ASSIGNMENT-7

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1 QUESTION No-2.72(k) (Quadratic forms)

Find the equation of the ellipse, with major axis on x-axis and passing through the points $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}6\\2\end{pmatrix}$

2 Solution

Given,

$$\mathbf{p} = \begin{pmatrix}4\\3\end{pmatrix}, \mathbf{q} = \begin{pmatrix}6\\2\end{pmatrix} \tag{2.0.1}$$

are the points on the ellipse. The general form of the conic is given by

$$\mathbf{x}^\top \mathbf{D}\mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}, \lambda_1, \lambda_2 > 0 \tag{2.0.2}$$

The points $\mathbf{p}$ and $\mathbf{q}$ satisfy (2.0.2), and thus we have

$$\mathbf{p}^\top \mathbf{D}\mathbf{p} = 1, \tag{2.0.3}$$
$$\mathbf{q}^\top \mathbf{D}\mathbf{q} = 1 \tag{2.0.4}$$

which can be further expressed as,

$$\begin{aligned}\mathbf{p}^\top \mathbf{P}\mathbf{d} &= 1,\\ \mathbf{q}^\top \mathbf{Q}\mathbf{d} &= 1\end{aligned} \tag{2.0.5}$$

where,

$$\mathbf{d} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}, \mathbf{P} = \begin{pmatrix}4 & 0\\0 & 3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}6 & 0\\0 & 2\end{pmatrix}. \tag{2.0.6}$$

(2.0.5) can then be expressed as,

$$\begin{pmatrix}\mathbf{p}^\top \mathbf{P}\\\mathbf{q}^\top \mathbf{Q}\end{pmatrix}\mathbf{d} = \begin{pmatrix}1\\1\end{pmatrix} \tag{2.0.7}$$

$$\begin{pmatrix}16 & 9\\36 & 4\end{pmatrix}\mathbf{d} = \begin{pmatrix}1\\1\end{pmatrix} \tag{2.0.8}$$

The augmented matrix is

$$\begin{pmatrix}16 & 9 & 1\\36 & 4 & 1\end{pmatrix} \tag{2.0.9}$$

and we perform row reduction,

$$\begin{pmatrix}16 & 9 & 1\\1 & 16 & 1\end{pmatrix} \xleftrightarrow{R_1\to\frac{R_1}{16}} \begin{pmatrix}1 & \frac{9}{16} & \frac{1}{16}\\36 & 4 & 1\end{pmatrix} \tag{2.0.10}$$

$$\xleftrightarrow{R_2\to R_2-36R_1} \begin{pmatrix}1 & \frac{9}{16} & \frac{1}{16}\\0 & \frac{-65}{4} & \frac{-5}{4}\end{pmatrix} \tag{2.0.11}$$

$$\xleftrightarrow{R_2\to\frac{-4}{65}R_2} \begin{pmatrix}1 & \frac{9}{16} & \frac{1}{16}\\0 & 1 & \frac{1}{13}\end{pmatrix} \tag{2.0.12}$$

$$\xleftrightarrow{R_1\to R_1-\frac{9}{16}R_2} \begin{pmatrix}1 & 0 & \frac{1}{52}\\0 & 1 & \frac{1}{13}\end{pmatrix} \tag{2.0.13}$$

$$\implies \mathbf{d} = \begin{pmatrix}\frac{1}{52}\\\frac{1}{13}\end{pmatrix}. \tag{2.0.14}$$

Thus we have,

$$\mathbf{D} = \begin{pmatrix}\frac{1}{52} & 0\\0 & \frac{1}{13}\end{pmatrix} \tag{2.0.15}$$

Hence equation of ellipse is given by,

$$\mathbf{x}^\top \begin{pmatrix}\frac{1}{52} & 0\\0 & \frac{1}{13}\end{pmatrix}\mathbf{x} = 1 \tag{2.0.16}$$

The plot of the ellipse is given below

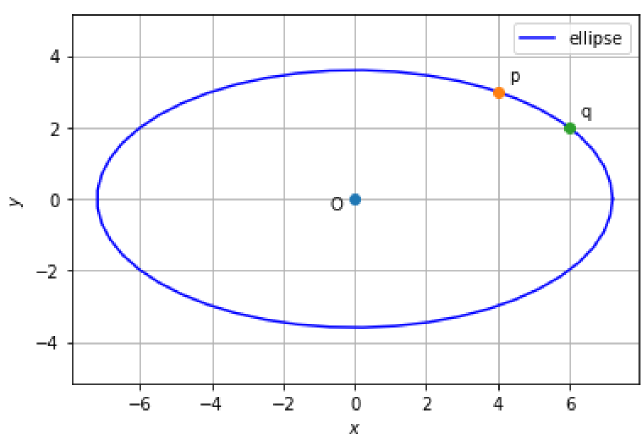

Fig. 2.1: Plot of standard ellipse

The center and axes of the ellipse is given as

$$\mathbf{c} = \mathbf{0}; \frac{1}{\sqrt{\lambda_1}} = \sqrt{52}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{13}. \tag{2.0.17}$$

Now let us consider the case when the ellipse is not in the standard form then we have the center to be $\mathbf{c} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}$.The equation is given by:

$$(\mathbf{x}-\mathbf{c})^\top \mathbf{D}(\mathbf{x}-\mathbf{c}) = 1 \tag{2.0.18}$$

where $\mathbf{D}$ is a diagonal matrix.

$\because$ $\mathbf{p}, \mathbf{q}$ satisfy (2.0.18), we have

$$(\mathbf{p}-\mathbf{c})^\top \mathbf{D}(\mathbf{p}-\mathbf{c}) = 1, \tag{2.0.19}$$

$$(\mathbf{q}-\mathbf{c})^\top \mathbf{D}(\mathbf{q}-\mathbf{c}) = 1, \tag{2.0.20}$$

which can be simplified as

$$2\left(\mathbf{p}-\mathbf{q}\right)^\top \mathbf{D}\mathbf{c} = \mathbf{p}^\top \mathbf{D}\mathbf{p} - \mathbf{q}^\top \mathbf{D}\mathbf{q} \tag{2.0.21}$$

Using the identity,

$$(\mathbf{p}^\top - \mathbf{q}^\top)\mathbf{D}(\mathbf{p}+\mathbf{q}) = \mathbf{p}^\top \mathbf{D}\mathbf{p} - \mathbf{q}^\top \mathbf{D}\mathbf{q} \tag{2.0.22}$$

in the equation (2.0.21)

$$2\left(\mathbf{p}-\mathbf{q}\right)^\top \mathbf{D}\mathbf{c} = (\mathbf{p}-\mathbf{q})^\top \mathbf{D}(\mathbf{p}+\mathbf{q}) \tag{2.0.23}$$

$$\implies (\mathbf{p}-\mathbf{q})^\top \mathbf{D}(2\mathbf{c}-(\mathbf{p}+\mathbf{q})) \tag{2.0.24}$$

Thus $\mathbf{c}$ can be expressed in parametric form as

$$\mathbf{c} = \frac{1}{2}((\mathbf{p}+\mathbf{q}) + k\mathbf{D}^{-1}\mathbf{m}) \tag{2.0.25}$$

where,

$$(\mathbf{p}-\mathbf{q})^\top \mathbf{m} = 0 \tag{2.0.26}$$

and k is a constant.Substituting numerical values in (2.0.26),

$$\mathbf{p}-\mathbf{q} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \implies \mathbf{m} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \tag{2.0.27}$$

and

$$\mathbf{p}+\mathbf{q} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} \tag{2.0.28}$$

substituting in (2.0.25),we get

$$\begin{pmatrix} \beta \\ 0 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 10 \\ 5 \end{pmatrix} + k\begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix}\begin{pmatrix} -1 \\ -2 \end{pmatrix}\right) \tag{2.0.29}$$

From the given information,the X-axis is the major axis.Hence,

$$\frac{\lambda_2}{\lambda_1} > 1 \implies \frac{20-4\beta}{5} > 1 \tag{2.0.30}$$

$$\beta < 3.75 \tag{2.0.31}$$

The possible ellipse satisfying the above conditions are plotted below

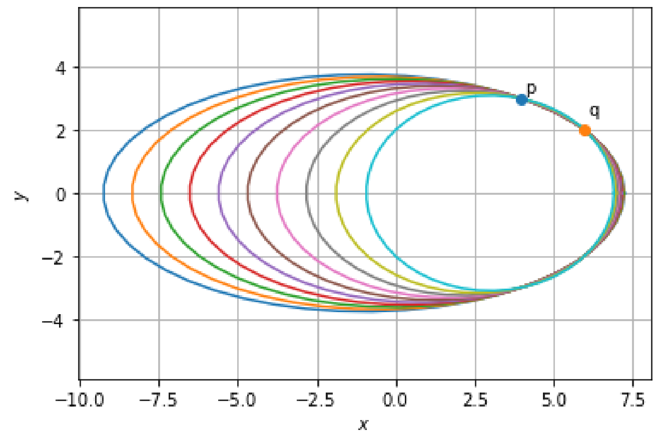

Fig. 2.2: Ellipses passing through the two points with X axis as major axis