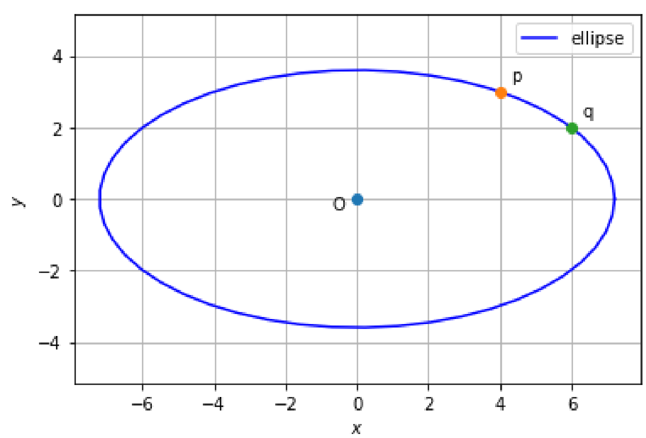

Fig. 2.1: Plot of standard ellipse

The center and axes of the ellipse is given as

$$\mathbf{c} = \mathbf{0}; \frac{1}{\sqrt{\lambda_1}} = \sqrt{52}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{13}. \tag{2.0.17}$$

Now let us consider the case when the ellipse is not in the standard form then we have the center to be $\mathbf{c} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}$.The equation is given by:

$$(\mathbf{x}-\mathbf{c})^\top \mathbf{M}(\mathbf{x}-\mathbf{c}) = 1 \tag{2.0.18}$$

where $\mathbf{M}$ is a diagonal matrix.

$$\mathbf{M} = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \tag{2.0.19}$$

Comparing (2.0.18) with the general equation of ellipse:

$$\mathbf{V} = \mathbf{M} \tag{2.0.20}$$

$$\mathbf{u} = -\mathbf{Mc} \tag{2.0.21}$$

$$f = \mathbf{c}^\top \mathbf{Mc} - 1 \tag{2.0.22}$$

The lengths of the major and minor axis of the Ellipse are:

$$\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{l_1}} = \frac{1}{\sqrt{l_1}} \tag{2.0.23}$$

$$\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{l_2}} = \frac{1}{\sqrt{l_2}} \tag{2.0.24}$$

$\because$ $\mathbf{p}, \mathbf{q}$ satisfy (2.0.18),

$$(\mathbf{p}-\mathbf{c})^\top \mathbf{M}(\mathbf{p}-\mathbf{c}) = 1, \tag{2.0.25}$$

$$(\mathbf{q}-\mathbf{c})^\top \mathbf{M}(\mathbf{q}-\mathbf{c}) = 1, \tag{2.0.26}$$

which can then be written as:

$$2 \times (\mathbf{p}^\top \mathbf{M} - \mathbf{q}^\top \mathbf{M})\mathbf{c} = \mathbf{p}^\top \mathbf{Mp} - \mathbf{q}^\top \mathbf{Mq} \tag{2.0.27}$$

Now we have:

$$(\mathbf{p}^\top - \mathbf{q}^\top)\mathbf{M}(\mathbf{p}+\mathbf{q}) \tag{2.0.28}$$

$$= \mathbf{p}^\top \mathbf{Mp} - \mathbf{q}^\top \mathbf{Mq} + \mathbf{p}^\top \mathbf{Mq} - \mathbf{q}^\top \mathbf{Mp} \tag{2.0.29}$$

$$= \mathbf{p}^\top \mathbf{Mp} - \mathbf{q}^\top \mathbf{Mq} \tag{2.0.30}$$

Let $\mathbf{m}$ satisfy $(\mathbf{p}^\top - \mathbf{q}^\top)\mathbf{m} = 0$.So,

$$(\mathbf{p}^\top - \mathbf{q}^\top)\mathbf{M}\mathbf{M}^{-1}\mathbf{m} = 0 \tag{2.0.31}$$

Now $\mathbf{c}$ can be expressed as:

$$2\mathbf{c} = (\mathbf{p}+\mathbf{q}) + K\mathbf{M}^{-1}\mathbf{m} \tag{2.0.32}$$

where K is an arbitrary constant Using values $\mathbf{c}$,$\mathbf{p}$,$\mathbf{q}$ and $\mathbf{M}$:

$$\begin{pmatrix} 2\beta \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} + \frac{K}{l_1 \times l_2}\begin{pmatrix} l_2 & 0 \\ 0 & l_1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.0.33}$$

For, X-axis as the major axis we have $\frac{l_2}{l_1} > 1$

$$\frac{2\beta - 10}{\frac{5}{2}} > 1 \tag{2.0.34}$$

$$\beta > 6.25 \tag{2.0.35}$$

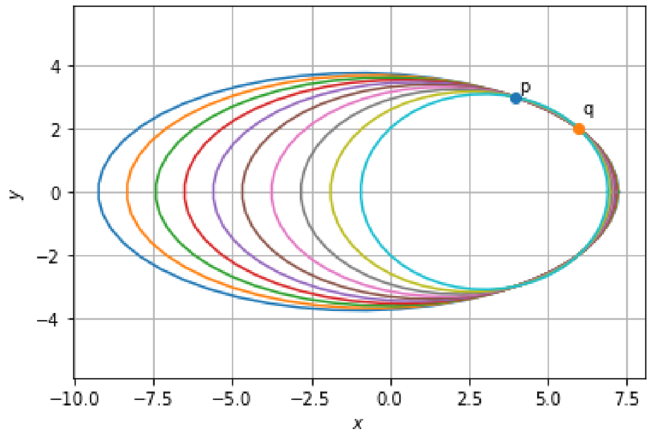

Fig. 2.2: Ellipses passing through the two points with X axis as major axis