ASSIGNMENT-7

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1 QUESTION No-2.72(k) (Quadratic forms)

Find the equation of the ellipse, with major axis on x-axis and passing through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 2 \end{pmatrix}$

2 Solution

Given,

$$\mathbf{p} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{2.0.1}$$

are the points on the ellipse. The general form of the conic is given by

$$\mathbf{x}^\top \mathbf{D}\mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1, \lambda_2 > 0 \tag{2.0.2}$$

The points $\mathbf{p}$ and $\mathbf{q}$ satisfy (2.0.2), and thus we have

$$\mathbf{p}^\top \mathbf{D}\mathbf{p} = 1, \tag{2.0.3}$$
$$\mathbf{q}^\top \mathbf{D}\mathbf{q} = 1 \tag{2.0.4}$$

which can be further expressed as,

$$\begin{aligned} \mathbf{p}^\top \mathbf{P}\mathbf{d} &= 1, \\ \mathbf{q}^\top \mathbf{Q}\mathbf{d} &= 1 \end{aligned} \tag{2.0.5}$$

where,

$$\mathbf{d} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}. \tag{2.0.6}$$

(2.0.5) can then be expressed as,

$$\begin{pmatrix} \mathbf{p}^T \mathbf{P} \\ \mathbf{q}^T \mathbf{Q} \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.7}$$

$$\begin{pmatrix} 16 & 9 \\ 36 & 4 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.8}$$

The augmented matrix is

$$\begin{pmatrix} 16 & 9 & 1 \\ 36 & 4 & 1 \end{pmatrix} \tag{2.0.9}$$

and we perform row reduction,

$$\begin{pmatrix} 16 & 9 & 1 \\ 1 & 16 & 1 \end{pmatrix} \xleftrightarrow{R_1 \to \frac{R_1}{16}} \begin{pmatrix} 1 & \frac{9}{16} & \frac{1}{16} \\ 36 & 4 & 1 \end{pmatrix} \tag{2.0.10}$$

$$\xleftrightarrow{R_2 \to R_2 - 36R_1} \begin{pmatrix} 1 & \frac{9}{16} & \frac{1}{16} \\ 0 & \frac{-65}{4} & \frac{-5}{4} \end{pmatrix} \tag{2.0.11}$$

$$\xleftrightarrow{R_2 \to \frac{-4}{65} R_2} \begin{pmatrix} 1 & \frac{9}{16} & \frac{1}{16} \\ 0 & 1 & \frac{1}{13} \end{pmatrix} \tag{2.0.12}$$

$$\xleftrightarrow{R_1 \to R_1 - \frac{9}{16} R_2} \begin{pmatrix} 1 & 0 & \frac{1}{52} \\ 0 & 1 & \frac{1}{13} \end{pmatrix} \tag{2.0.13}$$

$$\implies \mathbf{d} = \begin{pmatrix} \frac{1}{52} \\ \frac{1}{13} \end{pmatrix}. \tag{2.0.14}$$

Thus we have,

$$\mathbf{D} = \begin{pmatrix} \frac{1}{52} & 0 \\ 0 & \frac{1}{13} \end{pmatrix} \tag{2.0.15}$$

Hence equation of ellipse is given by,

$$\mathbf{x}^\top \begin{pmatrix} \frac{1}{52} & 0 \\ 0 & \frac{1}{13} \end{pmatrix} \mathbf{x} = 1 \tag{2.0.16}$$

The plot of the ellipse is given below

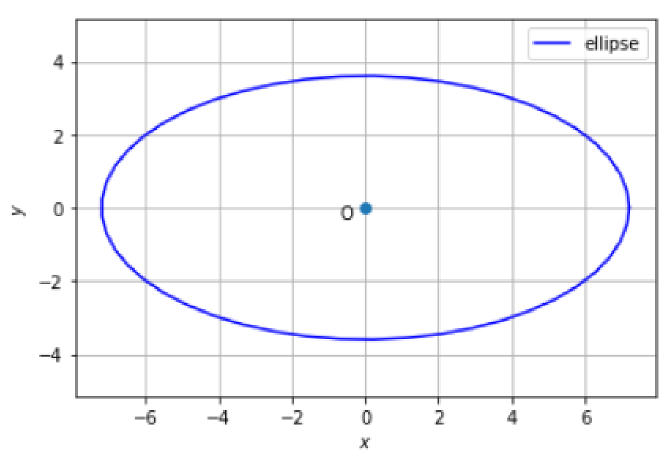

Fig. 2.1: Plot of the ellipse