зад. Нека $\mathcal{L} = \langle p \rangle$ FOL, $p \in Pred$, $\# p = 3$.
Нека $A = \langle \mathcal{P}(\mathbb{N}), p^{A} \rangle$ е структура за $\mathcal{L}$, като:

- $\mathcal{P}(\mathbb{N})$ - м-вото от всички подмн-ва на $\mathbb{N}$ т.е. $\mathcal{P}(\mathbb{N}) = \{B \mid B \subseteq \mathbb{N}\}$
- $\langle a, b, c \rangle \in p^{A} \iff a \cup b = c$

$a, b, c \in \mathcal{P}(\mathbb{N})$

Да се докаже, че:

a) $\{\emptyset\}$ е определимо

b) $\{\mathbb{N}\}$ е определимо

c) $\subseteq \leftrightharpoons \{\langle x, y \rangle \mid x, y \in \mathcal{P}(\mathbb{N}) \ \& \ x \subseteq y\}$ е определимо

d) $\cap \leftrightharpoons \{\langle x, y, z\rangle \mid x, y, z \in \mathcal{P}(\mathbb{N})$ & $x \cap y = z\}$

е определимо

e) $\overline{} \leftrightharpoons \{\langle x, y\rangle \mid x, y \in \mathcal{P}(\mathbb{N})$ & $y = \mathbb{N} \setminus x\}$

т.е. y е допълнението на x с точност до $\mathbb{N}$

е определимо.

f) Докажете, че за $a \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset, \mathbb{N}\}$, то $\{a\}$ е неопределимо.

a) $\{\emptyset\}$ е определимо

b) $\{\mathbb{N}\}$ е определимо

c) $\subseteq \leftrightharpoons \{\langle x, y\rangle \mid x, y \in \mathcal{P}(\mathbb{N})\ \&\ x \subseteq y\}$

е определимо

/ $\langle a, b, c\rangle \in p^{\mathcal{A}} \Longleftrightarrow \underbrace{a \cup b = c}_{}$ /

$\varphi_\emptyset(x) \leftrightharpoons \forall y\, p(x, y, y)$ // $\Leftrightarrow x = \emptyset$

Нека v е оценка в $\mathcal{A}$.

Д-во/ $\mathcal{A} \models_v \varphi_\emptyset(x) \Longleftrightarrow v(x) = \emptyset$

($\rightarrow$) Нека $\mathcal{A} \models_v \varphi_\emptyset(x)$ т.е. $\mathcal{A} \models_v \forall y\, p(x, y, y)$ т.е.

за вс Y $v(x) \cup Y = Y$

вкл. за $Y = \emptyset$

$v(x) \cup \emptyset = \emptyset$

$v(x) = \emptyset$

($\leftarrow$) Нека $v(x) = \emptyset$. Тогава

$\underbrace{\text{за вс. } y : v(x) \cup y = y}$

$A \models_v \forall y\, p(x, y, y)$.

b) $\{\mathbb{N}\}$ е определимо

e) $\subseteq \leftrightharpoons \{\langle x, y\rangle \mid x, y \in P(\mathbb{N})\ \&\ x \subseteq y\}$

е определимо

$e_{\mathbb{N}}(y) \leftrightharpoons \forall x\,(p(x, y, y))$

$A \models_v e_{\mathbb{N}}(y) \longleftrightarrow \underline{v(y) = \mathbb{N}}$

($\rightarrow$) $\underbrace{A \models_v \forall x\, p(x, y, y)}$

$\underbrace{\text{за вс. } x \subseteq \mathbb{N} \text{ то } x \cup v(y) = v(y)}$, $v(y) \subseteq \mathbb{N}$

$\underbrace{\text{нека за } x = \mathbb{N} \text{ то } \underbrace{\mathbb{N} \cup v(y) = v(y)}_{\mathbb{N} \subseteq v(y)}}$ 2)

$v(y) = \mathbb{N}$

е) $\underset{\sim}{\subseteq} \leftrightharpoons \{ \langle x, y \rangle \mid x, y \in \mathcal{P}(\mathbb{N}) \ \& \ x \subseteq y \}$

е определимо

$\varphi_{\subseteq}(x, z) \leftrightharpoons p(x, z, z).$

"м-вото x е подмн-во на м-вото z"

$\psi_{\subseteq}(x, z) \leftrightharpoons \exists y \, p(x, y, z)$

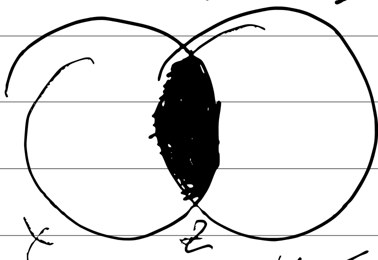
e) $\overline{} \leftrightharpoons \{\langle x, y \rangle \mid x, y \in \mathcal{P}(\mathbb{N})\ \&$
$y = \mathbb{N} \setminus x\}$

т.е. y е допълнението на x с точност до $\mathbb{N}$ е определимо. $\subseteq, \cup, \emptyset, \mathbb{N}$

$$x \cup \overline{x} = \mathbb{N}$$

$$\varphi_c(x, y) \leftrightharpoons \exists z \big(\varphi_{\mathbb{N}}(z)\ \&\ p(x, y, z)\ \&\ \forall k\,(\varphi_{\subseteq}(k, x)\ \&\ \varphi_{\subseteq}(k, y) \Rightarrow \varphi_{\emptyset}(k))\big)$$

d) $\cap \leftrightharpoons \{\langle x, y, z \rangle \mid x, y, z \in \mathcal{P}(\mathbb{N})\ \&$
$x \cap y = z\}$



$z \subseteq x$ и $z \subseteq y$

и ако да е $t \subseteq x$ и $t \subseteq y$ да вземем, то
$t \subseteq z$

$$\varphi_{\cap}(x, y, z) \leftrightharpoons \varphi_{\subseteq}(z, x)\ \&\ \varphi_{\subseteq}(z, y)\ \&\ \forall t\,(\varphi_{\subseteq}(t, x)\ \&\ \varphi_{\subseteq}(t, y) \Rightarrow \varphi_{\subseteq}(t, z))$$

$$x \cap y = \overline{\overline{x} \cup \overline{y}}$$

f) Докажете, че за $a \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset, \mathbb{N}\}$, то $\{a\}$ е неопределимо.

$a \neq \emptyset$ и $a \neq \mathbb{N}$

има поне един елем. (число) в него

има поне един елем. (число) извън него

Нека $n \in \mathbb{N}$, т.с. $\overset{2}{n \in a}$, нека $m \in \mathbb{N}$, т.с. $\overset{3}{m \in a}$.

$h_0 : \mathbb{N} \to \mathbb{N}$

$$h_0(x) \leftrightharpoons \begin{cases} n, & x = m \\ m, & x = n \\ x, & \text{else} \end{cases}$$

биекция

$h_0 = h_0^{-1}$

$\{2, 4, 5\}$

$h(\{2,4,5\}) = \{3,4,5\}$

$h(\{3,4,5\}) = \{2,4,5\}$

$h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$

$h(X) \leftrightharpoons \{h_0(y) \mid y \in X\}$, за $X \in \mathcal{P}(\mathbb{N})$

$h(\emptyset) = \emptyset$, $h(\mathbb{N}) = \mathbb{N}$, $h = h^{-1}$ биекция

$h(h(X)) = \{h_0(y) \mid y \in h(X)\} =$

$$= \{h_0(y) \mid y \in \{h_0(z) \mid z \in X\}\} =$$

$$= \{\underbrace{h_0(h_0}_{\text{[illegible]}}(z)) \mid z \in X\} = \{z \mid z \in X\} = X$$

Хом: $\langle a, b, c\rangle \in p^A \Leftrightarrow \langle h(a), h(b), h(c)\rangle \in p^A$

Дълго със случаи !!

Зад | $L = \langle p \rangle$, $\# p = 2$

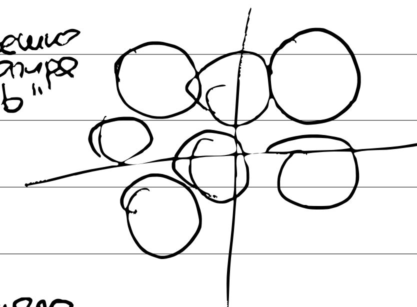
$A = \langle$ кръговете с ненулев радиус в $\mathbb{R}^2$, $p^A \rangle$, където

a) $p^A(a, b) \iff$ b "a вътрешно се допира до b"

b) $p^A(a, b) \iff$ a b

c) $p^A(a, b) \iff$ a b

d) $p^A(a, b) \iff$ b "a е изцяло вътре без допир в b"



Опр. a, b a b и останалите.

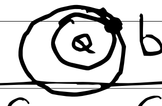
partial overlap

Невъзможно d)

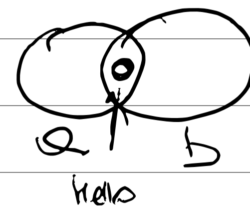
$$\varphi_=(x, y) \leftrightharpoons \forall z\, (p(x, z) \iff p(y, z))$$

$$\psi_=(x, y) \leftrightharpoons \forall z\, (p(z, x) \iff p(z, y))$$

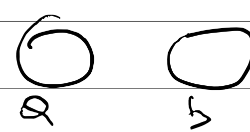
$$\varphi_\subseteq(x, y) \leftrightharpoons \forall z\, (p(z, x) \Rightarrow p(z, y))$$

Определете: 

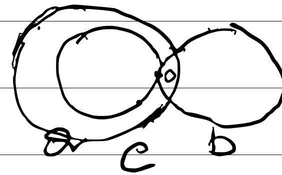
$$\varphi_{\odot}(x,y) \leftrightharpoons \neg \underbrace{P(x,y)}_{x \text{ не е собст. вътре в } y} \;\&\; \underbrace{\varphi_{\subseteq}(x,y)}_{x \text{ е изцяло вътре в } y} \;\&\; \neg \underbrace{\varphi_{=}(x,y)}_{x \text{ не съвпада с } y}.$$

Определете: 

$$\varphi_{\infty}(x,y) \leftrightharpoons \neg \varphi_{\subseteq}(x,y) \;\&\; \neg \varphi(y,x) \;\&\; \exists z(\varphi_{\subseteq}(z,x) \;\&\; \varphi_{\subseteq}(z,y)).$$

Определете: 

$\varphi_{\circ\circ}(x,y) \leftrightharpoons$ "не е нито едно от предните".

Определение: 

$$\wp_{\infty}(x,y) \leftrightharpoons \forall z \left(p(x,z) \Rightarrow \exists t \left(\wp_{\subseteq}(t,z) \,\&\, \wp_{\subseteq}(t,y) \right)\right) \,\&\, \neg \wp_{\subseteq}(x,y) \,\&\, \neg \wp_{\subseteq}(y,x).$$

Зад/ $\mathcal{L} = \langle p \rangle$, $p \in Pred_{\mathcal{L}}$, $\#(p) = 3$

$\mathcal{A} = \langle \mathbb{N}, p^{\mathcal{A}} \rangle$, където

$p^{\mathcal{A}}(k, n, m) \iff k + n = m + 2$

a) Док. че всеки синглетон е определим

b) Да се определят $=$ и $<$.

$\varphi_2(x) \leftrightharpoons p(x, x, x)$. // $\dfrac{x + x = x + 2}{x = 2}$

$\varphi_=(x, y) \leftrightharpoons \exists z(\varphi_2(z) \& p(x, z, y))$.

/ $x + \cancel{2} = y + \cancel{2}$

$x = y$.

/* $2k = m + 2$ $\to m + 2 \geq 2$

$k = \dfrac{m+2}{2} = \dfrac{m}{2} + 1$ $\underline{m = 2(k-1)}$ */

$\varphi_0(x) \leftrightharpoons \neg \exists y\, p(x, x, y)$

$\varphi_1(x) \leftrightharpoons \exists y(\varphi_0(y) \& p(x, x, y))$

$2x = 2 \iff x = 1$

300/ Нека $\mathcal{L} = \langle \underline{\text{ш}} \rangle$, $\underline{\text{ш}} \in Preds$, $\#(\underline{\text{ш}}) = 3$.

Нека $\mathcal{A} = \langle \mathbb{R}^2, \underline{\text{ш}}^{\mathcal{A}} \rangle$, където:

$$\underline{\text{ш}}^{\mathcal{A}}(A, B, C) \leftrightharpoons A \neq B \text{ и } A \neq C \text{ и } \angle BAC = 30^\circ$$

Да се док., че в стр. $\mathcal{A}$ са определими:

- $Eq \leftrightharpoons \{\langle A, A \rangle \mid A \in \mathbb{R}^2\}$
- $Col \leftrightharpoons \{\langle A, B, C \rangle \mid$ лежат на една права$\}$
- $Circ \leftrightharpoons \{\langle A, B, C \rangle \mid C$ лежи на окръжност с диаметър $AB\}$

Вярно ли е, че в $\mathcal{A}$ са опр. м-вата и защо?

- $Mid \leftrightharpoons \{\langle A, B, C \rangle \mid C$ е среда на отс. $AB\}$
- $Seg \leftrightharpoons \{\langle A, B, C \rangle \mid C$ лежи на отс. $AB\}$