Challenge Problem 1

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1 Question

Show that the matrix $(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)$ is of rank 1 for a parabola.

2 Solution

Method used : For a given nxn matrix A the rank equals the number of linearly independent rows or columns of A.

Solution: Let $\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$\|\mathbf{n}\|^2 = a^2 + b^2 \tag{2.0.1}$$

$$\mathbf{n}\mathbf{n}^T = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} \tag{2.0.2}$$

Now,

$$t = \frac{a^2 + b^2}{e^2} \tag{2.0.3}$$

$$\implies t\mathbf{I} = \begin{pmatrix} \frac{a^2+b^2}{e^2} & 0 \\ 0 & \frac{a^2+b^2}{e^2} \end{pmatrix} \tag{2.0.4}$$

$$(t\mathbf{I} - \mathbf{n}\mathbf{n}^T) = \begin{pmatrix} \frac{(1-e^2)a^2+b^2}{e^2} & -ab \\ -ab & \frac{a^2+(1-e^2)b^2}{e^2} \end{pmatrix} \tag{2.0.5}$$

For a parabola,$e = 1$.

$$(t\mathbf{I} - \mathbf{n}\mathbf{n}^T) = \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \tag{2.0.6}$$

Since row 2 can be expressed as a linear combination of row 1, that is

$$row2 = \frac{-a}{b}(row1) \tag{2.0.7}$$

Thus the number of linearly independent rows is 1 and hence we obtain the rank of the matrix as 1.