Algebraic Wheel Theory in Lean 4

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Chapter 1

Introduction

1.1 Definition of a Wheel

Algebraic wheels are structures generalising a commutative semiring, attempting to make sense of ‘division’ by zero.

Loosely speaking, given a semiring R and it’s associated monoids, one may extend the semiring in a variety of well-known ways. Considering an additive inverse extends a commutative semiring, to a structure with a given name: a commutative ring, and attempting the same succesfully for the multiplicative monoid yields a field.

Working backwards, given a set M with two monoids – one in additive notation and one in multiplicative. The path to obtain a semiring is clear, a wheel however generalises the semiring by removing the usual distributivity and defines a new unary map $wDiv$.

Definition 1 (Wheel). A Wheel W is an algebraic structure which has two binary operations $(+, *)$, like a ring. Similarly to a commutative ring, a Wheel is a commutative monoid in both operations. Additionally, there is a multiplicative unary map $wDiv$ which is an involution, as well as a few idiosyncratic properties in the interactions of the $+$,$*$ and $wDiv$.

1. Involution: $\forall w \in W, wDiv(wDiv(w)) = w$
2. Multiplicative automophism: $\forall w, v \in W, wDiv(wv) = wDiv(w)wDiv(v)$
3. Right distributivity rule 1: $\forall w, v, u \in W, (w + v)u + 0u = wu + vu$
4. Right distributivity rule 2: $\forall w, v, u \in W, (w + 0v)u + 0u = wu + 0v$
5. Right wDiv distributivity: $\forall w, v, u \in W, (w + uv)wDiv(u) = wDiv(u) + v + 0u$
6. Division by 0: $\forall w \in W, 0Div(0) + w = 0Div(0)$
7. Zero squared: $0 * 0 = 0$
8. Division rule: $\forall w, v \in W, wDiv(w + 0v) = wDiv(w) + 0v$

Whenever not specified, the notation for the monoids is assumed to be $(+, *)$ with neutral elements 0 and 1 respectively.

1.1.1 Basic results

Here we collate some very simple propositions that are straightforward given the Wheel definition. These are designed to be auxillary and thus somewhat assorted and perhaps trivial, however mechanisation demands specification of what is typically deemed trivial.

Define the notation $\backslash_a := wDiv$ for brevity.

Proposition 2 (**Unit preserving**). *Given a Wheel W , then $\backslash_a 1 = 1$ where 1 is the neutral element of the multiplicative commutative monoid.*

Proposition 3. *Given a Wheel W and any two elements $a, b \in W$, then:*

$$0 * a + 0 * b = 0 * a * b$$

Proposition 4. *Given a Wheel W and any element $a \in W$, then:*

$$(0 * \backslash_a 0) * a = 0 * \backslash_a 0$$

Proposition 5 (**Dividing by self**). *Given a Wheel W and any element $a \in W$, then:*

$$a * \backslash_a a = 1 + 0 * (a * \backslash_a a)$$

Proposition 6 (**Right cancellation**). *Given a Wheel W and any elements $a, b, c \in W$ such that $a * c = b * c$, then:*

$$a + 0 * c * \backslash_a c = b + 0 * c * \backslash_a c$$

Proposition 7 (**Monoid Automorphism**). *Given a Wheel W , $wDiv$ is a monoid automorphism for $(1, *)$.*

1.1.2 Unital interactions

This section examines how a Wheel W behaves when an element $x \in W$ happens to be a unit in the multiplicative monoid.

Proposition 8. *Given a Wheel W, and $x \in W$ a unit in the multiplicative monoid of W, then the unit and self Wheel division are related by:*

$$x^{-1} + 0 \backslash_a x = \backslash_a x + 0 x^{-1} \tag{1.1}$$

where $x^{-1} \in W$ is the associated two-sided multiplicative inverse of the unit x.

Chapter 2

References

[1] JESPER CARLSTRÖM. "Wheels – on division by zero". In: Mathematical Structures in Computer Science 14.1 (2004), pp. 143–184. doi: 10.1017/S0960129503004110.