Tutorial Problems 7 Abbreviated Solutions

1. Prove that a graph with n vertices where every vertex has degree at least $n/2$ is connected.

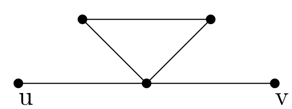
   Solution. Let $u, v \in V(G)$. If $uv \in E(G)$, then the edge is a u,v-path. Otherwise, let $N(u)$ and $N(v)$ be the set of neighbours of u and v respectively. Since every vertex has degree at least $n/2$, $|N(u)| \geq n/2$ and $|N(v)| \geq n/2$. But both sets are subsets of $V(G) \setminus \{u, v\}$ which has size $n-2$. So there must be at least one vertex $w \in N(u) \cap N(v)$. Then u, w, v is a u,v-path in G. Therefore, G is connected.

2. Prove that when $k \geq 2$, any k-regular graph contains a cycle of length at least $k+1$.

   Solution. Let $P = v_1, v_2, \ldots, v_m$ be a longeset path. All the neighbours of v_1 must be on the path, for otherwise P can be extended to form a longer path. Since the graph is k-regular, v_1 has degree k, so there are k vertices in P that are adjacent to v_1. In particular, the furthest one, say v_l, must satisfy $l \geq k+1$. Then $v_1, v_2, \ldots, v_l$ form a cycle of length at least $k+1$.

3. Prove or disprove: If there exists a u,v-walk in G of odd length, then there exists a u,v-path in G of odd length.

   Solution. False. For example, in the following graph, there exists a u,v-walk of length 5, but the only u,v-path has length 2.



4. Prove that the edges of a graph G can be partitioned into edge-disjoint cycles if and only if every vertex of G has even degree.

   Solution. If the edges can be partitioned into edge-disjoint cycles, then each cycle contributes 2 to the degree of each vertex it visits. So every vertex has even degree.

   For the other direction, we will use induction on the number of edges m. When $m = 0$, there are no edges, so this statement is trivially true. Let G be a graph with m edges. Remove all the isolated vertices. Since all the degrees are even and no vertex has degree 0, every vertex has degree at least 2. So G contains at least one cycle C. Remove C from G to get G'. In G', if C does not goes through a vertex, then the degree of the vertex does not change, so it is still even. If C goes through a vertex, then the degree of the vertex reduces by 2, so it is still even. So every vertex in G' has even degree, and G' has fewer than m edges. Therefore by induction hypothesis, the edges of G' can be partitioned into edge-disjoint cycles. Together with C, they form a partition of the edges of G into edge-disjoint cycles.

5. Let G be a graph and $X \subseteq V(G)$. Let C be a cycle in G. Prove that the number of edges in C that is in the cut induced by X is even.

   Solution. Let the vertices on the cycle be $v_1, v_2, \ldots, v_k, v_1$. Assume wlog that $v_1 \in X$. As we go along the cycle, whenever $v_i \in X$ and $v_{i+1} \notin X$, or when $v_i \notin X$ and $v_{i+1} \in X$, the edge $v_i v_{i+1}$ is in the cut induced by X. Since we start with $v_1 \in X$ and end with $v_1 \in X$, the number of times this change can happen is even. Hence there are even number of edges in the cut induced by X.