Challenge Problem 1

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1 Question

Show that the matrix $(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)$ is of rank 1 for a parabola.

2 Solution

Theorem 2.1. *Let A be an mxn matrix then rank(A) = 1 if and only if there exists colunm vectors u ϵ $\mathbb{R}^{\mathrm{m}}$ and w ϵ $\mathbb{R}^{\mathrm{n}}$ such that* $\mathbf{A} = \mathbf{u}\mathbf{w}^{\mathbf{T}}$.

Solution: Let $\mathbf{A} = t\mathbf{I} - \mathbf{n}\mathbf{n}^T$, where $\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $t = \frac{a^2+b^2}{e^2}$. We have,

$$\mathbf{A} = \begin{pmatrix} \frac{(1-e^2)a^2+b^2}{e^2} & -ab \\ -ab & \frac{a^2+(1-e^2)b^2}{e^2} \end{pmatrix} \tag{2.0.1}$$

For a parabola, $e = 1$.

$$\mathbf{A} = \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \tag{2.0.2}$$

We can express $\mathbf{A} = \mathbf{u}\mathbf{w}^{\mathbf{T}}$, where $\mathbf{u} = \begin{pmatrix} a \\ -b \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} a \\ -b \end{pmatrix}$.

That is $\mathbf{A} = \begin{pmatrix} a \\ -b \end{pmatrix}\begin{pmatrix} a & -b \end{pmatrix}$. Hence by Theorem 2.1 we have rank(A) = 1