Algebraic Wheel Theory in Lean 4

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Chapter 1

Introduction

Agebraic wheels are structures generalising a commutative semiring, attempting to make sense of 'division' by zero.

Loosely speaking, given a semiring R and it's associated monoids, one may extend the semiring in a variety of well-known ways. Considering an additive inverse extends a commutative semiring, to a structure with a given name: a commutative ring, and attempting the same succesfully for the multiplicative monoid yields a field.

Working backwards, given a set M with two monoids – one in additive notation and one in multiplicative. The path to obtain a semiring is clear, a wheel however generalises the semiring by removing the usual distributivity and defines a new unary map $wDiv$.

Definition 1 (**Wheel**)**.** A Wheel W is an algebraic structure which has two binary operations $(+, *)$, like a ring. Similarly to a commutative ring, a Wheel is a commutative monoid in both operations. Additionally, there is a multiplicative unary map $wDiv$ which is an involution, as well as a few idiosyncratic properties in the interactions of the $+$,$*$ and $wDiv$.

1. Involution: $\forall w \in W, wDiv(wDiv(w)) = w$
2. Multiplicative automophism: $\forall w, v \in W, wDiv(wv) = wDiv(w)wDiv(v)$
3. Right distributivity rule 1: $\forall w, v, u \in W, (w + v)u + 0u = wu + vu$
4. Right distributivity rule 2: $\forall w, v, u \in W, (w + 0v)u + 0u = wu + 0v$
5. Right wDiv distributivity: $\forall w, v, u \in W, (w + uv)wDiv(u) = wDiv(u) + v + 0u$
6. Division by 0: $\forall w \in W, 0Div(0) + w = 0Div(0)$
7. Zero squared: $0 * 0 = 0$
8. Division rule: $\forall w, v \in W, wDiv(w + 0v) = wDiv(w) + 0v$

Chapter 2

References:

[1] JESPER CARLSTRÖM. “Wheels – on division by zero”. In: Mathematical Structures in Computer Science 14.1 (2004), pp. 143–184. doi: 10.1017/S0960129503004110.