Optimal stopping and free-boundary problems

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Chapter 1

Introduction

1.1 An Overview

The mathematical theory of optimal stopping, also known as early stopping, is concerned with determining the optimal time to take a particular action in order to maximize an expected reward or minimize an expected cost. The history of optimal stopping and free boundary problems can be traced back to the early 20th century, when mathematicians began to study decision-making under uncertainty in more depth. The concept of optimal stopping was first introduced by the mathematician Joseph Leo Doob in the 1950s. The study of free boundary problems also dates back to the mid-20th century, with the pioneering work of mathematicians such as Fritz John, Jean Leray, and Oleksandr Oleinik.

Optimal stopping problems can arise in various fields, including statistics, economics, and mathematical finance, particularly in relation to the pricing of American options. One well-known example of an optimal stopping problem is the secretary problem. These problems can often be formulated using a Bellman equation and are typically solved using dynamic programming techniques.

1.2 Differences And Relations

Optimal stopping and free boundary problems are two closely related mathematical concepts that deal with decision-making under uncertainty.

Optimal stopping problems involve making a decision about when to stop a process in order to maximize a certain objective function. For example, suppose you are playing a game where you flip a coin repeatedly, and you can choose to stop the game at any time and receive a payout based on the number of heads you have flipped. An optimal stopping problem would involve finding the optimal time to stop the game in order to maximize your expected payout.

Free boundary problems are a more general class of problems that involve finding the boundary between two regions, where the boundary is determined by an underlying mathematical model. For example, in fluid mechanics, free boundary problems arise when trying to determine the shape of a liquid-air interface. In finance, free boundary problems arise in the pricing of American options, which allow the holder to exercise the option at any time up until expiration. The free boundary represents the optimal time to exercise the option.

Optimal stopping and free boundary problems are closely related because in many cases, the optimal stopping time corresponds to the location of the free boundary. For example, in the American option pricing problem, the optimal time to exercise the option corresponds to the location of the free boundary between the regions where it is optimal to exercise the option and where it is optimal to hold onto it.

Both optimal stopping and free boundary problems are important tools for modeling and analyzing decision-making under uncertainty in a wide range of fields, including finance, economics, engineering, physics, and more. They are used to optimize decisions and to gain insight into the underlying mathematical models governing the behavior of complex systems.

1.3 Our Case

If we know how things are going to change in the future, it will be much easier to make decisions about it, especially in the world of finance. In other words, if we know the future ups and downs of a financial product then we can decide whether to buy or sell it at the moment.

Unfortunately, even if we knew all the historical data about a financial product, we would not have complete insight into its future direction. This is because its price is influenced by many small factors, similar to the impact of many small particles on a particle in Brownian motion. Similar to the fact that the path of a Brownian motion is unpredictable, the price of a complex financial product is usually beyond our knowledge of prediction.

The amazing thing is that we are not helpless even in the face of completely unknown future, this theorem tells us that we can make the most informed choices even facing the complete random Brownian motion.

The optimal stopping problem we considered is of interest, for example, in financial mathematics and financial engineering, where an optimal decision (i.e., optimal stopping time) should be based on a prediction of the future behavior of the observable process (asset price, index, etc.). The argument also carries over to many other applied problems where such predictions play a role.

Below 1.4 we introduce the problem and explain it.

1.4 Our Problem

The problem we want to solve is below:

$$V_* = \inf_{\tau \in \mathcal{M}} \mathbf{E} \left(B_\tau - \max_{0 \leqq t \leqq 1} B_t \right)^2 \tag{1.1}$$

where V_* is some value we wish to calculate, and τ is the stopping time we wish to find out.

The solution to this problem is presented in the Theorem below.

Chapter 2

Related Work

The main content of this paper is based on *stopping brownian motion without anticipation as close as possible to its ultimate maximum* written by Graversen, Peskir, and Shiryaev [1]. My main job is to write some of the proof in detail, which means to explain the proof been taken for granted. My other jobs are to show the applications and finding other proofs of the equations or lemma in the proof.

Chapter 3

Prerequisites

3.1 Definition of Brownian Motion

A stochastic process

$$B = (B_t)_{t \geq 0}$$

defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a Wiener process (or a Brownian Motion Process) if the following four conditions are satisfied:

1. $t \mapsto B_t$ is continuous from $\mathbb{R}_+$to $\mathbb{R}$, and $B_0 = 0$ (both $\mathbb{P}$-a.s).

2. B has stationary increments, i.e.

$$B_t - B_s \sim B_{t-s} \sim N\left((t-s)\mu, (t-s)\sigma^2\right)$$

for any $0 \leq s < t$.

3. B has independent increments, i.e. $B_{t_1 - t_0}, B_{t_2 - t_1}, \ldots, B_{t_n - t_{n-1}}$ are independent random variables for any choice of $0 \leq t_0 < t_1 < \cdots < t_n$ with $n \geq 1$.

4. $B_t \sim N\left(\mu t, \sigma^2 t\right)$ for every $t > 0$ where $\mu \in \mathbb{R}$ and $\sigma > 0$ are given and fixed constants.

3.2 Definition of Square-Integrable Functions

In the following sections, we will make use of an important class of functions, *the square integrable functions.* A function $f : \mathbb{R} \to \mathbb{C}$ is square integrable if

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty.$$

The definition of square integrability can be localized to a bounded interval, such as $[a, b]$ for some $a \leq b$. Then f is said to be square integrable over $[a, b]$ means

$$\int_a^b |f(x)|^2 \, dx < \infty.$$

3.3 Ito's Formula

Theorem 3.3.1 (Itô (1994); Kunita & Watanabe (1967)). *Let $X = M + A$ be a continuous semimartingale, and let $F : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Then we have:*

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F'(X_s)\ dX_s + \frac{1}{2}\int_0^t F''(X_s)\ d\langle X, X\rangle_s \\ &= F(X_0) + \int_0^t F'(X_s)\ dM_s + \int_0^t F'(X_s)\ dA_s + \frac{1}{2}\int_0^t F''(X_s)\ d\langle M, M\rangle_s, \end{aligned} \tag{3.1}$$

for all $t \geq 0$.

We then have the following multidimensional extension of Theorem 3.3.1.

Theorem 3.3.2 (Multidimensional Itô formula). *Let $X = M + A$ be a continuous semimartingale with values in $\mathbb{R}^d$, i.e. $X_t^i = M_t^i + A_t^i$ where M_t^i is a continuous local martingale and A_t^i is a continous adapted process of BV for $1 \leq i \leq d$, and let $F : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function. Then we have:*

$$\begin{aligned} F(X_t) =& F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s)\ dX_s^i \\ &+ \frac{1}{2}\sum_{i=1}^d \sum_{i=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s)\ d\langle X^i, X^j\rangle_s \\ =& F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s)\ dM_s^i + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s)\ dA_s^i \\ &+ \frac{1}{2}\sum_{i=1}^d \sum_{i=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s)\ d\langle M^i, M^j\rangle_s \end{aligned} \tag{3.2}$$

for all $t \geq 0$.

3.4 Stochastic integral representation of the maximum process

In order to solve the main question we need to know the stochastic integral representation of the maximum process. The theorem is described in the following contents.

If $S_1 = \sup_{0 \leqq s \leqq 1} B_s$ is a square-integrable functional of the Brownian path on $[0, 1]$. The Itô-Clark representation theorem says there exists a unique $\mathbb{F}^B$-adapted process $H = (H_t)_{0 \leqq t \leqq 1}$ satisfying $\mathbf{E}\int_0^1 H_t^2 dt < \infty$ such that

$$S_1 = a + \int_0^1 H_t dB_t, \tag{3.3}$$

where $a = \mathbf{E}S_1$. And we also know from the theorem that the following explicit formula is known to be valid:

$$H_t = 2\left(1-\Phi\left(\frac{S_t-B_t}{\sqrt{1-t}}\right)\right) \tag{3.4}$$

for $0 \leqq t \leqq 1$

In order to show the above statment is correct, we begin with a standard Brownian motion with drift which takes the form of

$$B_t^\mu = B_t + \mu t,$$

where μ belongs to the real number set. The corresponding maximum process S^μ after defining B^μ is given by

$$S_t^\mu = \sup_{0\leqq s\leqq t} B_s^\mu.$$

3.4.1 Step One

To get the analogue of (3.1) and (3.2) in this case, we shall first note that stationary independent increments of B^μ imply

$$\mathbf{E}\left(S_1^\mu \mid \mathcal{F}_t^B\right) = \mathbf{E}\left(\sup_{0\leqq s\leqq t} B_s^\mu + \left(\sup_{t\leqq s\leqq 1} B_s^\mu - S_t^\mu\right)^+ \mid \mathcal{F}_t^B\right) \tag{3.5}$$

$$= S_t^\mu + \mathbf{E}\left(\left(\sup_{t\leqq s\leqq 1} B_s^\mu - S_t^\mu\right)^+ \mid \mathcal{F}_t^B\right) \tag{3.6}$$

$$= S_t^\mu + \mathbf{E}\left(\left(\sup_{t\leqq s\leqq 1} (B_s^\mu - B_t^\mu) - (S_t^\mu - B_t^\mu)\right)^+ \mid \mathcal{F}_t^B\right) \tag{3.7}$$

$$= S_t^\mu + \mathbf{E}\left(S_{1-t}^\mu - (z-x)\right)^+\Big|_{z=S_t^\mu, x=B_t^\mu}. \tag{3.8}$$

(3.5) holds since there is only two possibility, one is the maximum is taken in the interval of [0,t), the other is the maximum is taken in the interval of (t,1].
(3.6) holds since $\sup_{0\leqq s\leqq t} B_s^\mu = S_t^\mu$ is $\mathcal{F}_t^B$ measurable.
(3.7) is easy to see.
(3.8) holds since $\sup_{t\leqq s\leqq 1}(B_s^\mu - B_t^\mu) = \sup_{t\leqq s\leqq 1} B_{s-t}^\mu = \sup_{0\leqq s\leqq 1-t} B_s^\mu = S_{1-t}^\mu$.

Further, using the formula

$$\mathbf{E}(X-c)^+ = \int_c^\infty \mathbf{P}\{X > z\}dz, \tag{3.9}$$

(3.9) holds because

$$LHS = \int_c^{\infty} \mathbf{E}[(\mathbf{I}(X > x)]dx \tag{3.10}$$
$$= \mathbf{E}[\int_c^{\infty} \mathbf{I}(X > x)dx] \tag{3.11}$$
$$= \mathbf{E}[(X - c)^+]. \tag{3.12}$$

(3.11) holds because of the Fubini Theorem.
(3.12) holds since for ω s.t $X(\omega) = X \geq c$ the integral is $X - c$, while for ω s.t $X(\omega) = X < c$ the interal is 0. So the integral is $(X - c)^+$.

Then we can see that

$$\mathbf{E}\left(S_1^\mu \mid \mathcal{F}_t^B\right) = S_t^\mu + \mathbf{E}\left(S_{1-t}^\mu - (z - x)\right)^+\Big|_{z=S_t^\mu, x=B_t^\mu} \tag{3.13}$$
$$= S_t^\mu + \int_{z-x}^{\infty} \mathbf{P}\{S_{1-t}^\mu > m\}dm\Big|_{z=S_t^\mu, x=B_t^\mu} \tag{3.14}$$
$$= S_t^\mu + \int_{S_t^\mu - B_t^\mu}^{\infty} \mathbf{P}\{S_{1-t}^\mu > m\}dm \tag{3.15}$$
$$= S_t^\mu + \int_{S_t^\mu - B_t^\mu}^{\infty} [1 - \mathbf{P}\{S_{1-t}^\mu \leqq m\}]dm \tag{3.16}$$
$$= S_t^\mu + \int_{S_t^\mu - B_t^\mu}^{\infty} [1 - \mathbf{P}\left\{S_{1-t}^\mu \leqq z\right\}]dm \tag{3.17}$$
$$= S_t^\mu + \int_{S_t^\mu - B_t^\mu}^{\infty} \left(1 - F_{1-t}^\mu(z)\right) dz \tag{3.18}$$
$$\equiv f\left(t, B_t^\mu, S_t^\mu\right), \tag{3.19}$$

where we use the following notation:

$$F_{1-t}^\mu(z) = \mathbf{P}\left\{S_{1-t}^\mu \leqq z\right\}$$

and the map $f = f(t, x, s)$ is defined accordingly.

3.4.2 Step Two

Now we applying Itô's formula to f, and using that the left-hand side of (3.19) defines a continuous martingale, we find upon setting $a_\mu = \mathbf{E}S_1^\mu$ that

$$\mathbf{E}\left(S_1^\mu \mid \mathcal{F}_t^B\right) = a_\mu + \int_0^t \frac{\partial f}{\partial x}\left(s, B_s^\mu, S_s^\mu\right) dB_s \tag{3.20}$$
$$= a_\mu + \int_0^t \left(1 - F_{1-s}^\mu\left(S_s^\mu - B_s^\mu\right)\right) dB_s. \tag{3.21}$$

(3.20) holds since by the Ito's formula we get the following

$$\mathbf{E}\left(S_1^\mu \mid \mathcal{F}_t^B\right) = f\left(t, B_t^\mu, S_t^\mu\right) = a_\mu + M_t + A_t, \tag{3.22}$$

where $a_\mu = f(0, B_0^\mu, S_0^\mu) = f(0,0,0) = \int_0^\infty (1 - F_1^\mu(z))dz = \mathbf{E}S_1^\mu$, M_t is a continuous local martingale, A_t is of BV after a transform we get

$$\mathbf{E}\left(S_1^\mu \mid \mathcal{F}_t^B\right) - M_t = a_\mu + A_t \tag{3.23}$$

Notice $\mathbf{E}\left(S_1^\mu \mid \mathcal{F}_t^B\right)$ is a martingale, so it is also a local martingale. So LHS of (3.23) is a continuous local martingale. Notice RHS of (3.23) is a process of bounded variation. Both LHS and RHS is the same thing which is both of BV and is a continuous local martingale.

Corollary 3.4.1. *Any continuous BV local martingale is a constant. (This is shown in the lecture of Stochastic Calculus.)*

By the corollary above we have $a_\mu + A_t \equiv a_\mu + A_0 = a_\mu$, since $A_0 = \int_0^{t=0}(...)ds = 0$, hence $A_t = 0$ for all $t \geq 0$.
So RHS of equation (3.22) is $a_\mu + M_t$, where $M_t = \int_0^t \frac{\partial f}{\partial x} dB_s + \int_0^t \frac{\partial f}{\partial s} dS_s^\mu$, since

$$\begin{aligned} M_t^{orginal} &= \int_0^t \frac{\partial f}{\partial x} dB_s^\mu + \int_0^t \frac{\partial f}{\partial s} dS_s^\mu & (3.24)\\ &= \int_0^t \frac{\partial f}{\partial x} dB_s + \int_0^t \frac{\partial f}{\partial x} d(\mu s) + \int_0^t \frac{\partial f}{\partial s} dS_s^\mu & (3.25)\\ &= M_t + \int_0^t \frac{\partial f}{\partial x} d(\mu s). & (3.26) \end{aligned}$$

In order to prove (3.20) holds we only need to show $\int_0^t \frac{\partial f}{\partial s} dS_s^\mu = 0$, which is proved by the following arguments.

$$\begin{aligned} \int_0^t \frac{\partial f}{\partial s} dS_s^\mu &= \int_0^t F_{1-s}^\mu (S_s^\mu - B_s^\mu) dS_s^\mu & (3.27)\\ &= \int_0^t P(S_{1-s}^\mu \leq S_s^\mu - B_s^\mu) dS_s^\mu & (3.28)\\ &= 0 & (3.29) \end{aligned}$$

(3.29) holds since:
if $S_s^\mu = B_s^\mu$ (when $dS_s^\mu > 0$), we know $P(S_{1-s}^\mu \leq S_s^\mu - B_s^\mu) = 0$, so the integral equals to 0 in this case;
if $S_s^\mu > B_s^\mu$, then $dS_s^\mu = 0$, so the integral is 0 as well. Hence (3.29) holds.
(3.21) holds since $\frac{\partial f}{\partial x} = 1 - F_{1-t}^\mu (S_t^\mu - B_t^\mu)$, which is easy to verify.

As a nontrivial continuous martingale cannot have paths of bounded variation. This reduces the initial problem to the problem of calculating $F_{1-t}^\mu(z) = \mathbf{P}\left\{S_{1-t}^\mu \leqq z\right\}$.

3.4.3 Step Three

The following explicit formula [3] [6] is well known:

Corollary 3.4.2. *For $x \geq 0$, $\mu \in R$, $\sigma > 0$, we have $P(max_{s\in[0,t]}(\mu s+\sigma W_s) \leq x) = \Phi(\frac{x-\mu t}{\sigma\sqrt{t}}) - e^{\frac{2\mu t}{\sigma^2}}\Phi(\frac{-x-\mu t}{\sigma\sqrt{t}})$.*

In our case we have $F^\mu_{1-t}(z) = P(S^\mu_{1-t} \leq z) = P(sup_{s\in[0,1-t]}(B_s + \mu s) \leq z)$ hence in this case "$t = 1 - t$ $\mu = \mu$ $\sigma = \sigma$". So we have the following equality:

$$F^\mu_{1-t}(z) = \Phi\left(\frac{z-\mu(1-t)}{\sqrt{1-t}}\right) - e^{2\mu z}\Phi\left(\frac{-z-\mu(1-t)}{\sqrt{1-t}}\right).$$

Inserting this into (3.21) we finally obtain the representation

$$S^\mu_1 = a_\mu + \int_0^1 H^\mu_t dB_t,$$

where the process H^μ is explicitly given by

$$\begin{aligned} H^\mu_t =& 1 - \Phi\left(\frac{(S^\mu_t - B^\mu_t) - \mu(1-t)}{\sqrt{1-t}}\right) \\ &+ e^{2\mu(S^\mu_t - B^\mu_t)}\Phi\left(\frac{-(S^\mu_t - B^\mu_t) - \mu(1-t)}{\sqrt{1-t}}\right). \end{aligned}$$

Setting $\mu = 0$ in this expression, we recover (3.3) and (3.4).

$$\begin{aligned} S_1 &= a + \int_0^1 H_t dB_t \\ H_t &= 2\left(1 - \Phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right)\right). \end{aligned}$$

3.4.4 Remark

Note that the argument above extends to a large class of processes with stationary independent increments (including Lévy processes) by reducing the initial problem to calculating the analogue of $F^\mu_{1-t}(z) = \mathbf{P}\left\{S^\mu_{1-t} \leqq z\right\}$. In particular, the following "prediction" result deserves a special note. It is derived in exactly the same way as (3.13)-(3.19) above.

Let $X = (X_t)_{0\leqq t\leqq T}$ be a process with stationary independent increments started at 0, and let us denote $S_t = \max_{0\leqq s\leqq t} X_s$ for $0 \leqq t \leqq T$. If $\mathbf{E}S_T < \infty$, then the predictor $\mathbf{E}\left(S_T \mid \mathcal{F}^X_t\right)$ of S_T based on the observations $\{X_s \mid 0 \leqq s \leqq t\}$ is given by the following formula:

$$\mathbf{E}\left(S_T \mid \mathcal{F}^X_t\right) = S_t + \int_{S_t - X_t}^\infty (1 - F_{T-t}(z))\, dz,$$

where $F_{T-t}(z) = \mathbf{P}\{S_{T-t} \leqq z\}$.

3.5 Definition of Stopping Time

T is a stopping time w.r.t. $\left(\mathscr{F}^B_t\right)_{t\geq 0}$ if and only if:

$$\{T \leq t\} \in \mathscr{F}^B_t$$

for all $t \geq 0$. This T is a function from Ω into $\mathbb{R}_+$or $\overline{\mathbb{R}}_+ = [0, \infty]$. Here $\{T \leq t\} \equiv \{\omega \in \Omega \mid T(\omega) \leq t\}$.

3.6 Equal In Law

3.6.1 One Dimensional

By definition $X \sim Y$ if and only if $F_X = F_Y$.

3.6.2 Multi-dimensional

By definition $(X_1, X_2) \sim (Y_1, Y_2)$ if and only if $F_{X_1,X_2} = F_{Y_1,Y_2}$.

3.6.3 Continuous Case

By definition $(X_t)_{t\geq 0} \sim (Y_t)_{t\geq 0}$ if and only if $(X_{t_1}, X_{t_2}, ..., X_{t_n}) \sim (Y_{t_1}, Y_{t_2}, ..., Y_{t_n})$ for any $n > 0$, and for any $t_1 < t_2 < ... < t_n$.

3.6.4 Remark

$(X_t)_{t\geq 0} \sim (Y_t)_{t\geq 0}$ is different from for any $t \geq 0$ $X_t \sim Y_t$.

3.7 Reflection Principle

3.7.1 Part 1

Corollary 3.7.1. *Fix $a \in \mathbb{R}$. Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion and $\tau_a = \inf\{t > 0 : B_t = a\}$ be the first it hits a. The process $\widetilde{B} = \left(\widetilde{B}_t\right)_{t\geq 0}$, where*

$$\widetilde{B}_t := \begin{cases} B_t, & t < \tau_a \\ 2a - B_t, & t \geq \tau_a \end{cases}$$

is a standard Brownian motion.

Theorem 3.7.2. *Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion. Let $S_t := \sup_{s\leq t} B_s$ be the running supremum of B. Then, for all $a > 0$ and $t \geq 0$,*

$$\mathbb{P}\left(S_t \geq a\right) = \mathbb{P}\left(\tau_a \leq t\right) = 2\mathbb{P}\left(B_t \geq a\right) = \mathbb{P}\left(|B_t| \geq a\right).$$

3.7.2 Part 2

If $(B_t : t \geq 0)$ is a Wiener process, and $a > 0$ is a threshold (also called a crossing point), then the theorem states:

$$P(sup_{0\leq s\leq t} B_s \geq a) = 2P(B_t \geq a).$$

3.7.3 Conclusion

[3] For a fixed t, we have

$$S_t - B_t \sim |B_t| \sim S_t.$$

3.8 Stefan Problems

Free boundary problems refer to the type of partial differential equations (PDEs) that involve interfaces or boundaries that are not known in advance. These interfaces are often referred to as "free" since they are not prescribed by the problem, but rather determined by the solution. One classic example of such problems is the melting of ice to water, which is known as the Stefan problem. In this case, the free boundary refers to the interface between the liquid and solid phases, which is not initially known and is determined by the solution of the problem.

3.9 Ito-Tanaka Formula

Theorem 3.9.1. *For any $a \in \mathbb{R}$ there exists a continuous increasing process $(L_t^a)_{t\geqslant 0}$ such that*

$$\begin{aligned}
|X_t - a| &= |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}X_s + L_t^a \\
(X_t - a)_+ &= (X_0 - a)_+ + \int_0^t \mathbb{I}_{X_s > a}\ \mathrm{d}X_s + \frac{1}{2}L_t^a \\
(X_t - a)_- &= (X_0 - a)_- - \int_0^t \mathbb{I}_{X_s \leqslant a}\ \mathrm{d}X_s + \frac{1}{2}L_t^a
\end{aligned}$$

where $\operatorname{sgn}(x) = \mathbb{I}_{x>0} - \mathbb{I}_{x\leqslant 0}$

Remark 3.9.2. *This proves in particular that $|X_t - a|, (X_t - a)_\pm$ are semi-martingales. The process $(L_t^a)_{t\geqslant 0}$ it is called the local time of X at a.*

Proof. Each of the formulas derives from the previous lemma by computing the left derivative of the various convex functions. The missing point is to identify the various increasing processes $A^{\operatorname{sgn}(x-a)}, A^{(x-a)_+}, A^{(x-a)_-}$. Note that so we have

$$\begin{aligned}
X_t - a &= (X_t - a)_+ - (X_t - a)_- \\
&= X_0 - a + \int_0^t \underbrace{(\mathbb{I}_{X_s>a} + \mathbb{I}_{X_s\leqslant a})}_{=1}\ \mathrm{d}X_s + \frac{1}{2}\left(A_t^{(x-a)_+} - A_t^{(x-a)_-}\right),
\end{aligned}$$

and

$$\begin{aligned}
0 = X_t - X_0 - \int_0^t\ \mathrm{d}X_s &= \frac{1}{2}\left(A_t^{(x-a)_+} - A_t^{(x-a)_-}\right) \\
&\Rightarrow A_t^{(x-a)_+} = A_t^{(x-a)_-} =: L_t^a.
\end{aligned}$$

Moreover

$$|X_t - a| = (X_t - a)_+ + (X_t - a)_- + \int_0^t \underbrace{\left(\mathbb{I}_{X_s > a} - \mathbb{I}_{X_s \leqslant a}\right)}_{=\operatorname{sgn}(X_s - a)} \mathrm{d}X_s + \underbrace{\frac{1}{2\left(A_t^{(x-a)_+} + A_t^{(x-a)_-}\right)}}_{L_t^a}.$$

The increasing process $(L_t^a)_{t\geqslant 0}$ is associated with a measure $\mathrm{d}L_t^a$ on $\mathbb{R}_+$(times) which represents the time the process X "spent" in a up to time t. We are going to make this precise in the following.

By Ito formula wrt. the semimartingale $(|X_t - a|)_{t\geqslant 0}$ (with $X = M + V$).

$$\begin{aligned}(X_t - a)^2 &= (|X_t - a|)^2 \\ &= (|X_0 - a|)^2 + 2\int_0^t |X_s - a| \operatorname{sgn}(X_s - a)\,\mathrm{d}X_s + 2\int_0^t |X_s - a|\,\mathrm{d}L_s^a + [|X_\cdot - a|]_t \\ &= (X_0 - a)^2 + 2\int_0^t (X_s - a)\,\mathrm{d}X_s + 2\int_0^t |X_s - a|\,\mathrm{d}L_s^a + \int_0^t \underbrace{\operatorname{sgn}(X_s - a)^2}_{=1}\,\mathrm{d}[M]_s\end{aligned}$$

And by comparing with the standard Ito formula, we conclude that

$$(X_t - a)^2 = (X_0 - a)^2 + 2\int_0^t (X_s - a)\,\mathrm{d}X_s + \underbrace{[X]_t}_{=[M]_t}$$

$$\int_0^t |X_s - a|\,\mathrm{d}L_s^a = 0, \quad t \geqslant 0$$

which proves that the measure $(\mathrm{d}L_s^a)_{s\geqslant 0}$ is supported in the (random) set $\{s \in \mathbb{R} : X_s = a\}$ of times. The process L^a increases only when the process X visits a (in general this will be a "fractal-like" and with zero Lebesgue measure).

For Brownian motion is it true (we will not prove it) that the set $\{s \in \mathbb{R} : X_s = a\}$ is the support of the measure $(L_t^a)_{t\geqslant 0}$.

□

3.10 Normal Distribution

A normal distribution, also known as a Gaussian distribution in statistics, is a continuous probability distribution that applies to real-valued random variables. The formula for the probability density function of a normal distribution takes the following general form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right).$$

When a random variable follows a normal distribution, it is said to be normally distributed and can be referred to as a normal deviate.

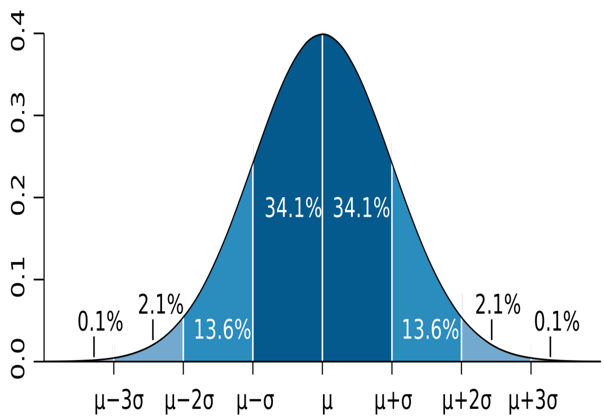

Figure 3.1: Caption

3.11 Definition of Time-homogeneous Markov Process

Setting Let $I \subseteq [0, \infty)$ and $(X_t)_{t \in I}$ be an $\mathbb{R}^n$-valued stochastic process vis-a-vis $(\mathcal{F}_t)_{t \in I}$, the natural filtration generated by $(X_t)_{t \in I}$.

Definition 3.11.1. We say that X has the Markov property (MP) iff for every $A \in \mathcal{B}(\mathbb{R}^n)$ and all $s, t \in I$ with $s \leq t$

$$\mathrm{P}[X_t \in A \mid \mathcal{F}_s] = \mathrm{P}[X_t \in A \mid X_s].$$

Definition 3.11.2. Let I be closed under addition and assume $0 \in I$. A stochastic process $X = (X_t)_{t \in I}$ is called a time-homogeneous Markov process with distributions $(\mathrm{P}_x)_{x \in \mathbb{R}^n}$ on the space $(\Omega, \mathcal{A})$ iff:
1. For every $x \in \mathbb{R}^n$, X is a stochastic process on the probability space $(\Omega, \mathcal{A}, \mathrm{P}_x)$ with $\mathrm{P}_x[X_0 = x] = 1$.
2. The map $\kappa : \mathbb{R}^n \times \mathcal{B}(\mathbb{R}^n)^{\otimes I} \to [0,1], (x, B) \mapsto \mathrm{P}_x[X \in B]$ is a stochastic kernel.
3. X has the time-homogeneous Markov property: For every $A \in \mathcal{B}(\mathbb{R}^n)$, every $x \in \mathbb{R}^n$ and all $s, t \in I$,
we have

$$\mathrm{P}_x[X_{t+s} \in A \mid \mathcal{F}_s] = \kappa_t(X_s, A)\, \mathrm{P}_x - \text{ a.s.}$$

Here, for every $t \in I$, the transition kernel $\kappa_t : \mathbb{R}^n \times \mathcal{B}(\mathbb{R}^n) \to [0,1]$ is the stochastic kernel defined for $x \in \mathbb{R}^n$ and $A \in \mathcal{B}(\mathbb{R}^n)$ by

$$\kappa_t(x, A) := \kappa\left(x, \left\{y \in \mathbb{R}^I : y(t) \in A\right\}\right) = \mathrm{P}_x[X_t \in A].$$

The family $(\kappa_t(x, A), t \in I, x \in \mathbb{R}^n, A \in \mathcal{B}(\mathbb{R}^n))$ is also called the family of transition probabilities of X.

3.12 Infinitesimal Generator

For a stochastic process X_t, the infinitesimal generator is given by

$$L = \mu(x)\frac{d}{dx} + \frac{\sigma^2(x)}{2}\frac{d^2}{dx^2},$$

where

$$\mu(x) = lim_{t\to 0}\frac{E_x[X_t - x]}{t}$$
$$\sigma^2(x) = lim_{t\to 0}\frac{E_x[(X_t - x)^2}{t}.$$

Chapter 4

Theorem

4.1 The Main Theorem

Consider the optimal stopping problem

$$V_* = \inf_{\tau \in \mathcal{M}} \mathbf{E} \left(B_\tau - \max_{0 \leqq t \leqq 1} B_t \right)^2$$

where $(B_t)_{0 \leqq t \leqq 1}$ is a standard Brownian motion. Then the value V_* is given by

$$V_* = 2\Phi\left(z_*\right) - 1 = 0.73\ldots, \tag{4.1}$$

where $z_* = 1.12\ldots$ is the unique root of

$$4\Phi\left(z_*\right) - 2z_*\varphi\left(z_*\right) - 3 = 0 \tag{4.2}$$

and the following corresponding stopping time is optimal:

$$\tau_* = \inf\left\{0 \leqq t \leqq 1 \mid S_t - B_t \geqq z_*\sqrt{1-t}\right\}, \tag{4.3}$$

where S_t is given by $S_t = \sup_{0 \leqq s \leqq t} B_s$.

In chapter Proof we start proving this main theorem.

Chapter 5

Proof

Since $S_1 = \sup_{0 \leqq s \leqq 1} B_s$ is a square-integrable functional of the Brownian path on $[0, 1]$, by the Itô-Clark representation theorem there exists a unique $\mathbb{F}^B$-adapted process $H = (H_t)_{0 \leqq t \leqq 1}$ satisfying $\mathbf{E} \int_0^1 H_t^2 dt < \infty$ such that

$$S_1 = a + \int_0^1 H_t dB_t,$$

where $a = \mathbf{E}S_1$. Moreover, the following explicit formula is known to be valid:

$$H_t = 2\left(1 - \Phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right)\right)$$

for $0 \leqq t \leqq 1$, (see reference [2] or [4]) see prerequisites 3.4 for a direct argument.

5.1 Step One

Associate with H the square-integrable martingale $M = (M_t)_{0 \leqq t \leqq 1}$ given by

$$M_t = \int_0^t H_s dB_s.$$

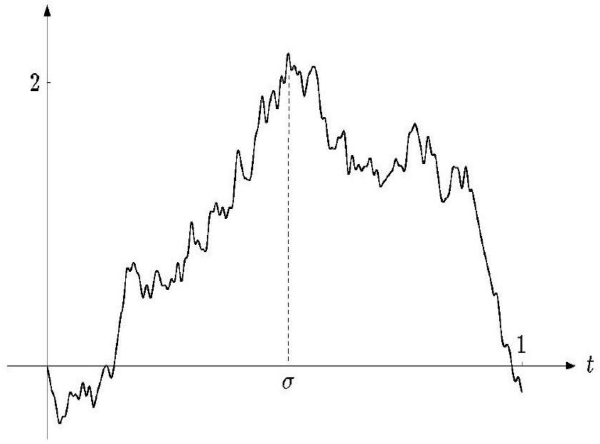

Figure 5.1: A computer simulation of a Brownian path $(B_t(\omega))_{0 \leqq t \leqq 1}$ with the maximum being attained at $\sigma = 0.51$.(The figure is taken from [1, Fig. 1])

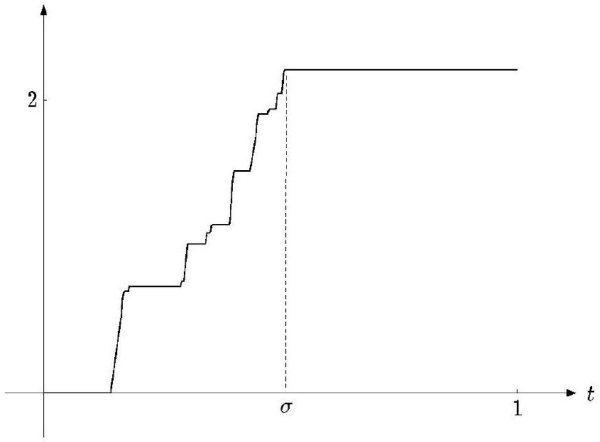

Figure 5.2: A computer drawing of the maximum process $(S_t(\omega))_{0 \leqq t \leqq 1}$ associated with the Brownian path from Figure 1.(The figure is taken from [1, Fig. 2])

By the martingale property of M and the optional sampling theorem, we obtain

$$\begin{aligned}
\mathbf{E}\,(B_\tau - S_1)^2 &= \mathbf{E}\,|B_\tau|^2 - 2\mathbf{E}\,(B_\tau M_1) + \mathbf{E}\,|S_1|^2 && (5.1)\\
&= \mathbf{E}\tau - 2\mathbf{E}\,(B_\tau M_\tau) + 1 && (5.2)\\
&= \mathbf{E}\left(\int_0^\tau (1 - 2H_t)\, dt\right) + 1 && (5.3)
\end{aligned}$$

for all $\tau \in \mathcal{M}$.

Explaining the above in detail:

(5.1) holds since $\mathbf{E}[B_\tau M_1] = \mathbf{E}[B_\tau S_1]$ (where $M_1 = S_1 - a$), which holds because that $\mathbf{E}[B_\tau M_1] = \mathbf{E}[B_\tau S_1] - \mathbf{E}[B_\tau a]$ where $\mathbf{E}[B_\tau] = 0$.
Notice that τ is bounded and B_t is a martingale hence $\mathbf{E}[B_\tau] = \mathbf{E}[B_0] = 0$ holds.

(5.2) holds since

$$\mathbf{E}\left|B_\tau\right|^2 = \mathbf{E}\tau \tag{5.4}$$

$$2\mathbf{E}\left(B_\tau M_1\right) = 2\mathbf{E}\left(B_\tau M_\tau\right) \tag{5.5}$$

$$\mathbf{E}\left|S_1\right|^2 = 1. \tag{5.6}$$

(5.4) is from basic golden martingale $B_t^2 - t$ and OST.

(5.5) is from the following argument:

Corollary 5.1.1. *M_t is a UI martingale and σ,τ are stopping times with $\sigma < \tau$ a.s. then $\mathbf{E}[M_\tau] < \infty$ and $\mathbf{E}[M_\tau|\mathscr{F}_\sigma^B] = M_\sigma$.*

Then we have

$$\mathbf{E}[M_1|\mathscr{F}_\tau^B] = M_\tau$$
$$\mathbf{E}[M_1|\mathscr{F}_\tau^B]B_\tau = M_\tau B_\tau$$
$$\mathbf{E}[M_1 B_\tau|\mathscr{F}_\tau^B] = M_\tau B_\tau$$

take expectation on both sides of the above equation, we get:

$$\mathbf{E}[M_1 B_\tau] = \mathbf{E}[M_\tau B_\tau]$$

.

(5.6) is from $\mathbf{E}\left|S_1\right|^2 = \mathbf{E}\left|B_1\right|^2 = 1$ since $S_1 = |B_1|$ by law.

(5.3) holds by following argument:

Corollary 5.1.2 (The Itô Isometry)**.** *If $H \in \mathcal{H}_2([0,t])$, then:*

$$\mathbf{E}\left[\left|\int_0^t H_s\ dB_s\right|^2\right] = \mathbf{E}\left[\int_0^t \left|H_s\right|^2\ ds\right].$$

Corollary 5.1.3. *Let $f, g \in L^2$ be progressively measurable, then:*

$$\mathbf{E}[(\int_0^\tau f(t)dB_t)(\int_0^\tau g(t)dB_t)] = \mathbf{E}[\int_0^\tau f(t)g(t)dt]. \tag{5.7}$$

Proof.

$$\mathbf{E}[(\int_0^\tau f(t)dB_t)(\int_0^\tau g(t)dB_t)] = \frac{1}{4}(\mathbf{E}[(\int_0^\tau f(t)+g(t)dB_t)^2] - \mathbf{E}[(\int_0^\tau f(t)-g(t)dB_t)^2]) \tag{5.8}$$

$$= \frac{1}{4}(\mathbf{E}[\int_0^\tau [f(t)+g(t)]^2 dt] - \mathbf{E}[\int_0^\tau [f(t)-g(t)]^2 dt]) \tag{5.9}$$

$$= \frac{1}{4}(\mathbf{E}[\int_0^\tau [f(t)+g(t)]^2 - [f(t)-g(t)]^2 dt]) \tag{5.10}$$

$$= \frac{1}{4}(\mathbf{E}[\int_0^\tau 4f(t)g(t)dt]) \tag{5.11}$$

$$= \mathbf{E}[\int_0^\tau f(t)g(t)dt]. \tag{5.12}$$

□

Once we have shown $\mathbf{E}[B_\tau M_\tau] = \mathbf{E}[\int_0^\tau H_t dt]$, we will get

$$\mathbf{E}[\tau - B_\tau M_\tau] = \mathbf{E}[\tau - \int_0^\tau 2H_t dt] = \mathbf{E}[\int_0^\tau 1 - 2H_t dt].$$

Below we show $\mathbf{E}[B_\tau M_\tau] = \mathbf{E}[\int_0^\tau H_t dt]$,

$$\mathbf{E}[B_\tau M_\tau] = \mathbf{E}[(\int_0^\tau 1 dB_t)(\int_0^\tau H_t dB_t)] = \mathbf{E}[\int_0^\tau H_t dt].$$

Hence we have proved (5.3) .

Next we inserting $H_t = 2\left(1 - \Phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right)\right)$ into

$$\mathbf{E}\left(B_\tau - S_1\right)^2 = \mathbf{E}\left(\int_0^\tau \left(1 - 2H_t\right) dt\right) + 1,$$

we see that $V_* = \inf_{\tau \in \mathcal{M}} \mathbf{E}\left(B_\tau - \max_{0 \leqq t \leqq 1} B_t\right)^2$ can be rewritten as

$$V_* = \inf_{\tau \in \mathcal{M}} \mathbf{E}\left(\int_0^\tau F\left(\frac{S_t - B_t}{\sqrt{1-t}}\right) dt\right) + 1, \tag{5.13}$$

where we denote $F(x) = 4\Phi(x) - 3$.

Since $S - B = (S_t - B_t)_{0 \leqq t \leqq 1}$ is a Markov process for which the natural filtration $\mathbb{F}^{S-B}$ coincides with the natural filtration $\mathbb{F}^B$, it follows from general theory of optimal stopping [5] that in (5.13) we need only consider stopping times which are hitting times for $S - B$.

Moreover, recalling that $S - B \overset{\text{law}}{=} |B|$ by Lévy's distributional theorem,

Corollary 5.1.4 (Lévy's distributional theorem)**.** *The two-dimensional process* $(S_t - B_t, S_t)$ *and* (B_t, L_t) *have the same law.*

so $S_t - B_t = B_t$ in law and once more appealing to general theory,

Remark 5.1.5. $S_t - B_t \sim |B_t|$ *is different from* $(S_t - B_t)_{t \geq 0} \sim (|B_t|)_{t \geq 0}$.

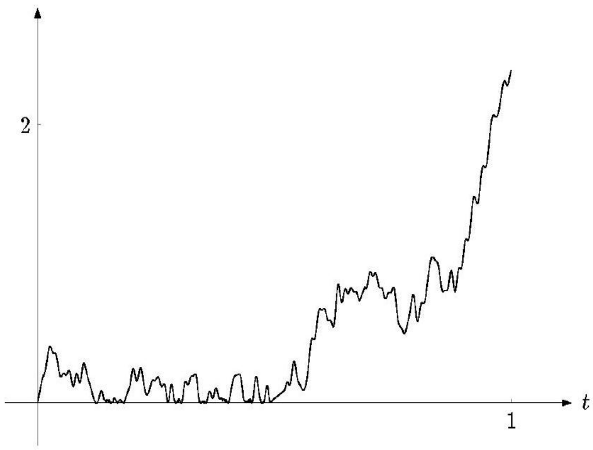

Figure 5.3: A computer drawing of the difference process $(S_t(\omega) - B_t(\omega))_{0 \leqq t \leqq 1}$ from Figures 1 and 2.(The figure is taken from [1, Fig. 3])

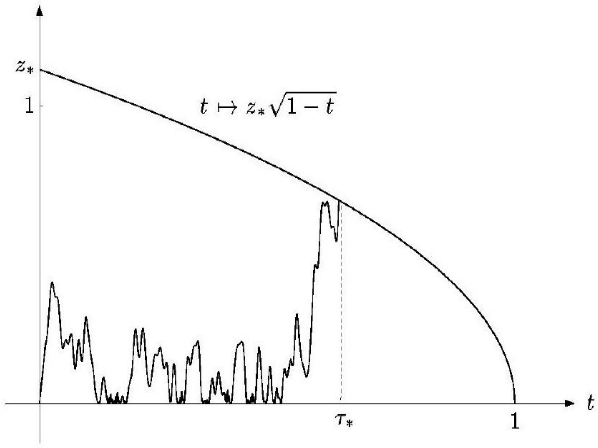

Figure 5.4: A computer drawing of the optimal stopping strategy for the Brownian path from Figures 1-3. It turns out that $\tau_* = 0.62$ in this case ($cf.$ Figure 1).(The figure is taken from [1, Fig. 4])

we see that (5.13) is equivalent to the optimal stopping problem

$$V_* = \inf_{\tau \in \mathcal{M}} \mathbf{E} \left(\int_0^\tau F \left(\frac{|B_t|}{\sqrt{1-t}} \right) dt \right) + 1. \tag{5.14}$$

In our treatment of this problem, we first make use of a deterministic change of time.

5.2 Step Two

Motivated by the form of (5.14), consider the process $Z = (Z_t)_{t \geqq 0}$ given by

$$Z_t = e^t B_{1-e^{-2t}}.$$

By Itô's formula we find that Z is a (strong) solution of the linear stochastic differential equation

$$dZ_t = Z_t dt + \sqrt{2} d\beta_t, \tag{5.15}$$

where $Z_0 = 0$ and the process $\beta = (\beta_t)_{0 \leqq t \leqq 1}$ is given by

$$\beta_t = \frac{1}{\sqrt{2}} \int_0^t e^s dB_{1-e^{-2s}} \tag{5.16}$$

$$= \frac{1}{\sqrt{2}} \int_0^{1-e^{-2t}} \frac{1}{\sqrt{1-s}} dB_s. \tag{5.17}$$

(5.15) holds since the following argument:
by two dimensional Ito formula

$$\begin{aligned} F(X_t) =& F(X_0) + \sum_{i=1}^2 \int_0^t \frac{\partial F}{\partial x_i}(X_s)\ dM_s^i + \sum_{i=1}^2 \int_0^t \frac{\partial F}{\partial x_i}(X_s)\ dA_s^i \\ &+ \frac{1}{2} \sum_{i=1}^2 \sum_{i=1}^2 \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s)\ d\langle M^i, M^j \rangle_s \end{aligned}$$

and by the transformation $u = 1 - e^{-2t}$.

$$Z_t = e^t B_{1-e^{-2t}} = \frac{1}{\sqrt{1-u}} B_u = F(u, B_u).$$

So

$$\begin{aligned} F &= F(0, B_0) + \int_0^u \frac{\partial F}{\partial u}\ ds + \int_0^u \frac{\partial F}{\partial B}\ dB_s + \frac{1}{2} \int_0^u \frac{\partial^2 F}{\partial B^2}\ d\langle M^i, M^j \rangle_s \\ &= 0 + \int_0^u \frac{1}{2}(1-s)^{-2/3} B_s\ ds + \int_0^u \frac{1}{\sqrt{1-s}}\ dB_s + \frac{1}{2} \int_0^u 0\ ds \\ &= \int_0^u \frac{1}{2}(1-s)^{-2/3} B_s\ ds + \int_0^u \frac{1}{\sqrt{1-s}}\ dB_s. \end{aligned}$$

Hence $dF = \frac{1}{2}(1-u)^{-2/3}B_u du + d(\int_0^u \frac{1}{\sqrt{1-s}}\, dB_s)$, then after backward transformation we get:

$$\begin{aligned}
dZ_t &= \frac{1}{2}(e^{-2t})^{-2/3}B_{1-e^{-2t}}d(1-e^{-2t}) + d(\int_0^{1-e^{-2t}} \frac{1}{\sqrt{1-s}}\, dB_s) \\
&= \frac{1}{2}(e^{3t})B_{1-e^{-2t}}d(1-e^{-2t}) + \sqrt{2}d(\frac{1}{\sqrt{2}}\int_0^{1-e^{-2t}} \frac{1}{\sqrt{1-s}}\, dB_s)
\end{aligned}$$

By $\frac{d(1-e^{-2t})}{dt} = 2e^{-2t}$, then

$$\begin{aligned}
dZ_t &= \frac{1}{2}(e^{3t})B_{1-e^{-2t}}2e^{-2t}dt + \sqrt{2}d(\frac{1}{\sqrt{2}}\int_0^{1-e^{-2t}} \frac{1}{\sqrt{1-s}}\, dB_s) \\
&= (e^{t})B_{1-e^{-2t}}dt + \sqrt{2}d(\frac{1}{\sqrt{2}}\int_0^{1-e^{-2t}} \frac{1}{\sqrt{1-s}}\, dB_s) \\
&= Z_t dt + \sqrt{2}d\beta_t
\end{aligned}$$

where $\beta_t = \frac{1}{\sqrt{2}}\int_0^{1-e^{-2t}} \frac{1}{\sqrt{1-s}}\, dB_s$ and Z_t mentioned above $Z_t = e^t B_{1-e^{-2t}}$.

Remark 5.2.1. *We noticed that by this way we do not need to know* (5.16) *to get it done!*

We need the following corollary [3] to get the following statement.

Corollary 5.2.2 (Lévy's characterization theorem)**.** *Let $M = (M^1, \dots, M^d)$ be a continuous process satisfy*

(i) M is adapted to $(\mathscr{F}_t)_{t\geq 0}$,

(ii) $M^i \in \mathcal{M}_{loc}^{C,0}$, and

(iii) $\langle M^i, M^j\rangle = \delta_{ij}\cdot t$,

for all $i, j \in \{1, \dots, d\}$. ***Then*** *M is a standard d-dimensional Brownian motion with respect to $(\mathscr{F}_t)_{t\geq 0}$.*

It can be shown that β is a continuous Gaussian martingale with mean zero and variance equal to t, since

(i) $\mathbf{E}[\int_0^t H_s dB_s] = \mathbf{E}[M_t] = \mathbf{E}[M_0] = \mathbf{E}[\int_0^0 H_s dB_s] = 0$ where $M_t = \int_0^t H_s dB_s$ is a martingale. Hence by the same principal we know that $\mathbf{E}[\beta_t] = 0$.

(ii) Because $(\beta_t^2 - \langle\beta,\beta\rangle_t)_{t\leqq 0}$ is a martingale. So we know $\mathbf{E}(\beta_t^2 - \langle\beta,\beta\rangle_t) = \mathbf{E}[\beta_0^2 - \langle\beta,\beta\rangle_0] = \mathbf{E}[0] = 0$. Hence

$$\mathbf{E}[\beta_t^2] = \mathbf{E}[\langle\beta,\beta\rangle_t]. \tag{5.18}$$

We noticed LHS of (5.18) is $\mathbf{VAR}(\beta_t)$, and

$$RHS = \frac{1}{2}\int_0^{1-e^{-2t}} \frac{1}{(\sqrt{1-s})^2}\, ds = t,$$

hence $\mathbf{VAR}(\beta_t) = t$.

It follows by Lévy's characterization theorem that β is a standard Brownian motion.
Now we have

$$dZ_t = Z_t dt + \sqrt{2} dB_t. \tag{5.19}$$

Corollary 5.2.3. *We present the standard calculation for a one-dimensional homogenous diffusion which has the following SDE form:*

$$dX_t = b\left(X_t\right) dt + \sigma\left(X_t\right) dB_t, \quad X_0 = x.$$

Here B_t is the standard Brownian motion. For convenience, we will assume that b and σ are bounded continuous functions (or we may need functions in C^2).
The infinitesimal generator in this case is

$$Af(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x).$$

We get this using Itô's lemma.

We thus may conclude that Z is a diffusion process with the infinitesimal generator given by

$$\mathbb{L}_Z = z\frac{d}{dz} + \frac{d^2}{dz^2}.$$

Substituting $t = 1 - e^{-2s}$ in $V_* = \inf_{\tau\in\mathcal{M}} \mathbf{E}\left(\int_0^\tau F\left(\frac{|B_t|}{\sqrt{1-t}}\right) dt\right) + 1$ and using $Z_t = e^t B_{1-e^{-2t}}$, we obtain

$$V_* = 2 \inf_{\tau\in\mathcal{M}} \mathbf{E}\left(\int_0^{\sigma_\tau} e^{-2s} F\left(|Z_s|\right) ds\right) + 1 \tag{5.20}$$

upon setting $\sigma_\tau = \log(1/\sqrt{1-\tau})$.
(5.20) holds since the simple method of substitution.

It is clear from $Z_t = e^t B_{1-e^{-2t}}$ that τ is a stopping time with respect to $\mathbb{F}^B$ if and only if σ_τ is a stopping time with respect to $\mathbb{F}^Z$. This shows that our initial problem (1.1) reduces to solve

$$W_* = \inf_{\sigma} \mathbf{E} \left(\int_0^{\sigma} e^{-2s} F\left(|Z_s|\right) ds \right), \tag{5.21}$$

where the infimum is taken over all $\mathbb{F}^Z$-stopping times σ with values in $[0, \infty]$. This problem belongs to the general theory of optimal stopping for time-homogeneous Markov processes.[5]

5.3 Step Three

To calculate (5.21) define

$$W_*(z) = \inf_{\sigma} \mathbf{E}_z \left(\int_0^{\sigma} e^{-2s} F\left(|Z_s|\right) ds \right) \tag{5.22}$$

for $z \in \mathbf{R}$, where $\mathbf{E}_z$ is an averaging with respect to the measure $\mathbf{P}_z$ of the process $Z = (Z_t)_{t \geqq 0}$ with differential $dZ_t = Z_t dt + \sqrt{2} dB_t$, $Z_0 = z$, and the infimum is taken as above. General theory combined with basic properties of the map $z \mapsto F(|z|)$ prompts that the stopping time

$$\sigma_* = \inf \{ t > 0 : |Z_t| \geqq z_* \}$$

should be optimal in (5.22), where $z_* > 0$ is a constant to be found.

Remark 5.3.1. *The above general theory is related to Hamilton-Jacobi-Bellman (HJB) equation. Unfortunately this is beyond my horizon of knowledge. However I can give some explanation of the HJB equation.*

The Hamilton-Jacobi-Bellman (HJB) equation is a fundamental part of optimal control theory. It is a partial differential equation which provides a necessary and sufficient condition for optimality of a control with respect to a certain criterion. The HJB equation is named after William Rowan Hamilton, Carl Gustav Jacob Jacobi, and Richard Bellman.

In its most general form, the HJB equation is nonlinear and can be very challenging to solve. However, it is extremely powerful because it allows for the determination of an optimal control policy by solving a single equation, as opposed to the potentially infinite number of equations that might be required in other situations.

In a more concrete way, you can think of the HJB equation as providing a way to find the best path from a starting point to an end point, where "best" is defined in terms of some criterion (like minimizing cost or maximizing reward), and the "path" is defined over a potentially complex and high-dimensional space. The HJB equation provides a mathematical framework to handle this in a systematic way.

To determine z_* and compute the value function $z \mapsto W_*(z)$ in (5.22), it is a matter of routine to formulate the following free-boundary (Stefan) problem:

$$(\mathbb{L}_Z - 2)\, W(z) = -F(|z|) \quad \text{for } z \in (-z_*, z_*)\,, \tag{5.23}$$
$$W\,(\pm z_*) = 0 \quad (\text{ instantaneous stopping}), \tag{5.24}$$
$$W'\,(\pm z_*) = 0 \quad (\text{ smooth fit}), \tag{5.25}$$

where $\mathbb{L}_Z$ is given by $\mathbb{L}_Z = z\frac{d}{dz} + \frac{d^2}{dz^2}$ above.

(5.23)(5.24) and (5.25) are "assumptions" or in other words necessary conditions for our question. (5.23) holds since the following corollary.

Corollary 5.3.2. *(Killed version) Given a continuous function $L : C \to \mathbb{R}$ consider*

$$F(x) = \mathrm{E}_x \int_0^{\tau_D} e^{-\lambda_t} L\,(X_t)\, dt$$

for $x \in E$ where $\lambda = (\lambda_t)_{t \geq 0}$ is given by

$$\lambda_t = \int_0^t \lambda\,(X_s)\, ds$$

for a measurable (continuous) function $\lambda : E \to \mathbb{R}_+$. The function F solves the (killed) Dirichlet/Poisson problem:

$$\mathbb{L}_X F = \lambda F - L \text{ in } C,$$
$$F|_{\partial C} = 0.$$

In our case, $\lambda_t = \int_0^t \lambda(X_s)ds = 2t$ which give us that $\lambda(X_s) = 2$ i.e. $\lambda = 2$, and $L(x) = F(|x|)$.

We shall extend the solution of (5.23)(5.24) and (5.25) by setting its value equal to 0 for $z \notin (-z_*, z_*)$, and thus the map so obtained will be C^2 everywhere on $\mathbf{R}$ except at $-z_*$ and z_*, where it is C^1.

Inserting $\mathbb{L}_Z$ into (5.23) leads to the following equation:

$$W''(z) + zW'(z) - 2W(z) = -F(|z|) \tag{5.26}$$

for $z \in (-z_*, z_*)$.

The form of (5.15) and (5.21) indicates that $z \mapsto W_*(z)$ should be even.

Thus we shall additionally impose

$$W'(0) = 0$$

since

Corollary 5.3.3. *The derivative of an even function is always an odd function.*

Corollary 5.3.4. *Any odd function is zero at $x = 0$.*

and consider (5.26) only for $z \in [0, z_*)$.

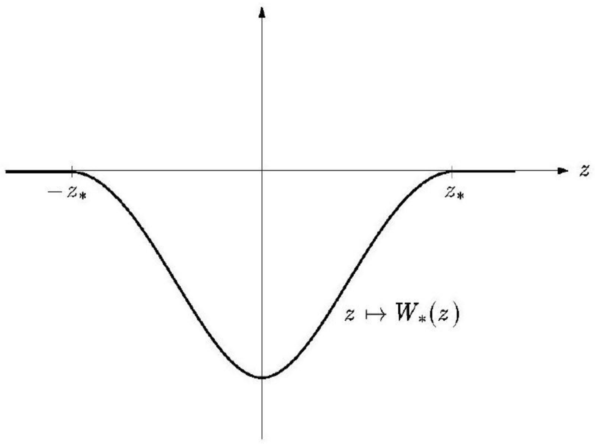

Figure 5.5: A computer drawing of the map (5.22). The smooth fit (5.25) holds at $-z_*$ and z_*. (The figure is taken from [1, Fig. 5])

The general solution of (5.26) for $z \geqq 0$ is given by

$$W(z) = C_1\left(1+z^2\right) + C_2\left(z\varphi(z) + \left(1+z^2\right)\Phi(z)\right) + 2\Phi(z) - \frac{3}{2}.$$

The three conditions $W(z_*) = W'(z_*) = W'(0) = 0$ uniquely determine constants C_1, C_2, and z_*; it is easily verified that $C_1 = \Phi(z_*), C_2 = -1$, and z_* is a unique root of $4\Phi(z_*) - 2z_*\varphi(z_*) - 3 = 0$.

Remark 5.3.5. *Recalling that $S - B \overset{law}{=} |B|$ (Prerequisite) we see that τ_* is identically distributed as the stopping time $\tilde{\tau} = \inf\left\{t > 0 : |B_t| = z_*\sqrt{1-t}\right\}$. This implies*

$$\mathbf{E}\tau_* = \mathbf{E}\widetilde{\tau} = \mathbf{E}\left|B_{\widetilde{\tau}}\right|^2 = z_*^2\mathbf{E}(1-\widetilde{\tau}) = z_*^2\left(1 - \mathbf{E}\tau_*\right),$$

and hence we obtain

$$\mathbf{E}\tau_* = \frac{z_*^2}{1+z_*^2} = 0.55\ldots.$$

Moreover, using the fact that $\left(B_t^4 - 6tB_t^2 + 3t^2\right)_{t\geqq 0}$ is a martingale, similar arguments show that

$$\mathbf{E}\tau_*^2 = \frac{z_*^6 + 5z_*^4}{(1+z_*^2)(3+6z_*^2+z_*^4)} = 0.36\ldots.$$

From (2.29) and (2.30) we find

$$\mathbf{D}\tau_* = \frac{2z_*^4}{(1+z_*^2)^2(3+6z_*^2+z_*^4)} = 0.05\ldots$$

where D is another notation of Variance.

Inserting this back into (5.26), we obtain the following candidate for the value (5.22) :

$$W(z) = \Phi(z_*)\left(1+z^2\right) - z\varphi(z) + \left(1-z^2\right)\Phi(z) - \frac{3}{2},$$

when $z \in [0, z_*]$, upon extending it to an even function on $\mathbf{R}$ as indicated above (see Figure 5.5).

To verify that this solution $z \mapsto W(z)$ coincides with the value function (5.22), and that σ_* from $\sigma_* = \inf\{t > 0 : |Z_t| \geqq z_*\}$ is an optimal stopping time, we shall note that $z \mapsto W(z)$ is C^2 everywhere but at $\pm z_*$, where it is C^1. Thus, by the Itô-Tanaka formula we find

$$\begin{aligned} e^{-2t}W(Z_t) =& W(Z_0) + \int_0^t e^{-2s}\left(\mathbb{L}_Z W(Z_s) - 2W(Z_s)\right)ds \\ &+ \sqrt{2}\int_0^t e^{-2s}W'(Z_s)\,d\beta_s. \end{aligned} \tag{5.27}$$

Hence, by (5.26) and the fact that $\mathbb{L}_Z W(z) - 2W(z) = 0 > -F(|z|)$ for $z \notin [-z_*, z_*]$, upon extending W'' to $\pm z_*$ as we please and using the fact that the Lebesgue measure of those $t > 0$ for which $Z_t = \pm z_*$ is zero, we get

$$e^{-2t}W(Z_t) \geqq W(Z_0) - \int_0^t e^{-2s}F(|Z_s|)\,ds + M_t, \tag{5.28}$$

where $M = (M_t)_{t\geqq 0}$ is a continuous local martingale given by

$$M_t = \sqrt{2}\int_0^t e^{-2s}W'(Z_s)\,d\beta_s.$$

Further, using the fact that $W(z) \leqq 0$ for all z, a simple application of the optional sampling theorem in the stopped version of (5.28) under $\mathbf{P}_z$ shows that $W_*(z) \geqq W(z)$ for all z. To prove equality one may note that the passage from (5.27) to (5.28) also yields

$$0 = W(Z_0) - \int_0^{\sigma_*} e^{-2s}F(|Z_s|)\,ds + M_{\sigma_*}$$

upon using (5.23) and (5.24). Since clearly $\mathbf{E}_z\sigma_* < \infty$ and thus $\mathbf{E}_z\sqrt{\sigma_*} < \infty$ as well, and $z \mapsto W'(z)$ is bounded on $[-z_*, z_*]$, we can again apply the optional sampling theorem and conclude that $\mathbf{E}_z M_{\sigma_*} = 0$. Taking the expectation under $\mathbf{P}_z$ on both sides in $4\Phi(z_*) - 2z_*\varphi(z_*) - 3 = 0$ enables one to conclude that $W_*(z) = W(z)$ for all z, and the proof of the claim is complete.

From (5.20)-(5.22) and $W(z) = C_1(1+z^2) + C_2\left(z\varphi(z) + (1+z^2)\Phi(z)\right) + 2\Phi(z) - \frac{3}{2}$ we find that $V_* = 2W_*(0) + 1 = 2(\Phi(z_*) - 1) + 1 = 2\Phi(z_*) - 1$. This establishes (4.1), which is $V_* = 2\Phi(z_*) - 1 = 0.73\ldots$.

Remark 5.3.6. *For the sake of comparison with (4.1) and the following case it is interesting to note that*

$$V_0 = \inf_{0 \leqq t \leqq 1} \mathbf{E}\left(B_t - \max_{0 \leqq s \leqq 1} B_s\right)^2 = \frac{1}{\pi} + \frac{1}{2} = 0.81\ldots \tag{5.29}$$

with the infimum being attained at $t = \frac{1}{2}$. *For this, recall from (2.8) and (2.6) that*

$$\mathbf{E}(B_t - S_1)^2 = \mathbf{E}\int_0^t F\left(\frac{S_s - B_s}{\sqrt{1-s}}\right) ds + 1,$$

where $F(x) = 4\Phi(x) - 3$. *Further, using the fact that* $S - B \overset{law}{=} |B|$, *elementary calculations show*

$$\begin{aligned}
\mathbf{E}(B_t - S_1)^2 &= 4\int_0^t \mathbf{E}\left(\Phi\left(\frac{|B_s|}{\sqrt{1-s}}\right)\right) ds - 3t + 1 &(5.30)\\
&= 4\int_0^t \left(1 - \frac{1}{\pi}\arctan\sqrt{\frac{1-s}{s}}\right) ds - 3t + 1 &(5.31)\\
&= -\frac{4}{\pi}\left(t\arctan\sqrt{\frac{1-t}{t}} + \frac{1}{2}\arctan\sqrt{\frac{t}{1-t}} - \frac{1}{2}\sqrt{t(1-t)}\right) + t + 1. &(5.32)
\end{aligned}$$

(5.31) *holds since*

$$
\begin{aligned}
\mathbf{E}\left(\Phi\left(\frac{|B_s|}{\sqrt{1-s}}\right)\right) &= \int_{-\infty}^{+\infty}\left[\left(\int_{-\infty}^{\frac{|y|}{\sqrt{1-s}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right)\frac{1}{\sqrt{2\pi s}}e^{-\frac{1}{2}\frac{y^2}{s}}\right]dy \\
&= \frac{1}{2\pi\sqrt{s}}\int_{-\infty}^{+\infty}[(\int_{-\infty}^{\frac{|y|}{\sqrt{1-s}}}e^{-\frac{z^2}{2}}dz)e^{-\frac{y^2}{2s}}]dy \\
&\overset{y=\sqrt{s}t}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\left(\int_{-\infty}^{\frac{\sqrt{s}}{\sqrt{1-s}}|t|}e^{-\frac{z^2}{2}}dz\right)e^{-\frac{t^2}{2}}\right]dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{0}\left(\int_{-\infty}^{-\frac{\sqrt{s}t}{\sqrt{1-s}}}e^{-\frac{z^2}{2}}dz\right)e^{-\frac{t^2}{2}}dt+\frac{1}{2\pi}\int_{0}^{+\infty}\left(\int_{-\infty}^{\frac{\sqrt{s}}{\sqrt{1-s}}t}e^{-\frac{z^2}{2}}dz\right)e^{-\frac{t^2}{2}}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\left(\int_{-\infty}^{+\infty}e^{-\frac{z^2}{2}}dz\right)e^{-\frac{t^2}{2}}\right]dt-\frac{1}{2\pi}\iint_D e^{-\frac{z^2}{2}-\frac{t^2}{2}}dzdt \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty}e^{-\frac{z^2}{2}}dz\right)^2-\frac{1}{2\pi}\left(\int_{0}^{+\infty}e^{-\frac{r^2}{2}}\cdot rdr\right)\left(\int_{\mathrm{artan}\sqrt{\frac{s}{1-s}}}^{\frac{\pi}{2}}d\theta+\int_{\frac{\pi}{2}}^{\pi-\arctan\sqrt{\frac{s}{1-s}}}d\theta\right) \\
&= 1-\frac{1}{2\pi}\left(\int_{0}^{+\infty}e^{-\frac{r^2}{2}}d\frac{r^2}{2}\right)\left(\pi-2arctan\sqrt{\frac{s}{1-s}}\right) \\
&= 1-\frac{1}{2\pi}(1)2\left(\frac{\pi}{2}-\arctan\sqrt{\frac{s}{1-s}}\right) \\
&= 1-\frac{1}{\pi}\arctan\sqrt{\frac{s}{1-s}},
\end{aligned}
$$

where D looks like the following graph.

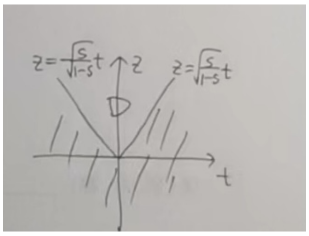

Figure 5.6: D

Hence (2.32) is easily verified by standard means. This is because that $\mathbf{E}\left(B_t-S_1\right)^2$ *is a deterministic function of t. Although it is complicated, we can solve it by taking first and second derivatives, and we omit the detailed process here.*

Remark 5.3.7. *Is the case $V_0 > V_*$ normal?*
Yes, it is. Since $\tau = t$ is a special case of the general stopping time. So in V_0 we are considering a smaller set, and we know that the inf of a smaller set is smaller. That explains the thing.

Transforming σ_* from $\sigma_* = \inf\{t > 0 : |Z_t| \geqq z_*\}$ back to the initial problem via the equivalence of (5.13), (5.14), and (5.20), we see that τ_* from $\tau_* = \inf\left\{0 \leqq t \leqq 1 \mid S_t - B_t \geqq z_*\sqrt{1-t}\right\}$ is optimal.

So far the proof is complete.

Remark 5.3.8. *The case of a general time interval $[0, T]$ easily reduces to the case of the unit time interval treated above by using the scaling property of Brownian motion. This implies*

$$\inf_{0 \leqq \tau \leqq T} \mathbf{E}\left(B_\tau - \max_{0 \leqq t \leqq T} B_t\right)^2 = T \inf_{0 \leqq \tau \leqq 1} \mathbf{E}\left(B_\tau - \max_{0 \leqq t \leqq 1} B_t\right)^2. \tag{5.33}$$

Remark 5.3.9. *The above equation holds since $B_{tT} = \sqrt{T}B_t$ in distribution. In detail we have:*

$$\begin{aligned}
\inf_{0 \leqq \tau \leqq T} \mathbf{E}\left(B_\tau - \max_{0 \leqq t \leqq T} B_t\right)^2 &= \inf_{0 \leqq \frac{\tau}{T} \leqq 1} \mathbf{E}\left(B_{\frac{\tau}{T}T} - \max_{0 \leqq \frac{t}{T} \leqq 1} B_{\frac{t}{T}T}\right)^2 \\
&= \inf_{0 \leqq \sigma \leqq 1} \mathbf{E}\left(B_{\sigma T} - \max_{0 \leqq s \leqq 1} B_{sT}\right)^2 \\
&= \inf_{0 \leqq \sigma \leqq 1} \mathbf{E}\left(\sqrt{T}B_\sigma - \max_{0 \leqq s \leqq 1} \sqrt{T}B_s\right)^2 \\
&= T \inf_{0 \leqq \sigma \leqq 1} \mathbf{E}\left(B_\sigma - \max_{0 \leqq s \leqq 1} B_s\right)^2.
\end{aligned}$$

Which further equals $T\left(2\Phi\left(z_*\right) - 1\ \right)$ by $V_* = 2\Phi\left(z_*\right) - 1 = 0.73\ldots$. Moreover, the same argument shows that the optimal stopping time in $V_0 = 0.81...$ is given by

$$\tau_* = \inf\left\{0 \leqq t \leqq T \mid S_t - B_t \geqq z_*\sqrt{T-t}\right\},$$

where z_* is the same as in the main theorem.

Remark 5.3.10. *The maximum functional in the argument above can be replaced by other functionals. The integral functional is an example which turns out to have a trivial solution.*

Here we swap the $max_{0 \leqq t \leqq 1} B_t$ i.e. S_1 to $I_1 = \int_0^1 B_t dt$.

Having $I_1 = \int_0^1 B_t dt$, we find by Itô's formula that the following analogue of the conclusion of the representation theorem is valid:

$$I_1 = \int_0^1 (1-t) dB_t \tag{5.34}$$

where $H_t = 1 - t$ and $a = 0$.

In order to prove that (5.34) holds, we need to choose $F(x)$ carefully. But we also noticed that $1 - t$ can not be expressed by $F(x)$ where $x = B_t$. The way out is too use the multidimensional case, which is $F = F(t, x)$ where $x = B_t$.

$$\begin{aligned}
F(t, B_t) = F(0, B_0) &+ \int_0^t \frac{\partial F}{\partial t}\, ds + \int_0^t \frac{\partial F}{\partial x}\, dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial t^2}\, d\langle t, t\rangle_s \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}\, d\langle B, B\rangle_s + \int_0^t \frac{\partial^2 F}{\partial t \partial x}\, d\langle t, B\rangle_s \\
&= F(0, B_0) + \int_0^t \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\, ds + \int_0^t \frac{\partial F}{\partial x}\, dB_s.
\end{aligned}$$

Hence by observation here we choose $F(t, x) = (1 - t)x$, then

$$F(t, B_t) = F(0, B_0) + \int_0^t (-B_s)\, ds + \int_0^t 1 - s\, dB_s.$$

Notice that if $t = 1$, we have $F(1, B_1) = (1 - 1)B_1 = 0$ and $F(0, B_0) = (1 - 0)B_0 = 0$. Then by the equation above, we have

$$\begin{aligned}
0 &= \int_0^1 (-B_t)\, dt + \int_0^1 1 - t\, dB_t \\
\int_0^1 B_t\, dt &= \int_0^1 1 - t\, dB_t.
\end{aligned}$$

Hence (5.34) holds.

Denoting

$$M_t = \int_0^t (1-s) dB_s,$$

it follows as in (5.1) to (5.3) that

$$\mathbf{E}\left(B_\tau - I_1\right)^2 = \mathbf{E}\left|B_\tau\right|^2 - 2\mathbf{E}\left(B_\tau M_1\right) + \mathbf{E}\left|I_1\right|^2 = \mathbf{E}\left(\tau^2 - \tau\right) + \frac{1}{3}$$

for all $\tau \in \mathcal{M}$.

We now need to consider the question $\inf_{\tau\in\mathcal{M}} \mathbf{E}\left(\tau^2-\tau\right)$, *we show this by the following two ways.*

Method One

Here, $\mathcal{M}$ *represents the set of all possible random variables that can be defined on the given probability space. We want to find the infimum of the expected value of the expression* $\tau^2-\tau$ *over all such random variables* τ.

To solve this problem, we can start by using the fact that for any random variable X, *we have* $\mathrm{Var}(X) = \mathbf{E}(X^2) - (\mathbf{E}(X))^2$. *Rearranging, we get* $\mathbf{E}(X^2) = \mathrm{Var}(X) + (\mathbf{E}(X))^2$.

Now, let's apply this to the expression $\tau^2-\tau$. *We have:*

$$\begin{aligned}\mathbf{E}(\tau^2-\tau) &= \mathbf{E}(\tau^2) - \mathbf{E}(\tau) && (5.35)\\ &= \mathrm{Var}(\tau) + (\mathbf{E}(\tau))^2 - \mathbf{E}(\tau) && (5.36)\\ &= \mathrm{Var}(\tau) + (\mathbf{E}(\tau)-1/2)^2 - 1/4 \qquad \geq -1/4, && (5.37)\end{aligned}$$

where the last inequality follows from the fact that the variance is always non-negative.

Therefore, we have $\inf_{\tau\in\mathcal{M}} \mathbf{E}(\tau^2-\tau) \geq -1/4$, *and this infimum is achieved when* τ *is a constant random variable taking the value* $1/2$ *with probability* 1, *which means* $\tau = 1/2$ *almost surely. To see this, note that in this case, we have* $\mathbf{E}(\tau) = 1/2$ *and* $\mathrm{Var}(\tau) = 0$, *so the expression* $\tau^2-\tau$ *simplifies to* $1/4 - 1/2 = -1/4$.

Method Two

We do the following transformation,

$$\begin{aligned}\inf_{\tau\in\mathcal{M}} \mathbf{E}\left(\tau^2-\tau\right) &= \inf_{\tau\in\mathcal{M}} \mathbf{E}[(\tau-\frac{1}{2})^2 - \frac{1}{4}]\\ &= \inf_{\tau\in\mathcal{M}} \mathbf{E}[(\tau-\frac{1}{2})^2] - \frac{1}{4}\\ &= -\frac{1}{4}.\end{aligned}$$

Since $\inf_{\tau\in\mathcal{M}} \mathbf{E}[(\tau-\frac{1}{2})^2] = 0$, *which holds by the following inequality,*

$$0 = \inf_{\tau\in\mathcal{M}}(0) \leqq \inf_{\tau\in\mathcal{M}} \mathbf{E}[(\tau-\frac{1}{2})^2] \leqq \mathbf{E}[(\tau-\frac{1}{2})^2]_{\tau=\frac{1}{2}} = 0, \tag{5.38}$$

where the left part of (5.38) *holds because of the following corollary.*

Corollary 5.3.11. *If* $A(x) \geq C$, *then* $inf_x A(x) \geq C$.

Therefore, the infimum of the expected value of the expression $\tau^2 - \tau$ *over all random variables* τ *is* $-1/4$*. Hence we see that*

$$\inf_{\tau \in \mathcal{M}} \mathbf{E} \left(B_\tau - I_1\right)^2 = \frac{1}{12} = 0.08\ldots$$

and that the infimum is attained at $\tau_* \equiv \frac{1}{2}$*.*

Remark 5.3.12. *From a mathematical statistics point of view, the "estimator"* B_τ *of* S_1 *is biased, since* $\mathbf{E}B_\tau = 0$ *for all* $0 \leqq \tau \leqq 1$ *but* $\mathbf{E}S_1 \neq 0$*. It is thus desirable to consider the values*

$$\widetilde{V}_* = \inf_{a \in \mathbf{R}, \tau \in \mathcal{M}} \mathbf{E} \left(a + B_\tau - S_1\right)^2, \tag{5.39}$$

$$\widetilde{V}_0 = \inf_{a \in \mathbf{R}, 0 \leqq t \leqq 1} \mathbf{E} \left(a + B_t - S_1\right)^2, \tag{5.40}$$

and compare them with the values from (1.1) and $\mathbf{E}\tau_* = \frac{z_*^2}{1+z_*^2} = 0.55\ldots$*. However, by using that* $\mathbf{E}B_\tau = 0$ *we also find at once that* $a_* = \mathbf{E}S_1$ *is optimal in (5.39) and (5.40) with* $\widetilde{V}_* = V_* - 2/\pi = 0.09\ldots$ *and* $\widetilde{V}_0 = V_0 - 2/\pi = 0.18\ldots$*.*

Chapter 6

Application

6.1 Financial derivatives pricing

Optimal stopping and free boundary problems are used to price financial derivatives such as American options, which allow the holder to exercise their option at any time up until expiration. The value of these options depends on the optimal stopping time or free boundary, which represents the optimal time to exercise the option.

6.2 Investment decisions

Optimal stopping is also used to determine the optimal time to invest in a project or asset. For example, a firm may need to decide when to invest in a new production facility or when to sell a current one. Optimal stopping can help determine the optimal time to make these decisions, based on factors such as the expected future value of the asset and the cost of delaying the investment.

6.3 Stochastic control

Free boundary problems are used in stochastic control theory to describe optimal control problems where the optimal control strategy depends on the state of the system. For example, in engineering, free boundary problems can be used to model the optimal design of a system with uncertain parameters, such as a chemical plant or a power grid.

6.4 Physical systems

Optimal stopping and free boundary problems are also used to model physical systems, such as the optimal stopping of random walks or the free boundary of a phase transition. These problems are important in the study of statistical physics and other areas of physics.

Overall, optimal stopping and free boundary problems are important tools for modeling and analyzing decision-making processes in a wide range of fields.

Chapter 7

Conclusion

In this dual-phase project, our primary objective was to address a specific stochastic control problem. This problem was tackled by transforming a non-Markovian element, which presents a challenge, into something more manageable that aligns with established solutions in the field. The transformation process, although complex, was navigated successfully with guidance from Professor Goran's research.

My role involved detailing the intricacies of this transformation journey, justifying equalities, and explaining the application of prerequisite theorems under specific circumstances. I also explored alternative methodologies to derive statements from the original paper, demonstrating that there are indeed multiple approaches leading to the same result ("All roads lead to Rome"). Additionally, my responsibilities included understanding and studying the applications of optimal stopping and free boundary problems.

Upon completion of this intensive project, I gained substantial knowledge in stochastic analysis and a deeper understanding of how theoretical constructs taught in lectures are applied to solve complex problems. Furthermore, it provided an intriguing glimpse into the captivating world of stochastic mathematics and probability, a discovery that goes beyond academic learning.

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