39. Нека $\mathcal{L} = \langle \underline{\amalg} \rangle$, $\underline{\amalg} \in Preds$, $\#(\underline{\amalg}) = 3$.
Нека $\mathcal{A} = \langle \mathbb{R}^2, \underline{\amalg}^{\mathcal{A}} \rangle$, където:

$$\underline{\amalg}^{\mathcal{A}}(A, B, C) \leftrightharpoons A \neq B \text{ и } A \neq C \text{ и } \angle BAC = 90^\circ$$

Да се докаже, че в стр. $\mathcal{A}$ са определими:

- $Eq \leftrightharpoons \{\langle A, A\rangle \mid A \in \mathbb{R}^2\}$
- $Col \leftrightharpoons \{\langle A, B, C\rangle \mid$ лежат на една права$\}$
- $Circ \leftrightharpoons \{\langle A, B, C\rangle \mid C$ лежи на окръжност с диаметър $AB\}$

Вярно ли е, че в $\mathcal{A}$ са опр. м-вата и защо?

- $Mid \leftrightharpoons \{\langle A, B, C\rangle \mid C$ е среда на отс. $AB\}$
- $Seg \leftrightharpoons \{\langle A, B, C\rangle \mid C$ лежи на отс. $AB\}$

- $E_Q \leftrightharpoons \{\langle A, A\rangle \mid A \in \mathbb{R}^2\}$
- $Col \leftrightharpoons \{\langle A, B, C\rangle \mid$ лежат на една права $\}$
- $Circ \leftrightharpoons \{\langle A, B, C\rangle \mid C$ лежи на окръжност с диаметър $AB\}$

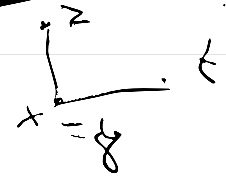
$\perp^{t}(A, B, C) \leftrightharpoons A \neq B$ и $A \neq C$ и $\angle BAC = 90^\circ$

$\cdot\, x = y$ $\qquad$ $x \cdot y$

$\varphi_=(x, y) \leftrightharpoons \forall z\, \neg \perp(z, x, y).$

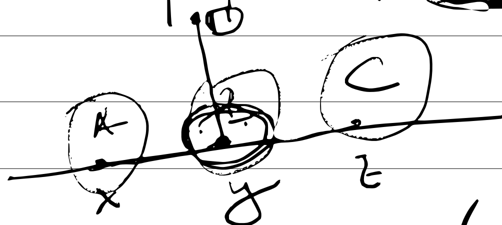
$\neg \perp(z, x, y) = z = x$ или $z = y$ или $\angle xzy \neq 90^\circ$

$\psi_=(x, y) \leftrightharpoons \forall z \forall t\, (\perp(x, z, t) \Leftrightarrow \perp(y, z, t)).$

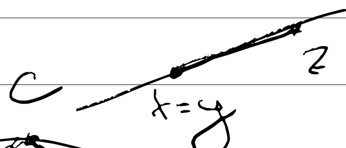


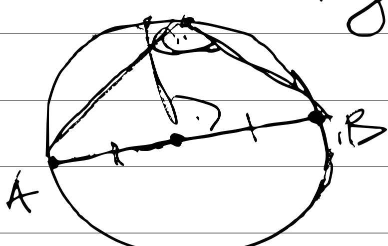
- $Col \leftrightharpoons \{\langle A, B, C\rangle \mid$ лежат на една права $\}$
- $Circ \leftrightharpoons \{\langle A, B, C\rangle \mid C$ лежи на окружност с диаметер $AB\}$

$\perp(A, B, C) \leftrightharpoons A \neq B$ и $A \neq C$ и $\angle BAC = 90^\circ$



$$\varphi_{Col}(x, y, z) \leftrightharpoons \big(\exists t\,(\perp(y, x, t) \,\&\, \perp(y, z, t)) \,\&\, \neg\varphi_{=}(x, z)\big) \vee \varphi_{=}(y, x) \vee \varphi_{=}(y, z).$$



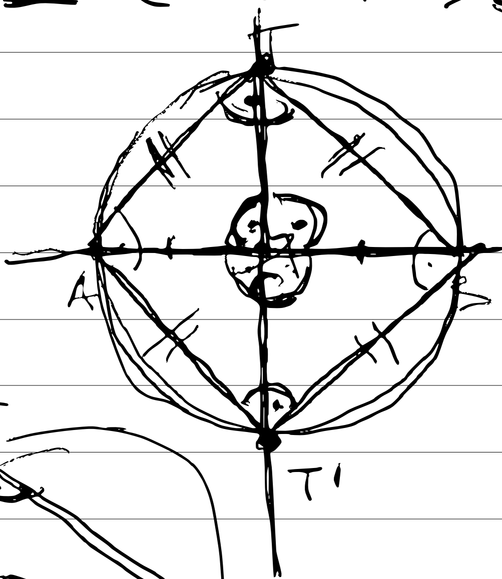


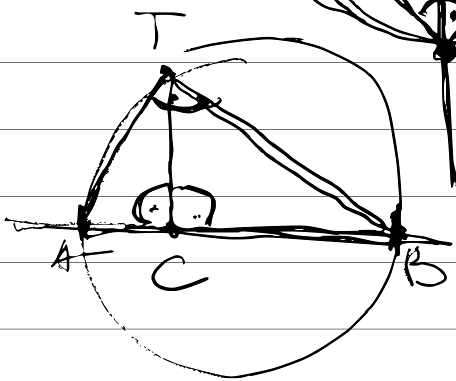
$$\varphi_{circ}(z, x, y) \leftrightharpoons \varphi_{=}(z, x) \vee \varphi_{=}(z, y) \vee \perp(z, x, y).$$

Вярно ли е, че в $\mathcal{A}$ са опр. м-вата и звезд?

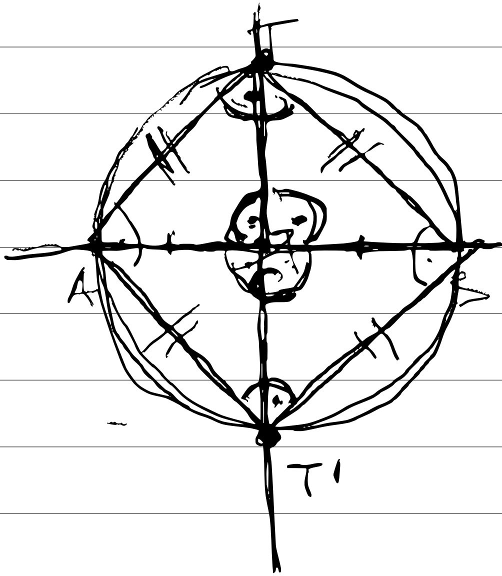
- $\mathrm{Mid} \leftrightharpoons \{\langle A\ B\ C\rangle \mid$ C е среда на отс. $AB\}$
- $\mathrm{Seg} \leftrightharpoons \{\langle A\ B\ C\rangle \mid$ C лежи на отс. $AB\}$

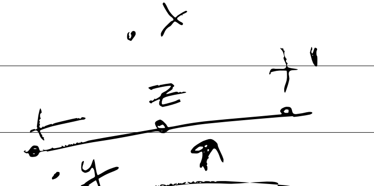
$\perp^{A}(A\ B\ C) \leftrightharpoons A \neq B$ и $A \neq C$ и $\angle BAC = 90^\circ$





$$\varphi_{Seg}(\overset{A}{x}, \overset{B}{y}, \overset{C}{z}) \leftrightharpoons \exists t\,(\varphi_{isc}(t, x, y)\ \&\ \perp(z, x, t)\ \&\ \perp(z, y, t)).$$

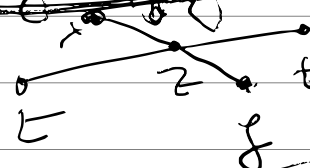


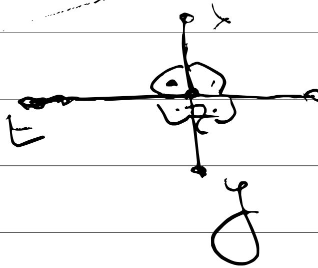


A B C

$\mathrm{Cmid}(x,y,z) \leftrightarrow \exists t \exists t' (\mathrm{Cseg}(z,t,t') \,\&$

$\mathrm{Cseg}(z,x,y) \,\&\; \underline{\parallel}(z,x,t) \,\&$

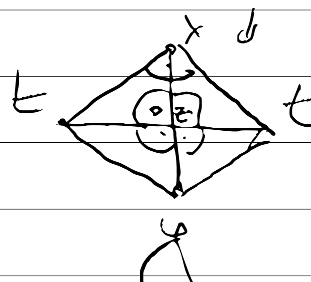


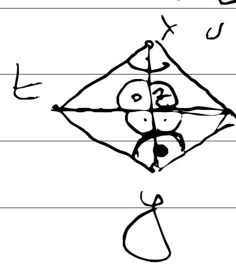


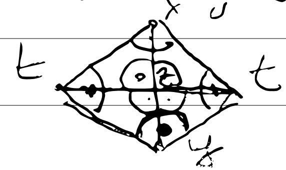
$\underline{\parallel}(x,t,t') \,\&$

$\underline{\parallel}(y,t,t') \,\&$

$\underline{\parallel}(t,x,y)).$







Зад 8

Числа на Фибоначи

Редицата $\{F_n\}_{n=0}^{\infty}$ се дефинира така:

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_{n+2} = F_n + F_{n+1} \quad \text{за } n \geq 0 \end{cases}$$

Разгл. език $\mathcal{L}$ без ф.р. и единст. нелог. символи: $Func_{\mathcal{L}} = \{f\}$, $Pred_{\mathcal{L}} = \{p\}$, $Const_{\mathcal{L}} = \emptyset$ като $\#(f) = 2$, $\#(p) = 1$.

Нека $S = \langle \mathbb{N}; f^S, p^S \rangle$ е стр. за $\mathcal{L}$ като:

$$f^S(x, y) = z \iff x + F_{y+1} = z$$

$$p^S(x) \iff x \in \{F_n\}_{n=0}^{\infty}$$

Да се док., че в S са определими:

a) $\{0\}$ ✓

b) $\{1\}$ ✓

c) $Eq = \{\langle a, a\rangle \mid a \in \mathbb{N}\}$ равенство ✓

d) Вярно ли е, че в S е опр. м-во

$Fibu = \{\langle F_n, F_{n+1}\rangle \mid n \in \mathbb{N}\}$

e) Да се намерят всички автоморф. на S с д-во.

$\varphi_0(x) \leftrightharpoons \forall y\, p(f(x,y))$ // $x + F_{y+1} \in \{F_k\}_{k=0}^{\infty}$

$\exists x . y$

$\varphi_1(x) \leftrightharpoons \neg \varphi_0(x) \& p(f(x,x))$. // $x \neq 0$ &

$x + F_{x+1} \in \{F_k\}_{k=0}^{\infty}$

0 1 1 2 3 5 8 . . .

0 1 2 3 4 5 . . .

$\varphi_=(x,y) \leftrightharpoons \forall z\, (p(f(x,z)) \iff p(f(y,z)))$.

$\psi_1(x) \leftrightharpoons \exists y\, (\varphi_0(y) \& \varphi_=(f(y,y),x))$.

- $\varphi_{+1}(x,y) \leftrightharpoons \exists z\, (\varphi_0(z) \& \varphi_=(f(x,z),y))$. // $y = x+1$

$F_{0+1} = F_1 = 1$

$i_x = z = t+1$

F_z

- $\varphi_{getIndex}(x, i_x) \leftrightharpoons p(x) \& ((\varphi_0(x) \& \varphi_0(i_x)) \vee (\neg\varphi_0(x) \& \exists t \exists e\, (\varphi_0(e) \& \varphi_=(f(e,t),x) \& \varphi_{+1}(t, i_x))))$.

$\mu = \{\langle F_n, n\rangle \mid n \in \mathbb{N}\}$

$0 + F_{t+1} = x$

- $Prev(x,y) \leftrightharpoons \exists i_x \exists i_y (\varphi_{getIndex}(x, i_x) \& \varphi_{+1}(i_x, i_y) \& \varphi_{getIndex}(y, i_y))$.

$Prev = \{\langle F_n, F_{n+1}\rangle \mid n \in \mathbb{N}\}$

$f^S(x,y) = z \iff x + F_{y+1} = z$

(base): φ_0

(i.h): Нека за $n>0$, то $\{n\}$ е опр. с φ_n.

(step): $\varphi_{n+1}(x) \leftrightharpoons \exists y\,(\varphi_n(y)\ \&\ \varphi_{+1}(y,x))$.

Т.е. всеки синглетон е опр: $(\forall n \in \mathbb{N})[\{n\}\text{ е опр.}]$

$Id_{\mathbb{N}} \in Aut(S) \neq \emptyset$

защо $Aut(S) \subseteq \{Id_{\mathbb{N}}\}$?

Нека $h \in Aut(S)$.

Т.к. h трябва да запазва определимите м-ва, т.е. за B опр. м-во то $h[B]=B$.

Но тук $(\forall n \in \mathbb{N})[\{n\}\text{ е опр.}] \to$

$(\forall n \in \mathbb{N})[h[\{n\}]=\{n\}] \to$

$(\forall n \in \mathbb{N})[h(n)=n] \to$

$\underline{h = Id_{\mathbb{N}}}$. $\to$ т.е. $Aut(S)=\{Id_{\mathbb{N}}\}$.

Задача за изпълнимост на м-во от формули

Формулировка:

Дадено: м-во от затворени ф-ли Φ

Търси се: структура, т.е. всички ф-ли в Φ да са верни в нея.

def/ Изпълнимост на ф-ла

Нека φ е ф-ла от език $\mathcal{L}$. φ е изпълнима, ако съществува структура $\mathcal{A}$ за $\mathcal{L}$ и оценка на инд. променливи v в $\mathcal{A}$, т.е. $\mathcal{A} \models_v \varphi$.

(*) Ако за всяка оценка v в $\mathcal{A}$, то $\mathcal{A} \models_v \varphi$, то пишем $\mathcal{A} \models \varphi$ и казваме, че стр. $\mathcal{A}$ е модел за φ.

(*) Ако φ е затворена, то верността на φ не зависи от оценката т.е. за затв. ф-ли е в сила, че:

φ e изпълнима в $\mathcal{A}$ ⟺ $\mathcal{A}$ e модел за φ

Задачата за намиране на модел на м-во от ф-ли е аналогична на задачата да напишем програма отговаряща на дадена формална спецификация

Зад.

$\varphi_1 \rightleftharpoons \forall x \neg p(x,x)$

$\varphi_2 \rightleftharpoons \forall x \exists y\, p(x,y)$

$\varphi_3 \rightleftharpoons \forall x \forall y \forall z\,(p(x,y) \,\&\, p(y,z) \Rightarrow p(x,z))$

1 нелогически символ: p

$\Gamma_1 \rightleftharpoons \{\varphi_1, \varphi_2\}$

$\Gamma_2 \rightleftharpoons \Gamma_1 \cup \{\varphi_3\}$

a) Докажете, че Γ_1 и Γ_2 имат модели.

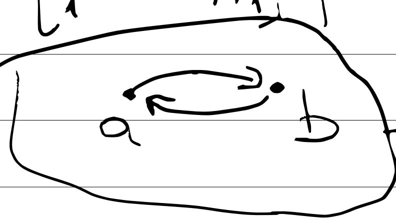
b) Γ_2 има ли краен модел (т.е. с краен универсиум)?

φ_1 – антирефлексивност по $p^{\mathcal{A}}$

φ_2 – за всеки връх v: $deg_{out}(v) \geq 1$ сериалност нагоре по $p^{\mathcal{A}}$

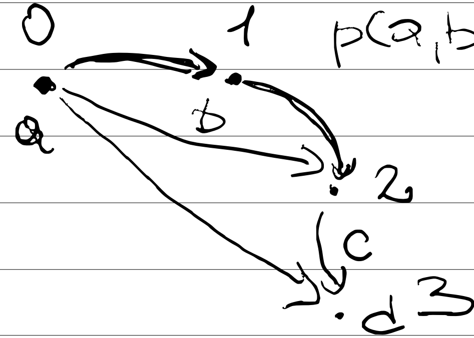
φ_3 – транзитивност по $p^{\mathcal{A}}$

$\to \Gamma_1$: $\mathcal{A}_1 = \langle A_1; p^{\mathcal{A}_1} \rangle$, $A_1 \neq \emptyset \to |A_1| > 0$



- $A_1 = \{a, b\}$
- $p^{\mathcal{A}_1} = \{\langle a,b \rangle, \langle b,a \rangle\}$

Γ_2:



$\underline{p(a,b) \,\&\, p(b,x) \Rightarrow p(a,x)}$

φ_1

$\mathcal{A}_2 = \langle A_2, p^{\mathcal{A}_2} \rangle$: $A_2 = \mathbb{N}$

$p^{\mathcal{A}_2} = <$ в списъка на ест. числа

б) Има ли $\mathcal{A} \models \Gamma_2$ и $\mathcal{A}$ да е крайна?

Доп., че има $\underbrace{\mathcal{A} \text{ крайна}}_{=}$, т.ч. $\mathcal{A} \models \Gamma_2$.

$(\exists n \in \mathbb{N})[\overline{\overline{A}} = n \,\&\, n \geq 1]$

Нека n_0 е свидетел т.е. $\overline{\overline{A}} = n_0$ и $n_0 \geq 1$.

Нека $a_0 \in A$ свидетел за $\underline{n_0 \geq 1} \to A \neq \emptyset$.

По φ_{2_0} същ. $a_1 \in A$ т.е. $p^{\mathcal{A}}(a_0, a_1)$.

и т.н. същото за $a_1, a_2, \ldots$

$p^A(a_0, a_1) \,\&\, p^A(a_1, a_2) \,\&\, p^A(a_2, a_3) \,\&\, \ldots \,\&\, p^A(a_{n_0-1}, a_{n_0})$

$\overline{\overline{A}} = n_0$, а имаме $a_0, \ldots, a_{n_0}$ редица с дълж. n_0+1 с елементи от A.

$\Rightarrow$ има / имаме a_i, a_j, $i \neq j$, $0 \leq i, j \leq n_0$, т.ч. $a_i = a_j$. БОО $i < j$

взели сме свидетели

$p^A(a_i, a_{i+1}) \,\&\, \ldots \,\&\, p^A(a_{j-1}, a_j)$

Сега по Ф3 прилагаме $(j-i+1)$ пъти:

$\underbrace{p^A(a_i, a_j)}_{\Downarrow}$

$p^A(a_i, a_i) \rightarrow a_i$ рефлексивно по p^A

Но по Ф1 нямаме рефл. точки ⚡

Допускането ни води до противоречие!

Значи всеки модел за Γ_2 е с безкр. унив.

HW) Нека няма $p_1 - p_n$ едноместни предик. символи!

С-во: Φ м-во от ф-ли в този език има модел, то Φ има задължително и краен модел.

3a)

$\varphi_1 \leftrightharpoons \exists x \forall y (x \neq y \Rightarrow \exists z \neg p(y, z))$

$\varphi_2 \leftrightharpoons \exists x \forall y\, p(x, y)$

$\varphi_3 \leftrightharpoons \forall x \forall y (p(x, y) \Rightarrow \exists z (p(x, z) \,\&\, p(z, y)))$

$\varphi_4 \leftrightharpoons \forall x \exists y\, p(x, y)$

322g

$\varphi_1 \Leftrightarrow \neg \exists x\, p(x,x)$

$\varphi_2 \Leftrightarrow \forall x \exists y \big(p(x,y) \,\&\, \neg \exists z (p(x,z) \,\&\, p(z,y))\big)$

$\varphi_3 \Leftrightarrow \exists x \neg \exists y\, p(y,x)$

$\varphi_4 \Leftrightarrow \exists x \big(\exists y\, p(y,x) \rightarrow \neg \exists y (p(y,x) \,\&\,$
$\neg \exists z (p(x,z) \,\&\, p(z,y))))\big)$

3a2)

$\varphi_1 \leftrightharpoons \neg \forall x \forall y \forall z (p(x,y) \& p(y,z) \Rightarrow p(x,z))$

$\varphi_2 \leftrightharpoons \forall y \exists x (p(x,y) \& \neg p(y,x) \& p(x,x))$

Зад

$\varphi_1 \leftrightharpoons \forall x \exists y \exists z (y \not\doteq z \;\&\; q(x,z) \;\&\; q(x,y))$

$\varphi_2 \leftrightharpoons \forall x \forall z ((q(x,z) \Rightarrow p(x,z)) \;\&\; (p(x,z) \Rightarrow \neg p(z,x)))$

$\varphi_3 \leftrightharpoons \forall x \forall y \forall z (p(x,y) \;\&\; p(y,z) \Rightarrow p(x,z))$

$\varphi_4 \leftrightharpoons \forall x \forall y (h(x) \doteq h(y) \Rightarrow \neg q(x,y))$

$\varphi_5 \leftrightharpoons \forall x \exists y (h(x) \doteq h(y) \;\&\; p(x,y))$

Зад 6

$$\varphi_1 \leftrightharpoons \forall x \forall y (p(x,y) \Rightarrow \neg \exists z (q(y,z) \,\&\, q(z,x)))$$

$$\varphi_2 \leftrightharpoons \exists x \exists y (\neg q(x,y) \,\&\, \neg q(y,x))$$

$$\varphi_3 \leftrightharpoons \forall z \exists x (q(x,z) \vee p(z,x))$$

Зад

$\varphi_1 \leftrightharpoons \forall x\, (x \not\doteq f(x)\ \&\ x \not\doteq g(x)).$

$\varphi_2 \leftrightharpoons \exists x\, (x \doteq f(g(x))\ \&\ x \doteq g(f(x))).$

$\varphi_3 \leftrightharpoons \exists x\, (f(x) \doteq g(x)).$

$\varphi_4 \leftrightharpoons \exists x\, (g(f(x)) \not\doteq f(g(x))).$

$\varphi_5 \leftrightharpoons \forall y \exists x\, (y \doteq g(x)).$

Зад

$\varphi_1 \rightleftharpoons \exists x \exists y (g(x) \doteq y \ \& \ f(x) \doteq y)$

$\varphi_2 \rightleftharpoons \forall x \forall y \forall z (f(x) \doteq y \ \& \ f(y) \doteq z \Rightarrow g(z) \doteq x)$

$\varphi_3 \rightleftharpoons \exists x \exists y \exists z (\neg(x \doteq y) \ \& \ \neg(y \doteq z) \ \& \ \neg(z \doteq x))$

$\varphi_4 \rightleftharpoons \forall x \neg(f(x) \doteq x)$

Док., че $\{\varphi_1, \varphi_2, \varphi_3\}$ и $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ са изпълними м-ва от фор-ли.

Зад. $\varphi_1 \leftrightharpoons \exists x (p(x,x) \,\&\, q(x,x))$

$\varphi_2 \leftrightharpoons \forall x (p(x,x) \Rightarrow p(a,x))$

$\varphi_3 \leftrightharpoons \forall x \exists y (q(x,y) \,\&\, q(y,b))$.

$\varphi_4 \leftrightharpoons \exists x (p(b,x) \,\&\, q(c,x))$.

$\varphi_5 \leftrightharpoons q(b,b) \,\&\, \neg p(c,c) \,\&\, \neg q(c,c)$.

(!) a, b и c са индивидни константи.

Зад 8

$\varphi_1 \leftrightharpoons f(f(x,y,a),z,a) \doteq f(x,f(y,z,a),a)$

$\varphi_2 \leftrightharpoons f(f(x,y,b),z,b) \doteq f(a,f(y,z,b),b)$

$\varphi_3 \leftrightharpoons f(f(x,y,a),z,b) \doteq f(f(x,z,b),f(y,z,b),a)$

$\varphi_4 \leftrightharpoons a \doteq b$

$\varphi_5 \leftrightharpoons \neg(a \doteq b)$

$\varphi_6 \leftrightharpoons \forall x \forall y \exists z \neg(f(x,y,z) \doteq y)$.

Да се докаже кои от м-вата:

- $\{\varphi_1, \varphi_2, \varphi_3\}$
- $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$
- $\{\varphi_1, \varphi_2, \varphi_3, \varphi_5\}$
- $\{\varphi_1, \varphi_2, \varphi_3, \varphi_5, \varphi_6\}$

са изпълними!