Exploration of 2D Elastic Wave Scattering Using Staggered Grid

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Background and motivation

In this project we explore the two dimensional elastic wave scattering with staggered grid method. In particular,we simulate P-SV wave scattering by 2-D parallel cracks using the finite difference method (FDM). We apply a standard FDM (second-order velocity-stress scheme with a staggered grid) to media including traction-free, infinitesimally thin cracks, which are expressed in a simple manner provided in the paper written by Yuji Suzuk in 2013 [3]. The standard staggered grid scheme is an implementation of Virieux's 1984 paper[1], and the subsequent rotated staggered scheme is an implementation of Saenger's 2000 paper[2].

We think this project is interesting because such simulations provide us a straightforward yet accurate way to simulate wave scattering in a more realistic setting which has a discontinuous nature. Staggered Grid has been proven to be very useful in simulating wave propagation in media with large material contrasts such as irregular land topography (i.e., irregular free surface), irregular ocean-bottom topography (i.e., irregular liquid-solid interface), and three dimensional heterogeneity including low-velocity soft sediments[3]. While in this project we only cover a small fraction of such scenarios, but the scalability of this numerical scheme can be really helpful in helping us further explore more complicated heterogeneity in the future. In a broader perspective, we are excited and intrigued especially about its capability in seismology in which the simulation of crack scattering can provide a lot of insights on disaster prevention.

Governing Equations

For 2D elastic wave case (P-SV system), force balance and constitutive equations can be written as:

$$\rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \tag{1}$$

$$\rho \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} \tag{2}$$

$$\tau_{xx} = (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_z}{\partial z} \tag{3}$$

$$\tau_{zz} = (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda \frac{\partial u_x}{\partial x} \tag{4}$$

$$\tau_{xx} = \mu \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) \tag{5}$$

[1]The standard staggered grid implementation references some part of the derivation and content provided by http://geodynamics.usc.edu/ becker/teaching/557/problem_sets/problem_set_fd_wave.pdf

Here, (u_x, u_z) are the particle displacements. For seismic waves, they are typically called radial and vertical components, respectively, if they are recorded at surface. Further, τ is the stress tensor, and λ and μ the elastic, Lamé coefficients (μ is shear modulus).

Typically, those equations are solved for particle velocities as $U = \frac{\partial u_x}{\partial t}$ and $V = \frac{\partial u_z}{\partial t}$. Then, the system is transformed into the frst-order hyperbolic system, introducing the abbreviations $\Sigma = \tau_{xx}$, $T = \tau_{zz}$, $\Lambda = \tau_{xz}$,

$$\frac{\partial U}{\partial t} = b\left(\frac{\partial \Sigma}{\partial x} + \frac{\partial \Lambda}{\partial z}\right) \tag{6}$$

$$\frac{\partial V}{\partial t} = b\left(\frac{\partial \Lambda}{\partial x} + \frac{\partial T}{\partial z}\right) \tag{7}$$

$$\frac{\partial \Sigma}{\partial t} = (\lambda + 2\mu)\frac{\partial}{\partial x} + \lambda\frac{\partial V}{\partial z} \tag{8}$$

$$\frac{\partial}{\partial t} = (\lambda + 2\mu)\frac{\partial}{\partial z} + \lambda\frac{\partial U}{\partial x} \tag{9}$$

$$\frac{\partial \Lambda}{\partial t} = \mu\left(\frac{\partial}{\partial z} + \frac{\partial V}{\partial x}\right) \tag{10}$$

$$\tag{11}$$

with $b = \frac{1}{\rho}$.

Stability Conditions

For a homogeneous medium, the stability condition is [1]

$$v_P S = v_P \frac{\delta t}{\delta h} < \frac{1}{\sqrt{2}} \tag{12}$$

$$\tag{13}$$

where

$$v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{14}$$

$$\tag{15}$$

is the P-wave velocity. The stability condition is independent of the S-wave velocity

$$v_S = \sqrt{\frac{\mu}{\rho}} \tag{16}$$

$$\tag{17}$$

because information will propagate at the P wave speed.

Staggered Grid System

Given our system of equations, we are able to allow the stress and particle velocity to be spatially interlaced on the grids as in Fig. 1. In particular, the staggered-grid scheme [4] allows the spatial derivative to be computed to a much higher accuracy.By staggering the stress and velocity field in time, we follow an explicit scheme and compute the discretization in the following section.

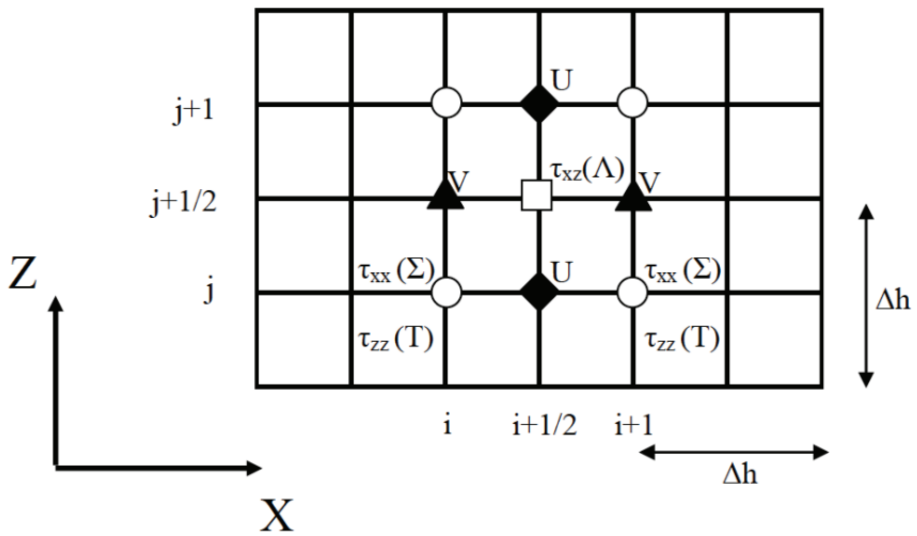

Figure 1: 2D staggered finite difference grid for wave propagation

Discretization

Using the staggered scheme, we are able to fully discretize our first order hyperbolic system in to the following coupled equations:

$$\begin{aligned}
U^{k+1/2}_{i+1/2,j} &= U^{k-1/2}_{1+1/2,j} + bS(\Sigma^k_{i+1,j} - \Sigma^k_{i,j}) + bS(\Lambda^k_{i+1/2,j+1/2} - \Lambda^k_{i+1/2,j-1/2}) \\
V^{k+1/2}_{i,j+1/2} &= V^{k-1/2}_{i,j+1/2} + bS(\Lambda^k_{i+1/2,j+1/2} - \Lambda^k_{i-1/2,j+1/2}) + bS({}^k_{i,j+1} - T^k_{i,j}) \\
\Sigma^{k+1}_{i,j} &= \Sigma^k_{i,j} + (\lambda+2\mu)S(U^{k+1/2}_{i+1/2,j} - U^{k+1/2}_{i-1/2,j}) + \lambda_{i,j}S(V^{k+1/2}_{i,j+1/2} - V^{k+1/2}_{i,j-1/2}) \\
T^{k+1}_{i,j} &= T^k_{i,j} + (\lambda+2\mu)S(V^{k+1/2}_{i,j+1/2} - V^{k+1/2}_{i,j-1/2}) + \lambda_{i,j}S(U^{k+1/2}_{i+1/2,j} - U^{k+1/2}_{i-1/2,j}) \\
\Lambda^{k+1}_{i+1/2,j+1/2} &= \Lambda^k_{i+1/2,j+1/2} + (\lambda+2\mu)S(V^{k+1/2}_{i+1/2,j+1/2} - V^{k+1/2}_{i,j+1/2}) + \lambda_{i,j}S(U^{k+1/2}_{i+1/2,j+1} - U^{k+1/2}_{i+1/2,j})
\end{aligned} \tag{18}$$

in which:

$$\begin{aligned}
S &= \frac{\Delta t}{\Delta h} \\
U &= \frac{\partial u_x}{\partial t} \\
V &= \frac{\partial v_x}{\partial t} \\
\Sigma &= \tau_{xx} \\
T &= \tau_{zz} \\
\Lambda &= \tau_{xz}
\end{aligned} \tag{19}$$

Width of Grid	Height of Grid	dz,dx	Time Steps	V_p	V_s	ρ	μ	λ	dt	Source
6000	6000	20	1000	5000	2000	2000	8×10^9	3.4×10^{10}	2.26×10^{-3}	$e^{-100(t-0.2)^2}$

Table 1: The parameters settings for each simulation.

Scattering Simulations using cracks

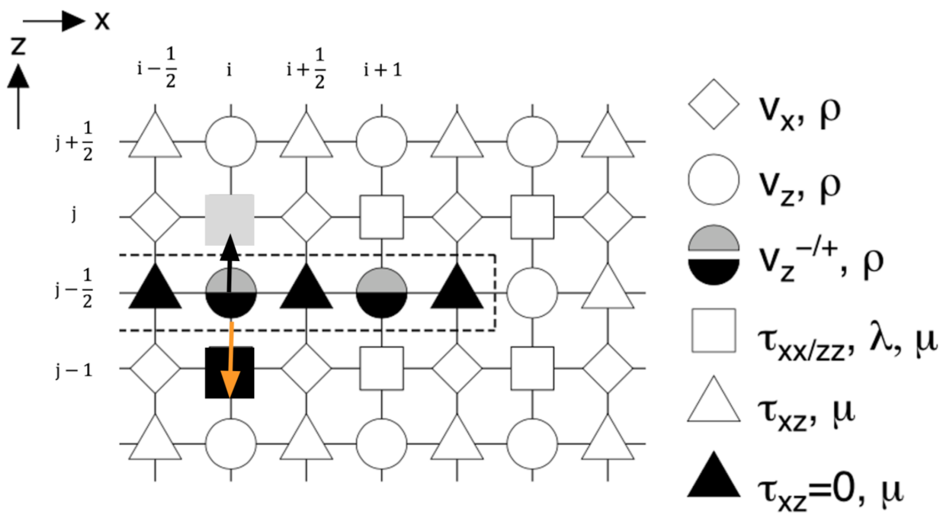

Figure 2: Staggered grid and implementation of cracks.

On the Figure 2, the region enclosed by the dotted line corresponds to a crack. And solid triangles denote the points with zero shear stress. Each pair of gray and black semi-circles denotes two close points located on the upper and lower crack faces. And separation between them is much smaller than the grid spacing.

On the graph, the crack is horizontal (parallel to the x-axis). We impose the traction-free boundary condition $\tau_{xz} = \tau_{zz} = 0$ on the crack plane. When dealing with cracks, first, we impose $\tau_{xz} = 0$, for example, the solid triangles, which define a crack plane. The crack tips are defined as the leftmost and rightmost points of the array. However, we cannot directly impose $\tau_{zz} = 0$ on the crack plane, because the crack is composed of grid points for $\tau_{xz} = 0$ and for v_z but not for $\tau_{zz} = 0$. Alternatively, we split here each grid point for v_z on the crack (the gray and black semi-circles) into two very close points, which represent the points on the upper and lower crack faces[3]. We assign the value of the semicircle in the crack to the adjacent T_{zz}. Then, we use the formula to update the value of semi-circle,

$$v_z^{\pm}(i, j + \frac{1}{2}, m + \frac{1}{2}) = v_z^{\pm}(i, j + \frac{1}{2}, m - \frac{1}{2}) \pm \frac{1}{\rho(i,j)} \frac{\triangle t}{3 \triangle h} [9\tau_{zz}(i, j \pm 1, m) - \tau_{zz}(i, j \pm 2, m)] \tag{20}$$

which is provided by the paper[3].

Simulation

For all simulation, we set the parameters as Table 1:

We also use the boundary condition that setting all $T_{xx} = T_{zz} = T_{xz} = 0$ at the boundary of the grid. The source used in our simulation is a Gaussian function, which shows on Table 1.

In the simulation of wave scattering in homogeneous medium, we set source at the middle of the grid $(3000, 3000)$. On the graph 3, we plot u_x.

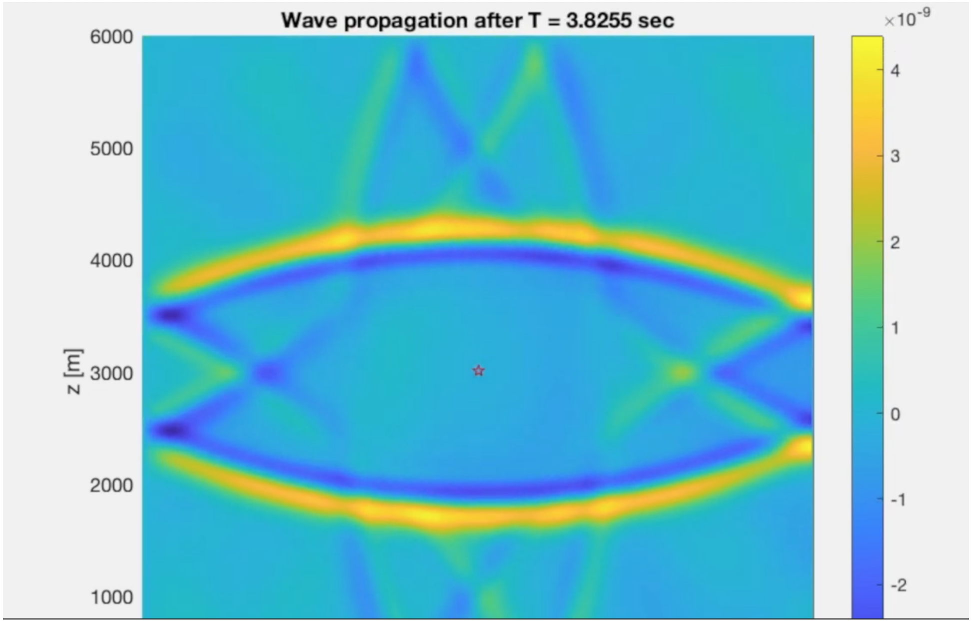

Figure 3: The simulation of wave scattering in homogeneous medium.

Then, we simulated the wave scattering with a horizontal crack. The source at $(3000, 3000)$ is normally injected. On the graph 4, we plot a crack at $x \in (2000, 4000), z = 3750$.

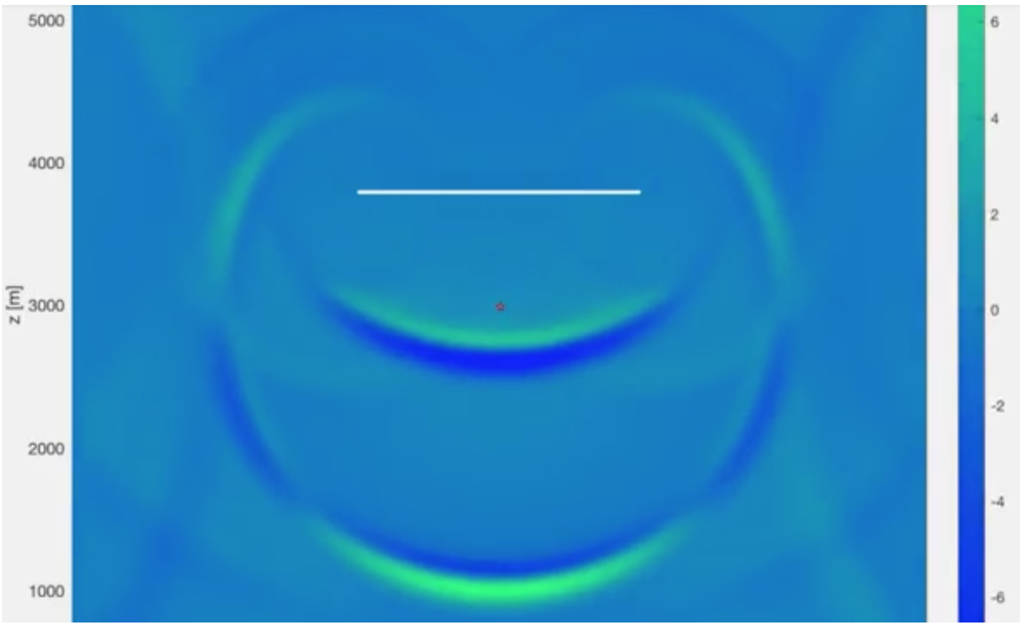

Figure 4: The simulation of wave scattering with a horizontal crack, and source is normally injected.

We also tried another simulation of the wave scattering with a horizontal crack. This time, the source is still at $(3000, 3000)$, but it is not normally injected. On the graph 5, we plot a crack at $x \in (2000, 3000), z = 3750$.

For observation of the diffraction effect, we placed the source at $(30, 3000)$ and between the two cracks. On the graph 6, we plot cracks at $x_1 \in (0, 2980), x_2 \in (3020, 6000), z = 800$.

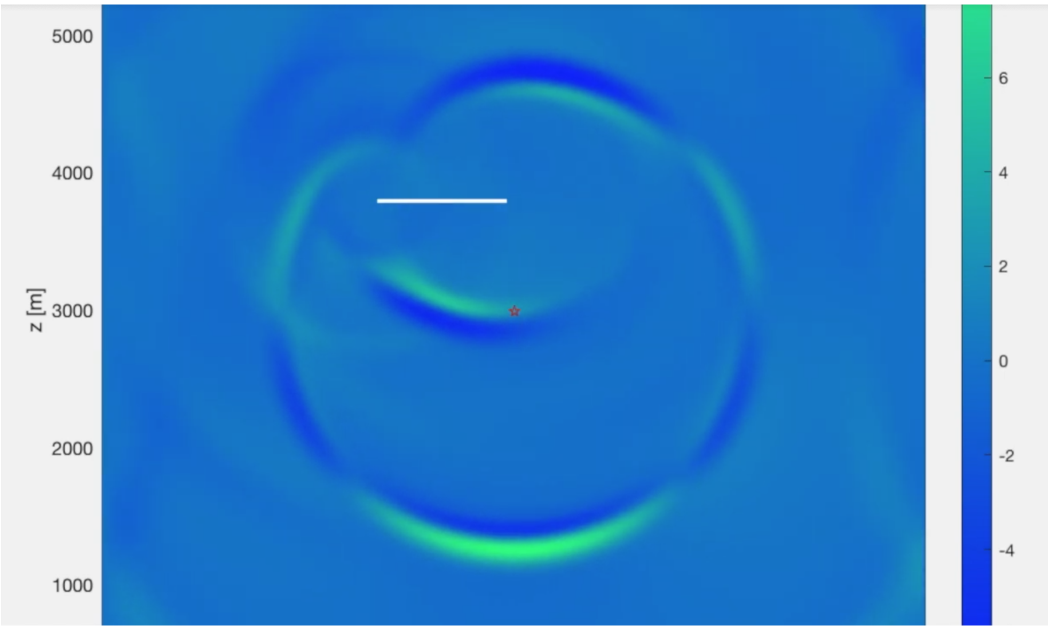

Figure 5: The simulation of wave scattering with a horizontal crack, and source is not normally injected.

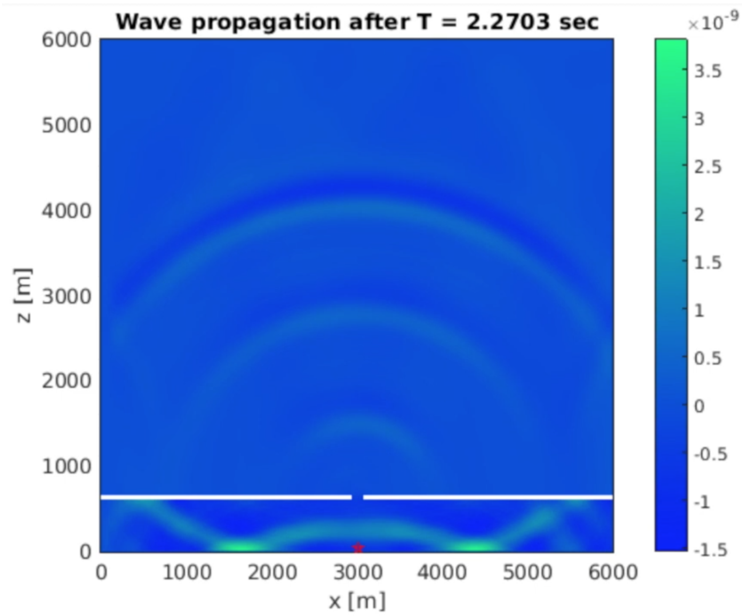

Figure 6: The simulation of the diffraction.

Then, we simulated the wave scattering with a vertical crack. The source at $(30, 3000)$ is normally injected. On the graph 7, we plot a crack at $x = 3000, z \in (450, 1650)$. Finally, we simulated the wave scattering with 9 cracks. The source at $(3000, 3000)$ is normally injected. On the graph 8, we plot 9 cracks.

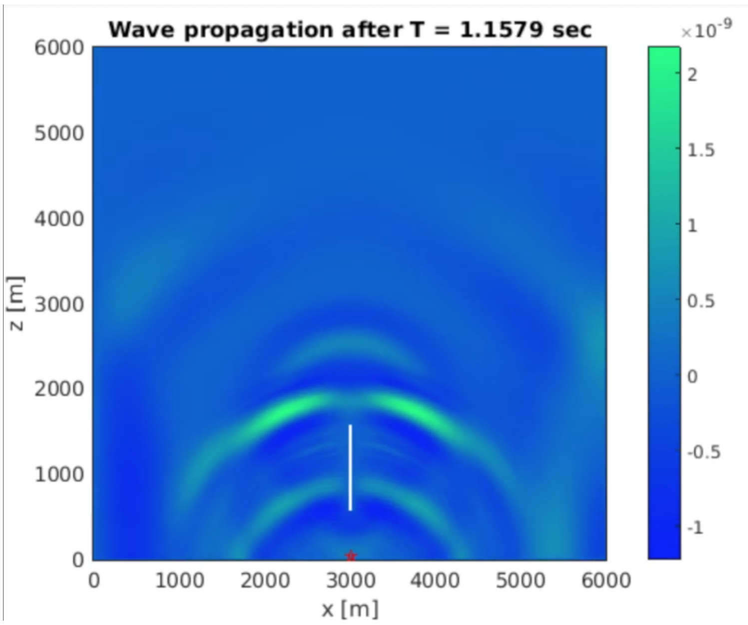

Figure 7: The simulation of wave scattering with a vertical crack, and source is normally injected.

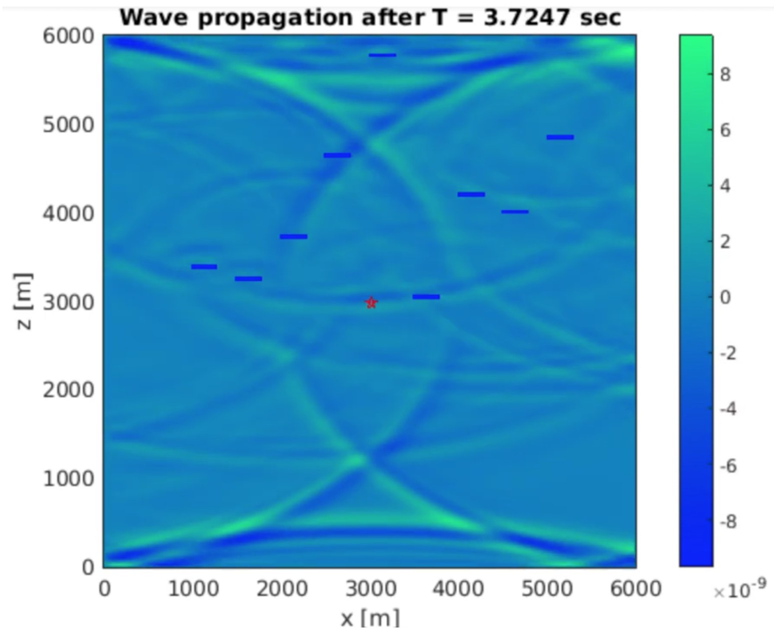

Figure 8: The simulation of wave scattering with 9 cracks.

Rotated Staggered Grid Method

One problem related to the standard staggered grid is that when the wave field hits the inhomogeneity with high density contrasts (e.g. cracks), stability problems can occur. These problems described above can be avoided by choosing another configuration of the grid using the rotated staggered grid.

As shown on the graph, instead of performing updates along the original axises, x_{old} and z_{old} in our conventional method, we perform the updates in a rotated axis shown here as x_{new} and z_{new}. In figure 9 for example, we will update our stress tensor Txx in the x direction using the Uxalong the diagonal as indicated by the red box, and the z direction with the other diagonal indicated

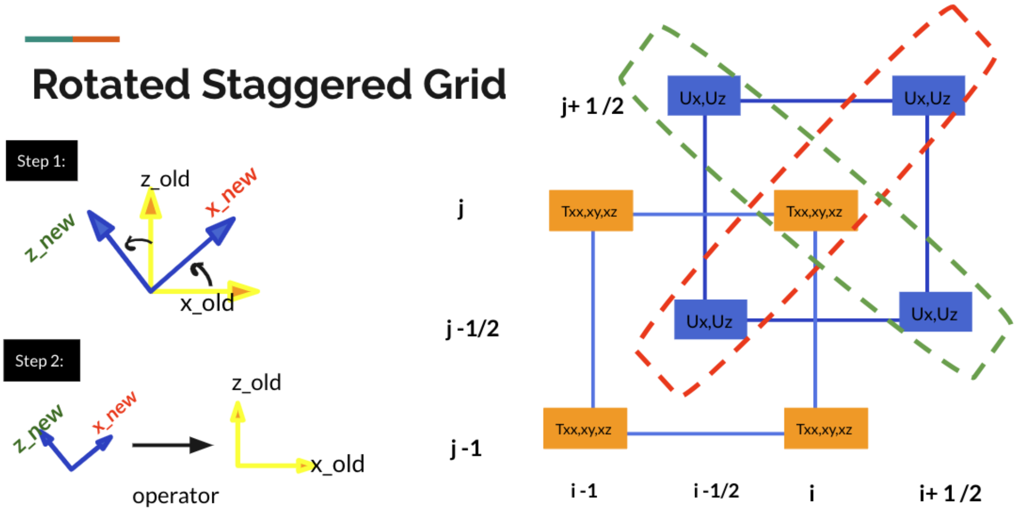

Figure 9: rotated stagger grid

by the green box. Then we just simply rotate this axis back to our original axis using rotational operators which essentially express our old coordinate system as a linear combination of the new ones [2] as defined below:

$$\begin{aligned} z_{new} &= \frac{\Delta h}{\Delta r}x + \frac{\Delta h}{\Delta r}z \\ x_{new} &= \frac{\Delta h}{\Delta r}x - \frac{\Delta h}{\Delta r}z \\ \frac{\partial}{\partial z} &= \frac{\Delta r}{2\Delta z}\left(\frac{\partial}{\partial z_{new}} - \frac{\partial}{\partial x_{new}}\right) \\ \frac{\partial}{\partial x} &= \frac{\Delta r}{2\Delta z}\left(\frac{\partial}{\partial z_{new}} + \frac{\partial}{\partial x_{new}}\right) \end{aligned} \tag{21}$$

in which:

$$\Delta r = \sqrt{\Delta h^2 + \Delta h^2} \tag{22}$$

At the same time, a tiny artificial viscosity of $0.7dx\nabla^2 u$ is applied in addition to the velocity updates to help damping the checkerboard pattern that would otherwise occur.

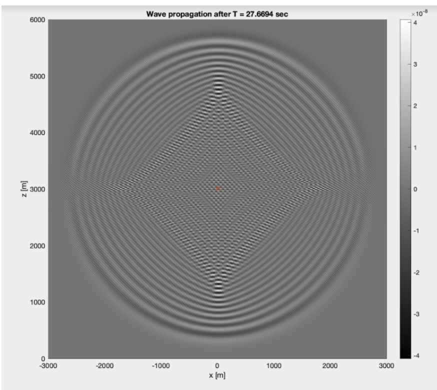

Figure 10: snapshot without damping term with rotated staggered grid

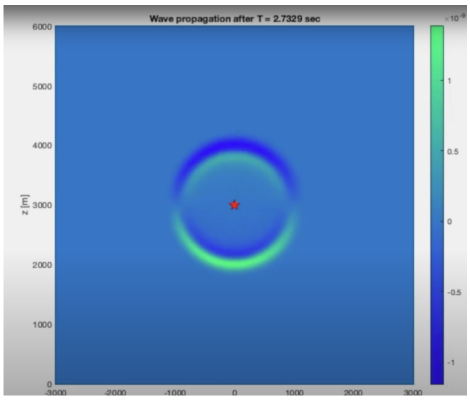

Figure 11: snapshot with damping term with rotated staggered grid

Further Steps

Due to time constraint, we haven't fully explore the scattering behaviour of due to more complicated cracks (such as intersected cracks). It would be interesting to explore more crack shapes and different heterogeneity, including but not restricted to different densities, velocities etc, using the grid method we have implemented and perform further analysis . It would be also interesting to explore possible simulations of cracks that evolve with time using this numerical methods and analyze the wave's behaviour that have more real world implications.

Also, full numerical analysis and accuracy test hasn't been thoroughly performed to our simulations by ourselves (though has been conducted extensively in the paper we reference). It is imperative for us to perform deeper numerical analysis on the comparisons between the two staggered grid method.

References

[1] Bernhard Hustedt, Stéphane Operto, and Jean Virieux. Mixed-grid and staggered-grid finite-difference methods for frequency-domain acoustic wave modelling. *Geophysical Journal International*, 157(3):1269–1296, 06 2004.

[2] E. H. Saenger, N. Gold, and S. Shapiro. Modeling the propagation of elastic waves using a modified finite-difference grid. *Wave Motion*, 31:77–92, 2000.

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[4] Jean Virieux. P-sv wave propagation in heterogeneous media: Velocity-stress finite-difference method. *Geophysics*, 51:889–901, 01 1984.