Зад. Нека $\mathcal{L} = \langle \star \rangle$ и е с ф.р. $\doteq$, $\#(\star)=2$

Нека $\mathcal{A} = \langle \mathbb{N}, \star^{\mathcal{A}} \rangle$ за

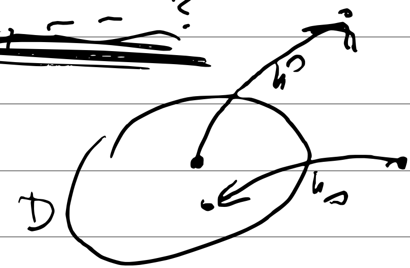
$x \star^{\mathcal{A}} y = z \iff z = x . y$

Зад: $m \star a = n$

Определете $\{0\}$ и $\{1\}$

- Опр. $Д_{n/m} = \{\langle n, m \rangle \in \mathbb{N} \times \mathbb{N} \mid \frac{n}{m} \in \mathbb{N}\}$
- Опр. м-вото на простите числа
- Определими ли са $\{2\}, \{3\}, \{4\}, \ldots$?

Да го завършим!



Крит. за неопр.

Нека $D \subseteq A^n$. D е неопр. м-во, ако има $h_0 \in Aut(\mathcal{A})$ и има $a_1, \ldots, a_n \in A$, т.е.

не $\left(\langle a_1, \ldots, a_n \rangle \in D \text{ т.с.т.к. } \langle h_0(a_1), \ldots, h_0(a_n) \rangle \in D \right)$

<u>О.Th.А.</u> / Нека $n \in \mathbb{N}_{>0}$. Тогава

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_i^{\alpha_i} \cdot \ldots \qquad \begin{matrix} p_1 = 2 \\ p_2 = 3 \\ \ldots \end{matrix}$$

$$\alpha_i \geq 0$$

<u>Зад</u> / Значи краен брой ненулеви <u>степенни показатели</u> $\underline{\alpha_i \ldots}$

<u>$\alpha \mathbb{Z} \mathbb{Z}$</u> $h \in Aut(A)$ т.ч. $\underline{h(2) \notin \{2\}}$

- $h_0 : \mathbb{U}_{prime} \to \mathbb{U}_{prime}$

$$\underline{h_0}(x) \leq \begin{cases} 3, & x = 2 \\ 2, & x = 3 \\ x, & \text{else} \end{cases}$$

$\mathbb{U}_{prime}$: 2, 3

$\underline{h_0(h_0(x)) = x}$ $\to$ т.е. $h_0 = h_0^{-1}$ <u>биекция</u>

- $h : \mathbb{N} \to \mathbb{N}$

Нека $x \in \mathbb{N}_{>0} \overset{\text{ОТНА}}{\Longrightarrow} x = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_i^{\alpha_i} \ldots \infty$

$$\underline{h(x)} = \begin{cases} 0, & x = 0 \\ \underline{h_0^{\alpha_1}(p_1) \cdot h_0^{\alpha_2}(p_2) \cdot \ldots \cdot h_0^{\alpha_i}(p_i) \cdot \ldots} = \prod_{i=1} h_0^{\alpha_i}(p_i), & x > 0 \end{cases}$$

* биекция $h(h(x)) = x \to \underline{h = h^{-1}}$

$x \in \mathbb{N}$ биекция

* хом

$a_1, a_2 \in \mathbb{N}$

$h(*^t(a_1, a_2)) = *^t(h(a_1), h(a_2))$

Th/ $f^{A \to B}$ биекция $\Leftrightarrow$ свущ. $g: B \to A$ т.ч.

$f \circ g = Id_B$ и

$g \circ f = Id_A$

- $a_1 > 0$ и $a_2 = 0$ или

$a_1 = 0$ и $a_2 > 0$ или

$a_1 = 0$ и $a_2 = 0$

очевидно, простейший

- $a_1 > 0,\ a_2 > 0 \xrightarrow{\text{OTHA}}$

$a_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i} \dots$

$a_2 = p_1^{\beta_1} p_2^{\beta_2} \dots p_i^{\beta_i} \dots$

- инекция
- сюрекция

$\to$ биекция

$p_1 = 2,\ p_2 = 3,\ p_3 = 5, \dots$

$*^t(h(a_1), h(a_2)) = *^t(h(2^{\alpha_1} 3^{\alpha_2} \dots p_i^{\alpha_i} \dots), h(2^{\beta_1} 3^{\beta_2} \dots p_i^{\beta_i} \dots)) =$

$= *^t(3^{\alpha_1} 2^{\alpha_2} \dots p_i^{\alpha_i} \dots, 3^{\beta_1} 2^{\beta_2} \dots p_i^{\beta_i} \dots) =$

$= 3^{\alpha_1+\beta_1} 2^{\alpha_2+\beta_2} \dots p_i^{\alpha_i+\beta_i} \dots =$

$= h(2^{\alpha_1+\beta_1} 3^{\alpha_2+\beta_2} \dots p_i^{\alpha_i+\beta_i}) =$

$h(*^t(2^{\alpha_1} 3^{\alpha_2} \dots p_i^{\alpha_i} \dots, 2^{\beta_1} 3^{\beta_2} \dots p_i^{\beta_i})) = h(*^t(a_1, a_2))$

Зач: $2\in\{2\}$, но $h(2)\notin\{2\}$, т.е. крит. за неопр., то $\{2\}$ не е опр. м-во

$6 = 2^1\cdot 3^1$, $h(6) = h_0^1(2)\cdot h_0^1(3) =$
$= 3^1\cdot 2^1 = 6$.

$15 = 3\cdot 5$

$2 = 2^1\cdot 3^0\cdot 5^0\cdot \ldots$

$\{n\} \to n = p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_i^{\alpha_i}\cdots$

Нека $\alpha_k \neq 0$ и $\alpha_j = 0$

краен брой — безкраен брой

$$h_0(x) = \begin{cases} p_j, & x = p_k \\ p_k, & x = p_j \\ x, & \text{else} \end{cases}$$

Зад. $\mathcal{L} = \langle * \rangle$ един двуместни функ. символи и ф.р. $\doteq$.

$\mathcal{A} = \langle \mathbb{R}, *^{\mathcal{A}} \rangle$

$$a *^{\mathcal{A}} b = c \Leftrightarrow a \cdot b = c$$

Кои синглетони са определими?

$$\varphi_0(x) \leftrightharpoons \forall y (x * y \doteq x)$$

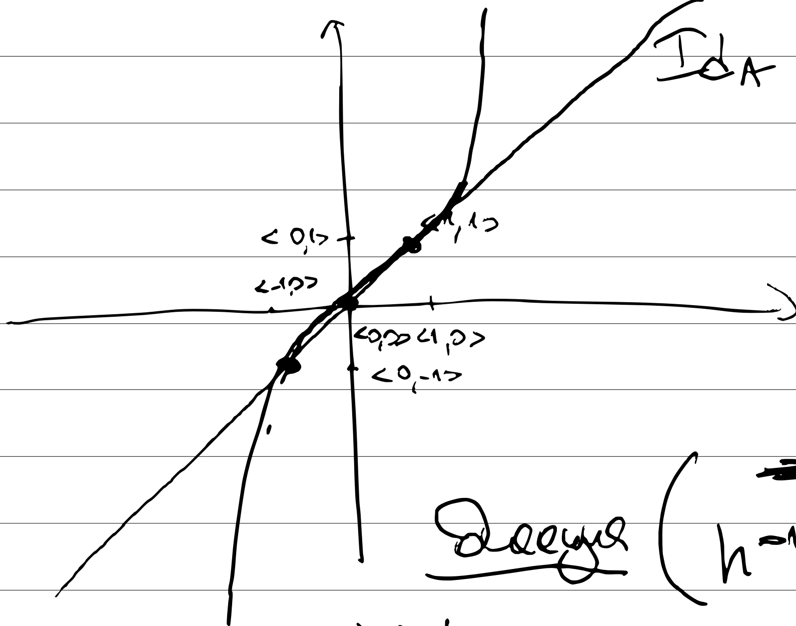
$$\varphi_1(x) \leftrightharpoons \forall y (x * y \doteq y)$$

$$\varphi_{-1}(x) \leftrightharpoons \forall y (x * x * y \doteq y) \,\&\, \neg\varphi_1(x)$$

$$\psi_{-1}(x) \leftrightharpoons \exists z (\varphi_1(z) \,\&\, x * x = z) \,\&\, \underline{\neg\varphi_1(x)}$$

Защо за вс. $r \in \mathbb{R} \setminus \{0, -1, 1\}$, то $\{r\}$ не е <u>определимо</u>?

h - биек., т.ч. $h(*^{\mathcal{A}}(a, b)) = *^{\mathcal{A}}(h(a), h(b))$



$h(x) = x^2$?, хом, но

не е биекция

не запазва -1

$h(x) = x^3$

биекция $\left(h^{-1}(x) = \sqrt[3]{x} = x^{1/3} \right.$

хом

$h(*^A(a,b)) = h(a.b) = (a.b)^3 =$

$= a^3 . b^3 = *^A(h(a), h(b))$

Зад. $\mathcal{A} = \langle * \rangle \doteq \#(*) = 2.$

Нека $A = \langle \mathbb{Q}, *^t \rangle.$

$a *^t b \doteq c \iff a \cdot b = c.$

$a, b, c \in \mathbb{Q}$

Кои синглетони са определими?

e_0, e_1, e_{-1} от предната задача $\to$ ✓.

Как да покажем, че за $q \in \mathbb{Q} \setminus \{0, -1, 1\}$, то

$\{q\}$ не е опр.

$h(x) = x^3$, хом, но не е биекция

$h^{-1}(2) = ?$ не е сюрекция

$h^{-1}(2) = \sqrt[3]{2} \notin \mathbb{Q}$

$\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \}$

$\mathbb{Z} = \mathbb{N} \cup \mathbb{N}^-$

$\quad$ '' $\{-n \mid n \in \mathbb{N} \setminus \{0\}\}$

$x = \frac{p}{q} \in \mathbb{Q}$ $\underline{|p|, |q| \in \mathbb{N}}$ OTHA $\Rightarrow$

$p \neq 0$

$$h(x) = \begin{cases} \operatorname{sign}\left(\frac{p}{q}\right) \cdot \frac{|p|}{|q|} & , p \neq 0 \\ 0 & , p = 0 \end{cases}$$

$$h(x) = \begin{cases} 1/x = x^{-1} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Зад | $\mathcal{L} = \langle f \rangle$, $\doteq$; $f \in Func_{\mathcal{L}}$, $\#(f) = 2$.

Нека Σ е крайна азбука.

Нека $A = \langle \Sigma^*, f^A \rangle$ където

$f^A(u, v) = w \overset{def}{\Leftrightarrow} u \circ v = w$

за произв. $u, v, w \in \Sigma^*$.

Да се докаже, че в A са определими:

- $Pref = \{ \langle u, v \rangle \in (\Sigma^*)^2 \mid u$ е префикс на $v \}$
- $Suff = \{ \langle u, v \rangle \in (\Sigma^*)^2 \mid u$ е суфикс на $v \}$
- $W_1 \leftrightharpoons \{ u \in \Sigma^* \mid |u| = 1 \}$
- $T_1 \leftrightharpoons \{ u \in \Sigma^* \mid |u| \leq 1 \}$
- за вс. $n \in \mathbb{N}$, W_n е опр.
- за вс. $n \in \mathbb{N}$, T_n е опр.
- $O \leftrightharpoons \{ \langle u, v, w \rangle \in (\Sigma^*)^3 \mid$ най-дългият общ префикс на u и v е суфикс на $w \}$
- $D \leftrightharpoons \{ \langle u, v, w \rangle \in (\Sigma^*)^3 \mid$ най-дългият общ суфикс на u и v е префикс на $w \}$

Да се изрази броят на автоморфизмите в $\mathcal{A}$ чрез броя на буквите от Σ.

- $Pref = \{ \langle u, v \rangle \in (\Sigma^*)^2 \mid u$ е префикс на $v \}$
- $Suff = \{ \langle u, v \rangle \in (\Sigma^*)^2 \mid u$ е суфикс на $v \}$

$f^{\mathcal{A}}(u, v) = u \circ v$

$\varphi_{Pref}(x, y) \leftrightharpoons \exists z (f(x, z) \doteq y).$ // $x \circ z = y$ за някое $z \in \Sigma^*$

$\varphi_{Suff}(x, y) \leftrightharpoons \exists z (f(z, x) \doteq y).$

- $W_1 \leftrightharpoons \{ u \in \Sigma^* \mid |u| = 1 \}$, $|u|$ - дължина на думата,
- $T_1 \leftrightharpoons \{ u \in \Sigma^* \mid |u| \leq 1 \}$

$\varphi_{\varepsilon}(x) \leftrightharpoons \forall y (f(x, y) \doteq y).$

$\varphi_{W_1}(x) \leftrightharpoons \neg \varphi_{\varepsilon}(x) \;\&\; \forall y \forall z (\varphi_{Pref}(y, x) \;\&\; \varphi_{Suff}(z, x) \;\&\; \neg \varphi_{\varepsilon}(y) \;\&\; \neg \varphi_{\varepsilon}(z) \Rightarrow y \doteq x \;\&\; z \doteq x)$

aaaa

$\varphi_{W_1}(x) \leftrightharpoons \neg \varphi_{\varepsilon}(x) \,\&\, \forall y(\varphi_{pref}(y,x) \Rightarrow \varphi_{\varepsilon}(y) \vee x = y)$

$\varphi_{T_1}(x) \leftrightharpoons \varphi_{\varepsilon}(x) \vee \varphi_{W_1}(x).$

$\varphi_{T_1}(x) \leftrightharpoons \forall y(Pref(y,x) \Rightarrow \varphi_{\varepsilon}(y) \vee x = y).$

- за вс. $n \in \mathbb{N}$, W_n е опр.
- за вс. $n \in \mathbb{N}$, T_n е опр.

база: $\varphi_{\varepsilon}, \varphi_{T_1}$

(i.h.): Нека за някое n, φ_{T_n} опр. T_n

(i.step): $\varphi_{T_{n+1}}(x) \leftrightharpoons \forall y(\varphi_{pref}(y,x) \Rightarrow x = y \vee \varphi_{T_n}(y))$

база: $\varphi_{\varepsilon}, \varphi_{W_1}$

(i.h.): Нека за вс. $k \leq n$, φ_{W_k} опр. W_k

(i.step): $\varphi_{W_{n+1}}(x) \leftrightharpoons \neg \varphi_{\varepsilon}(x) \,\&\, \neg \varphi_{W_1}(x) \,\&\, \dots \,\&\, \neg \varphi_{W_n}(x) \,\&\, \forall y(\varphi_{pref}(y,x) \Rightarrow \varphi_{\varepsilon}(y) \vee \varphi_{W_1}(x) \vee \dots \vee \varphi_{W_n}(x) \vee x = y)$

(i.h) За някое n, φ_{ω_n} опр. ω_n

(i.step) $\varphi_{\omega_{n+1}}(x) \leftrightharpoons \exists y \exists z \left(\varphi_{\omega_1}(y) \,\&\, \varphi_{\omega_n}(z) \,\&\, f(z,y) \doteq x \right)$

$O \leftrightharpoons \{ \langle u,v,w \rangle \in (\Sigma^*)^3 \mid$ най-дългият общ префикс на u и v е суфикс на $w \}$

$\varphi_{LCPref}(x,y,z) \leftrightharpoons \varphi_{Pref}(x,y) \,\&\, \varphi_{Pref}(x,z) \,\&$

"x е най-дългият общ префикс на y и z"

$\forall t \left(\varphi_{Pref}(t,y) \,\&\, \varphi_{Pref}(t,z) \Rightarrow \varphi_{Pref}(t,x) \right).$

$\varphi_O(x,y,z) \leftrightharpoons \exists t \left(\varphi_{LCPref}(t,x,y) \,\&\, \varphi_{Suff}(t,z) \right).$

Да се изрази броят на автоморфизмите в A чрез броя на буквите от Σ.

$$|\Sigma| = n, \quad n \in \mathbb{N}$$

$$\Sigma = \{a_1, \dots, a_n\}$$

$$|\{f \mid f : \Sigma \to \Sigma \text{ биекция}\}| = n!$$

$\in h_0$

$$h_0 : \Sigma \xrightarrow{\text{биек.}} \Sigma$$

$$h : \Sigma^* \to \Sigma^* : \; h(\omega) = h_0(b_1) \circ h_0(b_2) \circ \dots \circ h_0(b_k)$$

$$\omega = b_1 \dots b_k$$

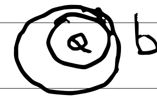
хом. → непосредствено

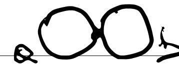
биекция: $h^{\text{(наобратни на циклите в } h_0)}(\omega) = \omega$

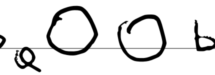
$$\begin{array}{cccccccc}
\Sigma: & a & b & c & d & e & f \\
\downarrow h_0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\Sigma: & b & c & a & d & f & e
\end{array}$$

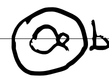
Зад. | $\mathcal{L} = \langle p \rangle$, $\#p = 2$.

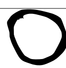
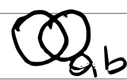
$\mathcal{A} = \langle$ окр. с ненулев радиус в $\mathbb{R}^2$, $p^{\mathcal{A}} \rangle$, където

a) $p^{\mathcal{A}}(a, b) \iff$ 

b) $p^{\mathcal{A}}(a, b) \iff$ 

c) $p^{\mathcal{A}}(a, b) \iff$ 

d) $p^{\mathcal{A}}(a, b) \iff$ 

Опр.  ^{a,b}  и останалите.

Зад/ $\mathcal{L} = \langle p \rangle$, $p \in Pred_{\mathcal{L}}$, $\#(p) = 3$.

$\mathcal{A} = \langle \mathbb{N}, p^{\mathcal{A}} \rangle$, където

$p^{\mathcal{A}}(k, n, m) \iff k + n = m + 2$

a) Док., че всеки синглетон е определим

b) Да се определят $=$ и $<$.

3ад. Нека $\mathcal{L} = \langle p \rangle$ FOL, $p \in Pred$, $\#p = 3$.
Нека $\mathcal{A} = \langle \mathcal{P}(\mathbb{N}), p^{\mathcal{A}} \rangle$ е структура за $\mathcal{L}$, като:
- $\mathcal{P}(\mathbb{N})$ - м-вото от всички подмн-ва на $\mathbb{N}$ т.е. $\mathcal{P}(\mathbb{N}) = \{B \mid B \subseteq \mathbb{N}\}$
- $\langle a, b, c \rangle \in p^{\mathcal{A}} \Longleftrightarrow a \cup b = c$

$a, b, c \in \mathcal{P}(\mathbb{N})$

Да се докаже, че:

a) $\{\emptyset\}$ е определимо

b) $\{\mathbb{N}\}$ е определимо

c) $\subseteq \leftrightharpoons \{\langle x, y \rangle \mid x, y \in \mathcal{P}(\mathbb{N}) \ \& \ x \subseteq y\}$ е определимо

d) $\cap \leftrightharpoons \{\langle x, y, z\rangle \mid x, y, z \in \mathcal{P}(\mathbb{N})\ \&\ x \cap y = z\}$

е определимо

e) $^{-} \leftrightharpoons \{\langle x, y\rangle \mid x, y \in \mathcal{P}(\mathbb{N})\ \&\ y = \mathbb{N} \setminus x\}$

т.е. y е допълнението на x

е определимо.

f) Докажете, че за $a \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset, \mathbb{N}\}$, то $\{a\}$ е неопределимо.

380/ Нека $\mathcal{L} = \langle \mathbb{L} \rangle$, $\mathbb{L} \in Preds$, $\#(\mathbb{L}) = 3$.

Нека $\mathcal{A} = \langle \mathbb{R}^2, \mathbb{L}^{\mathcal{A}} \rangle$, където:

$$\mathbb{L}^{\mathcal{A}}(A, B, C) \leftrightharpoons A \neq B \text{ и } A \neq C \text{ и } \angle BAC = 30^\circ$$

Да се док., че в стр. $\mathcal{A}$ са определими:

- $Eq \leftrightharpoons \{\langle A, A \rangle \mid A \in \mathbb{R}^2\}$
- $Col \leftrightharpoons \{\langle A, B, C \rangle \mid$ лежат на една права$\}$
- $Circ \leftrightharpoons \{\langle A, B, C \rangle \mid C$ лежи на окръжност с диаметър $AB\}$

Вярно ли е, че в $\mathcal{A}$ са опр. м-вата и защо?

- $Mid \leftrightharpoons \{\langle A, B, C \rangle \mid C$ е среда на отс. $AB\}$
- $Seg \leftrightharpoons \{\langle A, B, C \rangle \mid C$ лежи на отс. $AB\}$