Module 8: Tries and String Matching

CS 240 - Data Structures and Data Management

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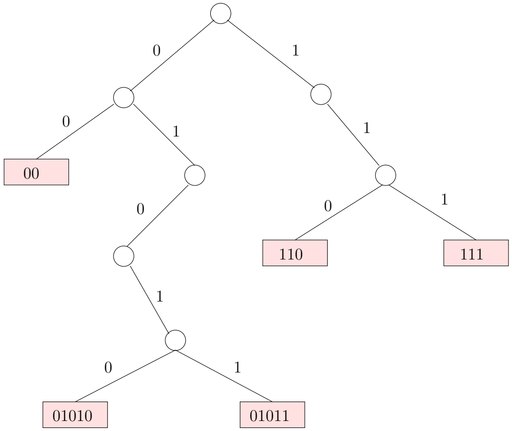
Tries

- Trie (Radix Tree): A dictionary for binary strings
  - Comes from retrieval, but pronounced "try"
  - A binary tree based on bitwise comparisons
  - Similar to radix sort: use individual bits, not the whole key
- Structure of trie:
  - Items (keys) are stored only in the leaf nodes
  - A left child corresponds to a 0 bit
  - A right child corresponds to a 1 bit
- Keys can have different number of bits
- prefix-free: no key is a prefix of another key
- A prefix of a string $S[0..n-1]$:
  a substring $S[0..i]$ of S for some $0 \leq i \leq n-1$

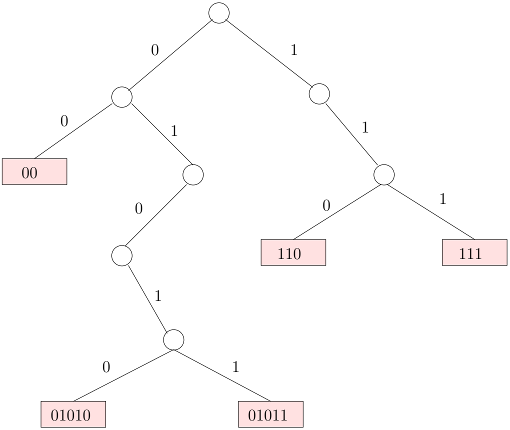
Tries

- Example: A trie for $S = \{00, 110, 111, 01010, 01011\}$



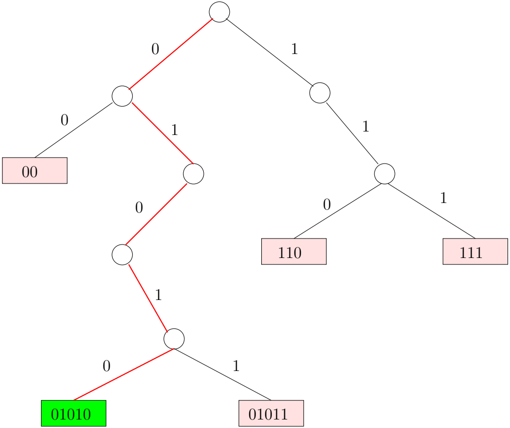
Tries: Search

- Search: start from the root, follow the relevant path using bitwise comparisons
- Example: Search(01010)



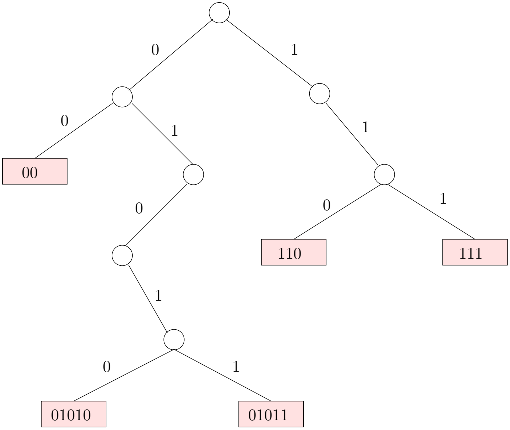
Tries: Search

- Search: start from the root, follow the relevant path using bitwise comparisons
- Example: Search(01010) successful



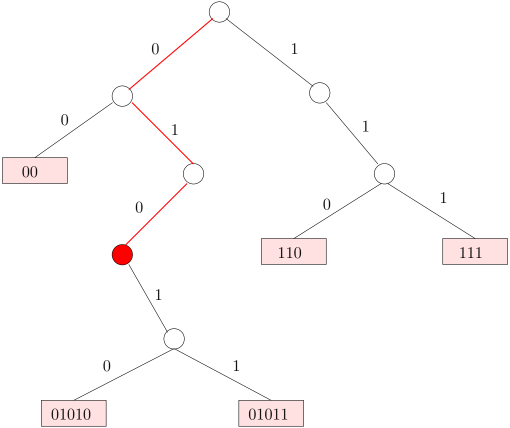
Tries: Search

- Search: start from the root, follow the relevant path using bitwise comparisons
- Example: Search(0100)



Tries: Search

- Search: start from the root, follow the relevant path using bitwise comparisons
- Example: Search(0100) unsuccessful

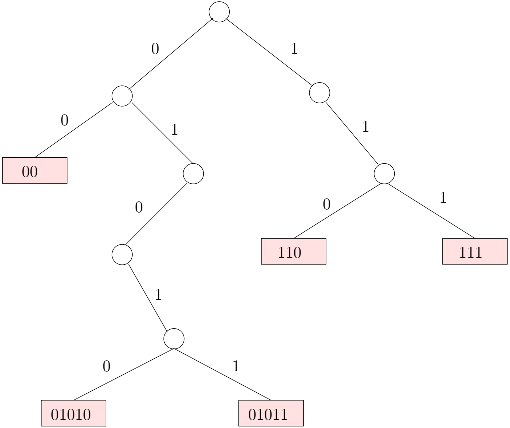


Tries: Insert

- Insert(x)
  - First search for x
  - If we finish at a leaf with key x, then x is already in trie: do nothing
  - If we finish at a leaf with a key $y \neq x$, then y is a prefix of x: not possible because our keys are prefix-free
  - If we finish at an internal node and there are no extra bits: not possible because our keys are prefix-free
  - If we finish at an internal node and there are extra bits: expand trie by adding necessary nodes that correspond to extra bits

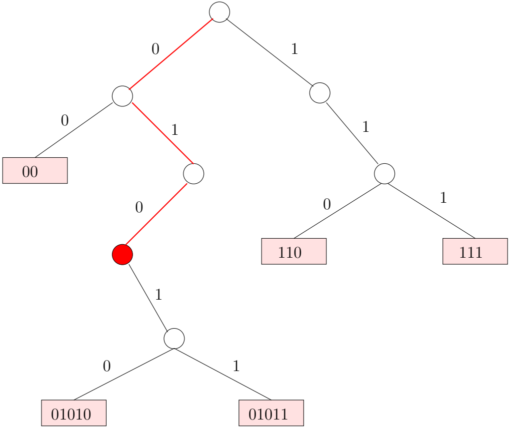
Tries: Insert

- Insert(x)
- Example: Insert(01000)



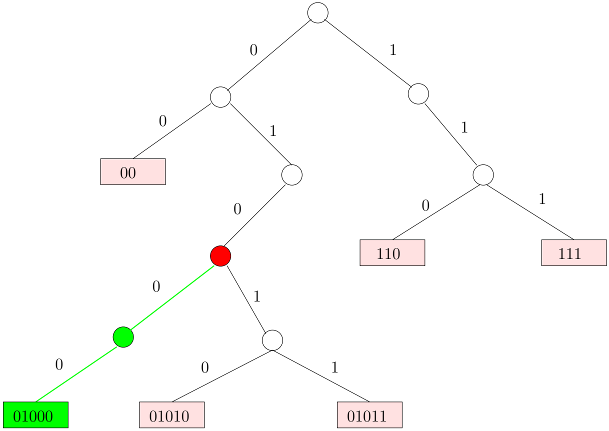
Tries: Insert

- Insert(x)
- Search(01000) unsuccessful Extra bits: 00



Tries: Insert

- Insert(x)
- Search(01000) unsuccessful Extra bits: 00

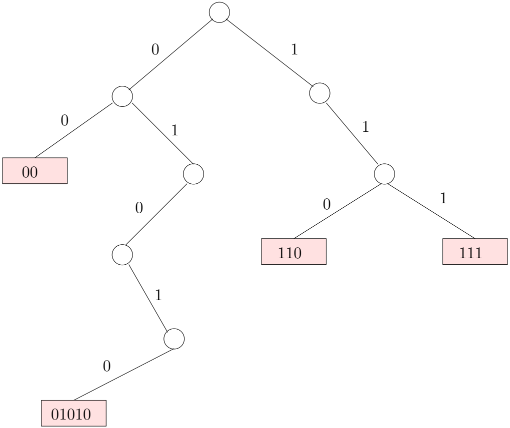


Tries: Delete

- Delete(x)
  - Search for x to find the leaf v_x
  - Delete v_x and all ancestors of v_x until we reach an ancestor that has two children

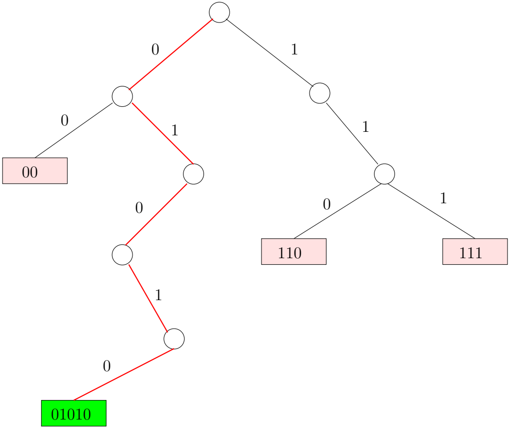
Tries: Delete

- Delete(x)
- Example: Delete(01010)



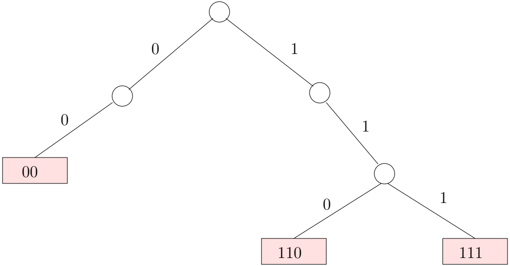
Tries: Delete

- Delete(x)
- Example: Delete(01010)
- Search(01010) successful



Tries: Delete

- Delete(x)
- Example: Delete(01010)



Tries: Operations

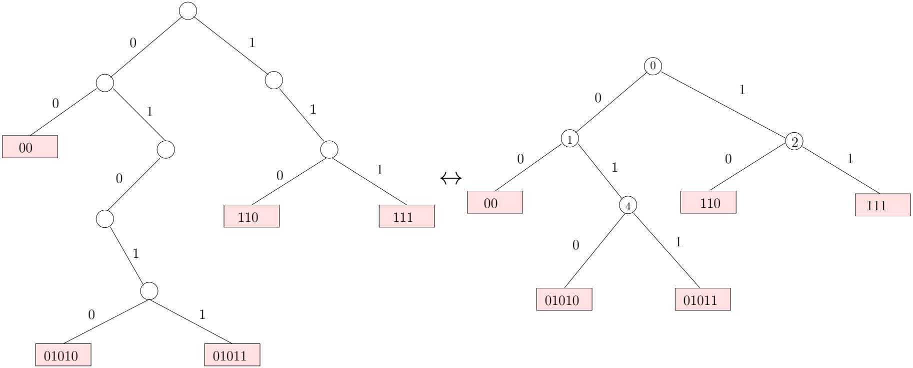
- Search(x)
- Insert(x)
- Delete(x)
- Time Complexity of all operations: $\Theta(|x|)$
  $|x|$: length of binary string x, i.e., the number of bits in x

Compressed Tries (Patricia Tries)

- Patricia: Practical Algorithm To Retrieve Information Coded in Alphanumeric
- Introduced by Morrison (1968)
- Reduces storage requirement: eliminate nodes with only one child
- Every path of one-child nodes is compressed to a single edge
- Each node stores an index indicating the next bit to be tested during a search
- A compressed trie storing n keys always has $n - 1$ internal (non-leaf) nodes

Compressed Tries (Patricia Tries)

- Each node stores an index indicating the next bit to be tested during a search
- Example: A trie and the equivalent compressed trie



Compressed Tries: Operations

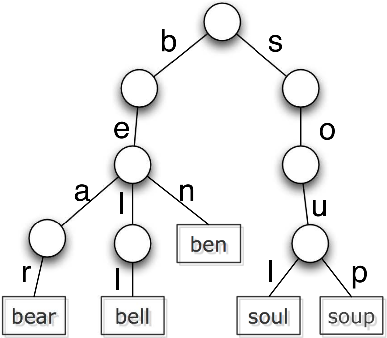
- Search(x):
  - Follow the proper path from the root down in the tree to a leaf
  - If search ends in an internal node, it is unsuccessful
  - In search ends in a leaf, we need to check again if the key stored at the leaf is indeed x
- Delete(x):
  - Perform Search(x) to find a leaf
  - Delete the leaf and its parent

Compressed Tries: Operations

- Insert(x):
  - Perform Search(x)
  - If the search ends at a leaf L with key y, compare x against y to determine the first index i where they disagree.
    Create a new node N with index i.
    Insert N along the path from the root to L so that the parent of N has index $< i$ and one child of N is either L or an existing node on the path from the root to L that has index $> i$.
    The other child of N will be a new leaf node containing x.
  - If the search ends at an internal node, we find the key corresponding to that internal node and proceed in a similar way to the previous case.

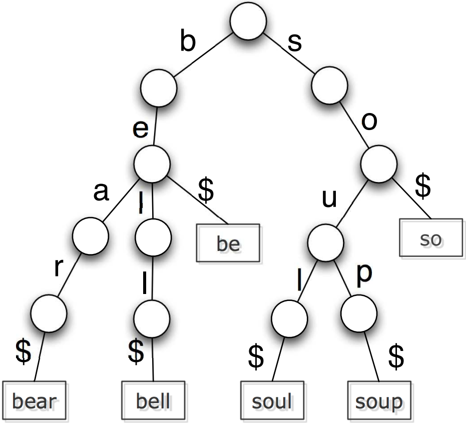
Multiway Tries

- To represent Strings over any fixed alphabet Σ
- Any node will have at most $|\Sigma|$ children
- Example: A trie holding strings {bear, bell, ben, soul, soup}



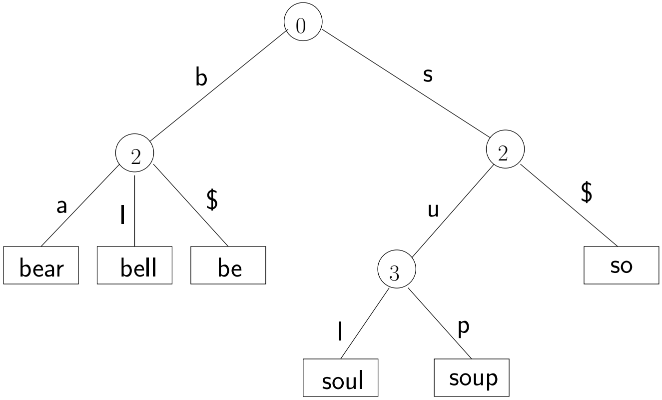
Multiway Tries

- Allow strings that are prefixes of other strings:
  Append a special end-of-word character, say \$, to all keys
- Example: A trie holding strings {bear, bell, be, so, soul, soup}



Multiway Tries

- Compressed multi-way tries
- Example: A compressed trie holding strings {bear, bell, be, so, soul, soup}



Pattern Matching

- Search for a string (pattern) in a large body of text
- $T[0..n-1]$ – The text (or haystack) being searched within
- $P[0..m-1]$ – The pattern (or needle) being searched for
- Strings over alphabet Σ
- Return the first i such that

$$P[j] = T[i+j] \quad \text{for} \quad 0 \leq j \leq m-1$$

- This is the first occurrence of P in T
- If P does not occur in T, return `FAIL`
- Applications:
  - Information Retrieval (text editors, search engines)
  - Bioinformatics
  - Data Mining

Pattern Matching

Example:

- $T =$ "Where is he?"
- $P_1 =$ "he"
- $P_2 =$ "who"

Definitions:

- Substring $T[i..j]$ $0 \leq i \leq j < n$: a string of length $j - i + 1$ which consists of characters $T[i], \ldots T[j]$ in order
- A prefix of T:
  a substring $T[0..i]$ of T for some $0 \leq i < n$
- A suffix of T:
  a substring $T[i..n-1]$ of T for some $0 \leq i \leq n-1$

General Idea of Algorithms

Pattern matching algorithms consist of guesses and checks:

- A **guess** is a position i such that P might start at $T[i]$.
  Valid guesses (initially) are $0 \leq i \leq n - m$.
- A **check** of a guess is a single position j with $0 \leq j < m$ where we compare $T[i + j]$ to $P[j]$. We must perform m checks of a single correct guess, but may make (many) fewer checks of an incorrect guess.

We will diagram a single run of any pattern matching algorithm by a matrix of checks, where each row represents a single guess.

Brute-force Algorithm

Idea: Check every possible guess.

```
BruteforcePM(T[0..n − 1], P[0..m − 1])
T: String of length n (text), P: String of length m (pattern)
    for i ← 0 to n − m do
        match ← true
        j ← 0
        while j < m and match do
            if T[i + j] = P[j] then
                j ← j + 1
            else
                match ← false
        if match then
            return i
    return FAIL
```

Example

- Example: $T =$ `abbbababbab`, $P =$ `abba`

a	b	b	b	a	b	a	b	b	a	b
a	b	b	a							
	a									
		a								
			a							
				a	b	b				
					a					
						a	b	b	a	

- What is the worst possible input?
  $P = a^{m-1}b$, $T = a^n$
- Worst case performance $\Theta((n-m+1)m)$
- $m \leq n/2 \Rightarrow \Theta(mn)$

Pattern Matching

More sophisticated algorithms

- **KMP** and **Boyer-Moore**
- Do extra preprocessing on the pattern P
- We eliminate guesses based on completed matches and mismatches.

KMP Algorithm

- Knuth-Morris-Pratt algorithm (1977)
- Compares the pattern to the text in left-to-right
- Shifts the pattern more intelligently than the brute-force algorithm
- When a mismatch occurs, what is the most we can shift the pattern (reusing knowledge from previous matches)?

$T =$	a	b	c	d	c	a	b	c	?	?	?
	a	b	c	d	c	a	b	a			
						a	b	c	d	c	a

- KMP Answer: the largest prefix of $P[0..j]$ that is a suffix of $P[1..j]$

KMP Failure Array

- Preprocess the pattern to find matches of prefixes of the pattern with the pattern itself
- The failure array F of size m: $F[j]$ is defined as the length of the largest prefix of $P[0..j]$ that is also a suffix of $P[1..j]$
- $F[0] = 0$
- If a mismatch occurs at $P[j] \neq T[i]$ we set $j \leftarrow F[j-1]$
- Consider $P =$ `abacaba`

j	$P[1..j]$	P	$F[j]$
0	—	abacaba	0
1	b	abacaba	0
2	ba	abacaba	1
3	bac	abacaba	0
4	baca	abacaba	1
5	bacab	abacaba	2
6	bacaba	abacaba	3

KMP Algorithm

KMP(T, P)

T: String of length n (text), P: String of length m (pattern)

```
F ← failureArray(P)
i ← 0
j ← 0
while i < n do
    if T[i] = P[j] then
        if j = m − 1 then
            return i − j //match
        else
            i ← i + 1
            j ← j + 1
    else
        if j > 0 then
            j ← F[j − 1]
        else
            i ← i + 1
return −1 // no match
```

KMP: Example

$P =$ abacaba
$T =$ abaxyabacabbaababacaba

0	1	2	3	4	5	6	7	8	9	10	11
a	b	a	x	y	a	b	a	c	a	b	b
a	b	a	c								
		(a)	b								
			a								
				a							
					a	b	a	c	a	b	a
									(a)	(b)	a

Exercise: continue with $T =$ abaxyabacabbaababacaba

Computing the Failure Array

```
failureArray(P)
P: String of length m (pattern)
    F[0] ← 0
    i ← 1
    j ← 0
    while i < m do
        if P[i] = P[j] then
            F[i] ← j + 1
            i ← i + 1
            j ← j + 1
        else if j > 0 then
            j ← F[j − 1]
        else
            F[i] ← 0
            i ← i + 1
```

KMP: Analysis

failureArray

- At each iteration of the while loop, either
  1. i increases by one, or
  2. the guess index $i - j$ increases by at least one ($F[j-1] < j$)
- There are no more than $2m$ iterations of the while loop
- Running time: $\Theta(m)$

KMP: Analysis

failureArray

- At each iteration of the while loop, either
  1. i increases by one, or
  2. the guess index $i - j$ increases by at least one $(F[j-1] < j)$
- There are no more than $2m$ iterations of the while loop
- Running time: $\Theta(m)$

KMP

- failureArray can be computed in $\Theta(m)$ time
- At each iteration of the while loop, either
  1. i increases by one, or
  2. the guess index $i - j$ increases by at least one $(F[j-1] < j)$
- There are no more than $2n$ iterations of the while loop
- Running time: $\Theta(n)$

KMP: Another Example

- $T =$abacaabaccabacabaabb
- $P =$abacab

Boyer-Moore Algorithm

Based on three key ideas:

- Reverse-order searching: Compare P with a subsequence of T moving backwards
- Bad character jumps: When a mismatch occurs at $T[i] = c$
  - If P contains c, we can shift P to align the last occurrence of c in P with $T[i]$
  - Otherwise, we can shift P to align $P[0]$ with $T[i+1]$
- Good suffix jumps: If we have already matched a suffix of P, then get a mismatch, we can shift P forward to align with the previous occurance of that suffix (with a mismatch from the actual suffix). Similar to failure array in KMP.
- Can skip large parts of T

Bad character examples

P = a l d o
T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								

P = m o o r e
T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				

P = m o o r e

T = b o y e r m o o r e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
											o

P = m o o r e

T = b o y e r m o o r e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
										d	o

P = m o o r e

T = b o y e r m o o r e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
									l	d	o

P = m o o r e

T = b o y e r m o o r e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
								a	l	d	o

P = m o o r e

T = b o y e r m o o r e

Bad character examples

P = a l d o
T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e
T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e

T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e
				e					

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e

T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e
				e					
				(r)	e				

Bad character examples

P = a l d o

T =	w	h	e	r	e	i	s	w	a	l	d	o
				o								
								o				
									a	l	d	o

6 comparisons (checks)

P = m o o r e

T =	b	o	y	e	r	m	o	o	r	e
					e					
					(r)	e				
						(m)				e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e

T = b o y e r m o o r e

				e					
				(r)	e				
					(m)			r	e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e

T = b o y e r m o o r e

				e					
				(r)	e				
					(m)	o	o	r	e

7 comparisons (checks)

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
								(e)	(l)	(l)	(s)							s

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o

Good suffix examples

P = sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

P = odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										

Good suffix examples

P = sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

P = odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										
				(e)						d								

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										
							(o)	(d)							d			

- Good suffix moves further than bad character for 2nd guess.

Good suffix examples

P = sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

P = odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										
							(o)	(d)							d			

- Good suffix moves further than bad character for 2nd guess.
- Bad character moves further than good suffix for 3rd guess.
- This is out of range, so pattern not found.

Last-Occurrence Function

- Preprocess the pattern P and the alphabet Σ
- Build the last-occurrence function L mapping Σ to integers
- $L(c)$ is defined as
  - the largest index i such that $P[i] = c$ or
  - -1 if no such index exists
- Example: $\Sigma = \{a, b, c, d\}, P = abacab$

c	a	b	c	d
$L(c)$	4	5	3	-1

- The last-occurrence function can be computed in time $O(m + |\Sigma|)$
- In practice, L is stored in a size-$|\Sigma|$ array.

Suffix skip array

- Again, we preprocess P to build a table.
- Suffix skip array S of size m: for $0 \leq i < m$, $S[i]$ is the largest index j such that $P[i+1..m-1] = P[j+1..j+m-1-i]$ **and** $P[j] \neq P[i]$.
- **Note**: in this calculation, any negative indices are considered to make the given condition true (these correspond to letters that we might not have checked yet).
- Similar to KMP failure array, with an extra condition.

Example: $P = $ bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$								

Suffix skip array

- Again, we preprocess P to build a table.
- Suffix skip array S of size m: for $0 \leq i < m$, $S[i]$ is the largest index j such that $P[i+1..m-1] = P[j+1..j+m-1-i]$ **and** $P[j] \neq P[i]$.
- **Note**: in this calculation, any negative indices are considered to make the given condition true (these correspond to letters that we might not have checked yet).
- Similar to KMP failure array, with an extra condition.

Example: $P =$ bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$								6

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Example: $P =$ bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$							2	6

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- Similar to KMP failure array, with an extra condition.

Example: $P =$ bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$						-1	2	6

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- **Note**: in this calculation, any negative indices are considered to make the given condition true (these correspond to letters that we might not have checked yet).
- Similar to KMP failure array, with an extra condition.

Example: $P =$ bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$				-3	2	-1	2	6

Suffix skip array

- Again, we preprocess P to build a table.
- Suffix skip array S of size m: for $0 \leq i < m$, $S[i]$ is the largest index j such that $P[i+1..m-1] = P[j+1..j+m-1-i]$ **and** $P[j] \neq P[i]$.
- **Note**: in this calculation, any negative indices are considered to make the given condition true (these correspond to letters that we might not have checked yet).
- Similar to KMP failure array, with an extra condition.

Example: $P =$ bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$	-6	-5	-4	-3	2	-1	2	6

- Computed similarly to KMP failure array in $\Theta(m)$ time.

Boyer-Moore Algorithm

```
boyer-moore(T,P)
1.    L ← last occurrance array computed from P
2.    S ← suffix skip array computed from P
3.    i ← m − 1,      j ← m − 1
4.    while i < n and j ≥ 0 do
5.        if T[i] = P[j] then
6.            i ← i − 1
7.            j ← j − 1
8.        else
9.            i ← i + m − 1 − min(L[T[i]], S[j])
10.           j ← m − 1
11.   if j = −1 return i + 1
12.   else return FAIL
```

Exercise: Prove that $i - j$ always increases on lines 9–10.

Boyer-Moore algorithm conclusion

- Worst-case running time $\in O(n + |\Sigma|)$
- This complexity is difficult to prove.
- What is the worst case?
- On typical English text the algorithm probes approximately 25% of the characters in T
- Faster than KMP in practice on English text.

Suffix Trees (Suffix Tries)

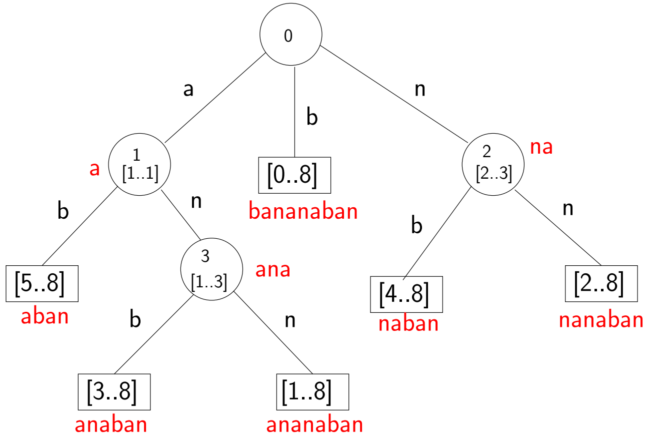
- What if we want to search for for many patterns P within the same fixed text T?
- Idea: Preprocess the text T rather than the pattern P
- Observation: P is a substring of T if and only if P is a prefix of some suffix of T.

Suffix Trees

- Build a compressed trie that stores all suffixes of text T
- Insert suffixes in decreasing order of length
- If a suffix is a prefix of another suffix, we do not insert it
- Store two indexes l, r on each node v (both internal nodes and leaves) where node v corresponds to substring $T[l..r]$

Suffix Trees: Example

T =`bananaban`



0	1	2	3	4	5	6	7	8
b	a	n	a	n	a	b	a	n

Suffix Trees: Pattern Matching

To search for pattern P of length m:

- Similar to Search in compressed trie with the difference that we are looking for a prefix match rather than a complete match
- If we reach a leaf with a corresponding string length less than m, then search is unsuccessful
- Otherwise, we reach a node v (leaf or internal) with a corresponding string length of at least m
- It only suffices, to check the first m characters of that string to see if there indeed is a match

Pattern Matching Conclusion

	Brute-Force	KMP	Boyer-Moore	Suffix trees
Preprocessing:	–	$O(m)$	$O(m + \lvert\Sigma\rvert)$	$O(n^2)$
Search time:	$O(nm)$	$O(n)$	$O(n)$ (often better)	$O(m)$
Extra space:	–	$O(m)$	$O(m + \lvert\Sigma\rvert)$	$O(n)$