Radar Signal Processing Mastery

Theory and Hands-On Applications with mmWave MIMO Radar Sensors

Date: 7-11 October 2024
Time: 9:00AM-11:00AM ET (New York Time)

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Website: https://radarmimo.com/

Outline

Time: 9:00AM-11:00AM ET (New York Time)

Lecture	Duration	Date
Lecture 1: Radar Systems Fundamental	**2 Hours**	October 7th , 2024
Lecture 2: Advanced Radar Systems	**2 Hours**	October 8th , 2024
Lecture 3: Practical Radar Signal Processing - Motion Detection	**2 Hours**	October 9th , 2024
Lecture 4: Practical Radar Signal Processing - Breathing and Heart Rate Estimation	**2 Hours**	October 10th , 2024
Lecture 5: Practical Radar Signal Processing – Angle estimation with MIMO radar	**2 Hours**	October 11th , 2024

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Lecture 2

Advanced Radar Systems

Lecture 2: Advanced Radar Systems

What we learn in Lecture 2

- **SISO Radar**
- **Phased Array Radar**
- **MIMO Radar**
- **Waveform Multiplexing**
- **Nonconvex optimization**

Scan the QR code for access to the codes

- ❑ Antenna beamwidth and aperture size
- ❑ Beam steering in phased array radars
- ❑ Virtual array in MIMO radars
- ❑ TDM, DDM, BPM, and FDM
- ❑ Coordinate Descent, Gradient Descent, and Majorization Minimization

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Recall from lecture 1

Unambiguous Range

$$R_{un} \leq \frac{cT_p}{2} = \frac{c}{2f_p}$$

Unambiguous Doppler

$$v_{r\,max} = \lambda \frac{f_{d\,max}}{2} \leq \frac{\lambda f_p}{4}$$

Range Resolution

$$\Delta R = \frac{c}{2B}$$

Doppler Resolution

$$\Delta f_d = \frac{1}{T_{CPI}}$$

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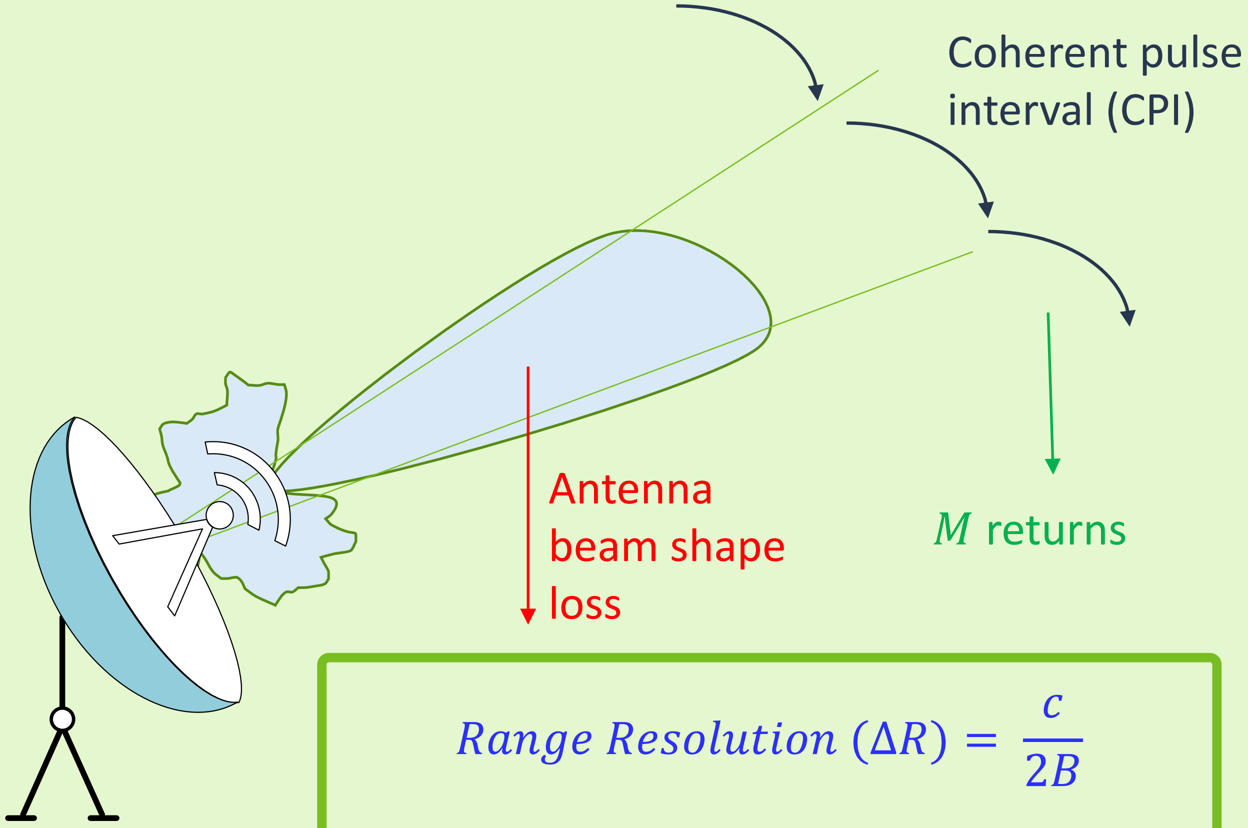
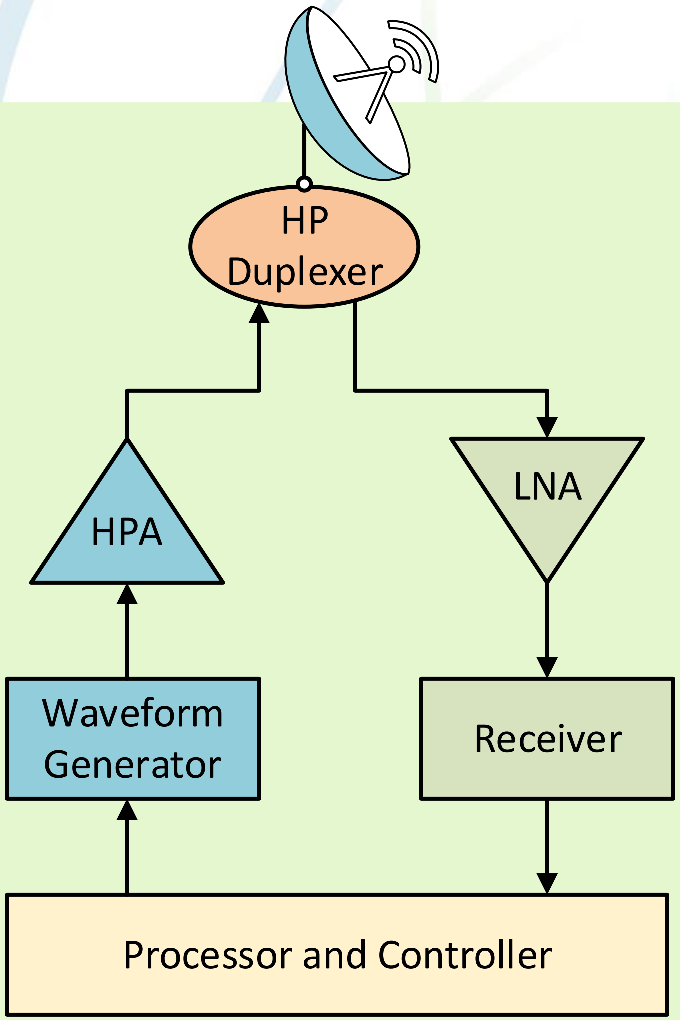
Single Input Single Output (SISO)

Mechanical Scanning



$$Range\ Resolution\ (\Delta R) = \frac{c}{2B}$$

$$Doppler\ Resolution\ (\Delta f_d) = \frac{1}{T_{CPI}}$$



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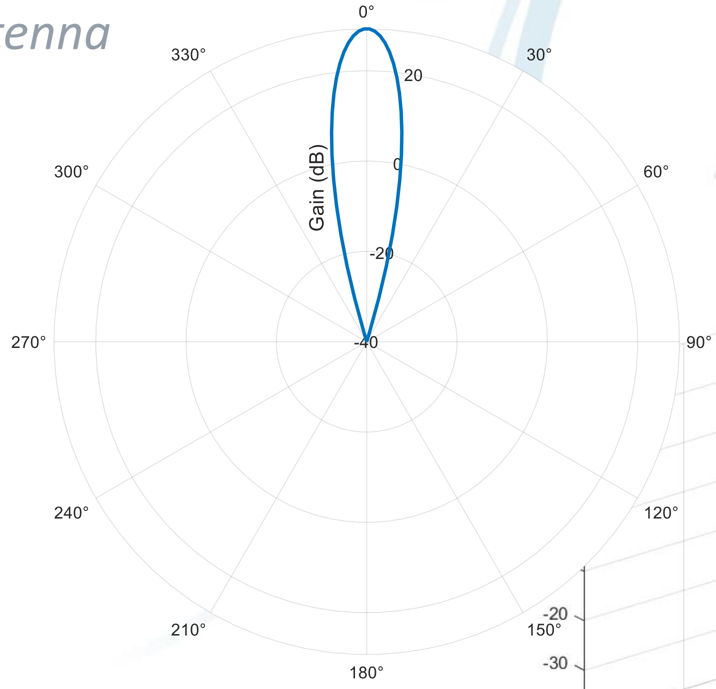
Single Input Single Output (SISO)

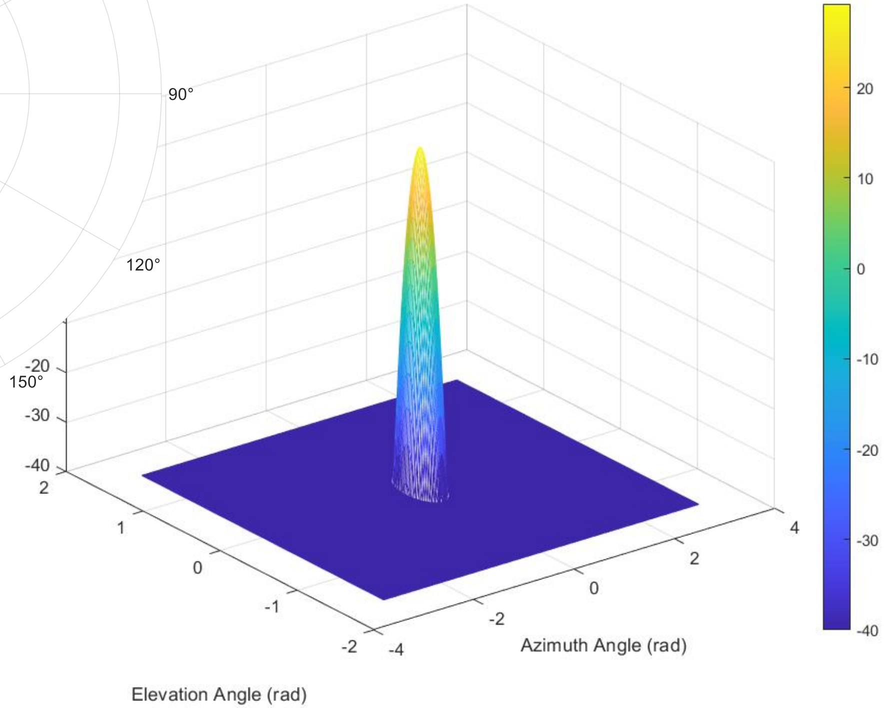
Mechanical Scanning – Parabolic antenna

$$\text{Beamwidth (deg)} \approx 70\frac{\lambda}{d}$$

$$G \approx \frac{4\pi}{\theta\phi}$$

$$f_c = 1\ \text{GHz}$$
$$d = 3\ \text{m}$$
$$\theta = \phi \approx 7\ \text{deg}$$
$$G_t = G_r \approx 29\ \text{dB}$$





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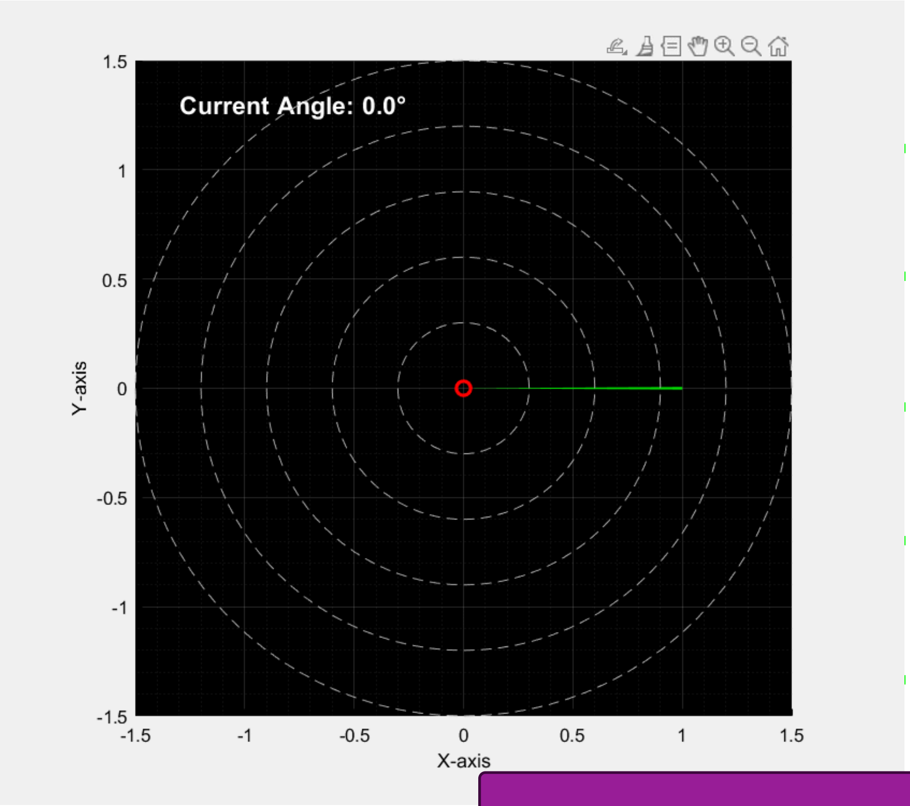
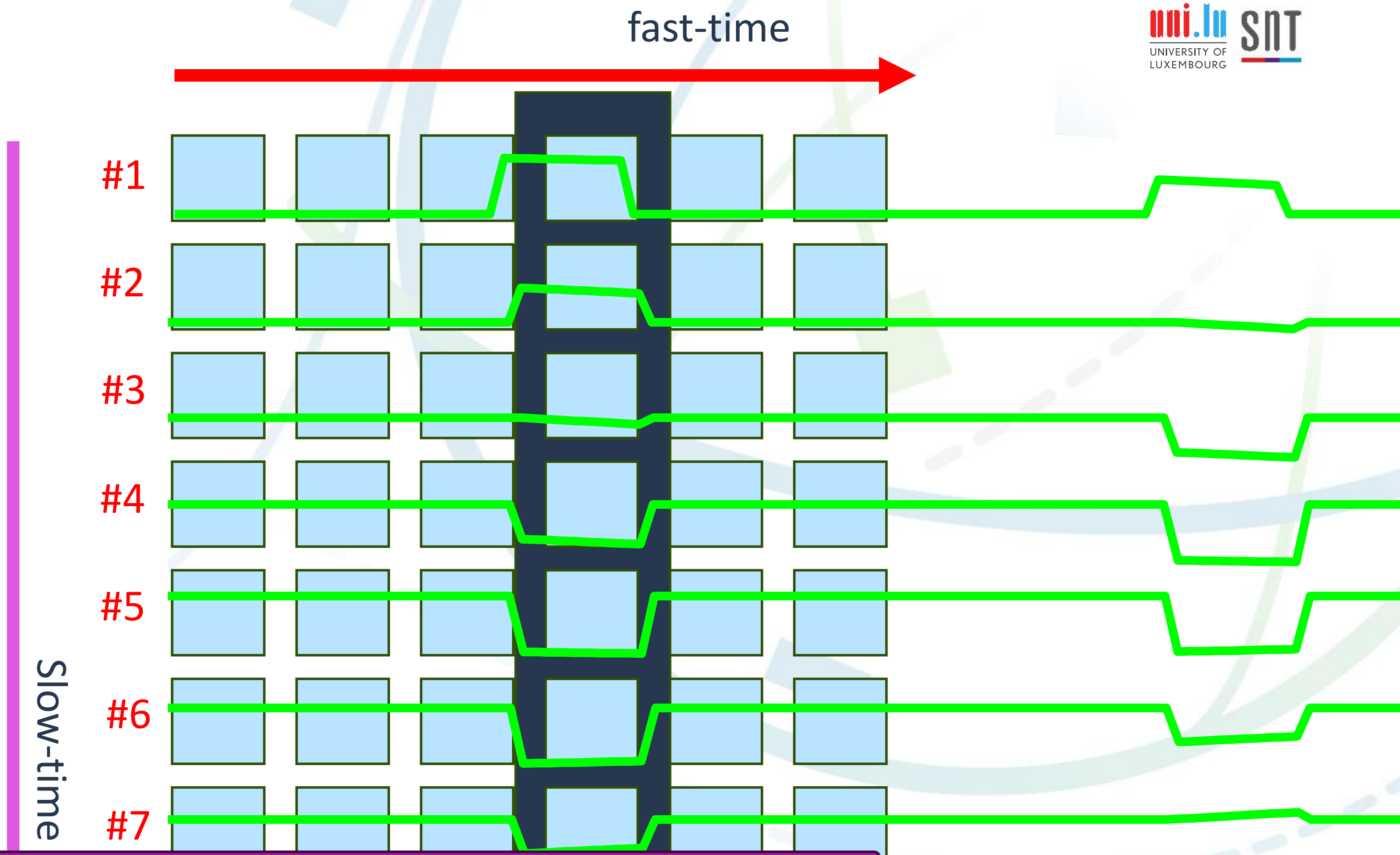
SISO - Mechanical Scanning – Parabolic antenna

Received signal





If there are two targets at the same range, radial speeds, but at different angles, they cab still be separated

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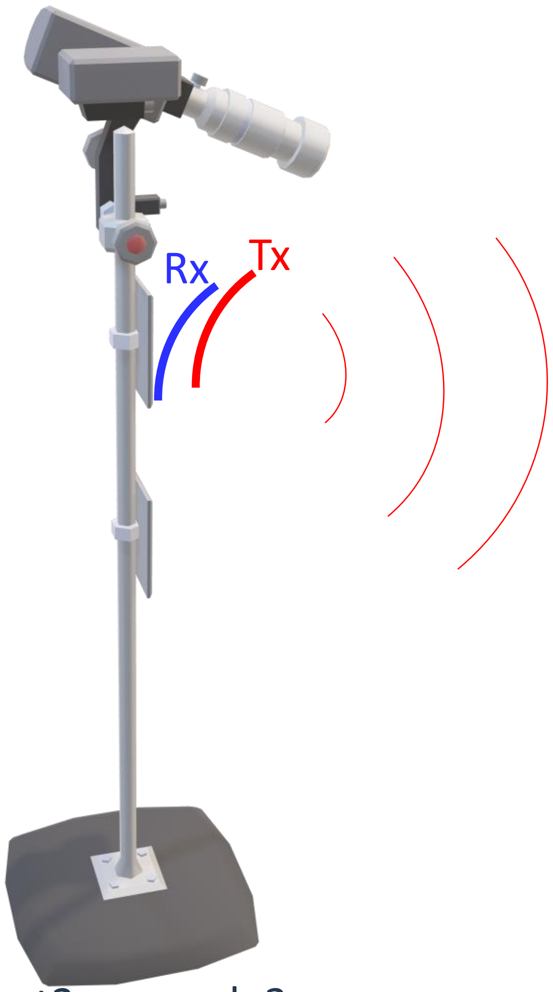
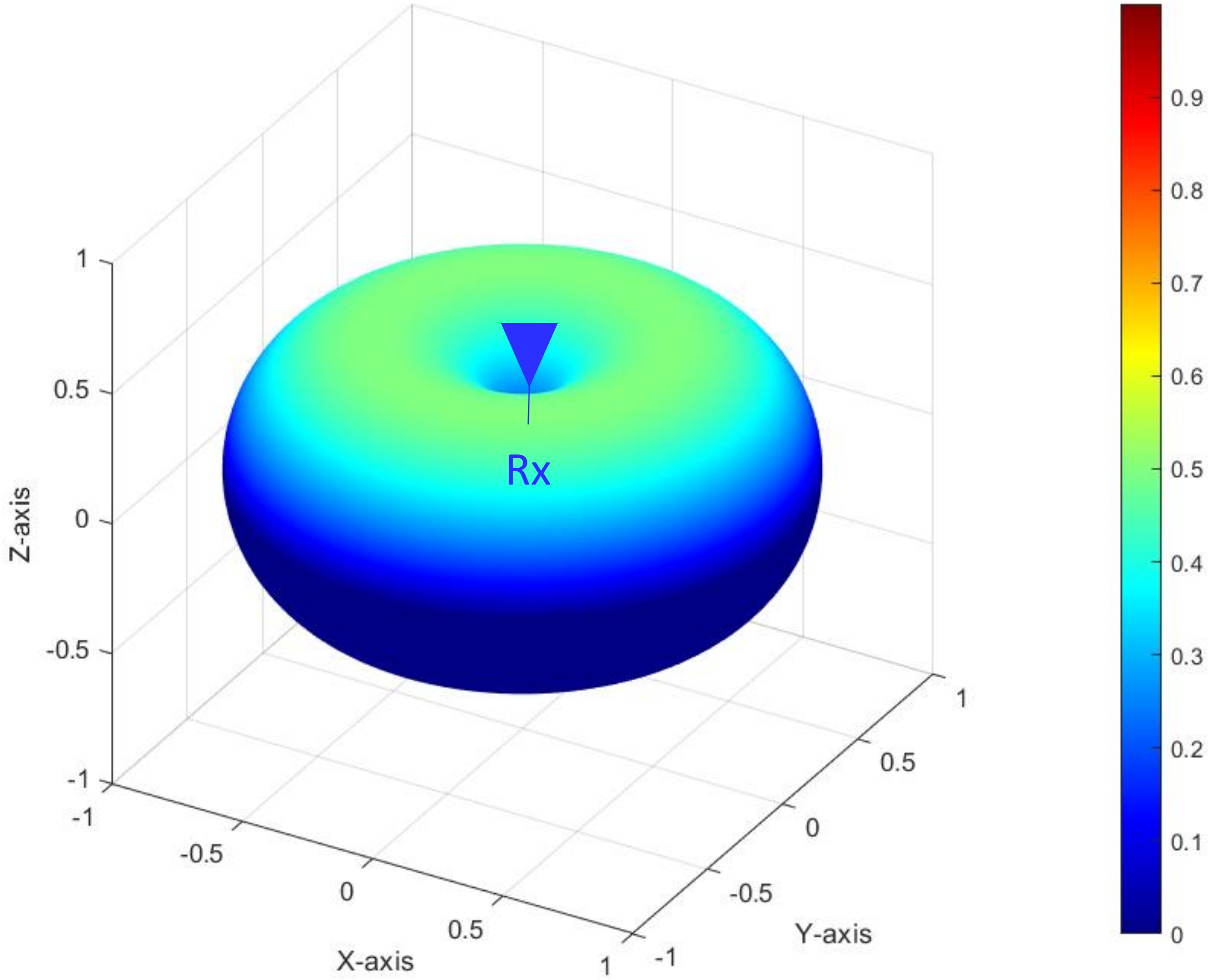
CW Radar - SISO

Electronical scanning





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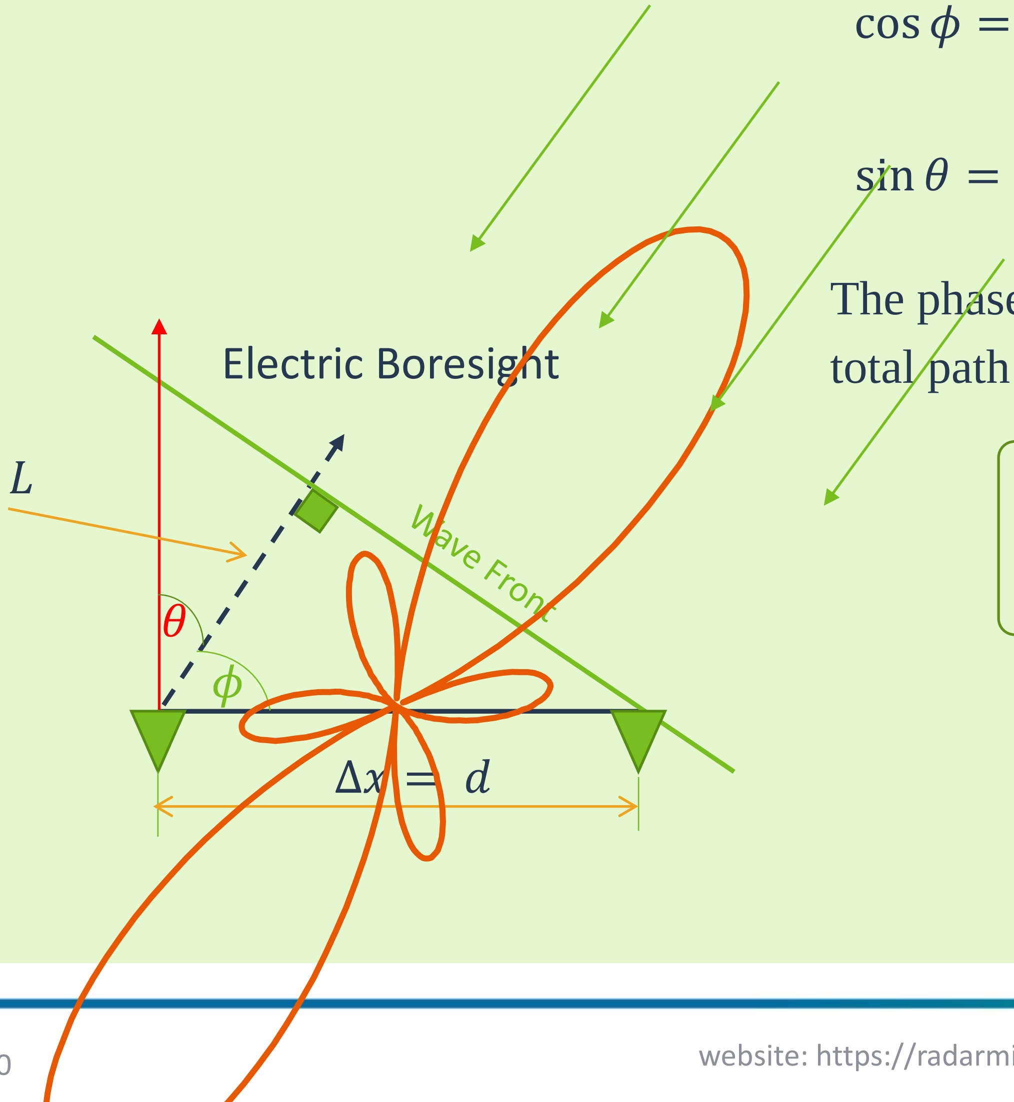
Angle of Arrival Estimation with Array of Antennas



$$\cos\phi = \frac{L}{d}, \qquad \theta + \phi = 90 \qquad \cos\phi = \cos(90 - \theta) = \sin\theta$$

$$\sin\theta = \frac{L}{d} \Rightarrow \qquad L = d\sin\theta$$

The phase variation across the array surface, or *aperture*, is the total path length variation times $\frac{2\pi}{\lambda}$

$$\boxed{\Delta\phi = \frac{2\pi d \sin\theta}{\lambda}}$$

$$\text{If } d = \frac{\lambda}{2} \Rightarrow \quad \Delta\phi = \pi\sin\theta$$

What happens if we increase d ?

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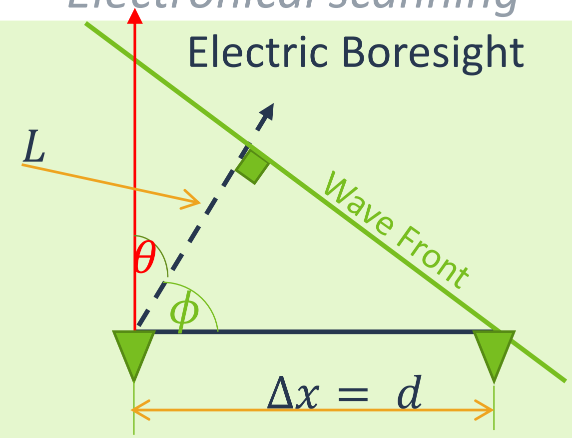
Angle of Arrival Estimation with Array of Antennas

Electronical scanning



1

$\exp -j\dfrac{2\pi(1)d\sin\theta}{\lambda}$

$\exp -j\dfrac{2\pi(2)d\sin\theta}{\lambda}$

$\vdots$

$\exp -j\dfrac{2\pi(N-1)d\sin\theta}{\lambda}$

$+$

$$\boxed{\text{(3dB) Beamwidth [rad]} \cong \frac{\alpha\lambda}{N\,d}}$$

α is the beamwidth factor and is determined by the aperture taper function

N is number of antennas

d is the distance between two antenna elements

$$AF(\theta) = \frac{1}{N}\sum_{n=0}^{N-1} \exp\left(-j\frac{2\pi}{\lambda} nd\sin\theta\right)$$

This expression is referred to as the array factor (AF)

If the element is assumed to be an isotropic radiator, which has no angular dependence, then the array factor and the phased array radiation pattern will be equal.

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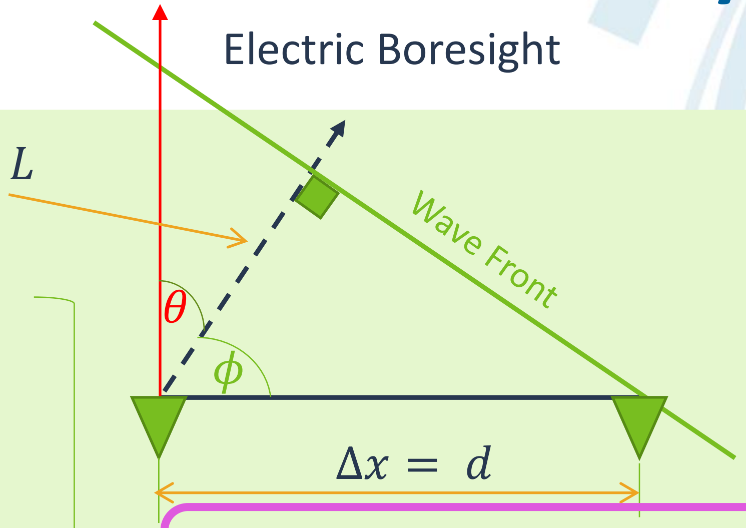
Angle of Arrival Estimation with Array of Antennas

Electronical scanning



$$1$$

$$\exp -j\frac{2\pi(1)\Delta x \sin\theta}{\lambda}$$

$$\exp -j\frac{2\pi(2)\Delta x \sin\theta}{\lambda}$$

$$\vdots$$

$$\exp -j\frac{2\pi(\mathrm{N}-1)\Delta x \sin\theta}{\lambda}$$

$$\boldsymbol{a}(\theta) = \left[1, \exp\left(-j\frac{2\pi\Delta x \sin\theta}{\lambda}\right), \ldots, \exp\left(-j\frac{2\pi(\mathrm{N}-1)\Delta x \sin\theta}{\lambda}\right)\right]^T \in \mathbb{C}^N$$

Steering vector

$$AF(\theta) = \boldsymbol{w}^H(\theta)\,\boldsymbol{a}(\theta)$$

Weights

Can include
amplitude and phase

✓ steer beam to a desired angle
✓ control the sidelobe levels

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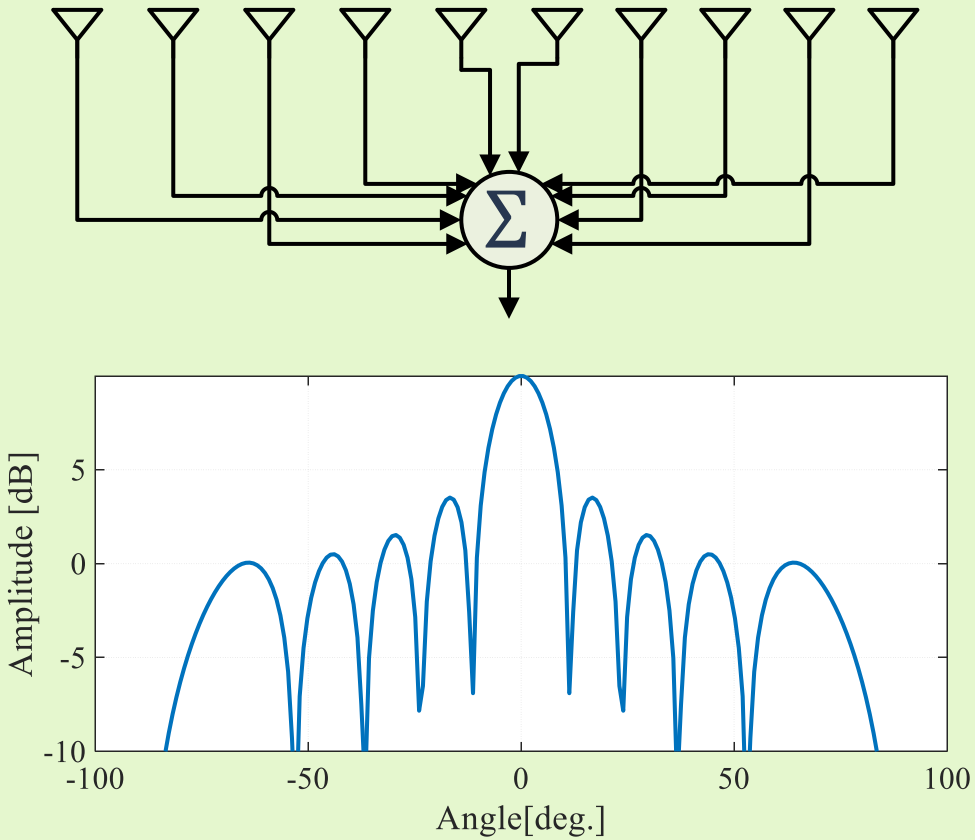
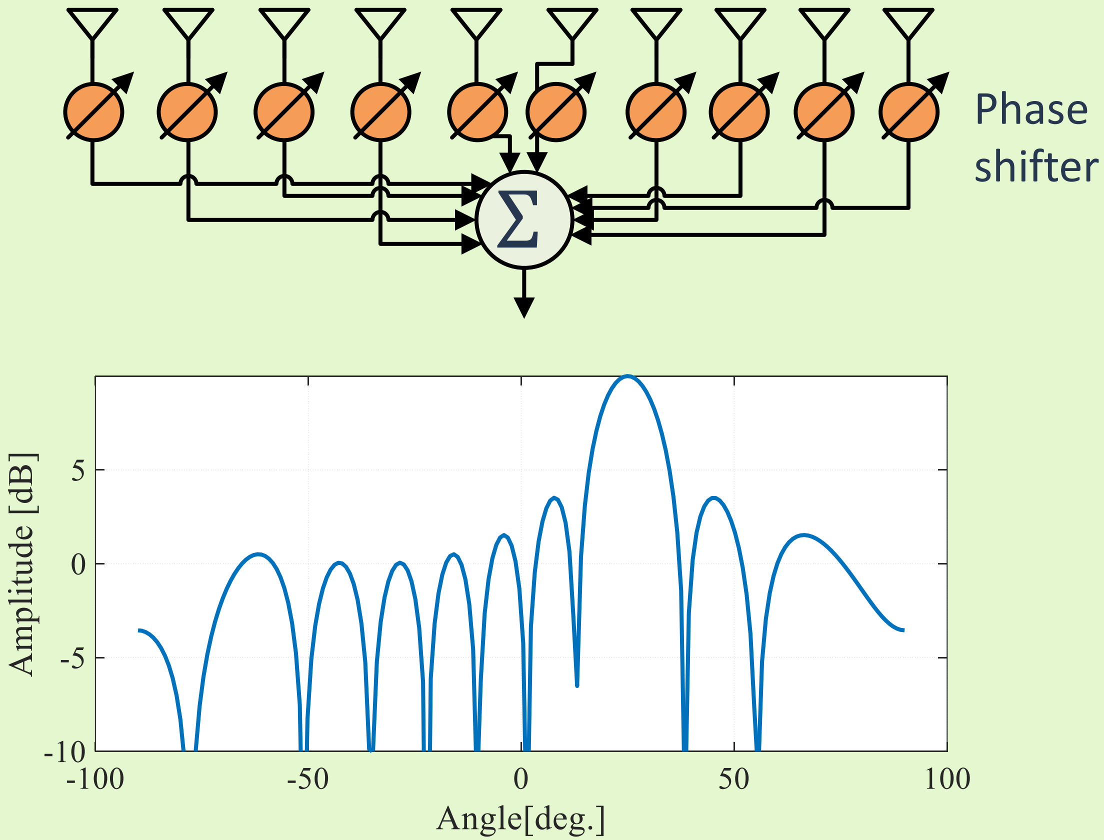
Angle of Arrival Estimation with Array of Antennas

Electronical scanning





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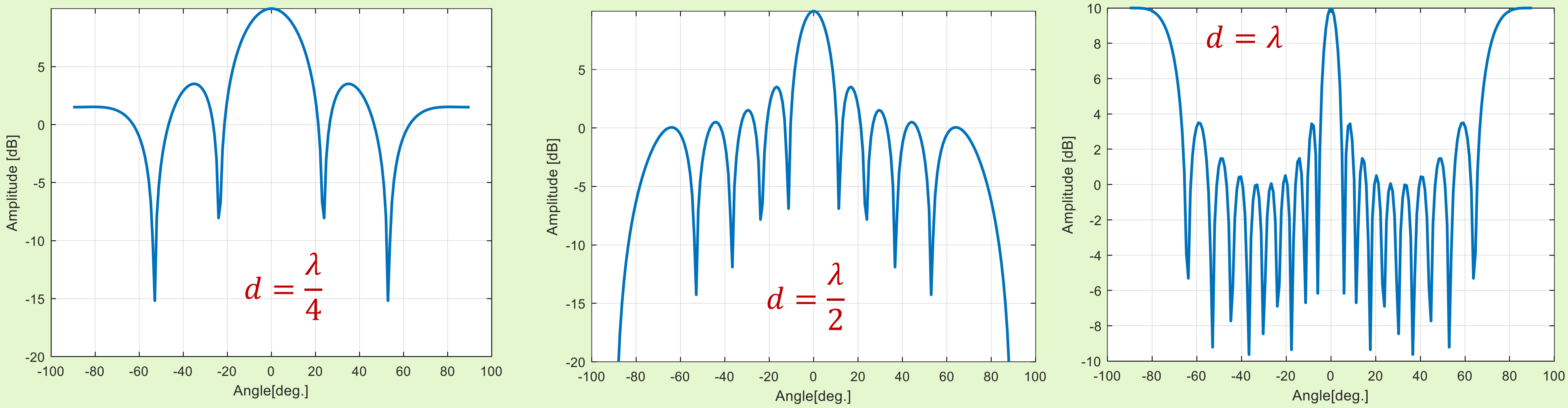
Angle of Arrival Estimation with Array of Antennas

Electronical scanning

$N = 10$ Isotropic Elements
No Phase Shifting



Limit element separation to $d < \lambda$ to prevent grating lobes for broadside array

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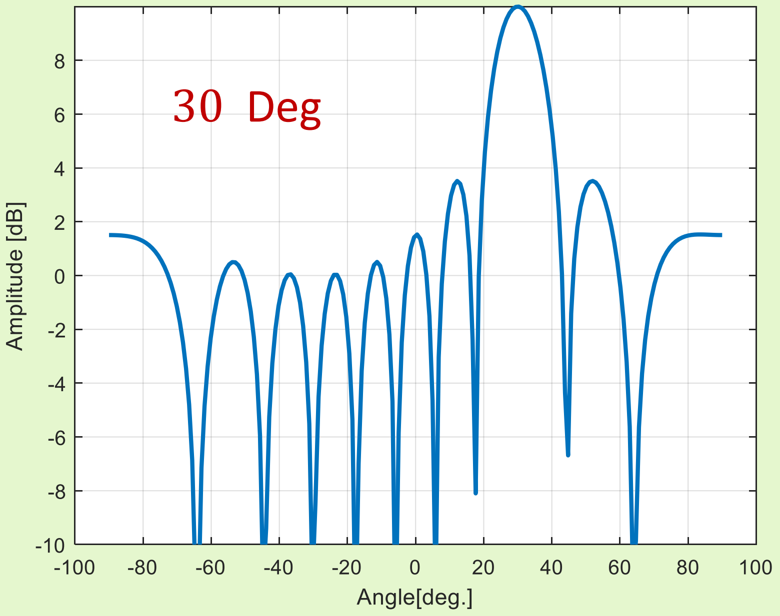
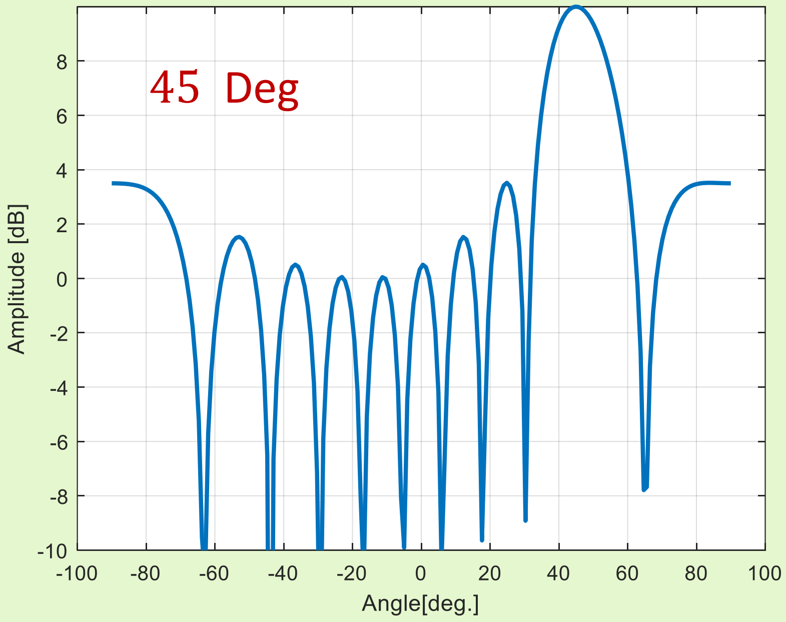
Angle of Arrival Estimation with Array of Antennas

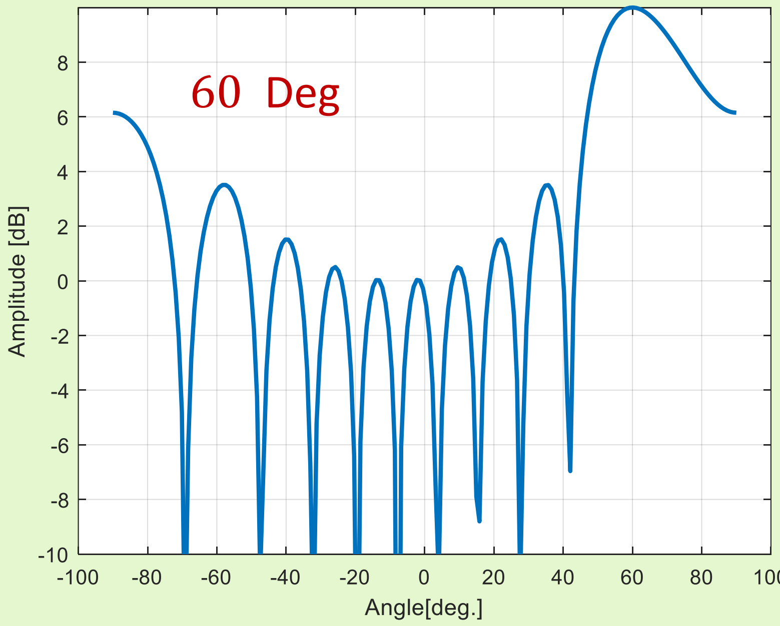
Electronical scanning

$N = 10$ Isotropic Elements

$d = \frac{\lambda}{2}$, Beam pointing direction = 30, 45 , 60







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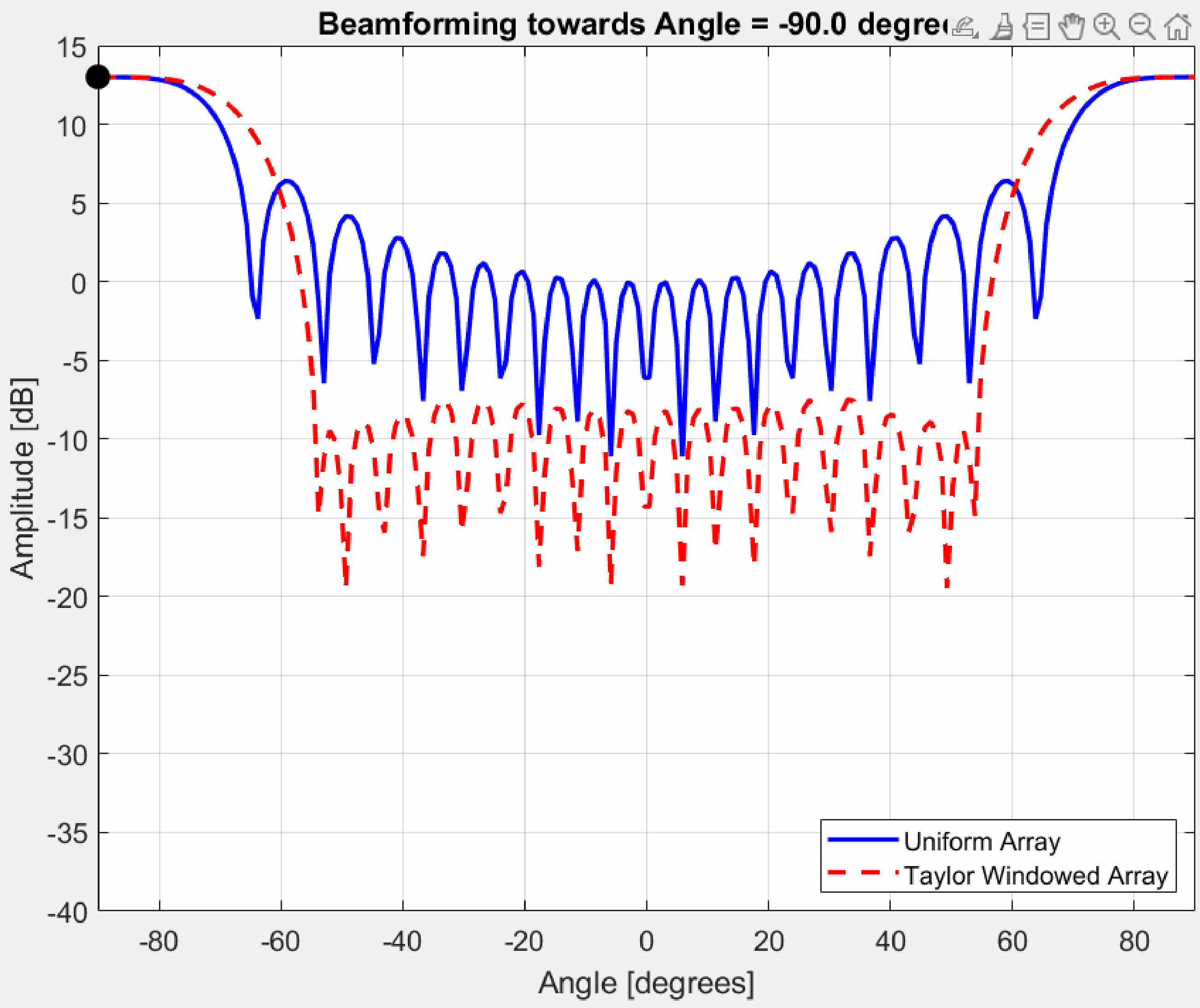
Phase Array Radars

$N = 20$ Isotropic Elements

$d = \frac{\lambda}{2}$



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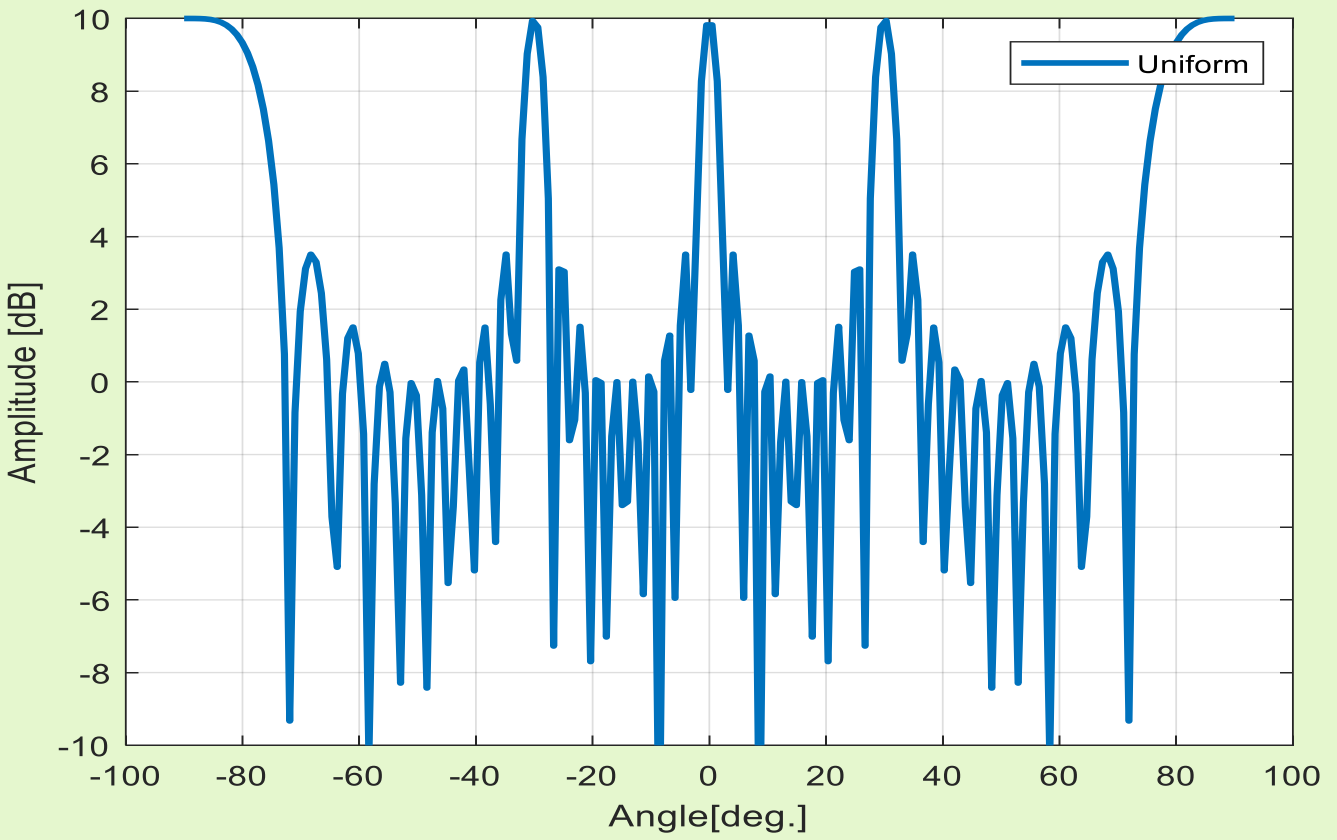
Angle of Arrival Estimation with Array of Antennas

Electronical scanning

$N \ = \ 10$ Isotropic Elements
No Phase Shifting

$d \ = 2\lambda$

What are side effects of grating lobes ?



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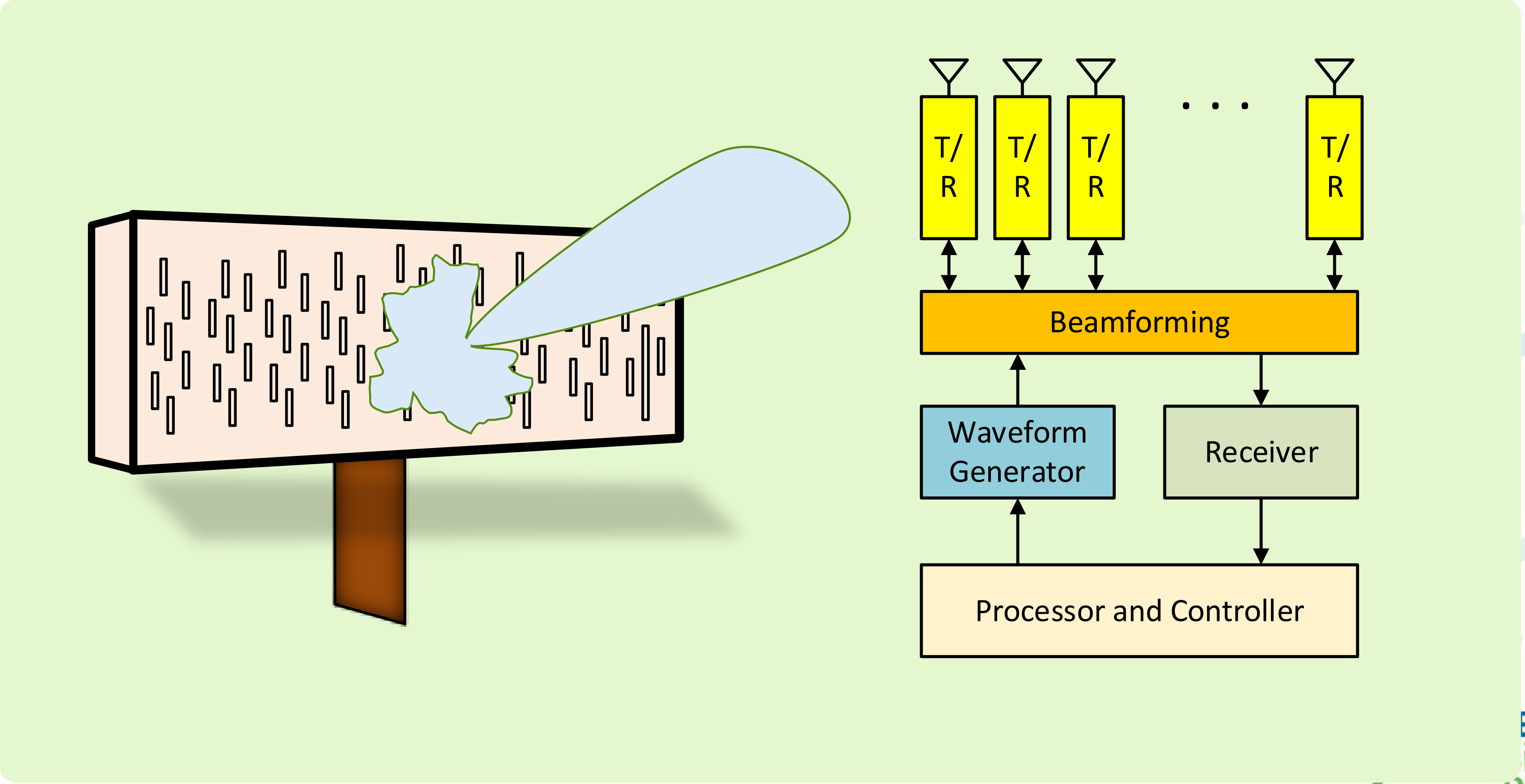
Phase Array Radars

Electronical scanning



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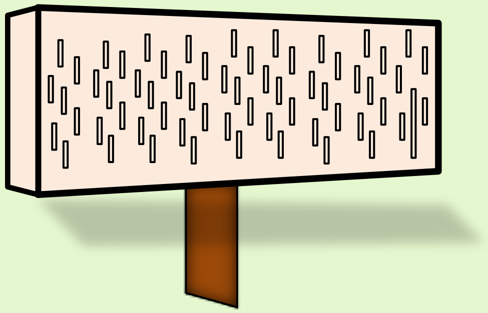
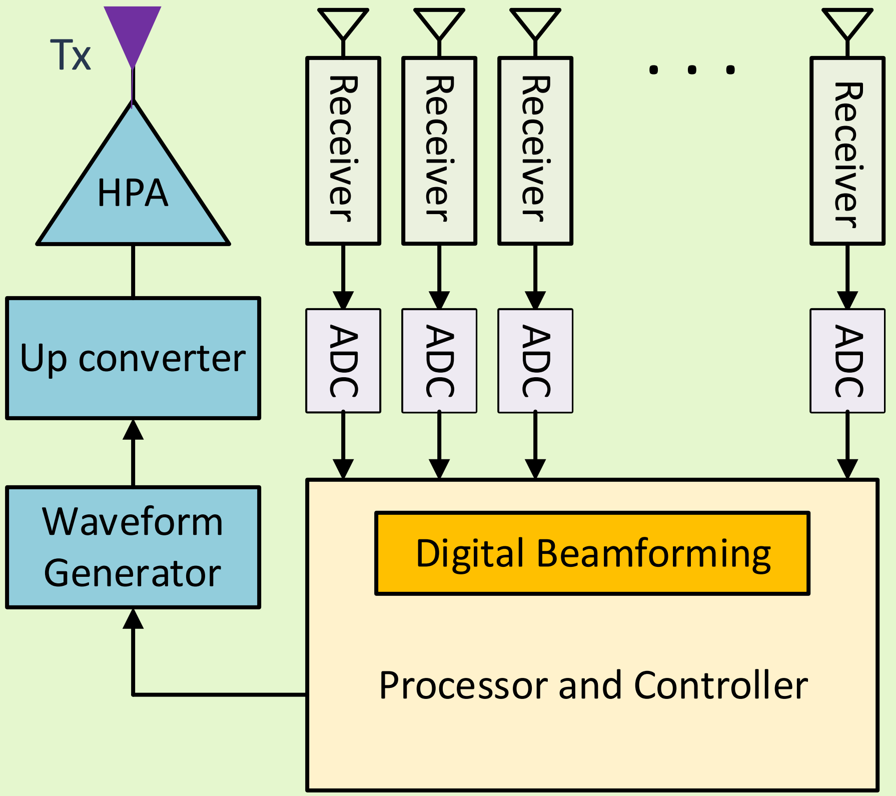
Digital Beamforming

Electronical scanning

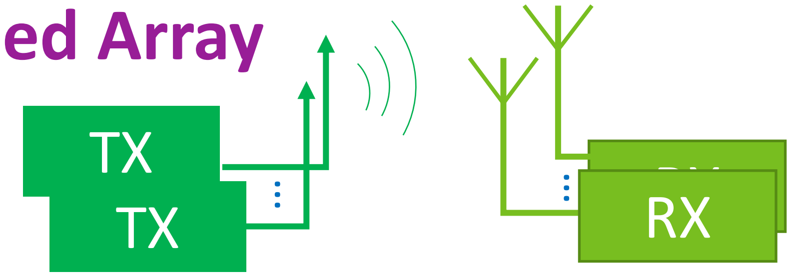




Phase Array Radars
with Digital Beamforming

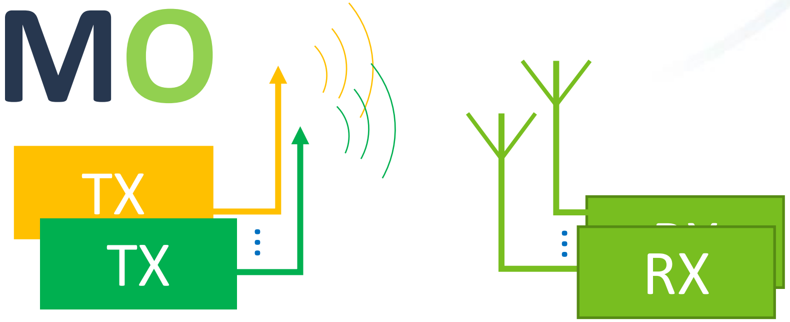
Phase Array and MIMO Radars

Phased Array



Single Input Multi Output

MIMO



Multi Input Multi Output

Waveform diversity

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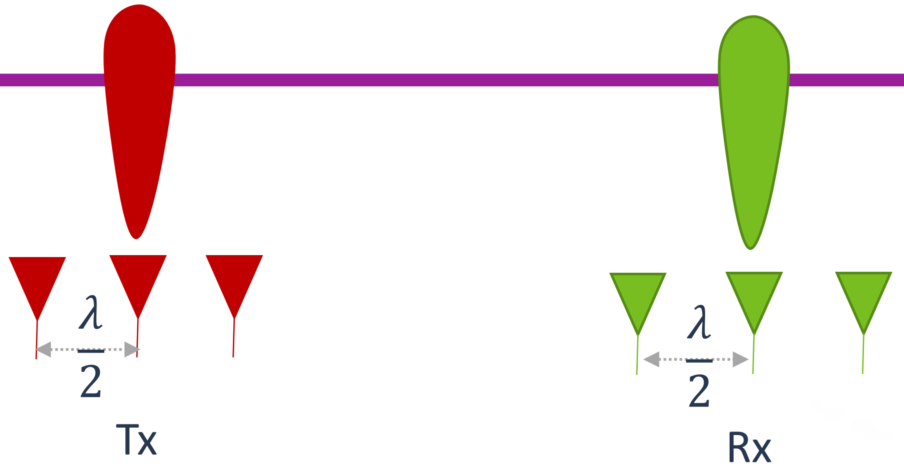
Phase Array and MIMO Radars

A

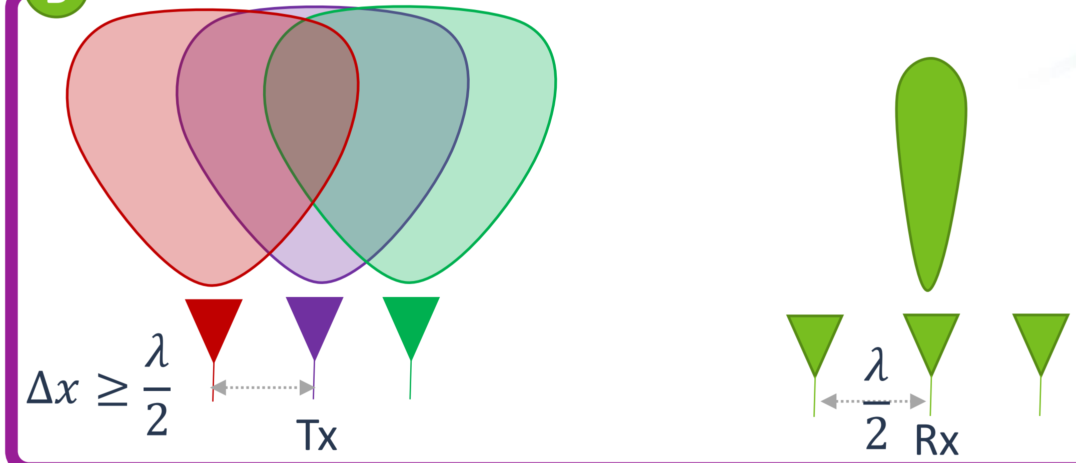


Phased Array

The Tx antennas transmit a single waveform.

$$\text{SNR} \propto G_t G_r$$

B



MIMO

Every antenna element will emit a different waveform. Why?

$$\text{SNR} \propto \frac{G_t}{N_t} G_r$$

MIMO SNR is N_t times lower

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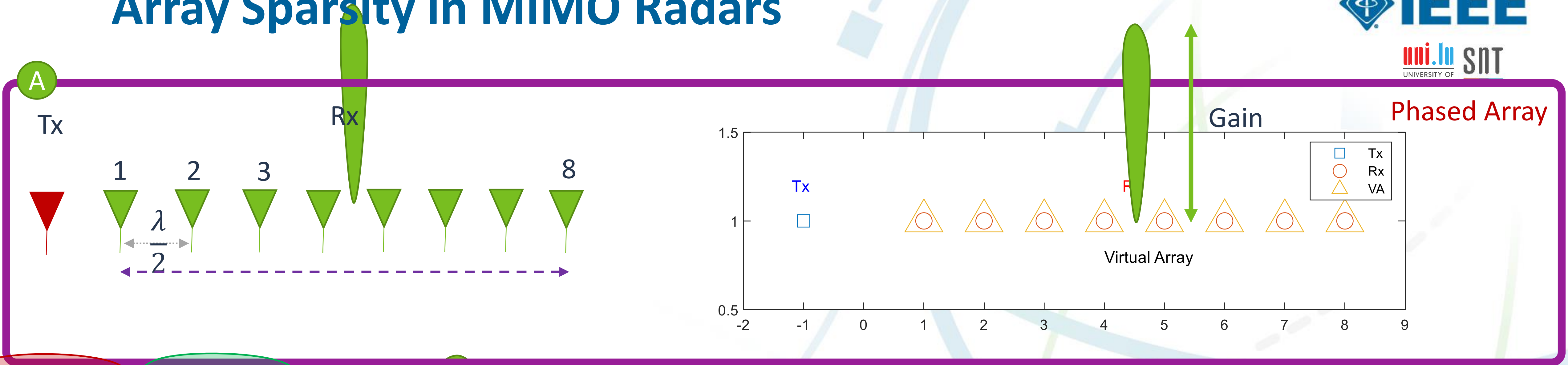
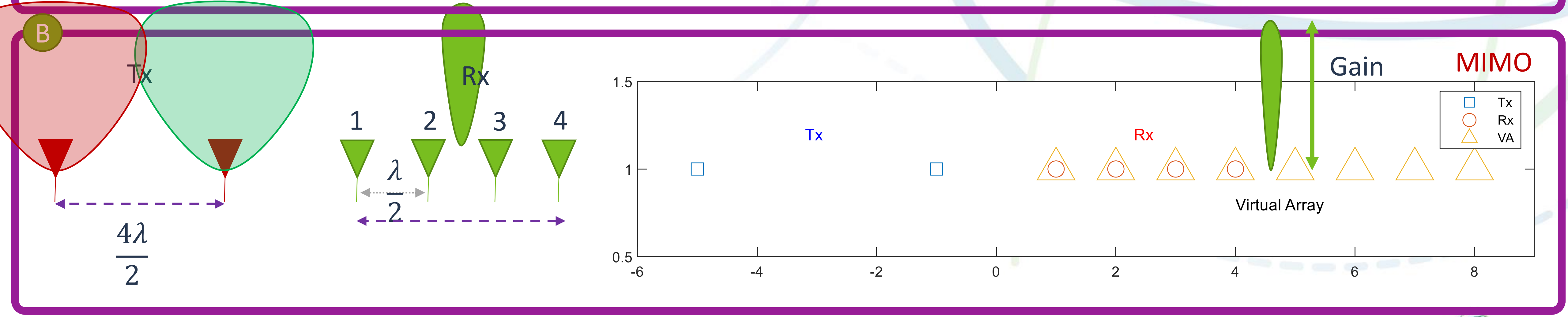
Array Sparsity in MIMO Radars





Same angular resolution but lower antenna gain for MIMO radar

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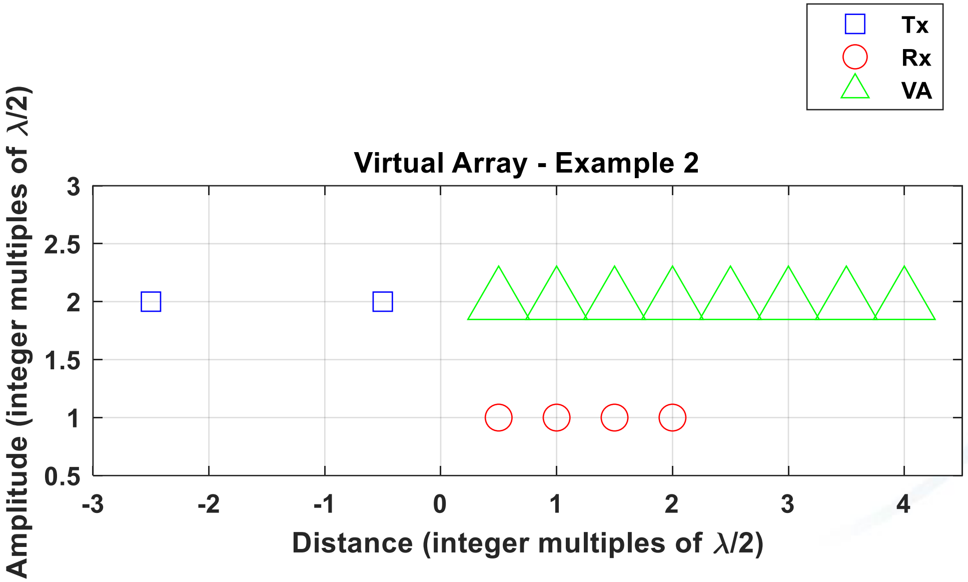
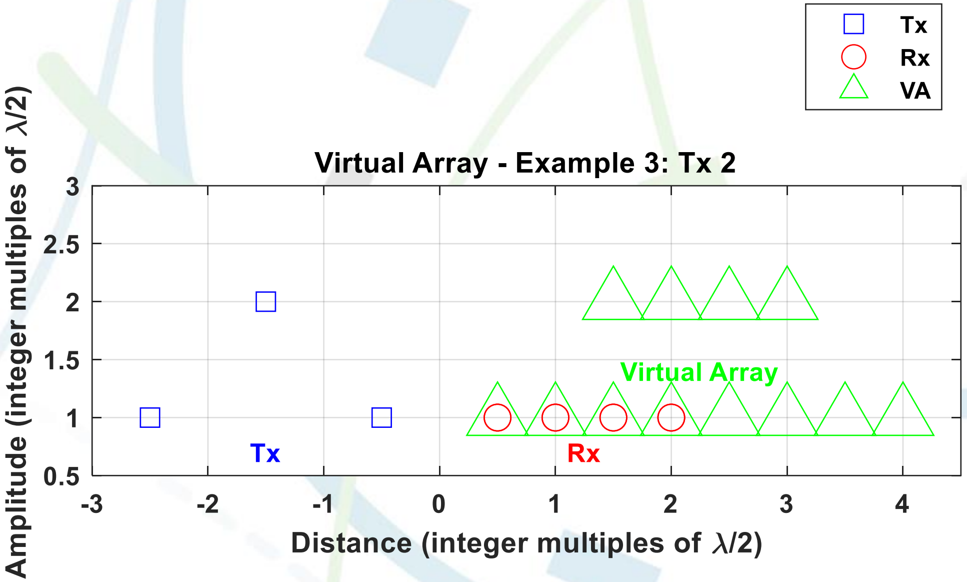
Virtual Array in MIMO Radars





2 + 4 = 6 => physical elements

2 × 4 = 8 => virtual elements

3 + 4 = 7 => physical elements

3 × 4 = 12 => virtual elements

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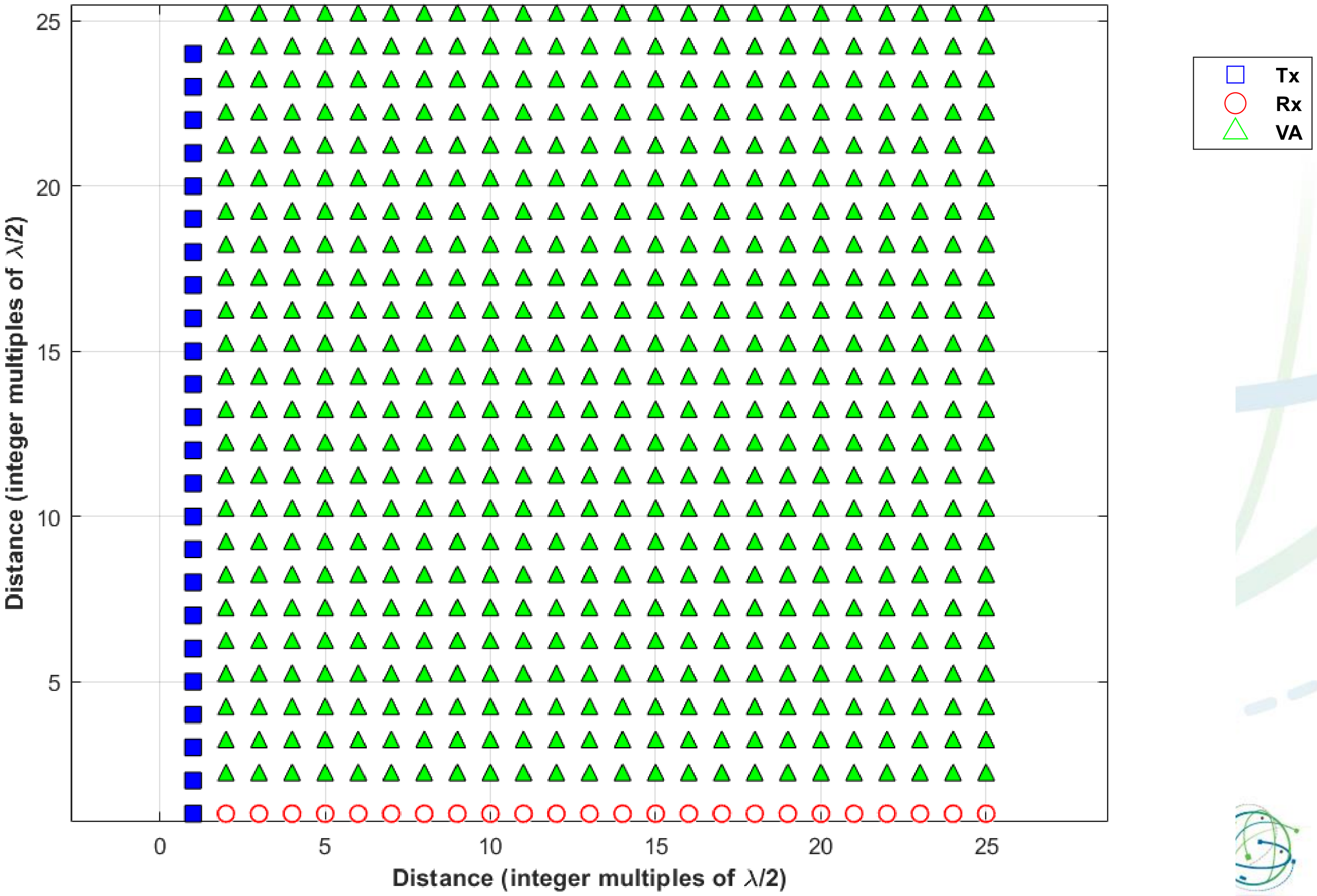
Virtual Array in MIMO Radars

24 + 24 = 48 => physical elements

24 × 24 = 576 => virtual elements

576
Antenna array

4D – imaging
MIMO Radar



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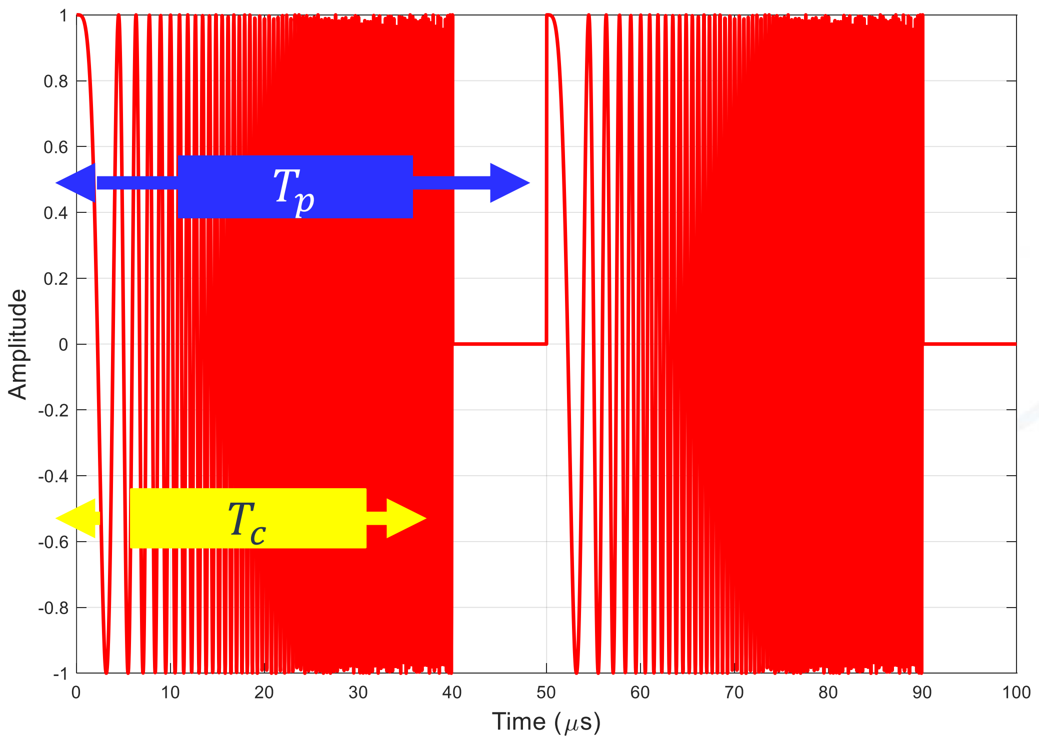
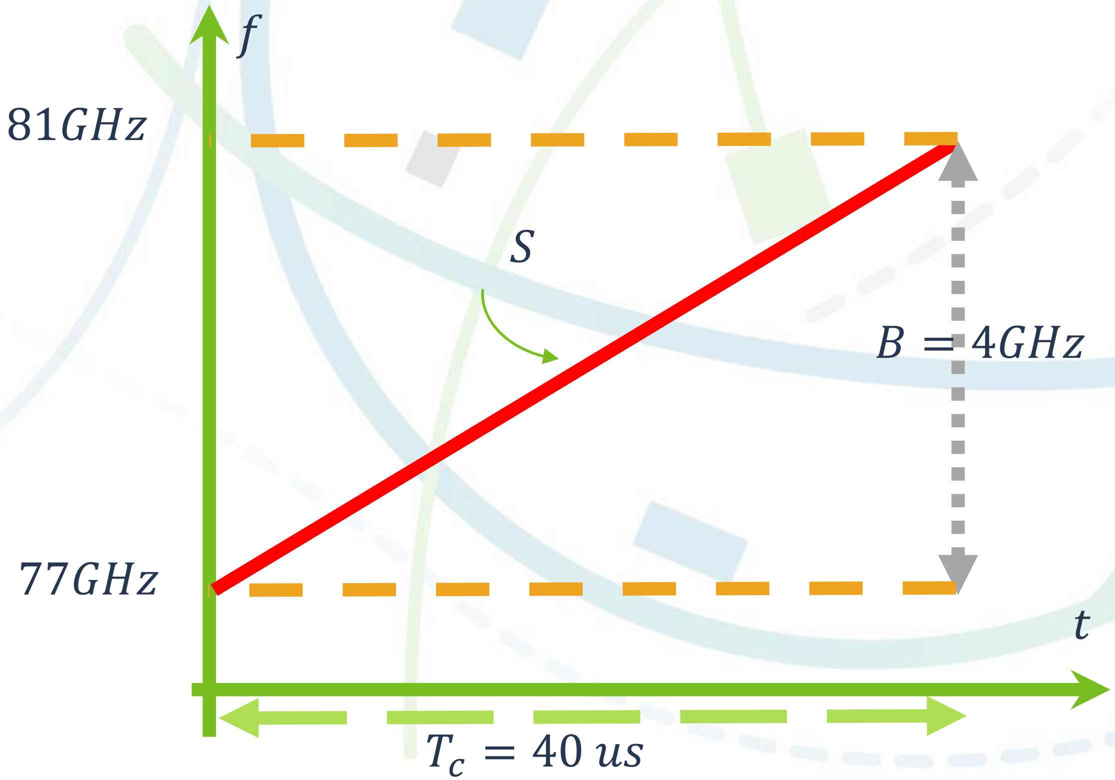
Chirp in MIMO Radars





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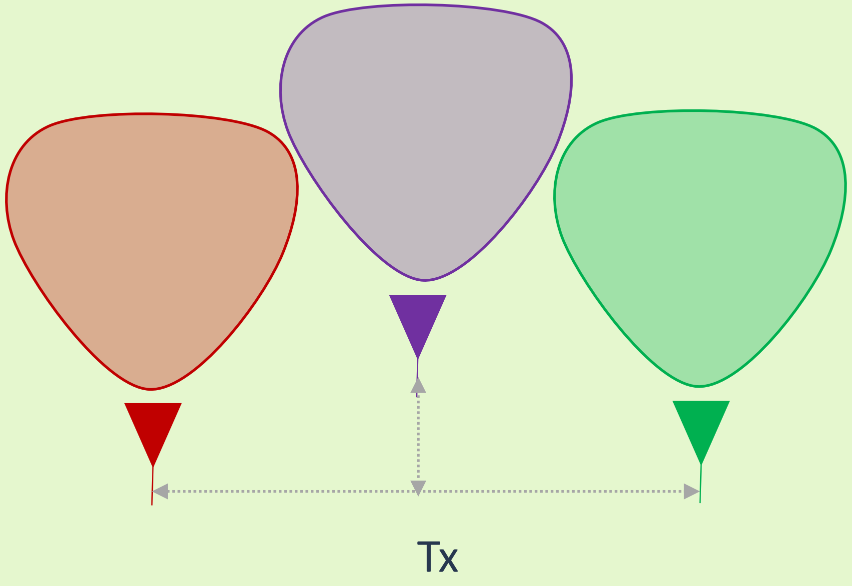
Waveforms in MIMO Radars

Inter-Pulse Modulation Techniques

- Time Division Multiplexing (TDM)
- Frequency Division Multiplexing (FDM)
- Doppler Division Multiplexing (DDM)
- Binary Phase Modulation (BPM)



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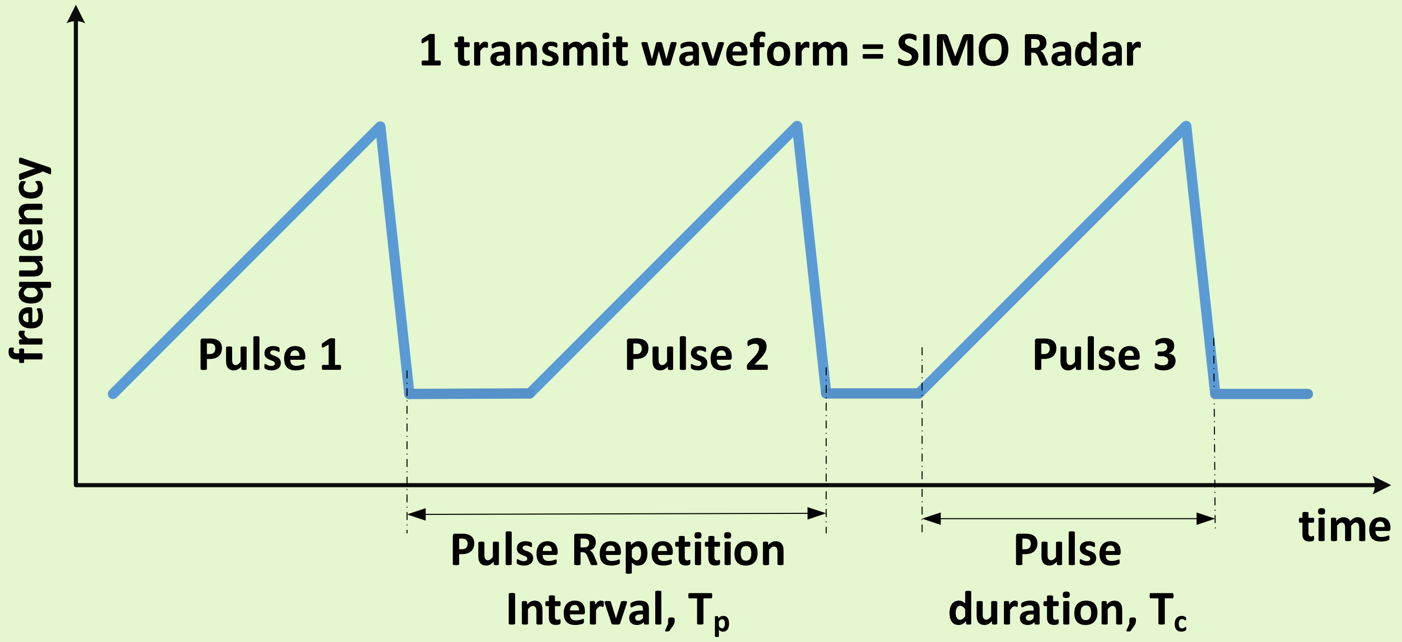
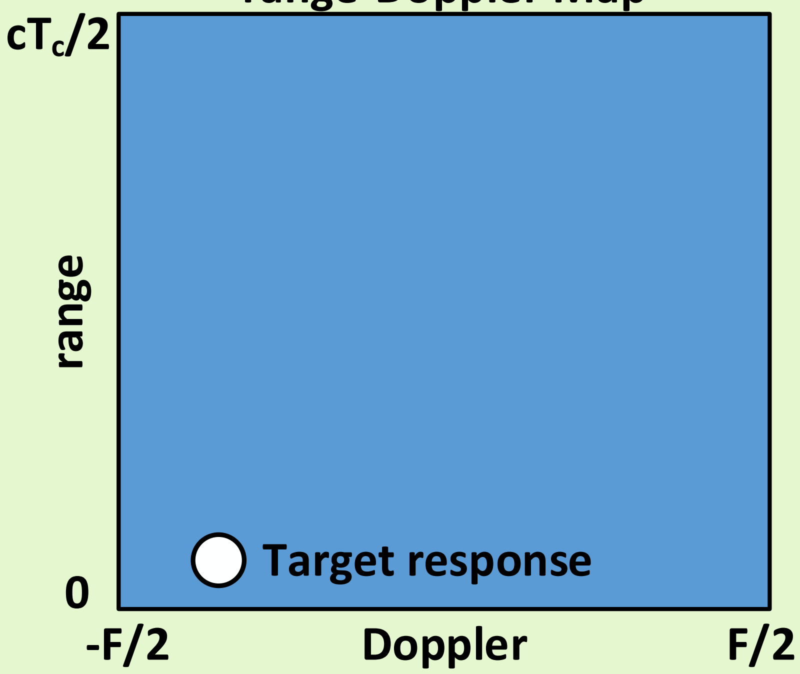
Time Division Multiplexing (TDM)

$$F = \frac{1}{T_R}$$





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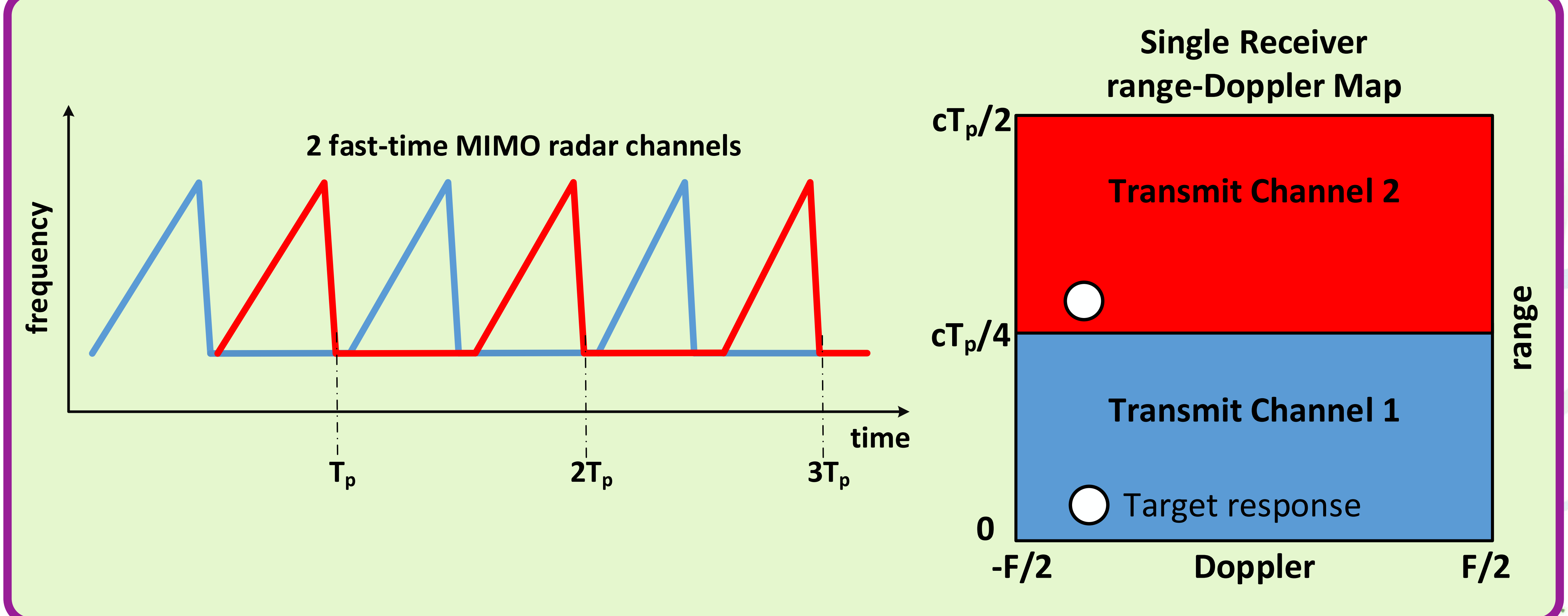
TDM-MIMO



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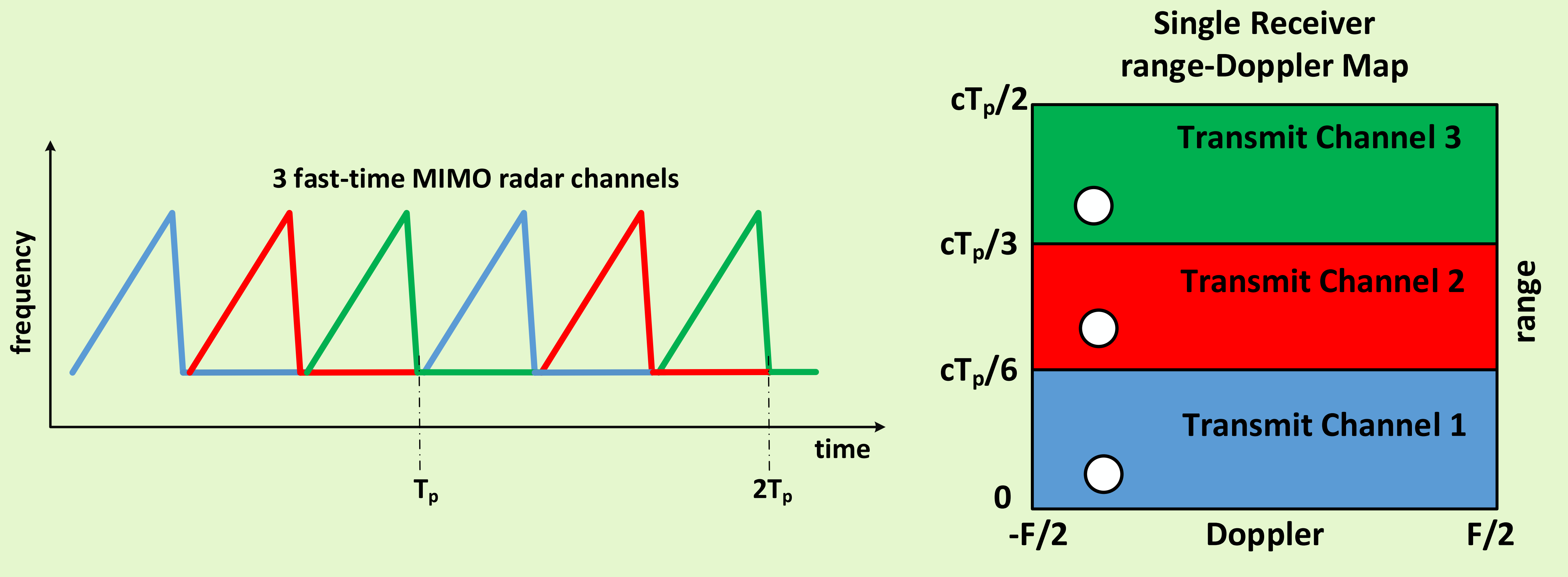
TDM-MIMO



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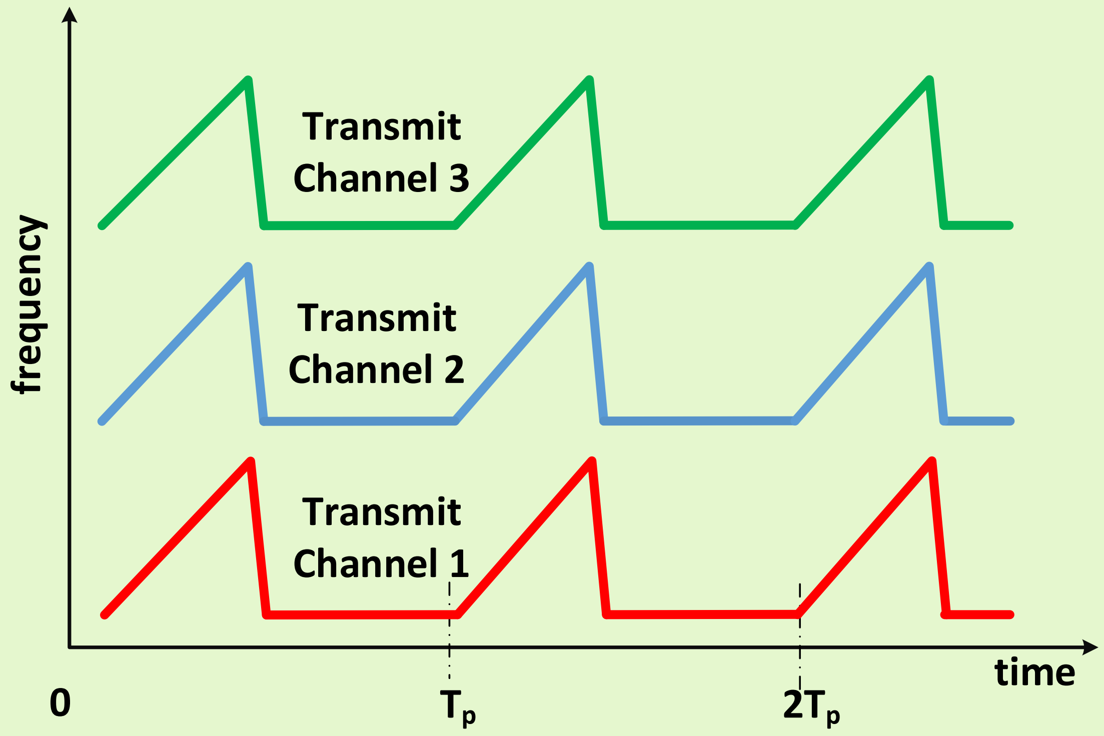
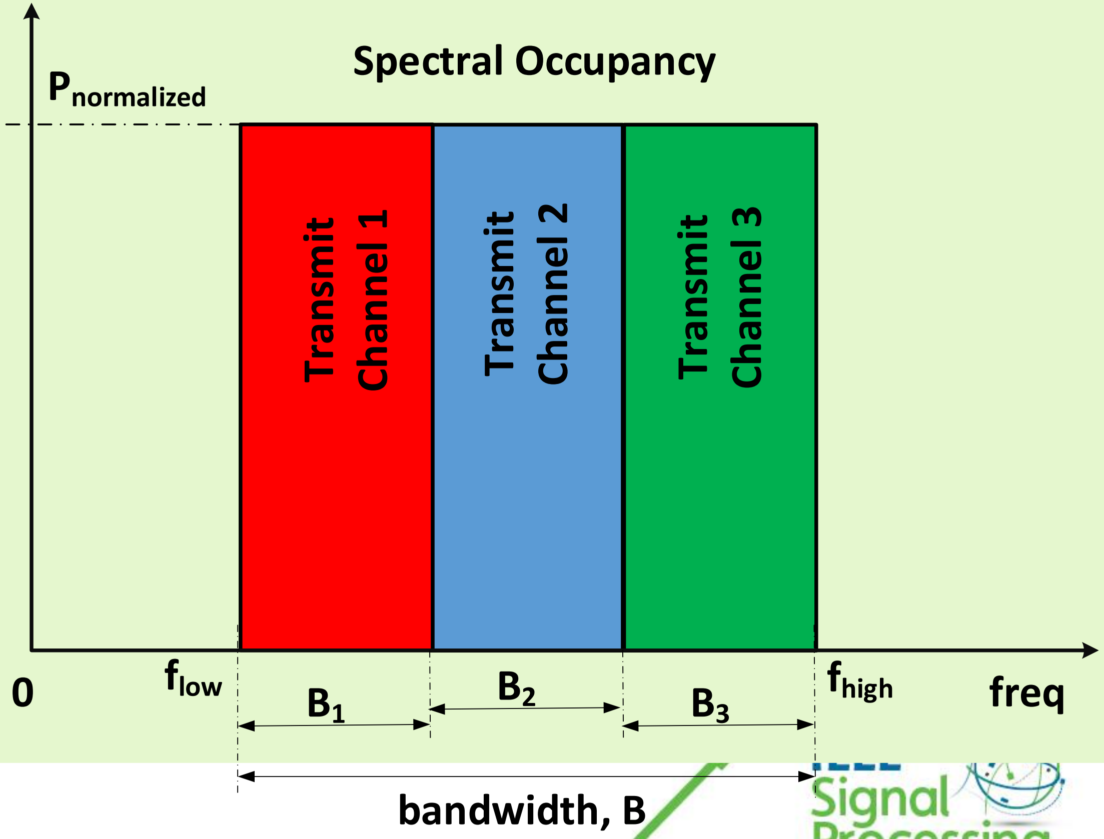
Frequency Division Multiplexing (FDM)

- Easy to implement with minimal hardware complexity
- range resolution compromised for more channels

$$\text{range resolution} = \frac{c}{2B},$$

$$\text{where } c = \text{speed of light}, \ B = \text{bandwidth}.$$





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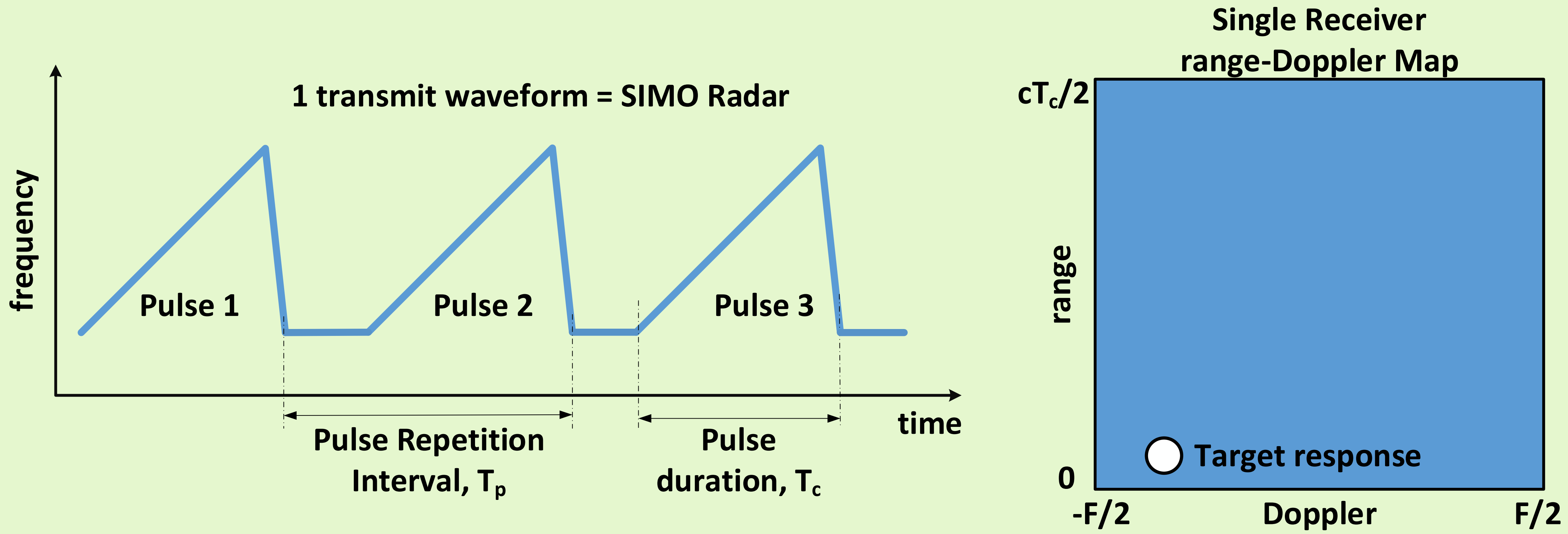
Doppler Division Multiplexing (DDM)



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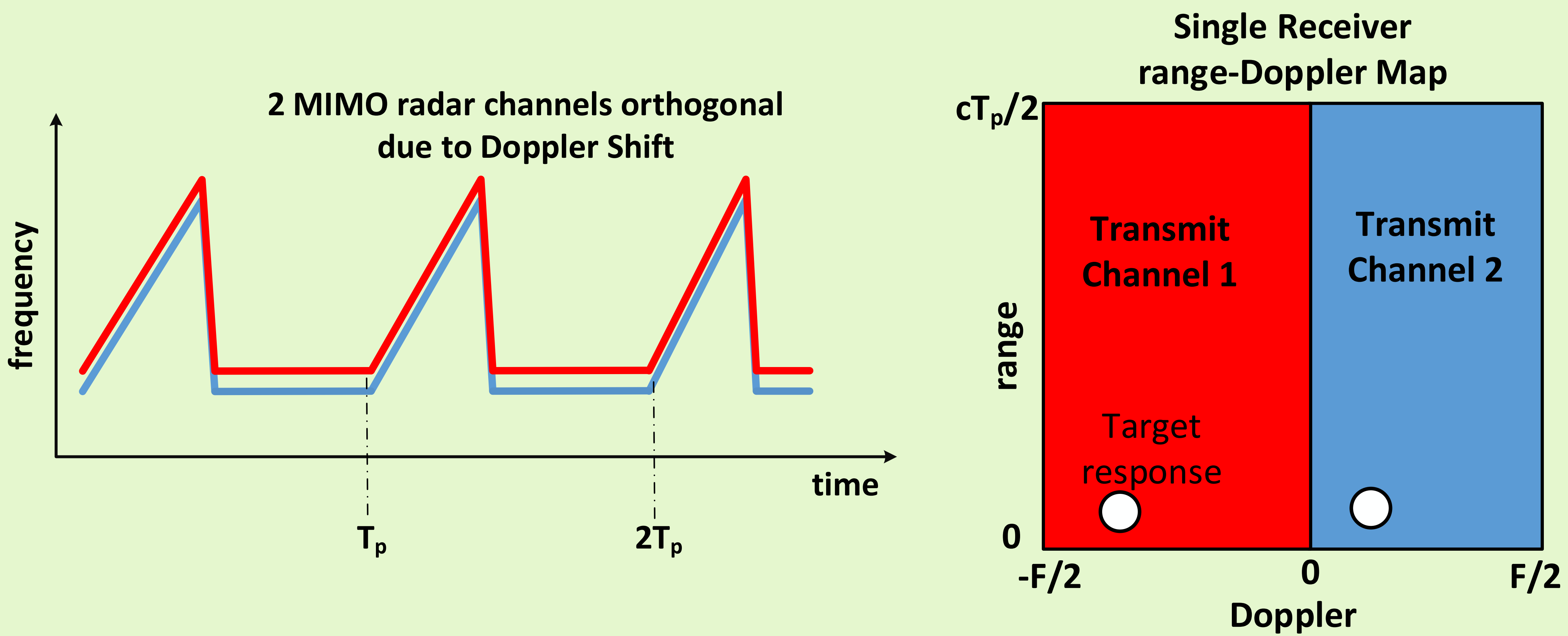
DDM - MIMO



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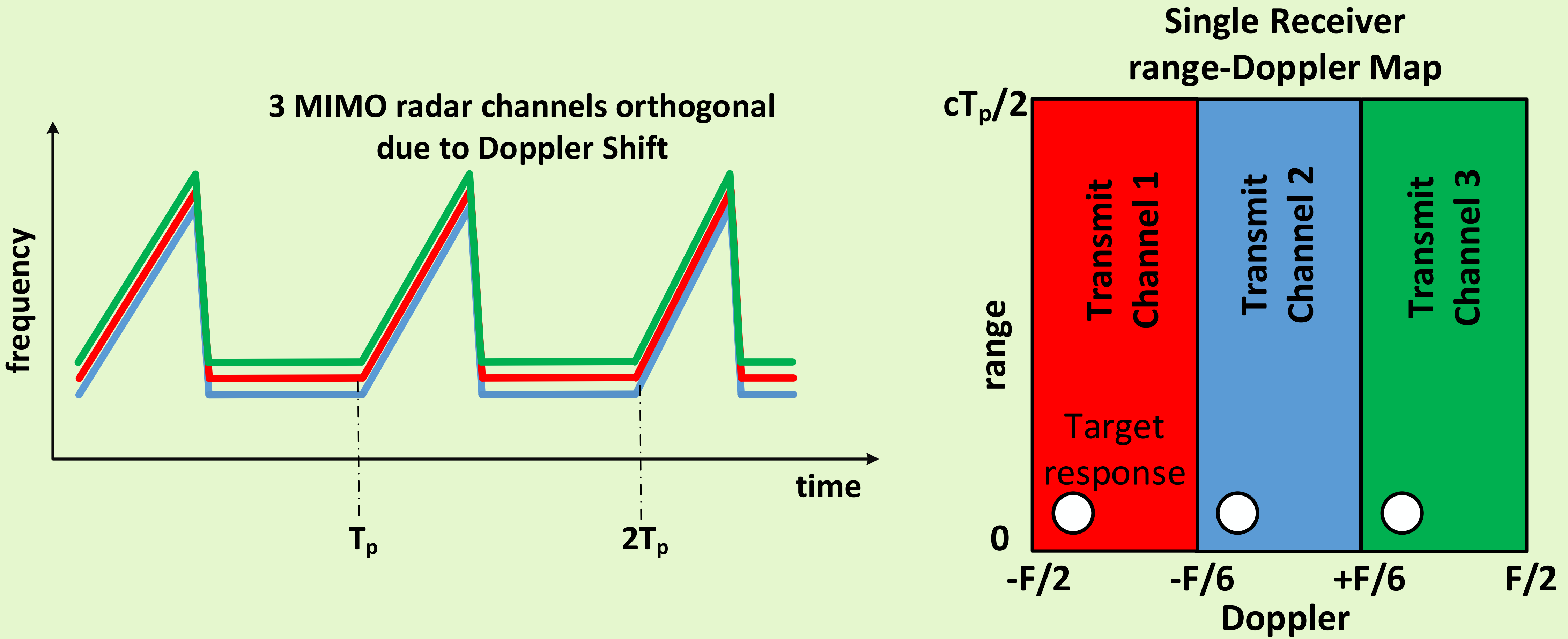
DDM - MIMO



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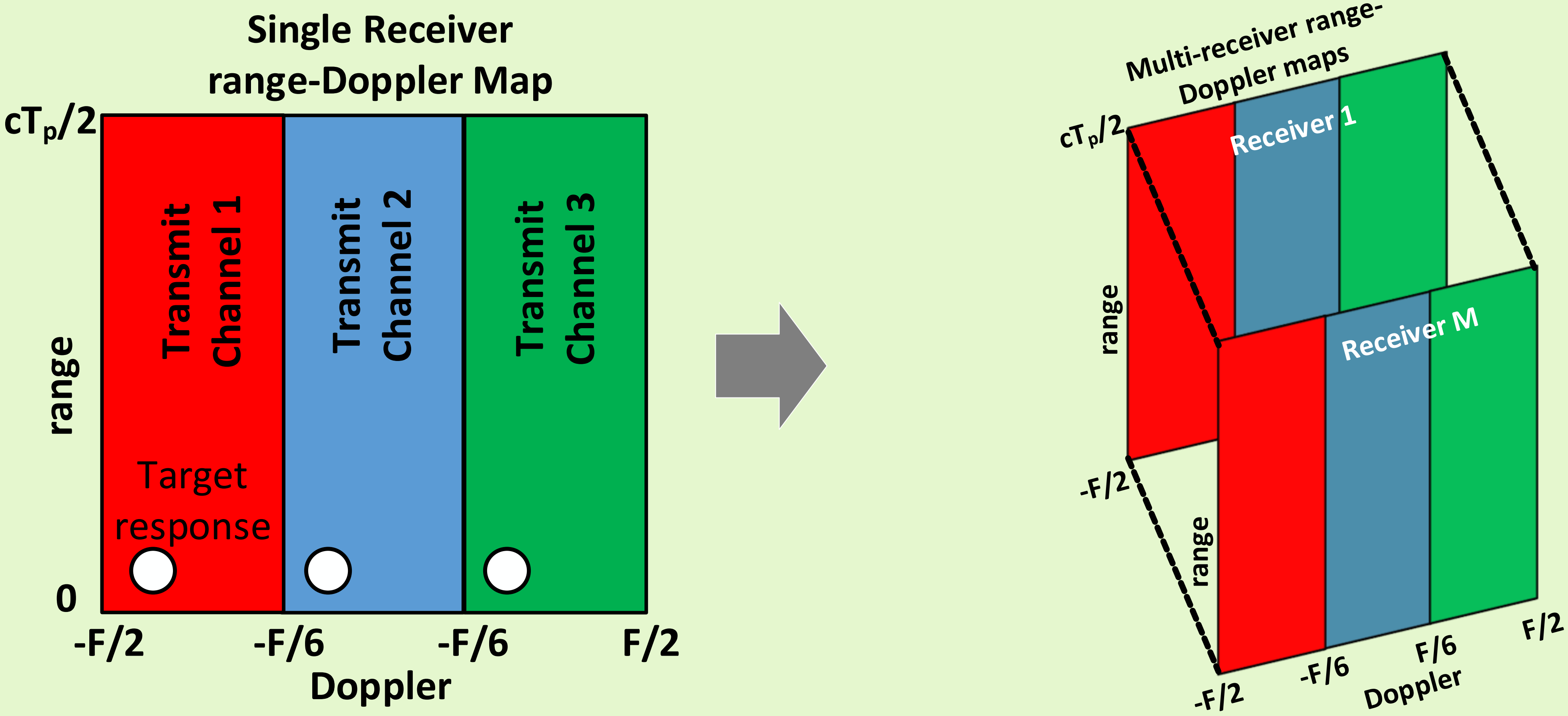
DDM - MIMO



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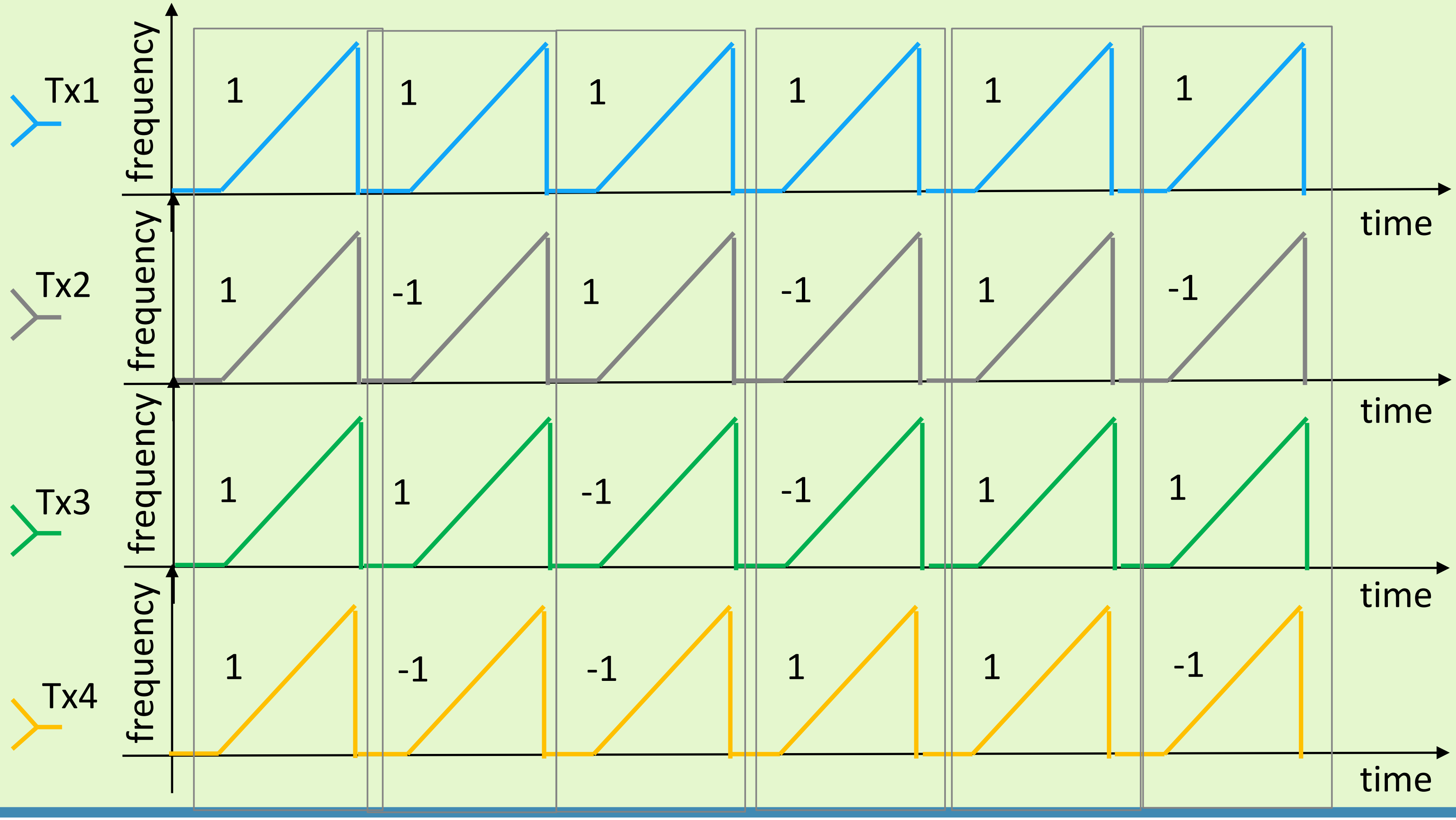
Binary Phase Modulation (BPM)



Hadamard Code

1	1	1	1
1	-1	1	-1
1	1	-1	-1
1	-1	-1	1

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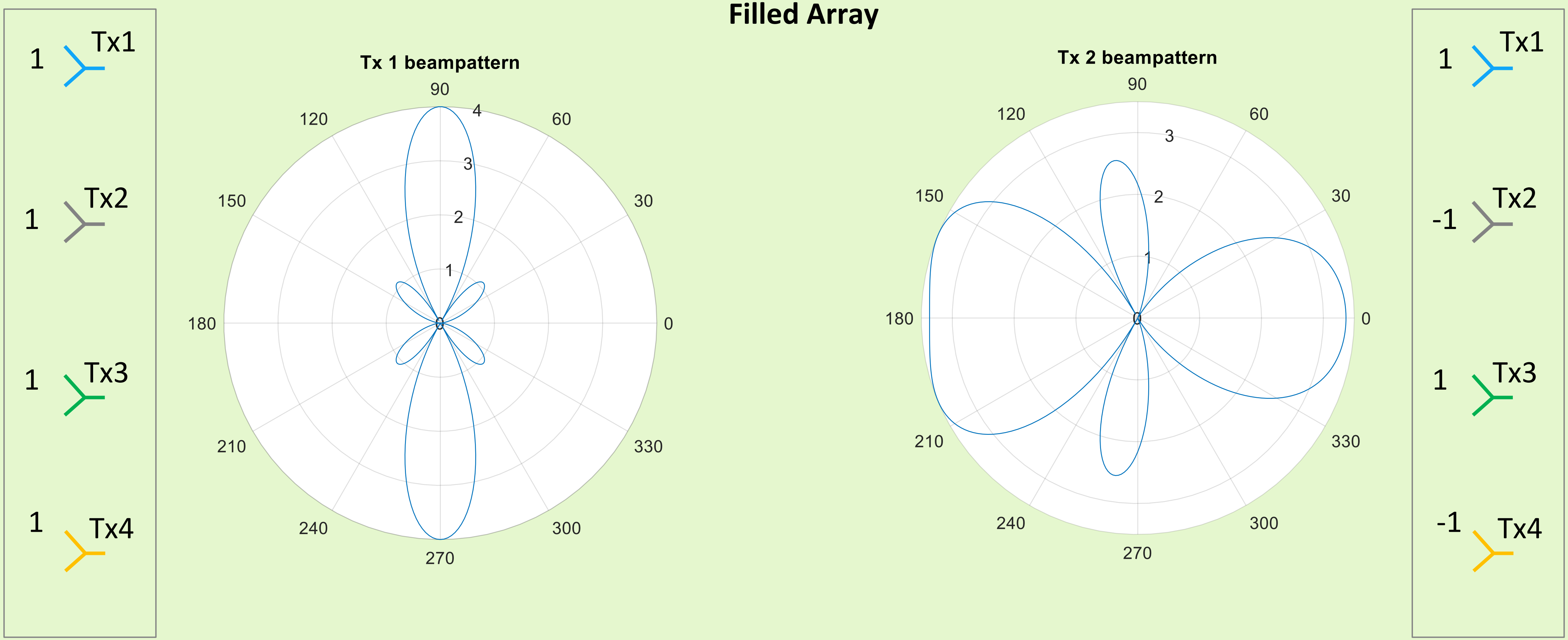
BPM - MIMO

Filled Array



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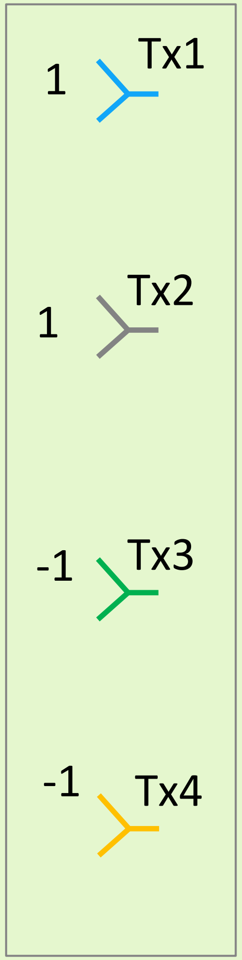
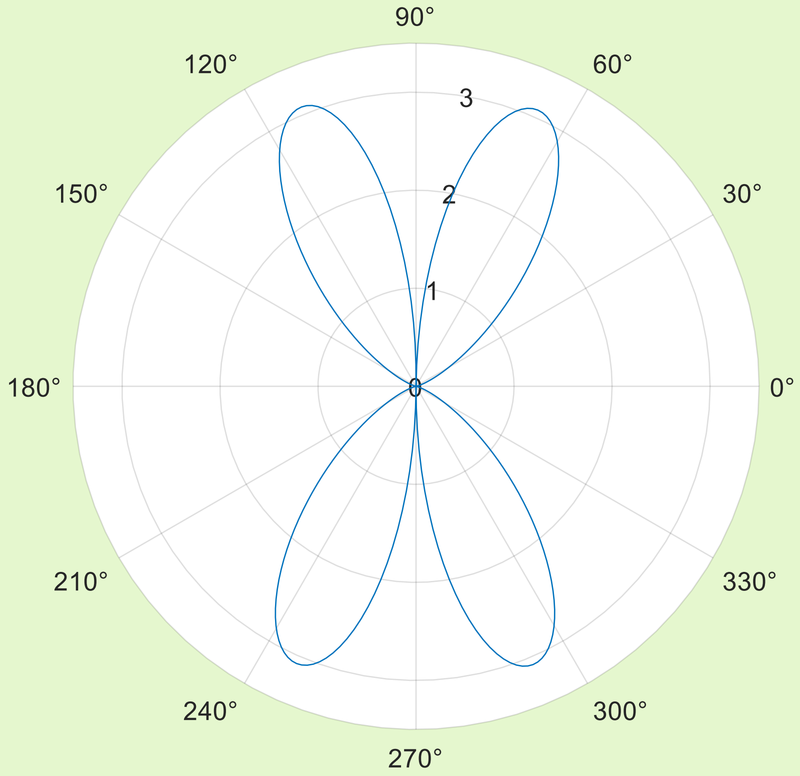
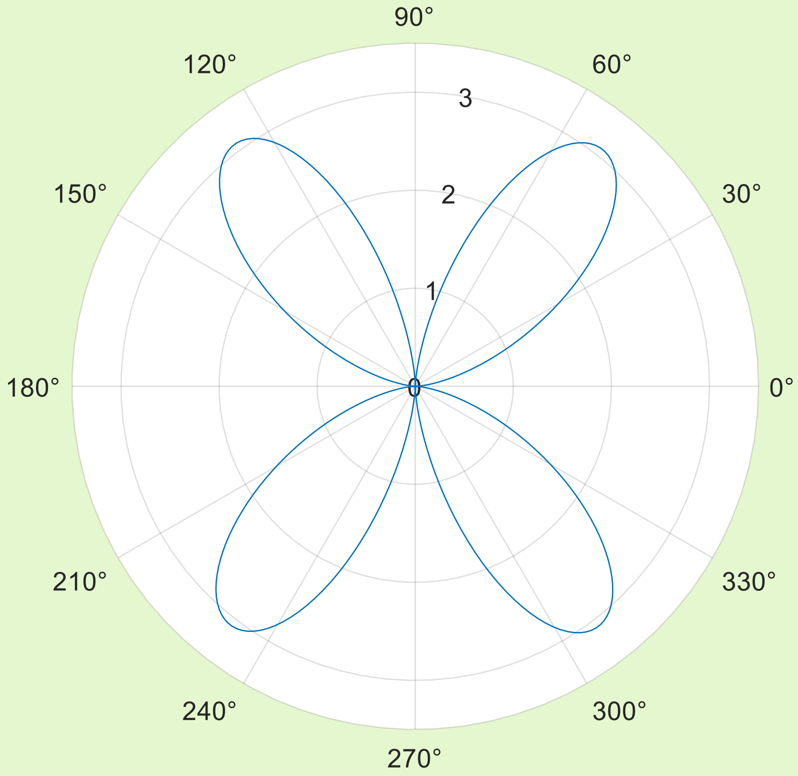
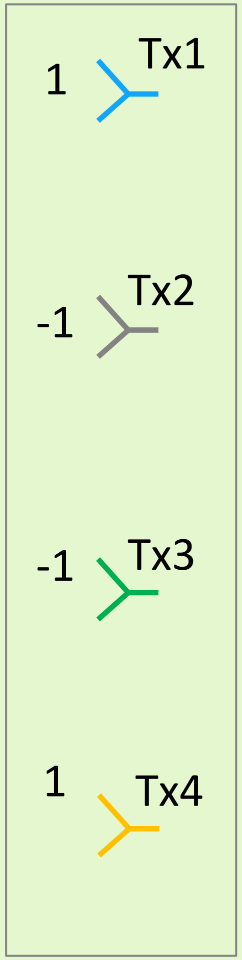
BPM - MIMO

Filled Array



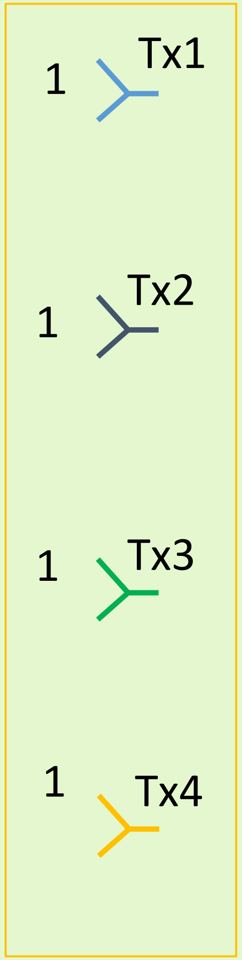


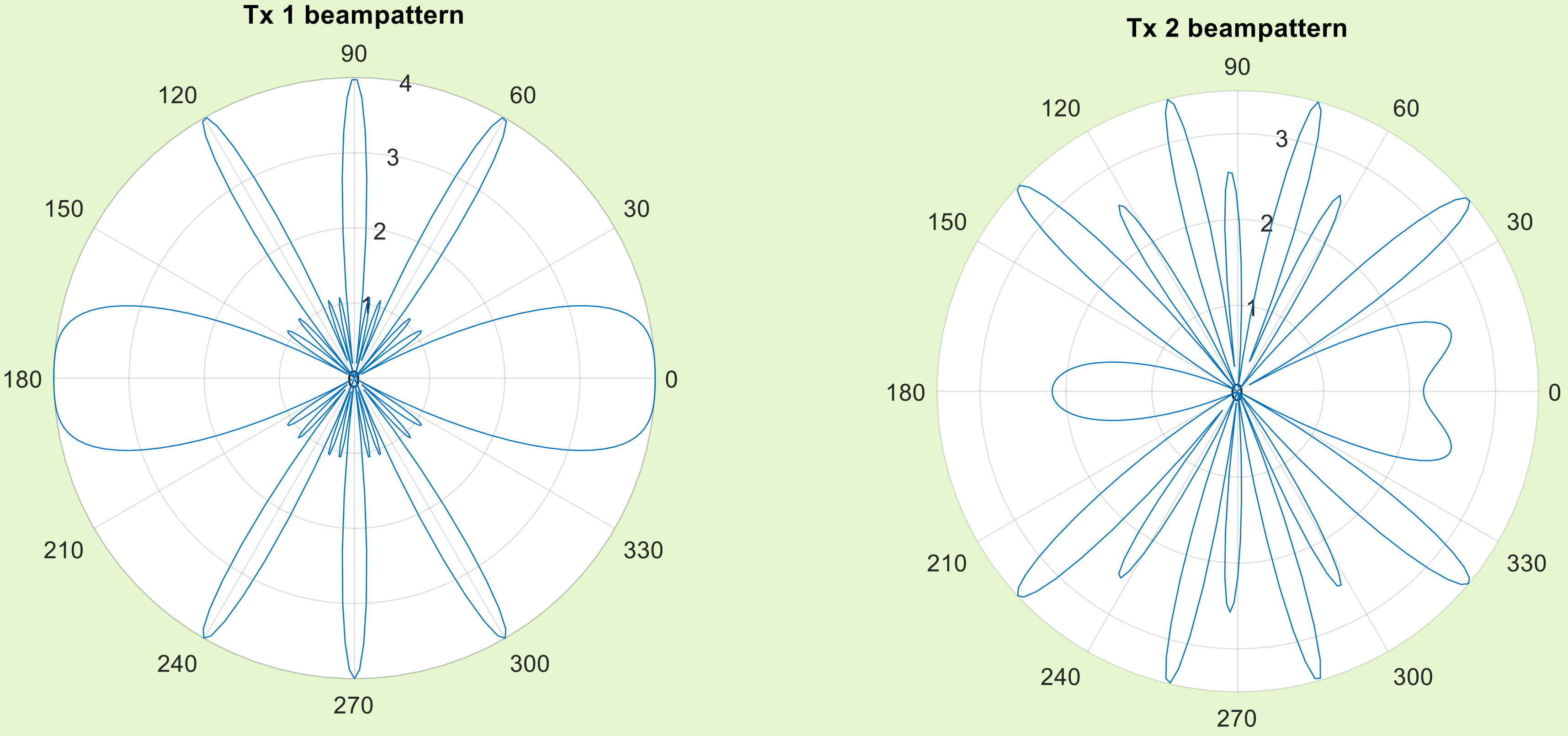


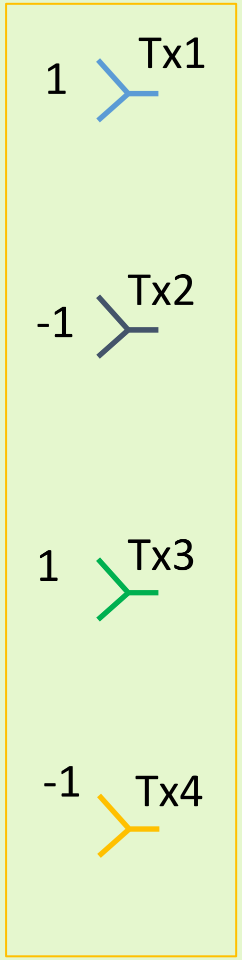


BPM - MIMO

Sparse Array







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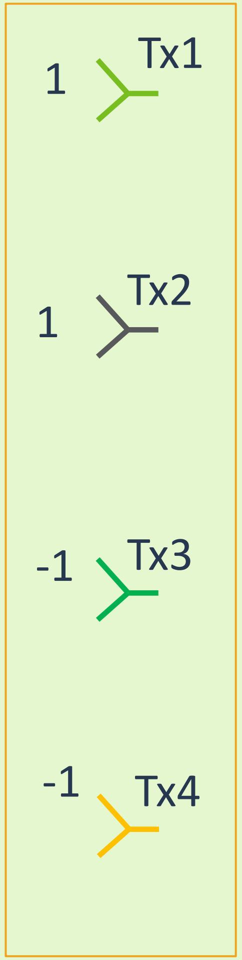
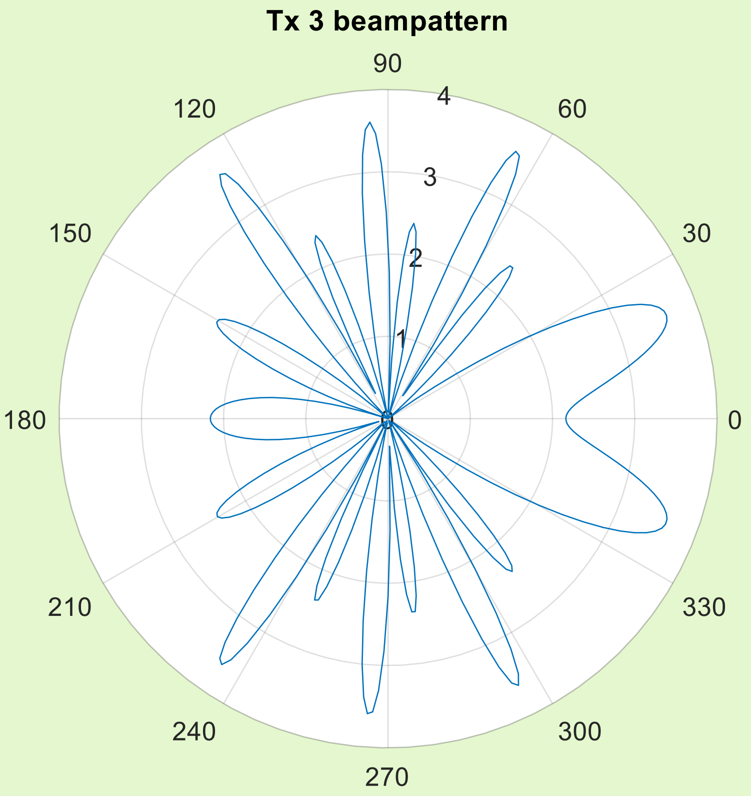
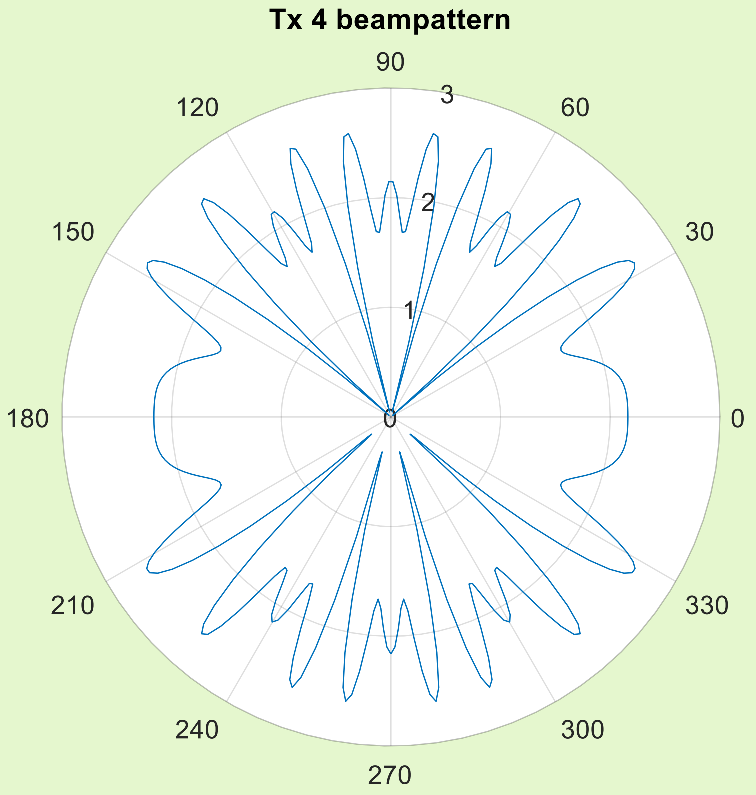
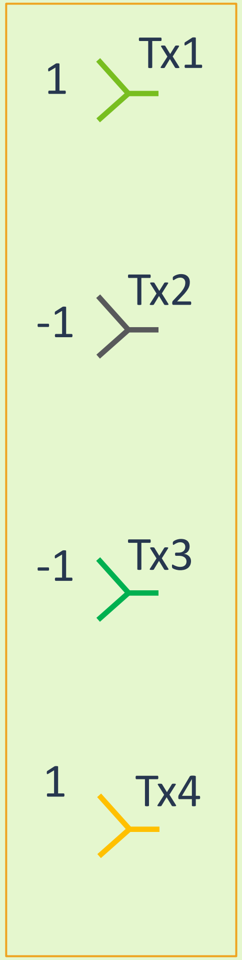
BPM - MIMO

Sparse Array









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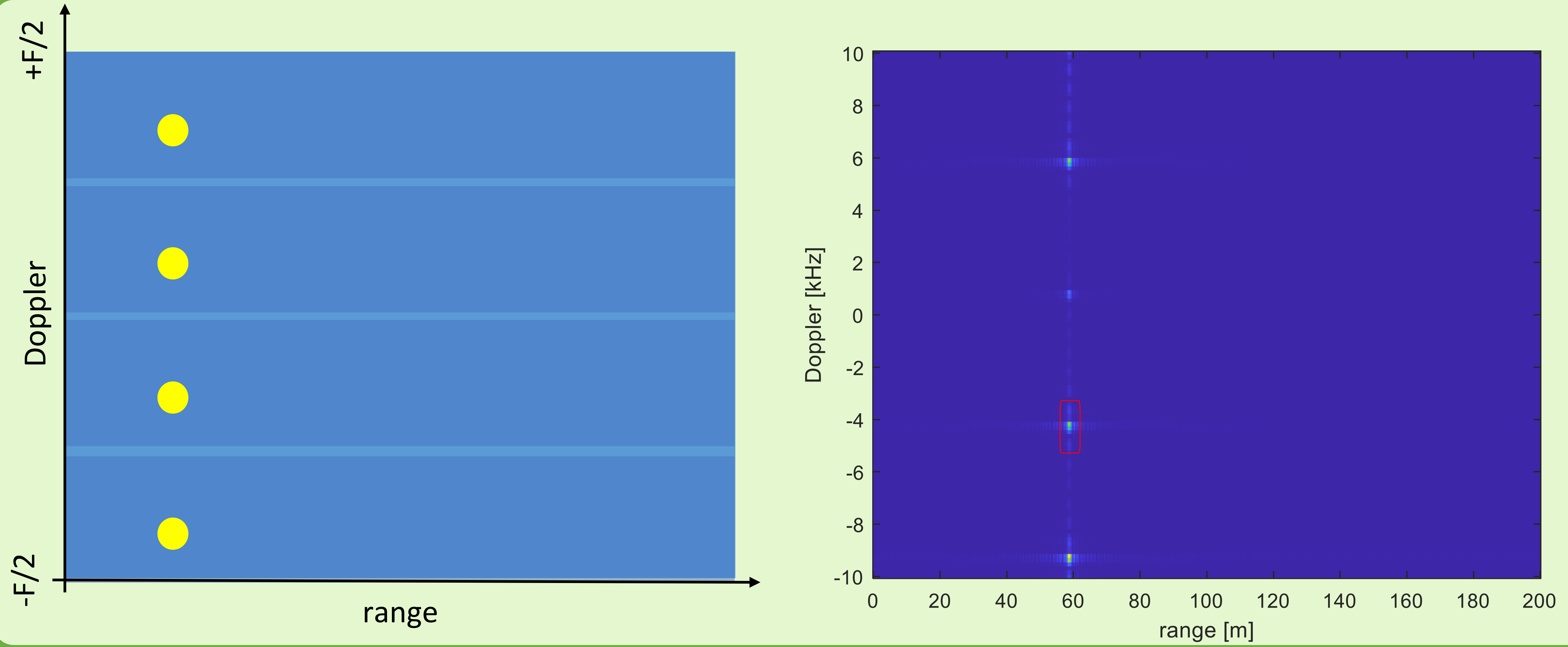
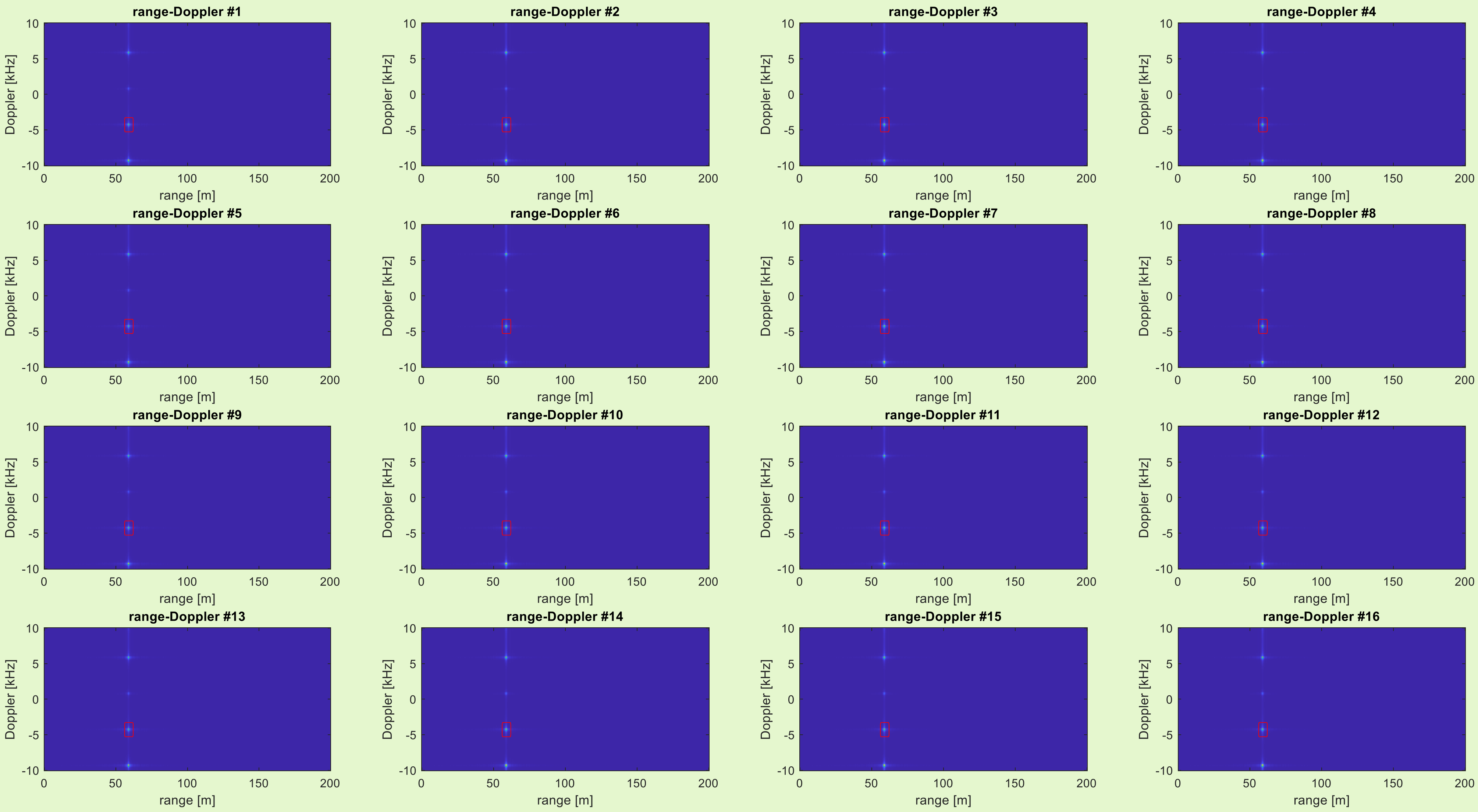
BPM - MIMO

Doppler aliased clutter and targets appear in other channels



BPM - MIMO

Q & A



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Waveform Design Techniques

Gradient-Descent Based Methods (GD)

Majorization-Minimization (MM)

Coordinate Descent (CD)

Block Successive Upper-bound Minimization (BSUM)

Alternating Direction Method of Multipliers (ADMM)

Several others ...

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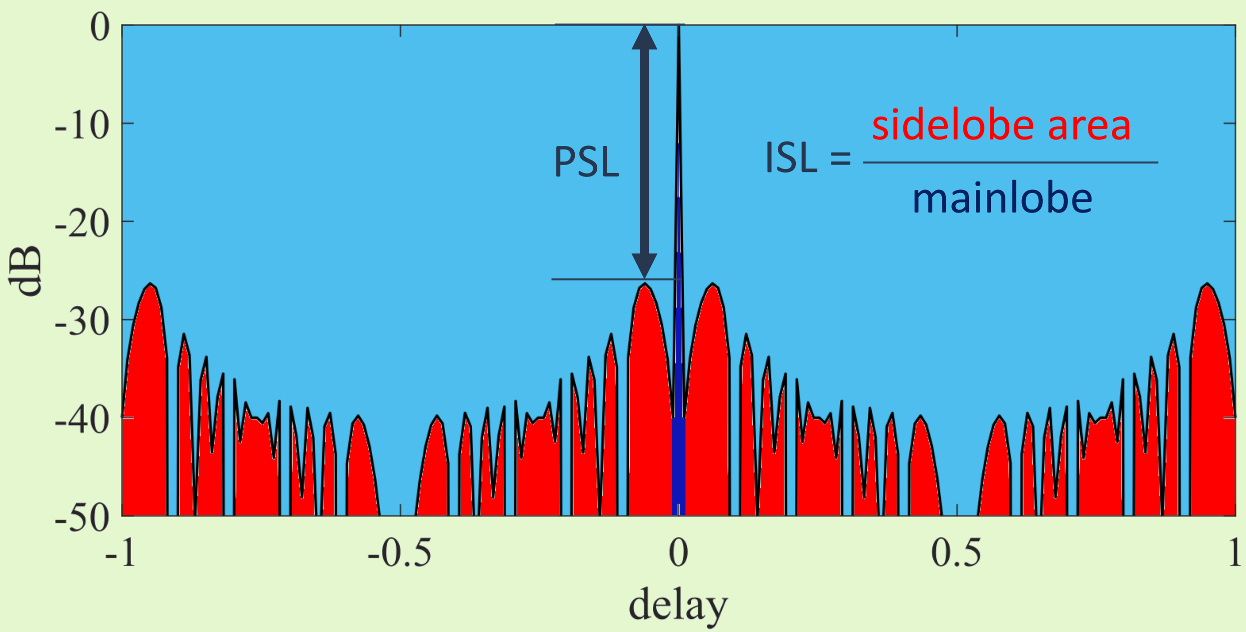
Metrics for Goodness of the Waveforms

Low PSL
- avoid masking of weak targets

Low ISL
- mitigate deleterious effects of distributed clutter

Good Waveform



Conceptual definition of PSL and ISL measured on autocorrelation function response of a Golomb sequence

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Mathematical Optimization

$$\boldsymbol{x} = [x_1, x_2, \dots, x_N]^T$$

Transmit waveform

Code length

$$r_k = \sum_{n=1}^{N-k} x_n^* x_{n+k}, \qquad k = 0.\dots N-1$$

$$\boxed{\mathrm{PSL} = \max_{k\neq 0} |r_k|}$$

$$\boxed{\mathrm{ISL} = \sum_{k=1}^{N-1} r_k^2}$$

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PSL Minimization Problem

$$\boldsymbol{x} = [x_1, x_2, \ldots, x_N]^T$$

$$\mathcal{P}_{\boldsymbol{x}} \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \max_{k \neq 0} |r_k| \\ \text{subject to} & x_n \in \psi_n \end{cases}$$

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Waveform Design Techniques

$$\boldsymbol{x} = [x_1, x_2, \ldots, x_N]^T$$

$$\mathcal{P}_{\boldsymbol{x}} \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \displaystyle\sum_{k=1}^{N-1} r_k^2 \\ \text{subject to} & x_n \in \psi_n \end{cases}$$

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Waveform Requirements

- Hardware perspective: costly, non-ideal
- Some factors need to consider when designing waveforms
- Two common waveform constraints: Unimodularity, finite phase value

Nonideal power amplifier:
limited linear region

Unimodular waveform: $|x_n| = 1, \forall n = 1, \ldots, N$

More general version

Peak to average power ration (PAPR):

$$\text{PAR}(\mathbf{x}) = \max_n \left\{ |x_n|^2 \right\} / \|\mathbf{x}\|_2^2 \leq \gamma$$

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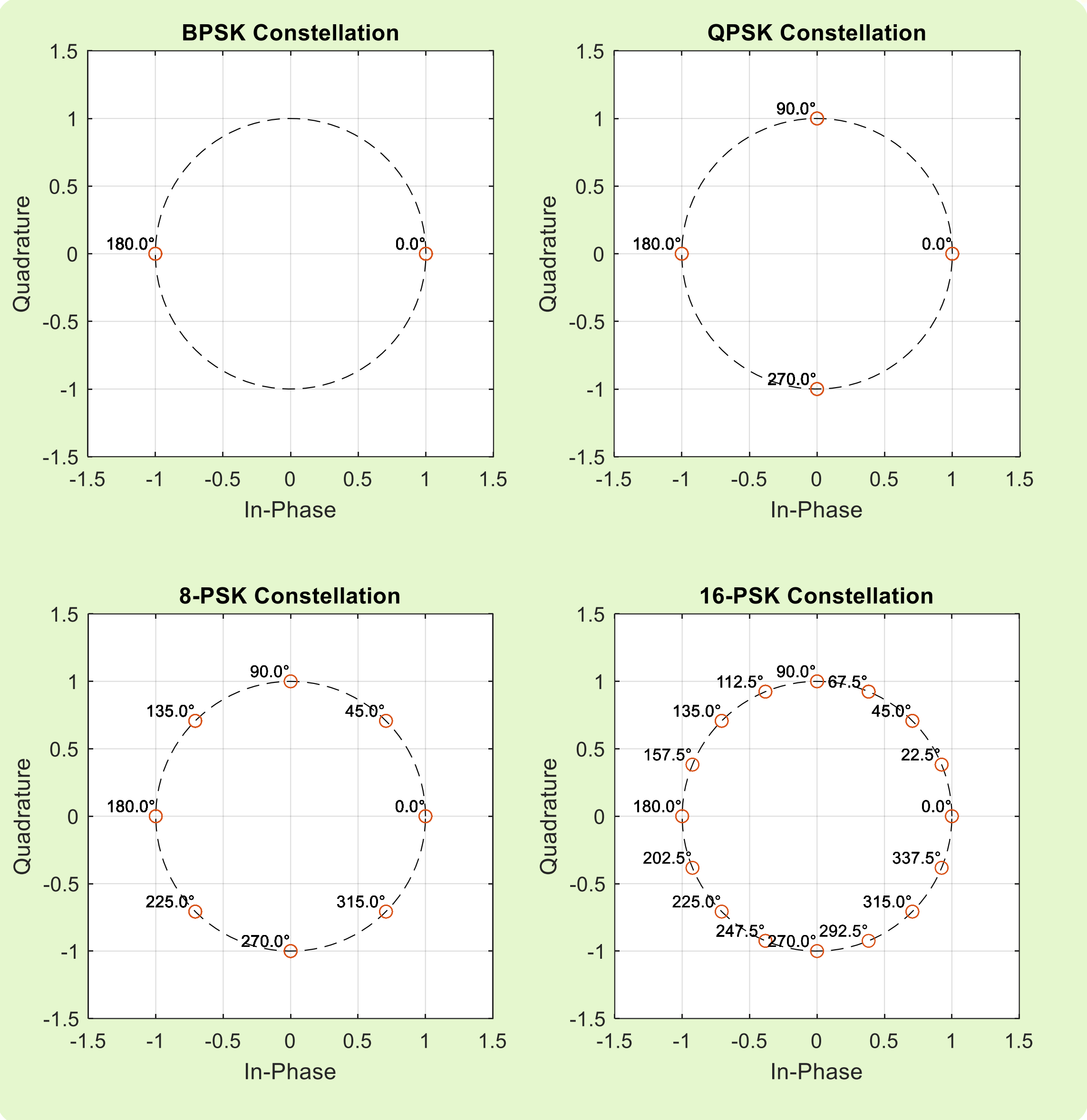
Waveform Requirements

- Generates finite phase values
- Phase quantization should be considered

Constraint on phase alphabet

$$x_n \in \Omega_M$$

$$\Omega_M = \left\{1, e^{\frac{j2\pi}{M}}, \ldots, e^{\frac{j2\pi(M-1)}{M}}\right\}$$



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How to solve the formulated waveform design problem?

objective function

Useful performance metrics:
PSL / ISL

Practical waveform requirement:
Unimodular, phase alphabet

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \text{ISL/PSL} \\ \text{subject to} & x_n \in \boldsymbol{\Omega_M} \end{cases}$$

constraints

How to solve the formulated waveform design problem?

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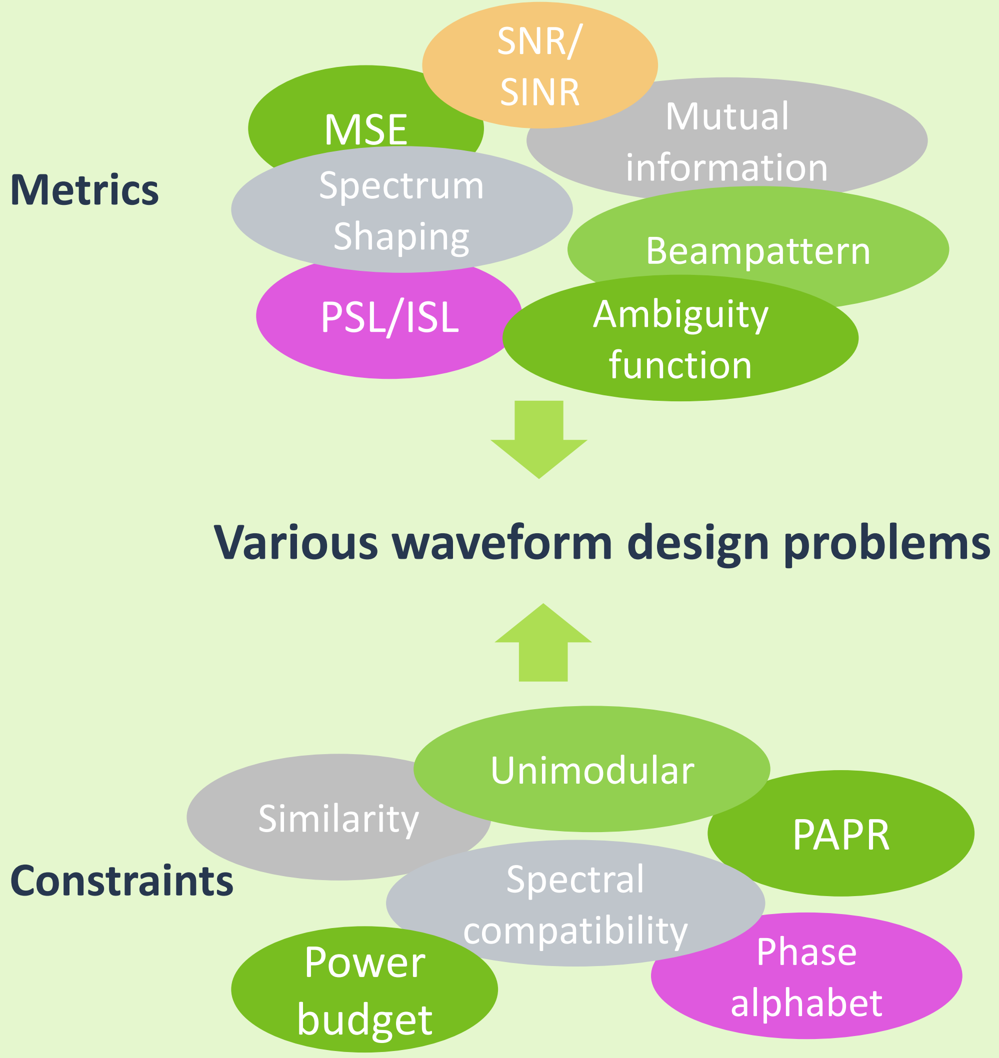
Beyond ISL/PSL



Algebraic construction: Frank sequence, Golomb sequence…

Heuristic constriction: exhaustive search, evolutionary algorithm, simulated annealing…

- Still cannot cover all needs
- Many problems are nonconvex and NP-hard
- High dimension if long sequence is needed
- Time efficiency matters

We focus on Optimization-based approach

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Recall ISL/PSL Problems

Waveform to be designed: $x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{C}^N$

	Unimodular	Phase alphabet
PSL	$\begin{cases} \underset{x}{\text{minimize}} & \max \{\lvert r_k \rvert\}_{k=1}^{N-1} \\ \text{subject to} & \lvert x_n \rvert = 1 \end{cases}$	$\begin{cases} \underset{x}{\text{minimize}} & \max \{\lvert r_k \rvert\}_{k=1}^{N-1} \\ \text{subject to} & x_n \in \Omega_M \end{cases}$
ISL	$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} \lvert r_k \rvert^2 \\ \text{subject to} & \lvert x_n \rvert = 1 \end{cases}$	$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} \lvert r_k \rvert^2 \\ \text{subject to} & x_n \in \Omega_M \end{cases}$

$\Omega_M = \left\{1, e^{\frac{j2\pi}{M}}, \ldots, e^{\frac{j2\pi(M-1)}{M}}\right\}$

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Gradient-Descent Based Methods (GD)

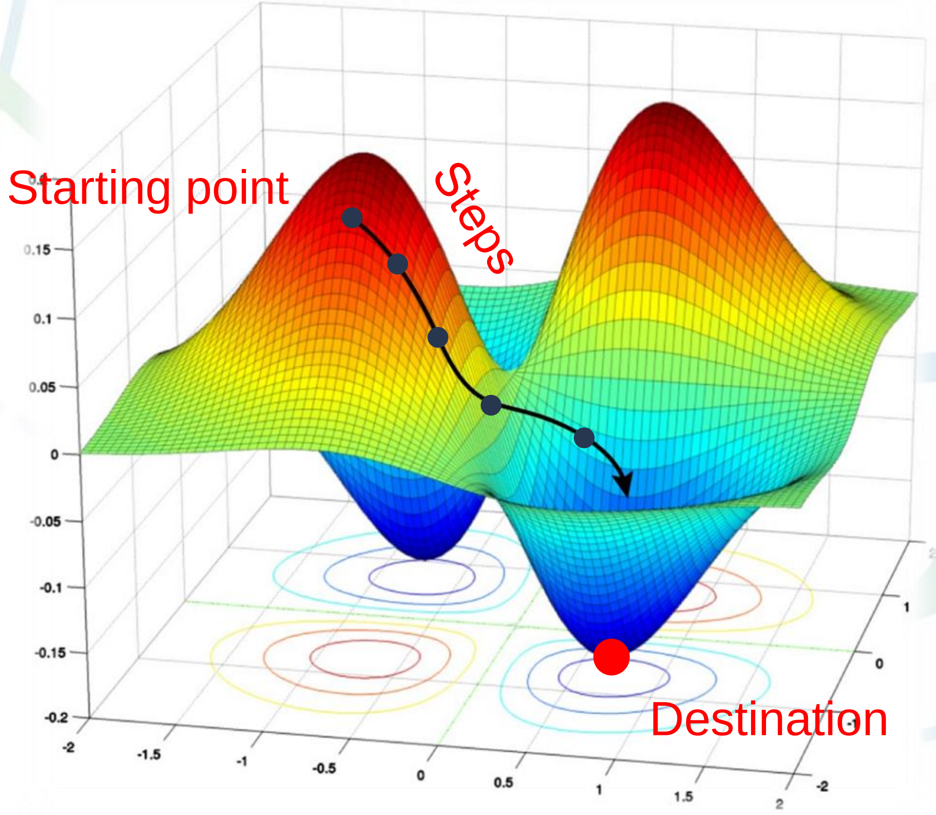
Unconstrained problem:

$$\underset{\boldsymbol{x}}{\text{minimize}} \qquad f(\boldsymbol{x})$$

Gradient descent (GD) is well-known

$$\underbrace{\mathbf{x}_{k+1}}_{\text{Updated point}} = \underbrace{\mathbf{x}_k}_{\text{Current point}} - \underbrace{\alpha}_{\text{Step-size}} \underbrace{\nabla f(\mathbf{x}_k)}_{\text{Direction}}$$

Iteratively repeat the update rule,
the sequence $\{\boldsymbol{x}_k\}$ **converge** at local optimum



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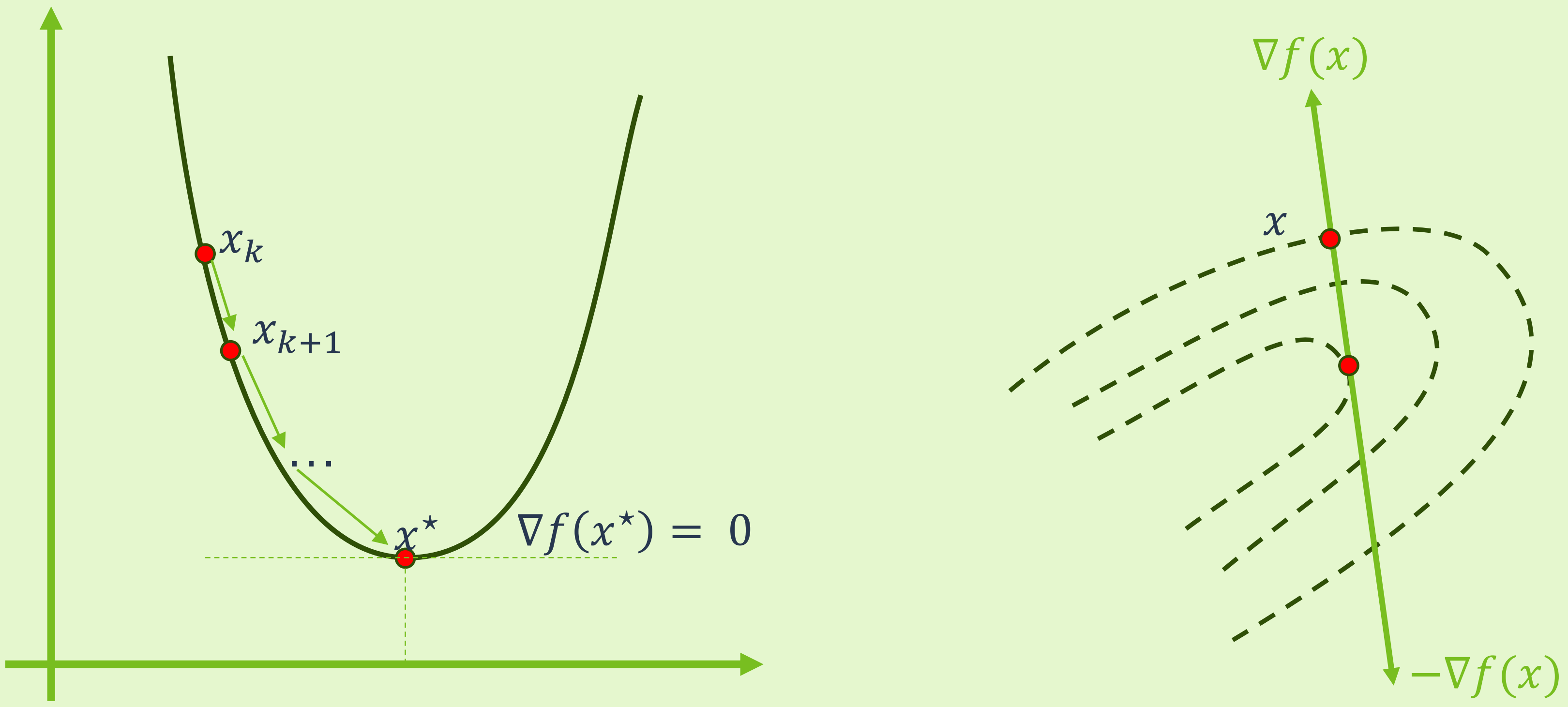
GD Based Methods

$$\underset{\boldsymbol{x}}{\text{minimize}} \qquad f(\boldsymbol{x})$$



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GD Based Methods – Algorithm

$$\underset{\boldsymbol{x}}{\text{minimize}} \qquad f(\boldsymbol{x})$$

1. Start with some guess $\boldsymbol{x}_0$
2. For each $k = 0, 1, \ldots$
   - $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha \nabla f(\boldsymbol{x}_k)$
   - Check when to stop (e. g. if $\nabla f(\boldsymbol{x}_{k+1}) = 0$)

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GD Based Methods

$$\underset{\boldsymbol{x}}{\text{minimize}} \qquad f(\boldsymbol{x})$$

$$\boxed{\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha \nabla f(\boldsymbol{x}_k), \qquad k = 0, 1, \ldots}$$

Stepsize $\alpha \geq 0$, usually ensures $f(\boldsymbol{x}_{k+1}) < f(\boldsymbol{x}_k)$

Numerous ways to select α

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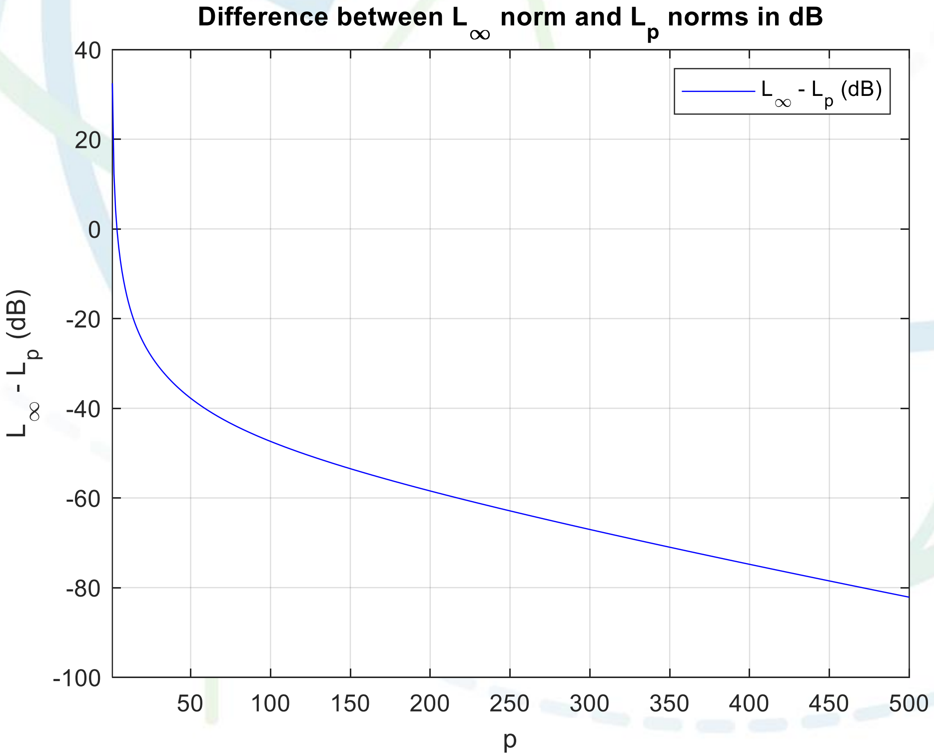
ℓ_p- Norm Minimization using GD

$$\ell_p \text{ norm: } \|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

$$\begin{cases} p = 2 : \|\mathbf{x}\|_2 = \sum_{n=1}^{N} |x_n|^2 & \text{ISL} \\ p = \infty : \|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i| & \text{PSL} \end{cases}$$

ℓ_∞ norm approximation: use a large value of p



A unified formulation:

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} |r_k|^p \\ \text{subject to} & x_n \in \Omega_M \end{cases}$$

How to solve this non-convex problem?

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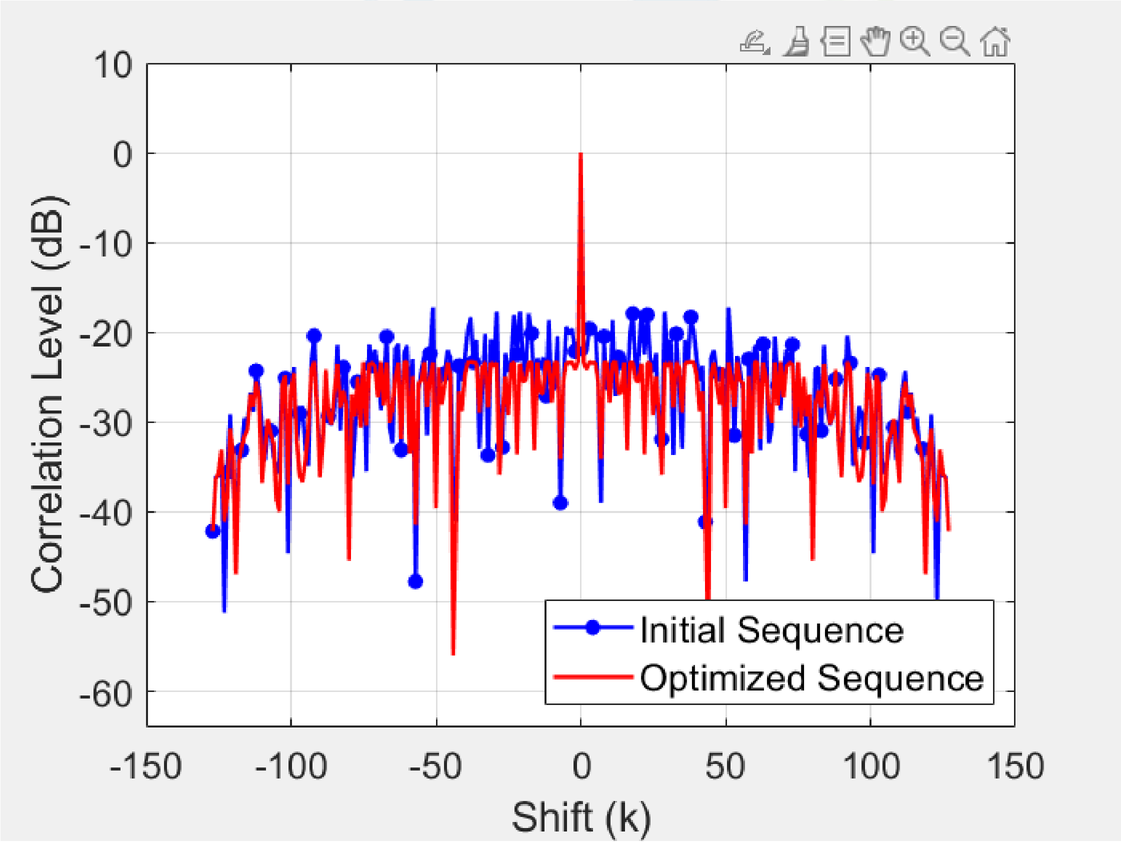
ℓ_p- Norm Minimization using GD

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \|r_k\|_p \\ \text{subject to} & |x_n| = 1 \end{cases}$$



Lect2_example7.m

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Not Easy for Waveform Design

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \boldsymbol{\mathcal{X}} \end{cases}$$

For waveform design problems,

- $f(\boldsymbol{x})$ can be complicated even non-differentiable
- $\boldsymbol{x}$ can be high-dimension → computational cost
- Some constraints to consider, i.e., $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$

We need more efficient optimization techniques

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Majorization-Minimization (MM)

An MM algorithm operates by creating a surrogate function that minorizes or majorizes the objective function. When the surrogate function is optimized, the objective function is driven uphill or downhill as needed.

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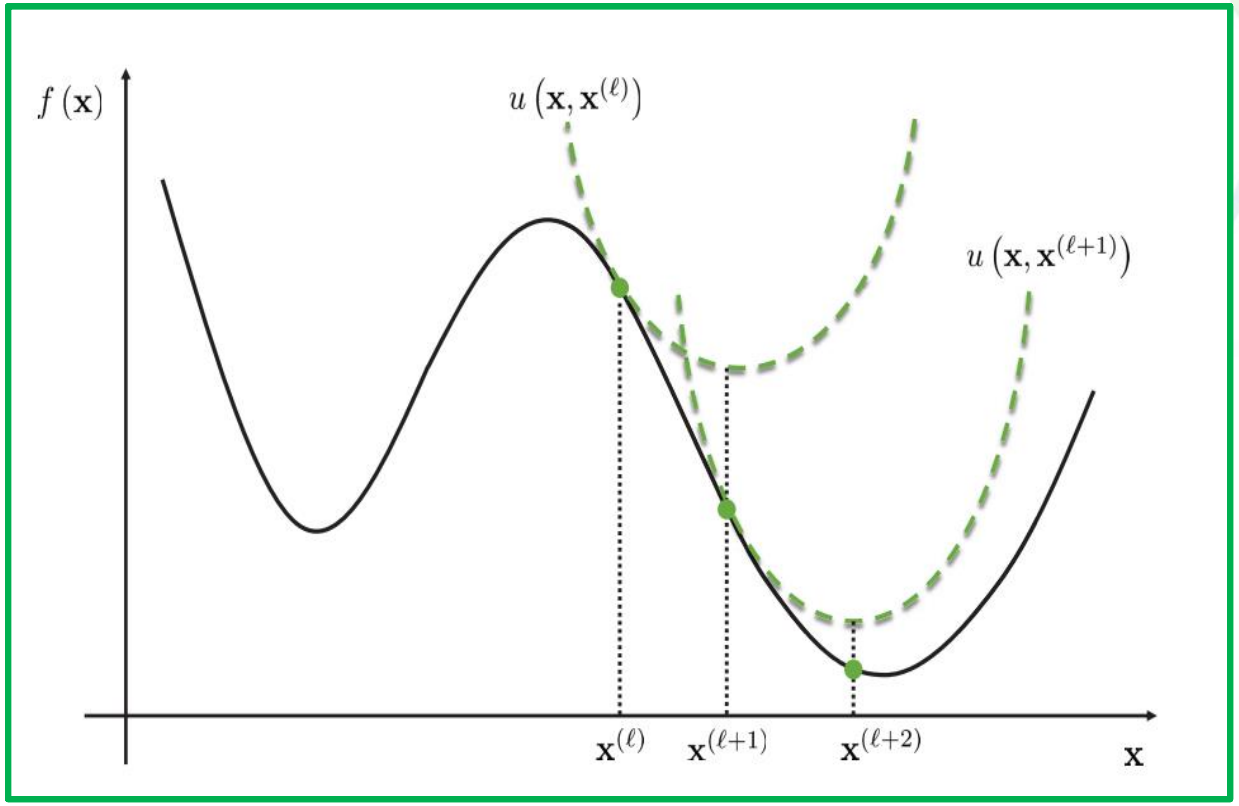
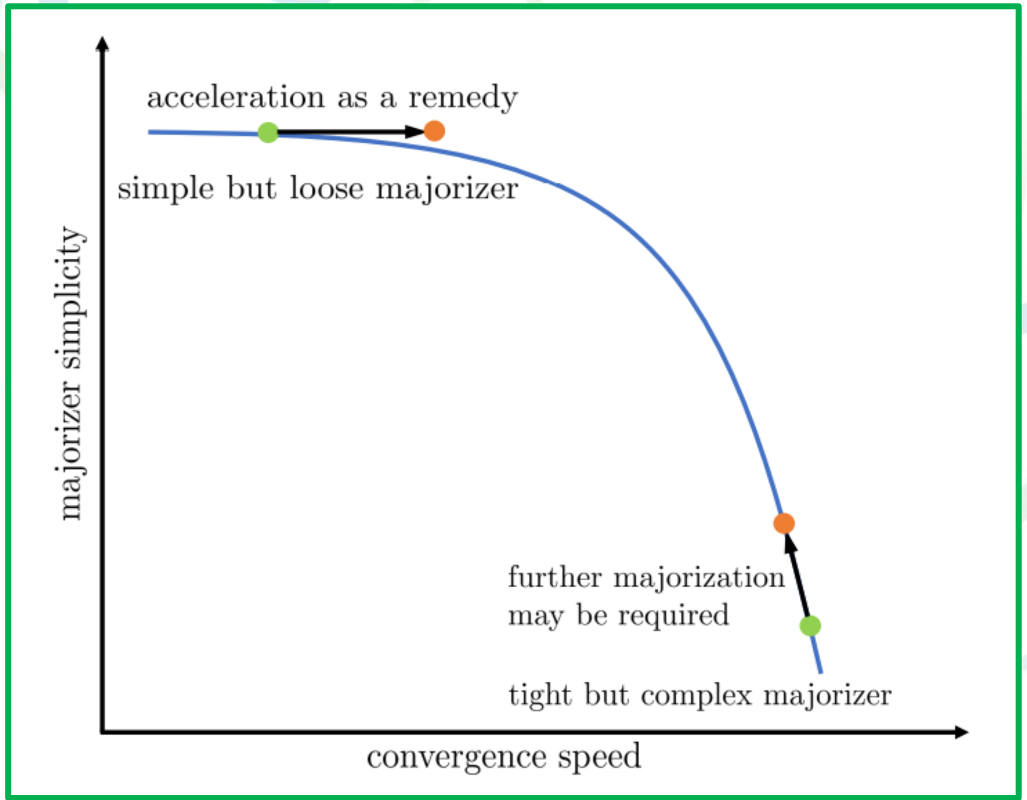
MM Method

Graphic illustration of MM



Simplicity versus convergence



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MM Example

Minimization of $\cos(x)$

Second order Taylor expansion

$$\cos(x) = \cos(x_n) - \sin(x_n)\,(x - x_n) - \frac{1}{2}\cos(z)\,(x - x_n)^2$$

Holds for some z between x and x_n

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Waveform Design Techniques

$$g(x|x_n) = \cos(x_n) - \sin(x_n)\,(x - x_n) + \frac{1}{2}(x - x_n)\text{^}2$$

Can be selected as majorizer that majorizes $f(x)$

Solving $\frac{d}{dx} g(x|x_n) = 0$ gives the MM algorithm

$$x_{n+1} = x_n + \sin(x_n)$$

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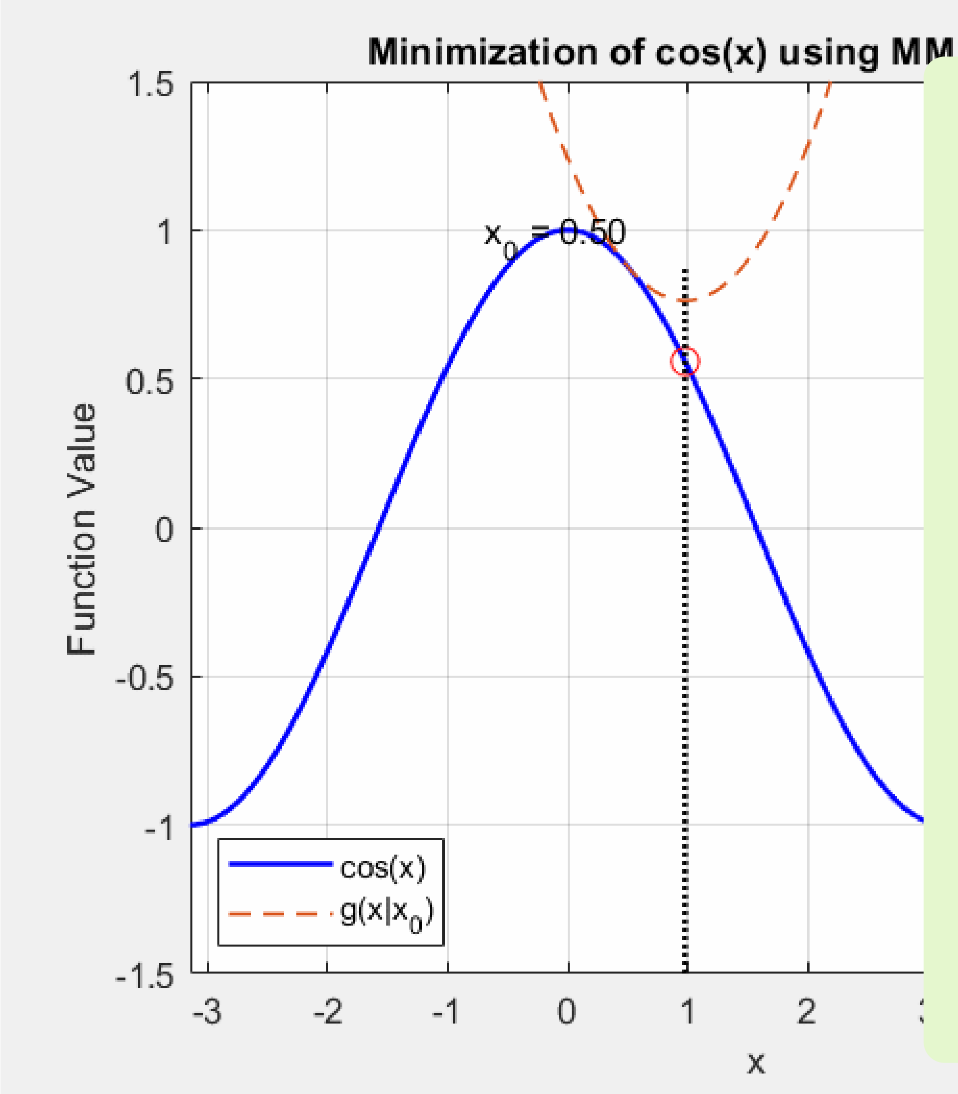
Minimum of cos(x) using MM



Input: $\boldsymbol{x}_0 \in \mathbb{C}^N$

1: **while** not converged **do**

2: Construct a surrogate function $g(\boldsymbol{x}|\boldsymbol{x}_n)$ of $f(\boldsymbol{x})$ at the current iteration

3: Minimize the surrogate to get the next iterate:

$$\boldsymbol{x}_{n+1} = \underset{\boldsymbol{x}}{\operatorname{argmin}}\, g(\boldsymbol{x}|\boldsymbol{x}_n)$$

4: $n \leftarrow n + 1$

5: **end while**

Output: The solution $\boldsymbol{x}_n$

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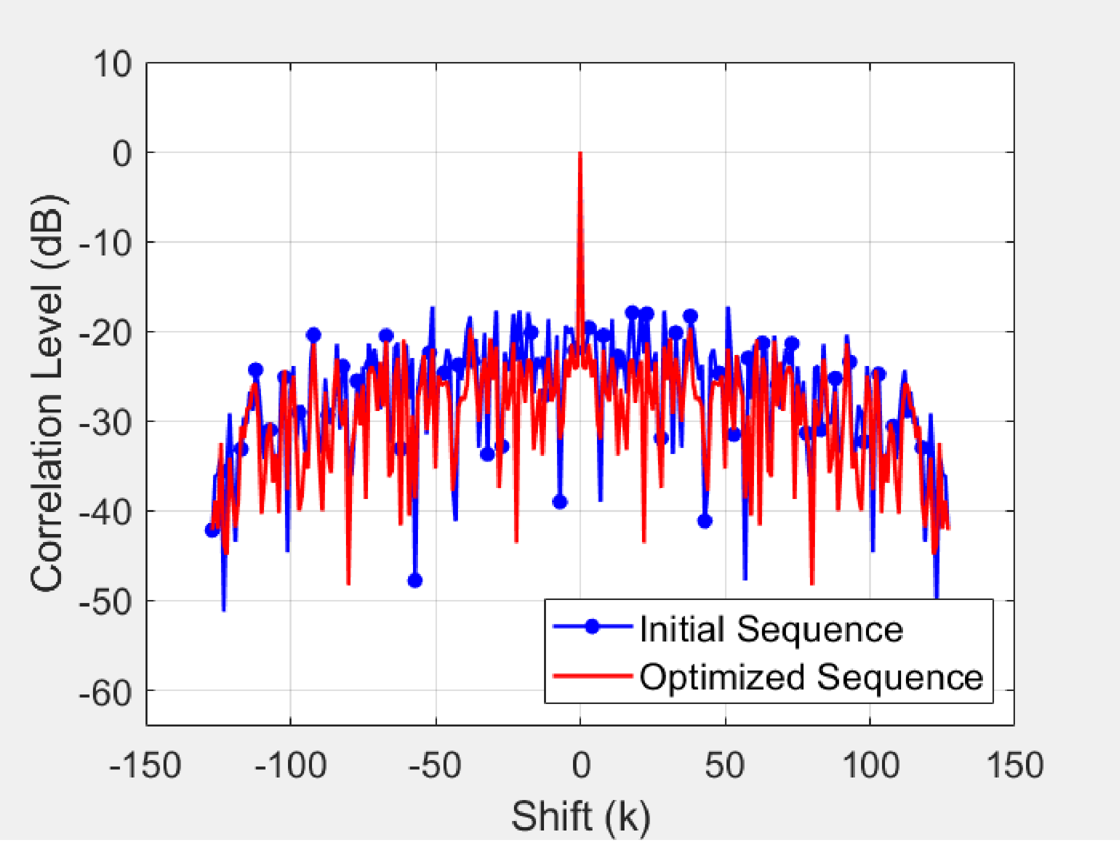
ISL Minimization Problem using MM

$$\mathrm{WISL} = \sum_{k=1}^{N-1} w_k |r_k|^2,$$

$$\begin{aligned} &\underset{x_n}{\text{minimize}} \quad \mathrm{WISL} \\ &\text{subject to} \quad |x_n| = 1, n = 1, \ldots, N, \end{aligned}$$



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Coordinate Descent (CD)

Successively minimizes along coordinate directions

Optimize each parameter separately, holding all the others fixed.

- ✓ Very simple and easy to implement
- ✓ Careful implementations can attain state-of-the-art
- ✓ Scalable, don't need to keep data in memory, low memory requirements
- ✓ Faster than gradient descent if iterations are N times cheaper

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CD idea

$$\boldsymbol{x} = [x_1, x_2, \dots, x_N]^T$$

$$\mathcal{P}_{\boldsymbol{x}} \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & x_n \in \psi_n \end{cases}$$

idea: optimize over individual coordinates

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CD Algorithm

$$x_1^{(k)} \in \arg\min_{x_1} f\left(x_1, x_2^{(k-1)}, x_3^{(k-1)}, \dots, x_N^{(k-1)}\right)$$

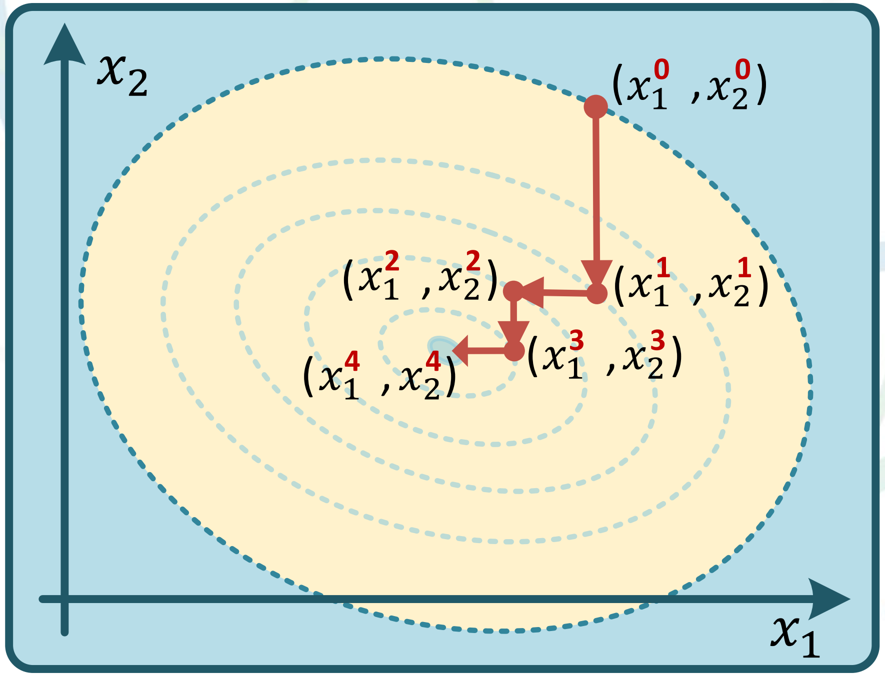
$$x_2^{(k)} \in \arg\min_{x_2} f\left(x_1^{(k)}, x_2, x_3^{(k-1)}, \dots, x_N^{(k-1)}\right)$$

$$x_3^{(k)} \in \arg\min_{x_3} f\left(x_1^{(k)}, x_2^{(k)}, x_3, \dots, x_N^{(k-1)}\right)$$

$$\vdots$$

$$x_N^{(k)} \in \arg\min_{x_N} f\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \dots, x_N\right)$$

Successively minimizes along coordinate directions



$$y = x_1^2 + 2\,x_2^2 - 9$$

No stepsize tuning!

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Variable update rule

Gauss-Seidel style (One-at-a-time)

$$x_i^{(k+1)} \leftarrow \arg\min_{\zeta} f(x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, \zeta, x_{i+1}^{(k)}, \ldots, x_N^{(k)})$$

Jacobi style (all-at-once ; easy to parallelize)

$$x_i^{(k+1)} \leftarrow \arg\min_{\zeta} f(x_1^{(k)}, \ldots, x_{i-1}^{(k)}, \zeta, x_{i+1}^{(k)}, \ldots, x_N^{(k)})$$

- **Maximum Block Improvement**
  - For differentiable f, pick the index that minimizes $\nabla f(x_i^k)$
- **Various update order:**
  - **Cyclic order:** $1, 2, \ldots, N, 1, \ldots$
  - **Double sweep**: $1, 2, \ldots, N$, then $N - 1, \ldots, 1$, repeat
  - **Cyclic with permutation**: random order each cycle
  - **Random sampling**: pick random index at each iteration

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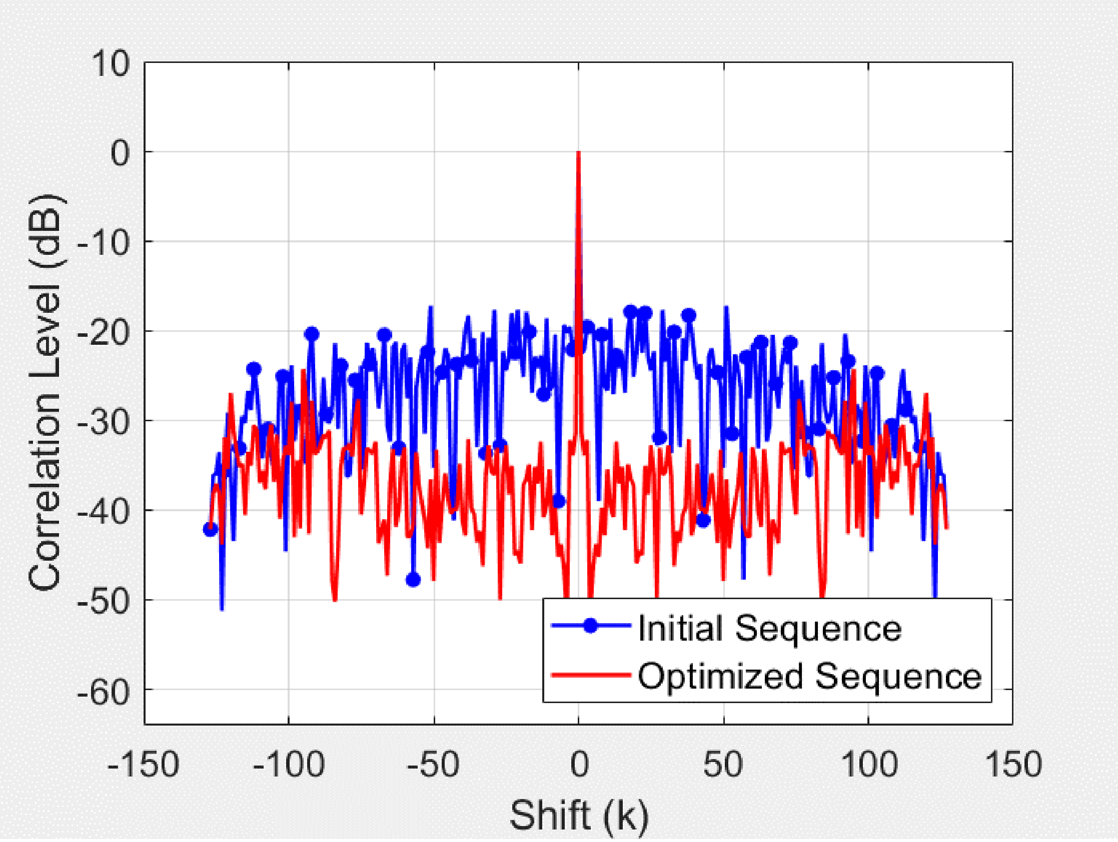
ISL Minimization Problem using CD

$$\boldsymbol{x} = [x_1, x_2, \ldots, x_N]^T$$

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} r_k^2 \\ \text{subject to} & x_n \in \psi_n \end{cases}$$



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CD Advantages

- ❑ Each iteration is usually cheap (single variable optimization)
- ❑ No extra storage vectors needed
- ❑ No stepsize tuning
- ❑ No other parameters that must be tuned
- ❑ In general, “derivative free”
- ❑ Simple to implement
- ❑ Works well for large-scale problems
- ❑ Currently quite popular; parallel version exist

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Alternative optimization

2 blocks CD is called alternative optimization

$$\boldsymbol{x} = [x_1, x_2]^T$$

$$x_1 \downarrow x \qquad x_2 \downarrow w$$

$$\mathcal{P}_{x,w}\begin{cases}\underset{x,w}{\text{minimize}} & f(x,w)\\ \text{subject to} & x \in \psi_1 , w \in \psi_2\end{cases}$$

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Block MM/BSUM

$$\boldsymbol{x} = [x_1, x_2, \ldots., x_N]^T$$

$$\mathcal{P}_{\boldsymbol{x}} \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & x_n \in \psi_n \end{cases}$$

$$x_i^{(k+1)} \leftarrow \arg\min_{\zeta} u_i(x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, \zeta, x_{i+1}^{(k)}, \ldots, x_N^{(k)})$$

Local approximation of the objective function

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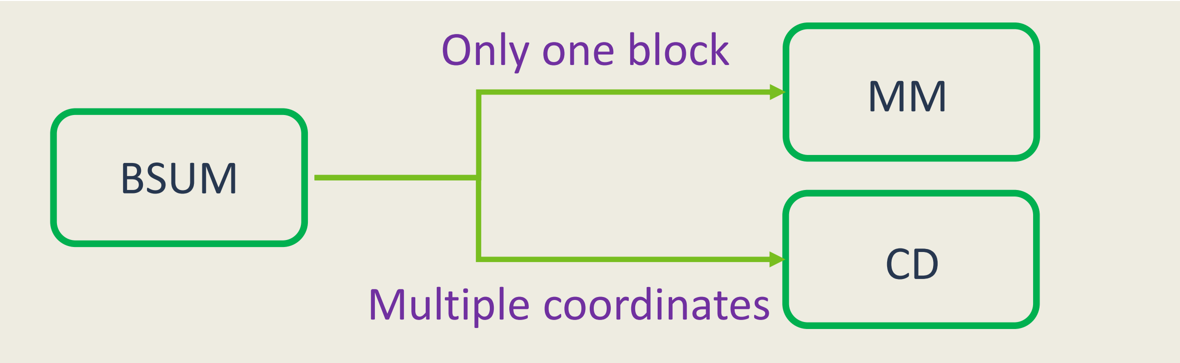
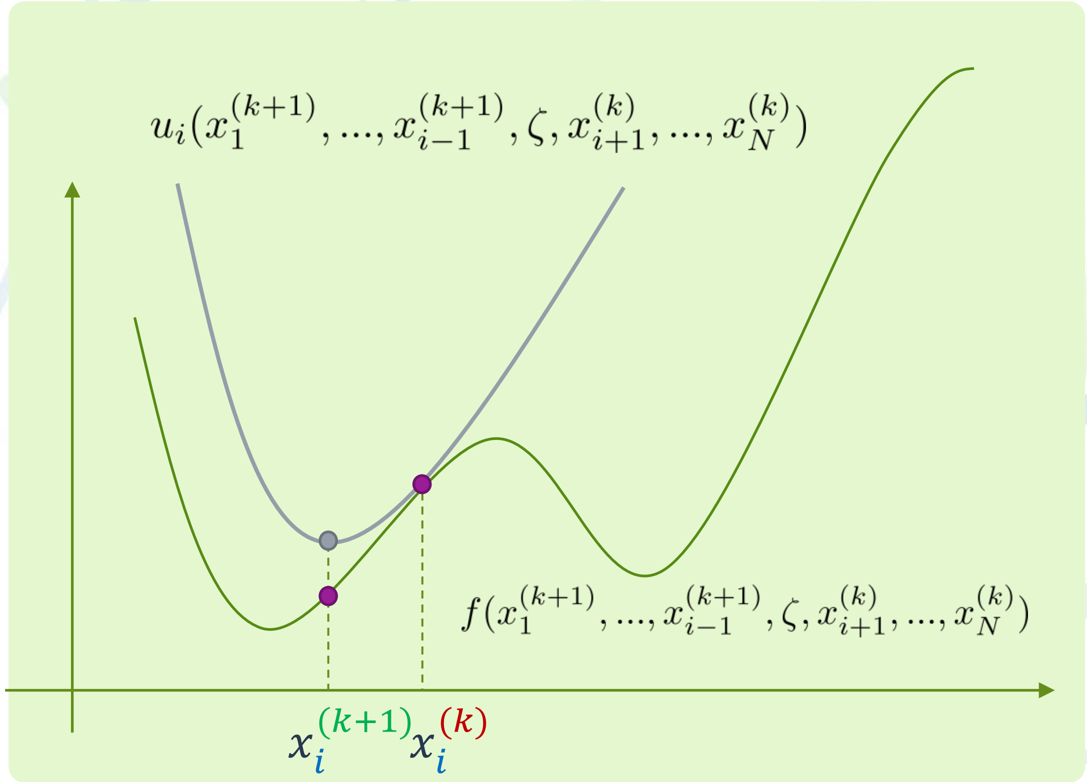
Block MM/BSUM

$$\begin{cases} \underset{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N}{\text{minimize}} & f\left(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N\right) \\ \text{subject to} & \boldsymbol{x}_n \in \mathcal{X}_n, n=1,\ldots,N \end{cases}$$



$$x_i^{(k+1)} \leftarrow \arg\min_{\zeta} u_i(x_1^{(k+1)},\ldots,x_{i-1}^{(k+1)},\zeta,x_{i+1}^{(k)},\ldots,x_N^{(k)})$$

Majorizer/upper bound of the objective function



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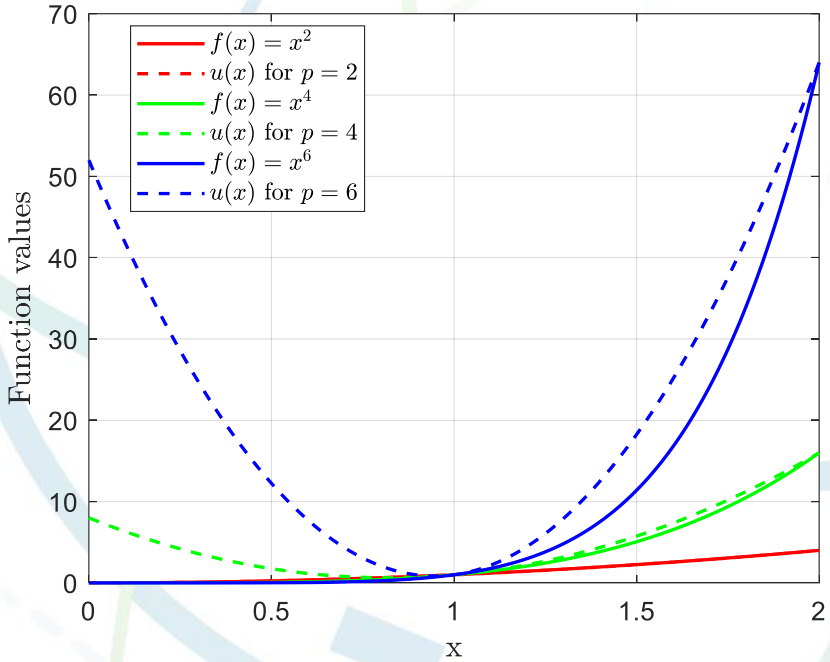
Block MM/BSUM for ℓ_p- Norm Minimization

Majorizer of $f(x) = x^p, \quad x \in [0, t]$ with $p \geq 2$

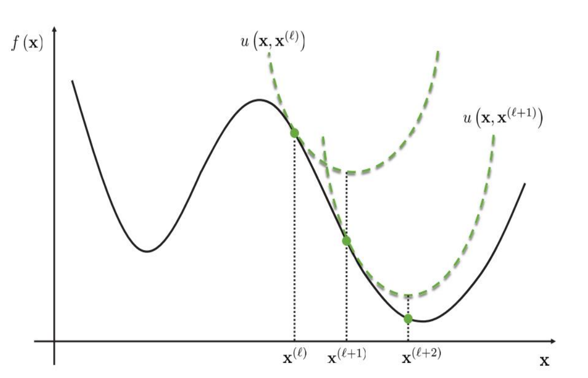
$$u(x) = ax^2 + \left(px_0^{p-1} - 2ax_0\right)x + ax_0^2 - (p-1)x_0^p$$

$$a = \frac{t^p - x_0^p - px_0^{p-1}(t - x_0)}{(t - x_0)^2}$$



At each iteration, we solve

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} |r_k|^p \\ \text{subject to} & x_n \in \Omega_M \end{cases} \Rightarrow \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} a_k |r_k|^2 + \sum_{k=1}^{N-1} b_k |r_k| \\ \text{subject to} & x_n \in \Omega_M \end{cases}$$



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Use CD for the Majorized Problem

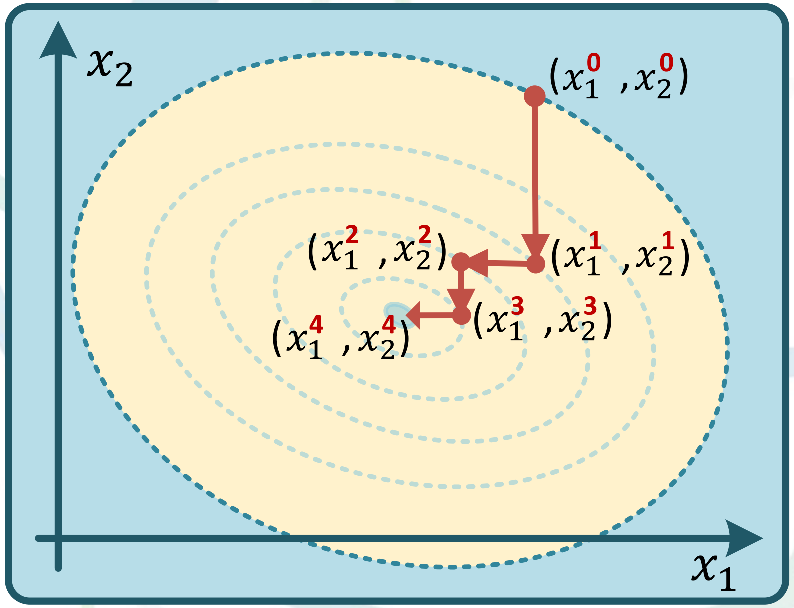
$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} a_k |r_k|^2 + \sum_{k=1}^{N-1} b_k |r_k| \\ \text{subject to} & x_n \in \Omega_M \end{cases}$$

$$x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{C}^N$$

x_d → Only variable to optimize

$$\mathbf{x}_{-d} = \left[x_1^{(i+1)}, \ldots, x_{d-1}^{(i+1)}, 0, x_{d+1}^{(i)}, \ldots x_N^{(i+1)}\right]^T \in \mathbb{C}^N$$

$$r_k(x_d) = a_{1k} x_d + a_{2k} x_d^* + a_{3k}$$



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Find the Optimal Phase

$$x_d \in \Omega_M$$
$$\Omega_M = \left\{1, e^{\frac{j2\pi}{M}}, \ldots, e^{\frac{j2\pi(M-1)}{M}}\right\}$$

$\longrightarrow$

$$x_d = e^{j\phi_d}$$
$$\widetilde{r}_k(\phi_d) = a_{1k}e^{j\phi_d} + a_{2k}e^{-j\phi_d} + a_{3k}$$

$$\widetilde{\mathcal{H}}_h^{(i+1)} \begin{cases} \min_{\phi_d} & \sum_{k=1}^{N-1} a_k \left|\widetilde{r}_k(\phi_d)\right|^2 + \sum_{k=1}^{N-1} b_k \operatorname{Re}\left\{\widetilde{r}_k(\phi_d)^* \frac{r_k^{(\ell)}}{\left|r_k^{(\ell)}\right|}\right\} \\ \text{s.t.} & \phi_d \in \Phi_M = \left\{0, \frac{2\pi}{M}, \frac{4\pi}{M}, \ldots, \frac{2\pi(M-1)}{M}\right\} \end{cases}$$

$$\beta_d = \tan\left(\frac{\phi_d}{2}\right) \qquad |\widetilde{r}_k(\phi_d)|^2 = \frac{\widetilde{p}_k(\beta_d)}{q(\beta_d)} \qquad \operatorname{Re}\left\{\widetilde{r}_k^*(\beta_d)\frac{r_k^{(i)}}{\left|r_k^{(i)}\right|}\right\} = \frac{\bar{p}_k(\beta_d)}{q(\beta_d)}$$

$$\begin{cases} \min\limits_{\beta_d} & \frac{1}{q(\beta_d)} \sum\limits_{k=1}^{N-1} a_k \widetilde{p}_k(\beta_d) + b_k \bar{p}_k(\beta_d) \\ s.t. & \beta_d \in B \end{cases}$$

$$\widetilde{p}_k(\beta_d) = \mu_{1k}\beta_d^4 + \mu_{2k}\beta_d^3 + \mu_{3k}\beta_d^2 + \mu_{4k}\beta_d + \mu_{5k}$$
$$\bar{p}_k(\beta_d) = \kappa_{1k}\beta_d^4 + \kappa_{2k}\beta_d^3 + \kappa_{3k}\beta_d^2 + \kappa_{4k}\beta_d + \kappa_{5k}$$
$$q(\beta_d) = (1 + \beta_d^2)^2$$

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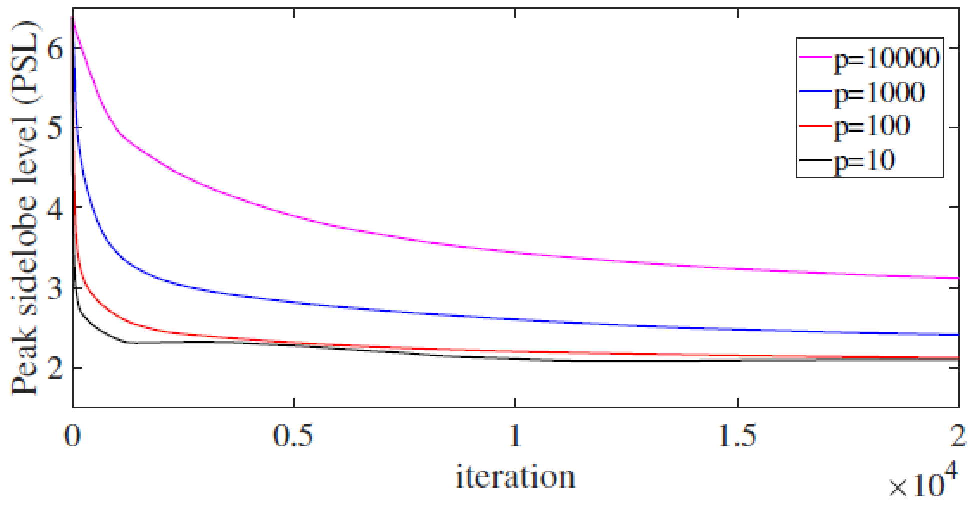
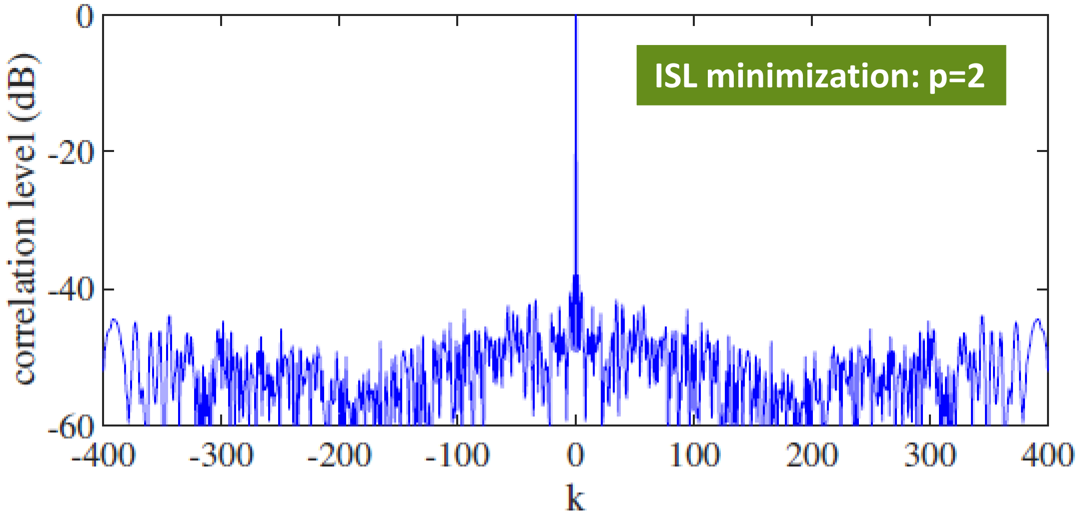
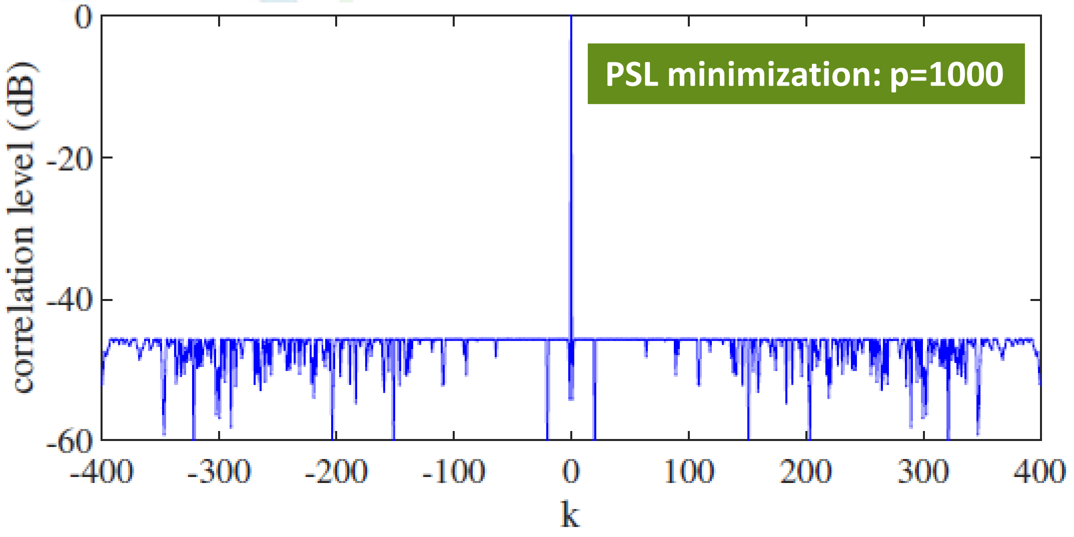
Block MM/BSUM for ℓ_p- Norm Minimization



- Monotonicity is ensured → know when to stop
- Both ISL and PSL ensure a low sidelobe level
- Slight difference in sidelobes between ISL and PSL





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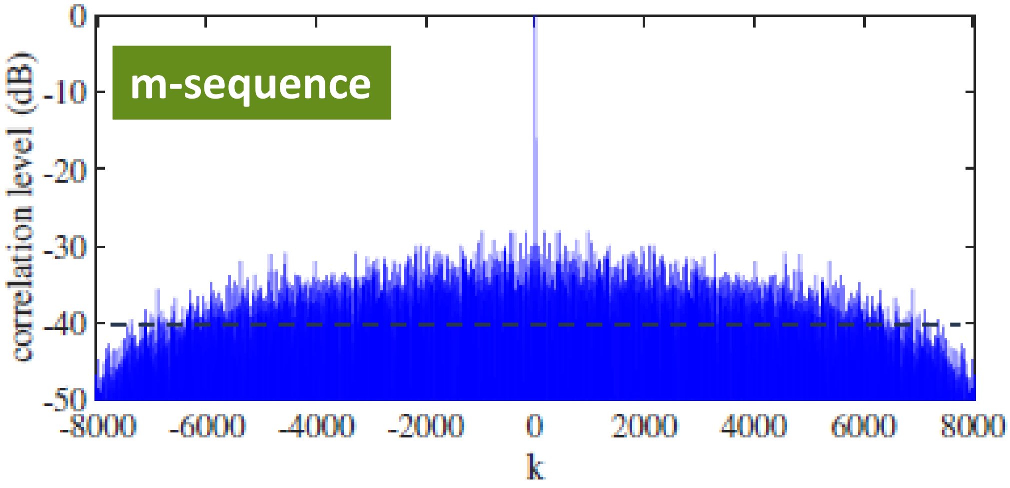
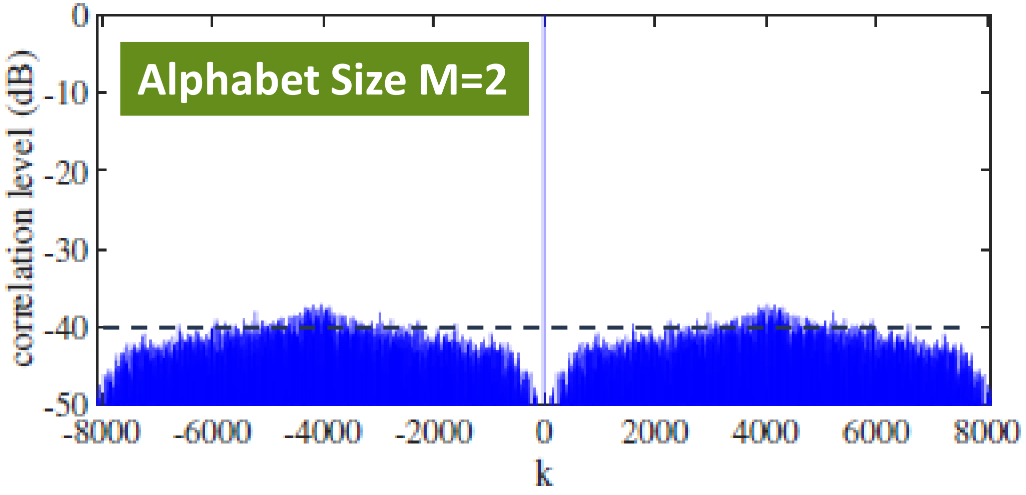
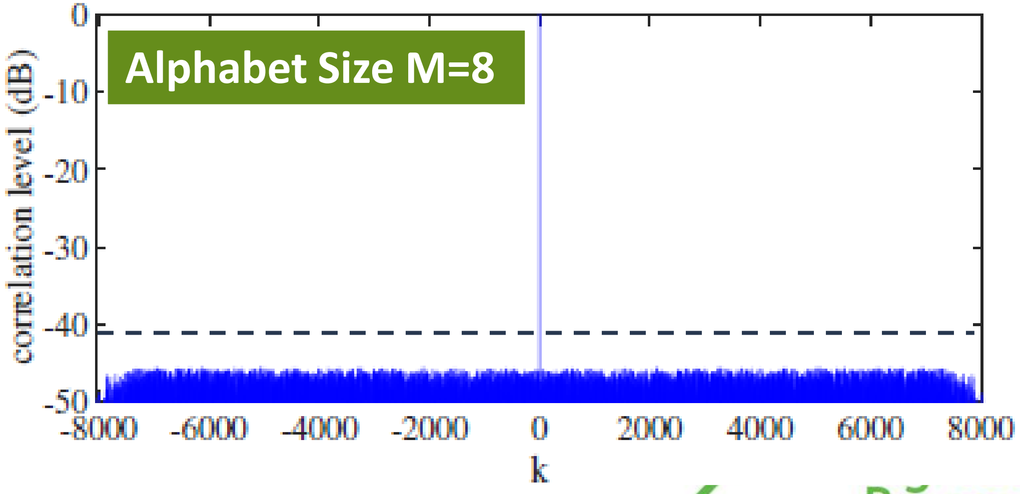
Block MM/BSUM for ℓ_p- Norm Minimization



- Optimized sequence is better than m-sequence
- Alphabet size ↗, sidelobe level ↘





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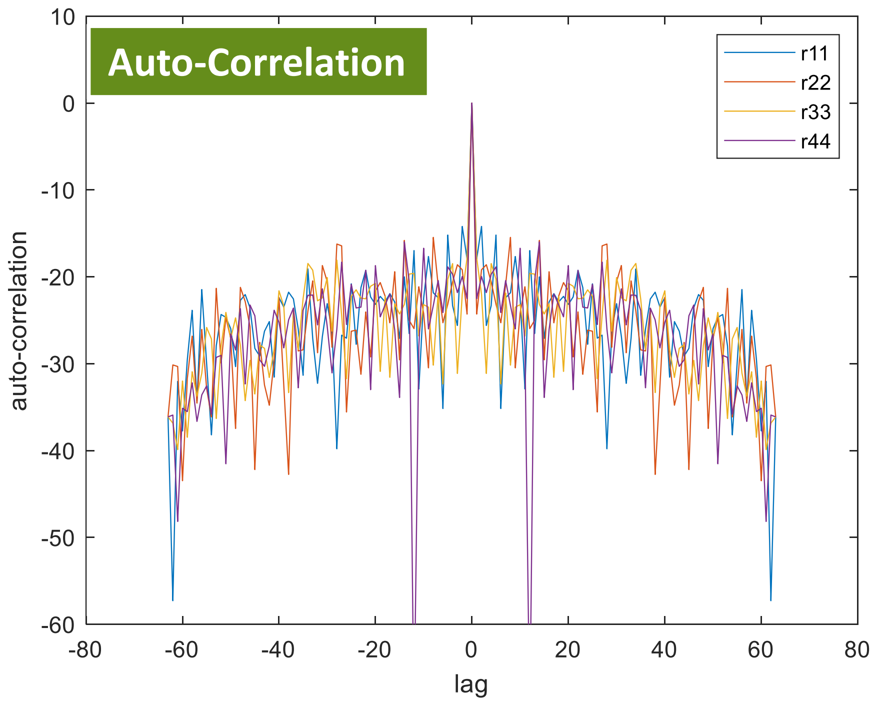
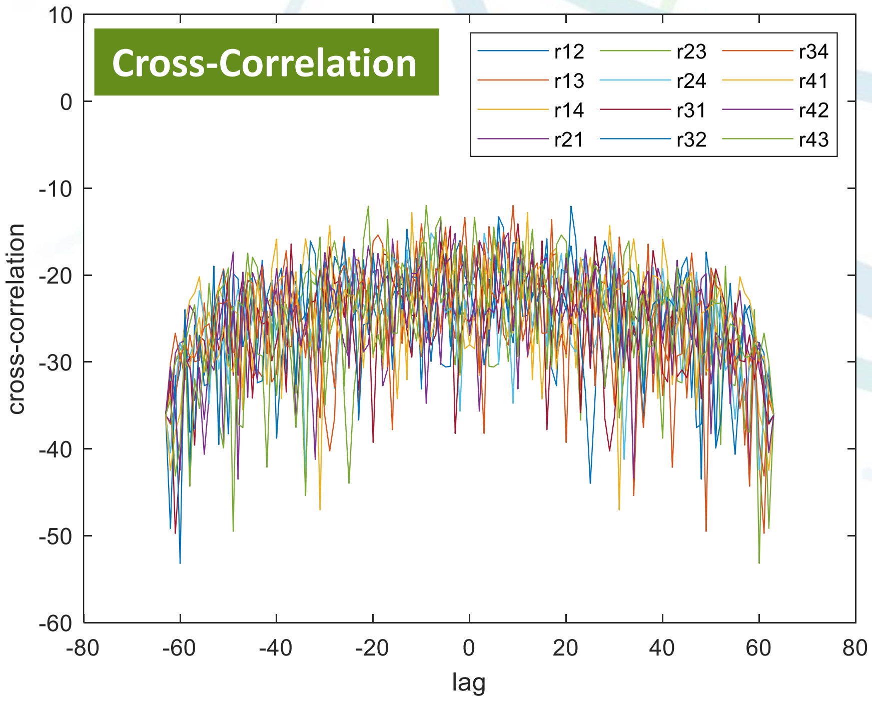
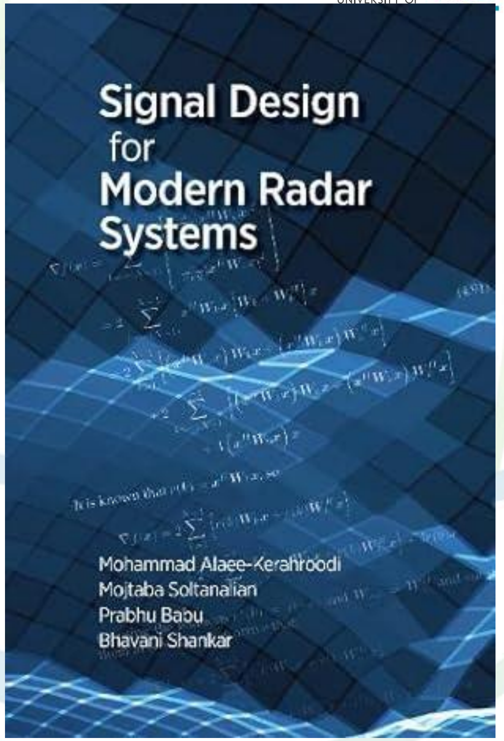
Extension: ISL/PSL for MIMO

$$\begin{cases} \min\limits_{\mathbf{x}\in\mathbb{C}^{N\times M}} & \text{ISL} = \sum_{m=1}^{N_T}\sum_{k=-N+1}^{N-1}|r_{mm}(k)|^2 + \sum_{m,l=1}^{N_T}\sum_{k=-N+1}^{N-1}|r_{ml}(k)|^2 \\ \text{subject to} & |x_{n,m}| = 1, \forall \begin{cases} n & = 1,\dots,N \\ m, m' & = 1,\dots,M. \end{cases} \end{cases}$$







Alaee-Kerahroodi, Mohammad, et al. *Signal design for modern radar systems*. Artech House, 2022.

What we learned from Lecture 2

- Lecture 2 introduced modern radar systems, including phased array, and MIMO radars. As a key parameter in modern radars, waveform diversity and optimization has been discussed and several optimization techniques such as gradient descent, majorization-minimization, and coordinate descent with the application on waveform design for radar has been illustrated

Scan the QR code for access to the codes

Q & A

Using a CW radar, how can we build a motion detector in practice?

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