$$T(n) = \frac{n}{n+1} \cdot T(n-1) + 1$$

От миналия път

Чрез развиване.

$$T(n-1) = \frac{n-1}{n} \cdot T(n-2) + 1$$

$$T(n-2) = \frac{n-2}{n-1} \cdot T(n-3) + 1$$

$$T(n-3) = \frac{n-3}{n-2} \cdot T(n-4) + 1$$

$$\frac{n+1}{n+1} =$$

$$T(n) = \frac{n}{n+1} \cdot \left(\frac{n-1}{n} \cdot T(n-2) + 1 \right) + 1 =$$

$$= \frac{\cancel{n}}{n+1} \cdot \frac{n-1}{\cancel{n}} \cdot T(n-2) + \frac{n}{n+1} + \frac{n+1}{n+1} =$$

$$= \frac{n-1}{n+1} \left(\frac{n-2}{n-1} \cdot T(n-3) + 1 \right) + \frac{n}{n+1} + \frac{n+1}{n+1} =$$

$$= \frac{n-2}{n+1} \cdot T(n-3) + \frac{n-1}{n+1} + \frac{n}{n+1} + \frac{n+1}{n+1} =$$

$$= \frac{n-2}{n+1}\left(\frac{n-3}{n-2}\cdot T(n-4)+1\right)+\frac{n-1}{n+1}+\frac{n}{n+1}+\frac{n+1}{n+1}=$$

$$= \frac{n-3}{n+1}\cdot T(n-4)+\frac{n-2}{n+1}+\frac{n-1}{n+1}+\frac{n}{n+1}+\frac{n+1}{n+1}= \dots =$$

$$= \underbrace{\overbrace{\frac{(n-(n-1))}{n+1}}^{=1}}\cdot T(\overbrace{n-n}^{=0}) + \underbrace{\overbrace{\frac{((n+1)-(n-1))}{n+1}}^{=2} + \frac{\overbrace{(n+1)-(n-2)}^{=3}}{n+1}} +$$

$$\dots + \frac{n-2}{n+1}+\frac{n-1}{n+1}+\frac{n}{n+1}+\frac{n+1}{n+1} = \frac{(n+1).(n+2)}{2}$$

$$= \frac{1}{n+1}\cdot T(0) + \sum_{i=2}^{n+1}\frac{i}{n+1} = \frac{1}{n+1}\cdot\left(T(0)+\overbrace{\sum_{i=1}^{n+1} i}^{=\frac{(n+1).(n+2)}{2}} - 1\right) =$$

$$= \frac{1}{n+1}\cdot\left(T(0)+\frac{(n+1).(n+2)}{2}-1\right) \simeq \frac{1}{\cancel{n+1}}\cdot\frac{\cancel{(n+1)}.(n+2)}{2} \simeq$$

$$\simeq n$$

Зад ①

```
f(n: unsigned int): int
  s <- 10
  while s <= n do        // s > n
     s <- 5.s + 3
  return s
```

a) Върнатата ст-т на $f(n) = ?$

b) Времева сложност $T(n) \asymp ?$

$s_0 = 10$

$s_{k+1} = 5 \cdot s_k + 3$

За хомогенната ч. $x^{k+1} = 5 \cdot x^k \quad // : x^k$

$x = 5 \rightarrow \{5\}_m$

От нехом. част $\dfrac{3 \cdot k^0 \cdot 1^k}{\deg(k^0)+1} \rightarrow \{1\}_m \quad \cup \quad = \{1, 5\}_m$

Общ. вид:

$$s_k = c_1 \cdot 1^k + c_2 \cdot 5^k$$

$s_0 = 10$

$s_1 = 5 \cdot s_0 + 3 = 5 \cdot 10 + 3 = 53$

$$\begin{cases} s_0 = 10 = c_1 \cdot 1^0 + c_2 \cdot 5^0 = c_1 + c_2 & \cdot(-1) \\ s_1 = 53 = c_1 \cdot 1^1 + c_2 \cdot 5^1 & (+) \end{cases}$$

$43 = c_2 \cdot 4 \iff c_2 = \frac{43}{4}$

$c_1 + \frac{43}{4} - 10 = 0 \iff c_1 = -\frac{3}{4}$

$$s_k = -\frac{3}{4} + \frac{43}{4} \cdot 5^k = \frac{43 \cdot 5^k - 3}{4} \quad (*)$$

$?k: S_k \le n \iff \frac{43.5^k - 3}{4} \le n \iff$

$\iff 5^k \le \frac{4.n+3}{43} \quad // \log_5$

$\iff k \le \log_5\left(\frac{4.n+3}{43}\right)$

Отговор: $f(n) = S_{\left\lfloor \log_5\left(\frac{4n+3}{43}\right)\right\rfloor} \ne t$

$T(n) \asymp \left\lfloor \log_5\left(\frac{4n+3}{43}\right)\right\rfloor \asymp \log n$

Зад ② $T(n) = 2 \cdot T(\sqrt{n}) + 1$, $\sqrt{n} = n^{1/2}$

Чрез заместване

Нека $n = 2^{2^m}$, където $m = \log_2 \log_2 n$

$n^{1/2} = \left(2^{2^m}\right)^{1/2} = 2^{\frac{2^m}{2}} = 2^{2^{m-1}}$

$T(n) = T\left(2^{2^m}\right) = 2 \cdot T\left(2^{2^{m-1}}\right) + 1$, $2^{2^m} = n$

$$\boxed{S(m) = 2 \cdot S(m-1) + 1}$$

$S(m-1) = 2 \cdot S(m-2) + 1$, $S(m-2) = 2 \cdot S(m-3) + 1$

$S(m) = 2^2 \cdot S(m-2) + 2 + 1 =$

$= 2^2 \cdot (2 \cdot S(m-3) + 1) + 2 + 1 = 2^3 \cdot S(m-3) + 2^2 + 2 + 1 =$

$= \ldots = 2^m \cdot S(0) + \sum_{i=0}^{m-1} 2^i \approx 2^m + \sum_{i=0}^{m-1} 2^i = \sum_{i=0}^{m} 2^i =$

$$= 2^{m+1} - 1 = 2^m \cdot 2 - 1 \simeq 2^m = 2^{\log_2 \log_2 n} = \log_2 n \simeq \log n$$

$$m = \log_2 \log_2 n$$

$$T(n) = 4 \cdot T(\sqrt[3]{\boxed{n}}) + n$$

$$n = 3^{3^{\log_3 \log_3 n}}$$

$$T(n) = 4 \cdot T(\sqrt[3]{n}) + n$$

$n^{1/3}$

Полагаме $n = 3^{3^m}$, $m = \log_3 \log_3 n$

$$n^{1/3} = \left(3^{3^m}\right)^{1/3} = 3^{3^{m-1}}$$

$$S(m) = 4 \cdot S(m-1) + 3^{3^m}$$

$$S(m-1) = 4 \cdot S(m-2) + 3^{3^{m-1}}$$

$$S(m) = 4^2 \cdot S(m-2) + 4 \cdot 3^{3^{m-1}} + 3^{3^m} = \ldots =$$

$$= 4^m S(0) + \sum_{i=0}^{m-1} 4^i \cdot 3^{3^{m-i}} =$$

$$= 4^m \cdot S(0) + 4^m \sum_{i=0}^{m-1} \frac{3^{3^{m-i}}}{4^{m-i}}$$

помислих

я или по-добре решение на др. стр.?

$$\sum_{i=0}^{m-1} \frac{3^{3^{m-i}}}{4^{m-i}} \le \sum_{i=0}^{m-1} 3^{3^{m-i}} \le \sum_{i=0}^{\infty} 3^{3^{m-i}} - \;\therefore$$

(*)
$$\lim_{i\to\infty} \frac{3^{3^{m-(i+1)}}}{3^{3^{m-i}}} = \lim_{i\to\infty} 3^{\left(3^{m-(i+1)} - 3^{m-i}\right)} =$$

$$3^{\left(3^{m-i}(3^{-1}-1)\right)} \qquad 3^{\left(3^{m-i}(3^{-1}-1)\right)}$$

$$= \lim_{i\to\infty} 3 \qquad = \lim_{i\to\infty} 3$$

<u>нов опит използвайки АИТ</u> в записките

към дъската и другият път

чрез направим поздня

Зад(3)

f(n: unsigned int): int

if $n = 0$

  return 0

$a \leftarrow n \cdot n \cdot (n+1) \cdot (n+1)/4$ $\qquad a = \dfrac{n^2 \cdot (n+1)^2}{4}$

for $k \leftarrow 1$ to $n-1$ do

  $a \leftarrow a - f(k) \neq a + (-f(k))$

return a

$f(n) = ?$ и $T(n) \asymp ?$

$$f(n) = \underbrace{\frac{n^2 \cdot (n+1)^2}{4}}_{a} - \underbrace{f(1) - f(2) - f(3) - \dots - f(n-1)}_{\text{цялата история}} =$$

$$= \frac{n^2 \cdot (n+1)^2}{4} - \sum_{i=1}^{n-1} f(i)$$

$$f(n) + \boxed{\sum_{i=1}^{n-1} f(i)} = \sum_{i=1}^{n} f(i) = \frac{n^2 \cdot (n+1)^2}{4} \quad (\#)$$

$n \mapsto n-1$

$$f(n-1) + \sum_{i=1}^{n-2} f(i) = \boxed{\sum_{i=1}^{n-1} f(i) = \frac{(n-1)^2 \cdot n^2}{4}} \quad (*)$$

Нека в (#) за (*):

$$f(n) + \frac{(n-1)^2 \cdot n^2}{4} = \frac{n^2 \cdot (n+1)^2}{4}$$

$$f(n) = \frac{n^2\left((n+1)^2 - (n-1)^2\right)}{4} = \frac{n^2 \cdot \cancel{4} n}{\cancel{4}} = n^3$$

Отговор: $\underline{\underline{f(n) = n^3}}$

$T(n) \asymp ?$

цялата история

$$T(n) = b + \underbrace{(T(1) + c) + (T(2) + c) + \dots + (T(n-1) + c)}$$

работата по инициализация

рекурсивните извиквания

работата на алгоритъма без рекурсия

$n \to n-1$

$$T(n-1) = b + (T(1) + c) + (T(2) + c) + \dots + (T(n-2) + c)$$

$$T(n) - T(n-1) = T(n-1) + c$$

$$T(n) = 2 \cdot T(n-1) + c \cdot n^0 \cdot 1^n$$

$x^n = 2 \cdot x^{n-1} \quad // : x^{n-1} \neq 0$

$x = 2 \to \{2\}_m$

$\{1\}_m$

$\cup$

$\to \{1, 2\}_m$

$$T(n) = c_1 \cdot 1^n + c_2 \cdot 2^n \asymp 2^n$$

Отговор:

$T(n) \asymp 2^n$

Решаване на рекурентни ур-я, за които Master Th е приложима

Master Theorem

Нека $a \geq 1$ и $b > 1$ са константи и нека $f(n)$ е положителна ф-я.

Нека

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

a — брой на рекурсивните извиквания

b — брой части, на които делим входните данни

$f(n)$ — всичката работа без рекурсията

/* Различните случаи, които ще опише тази Th не покриват всички възможности! */

Тогава асимптотиката на $T(n)$ е:

* Нека за улеснение $k = \log_b a$, т.е. $n^k = n^{\log_b a}$ е големината на дървото!!!

Сл.1 Ако $n^{k-\varepsilon} \succsim f(n)$ за някое $\varepsilon > 0$, то

$T(n) \asymp n^k$ $\qquad f(n) = O(n^{k-\varepsilon})$

Сл.2 Ако $n^k \asymp f(n)$, то $T(n) \asymp n^k . \log n$

$(\log n)$ $\qquad$ дълбочината на дървото е $\log_b n \asymp \log n$

Сл.3 Ако $n^{k+\varepsilon} \precsim f(n)$ за някое $\varepsilon > 0$ и

f е регулярна ф-я т.е.

$(\exists c \in (0;1))(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N})$

$\left[n \geq n_0 \Rightarrow a . f\left(\frac{n}{b}\right) \leq c . f(n) \right]$, то

$T(n) \asymp f(n)$

Сл.★ Специален случай

Ако $f(n) \asymp n^k \cdot \log^t n$ за някаква константа $t \geq 0$, то

$T(n) \asymp n^k \cdot \log^{t+1} n$

① $\varepsilon > 0$

$n^{k-\varepsilon} \geq f(n)$

$T(n) \asymp n^k$

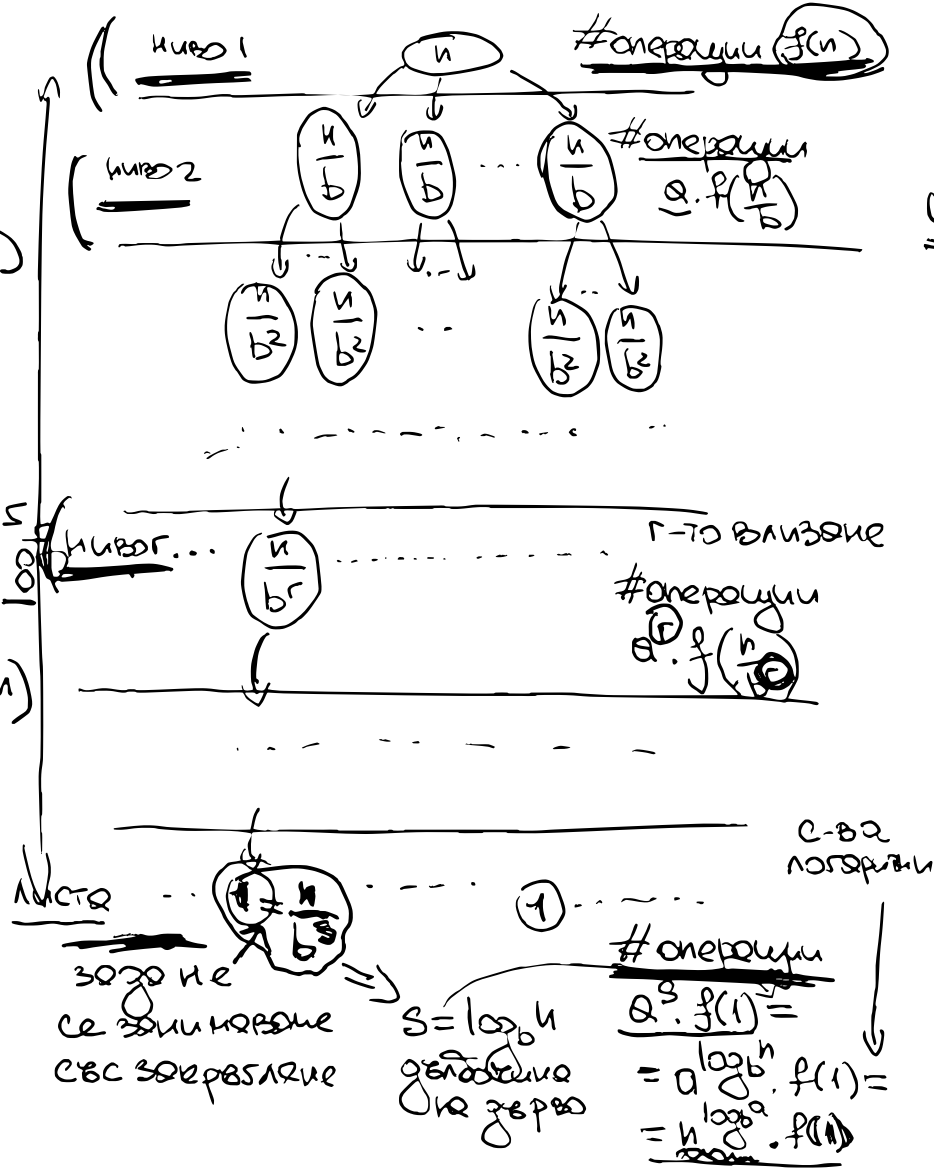
② $n^k \asymp f(n)$

$\log_b n \cdot n^k$

③ $\varepsilon > 0$ $n^{k+\varepsilon} \leq f(n)$

$T(n) \asymp f(n)$

n



$a^{\log_b n} = n^{\log_b a}$

$\left\lceil \frac{\lceil \frac{n}{b} \rceil}{b} \right\rceil - \frac{n}{b}$

$1 = \frac{n}{b^s} \Leftrightarrow$

$b^s = n \;//\log_b$

$S = \log_b n$

[illegible] 1 $T(n) = 128 \cdot T\left(\frac{n}{2}\right) + n^3 \cdot \log n$

$a = 128 = 2^7$

$b = 2$

$k = \log_b a = \log_2 2^7 = 7$

$n^k = n^7$, $f(n) = n^3 \cdot \log n$

$? \, \varepsilon > 0$ [illegible] $n^{k-\varepsilon} \succeq f(n)$

$\varepsilon = 1$

l'hôpital

$n^6 \succ n^3 \cdot \log n$

$$\lim_{n \to \infty} \frac{n^6}{n^3 \cdot \log n} = \lim_{n \to \infty} \frac{n^3}{\log n} = \lim_{n \to \infty} \frac{3 \cdot n^2}{1/n} =$$

$3 \cdot \lim_{n \to \infty} n^3 = \infty$

$n^6 \succ n^3 \cdot \log n \to$ I [illegible] $T(n) \simeq n^k = n^7$

Зад 2 $T(n) = \underline{10000} \cdot T\left(\frac{n}{10}\right) + n^4$

$a = 10000 = 10^4$

$b = 10$

$k = \log_b a = \log_{10} 10^4 = 4$

$n^k = n^4$, $f(n) = n^4$

$n^k \asymp f(n)$

Тогда по $\underline{II}$-му сл. МТ $\to T(n) \asymp n^k \cdot \log n =$

$= n^4 \cdot \log n$

Зад (3) $T(n) = 3,5 \cdot T\left(\frac{n}{2}\right) + n^5$

всяка ф-я по-голяма от n^k е автоматично отговор

$a = 3,5 = 7/2$

$b = 2$

$k = \log_b a = \log_2(7/2) \in (1; 2)$

$n^k = n^{\log_2(7/2)}$

$f(n) = n^5$

? $\varepsilon > 0$: $n^{k+\varepsilon} \preceq f(n)$

$\varepsilon = 0,1 \to \underline{n^{\log_2(7/2)+0,1} \prec n^5}$

$(\exists c \in (0;1))(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N})$

$\left[n \geq n_0 \Longrightarrow \underline{a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)} \right]$

$$\frac{7}{2} \cdot \left(\frac{n}{2}\right)^5 \leq c \cdot n^5 \quad // : n^5$$

$$(0;1) \ni \frac{7}{64} \leq c$$

$$c = \frac{7}{64}, \quad n_0 = ? \in \mathbb{N}$$

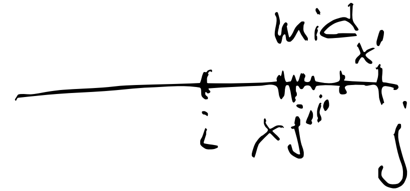
$$n_0 = 1$$

$$\forall n \geq n_0 \quad \boxed{\frac{7}{2} \cdot \left(\frac{n}{2}\right)^5 \leq c \cdot n^5}$$

III случай на МТ: $T(n) \asymp f(n) = n^5$

382 (4) BinarySearch(A[1...n] : sorted array of ints, key : int) : int

```
if n ≤ 0 then
    return -1
value ← A(⌈n/2⌉)
if value = key then
    return key
else if value < key
    return BinarySearch(A[1...⌈n/2⌉-1], key)
return BinarySearch(A[⌈n/2⌉+1...n], key)
```



$$T(n) = 1 \cdot T\left(\frac{n}{2}\right) + 1$$

$$n^k = n^{\log_2 1} = n^0$$

$$f(n) = 1 \longrightarrow n^k = n^0 = 1 \simeq n^0 = f(n)$$

IIa. $\Rightarrow T(n) \simeq n^0 \cdot \log n = \log n$

Зад 6 $T(n) = \underline{1 \cdot T\left(\frac{n}{4}\right)} + \log n$

$\left.\begin{array}{l} a = 1 \\ b = 4 \end{array}\right\} \log_b a = 0$

$n^k = n^0 = 1$

$f(n) = \log n$

$t \geq 0$ константа, т.е.

$f(n) = \log n \asymp n^k \cdot \log^t n = n^0 \cdot \log^t n = \log^t n$

$t = 1$

<u>по (*) случай ко</u> <u>2 Т</u>

$\underline{T(n) \asymp n^k \cdot \log^{t+1} n = \log^2 n}$

<u>Unfinished work for next time</u>

a) $T(n) = 3 \cdot T\left(\frac{n}{2}\right) + n \cdot \log n$

b) $T(n) = 2 \cdot T\left(\frac{n}{4}\right) + \sqrt{n}$

c) $T(n) = 4 \cdot T\left(\frac{n}{\sqrt{2}}\right) + n^4$

d) $T(n) = 2 \cdot T(n-1) + T(\log_2 n) + n$

Използвайки индукция ще го решим, че $\underline{T(n) \simeq 2^{n}}$.

e) $T(n) = 3 \cdot T(\sqrt[3]{n}) + \log_3 n$

f) $T(n) = T(\log_2 n) + 1$

След това почваме с Дизайн на алгоритми

HW /

a) $T(n) = T\left(\frac{9 \cdot n}{10}\right) + n$

b) $T(n) = 16 \cdot T\left(\frac{n}{2}\right) + n^2$

c) $T(n) = 8 \cdot T\left(\frac{n}{2}\right) + n^3$

d) $T(n) = T(n-1) + \sqrt{n}$

e) $T(n) = 2 \cdot T(\sqrt[4]{n}) + \log(\sqrt[4]{n})$

f)

```
sum(A[1..n]: array of nums): num
if n=0 then
    return 0
else if n = 1 then
    return A[1]
return sum(A[1..⌊n/2⌋]) +
          sum(A[⌊n/2⌋+1...n])
```

Формулирайте рекурентното уравнение и го решете.