# 势涡的共振和非共振输运与 $E \times B$ 台阶

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# 目录

# 研究背景 $E \times B$ 台阶: 实验结果

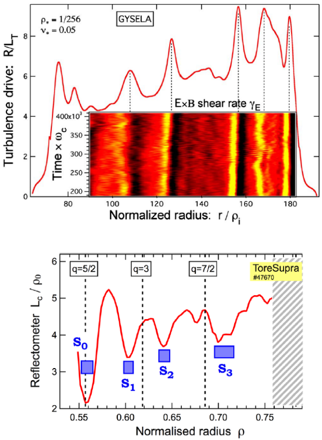


Fig. 1: 模拟中出现的平均温度分布褶皱和 $E \times B$ 剪切流的位置[Dif-Pradalier et al., 2010]; ToreSupra 中，反射计测量得到的相关长度[Dif-Pradalier et al., 2015]

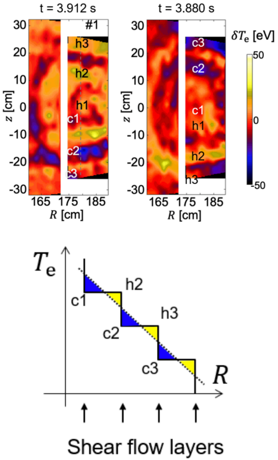


Fig. 2: KSTAR 中 ECEI 测量得到温度褶皱[Choi et al., 2019]

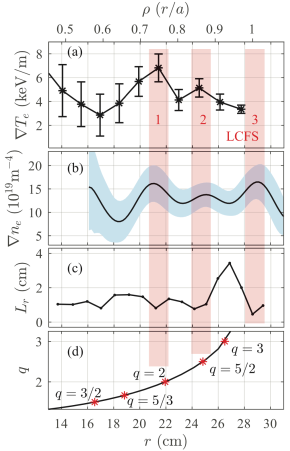


Fig. 3: HL-2A 中 ECE 和反射计测量得到温度台阶[Liu et al., 2021]

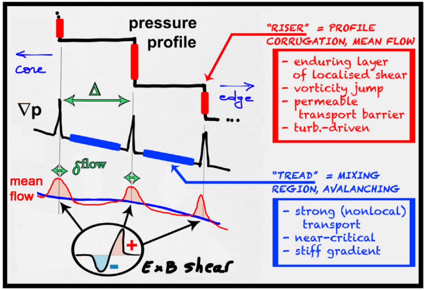


Fig. 4: $E \times B$ 台阶示意图[Dif-Pradalier et al., 2017]

- 台面尺寸给出了一个 **有效的混合长度** $\stackrel{?}{\Longrightarrow}$gyro-Bohm 定标的失效；
- 微输运垒融合为内输运垒？多级输运垒优于单级输运垒？[Ashourvan et al., 2019]

## $E \times B$ 台阶：剪切层和剖面褶皱的准规则模式（pattern）

- $E \times B$ 台阶中微输运垒（$E \times B$ 剪切层）将 DW 湍流的活跃区分隔（台面）
- 台面区为类雪崩输运
- 台面宽度大约 $20 - 40\rho_i$
- 剪切层位置和有理面没有明显关联
- 部分模拟中观察到台阶融合

## 研究意义

- 提高等离子体约束
- 先进运行模式
- 大规模回旋动理学模拟的简化

## 1、初值问题

任何热通量中的“惯性”都可能导致时间延迟 [Kosuga et al., 2014, Kosuga et al., 2013],

$$\partial_t Q = -\frac{1}{\tau}(Q - Q_0[\delta T]) \tag{1}$$

=====================================

$$\text{Large } \tau \Longrightarrow v_0 \sim \lambda \delta T_0 > v_{ph} \sim \sqrt{\frac{\chi_2}{\tau}} \Longrightarrow \text{Jam Inst.}$$

## 2、边值问题

基于 Hasegawa-Wakatani 模型，认为存在一个依赖于梯度的混合长度 $\Longrightarrow$ 双稳定（Bistable）[Ashourvan & Diamond, 2016, Ashourvan & Diamond, 2017].

$$l_{mix} = \frac{l_0}{1 + l_0^2[\partial_x(n-u)]^2/\mathcal{E}} \tag{2}$$

## 3、其他

- 剖面耦合 [Leconte & Kobayashi, 2021, Singh & Diamond, 2021]
- Wave Trapping[Garbet et al., 2021]

## 现有理论的主要不足

- 模型大多偏唯象、多讨论密度剖面、解释能力有限
- 没有从动理学出发给出的理论，缺少微观效应的讨论

如何理解台阶尺度的涌现（emergence）？
$E \times B$ 台阶需要更细致的理论研究

# 目录

## 关于 $\mathbf{E} \times \mathbf{B}$ 台阶的一个理论：

动理学 $\partial_t \bar{f} + \Omega_D E \partial_\alpha \bar{f} - [J_0\phi, \bar{f}] = 0$ $\xrightarrow{\text{求矩}}$ 扰动量 $\tilde{T}, \tilde{\phi}$ 演化 PV 守恒

$\xrightarrow[\text{准线性近似}]{\text{相关函数}}$ 强度量 $\langle \tilde{T}^2 \rangle_{\mathbf{k}}$, $\langle \tilde{\phi}^2 \rangle_{\mathbf{k}}$, 通量 $\langle \tilde{V}_\psi \tilde{T} \rangle_{\mathbf{k}}$

$\xrightarrow[\text{Green's Function}]{\text{Fourier 变换}}$ 时空演化 $\langle \tilde{\phi}^2 \rangle, \langle T \rangle, \langle \phi_Z \rangle$

$\xrightarrow[\text{解析 \& 数值}]{\text{Turbulence-Profile Evolution System}}$

1. Turbulence Spreading
2. 平均剖面 $\langle T \rangle$, $\overline{\Delta}\phi_Z$ 台阶

$$\text{波和粒子的共振} \xrightarrow{E\times B\text{平均流的调制}} \begin{cases} \text{非共振输运} \longrightarrow \text{峰化剖面} \\ \text{共振输运} \longrightarrow \text{平坦剖面} \end{cases} \xrightarrow{\text{自组织}} E \times B\text{台阶}$$

# 目录

对托卡马克中的低频湍流 ($\omega < \omega_b$, 回弹频率): $f(\vec{r}, \vec{p}, t) \xrightarrow[\text{Bounce-average}]{\text{Gyro-average}} \bar{f}(\psi, \alpha, E, t)$. 其中 $\psi$ 磁面坐标, $\alpha$ 环向, $E$ 是能量[Depret et al., 2000].

$$\begin{cases} \partial_t \bar{f} + \Omega_D E \partial_\alpha \bar{f} - [J_0 \phi, \bar{f}] = 0 \\ n_i = n_e \end{cases} \quad (3)$$

其中, $[F, G] = \partial_\alpha F \partial_\psi G - \partial_\psi F \partial_\alpha G$.

+

- $q\phi / T_{\text{eq}} \ll 1$, 平均、绝热和非绝热: $\bar{f} = \langle f \rangle - \frac{q_{i,e}\phi}{T_{i,e}} \langle f \rangle + h_{i,e}$.
- 扰动不响应带状电势: $\tilde{n}_{i,e}/n_0 = -q_{i,e}(\phi - \langle \phi \rangle_\alpha)/T_{i,e}$

非绝热扰动分布函数 $h_i$ 和准中性方程如下 (Darmet 模型 [Darmet et al., 2008, Sarazin et al., 2005]):

$$\partial_t h_i + \Omega_D E \partial_\alpha h_i - \left[\bar{\phi}, -\frac{q}{T_i}(\phi - \langle \phi \rangle_\alpha)\langle f_i \rangle + h_i\right] = \partial_t \left(\frac{q}{T_i}(\phi - \langle \phi \rangle_\alpha)\langle f_i \rangle\right) + \partial_\alpha (\overline{\phi - \langle \phi \rangle_\alpha}) \partial_\psi \langle f_i \rangle \quad (4)$$

$$C_{ad}(\phi - \langle \phi \rangle_\alpha) - C_i \Delta_{i+e} \phi = \frac{2}{n_0 \sqrt{\pi}} \int_0^\infty J h_i \sqrt{E} dE - \frac{2}{n_0 \sqrt{\pi}} \int_0^\infty J h_e \sqrt{E} dE \quad (5)$$

其中 $C_i = q/T_i, C_{ad} = C_i(1+\tau)/\sqrt{2\varepsilon_0}, \tau = T_i/T_e$。$\Delta_s = \rho_{0s}^2 \partial_\alpha^2 + \delta_{bs}^2 \partial_\psi^2$。可能是 DW 湍流的最小化动理学系统!

# 势涡守恒系统

涡度演化：

$$\left(\frac{\partial}{\partial t} + \tilde{\mathbf{V}} \cdot \nabla + \Omega_Z \cdot \nabla\right)(C_i \overline{\Delta} \tilde{\phi}) = \frac{3}{2}\Omega_D \partial_y \tilde{T}_i$$
$$- iC_e\left(\omega - \omega_E + \frac{\omega^i_{*n}}{\tau}\right)\tilde{\phi} - C_i \tilde{V}_x \partial_x (\overline{\Delta}\phi_Z) \quad (6)$$

$$\frac{\partial}{\partial t}\left[C_i \overline{\Delta}\phi_Z\right] = -C_i \partial_x \left\langle \tilde{V}_x \overline{\Delta}\tilde{\phi}\right\rangle_y \quad (7)$$

温度演化：

$$\sqrt{2\varepsilon_0}\left(\partial_t + \tilde{\mathbf{V}} \cdot \nabla + \Omega_Z \cdot \nabla\right)\tilde{T}_i$$
$$= -i(\omega - \omega_E - \omega^i_{*n} - \omega^i_{*T})\langle T_i \rangle C_i \tilde{\phi} \quad (8)$$

$$\frac{\partial}{\partial t}\left(\langle T_i \rangle + \langle T_i \rangle \ln \langle n_i \rangle\right) = -\sqrt{2\varepsilon_0}\,\partial_x \langle \tilde{V}_x \tilde{T}_i \rangle_y \quad (9)$$

Notation changed $(\psi, \alpha) \rightarrow (x, y)$

Eq. (6)-(9) 等价于一个势涡（Potential Vorticity, PV）演化系统（忽略密度梯度的贡献）[Yan & Diamond, 2021]:

$$\partial_t \langle q \rangle = -\partial_x \left\langle \tilde{V}_x \delta q \right\rangle_y \quad (10)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta q = -\frac{3}{2}\Omega_D \partial_y \tilde{T}_i - \tilde{\mathbf{V}} \cdot \nabla \langle q \rangle \quad (11)$$

PV $\langle q \rangle + \delta q$, $\Omega_Z \equiv \partial_x \phi_Z$.

$$\langle q \rangle = \frac{\tau}{\sqrt{2\varepsilon_0}} \ln \langle T_i \rangle - C_i \overline{\Delta}\phi_Z,$$

$$\delta q = \tau \frac{\tilde{T}_i}{\langle T_i \rangle} - C_i \overline{\Delta}\tilde{\phi}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \tilde{\mathbf{V}} \cdot \nabla + \Omega_Z \cdot \nabla$$

可类比于 Hasegawa-Wakatani 模型。

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$$\partial_t \ln\langle T\rangle = -\sqrt{2\varepsilon_0}\partial_x \underline{\langle \tilde{V}_x \tilde{T}\rangle_y} + \chi_{\rm neo}\partial_x^2 \ln\langle T\rangle$$

$$\downarrow$$

$$\langle \tilde{V}_x \tilde{T}\rangle_k$$

$$\downarrow$$

$$\sim R\left(\omega - k_y\Omega_Z - b_k\bar{\Omega}_D\right)\langle \tilde{V}_x^2\rangle_k\left[\partial_x\overline{\Delta}\phi_Z(..) - \partial_x \ln\langle T\rangle(..)\right]$$

$$\downarrow$$

$$(\chi_4^{\text{non-res}} + \chi_4^{\text{res}})\partial_x\overline{\Delta}\phi_Z - (\chi_3^{\text{non-res}} + \chi_3^{\text{res}})\partial_x \ln\langle T\rangle$$

$$\chi \text{ model}\downarrow$$

Equation (22)

$$\partial_t\left[\overline{\Delta}\phi_Z\right] = -\partial_x \underline{\langle \tilde{V}_x \overline{\Delta}\tilde{\phi}\rangle_y} + \nu_c\partial_x^2\overline{\Delta}\phi_Z$$

$$\underline{\langle \tilde{V}_x \overline{\Delta}\tilde{\phi}\rangle_k = -\langle \tilde{V}_x \delta q\rangle_k + \langle \tilde{V}_x \tilde{T}\rangle_k}$$

$$\downarrow$$

$$R\left(\omega - k_y\Omega_Z\right)R\left(\omega - k_y\Omega_Z - k_y b_k\bar{\Omega}_D\right)\langle \tilde{V}_x^2\rangle_k\partial_x \ln\langle T\rangle(..)$$
$$-R\left(\omega - k_y\Omega_Z - k_y b_k\bar{\Omega}_D\right)\langle \tilde{V}_x^2\rangle_k\partial_x\overline{\Delta}\phi_Z(..)$$

$$\downarrow$$

$$\chi_1^{\rm non-res}\frac{\partial_x \ln\langle T\rangle}{\sqrt{2\varepsilon_0}} - (\chi_2^{\rm non-res} + \chi_2^{\rm res})\partial_x\overline{\Delta}\phi_Z$$

$$\chi \text{ model}\downarrow$$

Equation (21)

$$\tilde{T}_{\mathbf{k}} = R(\ldots)\left[\partial_x\overline{\Delta}\phi_Z(\ldots) - \partial_x \ln\langle T\rangle(\ldots)\right]\tilde{V}_x(k)$$

$$\delta q_k = R(\omega - k_y\Omega_Z)\left[\tilde{T}_k(\ldots) - \partial_x\langle q\rangle(\ldots)\right]\tilde{V}_x(k)$$

$$C_i\overline{\Delta}\tilde{\phi} = \tau\tilde{T} - \delta q$$

$$\chi_3 = \Re\sum_k[\tilde{V}_x(k)]^2\frac{i}{\omega - k_y\left(\Omega_Z + b_k\bar{\Omega}_D\right)}$$

- 温度 and 涡度 梯度同时出现在热通量和涡量通量中[2]
- $\omega = \omega_R + i\gamma \Rightarrow$ 分离共振和非共振贡献
- 共振输运只出现在涡量通量中

[2] Connections to Ref.[Heinonen & Diamond, 2020]

数值求解色散函数并近似：$\omega_R = \dfrac{R_1 k_y \Omega_D}{1 + R_2(\rho^2 k_y^2 + \delta^2 k_x^2)} \simeq R k_y \Omega_D$， $R \approx 2.36$.

$$\chi_3^{\text{res}} = \sum_k \left[\tilde{V}_x(k)\right]^2 \pi\delta(\cdots) \longleftarrow \delta\left(\omega_R - k_y\Omega_Z - \frac{C_i\overline{\Omega}_D k_y}{\tau + \sqrt{2\epsilon_0}\left(\delta^2 k_x^2 + \rho^2 k_y^2\right)}\right) \tag{12}$$

共振发生：$\omega_{\mathrm{R}}$ (TIM), $E \times B$ mean flow $\Omega_Z$, trapped ion precession $\Omega_D$."Wave=Particle+Flow"

$$\Theta_{\text{res}} \equiv \delta^2 k_{x,\text{res}}^2 + \rho^2 k_{y,\text{res}}^2 \tag{13}$$

$$\Theta_{\max} \equiv \delta^2 k_{x,\max}^2 + \rho^2 k_{y,\max}^2 \lesssim 1 \tag{14}$$

其中 $C_i\overline{\Omega}_D = 3\Omega_D/2$, $\Omega_Z = \partial_x \phi_Z$.

$$\Theta_{\text{res}} = \frac{\Omega_D(R\tau - 1.5) - \tau\Omega_Z}{\sqrt{2\varepsilon_0}\left(\Omega_Z - R\Omega_D\right)} \tag{15}$$

只需 $\Omega_Z/\Omega_D \sim 1 - 1.5$，则存在 $k_x, k_y$ 满足共振条件！

共振条件转化为要求 $1 \lesssim \Omega_Z/\Omega_D \lesssim 1.5 \Longrightarrow$ 共振条件！

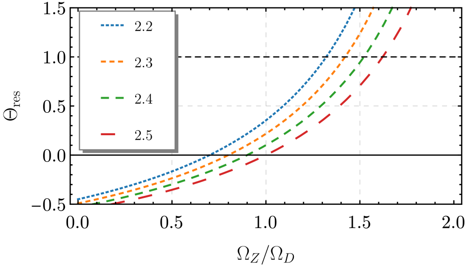



Fig. 5: $\Theta_{\text{res}}$ 和 $\Omega_Z$ 的关系。$\Theta_{\text{res}}$ 的定义决定其取值在 0 和 1 之间，这对应 $\Omega_Z/\Omega_D$ 需要在大概 1 到 1.5 之间。不同线对应的 $R$ 略有不同。

## 色散关系模型:

$$\omega_R = Rk_y\Omega_D, \quad \gamma = \gamma_0 R\Omega_D k_y(k_{y,\max} - k_y) \qquad (16)$$

其中实频率正比于 $k_y$；增长率近似模型中参数 $\Lambda \sim \rho/\sigma$, $k_{y,\max} = 1/(\mu_y\rho)$, $\mu_y^2\sigma^2 \sim 10$

## 设谱分布函数满足 Lorentz 分布[Diamond et al., 2010]:

$$|\tilde{\phi}|^2_{k_x,k_y} = \frac{|\tilde{\phi}_0|^2}{\pi^2\Delta k_x\Delta k_y}\frac{1}{\left[1+\left(\frac{k_x-k_{r0}}{\Delta k_x}\right)^2\right]\left[1+\left(\frac{k_y-k_{y0}}{\Delta k_y}\right)^2\right]} \qquad (17)$$

$$\chi_3^{\mathrm{non-res}} = \sum_k \frac{[\tilde{V}_x(k)]^2|\gamma|}{|\omega - k_y\Omega_Z - k_y b_k\bar{\Omega}_D|^2} \sim \frac{|\tilde{\phi}_0|^2\mu^2\sigma^2}{\Omega_D\rho\left[(\mu^2\sigma^2+1+\beta^2)^2\right]} \qquad (18)$$

$$\chi_3^{\mathrm{res}} = \sum_k [\tilde{V}_x(k)]^2\pi\delta(\omega_R - k_y\Omega_Z - k_y b_k\bar{\Omega}_D) \sim \frac{|\tilde{\phi}_0|^2 k_{y,\mathrm{res}}^2\beta^2}{\beta^2+\mu^2\Theta_{\mathrm{res}}}\frac{\mu_y\rho}{|\beta_y(\Omega_Z+\Omega_D)|} \qquad (19)$$

Define

- $\chi_3^{\text{non-res}} \equiv \chi^{\mathrm{n}}|\tilde{\phi}_0|^2$
- $\chi_3^{\mathrm{res}} \equiv \chi^{\mathrm{r}}|\tilde{\phi}_0|^2$

$$\chi^{\mathrm{r}}/\chi^{\mathrm{n}} \sim \frac{(\mu^2\sigma^2+1+\beta^2)^2}{\mu^2\sigma^2} \sim \mathcal{O}(10) - \mathcal{O}(10^2)$$
$$\sim \frac{1}{\Lambda^2 k_{y,\max}^2} \sim \frac{\Omega_D^2}{\Lambda^2\Omega_D^2 k_{y,\max}^2} \sim \frac{\omega_R^2}{\gamma^2} \qquad (20)$$

共振和非共振输运之间的比例大概为实频率和增长率之比的平方。构造一个关于 $\Omega_Z$ 的分段函数模型 $\chi \equiv (\chi^{\mathrm{n}} + \chi^{\mathrm{r}})|\tilde{\phi}_0|^2$。

$$\text{Eq.(6): } \tilde{U} \equiv \tilde{\phi} - \overline{\Delta\tilde{\phi}} \rightarrow \text{Eq.(36): } \partial_t\langle\tilde{U}^2\rangle \xrightarrow{\text{Green's function}} \langle\tilde{\phi}^2\rangle \sim |\tilde{\phi}_0|^2$$

$$\frac{\partial}{\partial t}\left(\overline{\Delta}\phi_Z\right)=-\frac{\partial}{\partial x}\left(\frac{1}{C_i}\vartheta\chi^{\mathrm{n}}\frac{\partial}{\partial x}\frac{\ln\langle T\rangle}{\sqrt{2\varepsilon_0}}\right)+\frac{\partial}{\partial x}\left[\vartheta\chi\frac{\partial}{\partial x}(\overline{\Delta}\phi_Z)\right]+\nu\frac{\partial^2}{\partial x^2}\overline{\Delta}\phi_Z \tag{21}$$

$$\frac{\partial}{\partial t}\ln\langle T\rangle=-\frac{\partial}{\partial x}\left[C_i\sqrt{2\varepsilon_0}(1-\vartheta)\chi\frac{\partial}{\partial x}\left(\overline{\Delta}\phi_Z\right)\right]+\frac{\partial}{\partial x}\left[\chi\frac{\partial}{\partial x}\ln\langle T\rangle\right]+\chi_{\mathrm{neo}}\frac{\partial^2\ln\langle T\rangle}{\partial x^2} \tag{22}$$

## 边界条件

$$\left.\frac{\partial}{\partial x}\overline{\Delta}\phi_Z\right|_{\mathrm{B}}=0 \tag{23}$$

$$\left.\frac{\partial}{\partial x}\ln\langle T\rangle\right|_{\mathrm{B}}\equiv\kappa_T^{\mathrm{B}}=\mathrm{Const.} \tag{24}$$

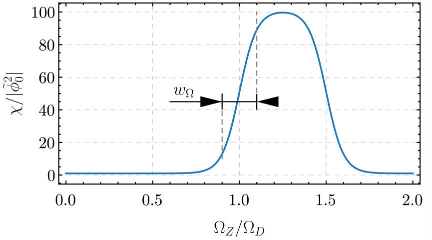
$$\left.\frac{\partial}{\partial x}\langle\tilde{U}^2\rangle\right|_{\mathrm{B}}=0,\quad\text{or}\quad\left.\frac{\partial}{\partial x}\langle\tilde{\phi}^2\rangle\right|_{\mathrm{B}}=0 \tag{25}$$

And: $\Omega_Z=\partial_x\phi_Z$, set B.C. for $\Omega_Z$ as:

$$\left.\Omega_Z\right|_{\mathrm{B}}=0 \tag{26}$$

Flux-driven System

## 分段输运系数模型 $\chi\equiv(\chi^{\mathrm{n}}+\chi^{\mathrm{r}})|\tilde{\phi}_0|^2$




Eq. (21), (22), (36) + 边界条件 (23-26) + 输运系数模型 $\Longrightarrow E\times B$ 台阶演化系统

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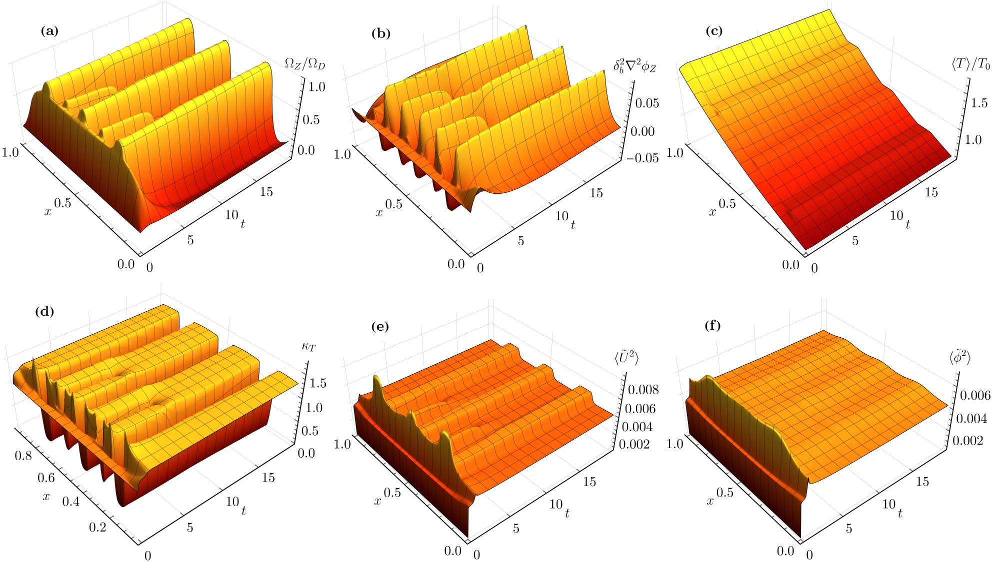
## 数值结果: 观察到 $E \times B$ 台阶形成



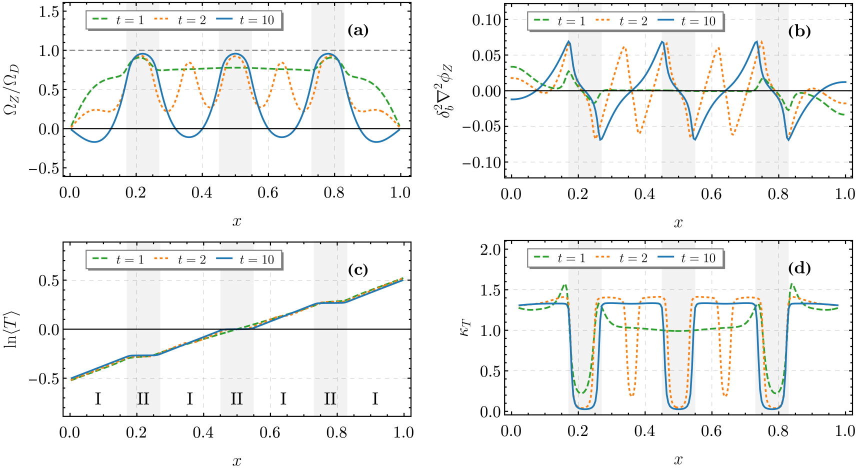



Fig. 6: 剖面演化。如 (c) 所示，将温度剖面几乎平坦的这些区域命名为 II-区，区域内平均涡量的剖面较平，同时由图 (a) 可知这些区域对应的 $\Omega/\Omega_D$ 高于了共振输运所需的阈值。所以在 II-区，共振输运占主导；与此同时在 I-区，剖面较为陡峭，非共振输运占主导。

$\Omega_Z \gtrsim \Omega_D$ resonant transport switched on

$$\frac{\partial}{\partial t}(\delta_b^2\Omega_Z) = -\frac{1}{C_i}\vartheta\chi^{\mathrm{n}}\frac{\partial}{\partial x}\frac{\ln\langle T\rangle}{\sqrt{2\varepsilon_0}} + \overbrace{\frac{1}{C_i}\left(\vartheta\chi^{\mathrm{n}}\frac{\partial}{\partial x}\frac{\ln\langle T\rangle}{\sqrt{2\varepsilon_0}}\right)\Bigg|_{\mathrm{B}}}^{\text{Drive Source}} + \vartheta\chi(\Omega_Z)\delta_b^2\frac{\partial^2}{\partial x^2}\Omega_Z + \nu_c\delta_b^2\frac{\partial^2}{\partial x^2}\Omega_Z$$

I: Local Gradient Balanced    II: Resonant Dissipation Balanced

2 feedback loops
↓
2 profile states emergent in space
↓
Staircase involving resonant transport!

## 准稳态 $\Omega_Z \rightarrow$ 需平衡边界驱动 $\kappa_T^{\mathrm{B}}$:

**I 局域梯度 $\kappa_T(t,x) \equiv \partial_x \ln\langle T\rangle$**

- $\Omega_Z/\Omega_D < 1$，通常的湍流输运
- $C_i = q\omega_0 L_\psi/T_{\mathrm{i,eq}} \ll 1$, $\kappa_T$ 起主要作用，且

  $\kappa_T|_{(t,x)} \sim (1-e^{-t/\tau_{\nabla T}})\kappa_T|^{\mathrm{B}}$, $\quad \tau_{\nabla T} \sim 1/(\pi^2\chi^{\mathrm{n}}m^2)$
- $\kappa_T^B - \kappa_T(t,x)$ 残余部分由 $\chi^{\mathrm{n}} + \nu_c$ 平衡
- 陡峭的温度剖面

**II $\Omega_Z$ 的共振输运**

- $\Omega_Z \gtrsim \Omega_D$ 触发 Eq.(22) 和 (21)中的共振输运
- 温度梯度仅有非共振贡献 $\chi^{\mathrm{n}}$
- $\kappa_T^B \rightleftarrows \Omega_Z$ 的共振输运
- $\kappa_T(x,t) \Rightarrow$ 平坦 $\Rightarrow$ Near Marginal (Hypothesis)

$$\frac{\partial}{\partial t}\left(\delta_b^2\Omega_Z\right) = -\frac{1}{C_i}\vartheta\chi^{\mathrm{n}}\frac{\partial}{\partial x}\frac{\ln\langle T\rangle}{\sqrt{2\varepsilon_0}} + \frac{1}{C_i}\left(\vartheta\chi^{\mathrm{n}}\frac{\partial}{\partial x}\frac{\ln\langle T\rangle}{\sqrt{2\varepsilon_0}}\right)\Bigg|_{\mathrm{B}} + \vartheta\chi\delta_b^2\frac{\partial^2}{\partial x^2}\Omega_Z + \nu_c\delta_b^2\frac{\partial^2}{\partial x^2}\Omega_Z$$

$$\frac{\partial}{\partial t}\ln\langle T\rangle = -\frac{\partial}{\partial x}\left[\sqrt{2\varepsilon_0}C_i(1-\vartheta)\chi\frac{\partial}{\partial x}\left(\Delta\phi_Z\right)\right] + \frac{\partial}{\partial x}\left[\chi\frac{\partial}{\partial x}\ln\langle T\rangle\right] + \chi_{\mathrm{neo}}\frac{\partial^2}{\partial x^2}\ln\langle T\rangle$$

Main Balance ⑤ ① ② ③ ④

## 台阶模式如何涌现?

$(x_1, x_2) \in$ Feedback loop II $\Rightarrow$ Heat flux $\uparrow$

$\Downarrow$

$(x_1, x_2)$ 的邻域: $\kappa_T(t,x) > \kappa_T^{\mathrm{B}}$

$\Downarrow$

邻域 $\in$ Feedback loop I

$\Downarrow$

触发共振输运 $\rightarrow$ 台阶模式自发形成

## II-区中的反馈循环

- ① $\partial_x \ln\langle T\rangle|_{\mathrm{B}}$ 驱动 $\Omega_Z$ 增长
- 若 $\Omega_Z \gtrsim \Omega_D$，共振输运触发 $\chi^{\mathrm{r}}$，② 流的输运和③ 局域温度的输运都被增强
- ④ 局域温度梯度变缓 *flattened* + 非共振输运。$\kappa_T(t,x)\downarrow$
- 边界热流驱动的作用更强 $(\kappa_T^{\mathrm{B}} - \kappa_T(t,x))\uparrow$
- ⑤ 在第②步中增强的 $\Omega_Z$ 的输运可以平衡驱动的增长
- 最终，$\Omega_Z$ 达到准稳态

Main Balance

$$\frac{\partial}{\partial t}\left(\delta_b^2 \Omega_Z\right) = -\frac{1}{C_i}\vartheta\chi^{\mathrm{n}}\frac{\partial}{\partial x}\frac{\ln\langle T\rangle}{\sqrt{2\varepsilon_0}} + \frac{1}{C_i}\left(\vartheta\chi^{\mathrm{n}}\frac{\partial}{\partial x}\frac{\ln\langle T\rangle}{\sqrt{2\varepsilon_0}}\right)\Bigg|_{\mathrm{B}} + \vartheta\chi\delta_b^2\frac{\partial^2}{\partial x^2}\Omega_Z + \nu_c\delta_b^2\frac{\partial^2}{\partial x^2}\Omega_Z$$

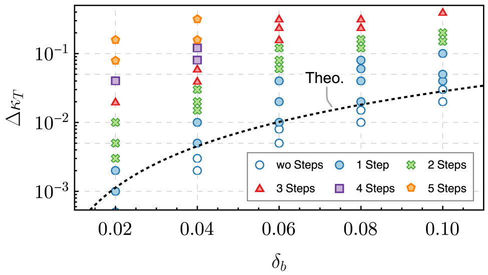
$$\frac{\partial}{\partial t}\ln\langle T\rangle = -\frac{\partial}{\partial x}\left[\sqrt{2\varepsilon_0}C_i(1-\vartheta)\chi\frac{\partial}{\partial x}\left\langle\Delta\phi_Z\right\rangle\right] + \frac{\partial}{\partial x}\left[\chi\frac{\partial}{\partial x}\ln\langle T\rangle\right] + \chi_{\mathrm{neo}}\frac{\partial^2}{\partial x^2}\ln\langle T\rangle$$

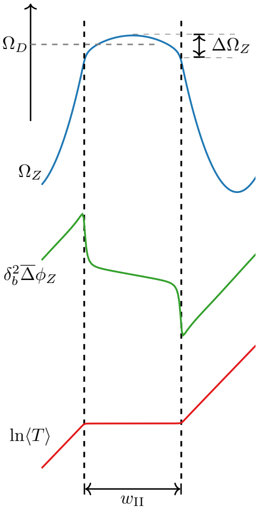
## 影响台阶模式的因素

- 触发所需的边界热通量阈值 $\Delta\kappa_T^{\mathrm{crit}}$
- 对我们的系统做数值扫描，结果显示符合理论给出的阈值
- 更小的 $\delta_b$，更小的触发临界值
- 相同参数下，更强的净驱动 $\Rightarrow$ 更多台阶

需要边界热流 $\kappa_T^{\mathrm{B}}$ 足够强 $\longrightarrow$ 在局域温度梯度达到边界值之前，$\Omega_Z$ 抵达共振阈值条件！

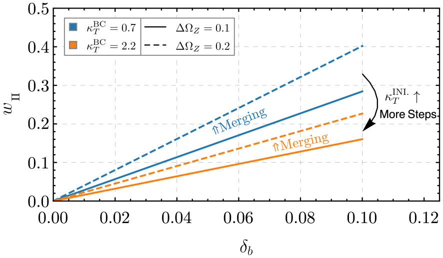
$$\kappa_T\Big|_{\mathrm{B}} - \kappa_T\Big|_{\mathrm{I}} > \Delta\kappa_T^{\mathrm{crit}} \equiv C_i\Omega_D\frac{8\delta_b^2\sqrt{2\varepsilon_0}}{\vartheta} \tag{27}$$






基于 $\Omega_Z$ 在 II-区的平衡：$\left(\frac{1}{C_i}\chi^{\mathrm{n}}\frac{\partial}{\partial x}\frac{\ln\langle T\rangle}{\sqrt{2\varepsilon_0}}\right)\Big|_{\mathrm{B}} \sim \chi\delta_b^2\frac{\partial^2}{\partial x^2}\Omega_Z$ 。定义了 $\Omega_Z^{\mathrm{Max}} - \Omega_Z^{\mathrm{Crit}}$ 为 $\Delta\Omega_Z$。台阶宽度的估计为：

$$w_{\mathrm{II}} \sim 2\sqrt{\frac{\Delta\Omega_Z}{K_{\mathrm{II}}}} \sim 2\delta_b\sqrt{\Delta\Omega_Z\frac{\chi^{\mathrm{r}}}{\chi^{\mathrm{n}}}\frac{C_i\sqrt{2\varepsilon_0}}{\kappa_T^{\mathrm{B}}}} \tag{28}$$




## 台阶宽度决定于：

- $\propto \delta_b$
- $\chi^{\mathrm{r}}/\chi^{\mathrm{n}} \sim \mathcal{O}(10) - \mathcal{O}(100)$
- 边界梯度（通量），越陡峭越窄
- 宽度越窄，台阶数量上限越高
- 更强的驱动，更多的台阶

# 总结

## 湍流-剖面演化系统

KE & QuasiNeutrality $\longrightarrow$ Darmet Model $\longrightarrow$ $\tilde{T}$、$\tilde{U} \equiv \tilde{\phi} - \overline{\Delta\tilde{\phi}}$ PV System

Quasi-linear Approx.

$\langle \tilde{V}_x \tilde{T} \rangle_y, \langle \tilde{V}_x \overline{\Delta\tilde{\phi}} \rangle_y$

Disper. Relation

Lorentzian Spectrum

$\chi^r, \chi^n$ model

$$\partial_t \langle T \rangle = -\sqrt{2\varepsilon_0}\partial_x \langle \tilde{V}_x \tilde{T} \rangle_y$$
$$\partial_t \left[\Delta\phi_Z\right] = -\partial_x \langle \tilde{V}_x \overline{\Delta\tilde{\phi}} \rangle_y$$

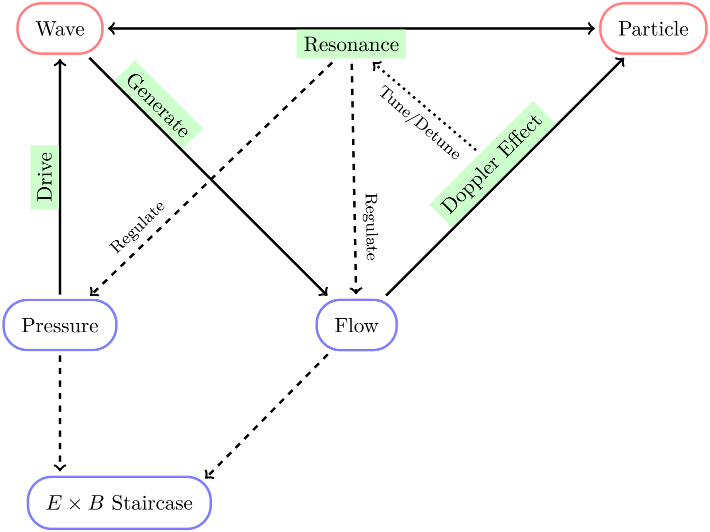
## 剖面模式（pattern）$--$ $E \times B$ 台阶

1. 共振: “Wave + Particle + Flow”
2. 涡度通量中仅温度梯度的非共振贡献
3. 流结构 $\Omega_Z$ 调制剖面状态:
   - 非共振态：陡峭的温度剖面
   - 共振态：(hypothesized) Near-marginal 温度剖面
4. 触发台阶的边界热通量条件 $\Delta\kappa_T^{\mathrm{crit}}$
5. 台阶宽度决定于：$\delta_b, \chi^r/\chi^n, \kappa_T^B$

## 可能的应用

- ZF 的无碰撞饱和[Li & Diamond, 2018]
- 模型推广到快离子和湍流相互作用
- ITB, F-ATB?
- kinetic DW 湍流中自组织和模式形成的范式?

[Yan & Diamond, 2022]



Wave
Particle
Resonance
Generate
Drive
Tune/Detune
Doppler Effect
Regulate
Regulate
Pressure
Flow
$E \times B$ Staircase

Thank You!

**Thank You!**

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Quasi-linear expressions for Fluctuation quantities.

$$\left(\frac{\partial}{\partial t}+\Omega_Z\cdot\nabla\right)\left(A\widetilde{\phi}-\overline{\Delta}\widetilde{\phi}\right)-\tilde{V}\cdot\nabla\left(\overline{\Delta}\widetilde{\phi}\right)=-\bar{\Omega}_D\partial_y\widetilde{T}+\delta_b^2\widetilde{V}_x\partial_x^3\phi_Z \tag{29}$$

$$\left(\frac{\partial}{\partial t}+ik_y\Omega_Z\right)\widetilde{\phi}_k=\frac{1}{A+\overline{k}_\perp^2}\sum_{\mathbf{k+p+q=0}}\hat{z}\cdot\mathbf{p}\times\mathbf{q}(q^2-p^2)\widetilde{\phi}_{\mathbf{p}}\widetilde{\phi}_{\mathbf{q}}-\frac{\bar{\Omega}_D}{A+\overline{k}_\perp^2}ik_y\widetilde{T}_{\mathbf{k}}-\frac{\delta_b^2\partial_x^3\phi_Z}{A+\overline{k}_\perp^2}ik_y\widetilde{\phi}_{\mathbf{k}} \tag{30}$$

The second equation above is a direct Fourier transformation of Eq.(29). Similarly for $\widetilde{T}_{\mathbf{k}}$ applying the Fourier Transformation and replacing $-i(\omega-\omega_Z)\tilde{\phi}$ with fluctuation vorticity equation.

$$\sqrt{2\varepsilon_0}\left(\partial_t+\widetilde{\mathbf{V}}\cdot\nabla+\Omega_Z\cdot\nabla\right)\widetilde{T}_i=-i(\omega-\omega_Z-\omega_{*n}^i-\omega_{*T}^i)\langle T_i\rangle\frac{q\widetilde{\phi}}{T_i} \tag{31}$$

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{T}_{\mathbf{k}}=&\sum_{\mathbf{p+q+k=0}}\hat{z}\cdot\mathbf{p}\times\mathbf{q}\left(\widetilde{\phi}_{\mathbf{p}}\widetilde{T}_{\mathbf{q}}-\widetilde{\phi}_{\mathbf{q}}\widetilde{T}_{\mathbf{p}}\right)+\frac{C_i}{\sqrt{2\varepsilon_0}}\frac{1}{A+\overline{k}_\perp^2}\sum_{\mathbf{p+q+k=0}}\hat{z}\cdot\mathbf{p}\times\mathbf{q}(q^2-p^2)\widetilde{\phi}_{\mathbf{p}}\widetilde{\phi}_{\mathbf{q}}\\&-ik_y\widetilde{T}_{\mathbf{k}}\left(\Omega_Z+\frac{C_i}{\sqrt{2\varepsilon_0}}\frac{\bar{\Omega}_D}{A+\overline{k}_\perp^2}\right)-\frac{ik_y\widetilde{\phi}_{\mathbf{k}}}{\sqrt{2\varepsilon_0}}\left(C_i\frac{\delta_b^2\partial_x^3\phi_Z}{A+\overline{k}_\perp^2}-\partial_x\ln\langle T\rangle\right)\end{aligned} \tag{32}$$

QL for fluctuation PV:

$$\delta q_k = \frac{C_i \bar{\Omega}_D k_y}{\omega - k_y \Omega_Z} \widetilde{T}_k - \frac{i\partial_x \langle q \rangle}{\omega - k_y \Omega_Z} \widetilde{V}_x(k)$$

Then,

$$\begin{aligned}
\langle \tilde{V}_x \delta q \rangle_k &\sim \tilde{V}_x(-k) \delta q_k \\
&= \frac{C_i \bar{\Omega}_D k_y}{\omega - k_y \Omega_Z} \tilde{V}_x(-k) \widetilde{T}_k - \frac{i\partial_x \langle q \rangle}{\omega - k_y \Omega_Z} \tilde{V}_x(-k) \widetilde{V}_x(k) \\
&= \frac{k_y}{\omega - k_y \Omega_Z} C_i \bar{\Omega}_D \langle \tilde{V}_x \widetilde{T} \rangle_k - \frac{i}{\omega - k_y \Omega_Z} \langle \tilde{V}_x^2 \rangle_k \partial_x \langle q \rangle \\
&= \frac{k_y}{\omega - k_y \Omega_Z} C_i \bar{\Omega}_D \langle \tilde{V}_x \widetilde{T} \rangle_k - \frac{i}{\omega - k_y \Omega_Z} \langle \tilde{V}_x^2 \rangle_k \left( \frac{1}{\sqrt{2\varepsilon_0}} \partial_x \ln \langle T \rangle - C_i \delta_b^2 \partial_x^3 \phi_Z \right)
\end{aligned}$$

$$\delta q = \tau\tilde{T} - C_i\overline{\Delta}\tilde{\phi}$$

$$\langle\tilde{V}_x\overline{\Delta}\tilde{\phi}\rangle =$$

$$\langle\tilde{V}_x\tilde{T}\rangle_k - \langle\tilde{V}_x\delta q\rangle_k = \left(1 - \frac{k_y}{\omega - k_y\Omega_Z}C_i\bar{\Omega}_D\right)\langle\tilde{V}_x\tilde{T}\rangle_k + \frac{i}{\omega - k_y\Omega_Z}\langle\tilde{V}_x^2\rangle_k\left(\frac{1}{\sqrt{2\varepsilon_0}}\partial_x\ln\langle T\rangle - C_i\delta_b^2\partial_x^3\phi_Z\right)$$

$$= \frac{\omega - k_y\Omega_Z - k_yC_i\bar{\Omega}_D}{\omega - k_y\Omega_Z}\frac{i}{\omega - k_y\left(\Omega_Z + \frac{C_i}{\sqrt{2\varepsilon_0}}\frac{\bar{\Omega}_D}{A+\bar{k}_\perp^2}\right)}\langle\tilde{V}_x^2\rangle_k\left(\frac{C_i}{\sqrt{2\varepsilon_0}}\frac{\delta_b^2\partial_x^3\phi_Z}{A+\bar{k}_\perp^2} - \frac{1}{\sqrt{2\varepsilon_0}}\partial_x\ln\langle T\rangle\right)$$

$$+ \frac{i}{\omega - k_y\Omega_Z}\langle\tilde{V}_x^2\rangle_k\left(\frac{1}{\sqrt{2\varepsilon_0}}\partial_x\ln\langle T\rangle - C_i\delta_b^2\partial_x^3\phi_Z\right)$$

$$= \frac{\partial_x\ln\langle T\rangle}{\sqrt{2\varepsilon_0}}\frac{ik_y\bar{\Omega}_Da_k\langle\tilde{V}_x^2\rangle_k}{(\omega - k_y\Omega_Z)(\omega - k_y\Omega_Z - k_yb_k\bar{\Omega}_D)} - \delta_b^2\partial_x^3\phi_Z\frac{ia_k\langle\tilde{V}_x^2\rangle_k}{\omega - k_y\Omega_Z - k_yb_k\bar{\Omega}_D}$$

here,

$$a_k \equiv 1 - \frac{1}{\sqrt{2\varepsilon_0}}\frac{1}{A+\bar{k}_\perp^2} = 1 - \frac{1}{\sqrt{2\varepsilon_0}}\frac{1}{\frac{\tau}{\sqrt{2\varepsilon_0}} + \bar{k}_\perp^2} > 0, \qquad b_k \equiv \frac{C_i}{\sqrt{2\varepsilon_0}}\frac{1}{A+\bar{k}_\perp^2}$$

Define:

$$\chi_1 = \Re \sum_k \left[\tilde{V}_x(k)\right]^2 \frac{i k_y \bar{\Omega}_D a_k}{(\omega - k_y \Omega_Z)(\omega - k_y \Omega_Z - k_y b_k \bar{\Omega}_D)}, \qquad \chi_2 = \Re \sum_k \left[\tilde{V}_x(k)\right]^2 \frac{i a_k}{\omega - k_y \Omega_Z - k_y b_k \bar{\Omega}_D}$$

$\Re$ represents the real part. By decomposition $\omega = \omega_R + i\gamma$, we have,

$$\chi_1^{\mathrm{non-res}} = \sum_k \left[\tilde{V}_x(k)\right]^2 a_k \frac{|\gamma| k_y \bar{\Omega}_D (2\omega_R - 2k_y \Omega_Z - k_y b_k \bar{\Omega}_D)}{|\omega - k_y \Omega_Z|^2 |\omega - k_y \Omega_Z - k_y b_k \bar{\Omega}_D|^2}$$

$$\chi_1^{\mathrm{res}} = \sum_k \left[\tilde{V}_x(k)\right]^2 a_k \frac{k_y \bar{\Omega}_D \pi \delta(\omega - k_y \Omega_Z - k_y b_k \bar{\Omega}_D)}{\omega - k_y \Omega_Z} + \sum_k \left[\tilde{V}_x(k)\right]^2 a_k \frac{k_y \bar{\Omega}_D \pi \delta(\omega - k_y \Omega_Z)}{\omega - k_y \Omega_Z - k_y b_k \bar{\Omega}_D} \sim 0$$

$$\chi_2^{\mathrm{non-res}} = \sum_k \left[\tilde{V}_x(k)\right]^2 a_k \frac{|\gamma|}{|\omega - k_y \Omega_Z - k_y b_k \bar{\Omega}_D|^2}, \qquad \chi_2^{\mathrm{res}} = \sum_k \left[\tilde{V}_x(k)\right]^2 a_k \pi \delta(\omega - k_y \Omega_Z - k_y b_k \bar{\Omega}_D)$$

QL for vorticity flux is:

$$\langle \tilde{V}_x \overline{\Delta} \tilde{\phi} \rangle_y = \chi_1^{\mathrm{non-res}} \frac{\partial_x \ln \langle T \rangle}{\sqrt{2\varepsilon_0}} - \left(\chi_2^{\mathrm{non-res}} + \chi_2^{\mathrm{res}}\right) \delta_b^2 \partial_x^3 \phi_Z \tag{33}$$

No temperature gradient contribution in the vorticity flux!

Backing to the profiles evolution, putting all those quasi-linear expression of fluxes, obtains

$$\frac{\partial}{\partial t}\left(\overline{\Delta\phi_Z}\right) = -\frac{\partial}{\partial x}\left(\frac{1}{C_i}\chi_1^{\mathrm{non-res}}\frac{\partial}{\partial x}\frac{\ln\langle T\rangle}{\sqrt{2\varepsilon_0}}\right) + \frac{\partial}{\partial x}\left[\left(\chi_2^{\mathrm{non-res}} + \chi_2^{\mathrm{res}}\right)\delta_b^2\frac{\partial^3}{\partial x^3}\overline{\phi_Z}\right] + \nu\frac{\partial^2}{\partial x^2}\overline{\Delta\phi_Z} \tag{34}$$

$$\frac{\partial}{\partial t}\ln\langle T\rangle = -\frac{\partial}{\partial x}\left[C_i(\chi_4^{\text{non-res}} + \chi_4^{\text{res}})\frac{\partial}{\partial x}\left(\overline{\Delta\phi_Z}\right)\right] + \frac{\partial}{\partial x}\left[(\chi_3^{\text{non-res}} + \chi_3^{\text{res}})\frac{\partial}{\partial x}\ln\langle T\rangle\right] \tag{35}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\langle\tilde{U}^2\rangle &= \frac{\partial}{\partial x}D_U\frac{\partial}{\partial x}\langle\tilde{U}^2\rangle - \frac{\bar{\Omega}_D}{\sqrt{2\varepsilon_0}}C_i(\chi_3^{\text{non-res}} + \chi_3^{\text{res}})\frac{\partial}{\partial x}\left(\overline{\Delta\phi_Z}\right) + \frac{\bar{\Omega}_D}{\sqrt{2\varepsilon_0}}(\chi_5^{\text{non-res}} + \chi_5^{\text{res}})\frac{\partial}{\partial x}\langle T\rangle \\ &\quad - \frac{1}{C_i}\chi_1^{\text{non-res}}\left(\frac{\partial}{\partial x}\frac{\ln\langle T\rangle}{\sqrt{2\varepsilon_0}}\right)\left[\frac{\partial}{\partial x}\left(\overline{\Delta\phi_Z}\right)\right] + (\chi_2^{\text{non-res}} + \chi_2^{\text{res}})\left[\frac{\partial}{\partial x}\left(\overline{\Delta\phi_Z}\right)\right]^2 - \frac{\nu_{\mathrm{NL}}}{l_x^2}\langle\tilde{U}^2\rangle^2\end{aligned} \tag{36}$$

To analyze the evolution, we need further simplification fo coefficients. Take $\chi_3$ as an example:

$$\chi_3^{\mathrm{non-res}} = \sum_k\left[\tilde{V}_x(k)\right]^2\frac{|\gamma|}{|\omega - k_y\Omega_Z - k_y b_k\bar{\Omega}_D|^2}, \qquad \chi_3^{\mathrm{res}} = \sum_k\left[\tilde{V}_x(k)\right]^2\pi\delta(\omega_R - k_y\Omega_Z - k_y b_k\bar{\Omega}_D)$$

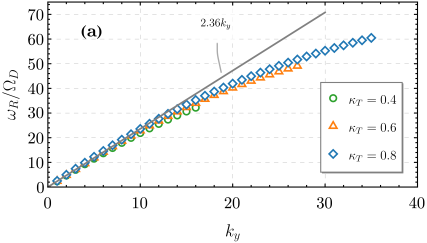
Directing calculating the integral above! Need dispersion relation $\omega(k)$ and turbulence spectrum $[\tilde{V}_x(k)]^2$.

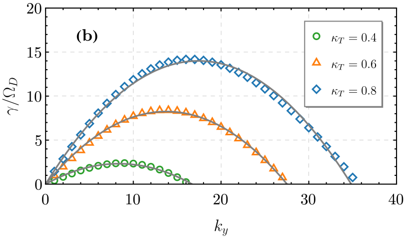
$$\langle\tilde{\phi}^2\rangle = \int_{-\infty}^{\infty} G_2(x-x')\langle\tilde{U}^2\rangle(x')\,\mathrm{d}x' \tag{37}$$

here the Green's function is defined as:

$$\begin{aligned} G_2(x) &= \mathscr{F}^{-1}\left\{\frac{1}{(A+k^2)^2}\right\} \\ &= \frac{1}{2}\sqrt{\frac{\pi}{2A^3}}\left(e^{-\sqrt{A}|x|} + \sqrt{A}|x|e^{-\sqrt{A}|x|}\right) \end{aligned} \tag{38}$$

$$D(k, z \equiv \frac{\omega}{k_y\Omega_D}) = (C_{ad}/C_i) + \rho_{i0}^2 k_y^2 + \delta_{b0}^2 k_x^2 + 2\left[z - \frac{\kappa_n}{C_i\Omega_D}(1 - \frac{3\eta_i}{2})\right][1 + \sqrt{z}Z_p(\sqrt{z})]$$
$$- \frac{\kappa_T}{C_i\Omega_D}\left[1 + 2z + 2z^{3/2}Z_p(\sqrt{z})\right] \quad (39)$$




Real frequency can be well fit by:

$$\omega_R = \frac{R_1 k_y \Omega_D}{1 + R_2(\rho^2 k_y^2 + \delta^2 k_x^2)} \quad (40)$$

where $R_1 \simeq 2.3675, R_2 \simeq 0.16 \ll 1$, so there is $\omega_R \simeq R k_y \omega_D = 2.36 k_y \omega_D$. Growth rate can be approximated as:

$$\gamma = \gamma_0 R \Omega_D k_y (k_{y,\mathrm{max}} - k_y) \quad (41)$$

Here $k_{y,\mathrm{max}}$ varies with $k_x$. It can be written as $k_{y,\mathrm{max}} = 1/(\mu_y \rho)$. Because $\Theta_{\mathrm{max}} \lesssim 1$, $\mu_y > 1$. $\gamma_0$ varies with $k_x$ and $\kappa_T$. We define $\gamma_0 = \rho/\sigma$. Numerical results show that $\sigma > 1$.

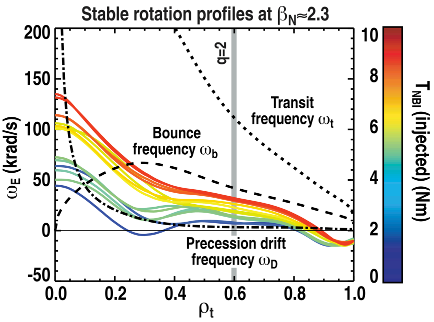



Fig. 7: Trapped ion precession frequency in DIII-D [*Reimerdes et al., 2011*]

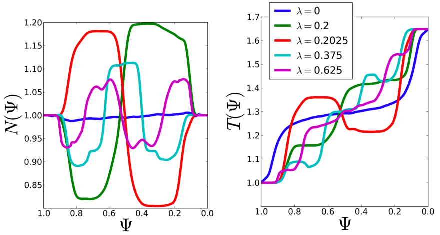



Fig. 8: Cartier-Michaud et al. obtained $E \times B$ staircase using TERESA[*Cartier-Michaud et al., 2014*]

Though the Darmet model is simple, it contains the physics for staircase formation.
Corollary: So does the PV conserving system derived from the Darmet model!

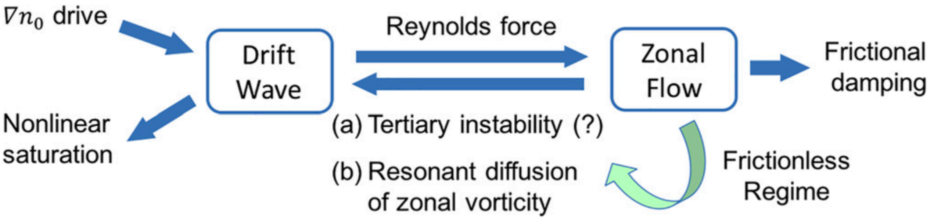
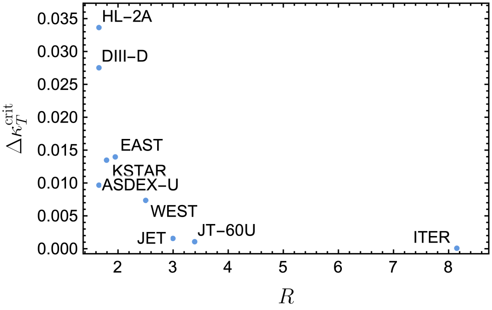
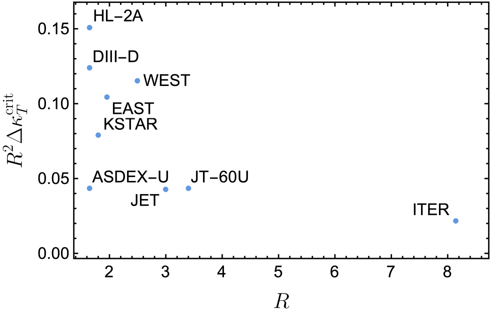



Fig. 9: [Li & Diamond, 2018]

$$\left.\frac{1}{L_T}\right|_{\mathrm{B}} - \left.\frac{1}{L_T}\right|_{\mathrm{I}} < -\frac{8\sqrt{2}}{\vartheta}\frac{1}{R_0}\sqrt{\frac{a}{R_0}}\left(\frac{\delta_b}{a}\right)^2 \tag{42}$$

Assuming $kT_i = 10$ keV,



# 附录 Nonlocality in Staircase

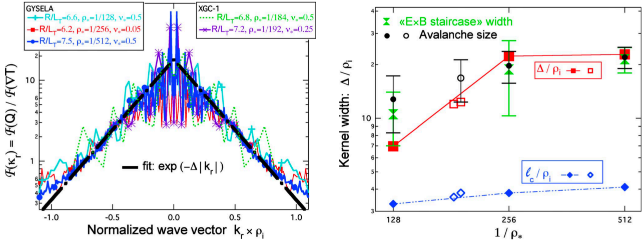
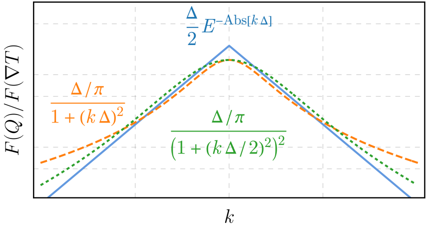



Fig. 10: [Dif-Pradalier et al., 2010]

$$Q(r) = -\int \mathcal{K}(r,r')\nabla T(r')\mathrm{d}r'$$

$$\begin{cases} \mathcal{F}(Q)/\mathcal{F}(\nabla T) \sim e^{-\Delta|k|} \\ \mathcal{K}(r,r') = \dfrac{\Lambda}{\pi}\dfrac{\Delta/2}{(\Delta/2)^2+(r-r')^2} \end{cases}$$




$$\begin{cases} \mathcal{F}(Q)/\mathcal{F}(\nabla T) \sim \dfrac{1}{1+(k\Delta)^2} \\ \mathcal{K}(r,r') = e^{-|r-r'|/\Delta} \end{cases}$$