Задача за изпълнимост на м-во от ф-ли

def/ Нека Γ е м-во от ф-ли за език $\mathcal{L}$.

i) **Γ е вярно в $\mathcal{A}$ при опр. v от $\mathcal{A}$** $\leftrightarrow$ $(\forall \varphi \in \Gamma)[\mathcal{A} \models_v \varphi]$.

Бел: $\mathcal{A} \models_v \Gamma$

ii) **$\mathcal{A}$ е модел за Γ**, ако $(\forall \varphi \in \Gamma)[\mathcal{A} \models_v \varphi]$.
$\qquad (\forall v$ от $\mathcal{A})$

Бел: $\mathcal{A} \models \Gamma$

iii) **Γ е изпълнимо**, ако Γ има модел.

Задачата за намиране на модел на м-во от ф-ли е аналогична на задачата дали съществува програма отговаряща на дадена формална спецификация.

(3.22) Да се док., че следния набор от затворени ф-ли е изпълним:

Γ: $\varphi_1 \leftrightharpoons \forall x \neg p(x,x)$. // иррефлексивност

$\varphi_2 \leftrightharpoons \forall x \exists y p(x,y)$. // $x \to y$? сериалност надясно

$\varphi_3 \leftrightharpoons \forall x \forall y \forall z (p(x,y) \& p(y,z) \Rightarrow p(x,z))$. // транзитивност

$\mathcal{A}_1 = \langle \mathbb{Z}, p^{\mathbb{Z}} \rangle$, $p^{\mathbb{Z}}(a,b) \leftrightarrow a < b$.

$\mathcal{A}_1 \models \Gamma$

$\mathcal{A}_1 \models \varphi_1$

Има ли краен модел Γ?

От деф. на структура знаем, че $A \neq \emptyset$.

Нека $a_0 \in A$ е свидетел за $A \neq \emptyset$.

$\mathcal{A} \models \neg p(x,x)[\![a_0]\!]$, $\mathcal{A} \models \exists y p(x,y)[\![a_0]\!]$

Нека $a_1 \neq a_0 \in A$, т.ч. $\mathcal{A} \models p(x,y)[\![a_0, a_1]\!]$.

$\mathcal{A} \models \exists y p(x,y)[\![a_1]\!]$.

Нека $a_2 \in A$, т.ч. $\mathcal{A} \models p(x,y)[\![a_1, a_2]\!]$

$a_0 \to a_1 \to a_2 \to a_3 \to a_4 \to \dots$ „Винаги има следващ"

Допускаме, че A е крайно, т.е. $(\exists n \in \mathbb{N})[|A| = n]$ и $A \in \Gamma$.
Нека $n_0 \in \mathbb{N}$ е т.ч. $|A| = n_0$.

$a_0 \to a_1 \to a_2 \to \dots \to a_{n_0-1} \to a_{n_0}$ Pigeonhole principle

Поне два елемента в редицата съвпадат.
Нека $i \neq j \in \mathbb{N}$, т.ч. $i, j \leq n_0$ и $a_i = a_j$.

$a_0 \to a_1 \to \dots \to a_i \to \dots \to a_j \to \dots a_{n_0}$

Но φ_3 значи $a_i \to a_j$, а също $a_i = a_j$.

$\Downarrow$

$a_i = a_j$

Но φ_1 твърдяме че няма рефлексивни точки. Абсурд!

(323) Да се док., че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \leftrightharpoons \forall y \exists x\, p(y,x,y).$

$\varphi_2 \leftrightharpoons \exists y \neg p(y,y,y)$

$\varphi_3 \leftrightharpoons \forall x \forall y\, p(x,y,x).$ $\vdash\!\dashv$ $\forall x \exists y \neg p(x,y,x).$

Нека $q(x,y) \leftrightharpoons p(x,y,x).$

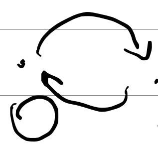
$\underline{y\ x\ \text{познава}\ y}$

$\forall y \exists x\, q(y,x).$ // имаме познат

$\exists y \neg q(y,y).$ // ние сме си непознати.

$\cancel{\forall x \exists y \neg q(x,y)}$ // имаме непознати

следствие от втората за $y = x.$



$\mathcal{A} = \langle \{0,1\}, p^{\mathcal{A}} \rangle$

$p^{\mathcal{A}} = \{ \langle 0,1,0 \rangle, \langle 1,0,1 \rangle \}$

(302) Да се док., че следния набор от затворени ф-ли е изпълним:

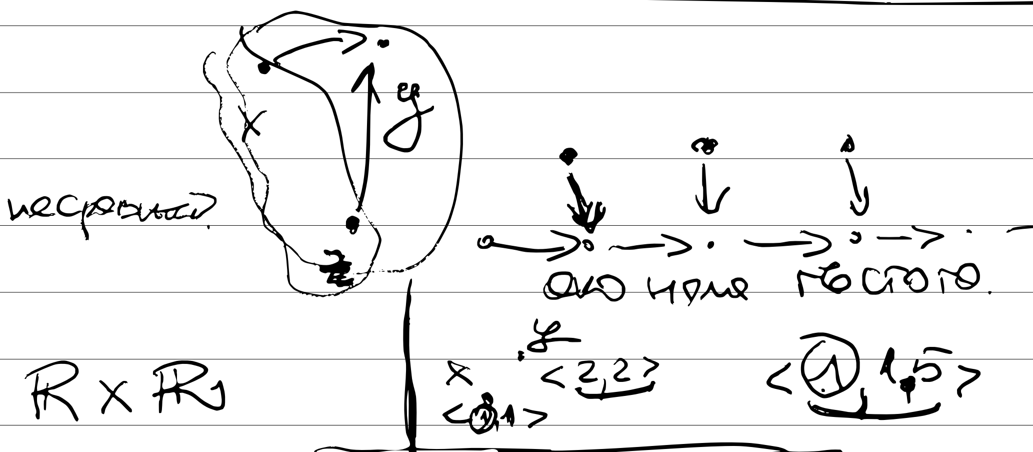
$\varphi_1 \leftrightharpoons \forall x \forall y \forall z (q(x,y) \,\&\, q(y,z) \Rightarrow q(x,z))$. //транзит.

$\varphi_2 \leftrightharpoons \exists x \exists y\, q(x,y)$. //непразна

$\varphi_3 \leftrightharpoons \forall x \neg q(x,x)$. //иррефл.

$\varphi_4 \leftrightharpoons \forall x \forall y (q(x,y) \Rightarrow \exists z (q(x,z) \,\&\, q(z,y)))$. //гъсто

$\varphi_5 \leftrightharpoons \forall x \forall y (q(x,y) \Rightarrow \exists z (q(z,y) \,\&\, \neg(z \doteq x) \,\&\, \neg q(x,z) \,\&\, \neg q(z,x)))$.



$p^A(\langle a,b\rangle, \langle c,d\rangle) \Leftrightarrow a < c \,\&\, b < d$

$a_1 < a \le c$

Нека $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in q^{\mathcal{A}}$.

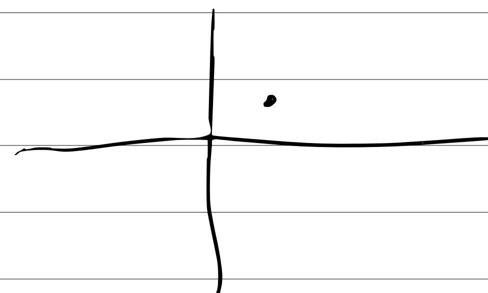
Тогава нека $z = \langle a_1, \frac{b+c}{2}\rangle \in \mathbb{R}\times\mathbb{R}$.

и $\langle z, \langle c,d\rangle\rangle \in q^{\mathcal{A}}$, но

$\langle z, \langle a,b\rangle\rangle \notin q^{\mathcal{A}}$ и

$\langle\langle a,b\rangle, z\rangle \notin q^{\mathcal{A}}$.

Тогава $\mathcal{A} \models \varphi_5$



$\mathcal{A} = \langle \mathbb{R}\times\mathbb{R}, q^{\mathcal{A}}\rangle$

$\mathbb{Q}\times\mathbb{Q}$

$\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in q^{\mathcal{A}} \iff a \le c$ и

$b < d$

(3.23) Да се док., че следния набор от затворени ф-ли е изпълним:

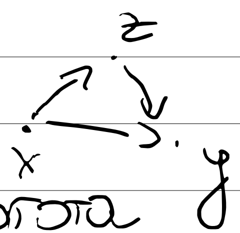
$\varphi_1 \rightleftharpoons \forall x(\neg p(f(x),x) \ \& \ \exists y\, p(f(x),y))$.

$\varphi_2 \rightleftharpoons \forall x \forall y(p(x,y) \Rightarrow \exists z(p(x,z) \ \& \ \neg p(f(z),f(z)) \ \& \ p(z,y)))$

$\varphi_3 \rightleftharpoons \neg\forall x \forall y \forall z(p(x,y) \ \& \ p(y,z) \Rightarrow \neg p(\underbrace{f(x)}_{x},z))$.

$\underbrace{\forall x \neg p(f(x),x)}_{\text{иррефл.}}$ $\underbrace{\forall x \exists y\, p(f(x),y)}_{\text{сериалност нагоре.}}$

$\underline{f^A = Id_A}$



гъстотата

$\models \exists x \exists y \exists z(p(x,y) \ \& \ p(y,z) \ \& \ p(\underbrace{f(x)}_{x},z))$

$A = \langle \mathbb{Q} ; f^A ; p^A \rangle$

$f^A \rightleftharpoons Id_{\mathbb{Q}}$, $p^A(a,b) \Leftrightarrow a < b$

(322) Да се док., че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \leftrightharpoons \forall x \forall y (p(x,y) \Leftrightarrow q(y,x))$.

$\varphi_2 \leftrightharpoons \forall x \exists y (p(x,y) \Leftrightarrow q(x,y))$.

$\varphi_3 \leftrightharpoons \forall x \forall y \forall z (p(x,y) \& p(y,z) \Rightarrow p(x,z))$.

$\varphi_4 \leftrightharpoons \exists x \exists y (\neg q(x,y) \& \neg q(y,x))$.

$\varphi_5 \leftrightharpoons \exists x \exists y \neg(x \doteq y)$.

→ има поне две несравними точки под q^{A}.

→ $p^{A} = (q^{A})^{-1}$

→ p^{A} е транзитивно.

→ поне за едно y: $p^{A}(x,y) \Leftrightarrow q^{A}(x,y)$.

$A = \langle \{0\}; p^{A}, q^{A} \rangle$, $p^{A} = q^{A} = \emptyset$.

→ $B = \langle \{0,1\}; p^{A}, q^{A} \rangle$, $p^{A} = q^{A} = \emptyset$.

(323) Да се док., че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \leftrightharpoons \forall x \exists y \exists z\, (p(x,y) \,\&\, p(x,z) \,\&\, \neg(y \doteq z))$.

$\varphi_2 \leftrightharpoons \neg p(a,b)$.

$\varphi_3 \leftrightharpoons \exists x \exists y \exists z \forall w\, (\neg(w \doteq x) \,\&\, \neg(y \doteq w) \Rightarrow w \doteq z)$.

a, b - константи, p - двуместен предик. символ.

Може ли $|A| = 1$? Ако гледаме само φ_3, да може. Т.е. $A_1 = \langle \{0\}, p^{A_1} = \emptyset \rangle$, $p^{A_1} = \emptyset$, то $x = y = z = 0$

$A_1 \models \varphi_3$.

Може ли $|A| = 2$? Гледаме φ_3.

$x = y = 0$, $z = 1$

$\langle 0, 1$

$\forall w\, (\neg(w \doteq 0) \Rightarrow w \doteq 1)$

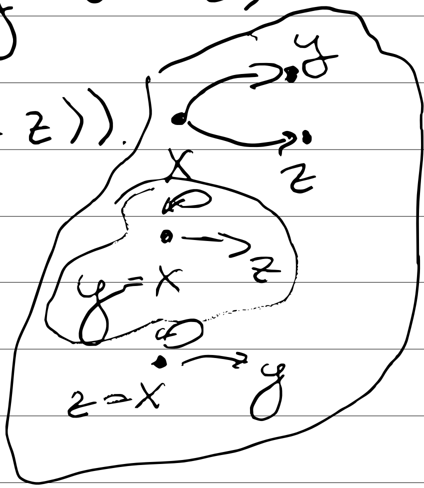
Може ли $|A| = 3$? Да за $x \neq y \neq z$.

Но може ли повече от 3 елем. $\rightarrow$ Не!

$\varphi_3 \leftrightharpoons \exists x \exists y \exists z \forall w (w \doteq x \vee w \doteq y \vee w \doteq z)$

$\varphi_1 \leftrightharpoons \forall x \exists y \exists z (p(x,y) \& p(x,z) \& \neg(y \doteq z))$.

$\varphi_2 \leftrightharpoons \neg p(a,b)$.



$\Rightarrow |A| > 1$

Ако $|A| = 2$.

но, ако следваме φ_2, то трябваше $\neg p(a,b)$.

Тогава остава $|A| = 3$.

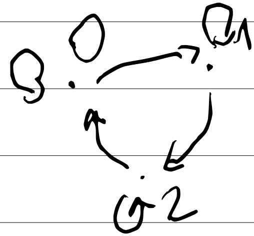
$a^A \leftrightharpoons 0$

$b^A \leftrightharpoons 1$

$A \models \neg p(a,b)$.

$A = \langle \{0,1,2\}; p^A \rangle$

$p^A = \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 2,1\rangle\}$.

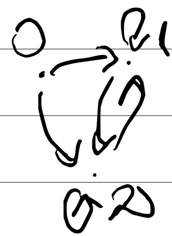


$a^A \leq 1$; $a^B \leq 0$

$b^A \leq 0$; $b^B \leq 2$

$\neg p(a,b)$

$\{\Box, \Diamond, \bigcirc\}$



$a^A \leq 0$

$b^A \leq 0$

$\neg p(a,b)$.

302) Да се докаже, че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \leftrightharpoons \forall x(\neg(x \doteq f(x)) \,\&\, \neg(x \doteq g(x)))$.

$\varphi_2 \leftrightharpoons \exists x(x \doteq f(g(x)) \,\&\, x \doteq g(f(x)))$.

$\varphi_3 \leftrightharpoons \exists x(f(x) \doteq g(x))$.

$\varphi_4 \leftrightharpoons \exists x \neg(f(g(x)) \doteq g(f(x)))$.

$\varphi_5 \leftrightharpoons \forall y \exists x(y \doteq g(x))$.

A, f^A, g^A, $Dom(f^A) = Dom(g^A) = A$

$Range(f^A) \subseteq A$

$Range(g^A) \subseteq A$

φ_5 – сюрекция g.

φ_1 – $f^A \neq id_A$, $g^A \neq id_A$

φ_4 – $f^A \neq g^A$

φ_3 – поне в една точка съвпадат.

$\mathbb{N}$, $\quad \mathcal{A} = \langle \mathbb{N}; f^{\mathcal{A}}, g^{\mathcal{A}} \rangle$, $\quad \mathcal{A} \models \Gamma$

$f^{\mathcal{A}}(x) \leftrightharpoons x+1$

$$g^{\mathcal{A}}(x) \leftrightharpoons \begin{cases} x-1, & x>0 \\ 1, & x=0 \end{cases} \qquad \Gamma = \{\varphi_1, \dots, \varphi_5\}.$$

$\varphi_1 \leftrightharpoons \forall x\,(\neg(x \doteq f(x)) \,\&\, \neg(x \doteq g(x)))$.

$\varphi_2 \leftrightharpoons \exists x\,(\underline{x \doteq f(g(x))} \,\&\, \underline{x \doteq g(f(x))})$.

$\varphi_3 \leftrightharpoons \exists x\,(f(x) \doteq g(x))$.

$\varphi_4 \leftrightharpoons \exists x \neg(f(g(x)) \doteq g(f(x)))$.

$\varphi_5 \leftrightharpoons \forall y \exists x\,(\underline{y \doteq g(x)})$.

$x=2: \quad 2 = (2-1)+1 \quad \& \quad 2 = ((2+1)-1)$

$\neg \cancel{x} = 0$

$$(\overset{g}{\underbrace{0}_{1}})+1 = 2 \neq 1 = (\underline{1+1})-1$$

$f^{\mathcal{A}}(x) \leftrightharpoons (x+1)\%3 \qquad \{0,1,2\}$

$g^{\mathcal{A}}(x) \leftrightharpoons (x+2)\%3$

Nope

$$f^{t}(x) \simeq \begin{cases} \frac{1}{x}, & x \neq 0, 1 \\ 1, & x = 0 \\ 0, & x = 1 \end{cases}$$

$$g^{t}(x) \simeq x + 1$$

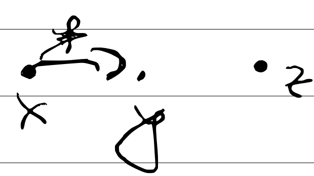
$x = 0$ и. $\varphi_3 : 0$

φ_1.

$\varphi_2 : 0$

НО: Има стр с 3 точки в унив, т.е. $\exists\, \mathfrak{B}$, т.ч. $\|\mathfrak{B}\| = 3$ и $\mathfrak{B} \models \Gamma$.

$\underline{f^{t}(x) = y}$



Зад. Нека имаме ф-лите:

$\varphi_1 \leftrightharpoons \forall x \exists y (\neg p(x,x) \,\&\, p(x,y))$.

$\varphi_2 \leftrightharpoons \forall x \forall z (\exists y (p(x,y) \,\&\, p(y,z)) \Leftrightarrow p(x,z))$.

$\varphi_3 \leftrightharpoons \exists x \exists y \exists z (\neg p(x,y) \,\&\, \neg p(y,x) \,\&\, \neg (p(x,z) \Leftrightarrow p(y,z)))$.

$\varphi_4 \leftrightharpoons \forall x \forall y (p(x,y) \Leftrightarrow p(x, f(x,y)) \,\&\, p(f(y,x), y))$.

$\varphi_5 \leftrightharpoons \forall x \forall y (p(x,y) \Leftrightarrow p(x, f(x,y)) \vee p(f(y,x), y))$.

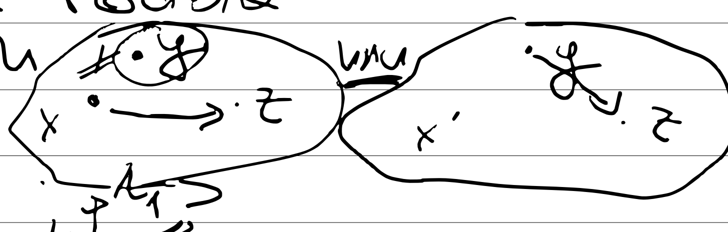
Нека $\Gamma_1 \leftrightharpoons \{\varphi_1, \varphi_2, \varphi_3\}$, $\Gamma_2 \leftrightharpoons \Gamma_1 \cup \{\varphi_4\}$, $\Gamma_3 \leftrightharpoons \Gamma_1 \cup \{\varphi_5\}$.

Да се докаже кои от множествата $\Gamma_1, \Gamma_2, \Gamma_3$ са изпълними.

иррефлекс. и сериалност надясно

транзитивност → и гъстота

две несравними и



$A_1 = \langle \{0,1\} \times \mathbb{Q}$

$1 \quad \mathbb{Q}$ ——————————

$0 \quad \mathbb{Q}$ ——————————

Γ_1

$p^{A_1}(\langle a,b\rangle, \langle c,d\rangle) \iff a=c \ \& \ b <_{\mathbb{Q}} d$

$\varphi_1 \leftrightharpoons \forall x \exists y (\neg p(x,x) \ \& \ p(x,y)).$

$\varphi_2 \leftrightharpoons \forall x \forall z (\exists y (p(x,y) \ \& \ p(y,z)) \iff p(x,z)).$

$\varphi_3 \leftrightharpoons \exists x \exists y \exists z (\neg p(x,y) \ \& \ \neg p(y,x) \ \& \ \neg (p(x,z) \iff p(y,z))).$

$x = \langle 0,0\rangle, \ y = \langle 1,0\rangle$

$z = \langle 0,1\rangle$

$\varphi_4 \leftrightharpoons \forall x \forall y (p(x,y) \iff (p(x, \underbrace{f(x,y)}_{\frac{x+y}{2}}) \ \& \ p(\underbrace{f(y,x)}_{\frac{x+y}{2}}, y))).$

Γ_2:

$$f^{A}(\langle a,b\rangle, \langle c,d\rangle) \leftrightharpoons \begin{cases} \frac{b+d}{2}, & a=c \\ \langle 0,0\rangle, & a \neq c \end{cases}$$

$A_2 = \langle \{0,1\} \times \mathbb{Q}, \ p^{A_2}, f^{A_2}\rangle$

$p^{A_2} \leftrightharpoons p^{A_1}$

$\varphi_5 \leq \forall x \forall y\, (p(x,y) \Leftrightarrow p(x, f(x,y)) \vee p(f(y,x), y))$

y

$f^{A_3}(x,y) \leq y$

$A_3 = \langle \{0,1\} \times \ldots ; p^{A_3}, f^{A_3} \rangle$

$p^{A_3} \leq p^{A_1}$

$$\frac{p(x,y) \vee p(x,y)}{p(x,y)}$$

$A_3 \vDash P_2$

$A_3 \vDash P_3$

(322) Да се докаже, че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \leftrightharpoons \forall x \forall y (\exists z (p(x,z) \Rightarrow p(z,y)) \Leftrightarrow \exists z (q(x,z) \Rightarrow q(z,y)))$.

$\varphi_2 \leftrightharpoons \exists x \exists y p(x,y) \& \exists x \forall y q(x,y)$.

$\varphi_3 \leftrightharpoons \forall x \forall y (q(x,y) \Rightarrow \neg q(y,x)) \& \forall x \forall y (p(x,y) \Rightarrow \neg p(y,x))$.

$\varphi_4 \leftrightharpoons \neg \forall x \forall y \forall z (p(x,y) \& p(y,z) \Rightarrow p(x,z))$.