Part 3

A Glimpse of Optimization on Radar Waveform Design

Introduction

Mathematical optimization:

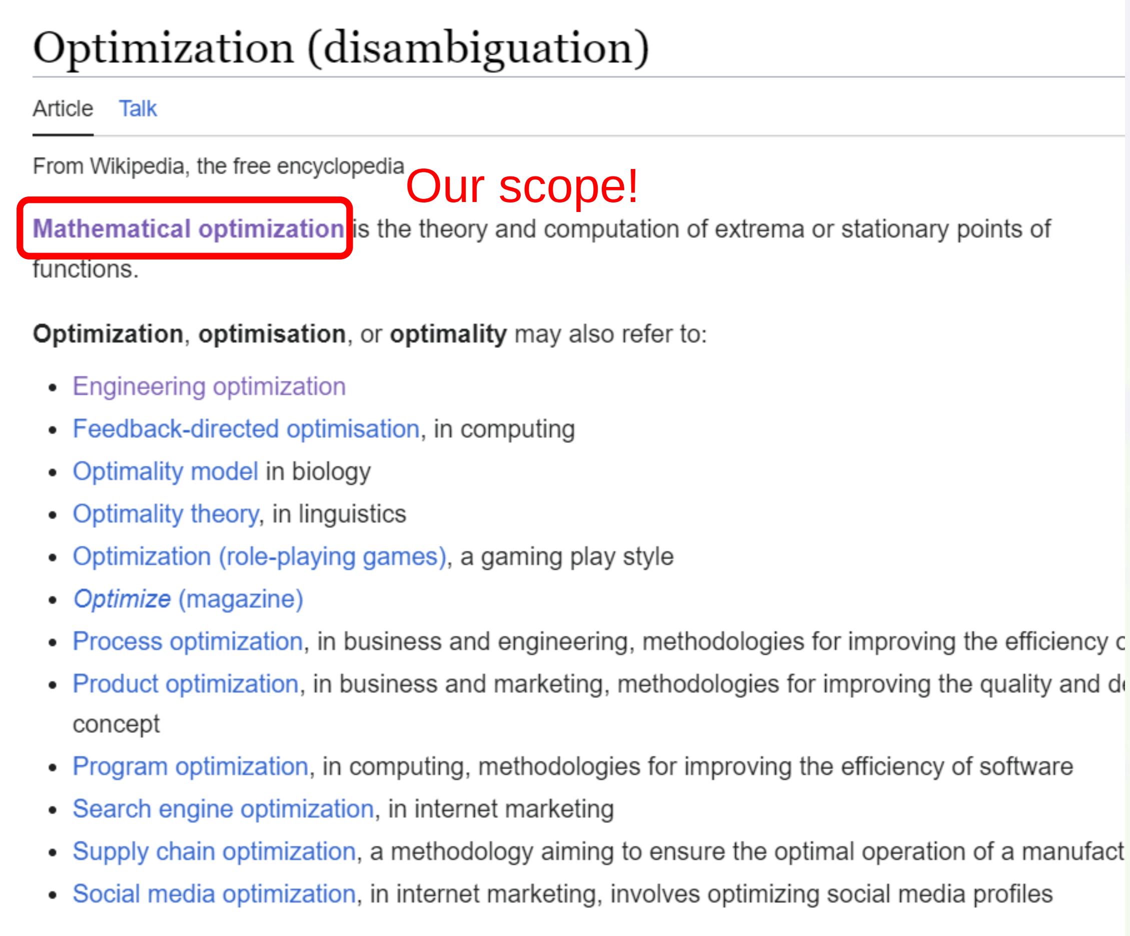
The selection of a best element,
with regard to some criterion,
from some set of available alternatives.

Simple example:

$$\begin{cases} \underset{x}{\text{minimize}} & x^2 + 1 \\ \text{subject to} & \boldsymbol{x} \in [-\mathbf{1}, \mathbf{1}] \end{cases} \longrightarrow$$

Optimal solution:
$$\boldsymbol{x}^\star = \mathbf{0}$$

If $\boldsymbol{x} \in [\mathbf{1}, \mathbf{2}]$, then $\boldsymbol{x}^\star = \mathbf{1}$


Optimization (disambiguation)

Article Talk

From Wikipedia, the free encyclopedia

Mathematical optimization is the theory and computation of extrema or stationary points of functions.

Optimization, **optimisation**, or **optimality** may also refer to:

- Engineering optimization
- Feedback-directed optimisation, in computing
- Optimality model in biology
- Optimality theory, in linguistics
- Optimization (role-playing games), a gaming play style
- *Optimize* (magazine)
- Process optimization, in business and engineering, methodologies for improving the efficiency c
- Product optimization, in business and marketing, methodologies for improving the quality and d concept
- Program optimization, in computing, methodologies for improving the efficiency of software
- Search engine optimization, in internet marketing
- Supply chain optimization, a methodology aiming to ensure the optimal operation of a manufact
- Social media optimization, in internet marketing, involves optimizing social media profiles

Our scope!

Radar Waveform Optimization

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \boldsymbol{\mathcal{X}} \end{cases}$$

$\boldsymbol{x}$: optimization variable

$f(\boldsymbol{x})$: objective function

$\boldsymbol{x} \in \chi$: constraint

$\boldsymbol{x}$: waveform

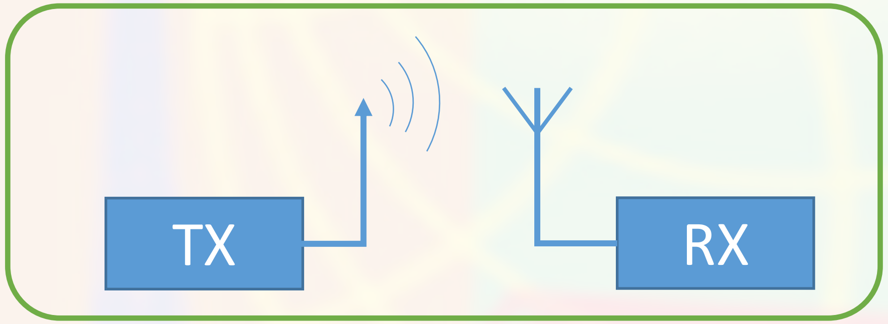
$f(\boldsymbol{x})$: performance metrics

$\boldsymbol{x} \in \chi$: requirements on waveform

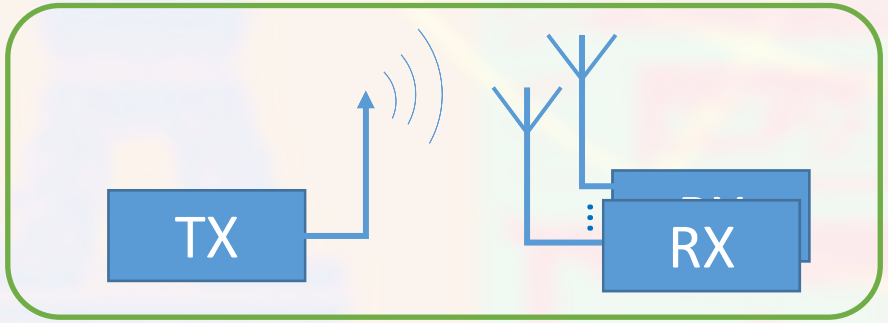
How to formulate the optimization problem for waveform design?

What is Performance Metrics?

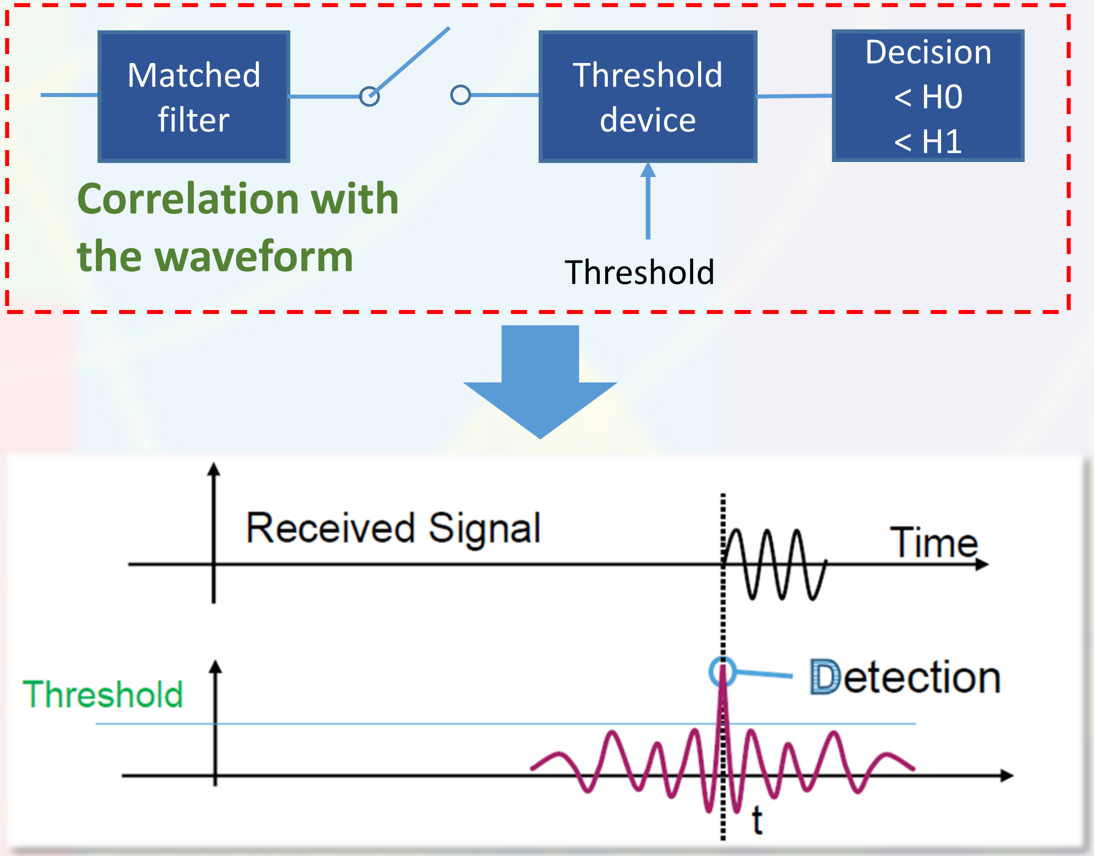
Single Input Single Output (SISO)



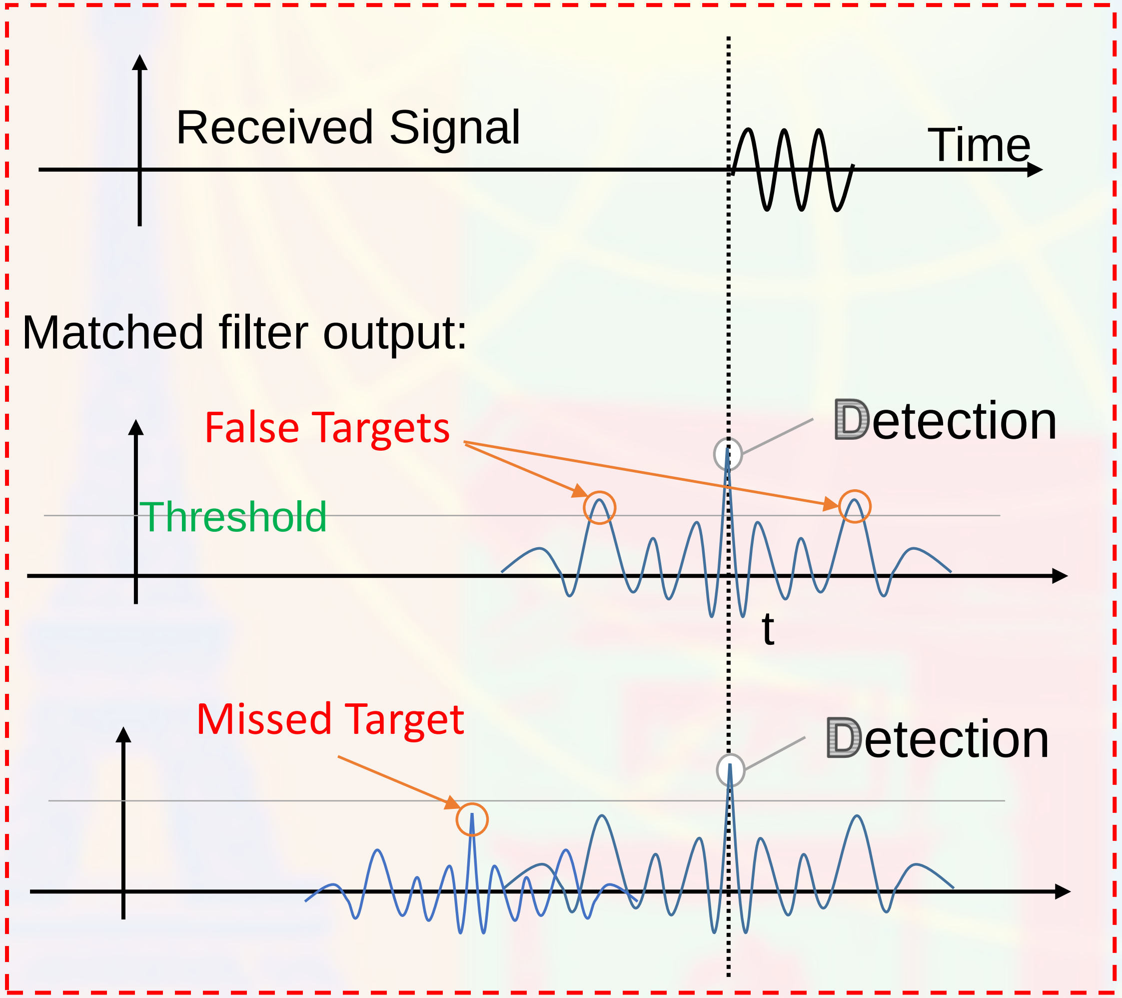
Single Input Multi Output (SIMO)



With the existence of white noise



PSL/ISL Matters in Detection



$$x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{C}^N$$

$$r_k = \sum_{n=1}^{N-k} x_n^* x_{n+k} . k = 0.....N-1$$

$$\mathrm{PSL} = \max_{k \neq 0} |r_k| \qquad \mathrm{ISL} = \sum_{k=1}^{N-1} |r_k|^2$$

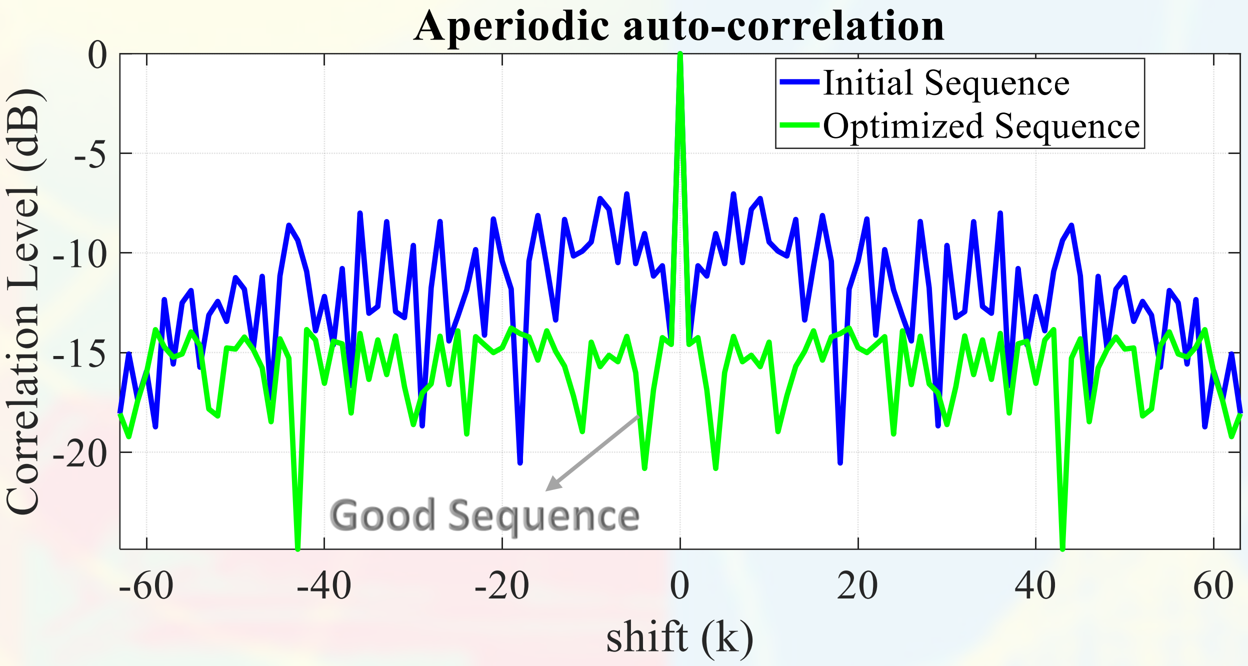
Peak Sidelobe Level (PSL)

avoid masking of weak targets in range sidelobes of a strong return

Integrated Sidelobe Level (ISL)

mitigate the deleterious effects of distributed clutter echoes which are close to the target of interest

ISL/PSL Minimization



$$x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{C}^N$$

$$\mathrm{PSL} = \max_{k \neq 0} |r_k|$$

$$\mathrm{ISL} = \sum_{k=1}^{N-1} |r_k|^2$$

objective function

Useful performance metrics: PSL / ISL

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{cases}$$

Waveform Requirements

- Hardware perspective: costly, non-ideal
- Some factors need to consider when designing waveforms
- Two common waveform constraints: Unimodularity, finite phase value

> In theory there is no difference between theory and practice. In practice there is.
>
> (Yogi Berra)

Unimodular Waveform

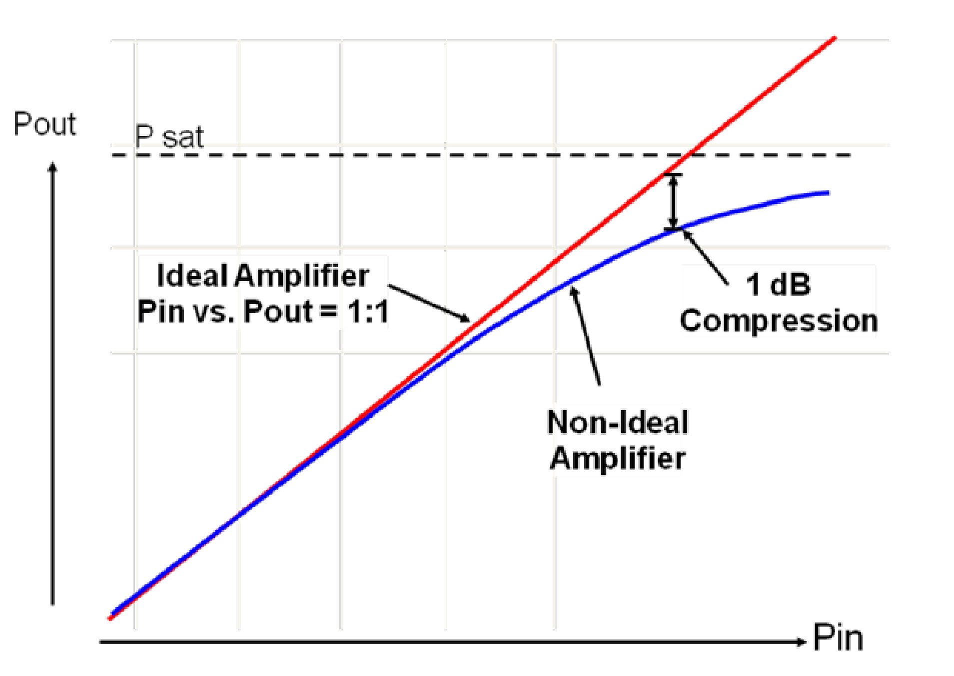
Nonideal power amplifier: limited linear region

Unimodular waveform: $|x_n| = 1, \forall n = 1, \ldots, N$

More general version

Peak to average power ration (PAPR):

$$\text{PAR}\,(\mathbf{x}) = \max_n \left\{ |x_n|^2 \right\} / \, \|\mathbf{x}\|_2^2 \leq \gamma$$



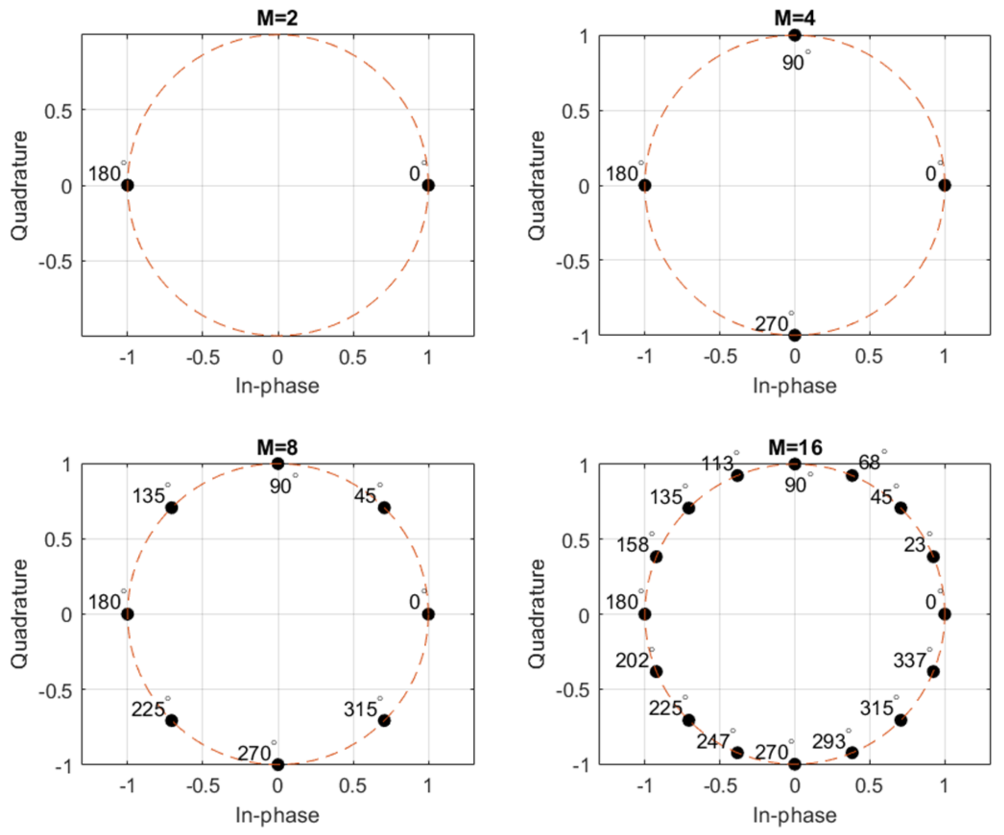
Phase Alphabet

- Phase shifter is expensive
- Generates finite phase values
- Phase quantization should be considered

Constraint on phase alphabet

$$x_n \in \Omega_M$$

$$\Omega_M = \left\{1, e^{\frac{j2\pi}{M}}, \ldots, e^{\frac{j2\pi(M-1)}{M}}\right\}$$



ISL/PSL Minimization

objective function

Useful performance metrics:
PSL / ISL

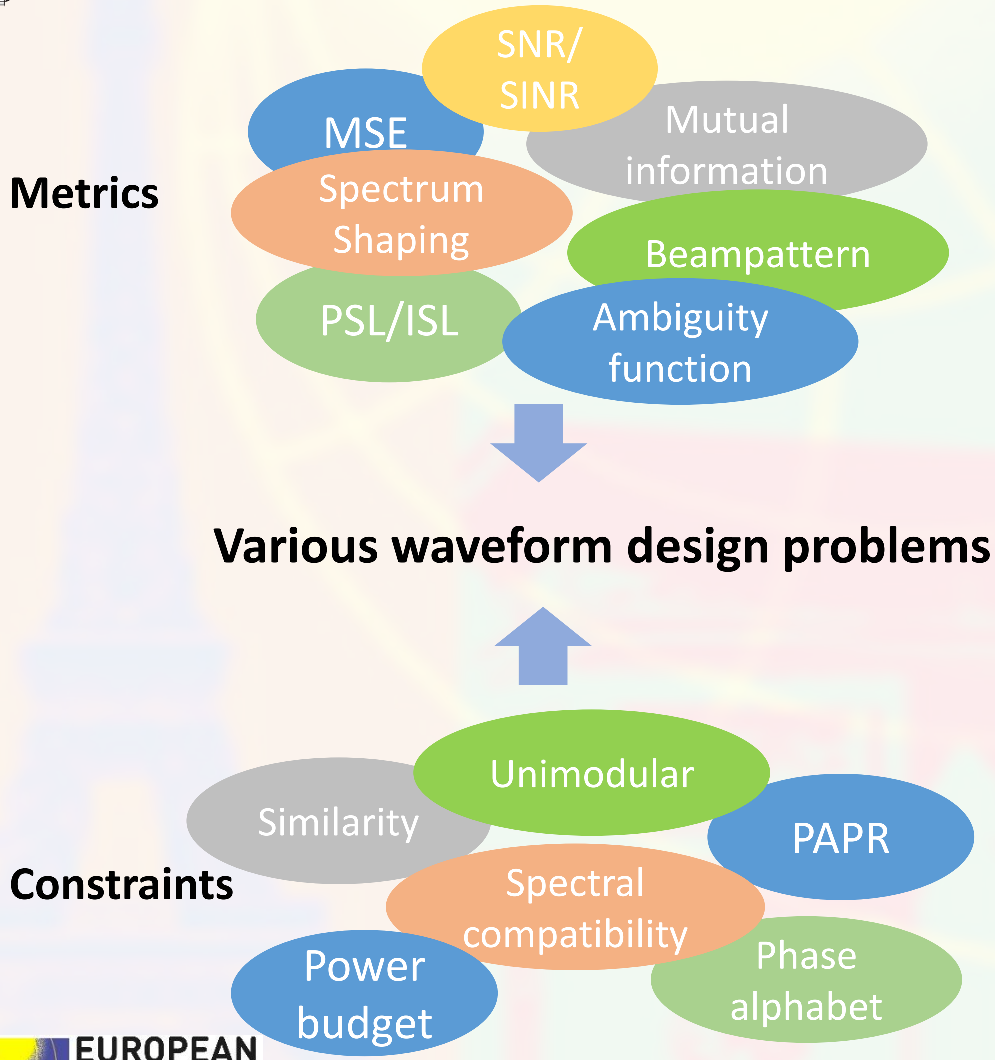
Practical waveform requirement:
Unimodular, phase alphabet

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \text{ISL/PSL} \\ \text{subject to} & x_n \in \boldsymbol{\Omega_M} \end{cases}$$

constraints

How to solve the formulated waveform design problem?

Beyond ISL/PSL



Algebraic construction: Frank sequence, Golomb sequence…

Heuristic constriction: exhaustive search, evolutionary algorithm, simulated annealing…

- Still cannot cover all needs
- Many problems are nonconvex and NP-hard
- High dimension if long sequence is needed
- Time efficiency matters

We focus on Optimization-based approach

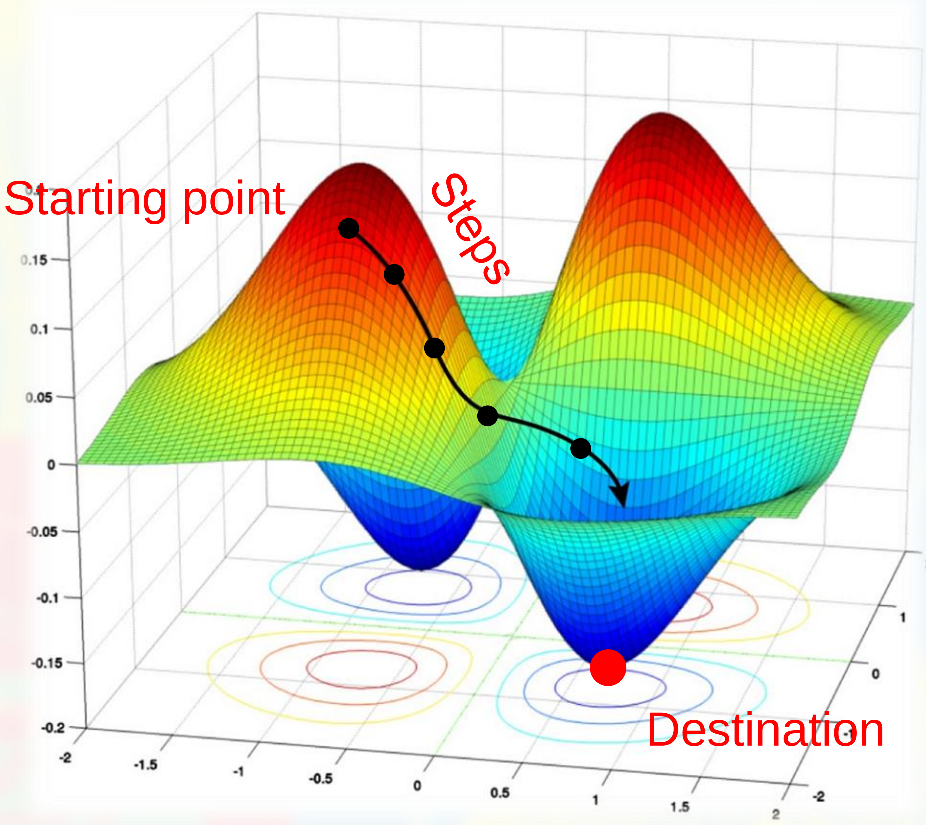
Basics: Iterative Method

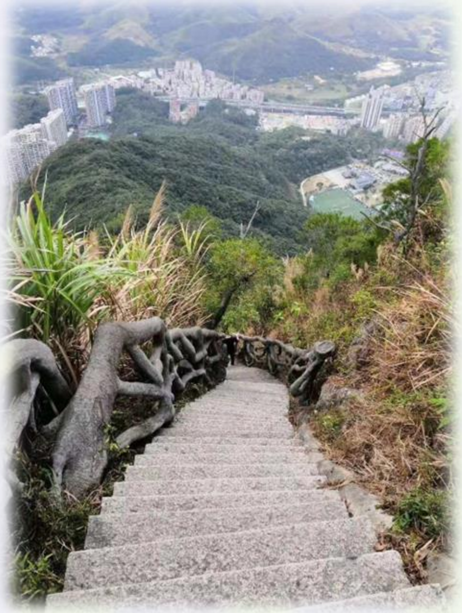
Unconstrained problem:

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad f(\boldsymbol{x})$$

Gradient descent (GD) is well-known

$$\underbrace{\mathbf{x}_{k+1}}_{\text{Updated point}} = \underbrace{\mathbf{x}_k}_{\text{Current point}} - \underbrace{\alpha}_{\text{Step-size}} \underbrace{\nabla f(\mathbf{x}_k)}_{\text{Direction}}$$





Iteratively repeat the update rule,
the sequence $\{\boldsymbol{x}_k\}$ converge at local optimum

Not Easy for Waveform Design

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \boldsymbol{\mathcal{X}} \end{cases}$$

For waveform design problems,

- $f(\boldsymbol{x})$ can be complicated even non-differentiable
- $\boldsymbol{x}$ can be high-dimension → computational cost
- Some constraints to consider, i.e., $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$

We need more efficient optimization techniques

Coordinate Descent (CD)

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{cases}$$

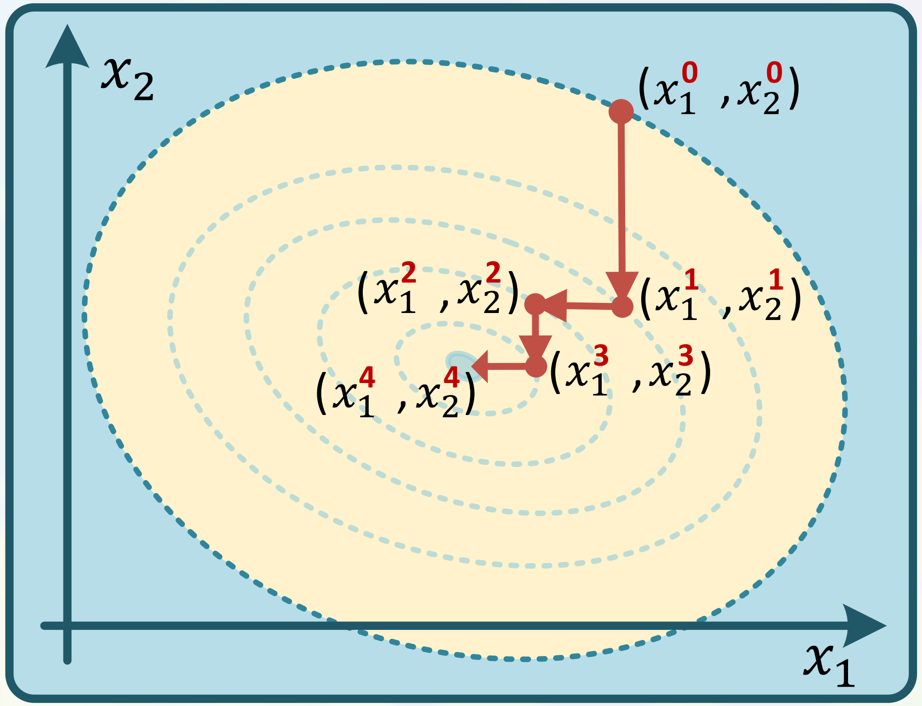
$$x_1^{(k)} \in \arg\min_{x_1} f(x_1, x_2^{(k-1)}, x_3^{(k-1)}, ..., x_N^{(k-1)})$$

$$x_2^{(k)} \in \arg\min_{x_2} f(x_1^{(k)}, x_2, x_3^{(k-1)}, ..., x_N^{(k-1)})$$

$$\vdots$$

$$x_N^{(k)} \in \arg\min_{x_N} f(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, ..., x_N)$$

Successively minimizes along coordinate directions



$$y = x_1^2 + 2x_2^2 - 9$$

Coordinate Descent (CD)

Gauss-Seidel style (One-at-a-time)

$$x_i^{(k+1)} \leftarrow \arg\min_\zeta f(x_1^{(k+1)}, ..., x_{i-1}^{(k+1)}, \zeta, x_{i+1}^{(k)}, ..., x_N^{(k)})$$

Jacobi style (all-at-once ; easy to parallelize)

$$x_i^{(k+1)} \leftarrow \arg\min_\zeta f(x_1^{(k)}, ..., x_{i-1}^{(k)}, \zeta, x_{i+1}^{(k)}, ..., x_N^{(k)})$$

- Maximum Block Improvement
  - For differentiable f, pick the index that minimizes $\nabla f(x_i^k)$
- Various update order:
  - Cyclic order: $1, 2, \dots, N, 1, \dots$
  - Double sweep: $1, 2, \dots, N$, then $N-1, \dots, 1$, repeat
  - Cyclic with permutation: random order each cycle
  - Random sampling: pick random index at each iteration

Majorization Minimization (MM)

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in X \end{cases}$$

First step: Majorization

Construct the majorizer satisfying

$$u\left(\mathbf{x}, \mathbf{x}^{(\ell)}\right) \geq f(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathcal{X},$$

$$u\left(\mathbf{x}^{(\ell)}, \mathbf{x}^{(\ell)}\right) = f\left(\mathbf{x}^{(\ell)}\right).$$

Second step: Minimization

$$\mathbf{x}^{(\ell+1)} \in \arg\min_{\mathbf{x} \in \mathcal{X}} u\left(\mathbf{x}, \mathbf{x}^{(\ell)}\right)$$

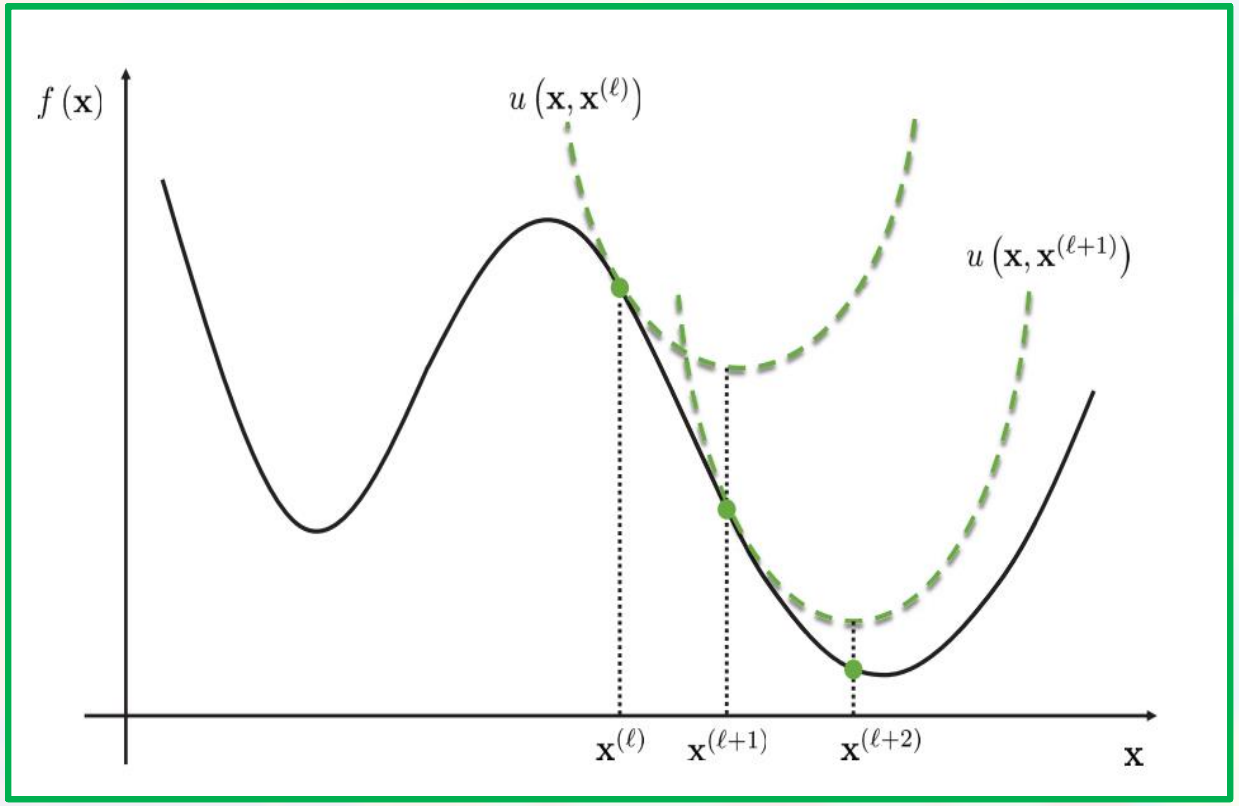
Algorithm 2: Sketch of the MM Method

Result: Optimized code vector $\mathbf{x}^\star$
initialization;
for $\ell = 0, 1, 2, \ldots$ **do**
  $\boxed{\mathbf{x}^{(\ell+1)} \in \operatorname{argmin}_{\mathbf{x} \in \mathcal{X}} u\left(\mathbf{x}, \mathbf{x}^{(\ell)}\right);}$
  Stop if convergence criterion is met;
  $\ell \leftarrow \ell + 1$
end

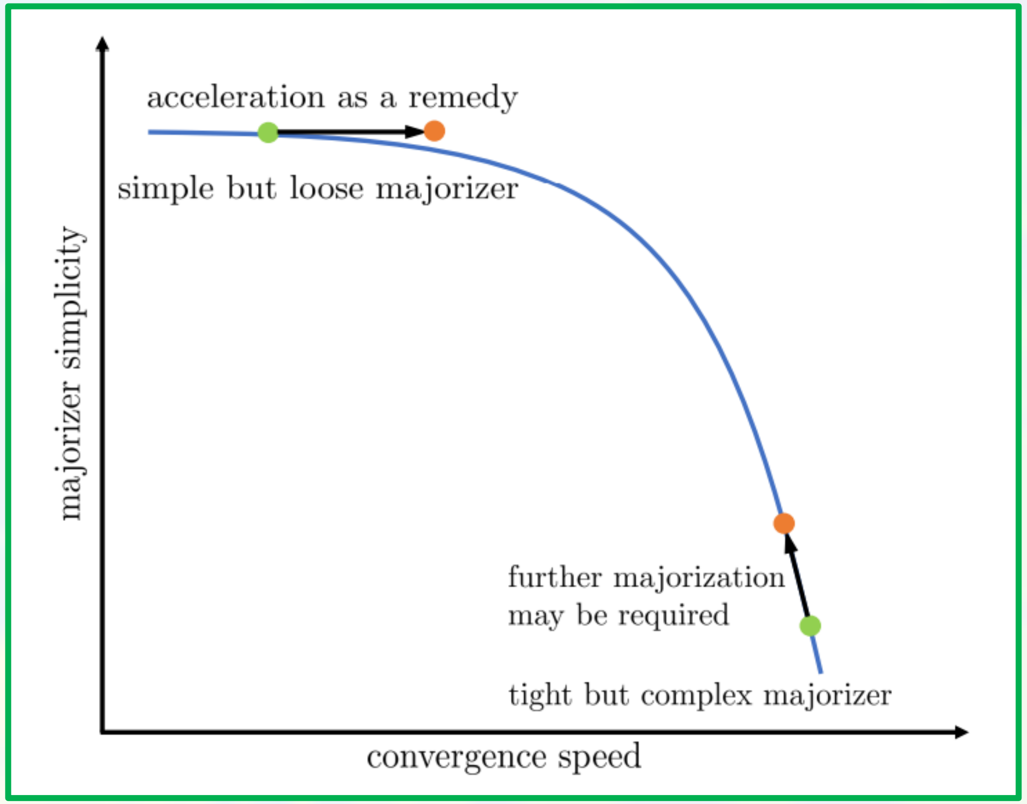
Note: For maximization problem, minorization maximization
Construct the minorizer and then maximize

Majorization Minimization (MM)

Graphic illustration of MM



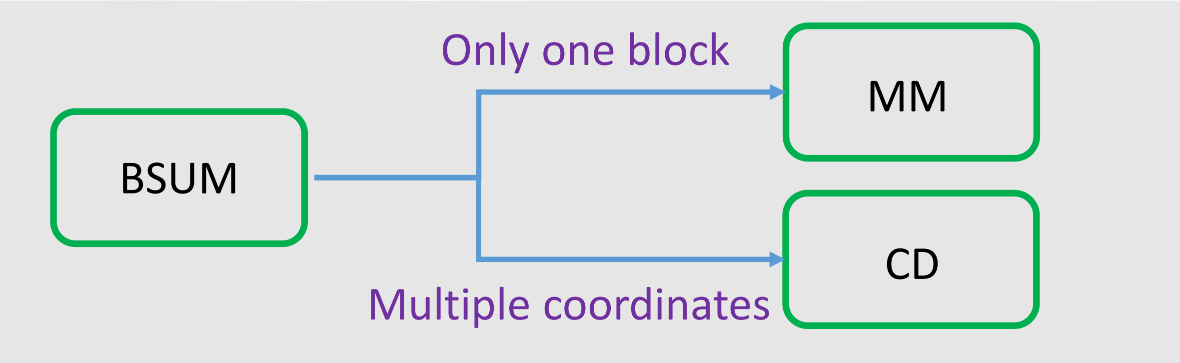
Simplicity versus convergence



Block MM/BSUM

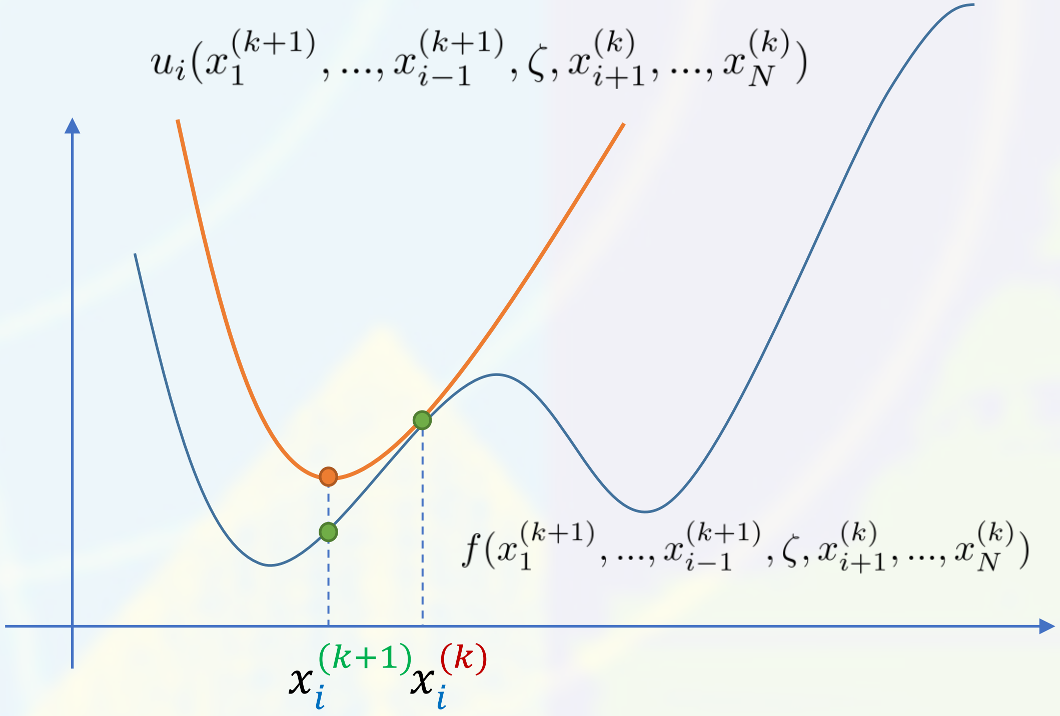
Block Successive Upper bound Minimization (BSUM)

$$\begin{cases} \underset{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N}{\text{minimize}} & f\left(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N\right) \\ \text{subject to} & \boldsymbol{x}_n \in \mathcal{X}_n, n=1,\ldots,N \end{cases}$$

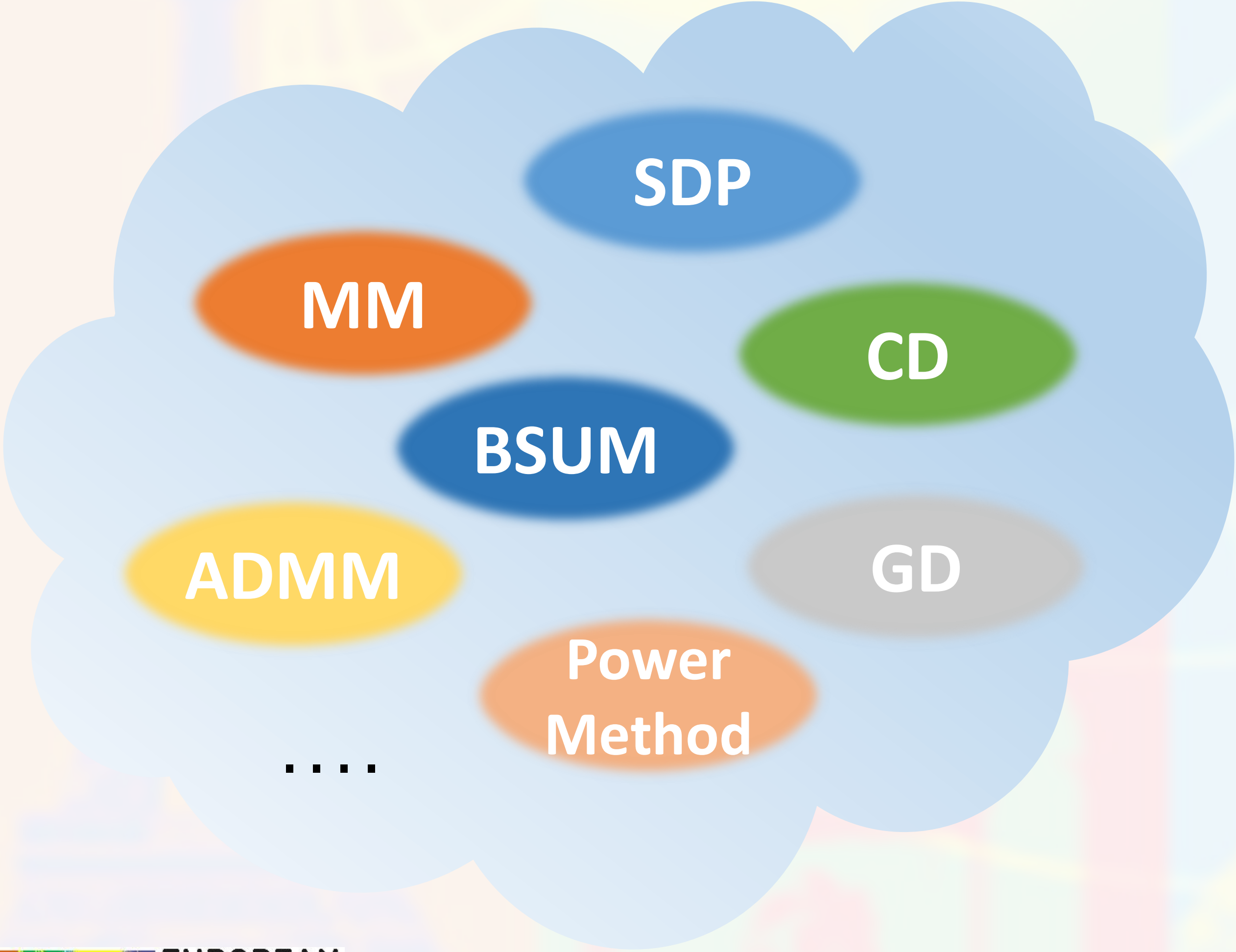


$$x_i^{(k+1)} \leftarrow \arg\min_{\zeta} u_i(x_1^{(k+1)},\ldots,x_{i-1}^{(k+1)},\zeta,x_{i+1}^{(k)},\ldots,x_N^{(k)})$$

Majorizer/upper bound of the objective function



Various Optimization Techniques



Depending on the problems

→ deployed solely or combined

Recall ISL/PSL Problems

Waveform to be designed: $x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{C}^N$

PSL	$\begin{cases} \underset{x}{\text{minimize}} & \max\{\lvert r_k \rvert\}_{k=1}^{N-1} \\ \text{subject to} & \lvert x_n \rvert = 1 \end{cases}$	$\begin{cases} \underset{x}{\text{minimize}} & \max\{\lvert r_k \rvert\}_{k=1}^{N-1} \\ \text{subject to} & x_n \in \Omega_M \end{cases}$
ISL	$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} \lvert r_k \rvert^2 \\ \text{subject to} & \lvert x_n \rvert = 1 \end{cases}$	$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} \lvert r_k \rvert^2 \\ \text{subject to} & x_n \in \Omega_M \end{cases}$
	Unimodular	Phase alphabet

$\Omega_M = \left\{1, e^{\frac{j2\pi}{M}}, \ldots, e^{\frac{j2\pi(M-1)}{M}}\right\}$

A Unified Problem Formulation

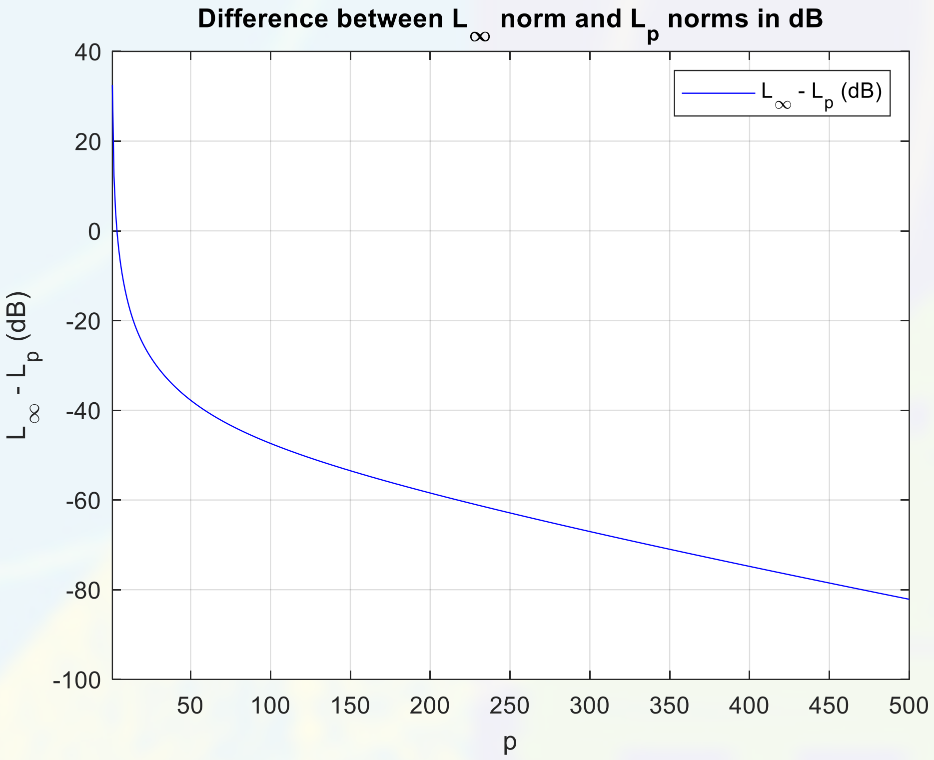
$$\ell_p \text{ norm: } \|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

$$\begin{cases} p = 2 : \|\mathbf{x}\|_2 = \sum_{n=1}^{N} |x_n|^2 & \text{ISL} \\ p = \infty : \|\mathbf{x}\|_\infty = \max_{1\le i \le n} |x_i| & \text{PSL} \end{cases}$$

A unified formulation:

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} |r_k|^p \\ \text{subject to} & x_n \in \Omega_M \end{cases}$$

ℓ_∞ norm approximation: use a large value of p



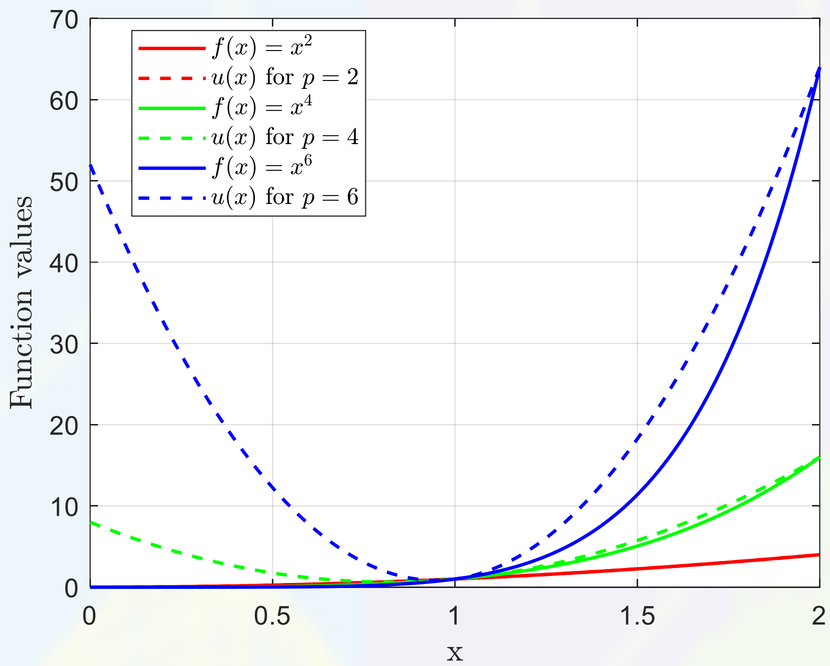
How to solve the non-convex problem?

Apply MM for a Simpler Problem

Majorizer of $f(x) = x^p, \;\; x \in [0,t]$ with $p \geq 2$

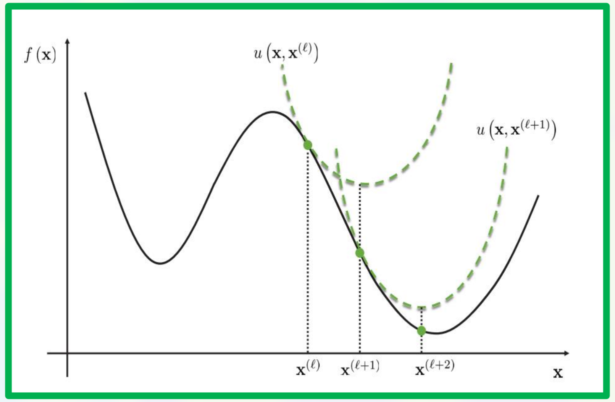
$$u(x) = ax^2 + \left(px_0^{p-1} - 2ax_0\right)x + ax_0^2 - (p-1)x_0^p$$

$$a = \frac{t^p - x_0^p - px_0^{p-1}(t-x_0)}{(t-x_0)^2}$$



At each iteration, we solve

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} |r_k|^p \\ \text{subject to} & x_n \in \Omega_M \end{cases} \quad \Rightarrow \quad \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} a_k |r_k|^2 + \sum_{k=1}^{N-1} b_k |r_k| \\ \text{subject to} & x_n \in \Omega_M \end{cases}$$



Use CD for the Majorized Problem

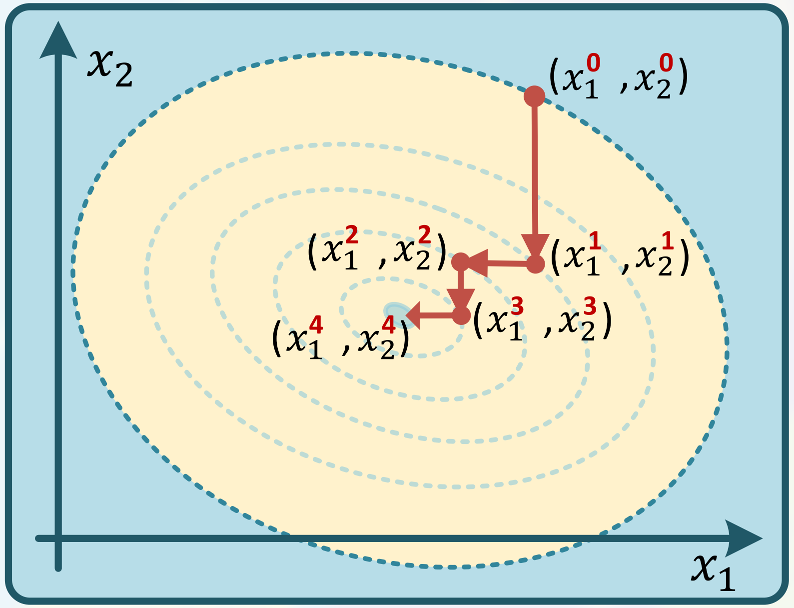
$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} a_k |r_k|^2 + \sum_{k=1}^{N-1} b_k |r_k| \\ \text{subject to} & x_n \in \Omega_M \end{cases}$$

$$x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{C}^N$$

x_d → Only variable to optimize

$$\mathbf{x}_{-d} = \left[x_1^{(i+1)}, \ldots, x_{d-1}^{(i+1)}, 0, x_{d+1}^{(i)}, \ldots x_N^{(i+1)}\right]^T \in \mathbb{C}^N$$

$$r_k(x_d) = a_{1k} x_d + a_{2k} x_d^* + a_{3k}$$



Find the Optimal Phase

$$x_d \in \Omega_M$$

$$\Omega_M = \left\{1, e^{\frac{j2\pi}{M}}, \ldots, e^{\frac{j2\pi(M-1)}{M}}\right\}$$

$$x_d = e^{j\phi_d}$$

$$\widetilde{r}_k(\phi_d) = a_{1k}e^{j\phi_d} + a_{2k}e^{-j\phi_d} + a_{3k}$$

$$\widetilde{\mathcal{H}}_h^{(i+1)} \begin{cases} \min_{\phi_d} & \sum_{k=1}^{N-1} a_k \left|\widetilde{r}_k(\phi_d)\right|^2 + \sum_{k=1}^{N-1} b_k \operatorname{Re}\left\{\widetilde{r}_k(\phi_d)^* \frac{r_k^{(\ell)}}{\left|r_k^{(\ell)}\right|}\right\} \\ \text{s.t.} & \phi_d \in \Phi_M = \left\{0, \frac{2\pi}{M}, \frac{4\pi}{M}, \ldots, \frac{2\pi(M-1)}{M}\right\} \end{cases}$$

$$\beta_d = \tan\left(\frac{\phi_d}{2}\right)$$

$$|\widetilde{r}_k(\phi_d)|^2 = \frac{\widetilde{p}_k(\beta_d)}{q(\beta_d)} \qquad \operatorname{Re}\left\{\widetilde{r}_k^*(\beta_d)\frac{r_k^{(i)}}{\left|r_k^{(i)}\right|}\right\} = \frac{\bar{p}_k(\beta_d)}{q(\beta_d)}$$

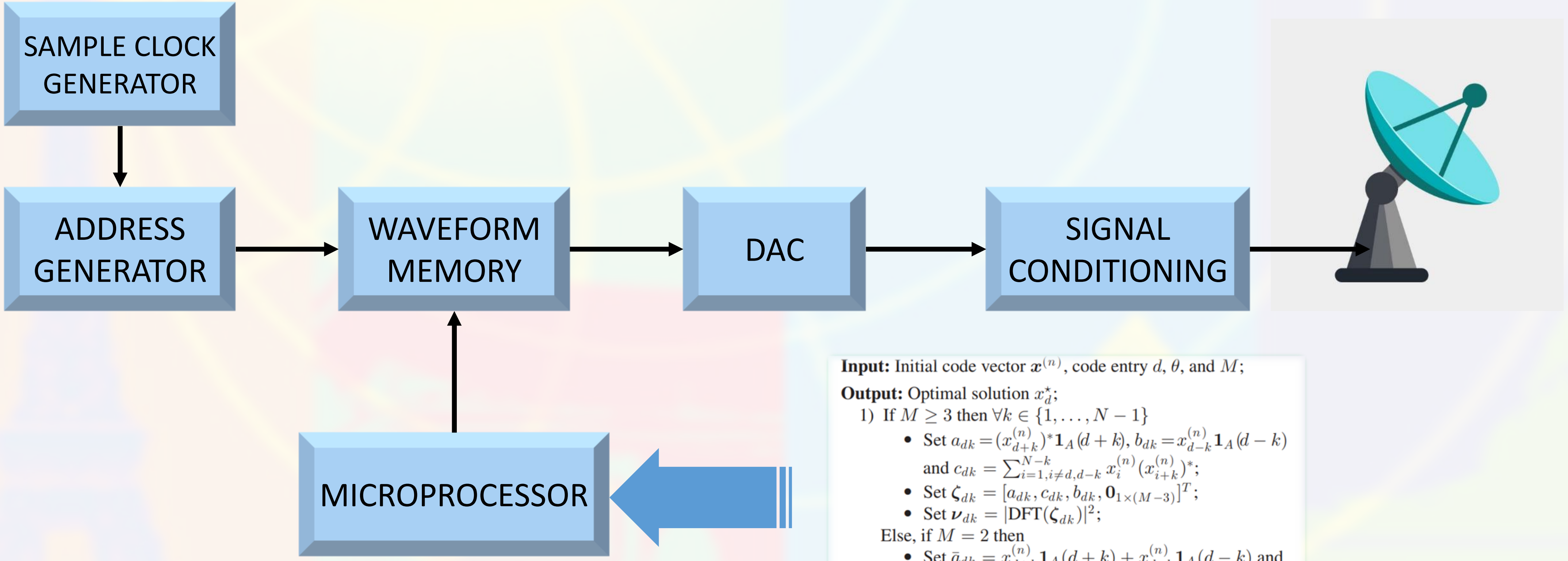
$$\begin{cases} \min\limits_{\beta_d} & \frac{1}{q(\beta_d)} \sum\limits_{k=1}^{N-1} a_k\widetilde{p}_k(\beta_d) + b_k\bar{p}_k(\beta_d) \\ s.t. & \beta_d \in B \end{cases}$$

$$\widetilde{p}_k(\beta_d) = \mu_{1k}\beta_d^4 + \mu_{2k}\beta_d^3 + \mu_{3k}\beta_d^2 + \mu_{4k}\beta_d + \mu_{5k}$$

$$\bar{p}_k(\beta_d) = \kappa_{1k}\beta_d^4 + \kappa_{2k}\beta_d^3 + \kappa_{3k}\beta_d^2 + \kappa_{4k}\beta_d + \kappa_{5k}$$

$$q(\beta_d) = (1+\beta_d^2)^2$$

Where the optimization works



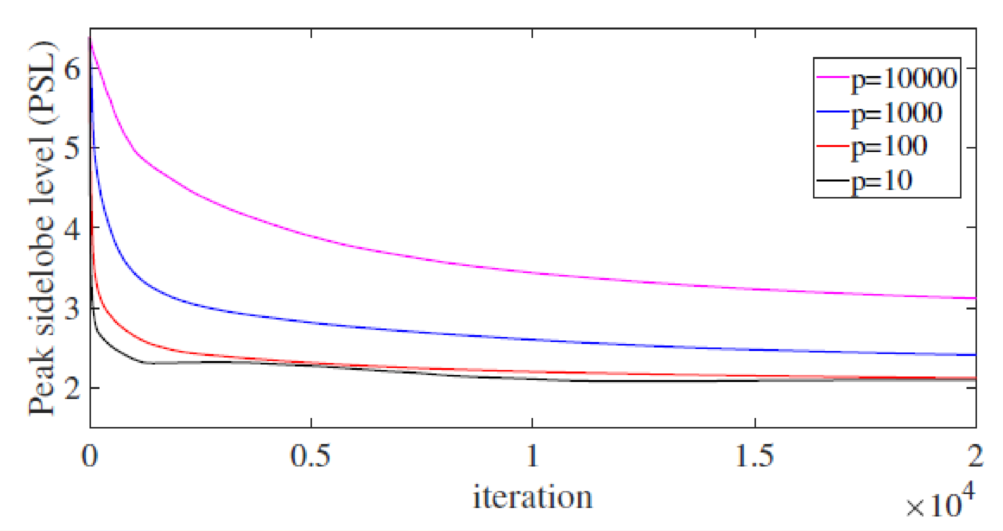
Input: Initial code vector $\boldsymbol{x}^{(n)}$, code entry d, θ, and M;

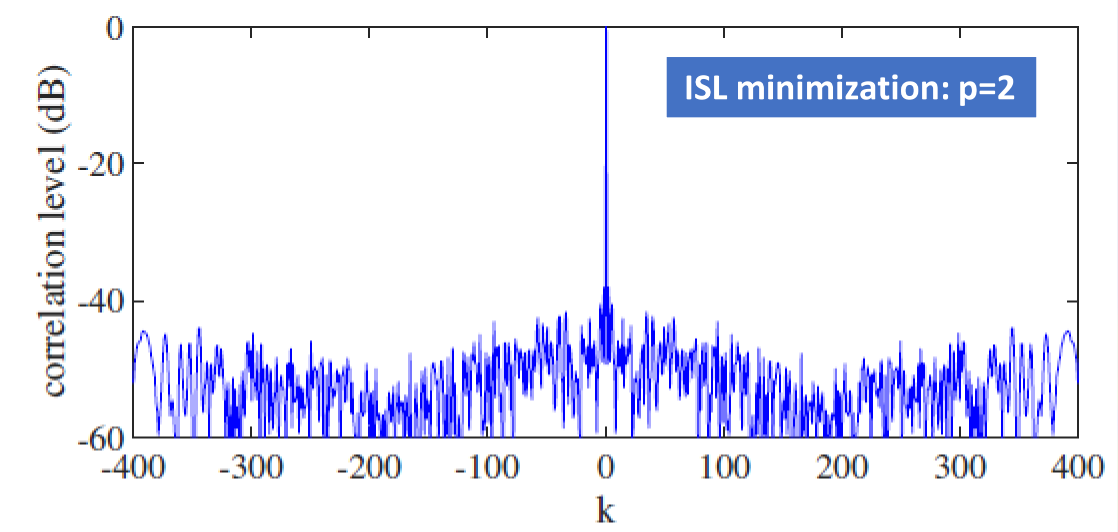
Output: Optimal solution $x_d^\star$;

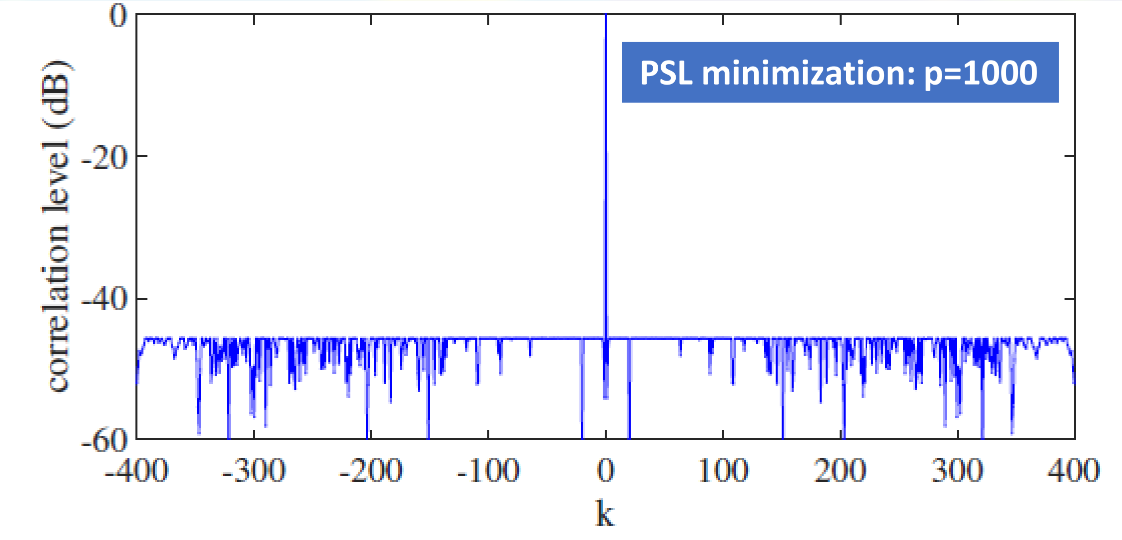
1) If $M \geq 3$ then $\forall k \in \{1, \ldots, N-1\}$
   - Set $a_{dk} = (x_{d+k}^{(n)})^* \mathbf{1}_A(d+k)$, $b_{dk} = x_{d-k}^{(n)} \mathbf{1}_A(d-k)$ and $c_{dk} = \sum_{i=1, i \neq d, d-k}^{N-k} x_i^{(n)} (x_{i+k}^{(n)})^*$;
   - Set $\boldsymbol{\zeta}_{dk} = [a_{dk}, c_{dk}, b_{dk}, \mathbf{0}_{1 \times (M-3)}]^T$;
   - Set $\boldsymbol{\nu}_{dk} = |\mathrm{DFT}(\boldsymbol{\zeta}_{dk})|^2$;

   Else, if $M = 2$ then
   - Set $\bar{a}_{dk} = x_{d+k}^{(n)} \mathbf{1}_A(d+k) + x_{d-k}^{(n)} \mathbf{1}_A(d-k)$ and $\bar{c}_{dk} = \sum_{i=1, i \neq d, d-k}^{N-k} x_i^{(n)} x_{i+k}^{(n)}$;
   - Set $\bar{\boldsymbol{\zeta}}_{dk} = [\bar{a}_{dk}, \bar{c}_{dk}]^T$;
   - Set $\boldsymbol{\nu}_{dk} = |\mathrm{DFT}(\bar{\boldsymbol{\zeta}}_{dk})|^2$;
2) Calculate $\boldsymbol{u}^k = \theta \boldsymbol{\nu}_{dk}^T + (1-\theta) \sum_{l=1}^{N-1} \boldsymbol{\nu}_{dl}^T \in \mathbb{R}^M, \quad k = 1, \ldots, N-1$ and $\boldsymbol{\omega}_d = [\max\{\boldsymbol{u}_1\}, \max\{\boldsymbol{u}_2\}, \ldots, \max\{\boldsymbol{u}_M\}]^T$;
3) Find the index $i^\star$ where $\boldsymbol{\omega}_d$ is minimum;
4) Set $x_d^\star = e^{\jmath \phi_d^\star}$ with $\phi_d^\star = \frac{2\pi(i^\star - 1)}{M}$.

Performance Evaluation

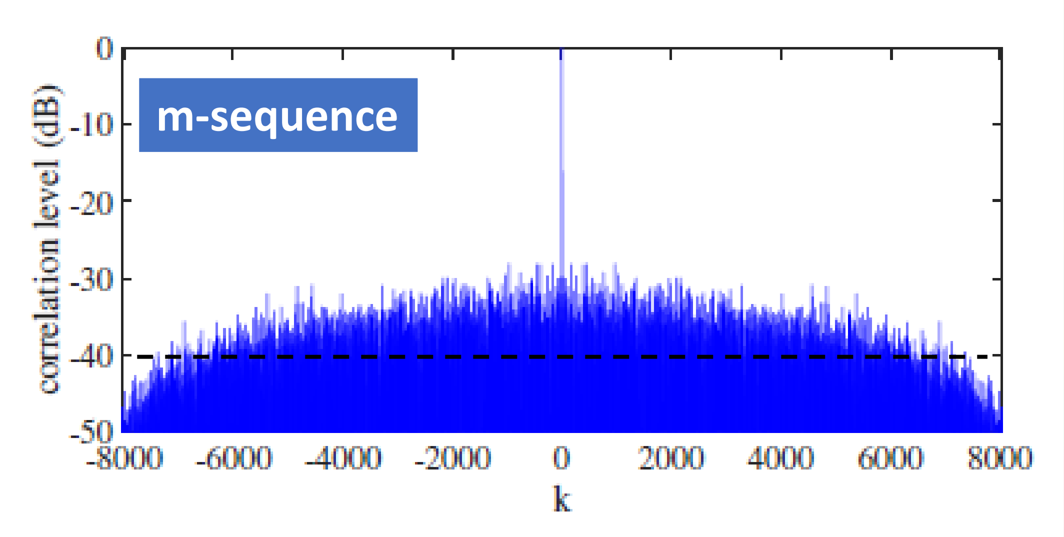




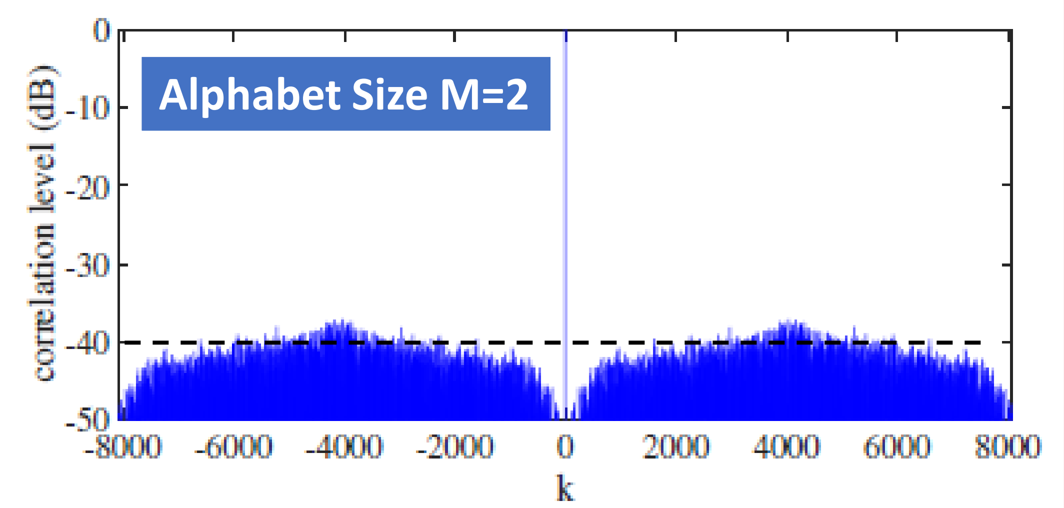


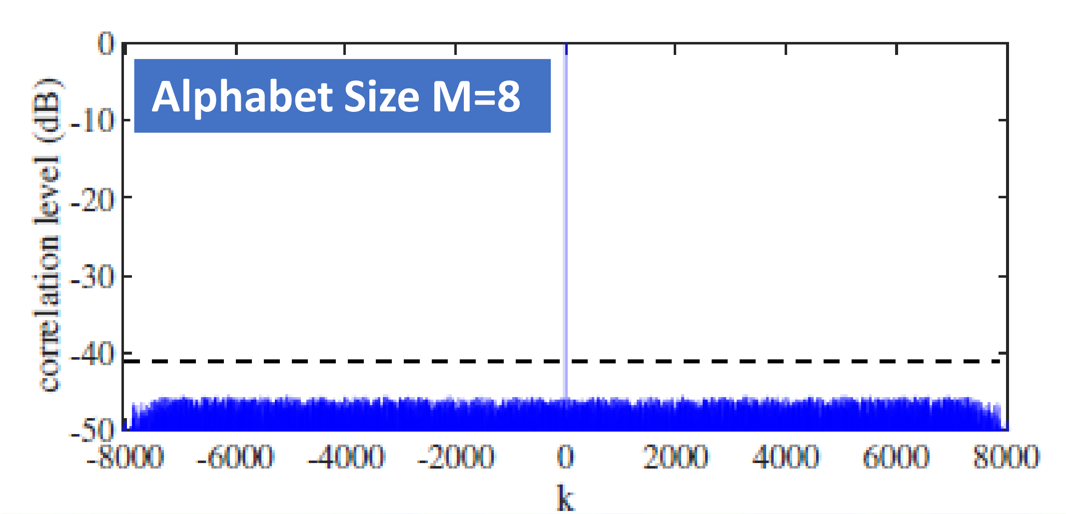
- Monotonicity is ensured → know when to stop
- Both ISL and PSL ensure a low sidelobe level
- Slight difference in sidelobes between ISL and PSL

Performance Evaluation

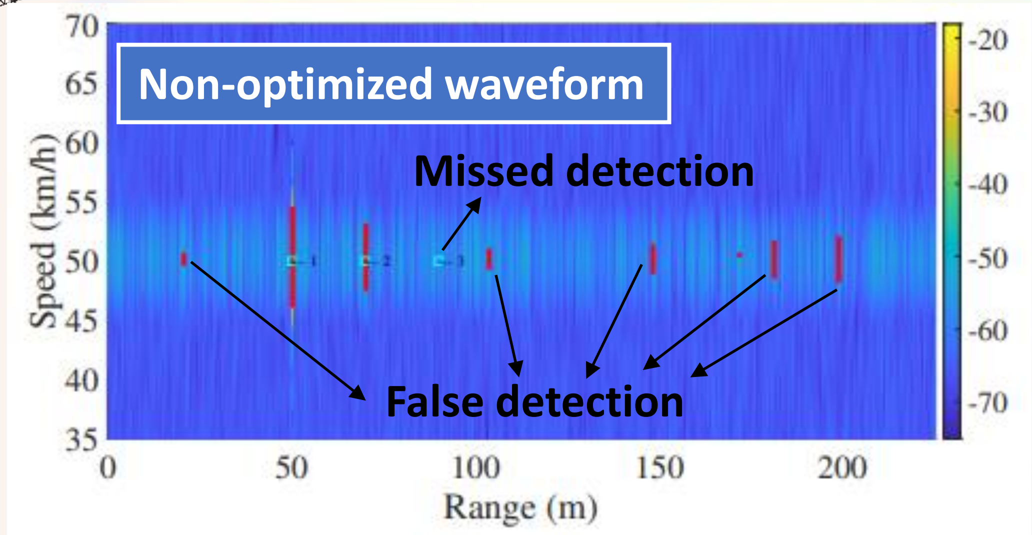


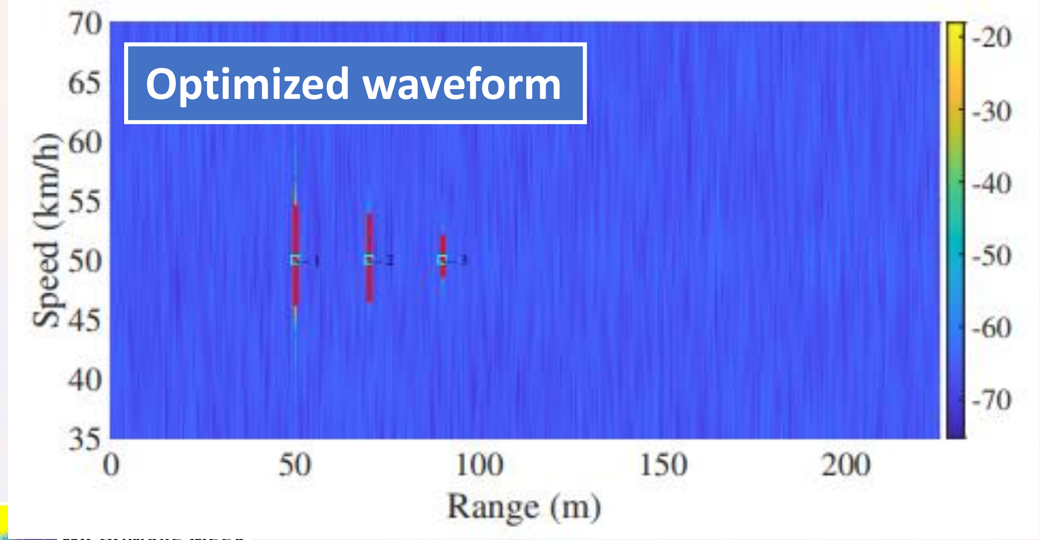
- Optimized sequence is better than m-sequence
- Alphabet size ↗, sidelobe level ↘





Performance Evaluation

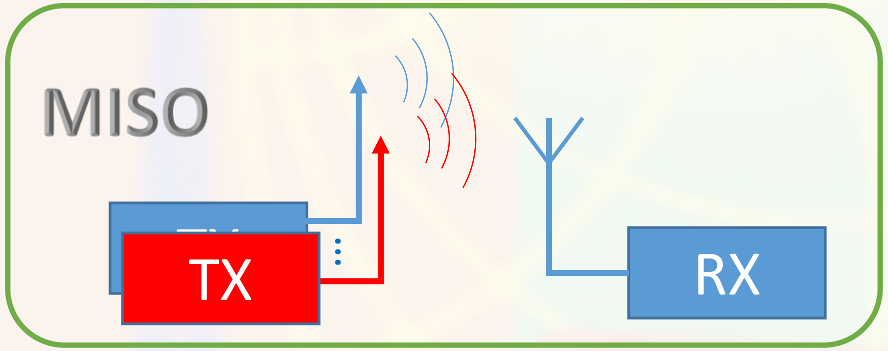




System parameters	Value
Operating frequency	79 (GHz)
Transmitting power	12 (dBm)
Antenna gain	10 (dB)
Maximum detection range	225 (m)
Bandwidth	300 (MHz)
Range resolution	0.5 (m)
Receiver noise figure	15 (dB)
Transmission time	27.3 (μs)
Inter-pulse duration	10.7 (μs)
PMCW code length	8191
Number of pulses	256
Doppler FFT size	512
Max umambiguous relative velocity	89 (km/h)
Total active frame time	7.68 (ms)

Extension: ISL/PSL for MIMO

Multi Input Single Output

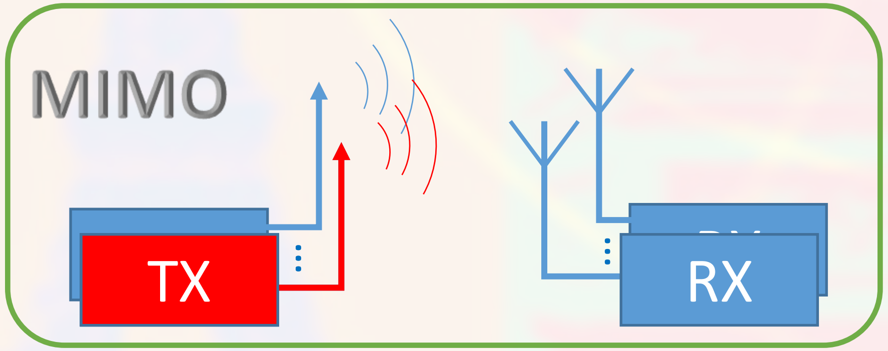


$$x_m = [x_m(1), x_m(2), \ldots., x_m(N)]^T \in \mathbb{C}^N$$

$$\mathbf{X} = [x_1, \quad x_2, \quad \ldots, x_M] \in \mathbb{C}^{N \times M}$$

$$r_{ml}(k) = \sum_{n=1}^{N-k} x_m(n) x_l^*(n+k) = r_{lm}^*(-k)$$

Multi Input Multi Output



$$\mathrm{PSL} = \max \left\{ \boxed{\max_m \max_{k \neq 0} |r_{mm}(k)|}, \boxed{\max_{m,l} \max_k |r_{ml}(k)|} \right\}$$

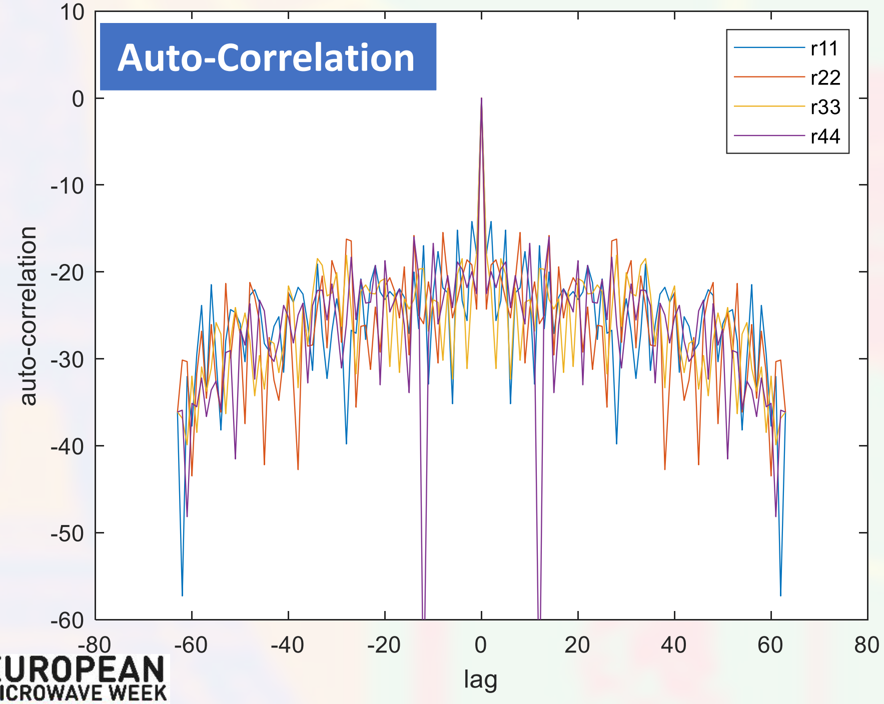
Intra Sequence (solved) Between Sequences

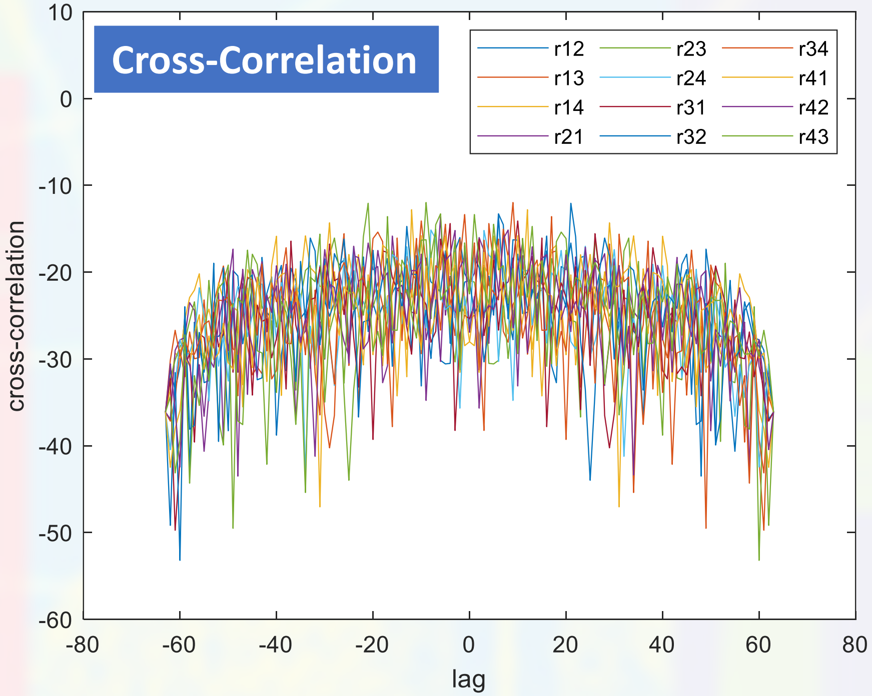
$$\mathrm{ISL} = \boxed{\sum_{m=1}^{N_T} \sum_{k=-N+1}^{N-1} |r_{mm}(k)|^2} + \boxed{\sum_{m,l=1}^{N_T} \sum_{k=-N+1}^{N-1} |r_{ml}(k)|^2}$$

How to design set of sequences with small PSL / ISL ?

Extension: ISL/PSL for MIMO

$$\begin{cases} \min_{\mathbf{X}\in\mathbb{C}^{N\times M}} \quad \text{ISL} = \sum_{m=1}^{N_T} \sum_{k=-N+1}^{N-1} |r_{mm}(k)|^2 + \sum_{m,l=1}^{N_T} \sum_{k=-N+1}^{N-1} |r_{ml}(k)|^2 \\ \text{subject to} \quad |x_{n,m}| = 1, \forall \begin{cases} n & = 1,\ldots,N \\ m, m' & = 1,\ldots,M. \end{cases} \end{cases}$$





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Thanks!