300/ Нека $\mathcal{L} = \langle \underline{ll} \rangle$, $\underline{ll} \in Preds$, $\#(\underline{ll}) = 3$.

Нека $\mathcal{A} = \langle \mathbb{R}^2, \underline{ll}^{\mathcal{A}} \rangle$, където:

$$\underline{ll}^{\mathcal{A}}(A, B, C) \leftrightharpoons \underline{A \neq B} \text{ и } \underline{A \neq C} \text{ и } \angle BAC = 90^\circ$$

Да се док., че в стр. $\mathcal{A}$ са определими:

- $Eq \leftrightharpoons \{\langle A, A \rangle \mid A \in \mathbb{R}^2\}$
- $Col \leftrightharpoons \{\langle A, B, C \rangle \mid$ лежат на една права$\}$
- $Circ \leftrightharpoons \{\langle A, B, C \rangle \mid C$ лежи на окръжност с диаметър $AB\}$

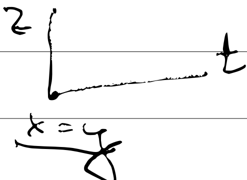
Вярно ли е, че в $\mathcal{A}$ са опр. м-вата и защо?

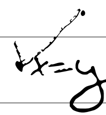
- $Mid \leftrightharpoons \{\langle A, B, C \rangle \mid C$ е среда на отс. $AB\}$
- $Seg \leftrightharpoons \{\langle A, B, C \rangle \mid C$ лежи на отс. $AB\}$

- $e_Q \leftrightharpoons \{\langle A, A\rangle \mid A \in \mathbb{R}^2\}$ равенство
- $Col \leftrightharpoons \{\langle A, B, C\rangle \mid$ лежат на една права$\}$
- $Circ \leftrightharpoons \{\langle A, B, C\rangle \mid \underline{C}$ лежи на окръжност с диаметър $AB\}$

$\perp(A, B, C) \leftrightharpoons A \neq B$ и $A \neq C$ и $\angle BAC = 90^\circ$

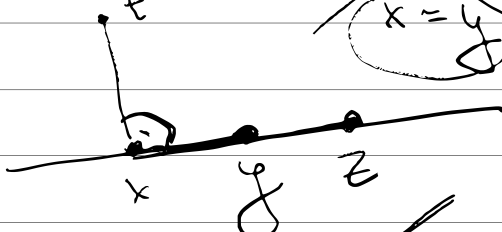
$\varphi_=(x, y) \leftrightharpoons \forall z \forall t\, (\perp(x, z, t) \Leftrightarrow \perp(y, z, t))$



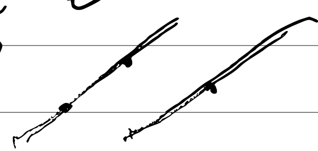


$\psi_=(x, y) \leftrightharpoons \forall z\, \neg \perp(x, y, z)$

$x = y$ или $x = z$ или $\angle yxz \neq 90^\circ$

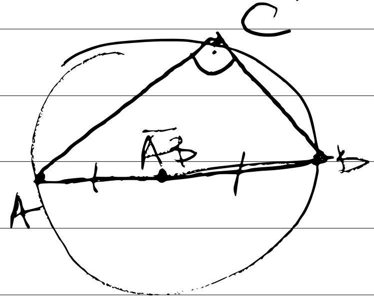


$\varphi_{Col}(x, y, z) \leftrightharpoons \exists t\, (\perp(x, t, y)\ \&\ \perp(x, t, z)) \vee \varphi_=(x, y) \vee \varphi_=(x, z)$

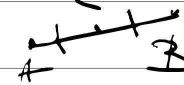


• $Circ \rightleftharpoons \{\langle A, B, C\rangle \mid C$ лежи на окръжност с диаметър $\overline{AB}\}$

$\perp(A, B, C) \rightleftharpoons A \neq B$ и $A \neq C$ и $\angle BAC = 90$

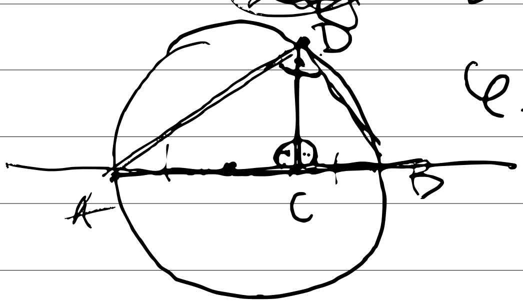


$$\varphi_{circ}(x, y, z) \rightleftharpoons \perp(z, x, y) \vee \varphi_{=}(z, x) \vee \varphi_{=}(z, y).$$



• $Mid \rightleftharpoons \{\langle A, B, C\rangle \mid C$ е среда на отс. $AB\}$

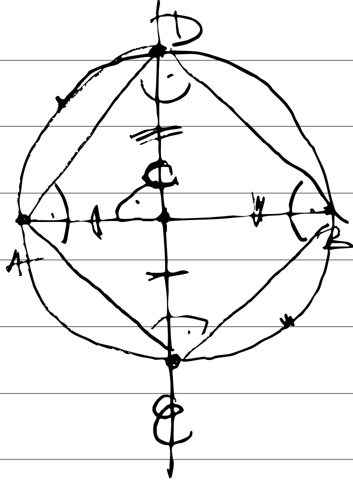
V • $Seg \rightleftharpoons \{\langle A, B, C\rangle \mid C$ лежи на отс. $AB\}$



$$\varphi_{seg}(x, y, z) \rightleftharpoons \varphi_{col}(x, y, z) \,\&\, \exists d(\varphi_{circ}(x, y, d) \,\&\, \perp(z, x, d)).$$

$\perp^t(A, B, C) \Leftrightarrow A \neq B$ и $A \neq C$ и $\angle BAC = 90^\circ$

• $\ell_{mid} \Leftrightarrow \{\langle A, B, C\rangle \mid$ C е средата на отс. $AB\}$



$$\ell_{mid}(A,B,C) \Leftrightarrow \exists D \exists E (\ell_{seg}(A,B,C) \;\&\; \ell_{seg}(D,E,C) \;\&\; \ell_{circ}(A,B,D) \;\&\; \ell_{circ}(A,B,E) \;\&\; \perp(C,A,D)).$$

зад

Числа на Фибоначи

Редицата $\{F_n\}_{n=0}^{\infty}$ се дефинира така:

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_{n+2} = F_n + F_{n+1} \quad \text{за } n \geq 0 \end{cases}$$

Разгл. език $\mathcal{L}$ без ф.р. и единст. нелог. символи: $Func_{\mathcal{L}} = \{f\}$, $Pred_{\mathcal{L}} = \{p\}$, $Const_{\mathcal{L}} = \emptyset$ като $\#(f) = 2$, $\#(p) = 1$.

Нека $S = \langle \mathbb{N}, f^S, p^S \rangle$ е стр. за $\mathcal{L}$ като:

$$f^S(x, y) = z \iff x + F_{y+1} = z$$

$$p^S(x) \iff x \in \{F_n\}_{n=0}^{\infty}$$

Да се док., че в S са определени:

a) $\{0\}$ ✓

b) $\{1\}$ ✓

c) $Eq = \{\langle a, a\rangle \mid a \in \mathbb{N}\}$ равенство ✓

d) Вярно ли е, че в S е опр. ф-во

$$Fib = \{\langle F_n, F_{n+1}\rangle \mid n \in \mathbb{N}\}$$

e) Да се намерят всички автоморф. на S с д-во.

$\varphi_0(x) \leftrightharpoons \forall y\, p(f(x,y))$ // $x + F_{y+1} \in \{F_k\}_{k=0}^{\infty}$

за вс. y

$\varphi_1(x) \leftrightharpoons \neg\varphi_0(x) \,\&\, p(f(x,x))$ // $x \neq 0$ &

$x + F_{x+1} \in \{F_k\}_{k=0}^{\infty}$

$f^A(x,y) = z \iff x + F_{y+1} = z$

$p^t(x)$ - когато x е число на fib.

0	1	1	2	3	5	8	...
0	1	2	3	4	5	...	

$\varphi_=(x,y) \leftrightharpoons \forall z\,(p(f(x,z)) \iff p(f(y,z)))$

$\varphi_1(x) \leftrightharpoons \exists y\,(\varphi_0(y) \,\&\, \varphi_=(f(y,y),x))$. $\quad 0 + F_{0+1} = \underset{1}{F_1} = x$

- $\varphi_{+1}(x,y) \leftrightharpoons \exists z\,(\varphi_0(z) \,\&\, \varphi_=(f(x,z),y))$

  // $y = x+1$ $\quad x + F_1 = y$ $\quad 0 + F_{i+1} = F_{i_x}$

- $\varphi_{getIndex}(x, i_x) \leftrightharpoons \cancel{p}(x) \,\&\, ((\varphi_0(x) \,\&\, \varphi_0(i_x)) \vee (\neg\varphi_0(x) \,\&\, \exists t \exists e\,(\varphi_0(e) \,\&\, \varphi_=(f(e,t),x) \,\&\, \varphi_{+1}(t,i_x))))$. $\quad t+1 = i_x$

  $M = \{\langle F_n, n\rangle \mid n \in \mathbb{N}\}$

- $Prev(x,y) \leftrightharpoons \exists i_x \exists i_y\,(\varphi_{getIndex}(x,i_x) \,\&\, \varphi_{+1}(i_x,i_y) \,\&\, \varphi_{getIndex}(y,i_y))$.

$Prev = \{\langle F_n, F_{n+1}\rangle \mid n \in \mathbb{N}\}$

$f^S(x,y) = z \iff x + F_{y+1} = z$

$Aut(S) \neq \emptyset$, $Id_{\mathbb{N}} \in Aut(S)$.

• Не може ли $Aut(S) = \{Id_{\mathbb{N}}\}$?

Опред. φ $\{n\}$ и опер. $+1$

(base:) φ_0

(i.h): φ_n опр. $\{n\}$, $n \geq 0$

(step): $\varphi_{n+1}(x) \leftrightharpoons \exists y(\varphi_n(y) \,\&\, \varphi_{+1}(y,x))$.

$(\forall n \in \mathbb{N})[\{n\}$ е опр.$]$

стр. е твърдо

$Aut(S) = \{Id_{\mathbb{N}}\}$.

Нека $h \in Aut(S)$ произв. Т.к. $(\forall n \in \mathbb{N})[\{n\}$ е опр.$]$, то

$(\forall n \in \mathbb{N})[h[\{n\}] = \{n\}] \Leftrightarrow$

$(\forall n \in \mathbb{N})[h(n) = n] \Rightarrow h = Id_{\mathbb{N}}$.

Т.к. h е произв., то $Aut(S) = \{Id_{\mathbb{N}}\}$.

Задача за изпълнимост на м-во от формули

Формулировка:

Дадено: м-во от затворени ф-ли Φ

Търси се: структура, т.е. всички ф-ли в Φ да са верни в нея.

def/ Изпълнимост на ф-ла

Нека φ е ф-ла от език $\mathcal{L}$. φ е изпълнима, ако съществува структура $\mathcal{A}$ за $\mathcal{L}$ и оценка на инд. променливи v в $\mathcal{A}$, т.е. $\mathcal{A} \models_v \varphi$.

(*) Ако за всяка оценка v в $\mathcal{A}$, то $\mathcal{A} \models_v \varphi$, то пишем $\mathcal{A} \models \varphi$ и казваме, че стр. $\mathcal{A}$ е модел за φ.

(*) Ако φ е затворена, то верността на φ не зависи от оценката т.е. за затв. ф-ли е в сила, че:

За φ затворена ф-ла

φ е изпълнима в $A \xLeftrightarrow{\text{по дефиниция}} A$ е модел за φ

Задачата за намиране на модел на м-во от ф-ли е аналогична на задачата да напишем програма отговаряща на дадена формална спецификация зад.

$$\varphi_1 \leftrightharpoons \forall x \neg p(x,x)$$
$$\varphi_2 \leftrightharpoons \forall x \exists y\, p(x,y)$$
$$\varphi_3 \leftrightharpoons \forall x \forall y \forall z\, (p(x,y) \,\&\, p(y,z) \Rightarrow p(x,z))$$

Нека $\Gamma_1 \leftrightharpoons \{\varphi_1, \varphi_2\}$, $\Gamma_2 \leftrightharpoons \Gamma_1 \cup \{\varphi_3\}$.

Док., че са изпълними. Γ_2 има ли краен модел?

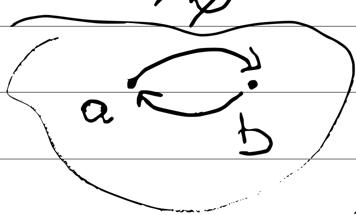
φ_1 – антирефл. по p^{A}

φ_2 – сериалност надясно $\quad x \to y$

φ_3 – транзитивност

Γ_1: $\mathcal{A}_1 = \langle A_1; p^{\mathcal{A}_1} \rangle$, т.е. имаме поне един индивид в A_1.

$A_1 \neq \emptyset$



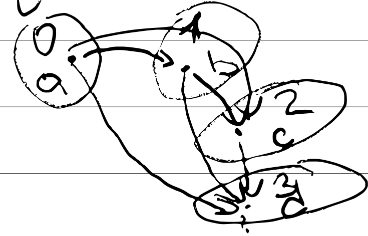
$A_1 = \{a, b\}$

$p^{\mathcal{A}_1} = \{\langle a,b \rangle, \langle b,a \rangle\}$.

$p^{\mathcal{A}_1}(a,b) \,\&\, p^{\mathcal{A}_1}(b,a) \xrightarrow[\varphi_1 \leftrightharpoons]{\varphi_3} p^{\mathcal{A}}(a,a)$

Γ_2: $\mathcal{A}_2 = \langle A_2, \rho^{\mathcal{A}_2} \rangle$

$\rho^{\mathcal{A}_2}$ - антирефл, транзитивно, сериалност надясно



$A_2 = \mathbb{N}$

$\rho^{\mathcal{A}_2} = <_{\mathbb{N}}$

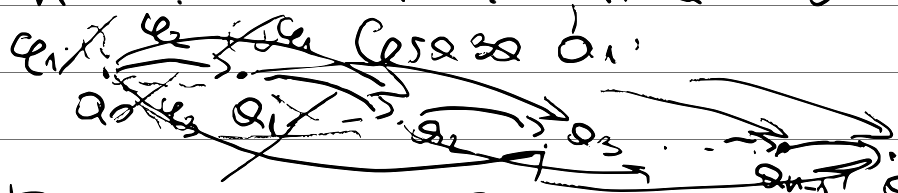
$\mathcal{A}_2 \models \Gamma_2$.

Защо не същ. $\mathcal{A}$ крайна т.ч. $\mathcal{A} \models \Gamma_2$

$(\exists n \in \mathbb{N})[\overline{\overline{A}} = n]$

Допускаме че $\mathcal{A}$ крайна и $\mathcal{A} \models \Gamma_2$. Нека $n \in \mathbb{N}_{>0}$ т.ч. $\overline{\overline{A}} = n$. Т.к. $A \neq \emptyset$ нека $a_0 \in A$ свидетел.

Свързва a_1

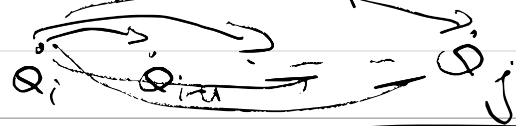


Редицата $a_0, \dots, a_n$ от $n+1$ точки, но $\overline{\overline{A}} = n$.

По П.Д. $i \neq j \in \mathbb{N}$ БОО $i < j$ т.ч. $0 \leq i < j \leq n$ и

$a_i = a_j$.

$p^A(a_i, a_{i+1})$ и $p^A(a_{i+1}, a_{i+2})$ и ... и $p^A(a_{j-1}, a_j)$



$p^A(a_i, a_j)$ → $p^A(a_i, a_i)$ примка в a_i

$a_i = a_j$

$\lightning$ с φ_1

Допускането ни води до противоречие. Така излиза, че всички модели за Γ_2 са с безкраен универсум.

HW) Нека имаме $p_1, \ldots, p_n$ едноместни предик. символи!

С-во: Φ м-во от формули в този език има модел, то Φ има задължително и краен модел.

3.a.7.

$\varphi_1 \leftrightharpoons \exists x \forall y (x \neq y \Rightarrow \exists z \neg p(y, z))$

$\varphi_2 \leftrightharpoons \exists x \forall y \, p(x, y)$

$\varphi_3 \leftrightharpoons \forall x \forall y (p(x, y) \Rightarrow \exists z (p(x, z) \& p(z, y)))$

$\varphi_4 \leftrightharpoons \forall x \exists y \, p(x, y)$

322g

$\varphi_1 \leftrightharpoons \neg \exists x\, p(x,x)$

$\varphi_2 \leftrightharpoons \forall x \exists y \big(p(x,y) \,\&\, \neg \exists z (p(x,z) \,\&\, p(z,y))\big)$

$\varphi_3 \leftrightharpoons \exists x \neg \exists y\, p(y,x)$

$\varphi_4 \leftrightharpoons \forall x \big(\exists y\, p(y,x) \rightarrow \neg \exists y (p(y,x) \,\&\,$
$\neg \exists z (p(x,z) \,\&\, p(z,y)))\big)$

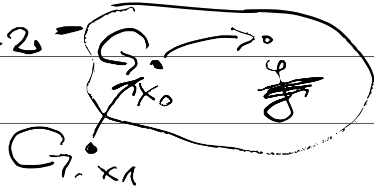
зад

$\varphi_1 \leftrightharpoons \neg \forall x \forall y \forall z (p(x,y) \,\&\, p(y,z) \Rightarrow p(x,z))$

$\varphi_2 \leftrightharpoons \forall y \exists x (p(x,y) \,\&\, \neg p(y,x) \,\&\, p(x,x))$

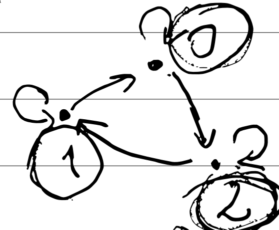
φ_1 - p^A не е транзитивна

φ_2 -



0 2 1

$\exists x \exists y \exists z (p(x,y) \,\&\, p(y,z) \,\&\, \neg p(x,z))$ $\models$



$A = \langle \{0,1,2\}, p^A \rangle$

$p^A = \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 1,0\rangle, \langle 0,2\rangle, \langle 2,1\rangle\}.$

Зад

$\varphi_1 \leftrightharpoons \forall x \exists y \exists z (y \not\doteq z \;\&\; q(x,z) \;\&\; q(x,y))$

$\varphi_2 \leftrightharpoons \forall x \forall z ((q(x,z) \Rightarrow p(x,z)) \;\&\; (p(x,z) \Rightarrow \neg p(z,x)))$

$\varphi_3 \leftrightharpoons \forall x \forall y \forall z (p(x,y) \;\&\; p(y,z) \Rightarrow p(x,z))$

$\varphi_4 \leftrightharpoons \forall x \forall y (h(x) \doteq h(y) \Rightarrow \neg q(x,y))$

$\varphi_5 \leftrightharpoons \forall x \exists y (h(x) \doteq h(y) \;\&\; p(x,y))$

Зад

$\varphi_1 \leftrightharpoons \forall x \forall y (p(x,y) = \neg \exists z (q(y,z) \,\&\, q(z,x)))$

$\varphi_2 \leftrightharpoons \exists x \exists y (\neg q(x,y) \,\&\, \neg q(y,x))$

$\varphi_3 \leftrightharpoons \forall z \exists x (q(x,z) \vee p(z,x))$

Зад

$\varphi_1 \leftrightharpoons \forall x\,(x \not\doteq f(x)\ \&\ x \not\doteq g(x)).$

$\varphi_2 \leftrightharpoons \exists x\,(x \doteq f(g(x))\ \&\ x \doteq g(f(x))).$

$\varphi_3 \leftrightharpoons \exists x\,(f(x) \doteq g(x)).$

$\varphi_4 \leftrightharpoons \exists x\,(g(f(x)) \not\doteq f(g(x))).$

$\varphi_5 \leftrightharpoons \forall y \exists x\,(y \doteq g(x)).$

(зад)

$\varphi_1 \leftrightharpoons \exists x \exists y (g(x) \doteq y \,\&\, f(x) \doteq y)$

$\varphi_2 \leftrightharpoons \forall x \forall y \forall z (f(x) \doteq y \,\&\, f(y) \doteq z \Rightarrow g(z) \doteq x)$

$\varphi_3 \leftrightharpoons \exists x \exists y \exists z (\neg(x \doteq y) \,\&\, \neg(y \doteq z) \,\&\, \neg(z \doteq x))$

$\varphi_4 \leftrightharpoons \forall x \neg(f(x) \doteq x)$

Док., че $\{\varphi_1, \varphi_2, \varphi_3\}$ и $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ са изпълними м-ва от ф-ли.

Зад 8 $\varphi_1 \leftrightharpoons \exists x(p(x,x) \,\&\, q(x,x))$

$\varphi_2 \leftrightharpoons \forall x(p(x,x) \Rightarrow p(a,x))$

$\varphi_3 \leftrightharpoons \forall x \exists y(q(x,y) \,\&\, q(y,b))$.

$\varphi_4 \leftrightharpoons \exists x(p(b,x) \,\&\, q(c,x))$.

$\varphi_5 \leftrightharpoons q(b,b) \,\&\, \neg p(c,c) \,\&\, \neg q(c,c)$.

(!) a, b и c са индивидни константи.

Зад 8

$\varphi_1 \leftrightharpoons f(f(x,y,a),z,a) \doteq f(x,f(y,z,a),a)$

$\varphi_2 \leftrightharpoons f(f(x,y,b),z,b) \doteq f(a,f(y,z,b),b)$

$\varphi_3 \leftrightharpoons f(f(x,y,a),z,b) \doteq f(f(x,z,b),f(y,z,b),a)$

$\varphi_4 \leftrightharpoons a \doteq b$

$\varphi_5 \leftrightharpoons \neg(a \doteq b)$

$\varphi_6 \leftrightharpoons \forall x \forall y \exists z \neg(f(x,y,z) \doteq y)$.

Да се докаже кои от м-вата:

- $\{\varphi_1, \varphi_2, \varphi_3\}$
- $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$
- $\{\varphi_1, \varphi_2, \varphi_3, \varphi_5\}$
- $\{\varphi_1, \varphi_2, \varphi_3, \varphi_5, \varphi_6\}$

са изпълними!