300/ Нека $\mathcal{L}=\langle p\rangle$ FOL, $p\in Pred$, $\#p=3$.
Нека $\mathcal{A}=\langle \mathcal{P}(\mathbb{N}), p^{\mathcal{A}}\rangle$ е структура за $\mathcal{L}$, като:
- $\mathcal{P}(\mathbb{N})$ - м-вото от всички подмн-ва на $\mathbb{N}$ т.е. $\mathcal{P}(\mathbb{N})=\{B \mid B\subseteq \mathbb{N}\}$
- $\langle a,b,c\rangle\in p^{\mathcal{A}} \Longleftrightarrow a\cup b=c$

$a,b,c\in\mathcal{P}(\mathbb{N})$

Да се докаже, че:

a) $\{\emptyset\}$ е определимо

b) $\{\mathbb{N}\}$ е определимо

c) $\subseteq \leq \{\langle x,y\rangle \mid x,y\in\mathcal{P}(\mathbb{N}) \,\&\, x\subseteq y\}$ е определимо

d) $\cap \leftrightharpoons \{\langle x, y, z\rangle \mid x, y, z \in \mathcal{P}(\mathbb{N})$ & $x \cap y = z\}$

е определимо

e) $^{-} \leftrightharpoons \{\langle x, y\rangle \mid x, y \in \mathcal{P}(\mathbb{N})$ & $y = \mathbb{N} \setminus x\}$

т.е. y е допълнението на x с точност до $\mathbb{N}$

е определимо.

f) Докажете, че за $Q \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset, \mathbb{N}\}$, то $\{Q\}$ е неопределимо.

a) $\{\emptyset\}$ е определимо

b) $\{\mathbb{N}\}$ е определимо

c) $\underset{\sim}{\subseteq} \leftrightharpoons \{\langle x,y\rangle \mid x,y \in \mathcal{P}(\mathbb{N})\ \&\ x \subseteq y\}$

е определимо

$\langle a,b,c\rangle \in p^t \iff a \cup b = c$

$\varphi_\emptyset(x) \leftrightharpoons \forall y\, p(x,y,y)$.

<u>За оценка v в $\mathcal{A}$, то:</u>

$\mathcal{A} \models_v \varphi_\emptyset(x) \iff v(x) = \emptyset$

$(\Leftarrow)$ Нека $v(x) = \emptyset$

Тогава $\underbrace{\mathcal{A} \models_v \forall y\, p(x,y,y)}$

$v_a^y(z) = \begin{cases} a & z = y \\ v(z), & z \neq y \end{cases}$

$\underbrace{\text{за вс. } a \in \mathcal{P}(\mathbb{N}) : \underset{\emptyset}{v(x)} \cup v_a^y(y) = v_a^y(y)}$

$\underbrace{\text{за вс. } a \in \mathcal{P}(\mathbb{N}) : \emptyset \cup \underset{a}{v_a^y(y)} = v_a^y(y) = a}$

за вс. $a \in \mathcal{P}(\mathbb{N})$: $\emptyset \cup a = a$ (V)

(→) Нека $v(x) \neq \emptyset$.

- $\varphi \to \psi$
- $\neg\psi \to \neg\varphi$

$$\| \forall y\, p(x, y, y) \|^{\mathcal{A}} [v] = ?$$

Нека $v^{y}_{\emptyset}$ въведем.

$$\text{Тогава } \underbrace{\| p(x, y, y) \|^{\mathcal{A}} [v^{y}_{\emptyset}] = \mathbb{T}}_{\langle v(x),\, v^{y}_{\emptyset}(y),\, v^{y}_{\emptyset}(y) \rangle \in p^{\mathcal{A}}}$$

$$v(x) \neq \emptyset, \quad v^{y}_{\emptyset}(y) = \emptyset, \quad v^{y}_{\emptyset}(y) = \emptyset$$

$$v(x) \cup \emptyset = \emptyset \quad ↯$$

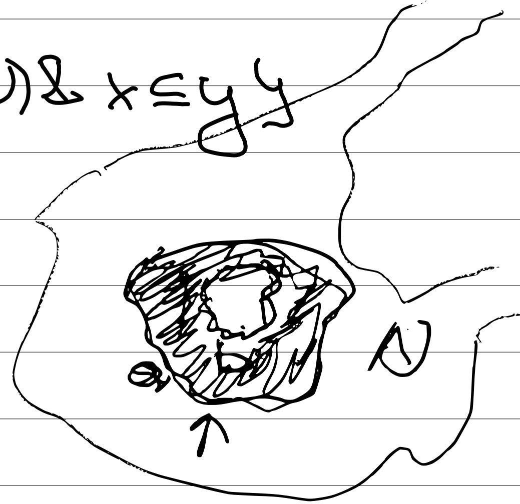
$$v(x) \neq \emptyset$$

Значи $\| \varphi_{\emptyset}(x) \|^{\mathcal{A}} [v] = \mathbb{F}$

b) $\{\mathbb{N}\}$ е определимо

c) $\subseteq \leftrightharpoons \{\langle x,y\rangle \mid x,y\in\mathcal{P}(\mathbb{N}) \,\&\, x\subseteq y\}$

е определимо



$\varphi_{\mathbb{N}}(x) \leftrightharpoons \forall y\, \varphi_{\subseteq}(y,x)$.

$\varphi_{\mathbb{N}}(x) \leftrightharpoons \forall y\, p(x,y,x)$

$\varphi_{\subseteq}(x,y) \leftrightharpoons \exists z\, p(x,z,y)$

$\varphi_{\subseteq}(x,y) \leftrightharpoons p(x,y,y)$ // $A\cup B = B \leftrightarrow A\subseteq B$

$\mathcal{A} \models_v \varphi_{\mathbb{N}}(x) \longleftrightarrow v(x)=\mathbb{N}$

$(\rightarrow)$ Нека $\mathcal{A}\models_v \varphi_{\mathbb{N}}(x)$ т.е. $\|\varphi_{\mathbb{N}}(x)\|^{\mathcal{A}}[v] = T$

т.е. за вс. $Q\in\mathcal{P}(\mathbb{N})$: $v(x)\cup v^{y}_{Q}(y) = v(x)$

Нека $Q=\mathbb{N}$, тогава $\underbrace{v(x)\cup \mathbb{N} = v(x)}$ $\rightarrow$ $v(x)=\mathbb{N}$

$\mathbb{N}\subseteq v(x)$, от ус. $v(x)\subseteq\mathbb{N}$

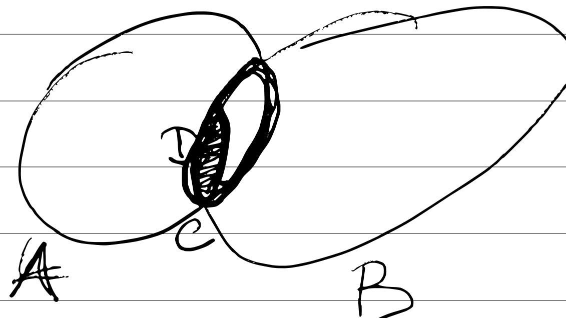
d) $\cap \leftrightharpoons \{\langle x, y, z\rangle \mid x, y, z \in \mathcal{P}(\mathbb{N}) \;\&\; x \cap y = z\}$

е определимо

e) $^{-} \leftrightharpoons \{\langle x, y\rangle \mid x, y \in \mathcal{P}(\mathbb{N}) \;\&\; y = \mathbb{N} \setminus x\}$

т.е. у е допълнението на х спрямо до $\mathbb{N}$

е определимо.



$C \subseteq A$ и $C \subseteq B$ и

$D \subseteq A$ и $D \subseteq B$, то $D \subseteq C$

защото е

$$\varphi_\cap(x, y, z) \leftrightharpoons \varphi_\subseteq(z, x) \;\&\; \varphi_\subseteq(z, y) \;\&\; \forall t\,(\varphi_\subseteq(t, x) \;\&\; \varphi_\subseteq(t, y) \Rightarrow \varphi_\subseteq(t, z))$$

e) $\overline{} \leftrightharpoons \{\langle x, y\rangle \mid x, y \in \mathcal{P}(\mathbb{N})\ \&\ y = \mathbb{N} \setminus x\}$

т.е. y е допълнението на x с точност до $\mathbb{N}$ е определимо.

$X \overset{\cap}{\cup} \overline{X}$

$$\varphi_c(x, y) \leftrightharpoons \exists z \exists t (\varphi_\emptyset(z)\ \&\ \varphi_{\mathbb{N}}(t)\ \&\ p(x, y, t)\ \&\ \varphi_\cap(x, y, z)).$$

f) Докажете, че за $a \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset, \mathbb{N}\}$, то $\{a\}$ е неопределимо.

Нека $a \subseteq \mathbb{N}$, т.е. $a \neq \emptyset$ и $a \neq \mathbb{N}$.

има $n \in \mathbb{N}$, т.е. $n \in a$

има $m \in \mathbb{N}$, т.е. $m \notin a$

$h_0 : \mathbb{N} \to \mathbb{N}$

$$h_0(x) = \begin{cases} n, & x = m \\ m, & x = n \\ x, & \text{else} \end{cases}$$

$h_0 = h_0^{-1}$

биекция

$h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$

$$h(X) = \{h_0(y) \mid y \in X\}, \quad X \subseteq \mathbb{N}.$$

$$h(h(X)) = \{h_0(y) \mid y \in h(X)\} =$$
$$= \{h_0(y) \mid y \in \{h_0(z) \mid z \in X\}\} =$$
$$= \{h_0(h_0(z)) \mid z \in X\} = \{z \mid z \in X\} = X.$$

$\langle a, b, c \rangle \in p^{A} \Leftrightarrow \langle h(a), h(b), h(c) \rangle \in p^{A}$

- $n, m \in a$, $n, m \in b$
  $n \notin a$, $m \notin a$, $n \notin b$, $m \notin b$)

$$\emptyset \subsetneq \overbrace{\{2, 5, 6, 8\}}^{A} \subsetneq \mathbb{N}$$

$5 \in$ $\qquad\qquad$ $0 \notin$

$$h_{0,A}(x) = \begin{cases} 0, & x = 5 \\ 5, & x = 0 \\ x, & \text{else} \end{cases}$$

$h(X) = \{h_0(y) \mid y \in X\}$

$h(\{2, 5, 6, 8\}) = \{0, 2, 6, 8\} \notin \{\{2, 5, 6, 8\}\}$

т.е. $\{\{2, 5, 6, 8\}\}$ не е определим.

Зад. | $L = \langle p \rangle$, $\#p = 2$

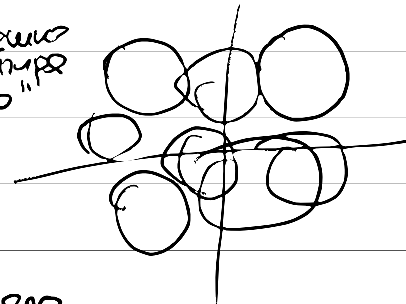
$A = \langle$ кръгове с ненулев радиус в $\mathbb{R}^2$, $p^A \rangle$, където

a) $p^A(a,b) \iff$ "a вътрешно се допира до b"

b) $p^A(a,b) \iff$ a b

c) $p^A(a,b) \iff$ a b

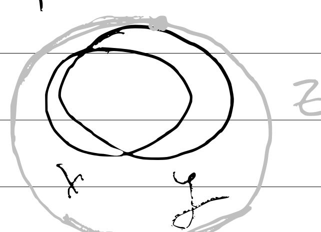
d) $p^A(a,b) \iff$ "a е изцяло вътре без допир в b"

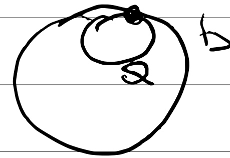


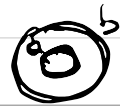
Опр. a, b и останалите.

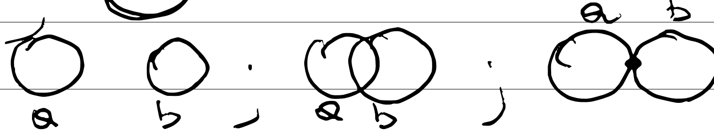
= , partial overlap) PO

$\varphi_{=}(x,y) \leftrightharpoons \forall z\,(p(x,z) \iff p(y,z))$

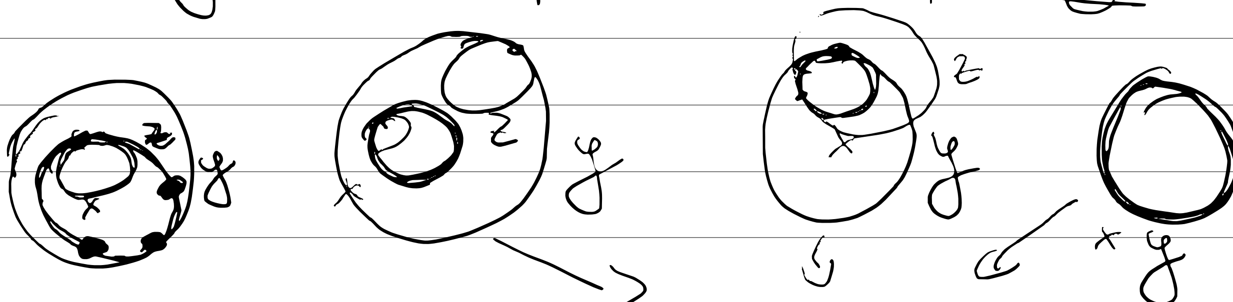


$p(a,b) \iff$ 

Done: = 

TODO: 

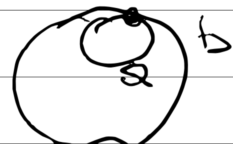
$$\cancel{e_{\subseteq}(x,y) \equiv \forall z\,(p(z,x) \Rightarrow p(z,y))}$$



$$e_{\subseteq}(x,y) \equiv \exists z\,(p(x,z)\ \&\ p(z,y))$$

$$e_{\circledcirc}(x,y) \equiv e_{\subseteq}(x,y)\ \&\ \neg e_{=}(x,y)\ \&\ \neg p(x,y)$$

$\overset{A}{p}(a,b) \Leftrightarrow$



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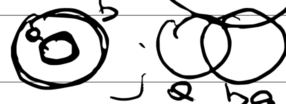
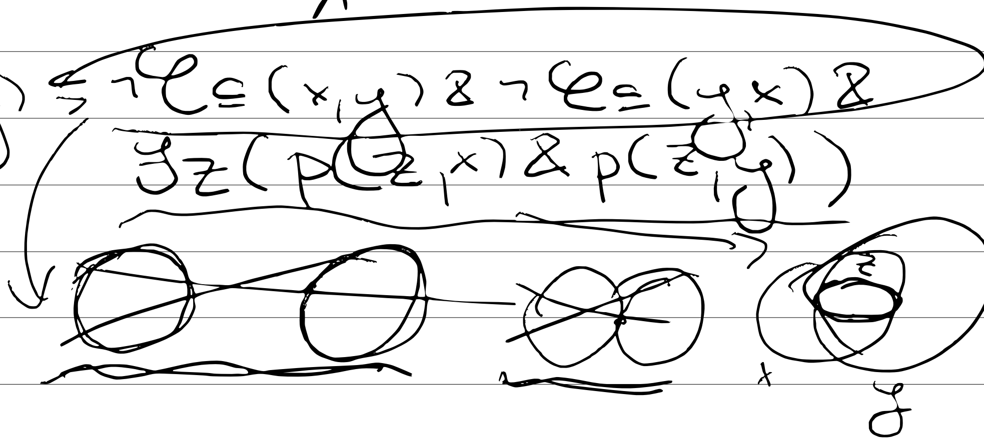


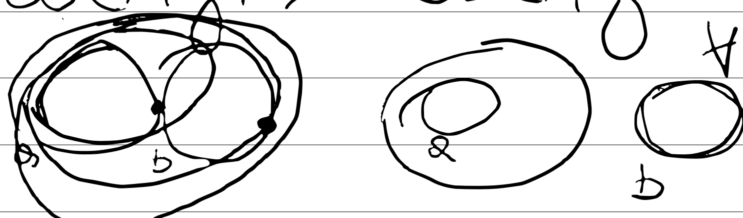
TODO:

$\varphi_{OO}(x,y) \leftrightharpoons$ "отру.ю всички останали"

$\varphi_{PO}(x,y) \leftrightharpoons \neg\varphi_{\subseteq}(x,y) \,\&\, \neg\varphi_{\subseteq}(y,x) \,\&\, \exists z(p(z,x) \,\&\, p(z,y))$



$\varphi_{OO}(x,y) \leftrightharpoons \neg\varphi_{\subseteq}(x,y) \,\&\, \neg\varphi_{\subseteq}(y,x) \,\&\, \neg\varphi_{PO}(x,y) \,\&\, \forall z(\varphi_{\subseteq}(x,z) \Rightarrow \varphi_{\subseteq}(y,z) \vee \varphi_{PO}(y,z))$



Зад/ $\mathcal{L} = \langle p \rangle$, $p \in Pred_{\mathcal{L}}$, $\#(p) = 3$

$\mathcal{A} = \langle \mathbb{N}, p^{\mathcal{A}} \rangle$, където

$$p^{\mathcal{A}}(k,n,m) \iff k+n = m+2$$

a) Док., че всеки синглетон е определим

b) Да се определят $=$ и $<$.

<u>3зд</u> Нека $\mathcal{L} = \langle \underline{\Pi} \rangle$, $\underline{\Pi} \in Preds$, $\#(\underline{\Pi}) = 3$.

Нека $\mathcal{A} = \langle \mathbb{R}^2, \underline{\Pi}^{\mathcal{A}} \rangle$, където:

$$\underline{\Pi}^{\mathcal{A}}(A, B, C) \leftrightharpoons A \neq B \text{ и } A \neq C \text{ и } \angle BAC = 30°$$

Да се док., че в стр. $\mathcal{A}$ са определими:

- $Eq \leftrightharpoons \{\langle A, A \rangle \mid A \in \mathbb{R}^2\}$
- $Col \leftrightharpoons \{\langle A, B, C \rangle \mid$ лежат на една права$\}$
- $Circ \leftrightharpoons \{\langle A, B, C \rangle \mid C$ лежи на окръжност с диаметър $AB\}$

Вярно ли е, че в $\mathcal{A}$ са опр. м-вата и защо?

- $Mid \leftrightharpoons \{\langle A, B, C \rangle \mid C$ е среда на отс. $AB\}$
- $Seg \leftrightharpoons \{\langle A, B, C \rangle \mid C$ лежи на отс. $AB\}$