Algebraic Wheel Theory in Lean 4

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Chapter 1

Introduction

1.1 Definition of a Wheel

Algebraic wheels are structures generalising a commutative semiring, attempting to make sense of 'division' by zero.

Loosely speaking, given a semiring R and it's associated monoids, one may extend the semiring in a variety of well-known ways. Considering an additive inverse extends a commutative semiring, to a structure with a given name: a commutative ring, and attempting the same succesfully for the multiplicative monoid yields a field.

Working backwards, given a set M with two monoids – one in additive notation and one in multiplicative. The path to obtain a semiring is clear, a wheel however generalises the semiring by removing the usual distributivity and defines a new unary map *wDiv*.

Definition 1 (Wheel). A Wheel W is an algebraic structure which has two binary operations $(+, *)$, like a ring. Similarly to a commutative ring, a Wheel is a commutative monoid in both operations. Additionally, there is a multiplicative unary map *wDiv* which is an involution, as well as a few idiosyncratic properties in the interactions of the $+$,$*$ and *wDiv*.

1. Involution: $\forall w \in W, wDiv(wDiv(w)) = w$
2. Multiplicative automophism: $\forall w, v \in W, wDiv(wv) = wDiv(w)wDiv(v)$
3. Right distributivity rule 1: $\forall w, v, u \in W, (w + v)u + 0u = wu + vu$
4. Right distributivity rule 2: $\forall w, v, u \in W, (w + 0v)u + 0u = wu + 0v$
5. Right wDiv distributivity: $\forall w, v, u \in W, (w + uv)wDiv(u) = wDiv(u) + v + 0u$
6. Division by 0: $\forall w \in W, 0Div(0) + w = 0Div(0)$
7. Zero squared: $0 * 0 = 0$
8. Division rule: $\forall w, v \in W, wDiv(w + 0v) = wDiv(w) + 0v$

Whenever not specified, the notation for the monoids is assumed to be $(+, *)$ with neutral elements 0 and 1 respectively.

1.1.1 Basic results

Here we collate some very simple propositions that are straightforward given the Wheel definition. These are designed to be auxillary and thus somewhat assorted and perhaps trivial, however mechanisation demands specification of what is typically deemed trivial.

Define the notation $\backslash_a := wDiv$ for brevity.

Proposition 2 (**Unit preserving**). *Given a Wheel W , then $\backslash_a 1 = 1$ where 1 is the neutral element of the multiplicative commutative monoid.*

Proposition 3. *Given a Wheel W and any two elements $a, b \in W$, then:*

$$0a + 0b = 0ab$$

Proposition 4. *Given a Wheel W and any element $a \in W$, then:*

$$(0\backslash_a 0)a = 0\backslash_a 0$$

Proposition 5 (**Dividing by self**). *Given a Wheel W and any element $a \in W$, then:*

$$a\backslash_a a = 1 + 0(a\backslash_a a)$$

Proposition 6 (**Right cancellation**). *Given a Wheel W and any elements $a, b, c \in W$ such that $ac = bc$, then:*

$$a + 0c\backslash_a c = b + 0c\backslash_a c$$

Proposition 7 (**Monoid Automorphism**). *Given a Wheel W , $wDiv$ is a monoid automorphism for $(1, *)$.*

1.1.2 Unital interactions

This section examines how a Wheel W behaves when an element $x \in W$ happens to be a unit in the multiplicative monoid.

Proposition 8. *Given a Wheel W, and $x \in W$ a unit in the multiplicative monoid of W, then the unit and self Wheel division are related by:*

$$x^{-1} + 0\backslash_a x = \backslash_a x + 0x^{-1} \tag{1.1}$$

where $x^{-1} \in W$ is the associated two-sided multiplicative inverse of the unit x.

Proposition 9. *Given a Wheel W, and $x \in W$ a unit in the multiplicative monoid of W, then zero enjoys the following identity:*

$$0\backslash_a x + 0\backslash_a x^{-1} = 0 \tag{1.2}$$

where $x^{-1} \in W$ is the associated two-sided multiplicative inverse of the unit x.

Proposition 10. *Given a Wheel W, and $x \in W$ a unit in the multiplicative monoid of W, then:*

$$x^{-1} = \backslash_a x + 0x^{-1}\backslash_a x^{-1} \tag{1.3}$$

where $x^{-1} \in W$ is the associated two-sided multiplicative inverse of the unit x.

Proposition 11. *Given a Wheel W, and $x \in W$ a unit in the multiplicative monoid of W, then:*

$$\backslash_a x = x^{-1} + 0x\backslash_a x \tag{1.4}$$

where $x^{-1} \in W$ is the associated two-sided multiplicative inverse of the unit x.

1.2 Algebraic sub-structures of a Wheel

Given a wheel W, certain sub-sets form familiar structures.

1.2.1 Induced semiring

Consider the subset:

$$\mathcal{R}_W := \{w \in W | 0 * w = 0\}$$

then $\mathcal{R}$ turns out to be a commutative unital semiring.

Remark 1.2.1. *For the purposes of this document, we assume semirings to be unital and commutative by default.*

Firstly,

Definition 12 ($\mathcal{R}_W$ **is a commutative magma in * and** $+$)**.** Given a wheel W, then $\mathcal{R}_W$ is a commutative magma in both the wheel operations.

This is primarily to address the 'closure' of the algebraic operations in the sub-set. To achieve a semi-ring, we further need two commutative monoids: one for each of the wheel operations. Having closure, what is left to prove is a neutral element and associativity.

Definition 13 ($\mathcal{R}_W$ **is a commutative monoid in * and** $+$)**.** Given a wheel W, then $\mathcal{R}_W$ is a commutative monoid in both the wheel operations.

Finally, the interaction between the two monoids must be distributive, and the semiring has been defined:

Definition 14 ($\mathcal{R}_W$ **is a semiring in * and** $+$)**.** Given a wheel W, then $\mathcal{R}_W$ is a semiring in * and $+$.

1.2.2 Induced commutative group

In this section, consider the subset of a wheel W:

$$\mathcal{S}_W := \{w \in W | 0 * w = 0 \wedge 0 * \backslash_a w = 0\},$$

note that this is also a subset of $\mathcal{R}_W$. Once more, begining with the closure of the binary operation * inherited from the wheel:

Definition 15 ($\mathcal{S}_W$ **is a commutative magma in *).** Given a wheel W, then $\mathcal{S}_W$ is a commutative magma in the * wheel operation.

Followed by the commutative monoid:

Definition 16 ($\mathcal{S}_W$ **is a commutative monoid in *).** Given a wheel W, then $\mathcal{S}_W$ is a commutative monoid in the * wheel operation.

And finally, the induced commutative group,

Definition 17 ($\mathcal{S}_W$ **is a commutative group in *)**. Given a wheel W, then $\mathcal{S}_W$ is a commutative group in *.

This shows that every element of $\mathcal{S}_W$ is $\backslash_a$-invertible. Furthermore, if an element of $\mathcal{R}_W$ is a $\backslash_a$-unit, then its inverse is contained in $\mathcal{S}_W$:

Proposition 18. *For a wheel W, if $x \in \mathcal{R}_W$ is $\backslash_a$-invertible, then $x^{-1} \in \mathcal{S}_W$.*

Chapter 2

References

[1] JESPER CARLSTRÖM. “Wheels – on division by zero”. In: Mathematical Structures in Computer Science 14.1 (2004), pp. 143–184. doi: 10.1017/S0960129503004110.