Algebraic Wheel Theory in Lean 4

Yan Yablonovskiy

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Chapter 1

Introduction

Agebraic wheels are structures generalising a commutative semiring, attempting to make sense of ‘division’ by zero.

Loosely speaking, given a semiring R and it’s associated monoids, one may extend the semiring in a variety of well-known ways. Considering an additive inverse extends a commutative semiring, to a structure with a given name: a commutative ring, and attempting the same succesfully for the multiplicative monoid yields a field.

The idea of a wheel, is to extend a commutative semiring by introducing a new unary operation /, to then have $a \cdot /b$ agree with $a * b^{-1}$.

Definition 1 (Wheel). A wheel is a unital commutative semiring W along with a unary map $wDiv : W \to W$ satisfying:

1. Involution: $\forall w \in W, wDiv(wDiv(w)) = w$
2. Multiplicative automophism: $\forall w, v \in W, wDiv(wv) = wDiv(w)wDiv(v)$
3. Addition psuedo-distributivity: $\forall w, v, u \in W, (w + uv)Div(u) = wDiv(u) + v$
4. Division by 0: $\forall w \in W, 0Div(0) + w = 0Div(0)$

Chapter 2

References:

[1] JESPER CARLSTRÖM. “Wheels – on division by zero”. In: Mathematical Structures in Computer Science 14.1 (2004), pp. 143–184. doi: 10.1017/S0960129503004110.