CS 341 Saturday Dec. 17, 2005

A. Lopez-Ortiz, A. Lubiw

FINAL EXAM

1. [10 marks; 2 each] Is each of the following True or False. Justify briefly.

   (a) 2^{n+3} is in $O(2^n)$

   (b) n is in $O(n \log n)$

   (c) $n^{1+\epsilon}$ is in $O(n \log n)$ if $\epsilon > 0$ is sufficiently small

   (d) $O(\max\{f(n), g(n)\}) = O(f(n) + g(n))$

   (e) $f(n)$ is in $\Omega(g(n))$ implies $f(n)$ is in $\Theta(g(n))$

2. [10 marks; 2 each] For each of the following, say whether it is proved to be true, proved to be false, or neither. Justify very briefly.

   (a) If a problem is NP-complete then there is no polynomial time algorithm for it.

   (b) The Church-Turing thesis.

   (c) If one NP-complete problem has an exponential time lower bound then they all do.

   (d) There is a decision problem that cannot be solved in polynomial time.

   (e) There is a branch-and-bound algorithm for the Halting Problem.

3. [15 marks] Give a divide-and-conquer algorithm to convert an n-bit binary number $b_{n-1}b_{n-2}\ldots b_0$ to a decimal number (i.e. base 10). Assuming that multiplication and and division of numbers that are at most n digits long takes $O(n^{1+\epsilon})$, prove that your algorithm's running time is $O(n^{1+\epsilon})$. (Note that addition of n-digit numbers takes $O(n)$.) Show this by stating the recurrence and solving it using the method of your choice (e.g. master method, guess and check, etc.).

4. [10 marks] Let $G = (V, E)$ be a directed graph with integer weights on the edges $w : E \to Z$. Note that the edge weights may be negative. Recall the dynamic programming algorithm to find shortest paths between all pairs of vertices in a directed graph. Order the vertices $1, \ldots, n$. Let $d_k(i, j)$ be the length of a shortest path from vertex i to vertex j using intermediate vertices from the set $\{1, \ldots, k\}$. Recall the formula

$$d_k(i, j) = \min\{d_{k-1}(i, j), d_{k-1}(i, k) + d_{k-1}(k, j)\}$$

   (a) Give the algorithm.

   (b) What goes wrong if the graph has a negative weight cycle? Be specific—give an example and show step by step what goes wrong.

   (c) Modify the algorithm so that it solves the following problem: Given a directed graph with integer weights on the edges, does the graph have a negative weight cycle?

5. [20 marks] Consider the following problem: Given two sets of natural numbers, A and B, are there non-empty subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\sum A' = \sum B'$? Note that $\sum A'$ means $\sum_{i \in A'} i$.

(a) [10 marks] Prove that the problem is NP-complete.

(b) [4 marks] If A has n numbers, and B has m numbers, what is a bound on the running time of a brute force algorithm that tries all possible subsets A' and all possible subsets B'?

(c) [6 marks] What advantages, if any, would a branch-and-bound algorithm have over the brute force algorithm? Is the worst case run time better?

6. [10 marks. This is a hard question, attempt last] Marvin would like to write a program T such that given a second program P, T tells us if P runs in time at most $O(n^2)$ on input size n, for all $n \geq 0$. Prove that this is undecidable and hence no such program T exists.