Зад | Нека $\mathcal{L} = \langle \cdot \rangle$ и е с ф.р.

Нека $\mathcal{A} = \langle \mathbb{N} \cup \{0\}, \cdot^{\mathcal{A}} \rangle$, за $x \cdot^{\mathcal{A}} y = z \iff z = x . y$

a) Опр. $\{0\}$ и $\{1\}$

b) Опр. $\mathcal{M}_{n/m} = \{\langle n, m \rangle \in \mathbb{N} \times \mathbb{N} \mid \frac{n}{m} \in \mathbb{N}\}$

т.е. $(\exists z \in \mathbb{N})[n/m = z]$

c) Опр. $\mathcal{M}_{prime} = \{n \in \mathbb{N} \mid 1|n$ и $n|n$ и само те$\}$

d) Опр. $\{2\}, \{3\}, \dots$

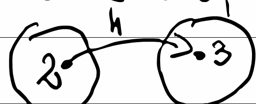
$\varphi_0(x) \leftrightharpoons \forall y (x \cdot y \doteq x)$

$\varphi_1(x) \leftrightharpoons \forall y (x \cdot y \doteq y)$

$\varphi_{\frac{n}{m}}(n, m) \leftrightharpoons \exists z (n \doteq z \cdot m)$

$\varphi_{prime}(x) \leftrightharpoons \forall y (\varphi_{n/m}(x, y) \Rightarrow \varphi_1(y) \vee x \doteq y)$

Но все $\{2\}, \{3\}$ не са определими.



def | Изоморфизъм

Нека h е биекция и $h: \mathcal{A} \to \mathcal{B}$ и

е хомоморфизъм т.е:

- $h(c^{\mathcal{A}}) = c^{\mathcal{B}}$
- $h(f^{\mathcal{A}}(a_1, \dots, a_n)) = f^{\mathcal{B}}(h(a_1), \dots, h(a_n))$
- $\langle a_1, \dots, a_n \rangle \in p^{\mathcal{A}} \iff \langle h(a_1), \dots, h(a_n) \rangle \in p^{\mathcal{B}}$

Следствие | h запазва определимите м-ва

Наричаме h автоморфизъм, ако е $h: A \to A$ и е изоморфизъм.

* $Aut(A) \neq \emptyset$, защото $Id_A \in Aut(A)$.

$\langle \mathbb{N}, pt \rangle$ е твърда структура

В предната задача с $\langle \mathbb{N}, p^k \rangle$ изх, че всички синглетони са определими, т.е. единствения автоморфизъм там е $Id_{\mathbb{N}}$. Тук нещата са по-различни.

От Алгебра знаем, че $\forall n \in \mathbb{N}, n > 1$ може да бъде представено в каноничен вид

$\underline{n = 2^{\alpha_1} 3^{\alpha_2} \dots p_m^{\alpha_m} \dots}$ $\to p_1 = 2, p_2 = 3, \dots$

Нека деф. ф-я $h: \mathbb{N}_{prime} \to \mathbb{N}_{prime}$

$$h(x) = \begin{cases} 2, & x = 3 \\ 3, & x = 2 \\ x, & else \end{cases}$$

Т.е. h е пермутация в/у простите числа $\to$

$\to h$ е биекция: $h(h(x)) = x \to h = h^{-1}$

Т.е. $h \circ h^{-1} = h^{-1} \circ h = Id_{\mathbb{N}prime}$

Нека конструираме $H: \mathbb{N} \to \mathbb{N}$ т.е.

$a \in \mathbb{N}: H(a) = H(2^{\alpha_1} \dots p_n^{\alpha_n} \dots) = h^{\alpha_1}(2) . h^{\alpha_2}(3) \dots h^{\alpha_n}(p_n) \dots$

$a = 2^{\alpha_1} \dots p_n^{\alpha_n} \dots$

$$= \prod_{i=1}^{\infty} h^{\alpha_i}(p_i)$$

$H(H(a)) = a \to H$ е биекция, $\underline{H = H^{-1}}$,
т.к. $H \circ H^{-1} = H^{-1} \circ H = Id_N$.

Защо е хом?

$H(\star^A(a,b)) \overset{?}{=} \star^A(H(a), H(b))$

Нека $a = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdots p_n^{\alpha_n} \cdots$

$b = 2^{\beta_1} \cdot 3^{\beta_2} \cdots p_n^{\beta_n} \cdots$

Тогава

$\star^A(H(2^{\alpha_1} 3^{\alpha_2} \cdots p_n^{\alpha_n} \cdots), H(2^{\beta_1} \cdot 3^{\beta_2} \cdots p_n^{\beta_n} \cdots)) =$

$= \star^A(3^{\alpha_1} \cdot 2^{\alpha_2} \cdots p_n^{\alpha_n} \cdots, 3^{\beta_1} 2^{\beta_2} \cdots p_n^{\beta_n} \cdots) =$

$= 2^{\alpha_2 + \beta_2} \cdot 3^{\alpha_1 + \beta_1} \cdots p_n^{\alpha_n + \beta_n} \cdots =$

$= H(\star^A(a,b))$ ✓.

т.е. $2 \in \{2\}$, но $H(2) = 3 \notin \{2\} \to \{2\}$ е неопр.

Реално, ако искаме да док., че $\{15\}$ не е определимо, то $15 = 3^1 \cdot 5^1$. Нека примерно менкаме 5 с 7 и док. неопр. за $\{15\}$ и т.н., а $60 = 5^1 3^1 2^2$, то може 3 с 2 да менкам

<u>HW</u> | $\mathcal{L} = \langle +, \star \rangle$ двуместни функ. символи и ф.р =
$\mathcal{A} = \langle \mathbb{R}, +^A, \star^A \rangle$, където

$a +^A b = c$
$a \star^A b = c$ $\Leftrightarrow$ $c = a + b$
$c = a \cdot b$

a) Опр $\{0\}$, $\{1\}$, $+1$, $\{2\}, \{3\}$... т.е. покажете, че $(\forall n \in \mathbb{N})[\{n\}$ е определимо$]$

b) $\{-n\}$ е опр. за $n \in \mathbb{N}$

c) $\{p/q\}$ за $q \neq 0$, $p, q \in \mathbb{Z}$ е опр.

d) $\leq$ и $<$ са определими

e) $\{-\sqrt[3]{\frac{13}{7}}\}$ е опр.

$\varphi_0(x) \leftrightharpoons \forall y (x \cdot y \doteq x)$

$\varphi_1(x) \leftrightharpoons \forall y (x \cdot y \doteq y)$

$\varphi_{x+1}(x, y) \leftrightharpoons \exists z (\varphi_1(z) \,\&\, x + z \doteq y)$

Така с индукция по n и база 0, 1, допускаме, че $\{n\}$ е определимо с ф-ла φ_n. Тогава за $\underline{n+1}$?

$\varphi_{n+1}(x) \leftrightharpoons \exists y (\varphi_n(y) \,\&\, \varphi_{x+1}(y, x))$

$\varphi_{-1}(x) \leftrightharpoons \exists y (\varphi_1(y) \,\&\, \neg \varphi_1(x) \,\&\, x \cdot x \doteq y)$.

$\varphi_{x-1}(x, y) \leftrightharpoons \exists z (\varphi_{-1}(z) \,\&\, x + z \doteq y)$

Отново с индукция по n определяме $\{-n\}$. Базата е 0, -1. Допускаме, че $\{-n\}$ е определимо с ф-ла φ_{-n}. А $-(n+1)$?

$\varphi_{-n-1}(x) \leftrightharpoons \exists y (\varphi_{-n}(y) \,\&\, \varphi_{-1}(y, x))$

Сега $\{p/q\}$ за $p, q \in \mathbb{Z}$.

$\varphi_{p/q}(x) \leftrightharpoons \exists y \exists z (\varphi_p(y) \,\&\, \varphi_q(z) \,\&\, \neg \varphi_0(z) \,\&\, x \cdot z \doteq y)$

$$\varphi_{\geq 0}(x) \Leftrightarrow \exists y\,(x \doteq y \cdot y)$$

$$\varphi_{>0}(x) \Leftrightarrow \neg \varphi_{0}(x) \,\&\, \varphi_{\geq 0}(x)$$

$$\varphi_{\leq}(x,y) \Leftrightarrow \exists z\,(\varphi_{\geq 0}(z) \,\&\, x + z = y)$$

$$\varphi_{<}(x,y) \Leftrightarrow \varphi_{\leq}(x,y) \,\&\, \neg(x \doteq y)$$

$$\varphi_{switchSign}(x,y) \Leftrightarrow \exists z\,(\varphi_{-1}(z) \,\&\, x * z \doteq y)$$

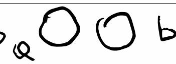
$$\varphi_{-\sqrt[3]{\frac{13}{7}}}(x) \Leftrightarrow \exists y \exists z\,(\varphi_{\frac{13}{7}}(y) \,\&\, z * z * z \doteq y \,\&\, \varphi_{switchSign}(z,x))$$

Зад | $\mathcal{L} = \langle p \rangle$, $\#p = 2$.

$\mathcal{A} = \langle$ окр. с ненулев радиус в $\mathbb{R}^2$, $p^{\mathcal{A}} \rangle$, където

a) $p^{\mathcal{A}}(a,b) \iff$ 

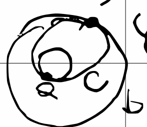
b) $p^{\mathcal{A}}(a,b) \iff$ 

c) $p^{\mathcal{A}}(a,b) \iff$ 

d) $p^{\mathcal{A}}(a,b) \iff$ 

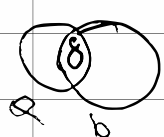
Опр. $\circledcirc^{a,b}$ ∞_{ab} и останалите.

a) $\varphi_=(x,y) \leftrightharpoons \forall z(p(x,z) \iff p(y,z))$ b), c), d)



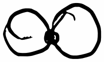
$\varphi_{\circledcirc}(x,y) \leftrightharpoons \exists z(p(x,z)\ \&\ p(z,y)\ \&\ \neg p(x,y)\ \&$
$\neg p(y,x)\ \&\ \neg \varphi_=(x,y))$

$\varphi_{\infty}(x,y) \leftrightharpoons \exists z(\varphi_{\circledcirc}(z,x)\ \&\ \varphi_{\circledcirc}(z,y)\ \&$
$\neg p(x,y)\ \&\ \neg p(y,x)\ \&\ \neg \varphi_=(x,y)\ \&$
$\neg \varphi_{\circledcirc}(x,y)\ \&\ \neg \varphi_{\circledcirc}(y,x))$

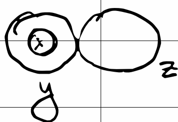


$\varphi_{OO}(x,y) \leftrightharpoons \forall z(\varphi_{\circledcirc}(y,z) \Rightarrow (p(x,z) \vee$
$\varphi_{\circledcirc}(x,z) \vee \varphi_{\infty}(x,z))$
$\&\ \neg p(x,y)\ \&\ \neg p(y,x)\ \&\ \neg \varphi_=(x,y)\ \&$
$\neg \varphi_{\infty}(x,y))$

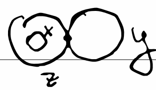
$\varphi_{oo}(x,y) \leftrightharpoons$ отр. които други **или**
$\exists z(\varphi_{\circledcirc}(x,z)\ \&\ \varphi_{OO}(z,y))$

b) p 

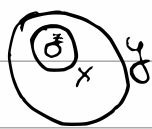
$\varphi_{\odot}(x,y) \leq \neg \varphi_{\subseteq}(x,y) \,\&\, \neg p(x,y) \,\&\, \forall z\,(p(y,z) \Rightarrow \neg p(z,x))$

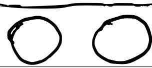


$\varphi_{\circ\circ}(x,y) \leq \exists z\,(\varphi_{\odot}(x,z) \,\&\, p(z,y))$

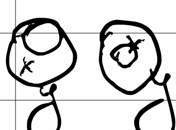


$\varphi_{\odot}(x,y) \leq \neg \varphi_{\odot}(x,y) \,\&\, \neg \varphi_{\odot}(y,x) \,\&\, \forall z\,(\varphi_{\odot}(z,x) \Rightarrow \varphi_{\odot}(z,y))$

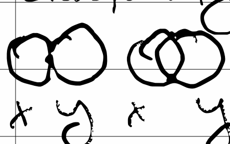


c)  p

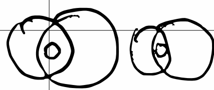
$\varphi_{\mathrm{Qcb}}(x,y) \leq \neg p(x,y) \,\&\, \neg \varphi_{=}(x,y) \,\&\, \forall z\,(p(y,z) \Rightarrow p(x,z))$



$\varphi_{1\,\mathrm{or}\,2\,\mathrm{points}}(x,y) \leq \neg p(x,y) \,\&\, \neg \varphi_{=}(x,y) \,\&\, \neg \varphi_{\mathrm{Qcb}}(x,y)$

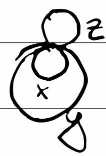


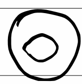
$\varphi_{\circ\!\!\circ}(x,y) \rightleftharpoons \varphi_{1 or 2 points}(x,y) \,\&\, \exists z\,(\varphi_{a\leq b}(z,x) \,\&\, \varphi_{a\leq b}(z,y))$



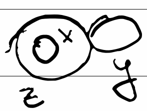
$\varphi_{\circ\circ}(x,y) \rightleftharpoons \varphi_{1 or 2 points}(x,y) \,\&\, \neg\varphi_{\circ\!\!\circ}(x,y)$

$\varphi_{\odot}(x,y) \rightleftharpoons \varphi_{a\leq b}(x,y) \,\&\, \exists z(\varphi_{\circ\circ}(x,z) \,\&\, \varphi_{\circ\circ}(y,z))$



d)  p

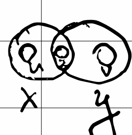
$\varphi_{\circ\circ\circ}(x,y) \rightleftharpoons \neg\varphi_{=}(x,y) \,\&\, \neg\exists z\,(p(z,x) \,\&\, p(z,y))$



$\varphi_{\circ\circ}(x,y) \rightleftharpoons \varphi_{\circ\circ\circ}(x,y) \,\&\, \exists z\,(p(x,z) \,\&\, \varphi_{\circ\circ\circ}(z,y))$

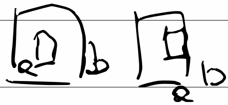
$\varphi_{\circ\circ}(x,y) \rightleftharpoons \varphi_{\circ\circ\circ}(x,y) \,\&\, \neg\varphi_{\circ\circ}(x,y)$

$\varphi_{\circ\!\!\circ}(x,y) \rightleftharpoons \exists z\,(p(z,x) \,\&\, p(z,y)) \,\&\, \exists u \exists v\,(p(u,x) \,\&\, p(v,y) \,\&\, \varphi_{\circ\circ}(x,v) \,\&\, \varphi_{\circ\circ}(y,u))$



НШ A < всички квадрати с ненулеви страни в $\mathbb{R}^2$, p^R > : плътни фигури

$p^R(a,b) \iff a$ и b имат поне една обща точка

a)  $a \subseteq b \to a \cap b = a$

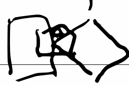
b) $a \cap b =$ точка $\to \varphi_\bullet$

c) $a \cap b =$ отсечка $\to \varphi_1$

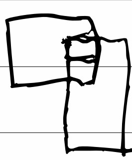
d) $a \cap b =$ квадрат $\to \varphi_\square$

$\varphi_\subseteq(x,y) \leftrightharpoons \forall z\,(p(z,x) \Rightarrow p(z,y))$

$\varphi_\bullet(x,y) \leftrightharpoons p(x,y) \,\&\, \forall z\,(p(z,x) \,\&\, p(z,y) \,\&\, \neg\varphi_\subseteq(z,x) \,\&\, \neg\varphi_\subseteq(z,y) \Rightarrow \exists w\,(\varphi_\subseteq(w,z) \,\&\, \neg p(w,x) \,\&\, \neg p(w,y)))$



$\varphi_\square(x,y) \leftrightharpoons \exists z\,(\varphi_\subseteq(z,x) \,\&\, \varphi_\subseteq(z,y) \,\&\, \forall t\,(\varphi_\subseteq(t,x) \,\&\, \varphi_\subseteq(t,y) \Rightarrow \varphi_\subseteq(t,z)))$



$\varphi_1(x,y) \leftrightharpoons p(x,y) \,\&\, \neg\varphi_\bullet(x,y) \,\&\, \neg\exists z\,(\varphi_\subseteq(z,x) \,\&\, \varphi_\subseteq(z,y))$

302g) $\mathcal{L} = \langle p \rangle$, $\Rightarrow$

$\mathcal{A} = \langle \{a,b\}^*, p^{\mathcal{A}} \rangle$

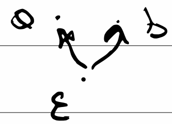
$p^{\mathcal{A}}(u,v) \Leftrightarrow ||u| - |v|| = 1$

a) $\{\varepsilon\}$

b) $\{w \mid |w| = 2\}$

c) Да, те $\{b\}$ е неопределимо.

$\varphi_\varepsilon(x) \leftrightharpoons \exists y_1 \exists y_2 (\neg(y_1 \doteq y_2) \,\&\, p(x,y_1) \,\&\, p(x,y_2)$
$\&\, \forall z (p(x,z) \Rightarrow z \doteq y_1 \vee z \doteq y_2))$



$\varphi_1(x) \leftrightharpoons \exists y(\varphi_\varepsilon(y) \,\&\, p(x,y))$

$\varphi_2(x) \leftrightharpoons \neg\varphi_\varepsilon(x) \,\&\, \neg\varphi_1(x) \,\&\, \forall y(\varphi_1(y) \Rightarrow p(x,y))$

...

$h : \{a,b\} \to \{a,b\}$ е т.ч. $h(a) = b$ и $h(b) = a$.

$h = h^{-1}$ биекция.

Нека $H(w) = H(a_1 \dots a_n) = h(a_1) \dots h(a_n)$

$w = a_1 \dots a_n$

$H = H^{-1}$ биекция. Хом?

$\langle a,b \rangle \in p^{\mathcal{A}} \Leftrightarrow \langle H(a), H(b) \rangle \in p^{\mathcal{A}}$ ✓

КШ1 $\mathcal{L} = \langle f \rangle$ с "$\doteq$"

$\mathcal{A} = \langle \mathbb{N}, f^{\mathcal{A}} \rangle$ и $f^{\mathcal{A}}(n) = \left\lfloor \frac{n}{2} \right\rfloor$

a) Опр. $\{0\}$ и $\{1\}$

b) Опр. ли е $\{n\}$ за $n \in \mathbb{N} \setminus \{0,1\}$?

$\varphi_0(x) \leftrightharpoons f(x) \doteq x$.

$\varphi_1(x) \leftrightharpoons \neg\varphi_0(x) \,\&\, \exists y(\varphi_0(y) \,\&\, f(x) \doteq y)$

Не е опр. $n \in \mathbb{N} \setminus \{0,1\}$.

Нека $h : \{0,1\} \to \{0,1\}$, т.е. $h(0) = 1$ и $h(1) = 0$.

$h = h^{-1}$ $\quad h \circ h^{-1} = h^{-1} \circ h = id_{\{0,1\}}$, h биекция

Нека $H : \mathbb{N} \to \mathbb{N}$.

Нека $n \in \mathbb{N} \setminus \{0,1\}$. Тогава n има двоичен запис

$n = 1.a_1 \ldots a_m$, $m \geq 1$, $m \in \mathbb{N}$, където $a_i \in \{0,1\}$ за $i \in \{1, \ldots, m\}$

(Разглеждаме n като низове от 0-ли и 1-ци).

Дефинираме $H(n) \leftrightharpoons 1.h(a_1).h(a_2). \ldots .h(a_m)$

и деф. $H(0) = 0$ и $H(1) = 1$.

$H(H(n)) = n$, т.е. $H = H^{-1}$ биекция. Хом?

$H(f(n)) \overset{?}{=} f(H(n))$

Как действа f в/у двоичния запис на числото n? $f(n) = \left\lfloor \frac{n}{2} \right\rfloor = 1a_1 \ldots a_{m-1}$, a

Н реално прави: $H(n) = 1\bar{a}_1\bar{a}_2 \ldots \bar{a}_m$ за $\bar{a}_i = 1 - a_i$

$a_i \in \{0,1\}$

т.е. H инвертира всички битове на числото и връщения му запис без старшия бит.

Сега:

$$f(H(n)) = f(H(1a_1 \dots a_m)) = f(1\bar{a}_1 \dots \bar{a}_m) =$$

$$\underset{m \geq 1}{=} 1\bar{a}_1 \dots \overline{a_{m-1}} = H(1a_1 \dots a_{m-1}) =$$

$$= H(f(1a_1 \dots a_{m-1}a_m)) =$$

$$= H(f(n)).$$

Зад/ $\mathcal{A} = \langle \mathbb{N}\setminus\{0\}, f^{\mathcal{A}} \rangle$ и $\mathcal{L} = \langle f \rangle$ с оф. р

Опр. фн-у за $\mathbb{N}\setminus\{0\}$.

$\varphi_1(x) \leftrightharpoons \forall z\,(f(x,z) \doteq x)$

addition and multiplication

$n^m \cdot n^k = n^{m+k}$ $\qquad (n^m)^k = n^{m\cdot k}$

$\varphi_*(x,y,z) \leftrightharpoons \forall w\,(f(f(w,x),y) \doteq f(w,z))$

$\varphi_+(x,y,z) \leftrightharpoons \forall w\,(\varphi_*(f(w,x), f(w,y), f(w,z)))$