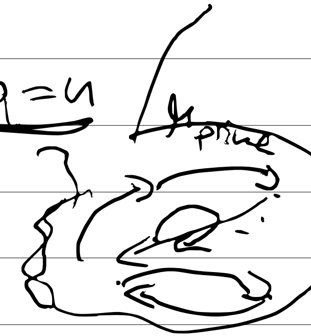
309 Нека $\mathcal{L} = \langle * \rangle$ и е с ф.р $\doteq$, $\#(*)=2$

Нека $A = \langle \mathbb{N}, *^A \rangle$ за

$x *^A y = z \iff z = x \cdot y$ $\exists q: m * q = n$



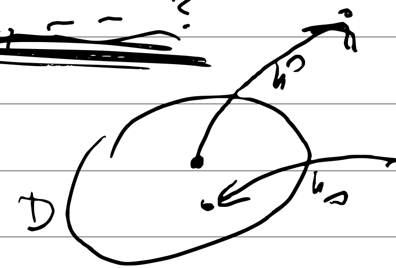
✓ Определете: $\{0\}$ и $\{1\}$

✓ • Опр. $Div_{n/m} = \{\langle n, m \rangle \in \mathbb{N} \times \mathbb{N} \mid \frac{n}{m} \in \mathbb{N}\}$

✓ • Опр. м-вото на простите числа

? • Определими ли са ~~$\{2\}, \{3\}, \{4\}, \ldots$~~ ?

Да го довършим!



Крит. за неопр.

Нека $D \subseteq A^n$. D е неопр. м-во, ако има $h_0 \in Aut(A)$ и има $a_1, \ldots, a_n \in A$, т.е.

не $\left(\langle a_1, \ldots, a_n \rangle \in D \text{ т.с.т.к. } \langle h_0(a_1), \ldots, h_0(a_n) \rangle \in D \right)$

O.Th.A. / Нека $n \in \mathbb{N}_{>0}$. Тогава

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_i^{\alpha_i} \cdots \qquad p_1 = 2,\quad p_2 = 3, \ldots$$

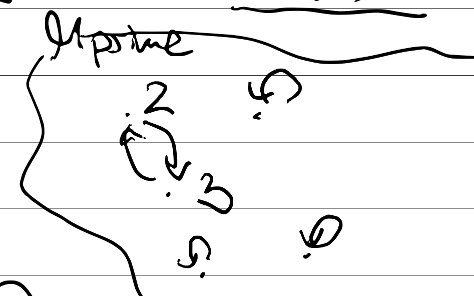
$$\alpha_i \geq 0$$

Заб / Винаги краен брой ненулеви степенни показатели α_i ..

$\{2\}$ $h \in Aut(A)$ i.e. $h(2) \notin \{2\}$

- $h_0 : \mathbb{M}_{prime} \to \mathbb{M}_{prime}$

$$h_0(x) = \begin{cases} 3, & x = 2 \\ 2, & x = 3 \\ x, & \text{else} \end{cases}$$



$h_0(h_0(x)) = x \Rightarrow h_0 = h_0^{-1}$ биекция

- $h : \mathbb{N} \to \mathbb{N}$

Нека $x \in \mathbb{N}_{>0} \overset{O.Th.A.}{\Longrightarrow} x = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_i^{\alpha_i} \cdots$

$$h(x) = \begin{cases} 0, & x = 0 \\ h_0^{\alpha_1}(p_1) \cdot h_0^{\alpha_2}(p_2) \cdots h_0^{\alpha_i}(p_i) \cdots = \prod_{i=1}^{\infty} h_0^{\alpha_i}(p_i), & x > 0 \end{cases}$$

$*$ биекция $\quad h(h(x)) = x \to \underline{h = h^{-1}}$ биекция
$\qquad\qquad\qquad\quad x \in \mathbb{N}$

$*$ хом

$a_1, a_2 \in \mathbb{N}$

$h(*^t(a_1, a_2)) = *^t(h(a_1), h(a_2))$

Th/ $f: A \to B$ биекция $\Leftrightarrow$ $\exists g: B \to A$ т.ч.
$f \circ g = Id_B$ и
$g \circ f = Id_A$

• $a_1 > 0$ и $a_2 = 0$ или
$a_1 = 0$ и $a_2 > 0$ или
$a_1 = 0$ и $a_2 = 0$
очевидно, проститель

• инекция
• сюрекция $\Big\} \to$ биекция

$p_1 = 2, p_2 = 3, p_3 = 5, \ldots$

• $a_1 > 0, a_2 > 0 \xrightarrow{OTHA}$ $a_1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i} \ldots$
$a_2 = p_1^{\beta_1} p_2^{\beta_2} \ldots p_i^{\beta_i} \ldots$

$*^t(h(a_1), h(a_2)) = *^t(h(2^{\alpha_1} 3^{\alpha_2} \ldots p_i^{\alpha_i} \ldots), h(2^{\beta_1} 3^{\beta_2} \ldots p_i^{\beta_i} \ldots)) =$
$= *^t(3^{\alpha_1} 2^{\alpha_2} \ldots p_i^{\alpha_i} \ldots, 3^{\beta_1} 2^{\beta_2} \ldots p_i^{\beta_i} \ldots) =$
$= 3^{\alpha_1+\beta_1} 2^{\alpha_2+\beta_2} \ldots p_i^{\alpha_i+\beta_i} \ldots =$
$= h(2^{\alpha_1+\beta_1} 3^{\alpha_2+\beta_2} \ldots p_i^{\alpha_i+\beta_i} \ldots) =$
$h(*^t(2^{\alpha_1} 3^{\alpha_2} \ldots p_i^{\alpha_i} \ldots, 2^{\beta_1} 3^{\beta_2} \ldots p_i^{\beta_i})) = h(*^A(a_1, a_2))$

Зоеи: $2 \in \{2\}$, но $h(2) \notin \{2\}$, т.е. крит. за неопр., то $\{2\}$ <u>не е опр. м-во</u>

$6 = \underline{2}^1 \cdot \underline{3}^1$, $h(6) = h_0^1(2) \cdot h_0^1(3) = 3^1 \cdot 2^1 = 6$.

$\underline{15 = 3 \cdot 5}$

$2 = 2^1 \cdot 3^0 \cdot 5^0 \cdot \ldots$

$\{n\} \to n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots p_i^{\alpha_i} \cdot \ldots$

Нека $\underset{\text{краен брой}}{\underline{\alpha_k \neq 0}}$ и $\underset{\text{безкраен брой}}{\underline{\alpha_j = 0}}$

$$h_0(x) = \begin{cases} p_j, & x = p_k \\ p_k, & x = p_j \\ x, & \text{else} \end{cases}$$

302) $\mathcal{L} = \langle * \rangle$ един двуместни функ. символи и ф.р. $\doteq$.

$\mathcal{A} = \langle \mathbb{R}, *^{\mathcal{A}} \rangle$

$$a *^{\mathcal{A}} b = c \Leftrightarrow a \cdot b = c$$

Кои синглетони са определими?

$\varphi_0(x) \leftrightharpoons \forall y (x * y \doteq x)$

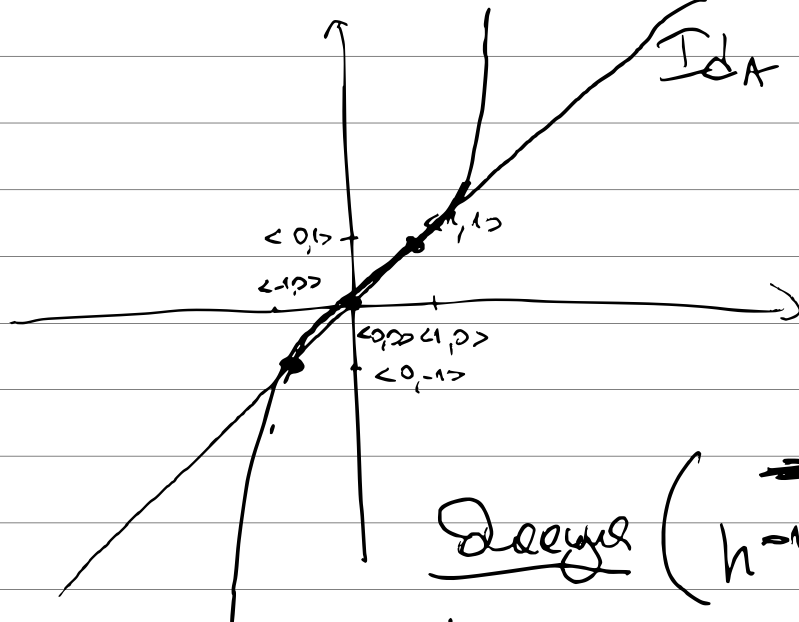
$\varphi_1(x) \leftrightharpoons \forall y (x * y \doteq y)$

$\varphi_{-1}(x) \leftrightharpoons \forall y (x * x * y \doteq y) \; \& \; \neg \varphi_1(x)$

$\psi_{-1}(x) \leftrightharpoons \exists z (\varphi_1(z) \; \& \; x * x = z) \; \& \; \underline{\neg \varphi_1(x)}$.

Защо за вс. $r \in \mathbb{R} \setminus \{0, -1, 1\}$, то $\{r\}$ не е <u>определимо</u>?

h-фнкц, т.ч. $h(*^{\mathcal{A}}(a, b)) = *^{\mathcal{A}}(h(a), h(b))$



$h(x) = x^2$?, хми, но

не е биекция

↑

не запазва -1

$h(x) = x^3$

Биекция $\left(h^{-1}(x) = \sqrt[3]{x} = x^{1/3} \right.$

хим

$h(\ast^{A}(a, b)) = h(a.b) = (a.b)^3 =$

$= a^3 . b^3 = \ast^{A}(h(a), h(b))$

Зад. $B = \langle * \rangle \doteq \#(*) = 2$.

Нека $A = \langle \mathbb{Q}, *^t \rangle$.

$a *^t b \doteq c \iff a \cdot b = c$.

$a, b, c \in \mathbb{Q}$

Кои синглетони са определими?

e_0, e_1, e_{-1} от предната задача $\to$ v.

Кои за останалите за $q \in \mathbb{Q} \setminus \{0, -1, 1\}$, то $\{q\}$ не е опр.

$h(x) = x^3$, хом, но не е биекция

$h^{-1}(2) = ?$ не е сюрекция

$h^{-1}(2) = \sqrt[3]{2} \notin \mathbb{Q}$

$\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \}$

$\mathbb{Z} = \mathbb{N} \cup \mathbb{N}^-$

$\mathbb{N}^- = \{ -n \mid n \in \mathbb{N} \setminus \{0\} \}$

$x = \frac{p}{q} \in \mathbb{Q}$, $|p|, |q| \in \mathbb{N}$ OTLA
$p \neq 0$

$$h(x) = \begin{cases} \operatorname{sign}\left(\frac{p}{q}\right) \cdot \frac{|p|}{|q|} \quad (i) \leftrightarrow (j), & p \neq 0 \\ 0, & p = 0 \end{cases}$$

$$h(x) = \begin{cases} 1/x = x^{-1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

3 зад. | $\mathcal{L} = \langle f \rangle, \doteq; \ f \in Func_{\mathcal{L}}, \ \#(f) = 2.$

Нека Σ е крайна азбука.

Нека $A = \langle \Sigma^*, f^A \rangle$, където

$f^A(u, v) = w \overset{def}{\iff} u \circ v = w$

за произв. $u, v, w \in \Sigma^*$.

Да се докаже, че в A са определими:

- $Pref = \{ \langle u, v \rangle \in (\Sigma^*)^2 \mid u$ е префикс на $v \}$
- $Suff = \{ \langle u, v \rangle \in (\Sigma^*)^2 \mid u$ е суфикс на $v \}$
- $W_1 \leftrightharpoons \{ u \in \Sigma^* \mid |u| = 1 \}$
- $T_1 \leftrightharpoons \{ u \in \Sigma^* \mid |u| \leq 1 \}$
- за вс. $n \in \mathbb{N}$, W_n е опр.
- за вс. $n \in \mathbb{N}$, T_n е опр.
- $O \leftrightharpoons \{ \langle u, v, w \rangle \in (\Sigma^*)^3 \mid w$ най-дългият общ префикс на u и v е суфикс на $w \}$
- $D \leftrightharpoons \{ \langle u, v, w \rangle \in (\Sigma^*)^3 \mid$ най-дългият общ суфикс на u и v е префикс на $w \}$

Да се изрази броят на автоморфизмите в $\mathcal{A}$ чрез броя на буквите от Σ.

- $Pref = \{\langle u, v\rangle \in (\Sigma^*)^2 \mid u$ е префикс на $v\}$
- $Suff = \{\langle u, v\rangle \in (\Sigma^*)^2 \mid u$ е суфикс на $v\}$

$f^{\mathcal{A}}(u, v) = u \circ v$

$\varphi_{Pref}(x, y) \leftrightharpoons \exists z (f(x, z) \doteq y)$. // $x \circ z = y$ за някое $z \in \Sigma^*$

$\varphi_{Suff}(x, y) \leftrightharpoons \exists z (f(z, x) \doteq y)$.

- $W_1 \leftrightharpoons \{u \in \Sigma^* \mid |u| = 1\}$, $|u|$ - дължина на думата.
- $T_1 \leftrightharpoons \{u \in \Sigma^* \mid |u| \leq 1\}$

$\varphi_\varepsilon(x) \leftrightharpoons \forall y (f(x, y) \doteq y)$.

$\varphi_{W_1}(x) \leftrightharpoons \neg\varphi_\varepsilon(x)\ \&\ \forall y \forall z (\varphi_{Pref}(y, x)\ \&\ \varphi_{Suff}(z, x)\ \&$
$\neg\varphi_\varepsilon(y)\ \&\ \neg\varphi_\varepsilon(z) \Rightarrow$

$y \doteq x\ \&\ z \doteq x)$

aaaa

$\varphi_{W_1}(x) \leftrightharpoons \neg \varphi_{\varepsilon}(x) \;\&\; \forall y(\varphi_{pref}(y,x) \Rightarrow \varphi_{\varepsilon}(y) \vee x = y)$

$\varphi_{T_1}(x) \leftrightharpoons \varphi_{\varepsilon}(x) \vee \varphi_{W_1}(x)$.

$\psi_{T_1}(x) \leftrightharpoons \forall y(Pref(y,x) \Rightarrow \varphi_{\varepsilon}(y) \vee x = y)$.

- за вс $n \in \mathbb{N}$, W_n е опр.
- за вс. $n \in \mathbb{N}$, T_n е опр.

base: $\varphi_{\varepsilon}, \varphi_{T_1}$

(i.h.): Нека за някое n, φ_{T_n} опр. T_n

(i.step): $\varphi_{T_{n+1}}(x) \leftrightharpoons \forall y(\varphi_{pref}(y,x) \Rightarrow x = y \vee \varphi_{T_n}(y))$

base: $\varphi_{\varepsilon}, \varphi_{W_1}$

(i.h): Нека за вс. $k \leq n$, φ_{W_k} опр. W_k

(i.step): $\varphi_{W_{n+1}}(x) \leftrightharpoons \neg \varphi_{\varepsilon}(x) \;\&\; \neg\varphi_{W_1}(x) \;\&\; \dots \;\&\; \neg \varphi_{W_n}(x) \;\&\; \forall y(\varphi_{pref}(y,x) \Rightarrow \varphi_{\varepsilon}(y) \vee \varphi_{W_1}(x) \vee \dots \vee \varphi_{W_n}(x) \vee x = y)$

(i.h) За някое n, φ_{W_n} опр W_n

(i.step) $\varphi_{W_{n+1}}(x) \leftrightharpoons \exists y \exists z \left(\varphi_{W_n}(y) \,\&\, \varphi_{W_n}(z) \,\&\, f(z, y) \doteq x \right)$

$O \leftrightharpoons \{ \langle u, v, w \rangle \in (\Sigma^*)^3 \mid$ най-дългият общ префикс на u и v е суфикс на $w \}$

$\varphi_{LCPref}(x, y, z) \leftrightharpoons \varphi_{Pref}(x, y) \,\&\, \varphi_{Pref}(x, z) \,\&$

"x е най-дългият общ префикс на y и z"

$\forall t \left(\varphi_{Pref}(t, y) \,\&\, \varphi_{Pref}(t, z) \Rightarrow \varphi_{Pref}(t, x) \right).$

$\varphi_O(x, y, z) \leftrightharpoons \exists t \left(\varphi_{LCPref}(t, x, y) \,\&\, \varphi_{Suff}(t, z) \right).$

Да се изрази броят на автоморфизмите в A чрез броя на буквите от Σ.

$|\Sigma| = n$, $n \in \mathbb{N}$

$\Sigma = \{a_1, \dots, a_n\}$

$|\{f \mid f: \Sigma \to \Sigma \text{ биекция}\}| = n!$

$h_0 \in$

$h_0: \Sigma \xrightarrow{\text{биек.}} \Sigma$

$h: \Sigma^* \to \Sigma^*$: $h(w) = h_0(b_1) \circ h_0(b_2) \circ \dots \circ h_0(b_k)$

$w = b_1 \dots b_k$

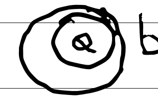
хом. -> непосредствено

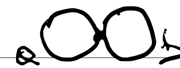
биекция: $h^{\text{НОК(дължини на цикли в } h_0\text{)}}(w) = w$

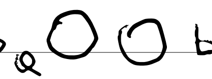
$$\begin{array}{cccccccc}
\Sigma: & a & b & c & d & e & f \\
\downarrow h_0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\Sigma: & b & c & a & d & f & e
\end{array}$$

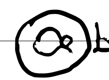
Зад. | $\mathcal{L} = \langle p \rangle$, $\# p = 2$.

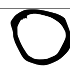
$\mathcal{A} = \langle$ окр. с ненулев радиус в $\mathbb{R}^2$, $p^{\mathcal{A}} \rangle$, където

a) $p^{\mathcal{A}}(a, b) \Leftrightarrow$ 

b) $p^{\mathcal{A}}(a, b) \Leftrightarrow$ 

c) $p^{\mathcal{A}}(a, b) \Leftrightarrow$ 

d) $p^{\mathcal{A}}(a, b) \Leftrightarrow$ 

Опр.  a, b  $a b$ и останалите.

Зад/ $\mathcal{L} = \langle p \rangle$, $p \in Pred_{\mathcal{L}}$, $\#(p) = 3$.

$\mathcal{A} = \langle \mathbb{N}, p^{\mathcal{A}} \rangle$, където

$p^{\mathcal{A}}(k, n, m) \Longleftrightarrow k + n = m + 2$

a) Док., че всеки синглетон е определим

b) Да се определят $=$ и $<$.

Зад. Нека $\mathcal{L} = \langle p \rangle$ FOL, $p \in Pred$, $\#p = 3$.

Нека $A = \langle \mathcal{P}(\mathbb{N}), p^A \rangle$ е структура за $\mathcal{L}$, като:

- $\mathcal{P}(\mathbb{N})$ - м-вото от всички подмн-ва на $\mathbb{N}$ т.е. $\mathcal{P}(\mathbb{N}) = \{B \mid B \subseteq \mathbb{N}\}$
- $\langle a, b, c \rangle \in p^A \iff a \cup b = c$

$a, b, c \in \mathcal{P}(\mathbb{N})$

Да се докаже, че:

a) $\{\emptyset\}$ е определимо

b) $\{\mathbb{N}\}$ е определимо

c) $\subseteq \leftrightharpoons \{\langle x, y \rangle \mid x, y \in \mathcal{P}(\mathbb{N}) \; \& \; x \subseteq y\}$ е определимо

d) $\cap \leftrightharpoons \{\langle x, y, z\rangle \mid x, y, z \in \mathcal{P}(\mathbb{N})\ \&\ x \cap y = z\}$

е определимо

e) $^{-} \leftrightharpoons \{\langle x, y\rangle \mid x, y \in \mathcal{P}(\mathbb{N})\ \&\ y = \mathbb{N} \setminus x\}$

т.е. y е допълнението на x

е определимо.

f) Докажете, че за $\underline{a \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset, \mathbb{N}\}}$, то

$\{a\}$ е <u>неопределимо</u>.

<u>300</u>/ Нека $\mathcal{L} = \langle \underline{\amalg} \rangle$, $\underline{\amalg} \in Preds$, $\#(\amalg) = 3$.

Нека $\mathcal{A} = \langle \mathbb{R}^2, \underline{\amalg}^{\mathcal{A}} \rangle$, където:

$$\underline{\amalg}^{\mathcal{A}}(A, B, C) \leftrightharpoons A \neq B \ \underline{\text{и}}\ A \neq C \ \text{и}\ \angle BAC = 30^\circ$$

Да се док., че в стр. $\mathcal{A}$ са определими:

- $Eq \leftrightharpoons \{\langle A, A\rangle \mid A \in \mathbb{R}^2\}$
- $Col \leftrightharpoons \{\langle A, B, C\rangle \mid$ лежат на една права$\}$
- $Circ \leftrightharpoons \{\langle A, B, C\rangle \mid C$ лежи на окръжност с диаметър $AB\}$

Вярно ли е, че в $\mathcal{A}$ са опр. м-вата и защо?

- $Mid \leftrightharpoons \{\langle A, B, C\rangle \mid C$ е среда на отс. $AB\}$
- $Seg \leftrightharpoons \{\langle A, B, C\rangle \mid C$ лежи на отс. $AB\}$