Sequential Elimination Contests with All-Pay Auctions

Yanwei Sun

Imperial College Business School

with **Fupeng Sun** (Imperial), **Chiwei Yan** (Berkeley), **Li Jin** (UM-SJTU)

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- *NSF Proposals:* National Science Foundation in the US filters potential low-quality submissions without going through the costly formal review process.

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- *Start-up competitions*: Y Combinator, a top incubator in the San Francisco Bay area, invite roughly top 7% of the applicants who fill out a short application form for on-site interviews to compete for a $500,000 funding.

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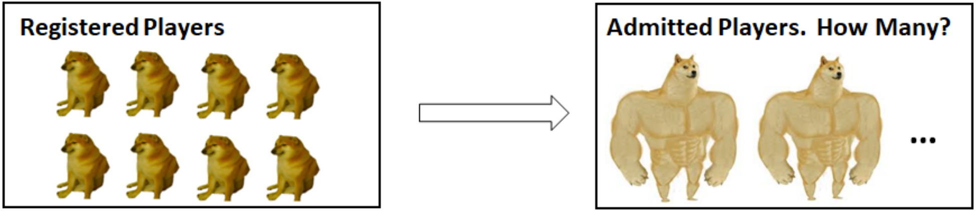
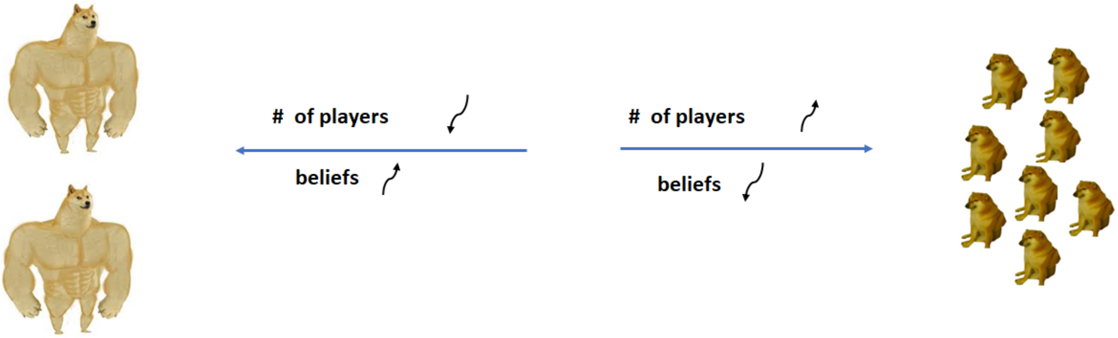
How many admitted players should be to maximize their efforts?



Number vs. Beliefs?

The contest designer might want to increase the competition, which comes from two parts:

- number of admitted players
- their beliefs about opponents' abilities



Model Setup

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Stage 4: Players update their beliefs and make their efforts

- admitted player i's decision variable: e_i
- ex-post utility of admitted player i:

$$u_i(e_i, e_{-i}) = V_\ell \mathbb{1}\{e_i \text{ is } \ell^{\text{th}} \text{ highest among } e\} - \frac{g(e_i)}{a_i}$$

Research Questions

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What is the optimal (n_2, V) in terms of

(i) expected *highest* equilibrium efforts?

(ii) expected *total* equilibrium efforts?

Bayesian Nash Equilibrium

A BNE is a tuple of *posterior beliefs* (densities) $[\beta_i(a_{-i} \mid s, a_i) : i \in \mathcal{I}]$ and *strategies* $[b_i(\cdot) : i \in \mathcal{I}]$ that satisfies the following conditions:

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(ii) (*Sequential Rationality*) For every admitted player $i \in \mathcal{I}$,

$$b_i(a_i) \in \arg\max_{e_i} \sum_{\ell=1}^{n_2} V_\ell P_{i\ell}(e_i) - \frac{g(e_i)}{a_i},$$

where $P_{i\ell}(e_i)$ is the probability that e_i ranks ℓ^{th} highest in

$$\{b_j(A_j) : j \in \mathcal{I}_{-i}\} \cup \{e_i\}$$

- the random variables $A_{-i} \sim \beta_i(a_{-i} \mid s, a_i)$

Equilibrium Analysis

Posterior Belief: Example $n_2 = 2$

When $n_2 = 2$, for any player $i \neq j \in \mathcal{I}$, player i's posterior belief (PDF) about player j's ability is

$$\beta_i\left(a_j \mid s, a_i\right) = \begin{cases} \frac{f(a_j)F^{n_1-2}(a_j)}{\frac{F^{n_1-1}(a_i)}{n_1-1}+1-F(a_i)}, & 0 < a_j < a_i, \\ \frac{f(a_j)}{\frac{F^{n_1-1}(a_i)}{n_1-1}+1-F(a_i)}, & a_i < a_j < 1 \end{cases}$$

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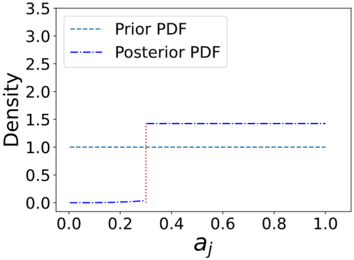
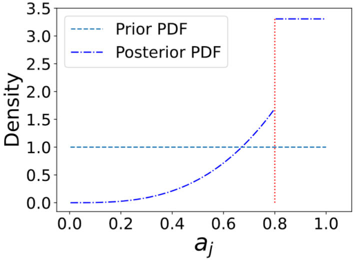
Remark 1: beliefs are *asymmetric* and <u>*private*</u>

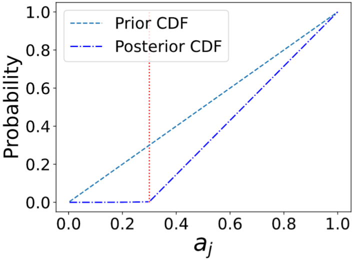
Example $n_1 = 5, n_2 = 2$ with uniform priors



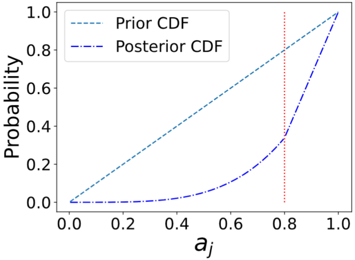
(a) PDF: $a_i = 0.3$



(b) PDF: $a_i = 0.8$



(c) CDF: $a_i = 0.3$



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Posterior Beliefs: General n_2

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Remark 3: One player's marginal posterior beliefs about different players' abilities can be correlated. Formally, $\beta_i \left(a_{-i} \mid s, a_i \right) \neq \prod_{j \in \mathcal{I}_{-i}} \beta_i \left(a_j \mid s, a_i \right)$ in general.

Equilibrium Strategy

Proposition

There exists a *unique symmetric* Bayesian Nash equilibrium strategy.

Optimal Mechanism

Equilibrium Dominance

Theorem (Equilibrium Dominance)

Fix any prize structure V, *all* admitted players exert weakly lower efforts compared with those under a regular one-round contest that admits all players.

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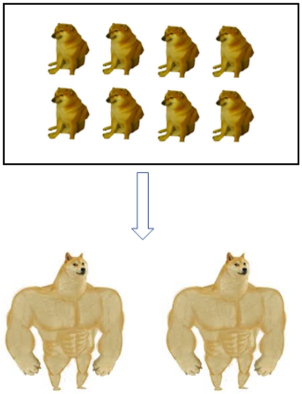
Proposition (Non-monotone)

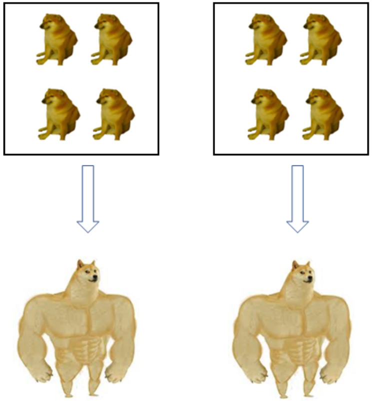
The equilibrium effort of admitted player i with ability a_i is increasing in n_2 when $n_2 \in [\lfloor (n_1+1)/2 \rfloor + 1, n_1]$, and is in general *not monotone* in n_2 when $n_2 \in [2, \lfloor (n_1+1)/2 \rfloor + 1]$.

Comparison: Sequential Elimination vs. Sub-elimination

Our setting: sequential elimination



Sub-elimination (Moldovanu and Sela 2006)



Admitting best players could backfire!

Limited Resources

Remark: When the designer cannot admit all registered n_1 players, numerical results show that the *optimal n_2 lies at the corner points* — either admitting 2 players or the maximum allowed.

Example: Expected Total Efforts

Prior distribution: $F(x) = x^{\theta}$; Cost function: $g(x) = x^5$

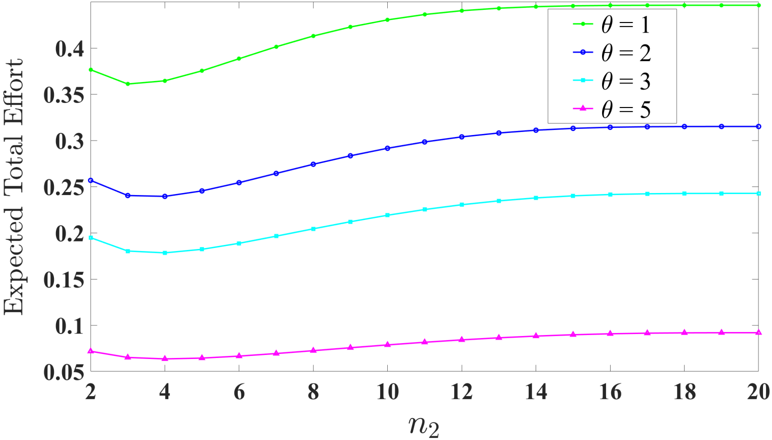
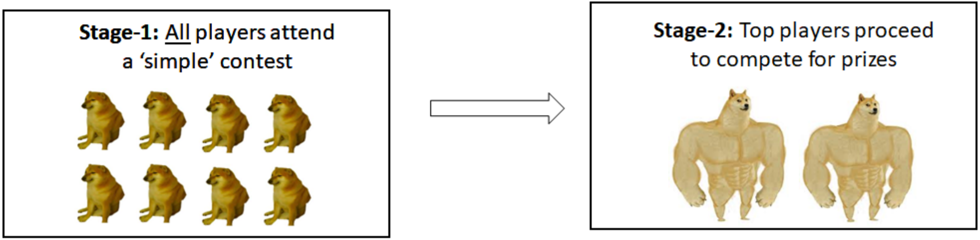


Figure: Expected Total Efforts ($n_1 = 20$)

Two-stage SEC

Two-stage Sequential Elimination Contests

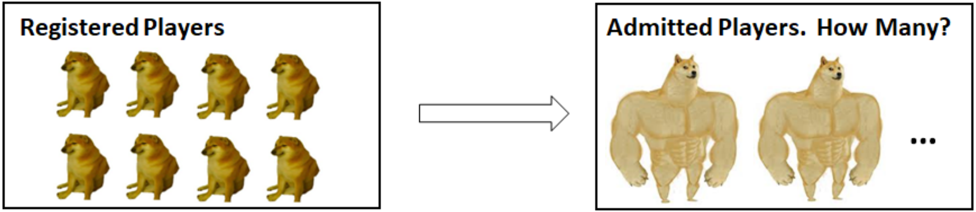
If the designer does not know the ranking of all registered players' abilities,



Proposition

There does *not* exist a symmetric and monotone Perfect Bayesian Equilibrium (PBE) in any two-stage sequential elimination contest.

Main Take-away



- **Equilibrium Dominance:** Admit all registers if you can.
- Otherwise, admit two players or the maximum allowed

References

Chawla, Shuchi, Jason D Hartline, and Balasubramanian Sivan (2019). “Optimal crowdsourcing contests”. In: *Games and Economic Behavior* 113, pp. 80–96.

Moldovanu, Benny and Aner Sela (2001). “The optimal allocation of prizes in contests”. In: *American Economic Review* 91.3, pp. 542–558.

Equilibrium Strategy

$$b(a_i) = g^{-1}\Bigg(\sum_{\ell=1}^{n_2-1} V_\ell \int_0^{a_i} \frac{x}{J(F(x), n_1, n_2)} \big(dF_{(\ell, n_1-1)}(x) - dF_{(\ell-1, n_1-1)}(x)\big)$$

$$- V_{n_2} \int_0^{a_i} \frac{(n_2-1)x}{I(F(x), n_1, n_2)} \big(1 - F(x)\big)^{n_2-2} dF(x) \Bigg),$$

where

$$J(x, n_1, n_2) := \binom{n_1 - 1}{n_2 - 1} \cdot I(x, n_1, n_2)$$

$$I(x, n_1, n_2) := \big(1 - x\big)^{n_2-1} + (n_2 - 1) \cdot B\big(x, n_1 - n_2 + 1, n_2 - 1\big)$$

$$B(x, p, q) := \int_0^x t^{p-1}(1-t)^{q-1} dt$$

Posterior Belief

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$$\beta_i(a_{-i} \mid s, a_i) = \begin{cases} \frac{1}{I(F(a_i), n_1, n_2)} \prod_{j \in \mathcal{I}_{-i}} f(a_j), & a_i < \min_{j \in \mathcal{I}_{-i}} a_j, \\ \frac{F^{n_1 - n_2}\left(\min_{j \in \mathcal{I}_{-i}} a_j\right)}{I(F(a_i), n_1, n_2)} \prod_{j \in \mathcal{I}_{-i}} f(a_j), & a_i > \min_{j \in \mathcal{I}_{-i}} a_j. \end{cases}$$

Marginal Posterior Belief:

$$\begin{aligned} &\beta_i\left(a_j \mid s, a_i\right) \\ &= \begin{cases} \frac{f(a_j)}{I(F(a_i), n_1, n_2)} \left((n_2 - 2) B\left(F(a_i), n_1 - n_2 + 1, n_2 - 2\right)\right) + \left(1 - F(a_i)\right)^{n_2 - 2}\Big), & a_j > a_i, \\ \frac{f(a_j)}{I(F(a_i), n_1, n_2)} \left((n_2 - 2) B\left(F(a_j), n_1 - n_2 + 1, n_2 - 2\right) + F^{n_1 - n_2}(a_j)(1 - F(a_j))^{n_2 - 2}\right), & a_j < a_i. \end{cases} \end{aligned}$$