Analysis of Stock Market Indices

Goal of the project

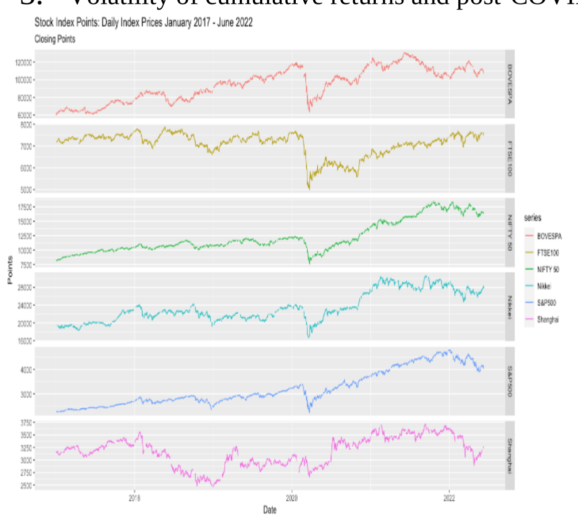
1. Analyze the dynamic performance of stock market indices in developed and emerging economies using time series techniques.
2. Analyze the stationarity and autocorrelation of the stock market indices.
3. Assess forecasts of various models under normal market conditions and abnormal market conditions.

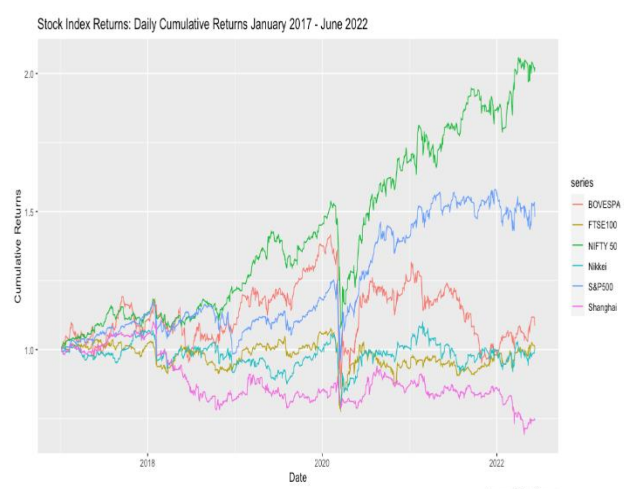
Data Description

Our data focuses on stocks, also known as equities because they are the most prominently featured in media, personal finance blogs, and trading platforms (e.g., Robinhood, AmeriTrade, among others). The number of company stocks is very large and heterogeneous, encompassing most sectors of the economy and company sizes. Fortunately, stock indices group diverse sets of individual stocks into a single benchmark that tracks the aggregate performance of the market. Among them, capitalization-weighted indices tend to reflect the overall market performance more closely. We chose to focus on these types of indices for six different markets on a daily basis; three in developed markets and three in emerging markets: S&P500 (USA), Nikkei (Japan), FTSE100 (UK), BOVESPA (Brazil), Shanghai Composite Index (China) and NIFTY50 (India).

Preliminary findings

1. All of the stock index points have a sharp drop (varying in magnitude) after the onset of the COVID-19 pandemic, followed by an increasing trend.
2. Unlike the other indices, the cumulative returns of Shanghai Composite Index dropped slightly when the Covid-19 pandemic was declared.
3. Volatility of cumulative returns and post-COVID-19 recovery vary widely across indices.





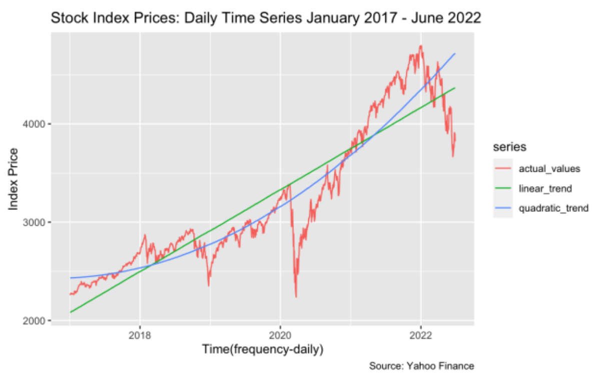
PART I

1. Transform S&P 500 to Stationarity

First, we focus on the USA stock market index-S&P 500 index. We plotted 1374 observations of the daily closing price of the S & P 500 index (from 2017-01-03 to 2022-06-29). It seems that the time series (Figure1) has an increasing time trend, and increasing variability over time, so we conclude that the data is non-stationary. We attempted to estimate the trend and then detrend the data to bring it to stationarity. Let x_t represent the S&P 500 index and let t represent the daily time, we assume that the S&P 500 data is of the form $x_t = \mu_t + y_t$ where, y_t is some stationary data and μ_t is the trend. We used 2 models to estimate the trend, Linear Trend ($\mu_t = \beta_0 + \beta_1 t$)

and Quadratic Trend ($\mu_t = \beta_0 + \beta_1 t + \beta_2 t^2$). We removed the trend from x_t to check the assumption of stationarity in y_t , where $y_t = x_t - \mu_t$.

Figure1 shows the time series with linear & quadratic trend, and Figure2 shows the detrended time series.



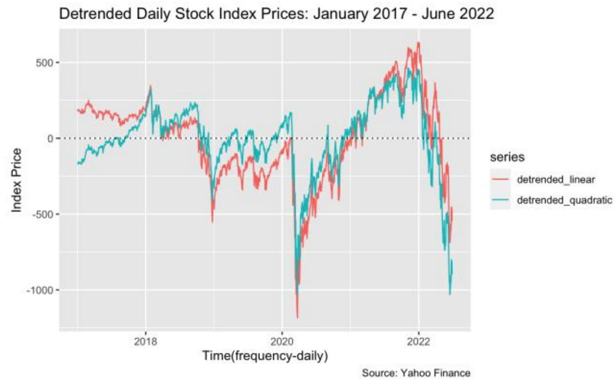
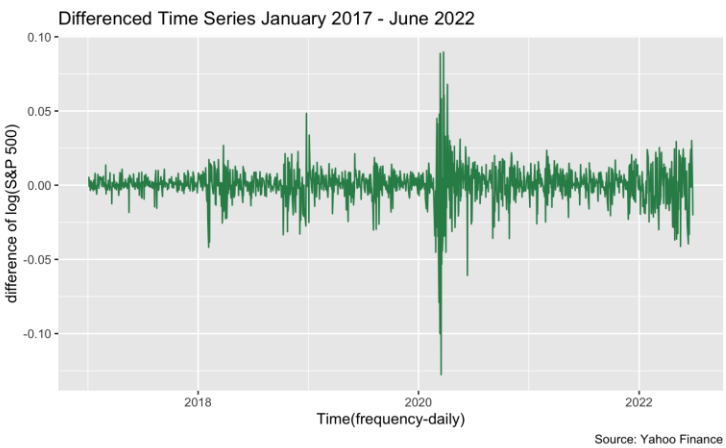


Figure 1: Time Series with trends

Figure2: Detrended Time Series

It can be observed from Figure2 that the detrended series for both linear trend and quadratic trend don't look close to white noise. So y_t is not stationary.

So, we log transformed the S&P 500 index and took the first difference of the series ($\nabla log(x_t)$) to bring it to stationarity. Figure (3a) shows the transformed time series.



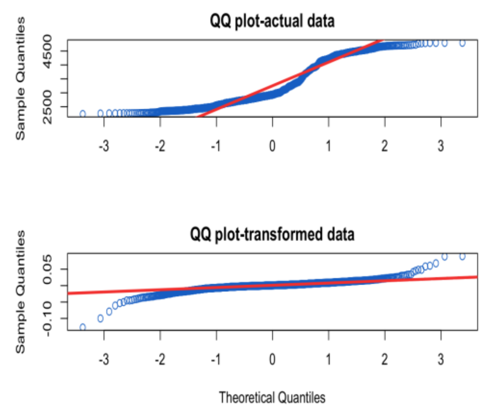


Figure (3a): Differenced (log(S&P500))

Figure (3b): Normal QQ Plot (actual & transformed)

It is evident from Figure (3a) that the transformed series seems to be stationary around mean zero, however the variance seems to be varying over time.

Figure (3b) shows the Normal Q-Q plots of the original (x_t) and the transformed series ($\nabla log(x_t)$), and it can be seen that the assumption of normality is also improved in the transformed series, however there is still significant non-normality towards the ends.

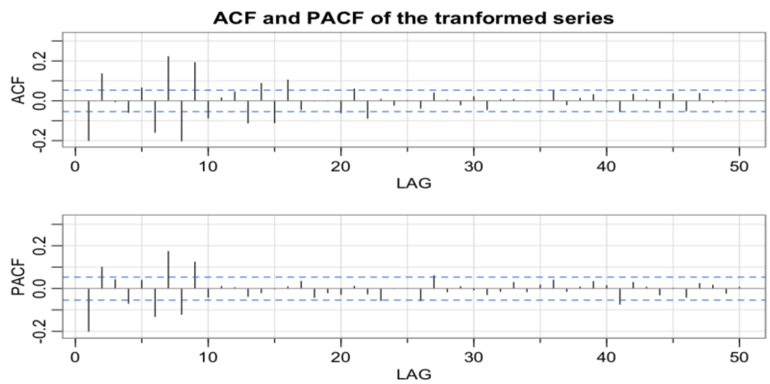


Figure 4: ACF and PACF of the transformed series

We also plotted in Figure 4 the Autocorrelation Function (ACF) and the Partial Autocorrelation Function (PACF) of the transformed series for the model selection. We see that values of ACF seem significant up to lag 9 and do not drop to 0 afterwards, which indicates an AR model may be suitable. Also, the values of PACF seem significant up to lag 9 which indicates an MA model may be suitable. So it may be suitable to select an ARMA model for the transformed data.

2. ARMA Models

Based on our preliminary data analysis above, we tried to estimate ARMA models for the transformed stock indices of S&P500, Nikkei, Nifty50, Shanghai, S&P500, BOVESPA and FTSE 100. However, we noted that the daily frequency depicted non-normal behavior as evident in Figure (3b), and highly fluctuating volatility as evident in Figure (3a) hence assumptions were not met for covariance stationarity. Furthermore, the high frequency of the data (daily) leads to a short forecast horizon before asymptotic behavior. Thus, we used weekly average data and made the series stationary using first difference to facilitate fitting the ARMA models. For this analysis, we used the sample period from June 2020 to June 2022, because at the onset of Covid-19 the series experienced a structural break that disfavored the analysis and estimation of models. We didn't find autoregressive (AR) or moving average (MA) component for S&P500, Nikkei, Nifty50, or Shanghai indices, but we found it in the case of BOVESPA and FTSE100 indices (see Figure 5). The ACF and PACF for the BOVESPA index suggest an ARMA (1, 1) could potentially fit the process, since they are both significant at lag 1 and flat zero afterwards. Similarly, the correlograms for FTSE100 index suggest an ARMA (2, 2) may work based on similar reasoning.

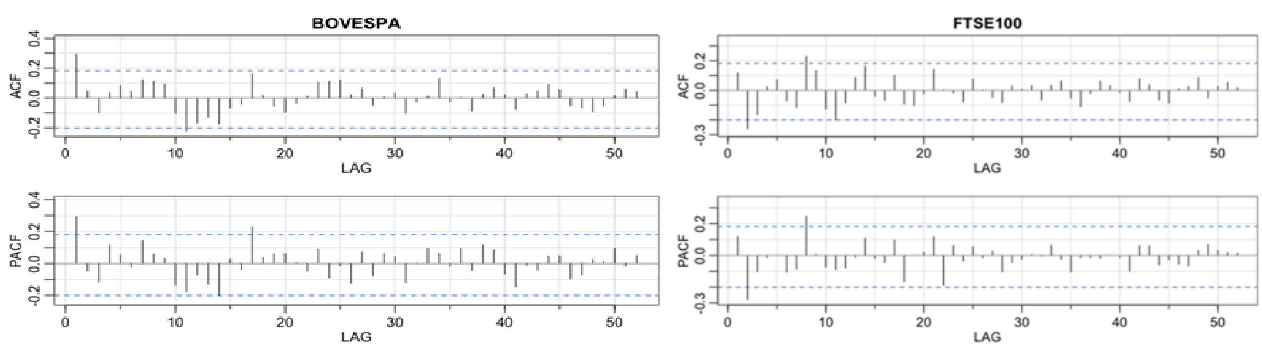


Figure 5: Correlograms for weekly BOVESPA and FTSE100 indexes (first difference)

We followed the following steps for fitting ARMA models for BOVESPA and FTSE100 indexes.

A. We used the R 'auto.arima' package to select the best fitting models and estimate the model parameters. The auto.arima() function in R uses a combination of unit root tests, minimization of the second order Akaike Information Criterion (AIC) and maximum likelihood estimation (MLE) to obtain an ARIMA model. The algorithm uses a stepwise search to traverse the model space to select the best model with the smallest Second-order Akaike Information Criterion (AICc). Figure 6 shows the auto.arima output for the indices.

auto.arima(BOVESPA, trace=TRUE)

```
 ARIMA(2,0,2) with non-zero mean : 1984.666
 ARIMA(0,0,0) with non-zero mean : 1989.476
 ARIMA(1,0,0) with non-zero mean : 1981.705
 ARIMA(0,0,1) with non-zero mean : 1982.293
 ARIMA(0,0,0) with zero mean     : 1987.516
 ARIMA(2,0,0) with non-zero mean : 1983.567
 ARIMA(1,0,1) with non-zero mean : 1983.705
 ARIMA(2,0,1) with non-zero mean : 1985.502
 ARIMA(1,0,0) with zero mean     : 1979.685
 ARIMA(2,0,0) with zero mean     : 1981.515
 ARIMA(1,0,1) with zero mean     : 1981.647
 ARIMA(0,0,1) with zero mean     : 1980.284
 ARIMA(2,0,1) with zero mean     : 1983.42

 Best model: ARIMA(1,0,0) with zero mean

Series: INDEX_diff
ARIMA(1,0,0) with zero mean

Coefficients:
         ar1
      0.2978
s.e.  0.0923

sigma^2 = 5193343:  log likelihood = -987.79
AIC=1979.57   AICc=1979.69   BIC=1984.94
```

auto.arima(FTSE100, trace=TRUE)

```
 ARIMA(2,0,2) with non-zero mean : 1321.247
 ARIMA(0,0,0) with non-zero mean : 1324.327
 ARIMA(1,0,0) with non-zero mean : 1324.883
 ARIMA(0,0,1) with non-zero mean : 1323.44
 ARIMA(0,0,0) with zero mean     : 1322.981
 ARIMA(1,0,2) with non-zero mean : Inf
 ARIMA(2,0,1) with non-zero mean : 1319.01
 ARIMA(1,0,1) with non-zero mean : 1324.063
 ARIMA(2,0,0) with non-zero mean : 1318.222
 ARIMA(3,0,0) with non-zero mean : 1318.952
 ARIMA(3,0,1) with non-zero mean : Inf
 ARIMA(2,0,0) with zero mean     : 1317.093
 ARIMA(1,0,0) with zero mean     : 1323.355
 ARIMA(3,0,0) with zero mean     : 1318.123
 ARIMA(2,0,1) with zero mean     : 1318.304
 ARIMA(1,0,1) with zero mean     : 1322.493
 ARIMA(3,0,1) with zero mean     : 1320.31

 Best model: ARIMA(2,0,0) with zero mean

Series: INDEX_diff
ARIMA(2,0,0) with zero mean

Coefficients:
         ar1      ar2
      0.1618  -0.2745
s.e.  0.0935   0.0928

sigma^2 = 11124:  log likelihood = -655.43
AIC=1316.86   AICc=1317.09   BIC=1324.91
```

Figure 6: R ‘auto.arima’ output for BOVESPA (left side) and FTSE100 (right side) indices

The output for BOVESPA (left side of Figure 6) suggests that the best fitting model would be an ARMA (1,0) i.e. a first order autoregressive model with zero mean. Also, an ARMA (1, 1) may work as per the correlograms but it yields higher AICc in results. Likewise, the output for the FTSE100 (right side of Figure 6) recommends an ARMA (2, 2), i.e. a second order autoregressive model with zero mean; also an ARMA (2, 2) may work as per the correlogram, but it yields higher AICc in results.

B. We checked the residuals for the AR (1) and AR (2) models for BOVESPA and FTSE100 respectively. We observed that the assumptions were met for the chosen models. For the BOVESPA model (left-side plot in Figure (7)), the residual plot suggests the residuals have a zero mean and constant variance, and the ACF suggests that there is no autocorrelation in residuals. The same conclusions hold in the case of the FTSE100 model (right-side plots in Figure (7)). We also observed that the residuals fit very well to a normal distribution in the histogram of residuals in Figure (7). Although we see outliers, they didn’t significantly impact the normality test.

BOVESPA **FTSE100**

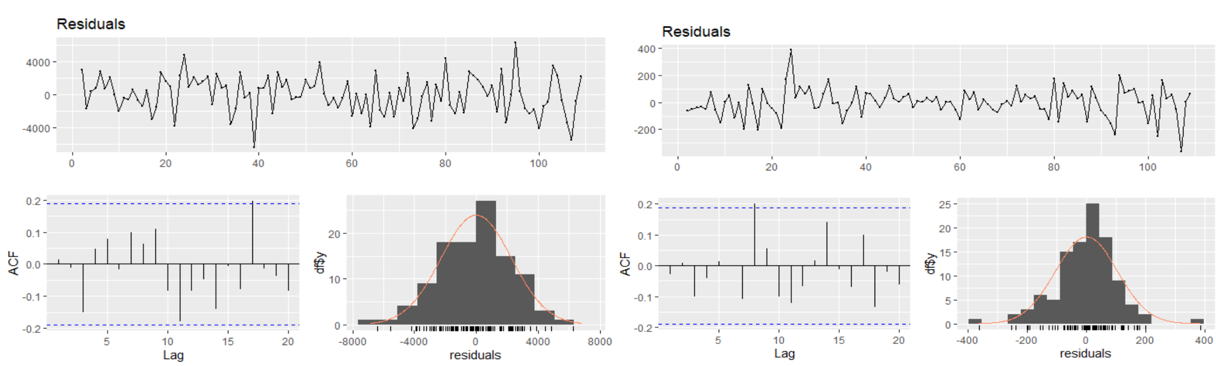


Figure (7a): Assumptions validation for BOVESPA **AR (1)** model (left side) and FTSE100 **AR (2)** model (right side)

C. We made some predictions based on the best fitting model specifications seen before. The final model regressions are given by (standard errors in parentheses):

BOVESPA

$$x_t = \underset{(0.0923)}{0.2978}\, x_{t-1} + u_t, \quad u_t \sim N(0, \sigma_u^2)$$

FTSE100

$$y_t = \underset{(0.0935)}{0.1618}\, y_{t-1} - \underset{(0.0928)}{0.2745}\, y_{t-2} + w_t, \quad w_t \sim N(0, \sigma_w^2)$$

Figure 8 shows the four week forecasts for BOVESPA and FTSE100 indexes, following the above models. In the case of the BOVESPA (left side), a gradual fall to zero is expected on average for the next four weeks, while for the FTSE100 (right side) a more pronounced fall is expected in the first week, followed by a rise in the second week and then a gradual drop to zero in the fourth week.

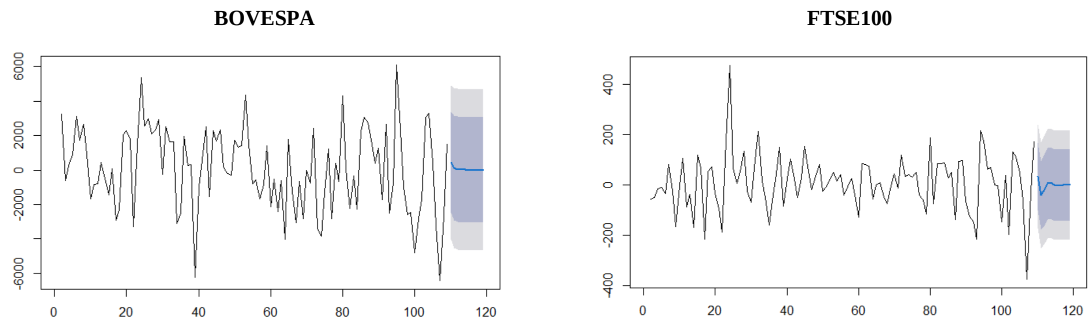


Figure 8: Four week forecasts for BOVESPA (left side) and FTSE100 (right side), using AR models

3. Regression on Term Spread and Covid Data

Further, we created regression models to include different variables that may be relevant when modeling and assessing the behavior of stock market indices. We analyzed the impact of Term Spread and Covid-19 on S&P500 Index. Term Spread is the difference between the interest rate of 2-Year and 10-Year US Treasury Constant Maturity, a higher spread indicates a positive market outlook. We analyzed daily term spread data for the period 2017-01-03 to 2022-06-29 (*Source: Federal Reserve Bank of St. Louis's*). For Covid-19 data (*Source: World Health Organization*) we created a dummy variable that captures the period before and after onset of Covid-19 in the USA. The first case of Covid-19 was reported on 03/23/2020 in the USA, so the dummy variable takes a value 1 after that date and 0 before that date.

The data of log(S&P500 Index), the Term Spread and the Covid-dummy for the period 2017-01-03 to 2022-06-29 is plotted in Figure 9. We considered log (S&P 500) to reduce the variance of the S&P500 and bring it to a comparable scale to the Term Spread. A visible correlation between log(S&P500) and Term Spread is evident in Figure 9, there seems to be a negative correlation before onset of Covid-19, and a positive correlation after Covid-19. We also plotted a scatter plot matrix of log(S&P500) vs Term Spread (see Figure 10), and there seems to be some non-linear correlation between these two variables.

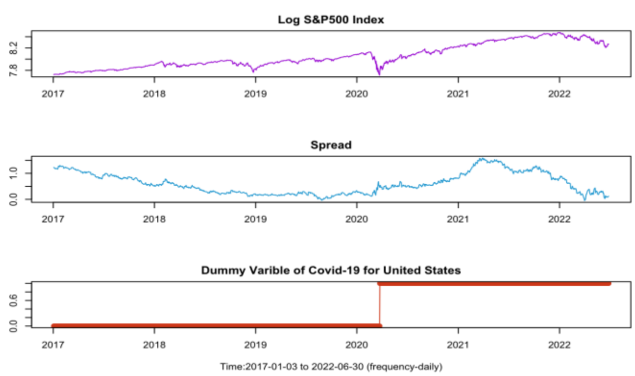


Figure 9: log(S&P500), Term Spread and Covid-dummy

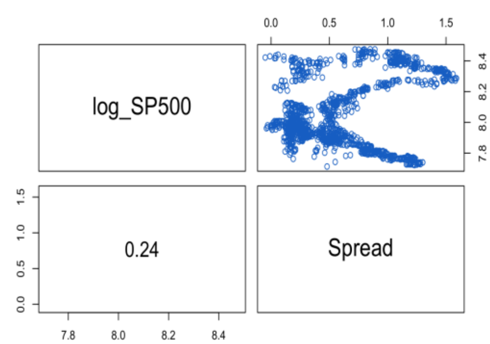


Figure10: Scatter Plot S&P500 vs Spread

However, both the log(S&P500) and the term spread seem to have some trend and the correlation may be spurious. Therefore, we transformed Term Spread & log(S&P500) to stationarity by taking first differences. The transformed series (see Figure 11) look stationary with mean zero, although the variance is not constant. We also observe that the volatility in the 2 series seems to be correlated and is highly evident around observation 800 (onset of Covid-19).

We plotted the CCF (see Figure 12) for the series and observed that there seems to be some negative correlation for lags:(-3, 1, 3, and 4) and some positive correlation for lags (-19, 0, 7 and 9). The ACF of the differenced spread series resembles white noise and since S&P 500 and spread are independent, we conclude that the CCF is significant.

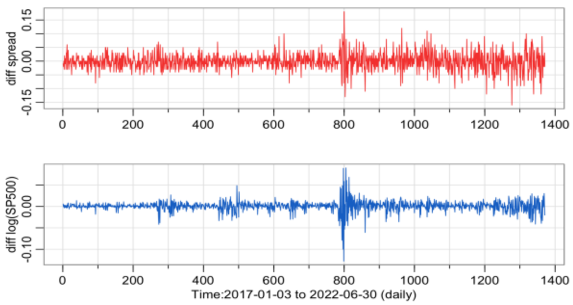


Figure 11: Differenced Term Spread and log(S&P500)

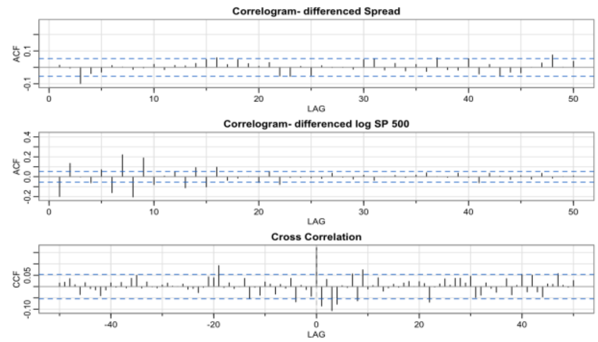


Figure12: ACF and ACF for differenced series

So, we tried various model specifications using the different combinations of regressors of the Covid dummy variable, the term spread and the lags of term spread. The summary of the models is presented in the table below.

Model	Model Specification	SSE	df	MSE	R_square	AIC	BIC
1	**Y~X**	**0.2114**	**1369**	**0.0002**	**0.0296**	**-7.7687**	**-7.7572**
2	Y~X+D+D*X	0.2111	1367	0.0002	0.0310	-7.7672	-7.7482
3	Y~X_L1+X_L2	0.2156	1349	0.0002	0.0089	-7.6515	-7.6363
4	Y~X_L1+X_L2+D+D*X_L1+D*X_L2	0.2152	1346	0.0002	0.0110	-7.6493	-7.6227
5	Y~X_L1	0.2156	1350	0.0002	0.0089	-7.6530	-7.6416
6	Y~X_L1+D+D*X_L1	0.2153	1348	0.0002	0.0103	-7.7331	-7.7139
7	Y~X+X_L1+X_L2	0.2154	1348	0.0002	0.0102	-7.7330	-7.7138
8	Y~X+X_L1+X_L2+D+ D*(X+X_L1+X_L2)	0.2150	1344	0.0002	0.0117	-7.7286	-7.6939
diff(logSP500)= Y, diff(spread)=X, lag(diff(spread), 3)=X_L1, lag(diff(spread), 19)=X_L2, Dummy=D							

Table 1: Summary of Models

As shown in the summary table, the best regression model is Model 1, with spread as a regressor. It has the lowest AIC value and R squared of nearly .03. We also observe that all the other models have AIC values very close to Model 1 however R squared is comparatively low.

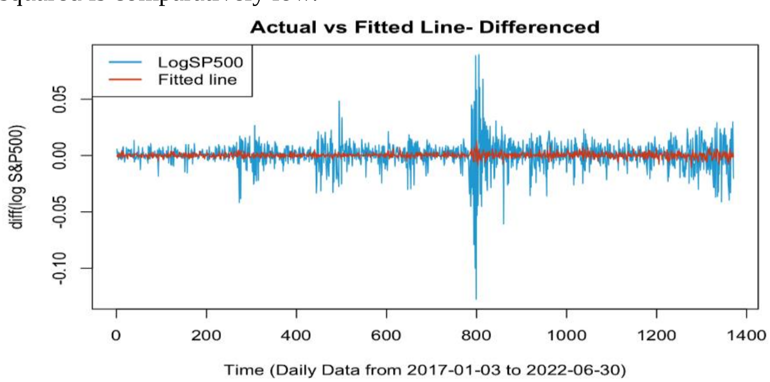


Figure 13: Regression Model 1

The plot of fitted values of Model 1 is shown in the figure 13.

PART II

1. ARIMA Models

Based on our preliminary data analysis, we first tried to estimate ARIMA models for the stock indices of S&P500, Nikkei, Nifty50, Shanghai, S&P500, BOVESPA and FTSE 100. However, we noted that the daily frequency depicted non-normal behavior, hence assumptions were not met for covariance stationarity. Furthermore, the high frequency of the data (daily) leads to a short forecast horizon before asymptotic behavior. Thus, we used weekly average data to facilitate fitting the ARIMA models. For this analysis, we used the sample period from June 2020 to June 2022, because at the onset of Covid-19 the series experienced a structural break that disfavored the analysis and estimation of models. We didn't find autoregressive (AR) or moving average (MA) component for S&P500, Nikkei, Nifty50, or Shanghai indices, but we found it in the case of BOVESPA and FTSE100 indices (see Figure 1). The ACF and PACF for the BOVESPA index suggest an ARIMA (1, 1, 1) could potentially fit the process, since they are both significant at lag 1 and flat zero afterwards. Similarly, the correlograms for FTSE100 index suggest an ARIMA (2, 1, 2) may work based on similar reasoning.

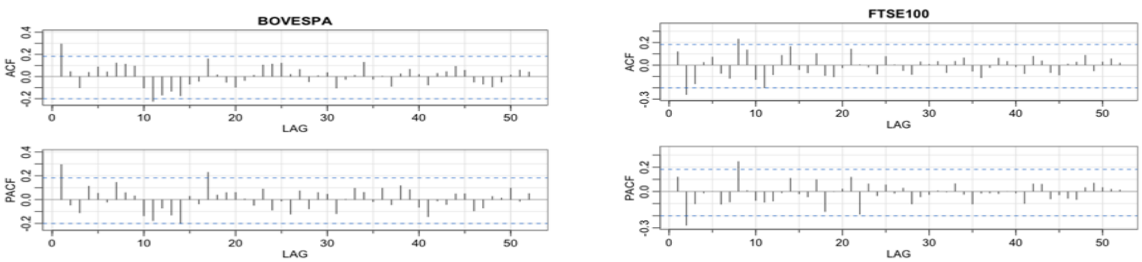


Figure 1: Correlograms for weekly BOVESPA and FTSE100 indexes (first difference)

We followed the following steps for fitting ARMA models for BOVESPA and FTSE100 indexes.

A. We used the R 'auto.arima' package to select the best fitting models and estimate the model parameters. The auto.arima() function in R uses a combination of unit root tests, minimization of the second order Akaike information criterion (AIC) and maximum likelihood estimation (MLE) to obtain an ARIMA model. The algorithm uses a stepwise search to traverse the model space to select the best model with the smallest Second-order Akaike Information Criterion (AICc). Figure 2 shows the auto.arima output for the indices.

auto.arima(BOVESPA, trace=TRUE)

```
 ARIMA(2,1,2) with drift         : 1984.666
 ARIMA(0,1,0) with drift         : 1989.476
 ARIMA(1,1,0) with drift         : 1981.705
 ARIMA(0,1,1) with drift         : 1982.293
 ARIMA(0,1,0)                    : 1987.516
 ARIMA(2,1,0) with drift         : 1983.567
 ARIMA(1,1,1) with drift         : 1983.705
 ARIMA(2,1,1) with drift         : 1985.502
 ARIMA(1,1,0)                    : 1979.685
 ARIMA(2,1,0)                    : 1981.515
 ARIMA(1,1,1)                    : 1981.647
 ARIMA(0,1,1)                    : 1980.284
 ARIMA(2,1,1)                    : 1983.42

 Best model: ARIMA(1,1,0)

Series: BOVESPA
ARIMA(1,1,0)

Coefficients:
         ar1
      0.2978
s.e.  0.0923

sigma^2 = 5193420:  log likelihood = -987.79
AIC=1979.57   AICc=1979.69   BIC=1984.94
```

auto.arima(FTSE100, trace=TRUE)

```
 ARIMA(2,1,2) with drift         : 1321.247
 ARIMA(0,1,0) with drift         : 1324.327
 ARIMA(1,1,0) with drift         : 1324.883
 ARIMA(0,1,1) with drift         : 1323.44
 ARIMA(0,1,0)                    : 1322.981
 ARIMA(1,1,2) with drift         : Inf
 ARIMA(2,1,1) with drift         : 1319.01
 ARIMA(1,1,1) with drift         : 1324.063
 ARIMA(2,1,0) with drift         : 1318.222
 ARIMA(3,1,0) with drift         : 1318.952
 ARIMA(3,1,1) with drift         : Inf
 ARIMA(2,1,0)                    : 1317.093
 ARIMA(1,1,0)                    : 1323.355
 ARIMA(3,1,0)                    : 1318.123
 ARIMA(2,1,1)                    : 1318.304
 ARIMA(1,1,1)                    : 1322.493
 ARIMA(3,1,1)                    : 1320.31

 Best model: ARIMA(2,1,0)

Series: FTSE100
ARIMA(2,1,0)

Coefficients:
         ar1      ar2
      0.1618  -0.2745
s.e.  0.0935   0.0928

sigma^2 = 11124:  log likelihood = -655.43
AIC=1316.86   AICc=1317.09   BIC=1324.91
```

Figure 2: R 'auto.arima' output for BOVESPA (left side) and FTSE100 (right side) indices

The output for BOVESPA (left side of Figure 2) suggests that the best fitting model would be an ARIMA(1,1,0) i.e. a first order autoregressive model with trend. Also, an ARIMA (1,1,1) may work as per the correlograms but it yields higher AICc in results. Likewise, the output for the FTSE100 (right side of Figure 2) recommends an ARMA(2,1,0), i.e. a second order autoregressive model trend; also an ARMA(2,1,2) may work as per the correlogram, but it yields higher AICc in results.

B. We checked the residuals for the ARIMA(1,1,0) and ARIMA(2,1,0) models for BOVESPA and FTSE100 respectively. We observed that the assumptions were met for the chosen models. For the BOVESPA model (left-side plot in Figure 3), the residual plot suggests the residuals have a zero mean and constant variance, and the

ACF suggests that there is no autocorrelation in residuals. The same conclusions hold in the case of the FTSE100 model (right-side plots in Figure 3). We also observe that the residuals fit very well to a normal distribution in the histogram of residuals in Figure 3. Although we see outliers, they didn't significantly impact the normality test.

BOVESPA **FTSE100**

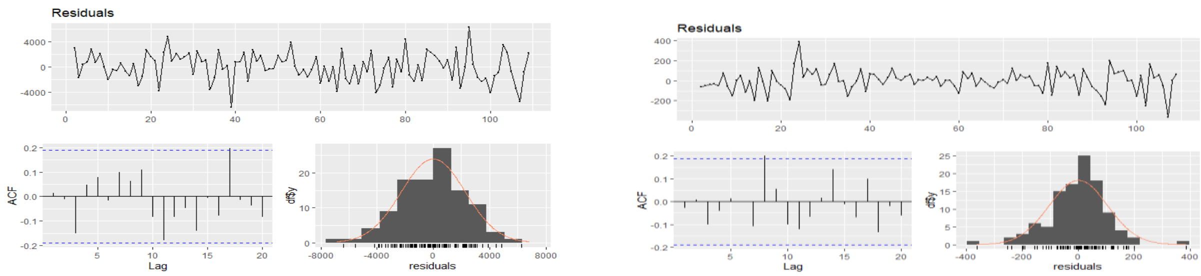


Figure 3: Assumptions validation for BOVESPA ARIMA(1,1,0) model (left side) and FTSE100 ARIMA(2,1,0) model (right side)

C. We made some predictions based on the best fitting model specifications seen before. The final model regressions are given by (standard errors in parentheses):

BOVESPA

$$x_t = 93.1360 + \underset{(0.0923)}{0.2973}\, x_{t-1} + u_t \qquad u_t \sim N(0, \sigma_u^2)$$

FTSE100

$$y_t = 9.1263 + \underset{(0.0932)}{0.1559}\, y_{t-1} - \underset{(0.0926)}{0.2813}\, y_{t-2} + w_t \qquad w_t \sim N(0, \sigma_w^2)$$

Figure 4 shows the four week forecasts for BOVESPA and FTSE100 indexes, following the above models. In the case of the BOVESPA (left side), a gradual linear increase is expected on average for the next four weeks, while for the FTSE100 (right side) a slight fall is expected in the first week, followed by a moderate rise from the second week onwards.

BOVESPA **FTSE100**

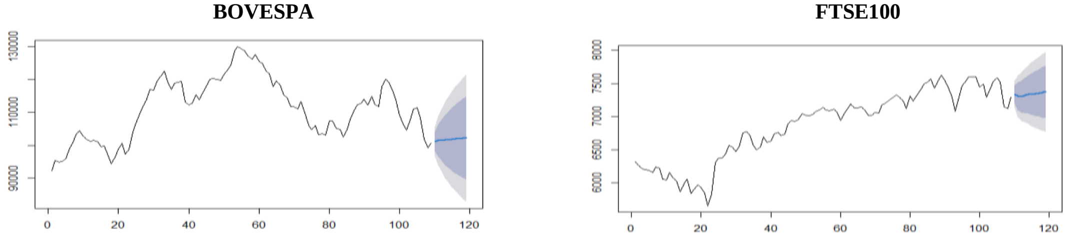


Figure 4: Four week forecasts for BOVESPA (left side) and FTSE100 (right side), using AR models

2. Regression with Autocorrelated Errors

In our previous work (report #2) we fitted several regressions to model the relationship between the S&P500 and potential predictors. The model that produced the best fit had a single feature: the spread between the 10 year and 2 year treasury bonds. Despite having the lowest BIC, the predictive power of the models was low (R-squared less than 0.1).

With the aim of improving the model, we tested linear regression with autocorrelated errors. The process we followed is the following:

1) Fit the returns of the S&P as a function of the spread of the US treasury yields.
2) Check the residual ACF and PACF plots and identify candidate model.
3) Test candidate model and validate assumptions.

We tested two models: one with a date range from 01/2017 to 06/2022 and another with a date range from 04/2020 to 06/2022. The rationale behind the second date range was to exclude excessive COVID-19 volatility from the time period and improve the model fit. The following are the ACF and PACF plots for the residuals of the model built in step 1.

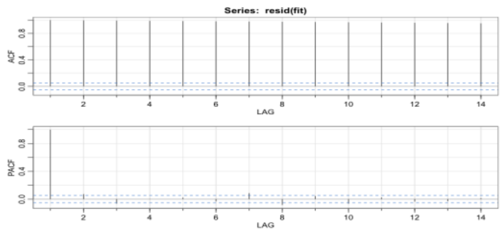


Figure 5: ACF and PACF plots for 01/17-06/22

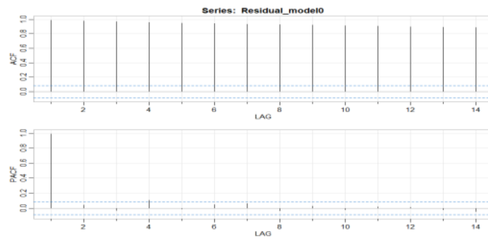


Figure 6: ACF and PACF plots for 04/20-06/22

Figures 5 and 6 clearly show the candidate ARIMA model under both date ranges is ARIMA(0,1,0), i.e., a first order autoregressive model first difference. However, after fitting the regression with autocorrelated errors, only the latter model (date range from 04/2020 to 06/2022) meets the assumptions: standardized residuals look like a random scatter, the ACF plot of the residuals does not show autocorrelation, the Normal Q-Q plot approximates a straight line (albeit with slight skew) and all the p-values for the Ljung-Box statistic are well above the significance threshold.

Therefore, we conclude that the advent of COVID-19 is impactful from a model building perspective. Depending on the date range selected, the degree of model fit and validity can change drastically. In practice the model would be redundant for predictive purposes: as new spread information gets generated, so do actual S&P500 index underlying prices.

3. ARCH and GARCH Models

3.1. S&P500

In earlier analysis, we tried to take log transformation and difference S&P 500 index, so the data would become stationary. However, the variance of the data is still not constant even after the transformation. We can see the time varying variance in the following plot:

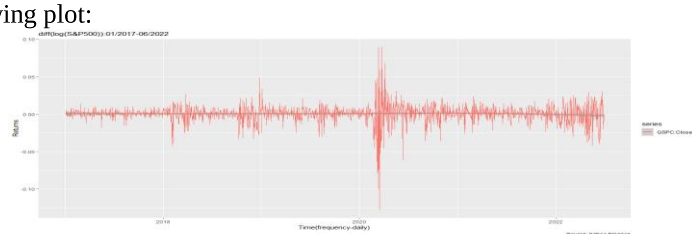
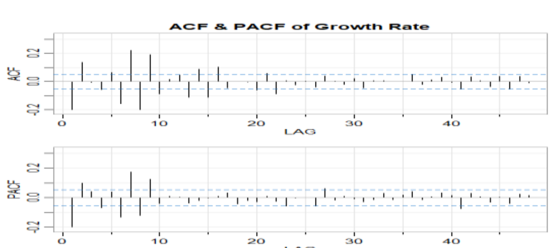


Figure 7. The Return of S&P500

When we took log transformation of S&P500 index and then difference it, we got the return or growth rate of it which is:

$$r_t = \frac{x_t - x_{t-1}}{x_{t-1}} \approx \nabla log(x_t)$$

In order to model the changing variance in S&P500 returns, we experimented with several ARCH and GARCH models here. We plotted the ACF and PACF of the returns and we found some of the ACF and PACF values are still significant and both of them are tailing off which suggests an ARMA (1, 1) model. The ACF and PACF values of the squared residuals of the fitted ARMA (1, 1) model are highly correlated, and both of them are tailing off which suggests that GARCH (1, 1) might be a good fit.



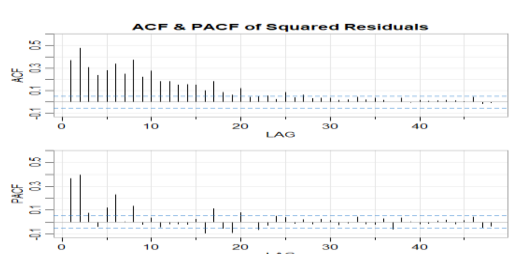


Figure 8. ACF&PACF of Growth Rate (left) and Squared Residuals of ARMA (1, 1) (right)

We fitted ARCH (1) and GARCH (1, 1) to the squared residuals of ARMA (1, 1) model, and we found that all parameters in ARCH (1) are significant, the residuals of the ARCH (1) are normally distributed. However, the Q-statistics associated with residuals and squared residuals are smaller than 0.05, so we reject the null hypothesis and conclude that the residuals are not white.

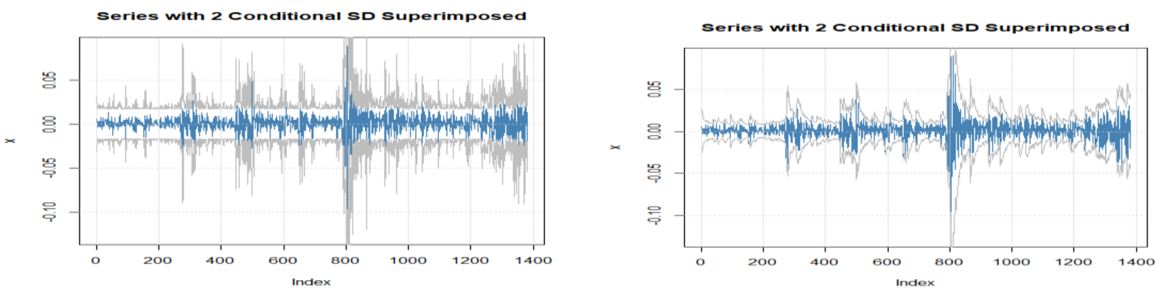


Figure 10. Fitted Series of ARMA (1, 1)-ARCH (1) (left) and ARMA (1, 1)-GARCH (1, 1) (right)

Figure 10 are fitted values of ARCH (1) and GARCH (1, 1) models. From the plots we can see that GARCH (1, 1) fits the data much better.

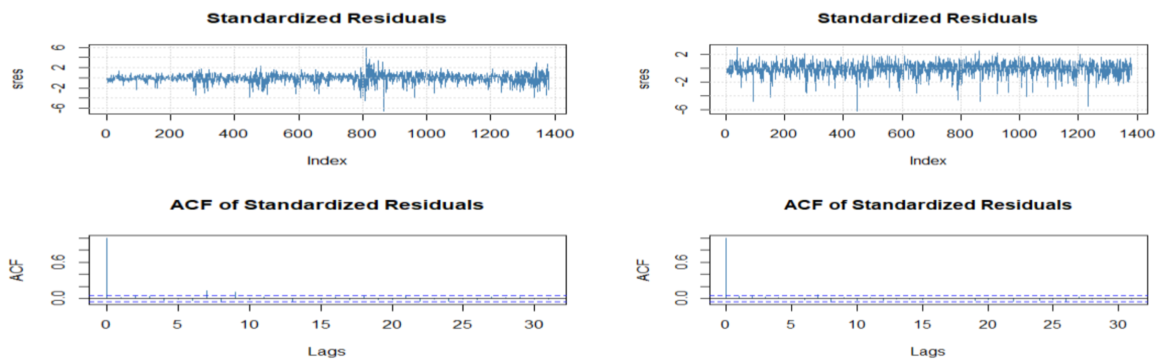


Figure 11. Residual Plots of ARMA (1, 1)-ARCH (1) (left) and ARMA (1, 1)-GARCH (1, 1) (right)

From the residual plot in Figure 11, we can see that there is still a small amount of variation and correlation in the residual of ARCH (1) model. In order to find the best model for the data, we used auto.arima() again to find the best combination of p and q for the ARMA model fitted to the S&P500 returns. It gave us a combination of p=0 and q=2. We fitted MA (2) to the returns and then fitted ARCH (1) and GARCH (1, 1) to the squared residuals. The coefficients of MA (2) are not significant in ARCH (1) model. The residuals are normally distributed, but the Q-statistics associated with residuals and squared residuals are smaller than 0.05. Therefore, we reject the null hypothesis and conclude that the residuals are not white.

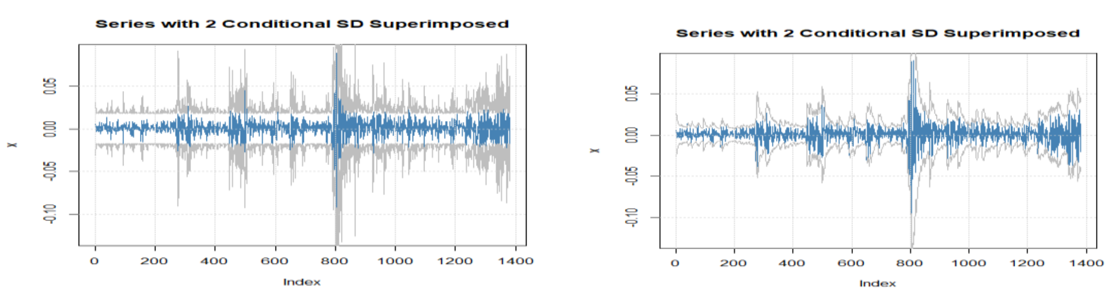


Figure 12. Fitted Series of MA (2)-ARCH (1) (left) and MA (2)-GARCH (1, 1) (right)

From the plots of the fitted series in Figure 12, we can see that GARCH (1, 1) fits the data better again. We may check their residual plots for more model diagnostics.

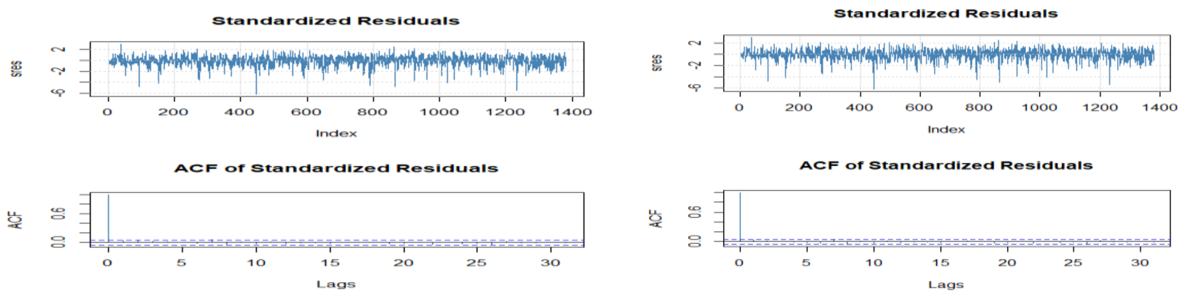


Figure 13. Residual Plots of MA (2)-ARCH (1) (left) and MA (2)-GARCH (1, 1) (right)

From the plots above, we can see that the residuals of MA (2)-ARCH (1) and MA (2)-GARCH (1, 1) look like white noise, and the ACF plots shows no correlation in the residuals. It's hard to tell which model is the best one from visual inspection, so we are going to use AIC and BIC to compare the four models we fitted to the returns of S&P 500.

Models	AIC	BIC
ARMA(1,1)-ARCH(1)	-6.518895	-6.496170
ARMA(1,1)-GARCH(1,1)	-6.734627	-6.708115
MA(2)- ARCH(1)	-6.513534	-6.490795
MA(2)- GARCH(1,1)	-6.731409	-6.704881

Table 1. Model Selection

From the table above, we can see that our best model to fit the returns of S&P500 is ARMA (1, 1)-GARCH (1, 1). It has the lowest AIC and BIC values. Therefore, we are going to use it to do predictions. Here is our prediction model:

$$\sigma_t{}^2 = 0.2197_{(0.036)} r_{t-1}{}^2 + 0.7972_{(0.028)} \sigma_{t-1}{}^2$$

The following plot shows our 30 days predictions and its confidence intervals.

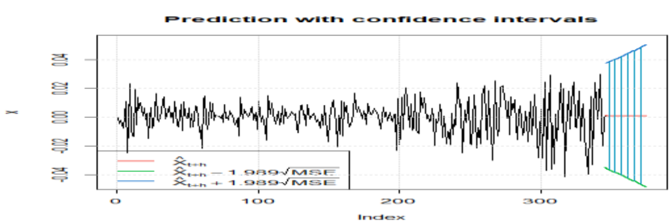


Figure 12. ARMA (1, 1)-GARCH (1, 1) 30 Days Forecast

3.2. FTSE100 Index

We first log transformed the FTSE100 index and then took the difference (See Figure 13). It is important to mention that we used daily data from June 2017 to June 2022.

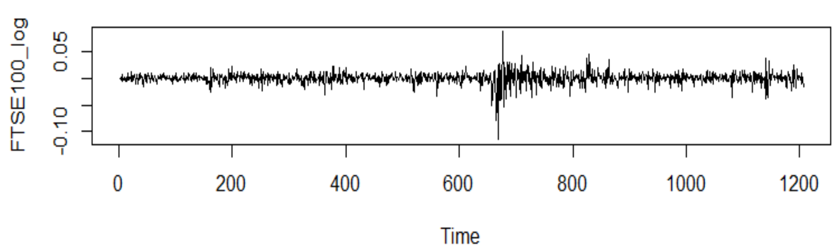


Figure 13. Series Plot of diff (log (FTSE100))

Additionally, we plotted the ACF and PACF (See Figure 14) to observe any patterns in the data. As shown in the plots there is some level of autocorrelation in the series itself. The returns show some classic GARCH features.

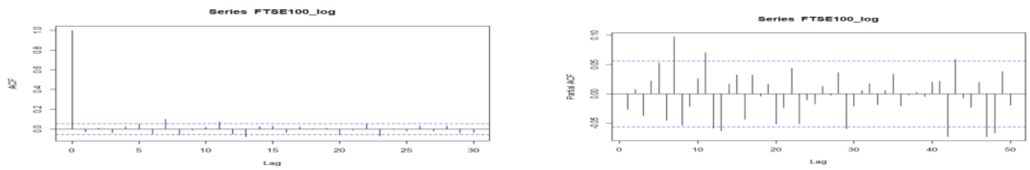


Figure 14. ACF & PACF Plots of diff (log (FTSE100))

Moreover, we fitted an ARMA (1, 1) since we observed some ARMA components. However, there is a high correlation in the squared residuals after fitting an ARMA (1, 1) (See Figure 15).

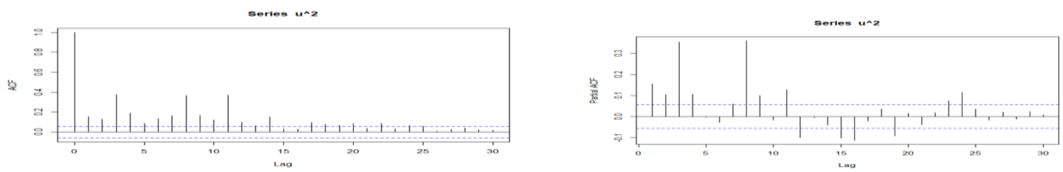


Figure 15. ACF & PACF of Squared Residuals of ARMA (1, 1)

Next, we decided to estimate both GARCH (1, 1) and ARMA (1, 1) models simultaneously. All parameters were significant at 95% level except for the constant mu (p=0.402). The AIC and BIC were -6.61 and -6.59, respectively. We can also see that the model fits the data well (See Figure 16).

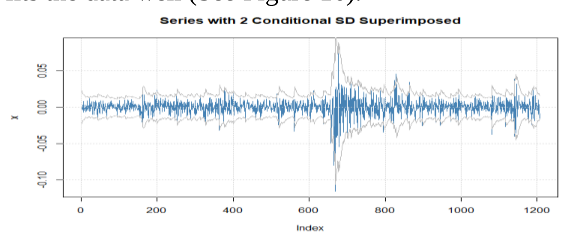


Figure 16. Fitted Series of ARMA (1, 1) – GARCH (1, 1)

A second model was built for comparison purposes. In this case, we modeled an ARMA (1, 1) and an ARCH (1) simultaneously. All parameters were significant at 95% level except for the constant mu (p=0.920). The AIC and BIC were -6.52 and -6.50, respectively. Since the second model has a bigger AIC and BIC than the first model, we disregarded this model and chose the first model for further analysis.

After selecting the best model, we proceeded to check if its residuals were uncorrelated. Fortunately, they were as shown below (See Figure 17).

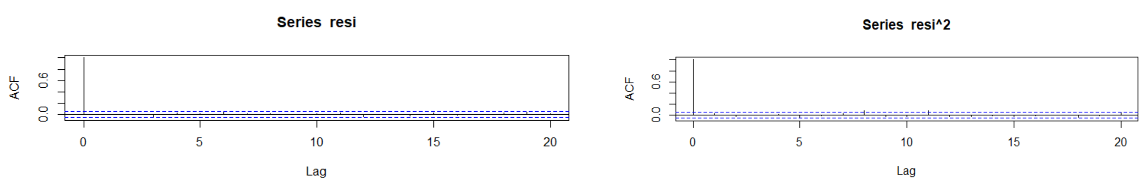


Figure 17. ACF Plots of Residuals and Squared Residuals of ARMA (1, 1) – GARCH (1, 1)

Finally, we predicted the mean and the variance 30 days ahead (See Figure 18).

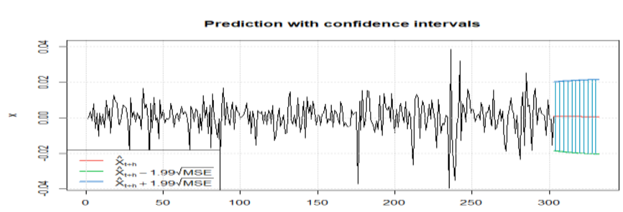


Figure 18. 30 Days Prediction of ARMA (1, 1) – GARCH (1, 1)

3.3. BOVESPA Index

Similar procedure was performed for the BOVESPA index (See Figure 19). We first log transformed the index and then took the difference. We used daily data from June 2017 to June 2022.

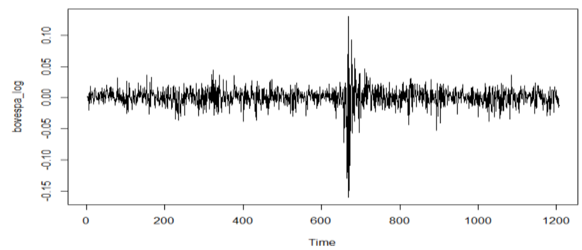


Figure 19. Series Plot of diff (log (BOVESPA))

Additionally, we plotted the ACF and PACF (See Figure 20) to observe any patterns in the data. As shown in the plots there is some autocorrelation in the series itself. The returns show some classic GARCH features.

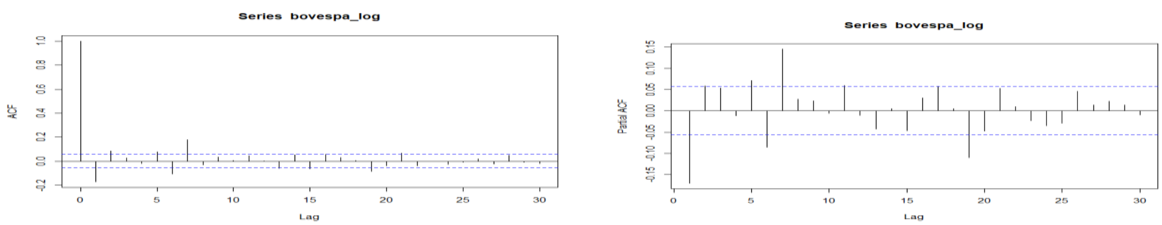


Figure 20. ACF & PACF Plots of diff (log (BOVESPA))

Moreover, we fitted an AR(1) since we observed some AR components. However, there is a high correlation in the squared residuals after fitting an AR(1) (See Figure 21).

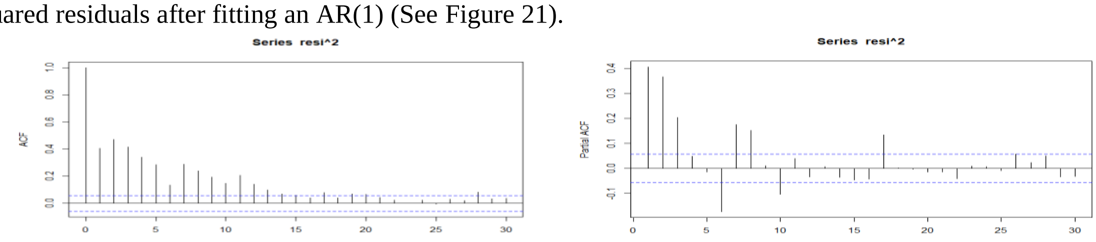


Figure 21. ACF & PACF Plots of Squared Residuals of AR (1)

Furthermore, we decided to estimate both GARCH (1, 1) and AR (1) models simultaneously (See Figure 22). All parameters were significant at 95% level, and the model fitted the data well. The AIC and BIC were -5.78 and -5.75 respectively.

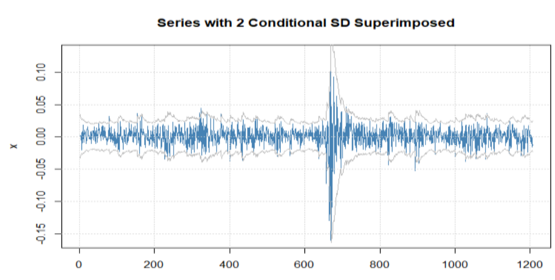


Figure 22. Fitted Series of AR (1) – GARCH (1, 1)

We also checked its residuals and they were uncorrelated as shown below (See Figure 23).

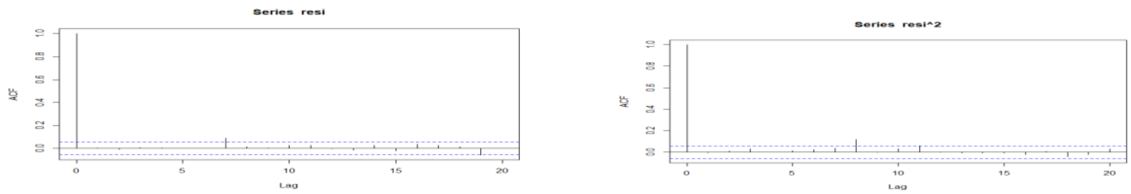


Figure 23. ACF Plots of Residuals and Squared Residuals of AR (1) – GARCH (1, 1)

A second model was built for comparison purposes. In this case, we modeled a MA(1) and a GARCH(1,1) simultaneously. All parameters were significant at 95% level, and the AIC and BIC were -5.78 and -5.76 respectively. Although both models had similar AIC and BIC, we decided to select the first model since unlike the second model, its squared residuals were uncorrelated at the 95% level (See Figure 24).

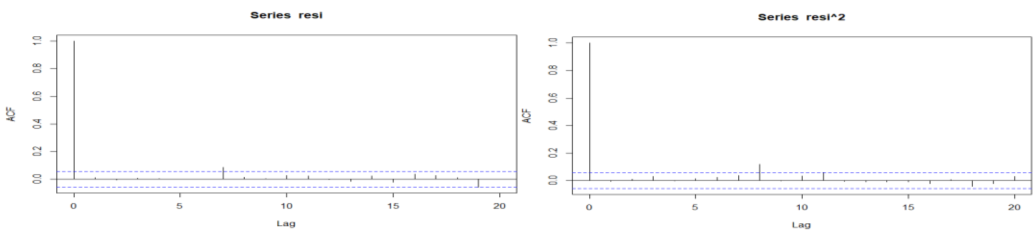


Figure 24. ACF Plots of Residuals and Squared Residuals of MA (1) – GARCH (1, 1)

Finally, we predicted the mean and the variance 30 days ahead (See Figure 25).

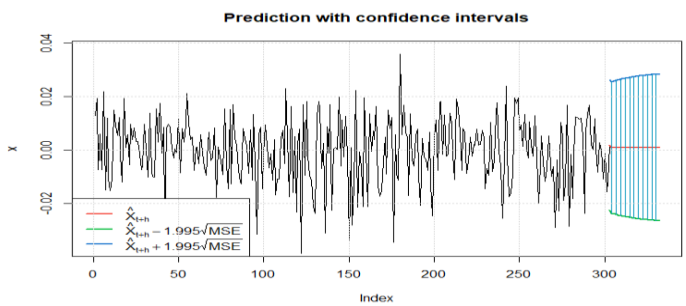


Figure 25. 30 Days Prediction of AR (1) – GARCH (1, 1)

4. Vector Auto-Regression (VAR) Models

We analyzed the six market indices-S&P 500, NIKKEI, FTSE, BOVESPA, SHANGHAI and NIFTY50- to investigate connectedness of the movements in market indices. Our aim was to detect whether changes in one stock index affected movement of other stock market indices. We used the log transformed and differenced daily stock index prices from 2017-01-03 to 2022-06-30 for our model. The data has been transformed to mean stationarity to remove the effect of trends in the respective markets and to ensure that spurious correlations were not generated. We used VAR (1) and VAR (2) models to determine the correlations.

4.1. VAR(1) Model

We fitted the following VAR (1) model to the six transformed indices. The results of the parameter estimates from the regression are presented below in Table 2. We noted that most of the regressions had insignificant lags of BOVESPA, FTSE100 and SHANGHAI indices. Also, we noted that the SHANGHAI index did not have significant correlation with its own lag. We also noted that the adjusted R-squared values were very low except for the regression of NIKKEI index. So we decided to reformulate our model and also drop the three indices-BOVESPA, FTSE100 and SHANGHAI.

			SP500.l1	Nikkei.l1	FTSE100.l1	BOVESPA.l1	Shanghai.l1	NIFTY50.l1	R.squared
1	SP500 = SP500.l1 + Nikkei.l1 + FTSE100.l1 + BOVESPA.l1 + Shanghai.l1 + NIFTY50.l1	Estimate	-0.13876	0.17896	-0.04623	-0.05136	-0.01732	-0.16261	0.06268
		Std. Error	0.03551	0.03923	0.04428	0.03591	0.03276	0.03774	
		p-values	9.75E-05	5.52E-06	0.297	0.153	0.597	1.75E-05	
		Significance	***	***				***	
2	Nikkei = SP500.l1 + Nikkei.l1 + FTSE100.l1 + BOVESPA.l1 + Shanghai.l1 + NIFTY50.l1	Estimate	0.31786	-0.07483	0.15493	-0.04283	-0.06012	-0.11017	0.2061
		Std. Error	0.02431	0.02686	0.03031	0.02459	0.02243	0.02584	
		p-values	< 2e-16	0.00541	3.64E-07	0.08169	0.00744	2.14E-05	
		Significance	***	**	***	.	**	***	
3	FTSE100 = SP500.l1 + Nikkei.l1 + FTSE100.l1 + BOVESPA.l1 + Shanghai.l1 + NIFTY50.l1	Estimate	0.10431	0.1584	-0.07199	0.04667	-0.06192	-0.20616	0.05959
		Std. Error	0.02774	0.03065	0.03459	0.02805	0.0256	0.02949	
		p-values	0.000177	2.71E-07	0.037602	0.096419	0.015682	4.16E-12	
		Significance	***	***	*	.	*	***	
4	BOVESPA = SP500.l1 + Nikkei.l1 + FTSE100.l1 + BOVESPA.l1 + Shanghai.l1 + NIFTY50.l1	Estimate	-0.07261	0.23549	0.0155	-0.12138	-0.02519	-0.13602	0.06063
		Std. Error	0.03292	0.03637	0.04105	0.03329	0.03038	0.03499	
		p-values	0.027572	1.31E-10	0.705719	0.000276	0.407093	0.000106	
		Significance	*	***		***		***	
5	Shanghai = SP500.l1 + Nikkei.l1 + FTSE100.l1 + BOVESPA.l1 + Shanghai.l1 + NIFTY50.l1	Estimate	0.149649	-0.014929	0.003127	0.029697	-0.025437	-0.065199	0.03084
		Std. Error	0.030585	0.033795	0.038139	0.030932	0.028222	0.032508	
		p-values	1.11E-06	0.6587	0.9347	0.3372	0.3676	0.0451	
		Significance	***					*	
6	NIFTY50 = SP500.l1 + Nikkei.l1 + FTSE100.l1 + BOVESPA.l1 + Shanghai.l1 + NIFTY50.l1	Estimate	0.21241	0.11822	-0.00304	0.04486	-0.06225	-0.1567	0.09406
		Std. Error	0.02788	0.03081	0.03477	0.0282	0.02573	0.02964	
		p-values	4.65E-14	0.00013	0.93034	0.11184	0.01566	1.43E-07	
		Significance	***	***			*	***	
	Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1								

Table 2. Parameter Estimates for VAR (1) Model

4.2. VAR(2) Model

We reformulated our model with the reduced data of the S&P 500, NIKKEI, and NIFTY50 indices. We fitted VAR(1) and VAR(2) models to our data and the VAR(2) model had better performance in terms of the R-squared and Information Criteria so we selected the VAR(2) model for further analysis. The VAR (2) model is presented below:

$$\begin{pmatrix} SP500 \\ Nikkei \\ NIFTY50 \end{pmatrix} = (\phi_1)_{3X3} \begin{pmatrix} SP500_{t-1} \\ Nikkei_{t-1} \\ NIFTY50_{t-1} \end{pmatrix} + (\phi_1)_{3X3} \begin{pmatrix} SP500_{t-2} \\ Nikkei_{t-2} \\ NIFTY50_{t-2} \end{pmatrix} + \left(\sum_{t} \right)_{3X1}$$

The results of the parameter estimates from regression are shown in Table 3 below. The estimates show that lag 1 of all the indices are significant in the three regressions. We also noted that lag 2 of S&P500 has a significant effect on the Nikkei Index and similarly lag 2 of Nikkei Index has a significant index on the S&P500 Index. We interpreted that this impact could be due to difference in the market operation timings, however this claim needs further investigation.

			SP500.l1	Nikkei.l1	NIFTY50.l1	SP500.l2	Nikkei.l2	NIFTY50.l2	R.squared
1	SP500 = SP500.l1 + Nikkei.l1 + NIFTY50.l1 + SP500.l2 + Nikkei.l2 + NIFTY50.l2	Estimate	**-0.128418**	**0.103591**	**-0.211229**	**0.089282**	**-0.006663**	**0.140946**	0.08231
		Std. Error	0.029689	0.041327	0.036261	0.032842	0.037386	0.035464	
		p-values	1.63E-05	0.0123	7.04E-09	0.00664	0.85857	7.41E-05	
		Significance	***	*	***	**		***	
2	Nikkei = SP500.l1 + Nikkei.l1 + NIFTY50.l1 + SP500.l2 + Nikkei.l2 + NIFTY50.l2	Estimate	**0.417586**	**-0.172156**	**-0.135904**	**0.176298**	**-0.006097**	**0.033476**	0.2307
		Std. Error	0.020222	0.028149	0.024698	0.022369	0.025464	0.024155	
		p-values	< 2e-16	1.24E-09	4.43E-08	6.39E-15	0.811	0.166	
		Significance	***	***	***	***			
3	NIFTY50 = SP500.l1 + Nikkei.l1 + NIFTY50.l1 + SP500.l2 + Nikkei.l2 + NIFTY50.l2	Estimate	**0.25688**	**0.06413**	**-0.18727**	**0.05689**	**0.09778**	**0.01432**	0.1051
		Std. Error	0.02342	0.0326	0.0286	0.0259	0.02949	0.02797	
		p-values	< 2e-16	0.049341	8.12E-11	0.028231	0.000936	0.608875	
		Significance	***	*	***	*	***		
	Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1								

Table 3. Parameter Estimates for VAR (2) Model

We obtained significant correlations between the different indices in the VAR (2) model, but to establish that the co-movements were causal we did the Granger Causality Test. The results (Table 4 below) show that the changes in market returns are causal to changes in the market returns of other indices.

	Null Hypothesis	Chi-Squared	F-test	p-value	RESULT
S&P 500	S&P 500 does not Granger Cause NIKKEI and NIFTY 50		5.0638	0.000453	CAUSAL
	No instantaneous Causality to NIKKEI and NIFTY 50	184.78		2.20E-16	CAUSAL
NIKKIEI	NIKKEI does not Granger Cause S&P500 and NIFTY 50		116.37	2.20E-16	CAUSAL
	No instantaneous Causality to S&P500 and NIFTY 50	249.59		2.20E-16	CAUSAL
NIFTY50	NIFTY50 does not Granger Cause NIKKEI and S&P500		17.063	6.84E-14	CAUSAL
	No instantaneous Causality to NIKKEI and S&P500	186.16		2.20E-16	CAUSAL

Table 4: Results of Granger Causality Test for VAR (2) Model

We made a prediction for the period 2022-07-01 to 2022-08-02 using the VAR (2) model and the prediction vs the realization of the market returns is plotted below in Figure 26.

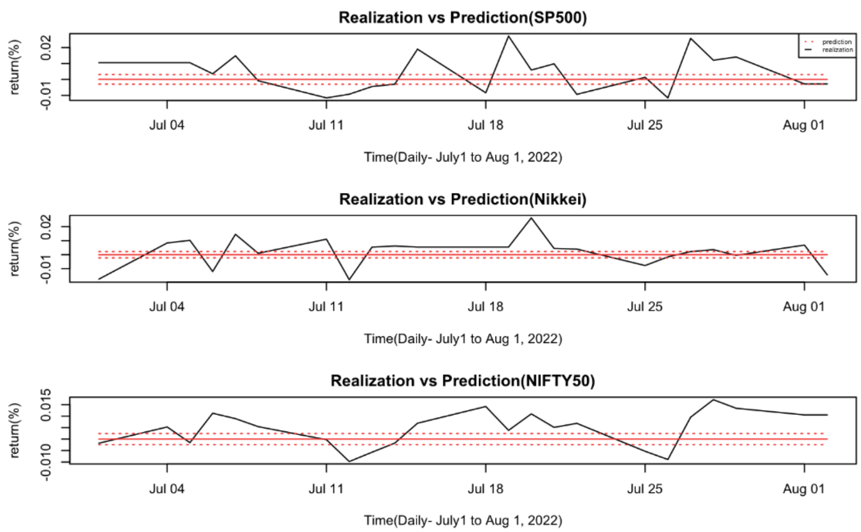


Figure 26: Parameter Estimates and Correlation Matrix of Residuals for VAR (2) model

We noted that the predictions are very close to the mean zero, which is expected because of the low R-squared values of the regression (highest R-squared is 0.23 for the Nikkei index). So, we concluded that the co-movements in the market indices are significant and causal but do not explain much variability. A limitation in our model is that we have time varying covariance in the return data and it violates the assumption of constant variance.