* $f(n) = n$ и $g(n) = n^{1+\sin}$ са несравними

Важни суми:

* $1+2+3+\dots+n = \frac{n.(n+1)}{2} \asymp n^2$
* $1^2+2^2+3^2+\dots+n^2 = \frac{n.(n+1).(2n+1)}{6} \asymp n^3$

по-генер.

Чрез интегриране се доказва, че

някои ф-ции: $\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx$

! $\sum_{k=0}^{n} k^m \asymp \begin{cases} n^{m+1}, & m > -1 \\ \log n, & m = -1 \text{ i.e. } 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \asymp \log_2 n \\ 1, & m < -1 \end{cases}$

хармоничен ред

$\sum_{k=1}^{n} k^{1/2} \asymp n.\sqrt{n}$; $\sum_{k=1}^{n} k^{-1/2} \asymp \sqrt{n}$

! $\sum_{k=1}^{n} a^k \asymp \begin{cases} a^n, & a > 1 \\ n, & a = 1 \\ 1, & 0 < a < 1 \end{cases}$

Апроксимация на Стирлинг

$$n! = \sqrt{2\pi n}.\frac{n^n}{e^n}.\left(1+\Theta\left(\frac{1}{n}\right)\right) \asymp \frac{n^{n+1/2}}{e^n}$$

Следствия:

1) $\lg n! \asymp n \lg n$

$$\lg n! \asymp \lg\left(\frac{n^{n+1/2}}{e^n}\right) = (n+1/2)\cdot \lg n - n \cdot \lg e \asymp n \lg n$$

2) $\binom{2n}{n} \asymp \frac{4^n}{\sqrt{n}}$ $\Rightarrow \binom{n}{\frac{n}{2}} \asymp \frac{2^n}{\sqrt{n}}$

$$\binom{2n}{n} = \frac{(2n)!}{n!\,(2n-n)!} = \frac{(2n)!}{(n!)^2} \asymp \frac{(2n)^{2n+1/2}\cdot \cancel{(e^n)^2}}{\cancel{e^{2n}}\,(n^{n+1/2})^2} =$$

$$= \frac{2^{2n}\cdot\sqrt{2}\cdot\cancel{n^{2n}}\cdot\cancel{\sqrt{n}}}{\cancel{n^{2n}}\cdot\cancel{n}\sqrt{n}} = \sqrt{2}\cdot\frac{4^n}{\sqrt{n}} \asymp \frac{4^n}{\sqrt{n}}$$

Зад. $\lg^*(n)$

if $(n \le 1) \to \lg^* n = 0$

else if $(\lg n \le 1) \to \lg^* n = 1$

else if $(\lg\lg n \le 1) \to \lg^* n = 2$

...

т.е. $\underbrace{\lg\lg\lg \dots \lg}_{} n \le 1$ $\quad \lg^* n = \min\{i \ge 0 \mid \lg^{(i)} n \le 1\}$

колко пъти да вземем lg докато всичко стане $\le$ на 1

$\lg^* 1 = 0$, $\lg^* \underset{2^1}{2} = 1$, $\lg^* \underset{2+1}{3} = 2$, $\lg^* \underset{2^2}{4} = 2$,

$\lg^* \underset{2^2+1}{5} = 3$ $\lg^* 6|7|\ldots|\underset{2^{2^2}}{16} = 3$

$\lg^* 17 = 4$, $17 = \underline{2^{2^2}+1}$

$\lg^*(2^{2^{2^2}}+1) = \lg^*(2^{16}+1) = \lg^*(65537) = 5$

Ако $\lg^* n = 6$, то $n \geq 2^{2^{2^{2^2}}}+1$.

Зад 1 $\frac{1}{n} \propto 1 - \frac{1}{n}$?

$2^{1/n} \propto 2^{1-1/n}$?

$$\lim_{n\to\infty} \frac{1/\cancel{n}}{\frac{\cancel{n}}{\cancel{n}} - 1/\cancel{n}} = \lim_{n\to\infty} \frac{1}{n-1} = 0 \quad N$$

$$\lim_{n\to\infty} \frac{2^{1/n}}{2^{1-1/n}} = \lim_{n\to\infty} \frac{1}{2^{1-\frac{2}{n}}} \xrightarrow[n\to\infty]{\infty} \frac{1}{2} > 0 \quad Y$$

$\frac{2}{n} \to 0$

Зад. Докаже:

$$1 \propto \lg\lg n \propto \lg n \propto n \propto n\lg n \propto n^2 \propto n^3 \propto 2^n \propto n! \propto n^n$$

Д-во

(1) $1 \propto \lg\lg n$ ✓

(2) $\lim\limits_{n\to\infty} \dfrac{\lg\lg n}{\lg n} \overset{\text{Лоп}}{=} \lim\limits_{n\to\infty} \dfrac{\frac{1}{\lg n}\cancel{(\lg n)'}}{\cancel{(\lg n)'}} = \dfrac{1}{\infty} = 0$

(3) $\lim\limits_{n\to\infty} \dfrac{\lg n}{n} \overset{\text{Лоп}}{=} \lim\limits_{n\to\infty} \dfrac{1/n}{1} = \dfrac{0}{1} = 0$

(4) $n \propto n\lg n$ ✓

(5) $n\lg n \propto n^2$

(6) $n^2 \propto n^3$

(7) $3\ln n \propto n\ln 2$

(8) $n\ln 2 \propto n\ln n$

(9) $\lim\limits_{n\to\infty} \dfrac{\cancel{n}\cdot(n-1)\cdots 1}{\cancel{n}\cdot n\cdots n} \overset{\lim\limits_{n\to\infty}}{=} \underbrace{\left(1-\frac{1}{n}\right)}_{\to 1}\underbrace{\left(1-\frac{2}{n}\right)}_{\to 1}\cdots\underbrace{\frac{1}{n}}_{\to 0} =$

$= \underline{0}$

A $n^{1/n} \asymp 1$ т.е. $\sqrt[n]{n} \asymp 1$

зад. Да се подредят по асимптотичен порядък ф-циите:

$(5n)!$, $n\sqrt{n}$, 2^{n^3}, $(\log n)^{15}$, $n^3\log n$,
$(2n)^{7n}$, $\frac{n^3}{\log n}$, $n^3\log(\log n)$, $4^n.n^7$, $5^n.n^2$,
$(\log n)^*$

Бавнорастящи $n^{\frac{1}{\lg n}}$, 1, $2^{\lg^* n}$, $\log^*(\log n)$, $\log(\log^* n)$ бързо

1 $\underbrace{\log..\log}$ $\underbrace{\log^? n}$ n^α $\underbrace{a^n}$ $\underbrace{n!}$ $\underbrace{n^n}$ $\underbrace{2^{n^2} 5^{7n}}$

полилогар. $\alpha\geq 0$ степенни експоненциални факториелни суперекспон.

1, $n^{\frac{1}{\lg n}}$

$\vee$ $\log(\log^* n)$
$\vee$ $\log^*(\log n)$
$\vee$ $\log^* n$
$\vee$ $2^{\lg^* n}$

$\vee$ $n^3\log n$
$\vee$ $n^3\log\log n$
$\vee$ $n^{3/2}$
$\vee$ $n^3.(\log n)^{-1}$

$4^n.n^7$
$5^n.n^2$

$(5n)!$

2^{n^3}
$(2n)^{7n}$

① $n^{\frac{1}{\lg n}} \asymp 2^{\ln n\cdot\frac{1}{\ln n}} = 2$ т.е. $n^{\frac{1}{\lg n}} \asymp 1$

② $1 \propto \log^* n$

③ $\log^* n$ и $\log(\log^* n)$

$m = \log^* n$, то $m \succ \log m$ $\lim\limits_{n\to\infty} \frac{m}{\lg m} \overset{L'Hop}{=} \lim\limits_{n\to\infty} \frac{1}{1/m} = \infty$

$\log(\log^* n) \propto \log^* n \to 1 \propto \log(\log^* n)$

(4) $\log^* n$ и $\log^*(\log n)$

$\log^*(\log n) = \log^* n - 1$

$\log^* n \simeq \log^*(\log n)$

(5) $2^{\log^* n}$ и $\log^* n$

$m = \log^* n$, то $m \propto 2^m$?

$\underbrace{\ln m \propto m \ln 2}$

$\log^* n \propto 2^{\log^* n}$

(6) $2^{\log^* n}$ и $n^{3/2}$ $\to$ $2^{\log^* n} \propto n^{3/2}$

$\ln^* n \simeq \log^* n \ln 2 \propto \frac{3}{2} \cdot \ln n \simeq \ln n$

Ако си мислим за n като кула от 2ки, то само взимаме ln я намаля само с 1-ца докато $\ln^*$ я намаля колкото трябва, за да стане резултат ≤ 1. $\to$ т.е. измерва височината на кулата т.е. $\ln^* n \propto \ln n$

(7) $n^{3/2}$? $\frac{n^3}{\log n}$ $\propto$

$\log n$? $\frac{n^3}{n^{3/2}} = n^{3/2}$ $\propto$

може да делим умножаваме положит. ф-ция
може и да степенуваме коренуваме и делим.

$\log(\log n)$? $3/2 \cdot \log n$ $\propto$ $\lim\limits_{n\to\infty} \frac{\log(\log n)}{\log n} \overset{?}{\underset{Лоп}{=}} \lim \frac{\frac{1}{\log n}\cancel{(\log n)'}}{\cancel{(\log n)'}} = \frac{1}{\infty} = 0$

$n^{3/2} \propto \frac{n^3}{\log n}$

(8) $\frac{n^3}{\log n} \underset{\alpha}{?} n^3.\log n$ //: n^3

$\frac{1}{\log n} \underset{\alpha}{?} \log n$ $m=\log n$

$\frac{1}{m} \underset{\alpha}{?} m$ $\lim_{m\to\infty} \frac{m}{\frac{1}{m}} = \lim_{m\to\infty} m^2 = \infty$

(9) $n^3.\log n \underset{u}{\alpha} n^3.\log(\log n)$ //: n^3 ✓

$\log(\log n) \alpha \log n$ ✓

$\frac{n^3}{\log n} \underset{✓}{\alpha} n^3.\log(\log n)$ //: n^3

$m=\log n$

$\frac{1}{m} \underset{✓}{\overset{?}{\alpha}} \log m$

$\lim_{m\to\infty} \frac{\log m}{\frac{1}{m}} = \lim_{m\to\infty} m\log m = \infty$

(10) $4^n n^7 \underset{\alpha}{?} 5^n n^2$ //: n^2

$4^n n^5 \underset{\alpha}{?} 5^n$ //: 4^n

$n^5 \underset{\alpha}{?} \left(\frac{5}{4}\right)^n \to 5\ln n \alpha n.\ln 5/4$

(11) $n^3.\log n \alpha 4^n.n^7$ //: n^3

$\log n \alpha 4^n.n^4$

$\lg\lg n \asymp \lg(\lg n) \prec n\lg 4 + 4.\lg n \asymp n$

(12) $5^n n^2$ и $(5n)!$

$n \asymp n\lg 5 + 2\lg 2 \prec 5n.\lg 5n \asymp n\lg n$

(15) 2^{n^3} и $(2n)^{7n}$

$n^3 \asymp n^3.\lg 2 \succ 7n.\lg 2n \asymp n\lg n$

(14) $(5n)!$ и $(2n)^{7n}$

С логаритмуване не става, но знаем, че

$$n! \asymp \frac{n^{n+1/2}}{e^n}$$

$$(5n)! \asymp \frac{(5n)^{5n+1/2}}{e^{5n}}$$

$$\lim_{n\to\infty} \frac{5^{5n}.5^{1/2}.\cancel{n^{5n}}.n^{1/2}}{e^{5n}.2^{7n}.n^{\cancel{7n}\,2n}} = \lim_{n\to\infty}\left(\frac{5^5}{2^7.e^5}\right)^n.\frac{\sqrt{5}.\sqrt{n}}{n^{2n}} =$$

$$= \lim_{n\to\infty} \sqrt{5}\left(\frac{3125}{128.e^5}\right)^n.\frac{1}{n^{2n-1/2}} = 0$$

$\left(\frac{3125}{128.e^5}\right)^n \xrightarrow[n\to\infty]{} 0$, $\frac{1}{n^{2n-1/2}} \to 0$

$3125 < 128.e^5 \approx 18997$

$(5n)! \prec (2n)^{7n}$

Сл1 $1 = n^{\frac{1}{\lg n}} \prec \lg(\lg^* n) \prec \lg^* n = \lg^*(\lg n) \prec 2^{\lg^* n} \prec$

$\prec n \cdot \sqrt{n} \prec \frac{n^3}{\log n} \prec n^3 \cdot \lg(\log n) \prec n^3 \cdot \log n \prec$

$\prec 4^n \cdot n^7 \prec 5^n \cdot n^2 \prec (5n)! \prec (2n)^{7n} \prec 2^{n^3}$

Не е правено в час. Не остана време.

стр. 15 в solutions manual

Да подредим по бързина на растене:

$\lg(\lg^* n)$, $2^{\lg^* n}$, $(\sqrt{2})^{\lg n}$, n^2, $n!$, $(\lg n)!$,
$(\frac{3}{2})^n$, n^3, $(\lg n)^2$, $\lg(n!)$, 2^{2^n}, $n^{\frac{1}{\lg n}}$,
$\ln\ln n$, $\lg^* n$, $n \cdot 2^n$, $n^{\lg\lg n}$, $\ln n$, 1
$2^{\lg n}$, $(\lg n)^{\lg n}$, e^n, $4^{\lg n}$, $(n+1)!$, $\sqrt{\lg n}$,
$\lg^*(\lg n)$, $2^{\sqrt{2\lg n}}$, n, 2^n, $n\lg n$, $2^{2^{n+1}}$

Стратегия: разделяме на групи:
полилогаритмични, полиноми, експоненти,
факториели,
Трябват ни 23 сравнения

бавнорастящи $\longrightarrow$ бързи

$\lg\ldots\lg$ | $\lg^{3} n$ логаритми | n^{α}, $\alpha \geq 0$ степени | a^{n} показател | $n!$ факториел | n^{n} | $\underbrace{2^{n^2};\ 5^{n}}$ суперекспоненти

① 1 и $\lg^{*} n$?

$$\lim_{n\to\infty} \frac{1}{\lg^{*} n} = 0 \quad \text{т.е.} \quad 1 \propto \lg^{*} n$$

② $n^{\frac{1}{\lg n}}$ експ. $2^{\log_2 n \cdot \frac{1}{\log_2 n}} \asymp 2 \rightarrow \underline{n^{\frac{1}{\lg n}} \asymp 1}$

③ $\lg(\underbrace{\lg^{*} n}_{m})$ и $\underbrace{\lg^{*} n}_{m}$, то $\lg m \propto m$!

$$\lg(\lg^{*} n) \propto \lg^{*} n$$

④ $\lg^* n$ и $\lg^*(\lg n)$

колко пъти от 2-ки се различават в 1

т.е. $\lg^*(\lg n) = \lg^* n - 1$

т.е. $\lg^* n \asymp \lg^*(\lg n)$

$n^{\frac{1}{\lg n}} \asymp 1 \propto \lg^* n \asymp \lg^*(\lg n)$

$1 \propto \lg(\lg^* n) \propto \lg^* n$

$1 \asymp n^{1/\lg n} \propto \lg(\lg^* n) \propto \lg^* n \asymp \lg^*(\lg n)$

⑤ $\lg^* n$ и $2^{\lg^* n}$

$\lg^* n \propto 2^{\lg^* n}$, защото $\lg^* n$ расте

⑥ $2^{\lg^* n}$ и $\ln(\ln(n)) \asymp \lg(\lg(n))$

$\lg(\lg(\lg(n))) \succ 9\lg^* n . \lg 2$

малка височина 3 от кулата

много по-малка височина

$2^{\lg^* n} \propto \ln(\ln(n))$

(7) $\sqrt{\lg n}$ и $\ln(\ln(n))$?

$(\lg n)^{1/2}$ и $\lg(\lg(n))$, нека $m = \lg(n)$

$\underline{m^{1/2} \succ \lg m}$ т.е. $\sqrt{\lg n} \succ \ln(\ln(n))$

(8) $\ln n$ и $\sqrt{\lg n}$

$\lg n$ $m = \lg n$

$m^1 \succ m^{1/2}$ т.е. $\ln n \succ \sqrt{\lg n}$

Засега:

$1 \asymp n^{\frac{1}{\lg n}} \prec \lg(\lg^* n) \prec \lg^* n \asymp \lg^*(\lg n) \prec 2^{\lg^* n} \prec$
$\ln(\ln(n)) \prec \sqrt{\lg n} \prec \ln n \prec \ln^2 n \prec 2^{\sqrt{2\lg n}}$

(9) $\ln n \prec \ln^2 n$

(10) $\ln^2 n$ и $2^{\sqrt{2\lg n}}$ // $\lg$

$\lg^2 n$ $2.\lg(\lg n) \prec \sqrt{2\lg n}.\lg 2$

$m = \lg n$

$\underline{2.\lg(m)} \prec \sqrt{2}.\lg 2.\,\underline{m^{1/2}}$

т.е. $\underline{\ln^2 n \prec 2^{\sqrt{2\lg n}}}$

(11) n и $2^{\sqrt{2\lg n}}$ $\underline{\lg n \succ \sqrt{2}.\sqrt{\lg n}.\lg 2}$

т.е. $n \succ 2^{\sqrt{2\lg n}}$

Задача: 1

$1 \asymp n^{\frac{1}{\lg n}} \prec \lg(\lg^* n) \prec \lg^* n \asymp \lg^*(\lg n) \prec 2^{\lg^* n} \prec$
$\ln(\ln(n)) \prec \sqrt{\lg n} \prec \ln n \prec \ln^2 n \prec 2^{\sqrt{2\lg n}} \prec n \asymp$
$\asymp 2^{\lg n} \prec n\lg n \prec \lg(n!) \prec n^2 \asymp 4^{\lg n} \prec n^3 \prec$
$\prec (\lg n)! \prec (\lg n)^{\lg n} \asymp n^{\lg(\lg n)}$

(12) n и $2^{\lg n} \asymp 2^{\log_2 n} = n$
т.е. $n \asymp 2^{\lg n}$

(13) $n \prec n\lg n$

(14) $\lg(n!) \asymp n\lg n$

(15) $n^2 \succ n\lg n$

(16) n^2 и $4^{\lg n}$, защ $2^{2\lg n} = 2^{\lg n^2} \asymp 2^{\log_2 n^2} = \underline{n^2}$
$n^2 \asymp 4^{\lg n}$

(17) $n^2 \prec n^3$

(18) n^3 и $(\lg n)!$
$3\lg n \prec \lg((\lg n)!) \asymp \lg n \cdot \lg(\lg n)$

(19) $(\lg n)!$ и $(\lg n)^{\lg n}$
Нека $m = \lg n \to m! \prec m^m$ ✓.

(20) $(\lg n)^{\lg n}$ и $n^{\lg(\lg n)}$
$\lg n^{\lg n} = n^{\lg(\lg n)}$ ✓
$\lg n \cdot \lg(\lg n) = \lg(\lg n) \cdot \lg n$

Задача: 1

$$1 \asymp n^{1/\lg n} \alpha \lg(\lg^* n) \alpha \lg^* n \asymp \lg^*(\lg n) \alpha 2^{\lg^* n} \alpha$$

$$\ln(\ln(n)) \alpha \sqrt{\lg n} \alpha \ln n \alpha \ln^2 n \alpha 2^{\sqrt{2\lg n}} \alpha n \asymp$$

$$\asymp 2^{\lg n} \alpha n\lg n \asymp \lg(n!) \alpha n^2 \asymp 4^{\lg n} \alpha n^3 \alpha$$

$$\alpha (\lg n)! \alpha (\lg n)^{\lg n} \asymp n^{\lg(\lg n)} \alpha \left(\frac{3}{2}\right)^n \alpha 2^n \alpha n.2^n \alpha$$

$$\alpha e^n \alpha n! \alpha (n+1)!$$

21) $n^{\lg(\lg n)}$ и $\left(\frac{3}{2}\right)^n$

$$\lg(\lg n).\lg n \quad \alpha \lg 2 \alpha \quad n.\lg\left(\frac{3}{2}\right)$$

22) $\left(\frac{3}{2}\right)^n \alpha 2^n$ $\quad \lim\limits_{n\to\infty} \frac{2^n}{\left(\frac{3}{2}\right)^n} = \lim\limits_{n\to\infty}\left(\frac{3}{4}\right)^n = \infty$

23) $2^n \alpha n.2^n$

24) $n.2^n \alpha e^n$

$$\lim_{n\to\infty} \frac{n.2^n}{e^n} = \lim_{n\to\infty} \frac{n}{\left(\frac{e}{2}\right)^n} = 0$$

25) $n! \alpha e^n$

26) $n! \alpha (n+1)!$

$$\lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Зосогз: 1

$1 \asymp n^{\frac{1}{\lg n}} \propto \lg(\lg^* n) \propto \lg^* n \asymp \lg^*(\lg n) \propto 2^{\lg^* n} \propto$
$\ln(\ln(n)) \propto \sqrt{\lg n} \propto \ln n \propto \ln^2 n \propto 2^{\sqrt{2\lg n}} \propto n \asymp$
$\asymp 2^{\lg n} \propto n\lg n \propto \lg(n!) \propto n^2 \asymp 4^{\lg n} \propto n^3 \propto$
$\propto (\lg n)! \propto (\lg n)^{\lg n} \asymp n^{\lg(\lg n)} \propto \left(\frac{3}{2}\right)^n \propto 2^n \propto n.2^n \propto$
$\propto e^n \propto n! \propto (n+1)! \propto 2^{2^n} \propto 2^{2^{n+1}}$

(27) $(n+1)!$ и 2^{2^n}

$\lg(n+1)! \asymp (n+1).\lg(n+1) \propto 2^n.\lg 2$

(28) 2^{2^n} и $2^{2^{n+1}} = 2^{2^n.2} = 2^{2^n}.2^{2^n}$

$2^{2^n} \propto 2^{2^{n+1}}$

(23) The missing piece:

★ $2^{\frac{\lg n}{2}} \propto 2^{\lg n}$ $\lim\limits_{n\to\infty} \left(\frac{\sqrt{2}}{2}\right)^{\lg n} = 0$

★ $2^{\sqrt{2\lg n}}$ (?) $2^{\frac{\lg n}{2}}$

$\sqrt{2\lg n}.\lg 2 \asymp (\lg n)^{1/2} \propto \frac{\lg n}{2}.\lg 2 \asymp \lg n$

thus $2^{\sqrt{2\lg n}} \propto 2^{\frac{\lg n}{2}}$