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ABSTRACT
In this paper, we used the semi-implicit extrapolation method to obtain numerical solution of systems of stiff ordinary differential equations. The method is based on linearizing the implicit Euler method and implicit midpoint rule. Some examples of system of initial value stiff ordinary differential equations were solved. Comparisons were made among the semi implicit extrapolation method, the exact solutions and Runge- Kutta method of order four to illustrate that the method works well.

Keywords*: Semi- implicit extrapolation method, Runge- Kutta method and System of stiff ordinary differential equations (ODEs).*

1. INTRODUCTION

A special problem arising in the numerical solutions of ordinary equations is **stiffness**. Stiffness occurs in differential equations where there are two or more different time scales of the independent variables on which the dependent variables are changing. Stiff differential equations are characterized as those whose exact solutions has a term of the form $e^{-\lambda t}$ where λ is a positive constant. This is usually a part of the solution called the **transcient solution**. The more important portion of the solution is called the **steady- state solution**. The transcient portion decay rapidly to zero as t increases.

Stiff systems derive their names from the motion of spring and mass systems that have large constants. The problem of stiffness occurs particularly in the study of vibrations, chemical reactions, control theory and electric circuit (Douglas and Burden, 1998). Although stiffness is usually associated with systems of differential equations. It also occurs in single ODEs (Binuyo et al, 2010). Systems of stiff ODEs can be easily solved analytically when the equations are linear. The computational labour of solving systems of non linear equations may be formidable even when the exact solutions exist. It is for this reason that numerical solutions are becoming much more preferable and popular.
The difficulty that arises in attempting to obtain numerical solutions to systems of stiff equations is that of numerical instability (Lambert, 2000). Conventional methods like explicit Euler method, explicit Runge- Kutta method, Taylor series and so on are usually restricted to a very small step size in order that the solution be stable. This means that a great deal of computational time and effort shall be required.

Hojjati et al., 2004, developed a multistep method for solving systems of stiff ordinary differential equations. Ahmad et al.,2004, presented an explicit Taylor- like method for stiff ODEs. Nie et al., 2006, presented a class of efficient semi- implicit schemes for stiff reaction-diffusion equations. A variable- step size algorithm for systems has been proposed by (Jannelli and Fazio, 2006) . Guzel and Bayram, 2005, presented a power series method for stiff systems. Al-Riyami and Ishak, 2006, applied a semi-numeric-analytic method based on Adomian decomposition method to stiff ordinary differential equations. Binuyo et al., 2010 used the semi-implicit extrapolation method (SIEM) to solve single stiff ordinary differential equations.

In this paper, we shall derive and apply the semi-implicit extrapolation method to systems of stiff differentials. Comparisons will be made with the exact solutions and Runge-Kutta method of order four.

2. METHOD OF SOLUTION

Let the system of stiff differential equation be given in the compact form

$$Y' = F(t, Y), \qquad Y(t_0) = Y_0 \tag{1}$$

We consider the interval [0, T] in (1) which is sub-divided into m individual sub-interval as;

$$0 = t_0 < t_1 < t_2 < t_3 < \ldots < t_{m-1} < t_m = T \tag{2}$$

Integrating (1) in the interval $[t_n, t_{n+1}]$ gives

$$Y_{n+1} = Y_n + hF(t_n, Y_n) \tag{3}$$

$$Y_{n+1} = Y_n + hY'_n \tag{4}$$

The above is the explicit Euler scheme. Resorting to implicit differencing; where the RHS of (3) is evaluated at the new Y location. This yield

$$Y_{n+1} = Y_n + hF(t_{n+1}, Y_{n+1}) \tag{5}$$

$$Y_{n+1} = Y_n + hF(Y_{n+1}) \tag{6}$$

$$Y_{n+1} = Y_n + hY'_{n+1} \tag{7}$$

Equation (7) is the implicit Euler scheme. Linearizing (7) by F(Yn), we have

$$Y_{n+1} = Y_n + h[F(Y_n) + \frac{\partial F}{\partial Y}(Y_{n+1} - Y_n)] \tag{8}$$

$$Y_{n+1} = Y_n + hF(Y_n) + h\frac{\partial F}{\partial Y}Y_{n+1} - h\frac{\partial F}{\partial Y}Y_n \tag{9}$$

$$Y_{n+1} - h\frac{\partial F}{\partial Y}Y_{n+1} = Y_n - h\frac{\partial F}{\partial Y}Y_n + hF(Y_n) \tag{10}$$

$$(I - h\frac{\partial F}{\partial Y})Y_{n+1} = (I - h\frac{\partial F}{\partial Y})Y_n + hF(Y_n) \tag{11}$$

$$Y_{n+1} = Y_n + h(I - h\frac{\partial F}{\partial Y})^{-1}F(Y_n) \tag{12}$$

Where

I is an identity matrix and $\frac{\partial F}{\partial Y}$ is Jacobian matrix. Equation (12) is the semi-implicit Euler method.

The implicit form of the midpoint rule is

$$Y_{n+1} - Y_{n-1} = 2hF\left(\frac{Y_{n+1} + Y_{n-1}}{2}\right) \tag{13}$$

To convert (13) into semi-implicit form; we linearize the RHS of (13) by $F(Y_n)$

$$F\left(\frac{Y_{n+1} + Y_{n-1}}{2}\right) = F(Y_n) + \frac{\partial F}{\partial Y}|Y_n\left(\frac{Y_{n+1} - Y_{n-1}}{2} - Y_n\right) \tag{14}$$

Hence (13) becomes

$$Y_{n+1} - Y_{n-1} = 2h\left[F(Y_n) + \frac{\partial F}{\partial Y}|Y_n\left(\frac{Y_{n+1} + Y_{n-1}}{2} - Y_n\right)\right] \tag{15}$$

$$Y_{n+1} - Y_{n-1} = 2h\,F(Y_n) + h\frac{\partial F}{\partial Y}Y_{n+1} + h\frac{\partial F}{\partial Y}Y_{n-1} - 2h\frac{\partial F}{\partial Y}Y_n \tag{16}$$

$$Y_{n+1} - h\frac{\partial F}{\partial Y}Y_{n+1} = Y_{n-1} + h\frac{\partial F}{\partial Y}Y_{n-1} + 2h\,F(Y_n) - 2h\frac{\partial F}{\partial Y}Y_n \tag{17}$$

$$(I - h\frac{\partial F}{\partial Y})Y_{n+1} = (I + h\frac{\partial F}{\partial Y})Y_{n-1} + 2h[F(Y_n) - \frac{\partial F}{\partial Y}Y_n] \tag{18}$$

For practical implementation, it is better to re-write the equations of the semi-implicit extrapolation using:

$$\Delta_n = Y_{n+1} - Y_n \tag{19}$$

where $h = \frac{H}{m}$

H is the step size and m is the number of sub interval of extrapolation.

$$Y_{n+1} = \Delta_n + Y_n \tag{20}$$

such that for n = 0,

$$\Delta_0 = Y_1 - Y_0 \text{ or } Y_1 = Y_0 + \Delta_0 \tag{21}$$

From (12)

$$Y_{n+1} - Y_n = h\left(I - h\frac{\partial F}{\partial Y}\right)^{-1} F(Y_n) \tag{22}$$

Using equations (19) and (21), we have

$$\Delta_n = h\left(I - h\frac{\partial F}{\partial Y}\right)^{-1} F(Y_n) \tag{23}$$

$$\Delta_0 = h\left(I - h\frac{\partial F}{\partial Y}\right)^{-1} F(Y_0) \tag{24}$$

Likewise for n = 1, . . . , m-1

$$\Delta_{n-1} = Y_n - Y_{n-1} \tag{25}$$

$$Y_{n-1} = Y_n - \Delta_{n-1} \tag{26}$$

$$Y_n = Y_{n-1} + \Delta_{n-1} \tag{27}$$

Equation (18) becomes

$$\left(I - h\frac{\partial F}{\partial Y}\right)(\Delta_n + Y_n) = \left(I + h\frac{\partial F}{\partial Y}\right)(Y_n - \Delta_{n-1}) + 2h\left[F(Y_n) - \frac{\partial F}{\partial Y}(Y_{n-1} + \Delta_{n-1})\right] \tag{28}$$

$$\Delta_n = \Delta_{n-1} + 2\left(I - h\frac{\partial F}{\partial Y}\right)^{-1}\left(hF(Y_n) - \Delta_{n-1}\right) \tag{29}$$

Equation (29) is the semi- implicit midpoint rule.

Finally for n = m, to smoothen the last step of the computation, we compute using

$$\Delta_m = h\left(I - h\frac{\partial F}{\partial Y}\right)^{-1}\left(hF(Y_m) - \Delta_{m-1}\right) \tag{30}$$

such that

$$\overline{Y_m} = Y_m + \Delta_m \tag{31}$$

$\overline{Y_m}$ gives the approximate solution to equation (1). The semi- implicit form of the mid point rule together with a special first step of the semi- implicit Euler method and a smoothing last step forms the basic algorithm of the SIEM.

Equations (24) and (21) are used to compute the approximate solution for (1) at the initial points of the sub-interval while equation (29) and (20) are used to compute the extrapolated values at the points between the sub-intervals i.e. for n= 1, 2, . . ., m-1. Equations (30) and (31) are used to give the approximate solutions to (1) at the end points of the sub-interval i.e. for n = m. The sub-interval is divided into m-1 equal intervals where point m is the end point of the sub-intervals.

3. NUMERICAL EXAMPLES

In this section, we shall demonstrate how well the SIEM compares with the exact solutions and the popular Runge- Kutta method of order four. Numerical results were presented in tabular form. The calculations were done with the aid of Maple 13 software. **N** is the number of partitions of the interval.

3.1 Problem 1

Consider the system of equations

$X'(t) = 998\, X(t) + 1998\, Y(t),$ $\quad X(0) = 1,$
$Y'(t) = -999\, X(t) - 1999\, Y(t),$ $\quad Y(0) = 0.$

The exact solutions are

$X(t) = 2e^{-t} - e^{-1000t},$
$Y(t) = e^{-1000t} - e^{-t}.$

The term e^{-1000t} in the exact solution makes the equation a stiff one. The results are displayed in Tables 3.1. 1 and 3.1. 2.

Table 3.1. 1: Exact solution (X(t)), Semi-implicit Extrapolation (X(t)), Runge- Kutta method of order four (X(t)) at various values of step length.

T	Exact X(t)	SIEM X(t) H=0.1,N=10,h=0.01	RK4 X(t),N=1000 h= 0.001	RK4 X(t) h= 0.01, N= 100
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.2	1.637461506	1.637764366	1.637461506	$-1.896093654\ 10^{49}$
0.4	1.340640092	1.341158616	1.340640092	$-3.595171122\ 10^{98}$
0.6	1.097623272	1.098260009	1.097623272	$-6.816781148\ 10^{147}$
0.8	0.8993530503	0.8993530503	0.8986579282	$-1.292525540\ 10^{197}$
1.0	0.7364703898	0.7364703898	0.7357588823	$-2.450749432\ 10^{246}$

Table 3.1. 2: Exact solution (Y(t)), Semi-implicit Extrapolation (Y(t)), Runge- Kutta method of order four (Y(t)) at various values of step length.

T	Exact Y(t)	SIEM Y(t) H=0.1,N=10,h=0.01	RK4 Y(t),N=1000 h= 0.001	RK4 Y(t) h= 0.01, N= 100
0.0	0.000000000	0.00000000000	0.000000000	0.000000000
0.2	-0.8187307531	-0.8188753252	-0.8187307505	$1.896093654\ 10^{49}$
0.4	-0.6703200460	-0.6705793086	-0.6703200443	$3.595171122\ 10^{98}$
0.6	-0.5488116361	-0.5491300054	-0.5488116489	$6.816781148\ 10^{147}$
0.8	-0.4493289641	-0.4496765252	-0.4493289528	1.292525540 10197
1.0	-0.3678794412	-0.3682351949	-0.3678794296	$2.450749432\ 10^{246}$

From Tables (3.1. 1) and (3.1. 2) we can see that the SIEM gives accurate result at step size H= 0.1 but RK4 gives unstable even with a smaller step size h=0.01. Accurate result with RK4 was obtained at a much smaller step size. This takes more computational time and effort.

3. 2 Problem 2

Consider the system of equations

$$U'(t) = 1195\,U(t) - 1995\,V(t), \qquad U(0) = 2,$$
$$V'(t) = 1197\,U(t) - 1997\,V(t), \qquad V(0) = -2.$$

The exact solutions are

$$U(t) = 10\,e^{-2t} - 8e^{-800t},$$
$$V(t) = 6\,e^{-2t} - 8e^{-800t}.$$

The term e^{-800t} in the exact solution is responsible for the stiffness. The results are shown in tabular form below.

Table 3.2. 1: Exact solution (U(t)), Semi- implicit Extrapolation (U(t)), Runge- Kutta method of order four (U(t)) at various values of step length.

T	Exact U(t)	SIEM U(t) H=0.1,N=10,h=0.01	RK4 U(t),N=1000 h= 0.001	RK4 U(t) h= 0.01, N= 100
0.0	2.000000000	2.000000000	2.000000000	2.000000000
0.2	6.703200460	6.708045348	6.703200385	$-5.717744743\ 10^{41}$
0.4	4.493289641	4.500006147	4.493289690	$-4.086575664\ 10^{82}$
0.6	3.011942119	3.018697802	3.011942122	$-2.920749561\ 10^{123}$
0.8	2.018965180	2.025005320	2.018965157	$-2.087512579\ 10^{164}$
1.0	1.353352832	1.358415708	1.353352842	$-1.491983020\ 10^{205}$

Table 3.2. 2: Exact solution (V(t)), Semi- implicit Extrapolation (V(t)), Runge- Kutta method of order four (V(t)) at various values of step length.

T	Exact V(t)	SIEM V(t) H=0.1,N=10,h=0.01	RK4V(t), N=1000 h= 0.001	RK4 V(t), h= 0.01, N= 100
0.0	-2.000000000	-2.000000000	-2.000000000	-2.000000000
0.2	4.021920276	4.024761892	4.021920230	$-5.717744743\ 10^{41}$
0.4	2.695973785	2.700003687	2.695973814	$-4.086575664\ 10^{82}$
0.6	1.807165271	1.811218682	1.807165274	$-2.920749561\ 10^{123}$
0.8	1.211379108	1.215003191	1.211379096	$-2.087512579\ 10^{164}$
1.0	0.8120116992	0.8150494242	0.8120117042	$-1.491983020\ 10^{205}$

From Tables (3.2. 1) and (3.2. 2) we can see that the SIEM gives accurate result at step size H= 0.1 but RK4 gives unstable even with a smaller step size h=0.01. Accurate result with RK4 was obtained at a much smaller step size h = 0.001which obviously takes more computational time and effort.

3. 3 Problem 3

Consider the system of equations

$$U'(t) = -2000U(t) + 999.75V(t) + 1000.25, \qquad U(0) = 0,$$
$$V'(t) = U(t) - V(t), \qquad V(0) = -2.$$

The exact solutions are

$$U(t) = \frac{3999}{8000} e^{-\frac{4001}{2}t} - \frac{11999}{8000} e^{-\frac{1}{2}t} + 1,$$
$$V(t) = \frac{-1}{4000} e^{-\frac{4001}{2}t}.$$

The term $e^{-\frac{4001}{2}t}$ in the exact solution is responsible for the stiffness. The numerical results are shown in the tables below.

Table 3.3. 1: Exact solution (U(t)), Semi- implicit Extrapolation (U(t)), Runge- Kutta method of order four (U(t)) at various values of step length.

T	Exact U(t)	SIEM U(t) H=0.1,N=10,h=0.01	RK4 U(t),N=1000 h= 0.001	RK4 U(t) h= 0.01, N= 100
0.1	-0.426725233	-0.4260010365	-0.426725233	$1.31319347\ 10^{37}$
0.2	-0.357143022	-0.3572085970	-0.357143022	$3.449816672\ 10^{74}$
0.3	-0.290954376	-0.2910495871	-0.290954376	$9.062819240\ 10^{111}$
0.4	-0.227993788	-0.2281145488	-0.227993788	$2.380842223\ 10^{149}$
0.5	-0.168103825	-0.1682474152	-0.168103825	$6.254576574\ 10^{186}$
0.6	-0.111134729	-0.1112986367	-0.111134729	$1.643104614\ 10^{224}$
0.7	-0.056944049	-0.05712594952	-0.-056944049	$4.316507654\ 10^{261}$
0.8	-0.005396279	-0.005594029526	-0.005396279	$1.133965432\ 10^{299}$
0.9	0.0436374761	0.04342585443	0.0436374761	$2.978976761\ 10^{336}$
1.0	0.0902798268	0.09005615644	0.0902798268	$7.825902168\ 10^{373}$

Table 3.3. 2: Exact solution (V(t)), Semi- implicit Extrapolation (V(t)), Runge- Kutta method of order four (V(t)) at various values of step length.

T	Exact V(t)	SIEM V(t) H=0.1,N=10,h=0.01	RK4V(t), N=1000 h= 0.001	RK4 V(t), h= 0.01, N= 100
0.1	-1.853450466	-1.853520994	-1.853450481	$-6.567609297\ 10^{33}$
0.2	-1.714286045	-1.714419501	-1.714286043	$-1.725339669\ 10^{71}$
0.3	-1.581908752	-1.582099178	-1.581908755	$-4.532542756\ 10^{108}$
0.4	-1.455987577	-1.456229098	-1.455987569	$-1.190718793\ 10^{146}$
0.5	-1.336207649	-1.336494830	-1.336207644	$-3.128070303\ 10^{183}$
0.6	-1.222269458	-1.222597273	-1.222269454	$-8.217577460\ 10^{220}$
0.7	-1.1138888097	-1.114251899	-1.113888086	$-2.158793525\ 10^{258}$
0.8	-1.010792558	-1.011188059	-1.010792546	$-5.671244971\ 10^{295}$
0.9	-0.912725048	-0.9131482911	-0.9127250337	$-1.489860846\ 10^{333}$
1.0	-0.819440346	-0.8198876871	-0.8194403329	$-3.913929566\ 10^{370}$

In Tables (3.3. 1) and (3. 3. 2), we can see that the SIEM gives accurate result at step size H= 0.1 but RK4 gives unstable result even with a smaller step size h= 0.01. Accurate result with RK4 was obtained at a much smaller step size h=0.001.

3. 4 Problem 4

We consider the popular Van der Pol equation which is a second order non linear differential equation. The equation is written as a system of two first order differential equations. The equation is written as a system of two first order differential equations. Van der Pol equation is stiff when the value of μ is up to 1000.

$$U'(t) = V(t), \qquad U(0) = 2,$$
$$V'(t) = \mu(1 - U(t)^2)\, V(t) - U(t), \qquad V(0) = 0.$$

The numerical results are shown in tabular form below.

Table 3.4. 1: Semi- implicit Extrapolation (U(t)), Runge- Kutta method of order four (U(t)) at various values of step length.

T	SIEM U(t) H=0.001,N=100,h=0.0001	RK4 U(t) h= 0.001, N= 1000
0.01	1.999975498	1.999993604
0.02	1.999949704	1.999987004
0.03	1.999923909	1.999980404
0.04	1.999898119	1.999973804
0.05	1.999872333	1.999967204
0.06	1.999846543	1.999960604
0.07	1.999820762	1.999954004
0.08	1.999794992	1.999947404
0.09	1.999769212	1.999940804
0.10	1.999743439	1.999934204

Table 3.4. 2: Semi- implicit Extrapolation (V(t)), Runge- Kutta method of order four (V(t)) at various values of step length.

T	SIEM V(t) H=0.001,N=100, h=0.0001	RK4 V(t) h= 0.001, N= 1000
0.01	-0.0006634837242	-0.0006666700977
0.02	-0.0006635357105	-0.0006666737645
0.03	-0.0006635502325	-0.0006666774313
0.04	-0.0006635647496	-0.0006666810980
0.05	-0.0006635792689	-0.0006666847649
0.06	-0.0006635937847	-0.0006666884317
0.07	-0.0006636083023	-0.0006666920988
0.08	-0.0006636228107	-0.0006666957657
0.09	-0.0006636373266	-0.0006666994328
0.1	-0.0006636518388	-0.0006667031000

4. CONCLUSION

In this paper, we presented the numerical solution of systems of initial-value stiff ordinary differential equations. We have used the method of semi-implicit extrapolation and Runge- Kutta method of order four. The SIEM compares reasonably well with the exact solutions were available and Runge- Kutta method of order four. The method prevent the approximate solutions from diverging from the exact solutions. The SIEM works well and does not require an extremely small value of the step length which obviously increases the computational time and effort. We also give the various step sizes and results at which the Runge- Kutta method of order four fails to yield reliable solutions to illustrate the concept of stiffness.

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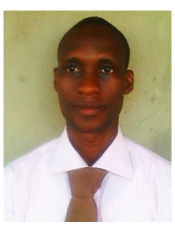
Author's Biography



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