Задача за изпълнимост на м-во от ф-ли

def / Нека Γ е м-во от ф-ли за език $\mathcal{L}$.

i) Γ е вярно в $\mathcal{A}$ при оц. v от $\mathcal{A}$ $\leftrightarrow$ $(\forall \varphi \in \Gamma)[\mathcal{A} \models_v \varphi]$.

Бел: $\mathcal{A} \models_v \Gamma$

ii) $\mathcal{A}$ е модел за Γ, ако $(\forall \varphi \in \Gamma)(\forall v$ от $\mathcal{A})[\mathcal{A} \models_v \varphi]$.

Бел: $\mathcal{A} \models \Gamma$

iii) Γ е изпълнимо, ако Γ има модел.

Задачата за намиране на модел на м-во от ф-ли е аналогична на задачата дали съществува програма отговаряща на дадена формална спецификация.

(Зад) Да се док, че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \leftrightharpoons \forall x\, \neg p(x,x)$. // иррефл.

$\varphi_2 \leftrightharpoons \forall x \exists y\, p(x,y)$. // $d_0(x) > 0$ $\quad x \circlearrowleft$ или $x \to y$

$\varphi_3 \leftrightharpoons \forall x \forall y \forall z\, (p(x,y) \,\&\, p(y,z) \Rightarrow p(x,z))$. // транзитивност

$\cancel{\exists y\, p(x,y)}$ не е затворена ф-ла

Да намерим стр. $\mathcal{A}$, т.ч. $\mathcal{A} \models \{\varphi_1, \varphi_2, \varphi_3\}$

$\varphi_1, \varphi_2, \varphi_3$ са верни в $\mathcal{A}$.

c - константи, $c^{\mathcal{A}} \in A$

// сериалност надясно

$\mathcal{A} = \langle A, p^{\mathcal{A}} \rangle$ т.ч. $A = ?$

$\mathcal{A} \models \{\varphi_1, \varphi_2, \varphi_3\}$ $\quad p^{\mathcal{A}} = ?$

$\mathcal{A} = \langle \mathbb{N}, p^{\mathcal{A}} \rangle$ за $p^{\mathcal{A}} \leftrightharpoons <_{\mathbb{N}}$

Да допуснем, че има краен модел и нека A_0 е свидетел. Тогава има $\underline{n}$ ест.ч., т.ч. $\underline{|A_0| = n}$.

$\underline{A_0 \neq \emptyset}$, т.е. $\underline{n > 0}$.

$$a_0 \to a_1 \to a_2 \to \dots \to a_{n-1} \to a_n$$

$\underline{n+1 \text{ точки засега}}$

По PH principle: има $0 \le i < j \le n$, т.ч. $\underline{a_i = a_j}$.

φ_1 - иррефл.; φ_2 - сериалност надясно; φ_3 - транзит.

$$a_j = a_i \to a_{i+1} \to \dots \to a_j$$

по φ_3

a_j

по φ_3 ↯ с φ_1.

Допускането ни води до против., т.е. $\{\varphi_1, \varphi_2, \varphi_3\}$ няма краен модел.

$\varphi \leftrightharpoons \varphi_1 \& \varphi_2 \& \varphi_3$

форсира универсума да е безкраен.

(503) Да се док., че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \leftrightharpoons \underline{\forall y \exists x\, p}(y, x, y). \models \forall x \exists y\, p(x, y, x)$

$\varphi_2 \leftrightharpoons \exists y \neg p(y, y, y).$

$\varphi_3 \leftrightharpoons \forall x \neg \forall y\, p(x, y, x). \models \forall x \exists y \neg p(x, y, x).$

$\mathcal{A} = \langle \underbrace{\mathbb{N} \setminus \{0\}}_{\mathbb{N}^+}, p^{\mathcal{A}} \rangle$

$p^{\mathcal{A}}(a, b, c) \longleftrightarrow a.b = c$

$\mathcal{B}$:

$\mathcal{B} = \langle \{a_0, a_1\}, p^{\mathcal{A}} \rangle$

$p^{\mathcal{A}} = \{\langle a_0, a_1, a_0 \rangle, \langle a_1, a_0, a_1 \rangle\}$

$q(x, y) \leftrightharpoons p(x, y, x).$

$\varphi_1' \leftrightharpoons \forall x \exists y\, q(x, y)$ сериалност надясно

$\varphi_2' \leftrightharpoons \exists y \neg q(y, y)$

$\varphi_3' \leftrightharpoons \forall x \exists y \neg q(x, y)$

Изпълнимо ли е следните м-ва от затв. ф-ли:

$\varphi_1 \leftrightharpoons \forall x \forall y\, (p(x,y) \vee p(y,x) \vee x \doteq y)$.

$\varphi_2 \leftrightharpoons \neg \exists x \exists y \exists z\, (p(x,y) \,\&\, p(y,z) \,\&\, p(z,x))$.

$\varphi_3 \leftrightharpoons \forall x \exists y \exists z \exists t\, (p(x,y) \,\&\, p(y,z) \,\&\, p(z,t) \,\&\, p(t,x))$

$\varphi_4 \leftrightharpoons \forall x \forall y \exists z\, (p(x,z) \,\&\, p(z,y))$

$\Phi_1 \leftrightharpoons \{\varphi_1, \varphi_2\}$, $\Phi_2 \leftrightharpoons \Phi_1 \cup \{\varphi_3\}$, $\Phi_3 \leftrightharpoons \Phi_1 \cup \{\varphi_4\}$

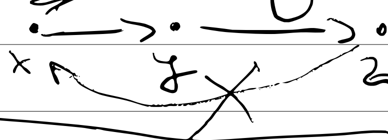
$\Phi_4 \leftrightharpoons \Phi_1 \cup \Phi_2 \cup \Phi_3$

Изпълними ли са $\Phi_1, \Phi_2, \Phi_3, \Phi_4$?

Докажете го.

За Φ_1: φ_1 – трихотомия

$\varphi_2 \models \forall x \forall y \forall z\, (p(x,y) \,\&\, p(y,z) \Rightarrow \neg p(z,x))$.

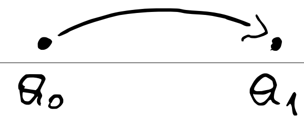


$y = z$

$p(x,y) \,\&\, p(y,x) \Rightarrow \neg p(x,x)$

$\mathcal{A}$: a

$\mathcal{A} = \langle \{a\}; p^{\mathcal{A}} \rangle$ $p^{\mathcal{A}} = \emptyset$

$\mathcal{A}_1$: 

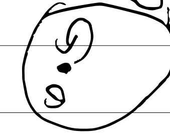
$\mathcal{A}_1 = \langle \{a_0, a_1\}; p^{\mathcal{A}} \rangle$

$p^{\mathcal{A}_1} = \{\langle a_0, a_1 \rangle\}$

$\mathcal{A}_2 = \langle \mathbb{N}; p^{\mathcal{A}} \rangle$, $p^{\mathcal{A}_2} \leftrightharpoons <_{\mathbb{N}}$

3a Φ_2: $\forall x \forall y \forall z \, (p(x,y) \,\&\, p(y,z) \Rightarrow \neg p(z,x))$

$$\left\{ \begin{array}{l} \varphi_1 \leftrightharpoons \forall x \forall y \, (p(x,y) \vee p(y,x) \vee x \doteq y). \\ \varphi_2 \leftrightharpoons \neg \exists x \exists y \exists z \, (p(x,y) \,\&\, p(y,z) \,\&\, p(z,x)). \\ \varphi_3 \leftrightharpoons \forall x \exists y \exists z \exists t \, (p(x,y) \,\&\, p(y,z) \,\&\, p(z,t) \,\&\, p(t,x)) \end{array} \right\}$$



$\mathcal{A}$: 

$\mathcal{A} = \langle \{x, y\}; p^{\mathcal{A}} \rangle : p^{\mathcal{A}} = \{\langle x,y \rangle, \langle y,x \rangle\}$

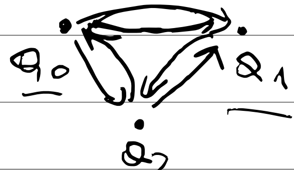
$p(x,y) \,\&\, p(y,y) \Rightarrow \neg p(y,x)$

$\varphi_1 \leftrightharpoons \forall x \forall y\, (p(x,y) \vee p(y,x) \vee x \doteq y)$.

$\varphi_2 \leftrightharpoons \neg \exists x \exists y \exists z\, (p(x,y) \,\&\, p(y,z) \,\&\, p(z,x))$.

$\varphi_4 \leftrightharpoons \forall x \forall y \exists z\, (p(x,z) \,\&\, p(z,y))$

Φ_3: $\forall x \forall y \forall z\, (p(x,y) \,\&\, p(y,z) \Rightarrow \neg p(z,x))$.



Доп., че $A \models \Phi_3$. Знаем, че $A \neq \emptyset$. Нека $a_0 \in A$ е свидетел. Тогава φ_4 за $x = y = a_0$ гласи $\exists z (p^A(a_0, z) \,\&\, p^A(z, a_0))$. Нека $z_0 \neq a_0$ изб.

От φ_2 нек. за $x = y = z$ имаме:

$\forall x\, (p^A(x,x) \Rightarrow \neg p^A(x,x))$ } p^A е антирефл.

$x = z$ $\forall x \forall y\, (p^A(x,y) \,\&\, p^A(y,x) \Rightarrow \neg p^A(x,x))$

Невъзможно →

Доп., че $A \models \Phi_3$.

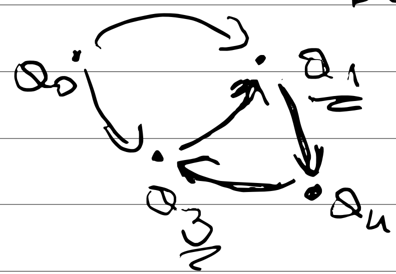
От φ_4, φ_2 $\rightsquigarrow$ $|A| \geq 2$

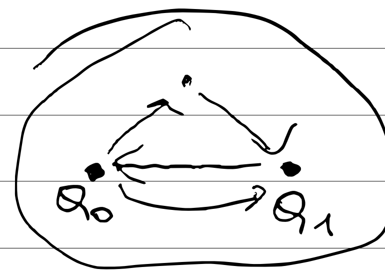
Нека $a_0, a_1 \in A$ са свидетели.

По φ_1: $p^A(a_0, a_1)$ или $p^A(a_1, a_0)$

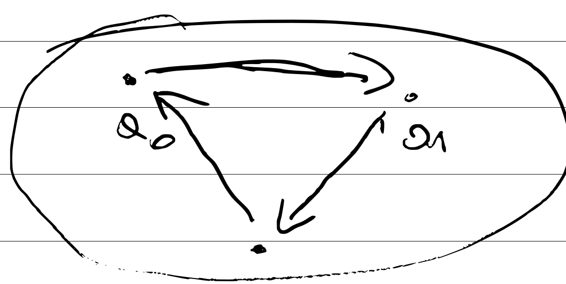
БОО

По φ_4:





По φ_4: и равни цикли



$\varphi_1, \varphi_2, \varphi_4$

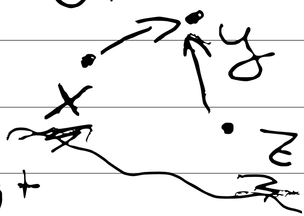
В закл. Φ_3 е неизпълнимо.

Т.к. $\Phi_3 \subsetneq \Phi_4$ и Φ_4 е неизпълнимо.

(303) Да се док., че следния набор от затворени ф-ли е изпълним:

$$\begin{cases} \varphi_1 \leftrightharpoons \forall x \forall y \forall z (q(x,y) \,\&\, q(y,z) \Rightarrow q(x,z)). & \text{// транз.} \\ \varphi_2 \leftrightharpoons \exists x \exists y\, q(x,y). & \text{// } q^t \neq \emptyset \\ \varphi_3 \leftrightharpoons \forall x \neg q(x,x). & \text{// антирефл.} \\ \varphi_4 \leftrightharpoons \forall x \forall y (q(x,y) \Rightarrow \exists z (q(x,z) \,\&\, q(z,y))). & \text{// гъсто} \end{cases}$$

$\varphi_5 \leftrightharpoons \forall x \forall y (q(x,y) \Rightarrow \exists z (q(z,y) \,\&\, \neg(z \doteq x) \,\&\, \neg q(x,z) \,\&\, \neg q(z,x))).$

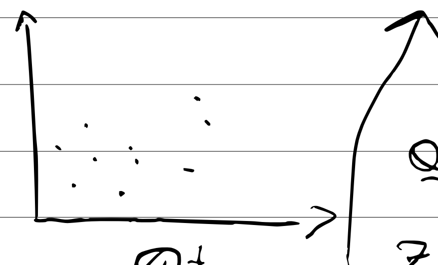


$\mathbb{Q}^+$

$\overbrace{\;}^{x}\;\overbrace{\;}^{y}$

$q^t(\langle a,b\rangle, \langle c,d\rangle) \Longleftrightarrow$ $a < c$ и $b < d$

$\mathbb{Q}^+$



$a < c$ и $b < d$ $\quad z = \langle e,f\rangle$

$\langle a,b\rangle$ $\quad z = \langle \underbrace{\frac{a+c}{2}}_{e}, \underbrace{b}_{f}\rangle \quad e < c,\ f < d$

$\mathbb{Q}^+$

$A = \langle \mathbb{Q}^+ \times \mathbb{Q}^+, q^t\rangle$

$$\mathcal{A} = \langle \{0\}\times\mathbb{Q} \cup \{1\}\times\mathbb{Q},\ q^{\mathcal{A}} \rangle$$

$$q^{\mathcal{A}}(\langle a, b\rangle, \langle c, d\rangle) \iff b < d$$

$\mathcal{B}$: факторизация на $\mathbb{Q}$ спрямо без позиция гъстотата.

(323) Да се док., че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \leftrightharpoons \forall x(\neg p(f(x),x) \& \exists y\, p(f(x),y))$

$\varphi_2 \leftrightharpoons \forall x \forall y(p(x,y) \Rightarrow \exists z(p(x,z) \& \neg p(f(z),f(z)) \& p(z,y)))$

$\varphi_3 \leftrightharpoons \neg\forall x \forall y \forall z(p(x,y) \& p(y,z) \Rightarrow \neg p(f(x),z))$

$\exists x \exists y \exists z(p(x,y) \& p(y,z) \& p(f(x),z))$

$1+2<3$

$<(+(1,2),3)$

$\varphi(x,y) \leftrightharpoons \_\_\_$

$\varphi(y) \leftrightharpoons \forall x\, p(x,y)$ $\quad \exists z(\ldots \& \varphi(z))$

$\varphi(x) \leftrightharpoons \forall x_1\, p(x_1,y)$ $\quad z = f(x)$

$\cancel{\varphi(f(x)) \leftrightharpoons \forall x\, p(x,f(x))}$

$\varphi(f(x)) \leftrightharpoons \forall x_1\, p(x_1,f(x))$

$\varphi(f(x,z)) \leftrightharpoons \forall x_1\, p(x_1,f(x,z))$

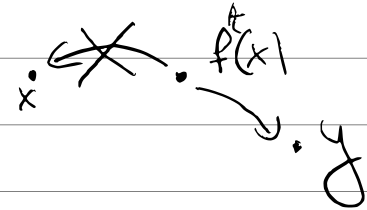
$$\varphi_1 \leftrightharpoons \forall x\,(\neg p(f(x),x)\ \&\ \exists y\, p(f(x),y))$$

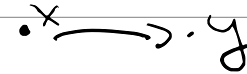
$$\varphi_2 \leftrightharpoons \forall x \forall y\,(p(x,y) \Rightarrow \exists z\,(p(x,z)\ \&\ \neg p(f(z),f(z))\ \&\ p(z,y)))$$

$$\varphi_3 \leftrightharpoons \neg \forall x \forall y \forall z\,(p(x,y)\ \&\ p(y,z) \Rightarrow \neg p(f(x),z))$$

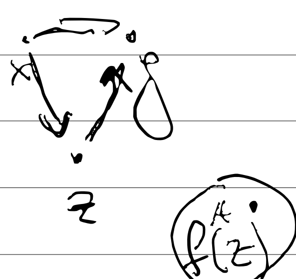
$$\hookrightarrow \exists x \exists y \exists z\,(p(x,y)\ \&\ p(y,z)\ \&\ p(f(x),z))$$

φ_1:

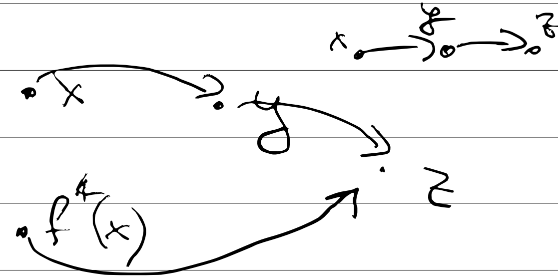




φ_2:

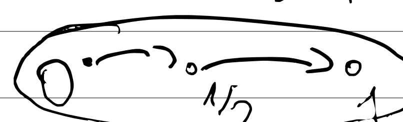


φ_3:



$A = \langle \emptyset ; p^A , f^A \rangle$ $f^A(x) \leftrightharpoons x$

$p^A \leftrightharpoons \langle^{\emptyset} \setminus \{\langle 0,1\rangle\}$



$\varphi_1 \Leftrightarrow \forall x\,(\neg p(f(x),x)\ \&\ \exists y\, p(f(x),y))$

$\varphi_2 \Leftrightarrow \forall x \forall y\,(p(x,y) \Rightarrow \exists z\,(p(x,z)\ \&\ \neg p(f(z),f(z))\ \&\ p(z,y)))$

$\varphi_3 \Leftrightarrow \neg \forall x \forall y \forall z\,(p(x,y)\ \&\ p(y,z) \Rightarrow \neg p(f(x),z))$

$\rightarrow \exists x \exists y \exists z\,(p(x,y)\ \&\ p(y,z)\ \&\ p(f(x),z))$

$\mathcal{A} = \langle \mathbb{Q}\,;\ p^{\mathcal{A}}\,;\ f^{\mathcal{A}} \rangle \qquad f^{\mathcal{A}}(x) \Leftrightarrow x+1$

$p^{\mathcal{A}} \Leftrightarrow <_{\mathbb{Q}}$

φ_3: $0 \quad 1/4 \quad 1/2 \quad 1$

$f(0)$

(302) Да се док., че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \leftrightharpoons \forall x \forall y (p(x,y) \Leftrightarrow q(y,x))$. // $p^A = (q^A)^{-1} \rightsquigarrow (q^A)^{-1}$ е транзит.

$\varphi_2 \leftrightharpoons \forall x \exists y (p(x,y) \Leftrightarrow q(x,y))$. // $x \bullet \xrightarrow[q^A\,?]{p^A} \bullet\, y$

$\varphi_3 \leftrightharpoons \forall x \forall y \forall z (p(x,y) \;\&\; p(y,z) \Rightarrow p(x,z))$.

$\varphi_4 \leftrightharpoons \exists x \exists y (\neg q(x,y) \;\&\; \neg q(y,x))$

$\varphi_5 \leftrightharpoons \exists x \exists y \neg (x \doteq y)$

→ има поне два индивида

→ има несравними по q^A

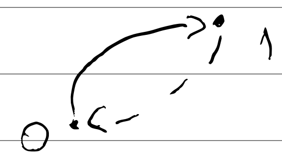
→ транзитивност по p^A

φ_2 е следствие от φ_1.

Удовл. $\varphi_1, \varphi_3, \varphi_4, \varphi_5$

p^A ———

q^A - - - -



A: $\bullet 1$ $\quad 0 \bullet$

$A = \langle \{0,1\}; p^A, q^A \rangle$

$p^A = q^A = \emptyset$

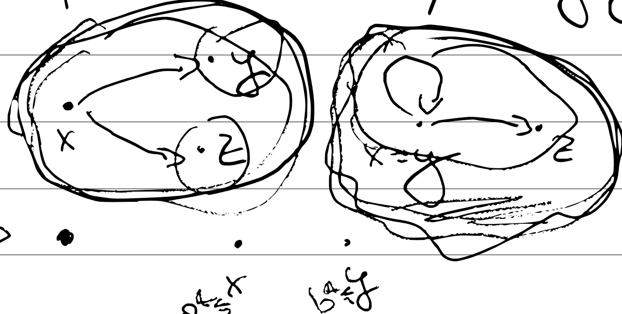
(3.2) Да се док., че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \rightleftharpoons \forall x \exists y \exists z \, (p(x,y) \,\&\, p(x,z) \,\&\, \neg(y \doteq z))$.

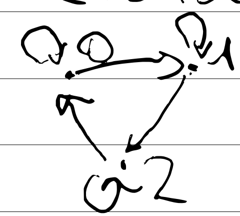
$\varphi_2 \rightleftharpoons \neg p(a,b)$

$\varphi_3 \rightleftharpoons \exists x \exists y \exists z \forall w \, (\neg(w \doteq x) \,\&\, \neg(y \doteq w) \Rightarrow w \doteq z)$

a, b - константи, p - двуместен предик. символ.

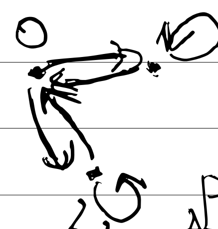


Има поне 2 точки



Имаме най-много 3 елемента в света

$\mathcal{A}$:



$\mathcal{A} = \langle \{0,1,2\}, p^{\mathcal{A}} \rangle$

$p^{\mathcal{A}} \rightleftharpoons \{ \dots \}$

$a^{\mathcal{A}} \rightleftharpoons 0 \quad a^{\mathcal{A}} \rightleftharpoons 1 \quad a^{\mathcal{A}} \rightleftharpoons 2$

$b^{\mathcal{A}} \rightleftharpoons 0 \quad b^{\mathcal{A}} \rightleftharpoons 2 \quad b^{\mathcal{A}} \rightleftharpoons 1$

(30?) Да се док., че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \rightleftharpoons \forall x(\neg(x \doteq f(x)) \,\&\, \neg(x \doteq g(x)))$. // f^A и g^A са ирефлексивни

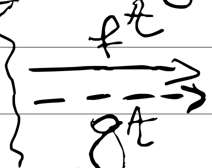
$\varphi_2 \rightleftharpoons \exists x(x \doteq f(g(x)) \,\&\, x \doteq g(f(x)))$. //

$\varphi_3 \rightleftharpoons \exists x(f(x) \doteq g(x))$. // има т. x

$\varphi_4 \rightleftharpoons \exists x \neg(f(g(x)) \doteq g(f(x)))$. // има точка, т.ч. f^A и g^A не са обратни една друга в нея.

$\varphi_5 \rightleftharpoons \forall y \exists x(y \doteq g(x))$. // g^A е сюрекция

$\forall x(\varphi \,\&\, \psi) \models \forall x \varphi \,\&\, \forall x \psi$



$\mathcal{A} = \langle \mathbb{N}, f^A, g^A \rangle \qquad g^A(x) \rightleftharpoons \begin{cases} x-1, & x>0 \\ 1, & x=0 \end{cases}$

$f^A(x) \rightleftharpoons x+1$

$\varphi_2 : g^A(\overbrace{f^A(1)}^{2}) = 1 = f^A(\underbrace{g^A(1)}_{0})$

$\varphi_4 : f^A(\underbrace{g^A(0)}_{1}) = 2 \;\cancel{=}\; g^A(\underbrace{f^A(0)}_{1}) = 0$

$\varphi_3 : g^A(0) = f^A(0) = 1$

$\varphi_1 \leftrightharpoons \forall x(\neg(x \doteq f(x)) \,\&\, \neg(x \doteq g(x)))$.

$\varphi_2 \leftrightharpoons \exists x(x \doteq f(g(x)) \,\&\, x \doteq g(f(x)))$.

$\varphi_3 \leftrightharpoons \exists x(f(x) \doteq g(x))$. // има т.

$\varphi_4 \leftrightharpoons \exists x \neg(f(g(x)) \doteq g(f(x)))$.

$\varphi_5 \leftrightharpoons \forall y \exists x(y \doteq g(x))$. //

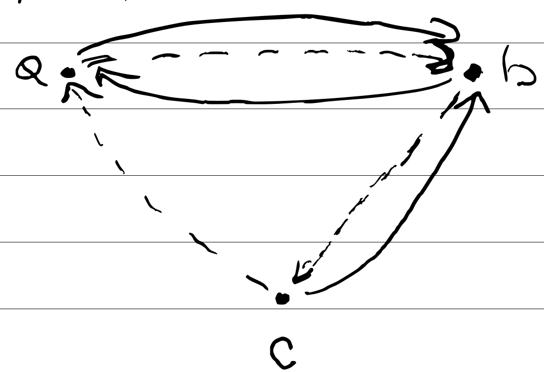
$\mathcal{A} = \langle \{a, b, c\}, f^{\mathcal{A}}, g^{\mathcal{A}} \rangle$

$f^{\mathcal{A}} \longrightarrow$

$g^{\mathcal{A}} \dashrightarrow$

$Dom(f^{\mathcal{A}}) = Dom(g^{\mathcal{A}}) = A$

$\varphi_1, \varphi_2, \varphi_3, \varphi_5$ са верни в този стр., но не и φ_4



$f^{\mathcal{A}}(a) \leftrightharpoons b$ $\quad g^{\mathcal{A}}(a) \leftrightharpoons b$

$f^{\mathcal{A}}(b) \leftrightharpoons a$ $\quad g^{\mathcal{A}}(b) \leftrightharpoons c$

$f^{\mathcal{A}}(c) \leftrightharpoons b$ $\quad g^{\mathcal{A}}(c) \leftrightharpoons a$

φ_1

φ_2

φ_3

φ_5

φ_4: $\underbrace{f^{\mathcal{A}}(\underbrace{g^{\mathcal{A}}(c)}_{a})}_{b} \neq g^{\mathcal{A}}(\underbrace{f^{\mathcal{A}}(c)}_{b})$

зад) Нека имаме ф-лите:

$\varphi_1 \leftrightharpoons \forall x \exists y (\neg p(x,x) \,\&\, p(x,y))$

$\varphi_2 \leftrightharpoons \forall x \forall z (\exists y (p(x,y) \,\&\, p(y,z)) \Leftrightarrow p(x,z))$

$\varphi_3 \leftrightharpoons \exists x \exists y \exists z (\neg p(x,y) \,\&\, \neg p(y,x) \,\&\, \neg (p(x,z) \Leftrightarrow p(y,z)))$.

$\varphi_4 \leftrightharpoons \forall x \forall y (p(x,y) \Leftrightarrow p(x, f(x,y)) \,\&\, p(f(y,x), y))$.

$\varphi_5 \leftrightharpoons \forall x \forall y (p(x,y) \Leftrightarrow p(x, f(x,y)) \vee p(f(y,x), y))$.

$\Phi_1 \leftrightharpoons \{\varphi_1, \varphi_2, \varphi_3\}$

$\Phi_2 \leftrightharpoons \Phi_1 \cup \{\varphi_4\}$

$\Phi_3 \leftrightharpoons \Phi_1 \cup \{\varphi_5\}$

$\to$ изпълними ли Φ_i?

Зад Да се док., че следния набор от затворени ф-ли е изпълним:

$\varphi_1 \leftrightharpoons \forall x \forall y (\exists z (p(x,z) \Rightarrow p(z,y)) \Leftrightarrow \exists z (q(x,z) \Rightarrow q(z,y)))$.

$\varphi_2 \leftrightharpoons \exists x \exists y\, p(x,y)\ \&\ \exists x \exists y\, q(x,y)$.

$\varphi_3 \leftrightharpoons \forall x \forall y (q(x,y) \Rightarrow \neg q(y,x))\ \&\ \forall x \forall y (p(x,y) \Rightarrow \neg p(y,x))$.

$\varphi_4 \leftrightharpoons \neg \forall x \forall y \forall z (p(x,y)\ \&\ p(y,z) \Rightarrow p(x,z))$.

(3зад) Нека p е едноместен П.С., а r е двуместен П.С. Дадени са следните формули:

$\varphi_1 \leftrightharpoons \forall x \forall y \forall z (r(x,y) \& r(y,z) \Rightarrow r(x,z))$.

$\varphi_2 \leftrightharpoons \forall x (p(x) \Rightarrow \exists y (r(x,y) \& \forall z (r(x,z) \Rightarrow \neg r(z,y))))$.

$\varphi_3 \leftrightharpoons \forall x (\neg p(x) \Rightarrow \exists y (r(y,x) \& \neg \exists z (r(y,z) \& r(z,x))))$.

$\varphi_4 \leftrightharpoons \forall x \forall y (p(x) \& \neg p(y) \Rightarrow r(x,y))$.

$\varphi_5 \leftrightharpoons \exists x \, p(x) \Leftrightarrow \exists x \neg p(x)$.

$\Gamma_0 \leftrightharpoons \{\varphi_1, \varphi_2, \varphi_3\}$, $\Gamma_1 \leftrightharpoons \Gamma_0 \cup \{\varphi_4\}$, $\Gamma_2 \leftrightharpoons \Gamma_1 \cup \{\varphi_5\}$,

$\Gamma \leftrightharpoons \Gamma_1 \cup \{\neg \varphi_5\}$.

Да се докаже кои от м-вата $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ са изпълними и кои не са изпълними.