ECE 205

NOTES ON LAPLACE TRANSFORMS

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If all of our signals were exponentials $y = e^{st}$, then differentiation would amount to simple multiplication

$$
\begin{aligned}
D y &= D e^{st} \\
&= s e^{st} \\
&= s y
\end{aligned}
$$

The next best thing would be to be able to express our signals as linear combinations of exponentials.

This can be done using the <u>Laplace transform</u>.

The Laplace transform

The Laplace transform (Pierre-Simon Laplace, 1749-1827) can be interpreted as a means of representing a function as a weighted sum of exponentials. The transform is itself the weighting function:

$$F(s) := \mathcal{L}\{f(t)\} := \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

- the transform is defined only if the above integral converges.

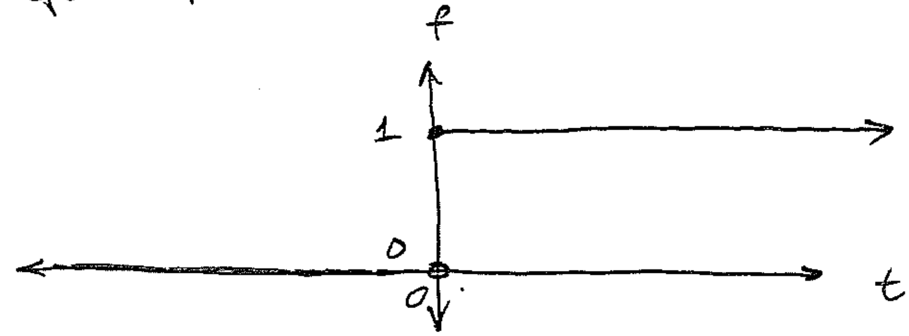
To ensure convergence, we shall suppose that, for some real α, the integral

$$\int_{-\infty}^{\infty} |f(t)|\, e^{-\alpha t}\, dt$$

converges.

Example:

Suppose $f(t)$ is the unit step function:



Then its transform is

$$F(s) := \mathcal{L}\{f(t)\} := \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

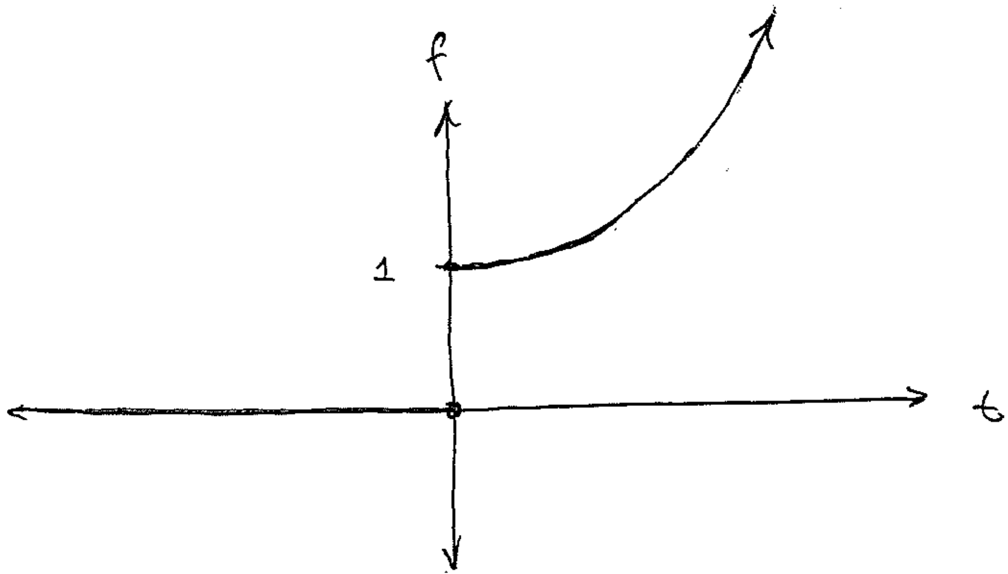
$$= \int_0^{\infty} e^{-st}\, dt$$

$$= \frac{1}{-s} e^{-st} \Big|_0^{\infty}$$

$$= \begin{cases} \frac{1}{s}, & \text{if } \mathrm{Re}(s) > 0 \\ \infty, & \text{otherwise} \end{cases}$$

<u>Example</u>:

$$f(t) = \begin{cases} e^{\alpha t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



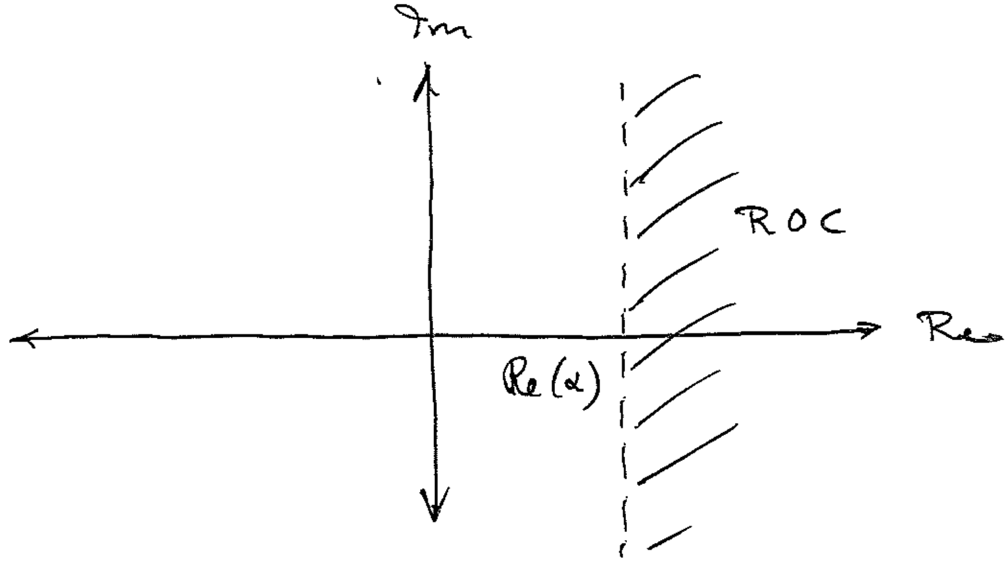
$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{\alpha t} e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s-\alpha)t}\, dt$$

$$= \frac{-1}{s-\alpha}\, e^{-(s-\alpha)t} \Big|_{0}^{\infty}$$

Hence,

$$F(s) = \frac{1}{s-\alpha}, \text{ provided } Re(s) > Re(\alpha)$$



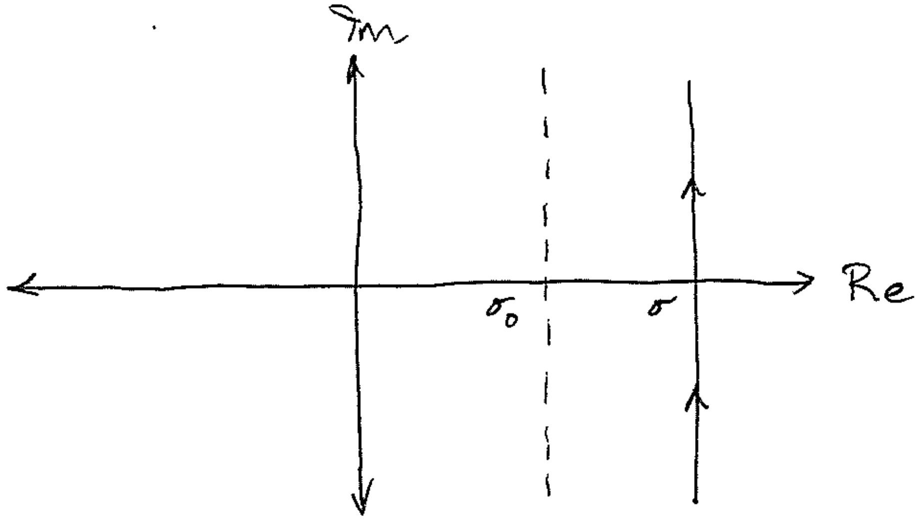
- Both of these examples illustrate how the factor e^{-st} can make the integral converge (in some cases) for sufficiently large $Re(s)$.

The transform can be inverted by means of the following <u>inversion integral</u>:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds$$

- This is a "contour integral" on the complex plane...

... the contour of integration, the straight line $Re(s) = \sigma$, must lie within the ROC of $F(s)$:



- As a practical matter, we won't compute this integral...

... instead, we'll apply table look-up — after a suitable partial-fractions decomposition.

- But, given that an integral is just a sum, the inversion formula shows that $f(t)$ is a sum of exponentials e^{st}, weighted by $F(s)$.

For this reason, it will greatly simplify the solution of linear ODEs with constant coefficients.

Example:

Suppose $F(s) = \dfrac{1}{s(s+10)}$

(for $Re(s) > 10$).

- partial fractions:

$$\frac{1}{s(s+10)} = \frac{1/10}{s} + \frac{-\frac{1}{10}}{s+10}$$

- therefore, the inverse transform of $F(s)$ is

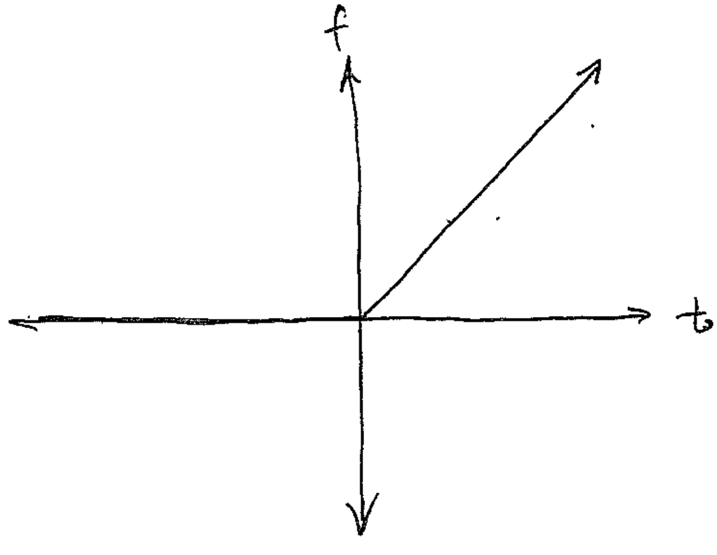
$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+10)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/10}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-1/10}{s+10}\right\}$$

(by linearity of $\mathcal{L}\{\cdot\}$)

$$= \begin{cases} \frac{1}{10}\left[1 - e^{-10t}\right], & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Example:

Suppose $f(t) = \begin{cases} t, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$



Then

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt = \int_{0}^{\infty} t\, e^{-st}\, dt$$

- Use integration by parts, with

$$u = t \quad , \quad v = \left(\frac{-1}{s}\right) e^{-st}$$

Thus,

$$\int_0^\infty t\, e^{-st}\, dt = \left(\frac{-1}{s}\right) t\, e^{-st} \Big|_0^\infty - \int_0^\infty \left(\frac{-1}{s}\right) e^{-st}\, dt$$

$$= 0 + \frac{1}{s^2}, \text{ provided } \operatorname{Re}(s) > 0$$

So $F(s) = \dfrac{1}{s^2}$.

We will mainly be interested in "one-sided" functions ...

... that is, in functions $f(t)$ that have the value 0 for $t < 0$, ..

We'll therefore mainly use the 'one-sided' Laplace transform:

$$F(s) := \mathcal{L}\{f(t)\} := \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

<u>Notation</u>: Let the unit-step function of our first example be denoted $u_{-1}(t)$ (because its transform is s^{-1}).

- This will facilitate the definition of functions that have the value 0 when $t < 0$:

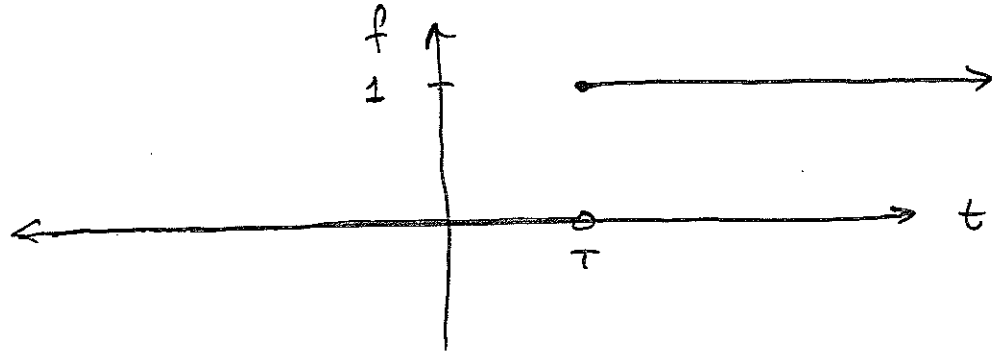
- Ex.

$$\text{Let} \quad f(t) = \begin{cases} 0, & t < 0 \\ t, & \text{otherwise.} \end{cases}$$

$$\text{Then} \quad f(t) = t\, u_{-1}(t), \quad \forall t$$

Example

$$f(t) = u_{-1}(t-T)$$



$$F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt = \int_{T}^{\infty} e^{-st}\, dt$$

$$= \left. \frac{-e^{-st}}{s} \right|_{T}^{\infty}$$

$$= e^{-sT}\,\frac{1}{s},$$

provided $\mathrm{Re}(s) > 0$

Example:

$$f(t) = (\sin \omega t)\, u_{-1}(t)$$

$$= \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \cdot u_{-1}(t)$$

$$= \frac{1}{2j} e^{j\omega t} u_{-1}(t) - \frac{1}{2j} e^{-j\omega t} u_{-1}(t)$$

So

$$F(s) = \frac{1}{2j} \frac{1}{s - j\omega} - \frac{1}{2j} \frac{1}{s + j\omega}$$

$$(\mathrm{Re}(s) > 0)$$

$$= \frac{1}{2j} \frac{2j\omega}{s^2 + \omega^2}$$

$$= \frac{\omega}{s^2 + \omega^2}$$

Example:

By a previous example,

$$\mathcal{L}\{(\sin t)\, u_{-1}(t)\} = \frac{1}{s^2+1} .$$

Hence, for $\omega > 0$,

$$\mathcal{L}\{(\sin \omega t)\, u_{-1}(t)\} = \frac{1}{\omega} \frac{1}{\left(\frac{s}{\omega}\right)^2 + 1}$$

$$= \frac{1}{\omega} \frac{\omega^2}{s^2+1}$$

$$= \frac{\omega}{s^2+1} ,$$

as we have already calculated.

Key properties:

1. Linearity

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$$

(the ROC is the intersection of those of $F(s)$ and $G(s)$).

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

2. Time-scaling: for $c > 0$,

$$\mathcal{L}\{f(ct)\} = \int_{\infty}^{\infty} f(ct)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\, e^{-s\frac{\tau}{c}}\, \frac{1}{c}\, d\tau$$

$$= \frac{1}{c}\, F\left(\frac{s}{c}\right)$$

3. <u>Exponential modulation</u>

$$\mathcal{L}\{e^{\alpha t} f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-(s-\alpha)t}\, dt$$

$$= F(s-\alpha)$$

Example:

$$\mathcal{L}\{e^{\alpha t} u_{-1}(t)\} = \frac{1}{s-\alpha}$$

(as we have already calculated).

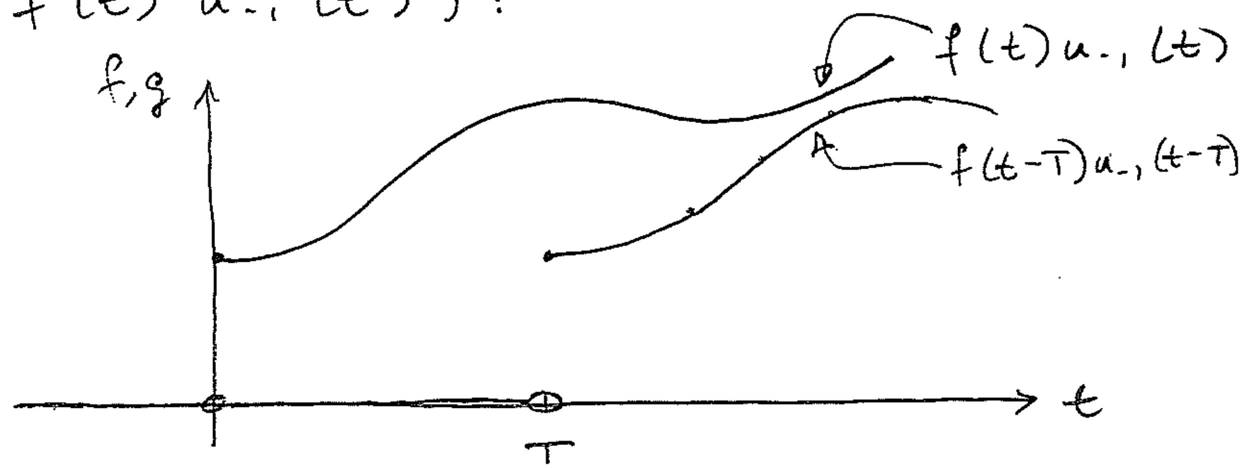
4. Time-shifting

Suppose $F(s) = \mathcal{L}\{f(t)\,u_{-1}(t)\}$

(that is, $F(s)$ is the one-sided transform of $f(t)$).

Let $g(t) = f(t-T)\cdot u_{-1}(t-T)$

(that is, a 'delayed' version of $f(t)\,u_{-1}(t)$):



Then

$$G(s) = \int_{-\infty}^{\infty} g(t)\,e^{-st}\,dt = \int_{-\infty}^{\infty} f(t-T)\,u_{-1}(t-T)\,e^{-st}\,dt$$

$$= \int_{T}^{\infty} f(t-T)\,e^{-st}\,dt$$

Then

$$G(s) = \int_{-\infty}^{\infty} g(t) \cdot e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} f(t-T)\, u_{-1}(t-T) \cdot e^{-st}\, dt$$

$$= \int_{T^-}^{\infty} f(t-T)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} f(\tau)\, e^{-s(\tau+T)}\, d\tau$$

$$= e^{-sT} \int_{0^-}^{\infty} f(\tau)\, e^{-s\tau}\, d\tau$$

$$= e^{-sT}\, F(s)$$

5. Multiplication by t:

$$\mathcal{L}\{t \cdot f(t)\} = \int_{-\infty}^{\infty} t \cdot f(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} t\, e^{-st} f(t)\, dt$$

$$= \int_{-\infty}^{\infty} -\frac{\partial}{\partial s}\, e^{-st} f(t)\, dt$$

$$= -\frac{d}{ds} \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

(this can be shown to follow from absolute convergence)

$$= -\frac{d}{ds} F(s)$$

Example:

We've already seen that

$$\mathcal{L}\{t \cdot u_{-1}(t)\} = \frac{1}{s^2}$$

$$= -\frac{d}{ds}\frac{1}{s}$$

$$= -\frac{d}{ds}\mathcal{L}\{u_{-1}(t)\}$$

It follows from the above property that

$$\mathcal{L}\{t^{n-1} \cdot u_{-1}(t)\} = (-1)^{n-1}\frac{d^{n-1}}{ds^{n-1}}\frac{1}{s}$$

$$= (-1)^{n-1}(-1)^{n-1}\frac{(n-1)!}{s^n}$$

$$= \frac{(n-1)!}{s^n}$$

6. Differentiation / Integration

Suppose that there exists a real α such that

$$\int_{0^-}^{\infty} |f(t)| e^{-\alpha t}\, dt$$

converges, and that there exists a function $f'(t)$ such that, for $t \geq 0$,

$$f(t) = f(0^-) + \int_{0^-}^{t} f'(\tau)\, d\tau$$

and there exists a real β such that

$$\int_{0^-}^{\infty} |f'(t)| e^{-\beta t}\, dt$$

converges. Then both $f(\cdot)$ and $f'(\cdot)$ must have Laplace transforms. (One-sided Laplace transforms, at least.)

$$\int_{0^-}^{\infty} f(t)e^{-st}\,dt = \int_{0^-}^{\infty}\left[f(0^-) + \int_{0^-}^{t} f'(\tau)\,d\tau\right]e^{-st}\,dt$$

$$= \frac{1}{s}f(0^-) + \int_{0^-}^{\infty}\int_{0^-}^{t} f'(\tau)\,d\tau\, e^{-st}\,dt$$

Integrating by parts,

$$\int_{0^-}^{\infty}\int_{0^-}^{t} f'(\tau)\,d\tau\, e^{-st}\,dt$$

$$= \left(\frac{-1}{s}\right)\int_{0^-}^{t} f'(\tau)\,d\tau\, e^{-st}\Bigg|_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty} f'(t)e^{-st}\,dt$$

The second term on the right is $\frac{1}{s}\mathcal{L}\{f'(t)\}$; the first term is zero . . .

To see why, recall that, for some real β — and hence, for some real $\beta > 0$ — the integral

$$\int_{0^-}^{\infty} |f'(t)| e^{-\beta t} \, dt$$

converges. Now,

$$\left| \int_{0^-}^{t} f'(\tau) d\tau \right| = \left| \int_{0^-}^{t} f'(\tau) e^{\beta\tau} e^{-\beta\tau} d\tau \right|$$

$$\leq e^{\beta t} \left| \int_{0^-}^{t} f'(\tau) e^{-\beta\tau} d\tau \right|$$

$$\leq e^{\beta t} \int_{0^-}^{\infty} |f'(\tau)| e^{-\beta\tau} d\tau$$

Hence,

$$\lim_{t\to\infty} \int_{0^-}^{t} f'(\tau) d\tau \cdot e^{-st} = 0,$$

provided $\mathrm{Re}(s) > \beta$.

Going back to our first equation, we therefore have, for $Re(s) > \beta$,

$$F(s) = \frac{1}{s} f(0^-) + \frac{1}{s} \mathcal{L}\{f'(t)\}.$$

or

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-)$$

This property gives us a powerful tool for solving differential equations — by converting them into algebraic equations.

Example:

$$\dot{y} + y = t + e^{t}, \quad y(0^-) = 1$$

$$\Longleftrightarrow \quad sY(s) - y(0^-) + Y(s) = \frac{1}{s^2} + \frac{1}{s-1}$$

$$\Longleftrightarrow \quad (s+1)Y(s) = 1 + \frac{1}{s^2} + \frac{1}{s-1}$$

$$\Longleftrightarrow \quad Y(s) = \frac{1}{s+1} + \frac{1}{s^2(s+1)} + \frac{1}{(s+1)(s-1)}$$

$$= \frac{3/2}{s+1} + \frac{1}{s^2} - \frac{1}{s} + \frac{1/2}{s-1}$$

(by partial fractions)

$$\Longleftrightarrow \quad y(t) = \frac{3}{2}e^{-t} + t - 1 + \frac{1}{2}e^{t}$$

Owing to the previous property, we'll be solving ODES by doing algebra in the Laplace domain.

Algebra is based on the operations of addition and multiplication. By linearity, we know that addition in the Laplace domain corresponds to addition in the time domain. What about multiplication?

It turns out that the counterpart of multiplication is an operation called <u>convolution</u>. Naturally, convolution therefore bears an important relationship to ODEs.

Given two functions $f(t)$ and $g(t)$, their <u>convolution</u> is the function

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

Consider integrating instead with respect to $u = t - \tau$ $\iff \tau = t - u$. We have

$$(f*g)(t) = \int_{+\infty}^{-\infty} f(t-u)\, g(u)\, (-du)$$

$$= \int_{-\infty}^{\infty} g(u)\, f(t-u)\, du$$

$$= (g*f)(t)$$

So convolution is commutative.

When we compute $f*g = g*f$, we say that we are <u>convolving</u> the functions.

<u>If the functions being convolved are one-sided</u>, we can simplify the integral:

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} f(\tau)\, g(t-\tau)\, d\tau \qquad (\because f \text{ is 1-sided})$$

$$= \int_{0}^{t} f(\tau)\, g(t-\tau)\, d\tau \qquad (\because g \text{ is 1-sided})$$

In this case, the convolution is also a one-sided function.

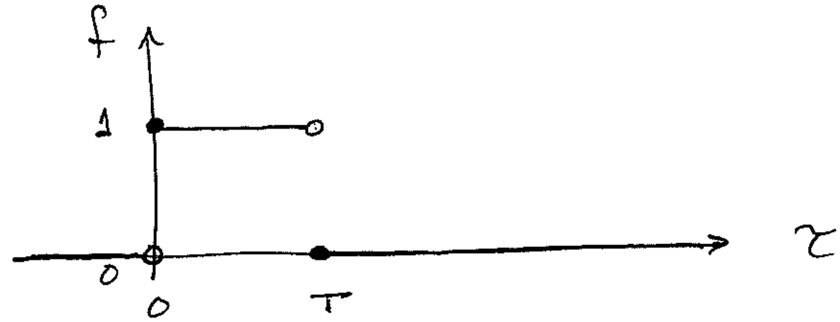
Example:

Suppose $f(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$

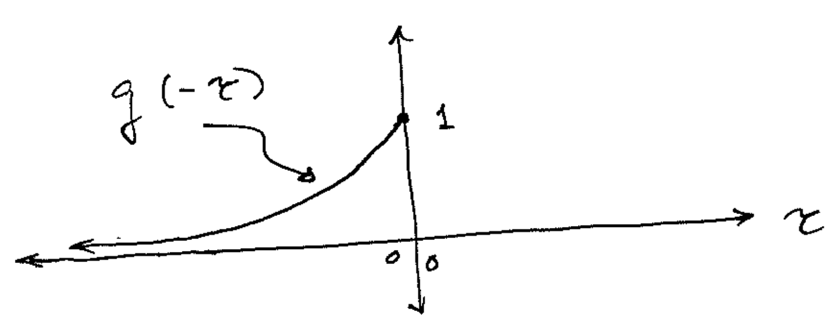
and $g(t) = e^{-t} u_{-1}(t)$. Then

$$f * g\,(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

It's easy to see that $f(\tau)$ has this graph:

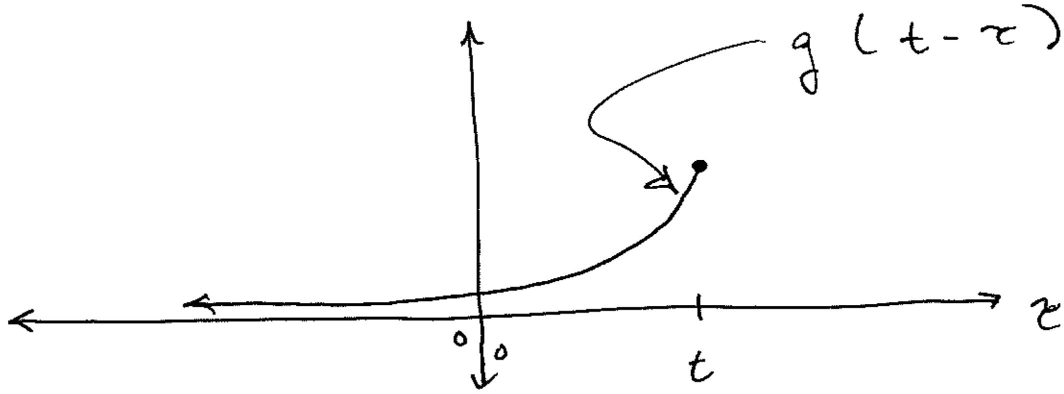


To plot $g(t-\tau)$, consider first $g(-\tau)$:

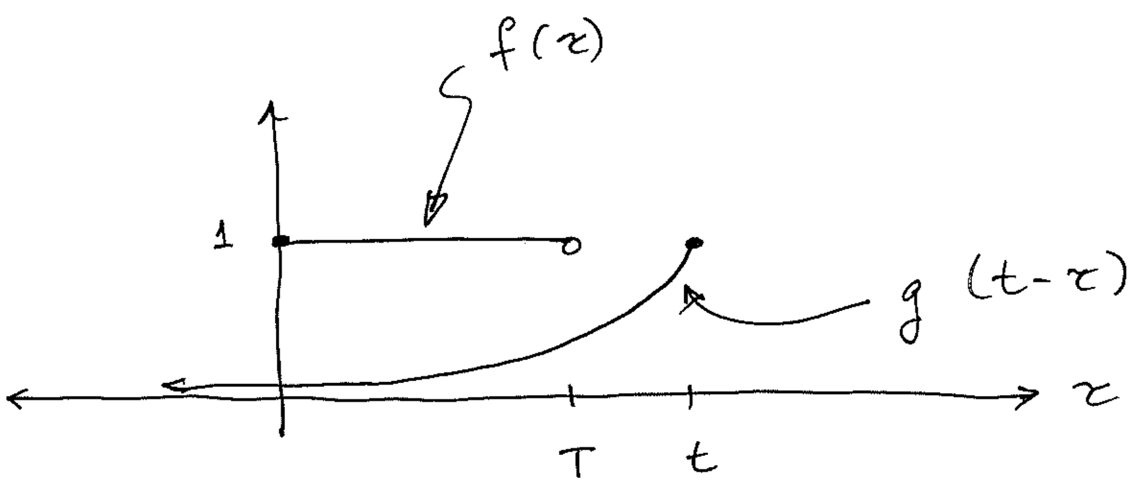


So $g(t-\tau) = g(-(\tau - t))$

looks the same, but shifted right by an interval t:



To find the convolution, multiply the two functions and integrate from 0 to t:



It can be seen from the picture that, if $0 \le t < T$, the value of the convolution is the area under the curve $e^{-\tau}$ from $\tau = 0$ to $\tau = t$.

That is, for $0 \le t < T$,

$$(f * g)(t) = \int_0^t e^{-\tau} \, d\tau$$

$$= -e^{-\tau} \Big|_0^t$$

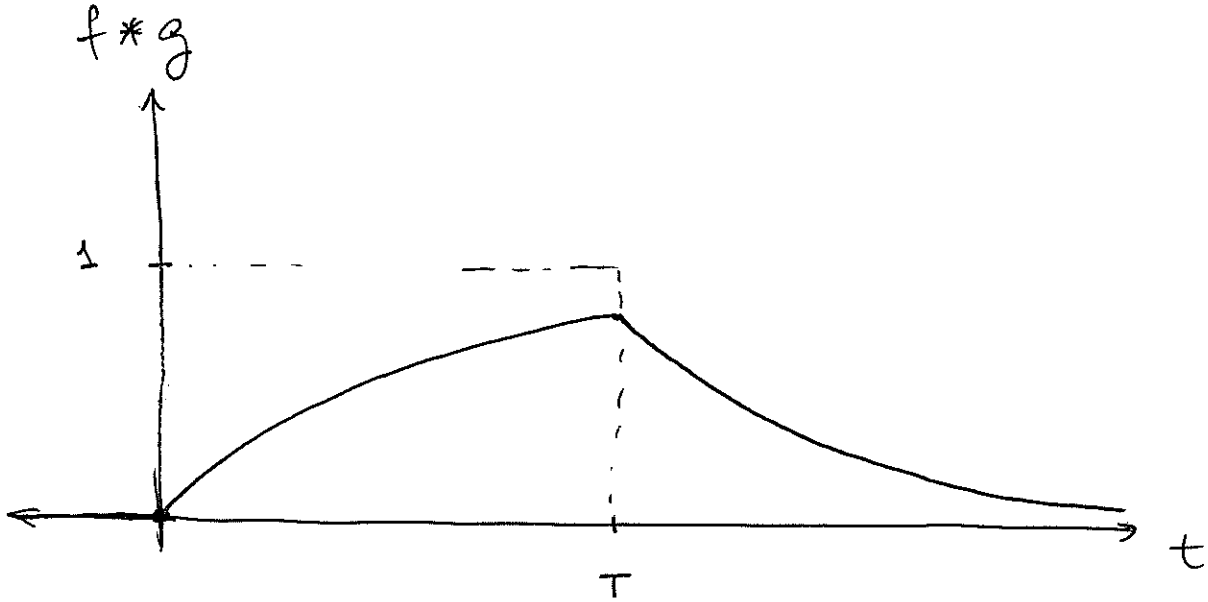
$$= 1 - e^{-t}$$

For $t \geq T$, $(f * g)(t)$ is the area under $e^{-\tau}$ between $\tau = t - T$ and $\tau = t$:

$$(f * g)(t) = \int_{t-T}^{t} e^{-\tau} \, d\tau$$

$$= -e^{-\tau} \Big|_{t-T}^{t} = e^{T-t} - e^{-t}$$

So the plot of $f * g\ (t)$ looks like this:



We'll discuss convolution further a little later. For now, let's continue establishing the properties of the Laplace transform.

7. Convolution

Suppose that, for some real α and β, the integrals

$$\int_{-\infty}^{\infty} |f(t)| e^{-\alpha t}\, dt \quad \& \quad \int_{-\infty}^{\infty} |g(t)| e^{-\beta t}\, dt$$

converge.

We'll show that this means that $(f*g)(t)$ has a Laplace transform, and that

$$\mathcal{L}\{(f*g)(t)\} = F(s)\,G(s)$$

Suppose that $\gamma \geq \alpha, \beta$, and consider the product of the two convergent integrals:

$$\int_{-\infty}^{\infty} |f(t)| e^{-\gamma t} \, dt \int_{-\infty}^{\infty} |g(\tau)| e^{-\gamma \tau} \, d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t)| e^{-\gamma t} |g(\tau)| e^{-\gamma \tau} \, dt \, d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau)| e^{-\gamma (u-\tau)} |g(\tau)| e^{-\gamma \tau} \, du \, d\tau$$

$(u = t + \tau)$.

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau)| e^{-\gamma (u-\tau)} |g(\tau)| e^{-\gamma \tau} \, d\tau \, du$$

(by "Fubini's Theorem")

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau)| |g(\tau)| \, d\tau \, e^{-\gamma u} \, du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | f(u-\tau)\, g(\tau) | \, d\tau \; e^{-\gamma u} \, du$$

$$\geq \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(u-\tau)\, g(\tau) \, d\tau \right| e^{-\gamma u} \, du$$

$$= \int_{-\infty}^{\infty} | (f * g)(u) | \; e^{-\gamma u} \, du$$

So the last integral converges, and $f * g$ has a Laplace transform.

Now,

$$\mathcal{L}\{(f*g)(t)\} = \int_{-\infty}^{\infty} (f*g)(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, e^{-st}\, dt\, d\tau$$

(by the previous proof & "Fubini's Thm")

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, g(\tau)\, e^{-s(u+\tau)}\, du\, d\tau$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-su}\, du \int_{-\infty}^{\infty} g(\tau)\, e^{-s\tau}\, d\tau$$

$$= F(s)\, G(s)$$

8. The initial-value theorem

If $f(\cdot)$ is piecewise-continuous and $\int_{0^-}^{\infty} |f(t)| e^{-\alpha t}\, dt$ converges for some real α, then

$$f(0^+) = \lim_{s \to \infty} s F(s)$$

This gives a means of computing $f(0^+)$ (the "initial value of f") without the need to invert the transform.

Instead, we compute the limit on the right-hand side.

What does this limit mean?

- think of the real part of s as tending to $+\infty$.

- There's a different way of defining the "limit at infinity" of a function of a complex variable, but it doesn't apply, for example, to complex exponentials.

- Depending on the direction in which s "goes to infinity" on the complex plane, e^{-st} may not have a limit.

- It's said to have an "essential singularity" at ∞.

Proof of the initial-value theorem:

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\int_{0^-}^{\infty} f(t)\,e^{-st}\,dt$$

$$= \lim_{s\to\infty} s\int_{0^-}^{\epsilon} f(t)\,e^{-st}\,dt + \lim_{s\to\infty}\int_{\epsilon}^{\infty} f(t)\,s\,e^{-st}\,dt$$

$$= \lim_{s\to\infty} s\int_{0^-}^{\epsilon} f(t)\,e^{-st}\,dt + \int_{\epsilon}^{\infty} \lim_{s\to\infty} f(t)\,s\,e^{-st}\,dt$$

(as $s\to\infty$, the integrands "converge uniformly" on the interval $[\epsilon,\infty)$ if $\epsilon>0$.)

$$= \lim_{s\to\infty} s\int_{0^-}^{\epsilon} f(t)\,e^{-st}\,dt$$

So the value of this limit (if it exists) must be independent of ϵ. Moreover, as ϵ approaches 0,

$$\int_{0^-}^{\epsilon} f(t)\,e^{-st}\,dt \quad \text{behaves like} \quad \int_{0^-}^{\epsilon} f(0^+)\,e^{-st}\,dt$$

$$= \frac{f(0^+)}{s}\left[1-e^{-s\epsilon}\right]$$

So

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \cancel{s}\,\frac{f(0^+)}{\cancel{s}}\left[1-e^{-s\epsilon}\right]$$

$$= f(0^+)$$

 $\square$

Examples:

(1) $F(s) = e^{-sT}\frac{1}{s^2}$, $T > 0$

$$\lim_{s \to \infty} s F(s) = 0$$

CHECK: $f(t) = \mathcal{L}^{-1}\{F(s)\} = (t-T)\,u_{-1}(t-T)$

$$f(0^+) = \lim_{t \downarrow 0} f(t) = 0 \checkmark$$

(2) $F(s) = \frac{1}{s}$

$$\lim_{s \to \infty} s F(s) = 1$$

CHECK: $f(t) = u_{-1}(t)$, so $f(0^+) = 1 \checkmark$

(3) $F(s) = \dfrac{s}{s^2 + \omega^2}$

$$\lim_{s \to \infty} s F(s) = 1$$

CHECK: $f(t) = \cos \omega t$,

$$f(0^+) = \lim_{t \downarrow 0} \cos \omega t = 1 \checkmark$$

Rational functions

Many of the Laplace transforms that we've seen so far take the form of rational functions – that is, functions represented as ratios of polynomials:

– e.g. $F(s) \quad \dfrac{s}{s^2 + 2s + 2}$

Just as with rational numbers, common factors in the numerator and denominator cancel out:

$$\frac{s\,(s+3)}{(s^2+2s+2)(s+3)}$$ is considered equivalent to $\dfrac{s}{s^2+2s+2}$ —

just as $\frac{2}{4}$ is equivalent to $\frac{1}{2}$.

Moreover, just as the rational numbers extend the integers to a <u>field</u>, so the rational functions extend the polynomials to a field (by ensuring that every element has a multiplicative inverse).

Indeed, all of the transforms that we have seen consist of rational functions in s, possibly multiplied by exponentials in s.

The roots of the numerator of a rational function are called the function's (finite) <u>zeros</u>; the roots of the denominator are called its (finite) <u>poles</u>.

The function $\dfrac{s}{s^2+2s+2}$ has one finite zero at $s=0$, and two finite poles at $s = -1 \pm j$.

This function is also said to have a <u>zero at infinity</u>, because, in the theory of complex analysis, it tends to zero as s tends to infinity. The reciprocal of the function is said to have a pole at infinity.

If we don't specify which type of pole or zero we're speaking of, assume we're referring to finite ones.

A rational function is <u>proper</u> if the degree of its numerator is less than or equal to that of its denominator; it is <u>strictly proper</u> if the degree of the numerator is strictly less than that of the denominator.

- ex.

$$\frac{s}{s^2 + 2s + 2}$$

is not only proper but strictly proper.

- Strictly proper functions have zeros at infinity.

Exercise

Write your own proof of the initial-value theorem, for the special case where $F(s)$ is a proper rational function.

If $F(s)$ is a proper rational function — possibly multiplied by a complex exponential in s — then we can prove a "final-value theorem" ...

9. The final-value theorem

Let $F(s)$ be a proper rational function, all of whose poles have real parts that are strictly negative, with the possible exception of a single pole at $s=0$.
(alternatively, $F(s)$ may consist of the product of such a rational function with a complex exponential e^{sT}) Then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)$$

Moreover, if the poles of the rational function do not satisfy the above condition, then the limit

$$\lim_{t \to \infty} f(t)$$

does not exist.

When it applies, this result lets us calculate the "final value" $\lim_{t \to \infty} f(t)$ without inverting the transform $F(s)$.

What does the right-hand limit mean?

A function $G(z)$ of a complex variable z is said to have a <u>limit</u> $\lambda \in \mathbb{C}$ as z approaches $z_0 \in \mathbb{C}$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|G(z) - \lambda| < \epsilon$$

whenever $|z - z_0| < \delta$.

Example:

$$F(s) = \frac{s}{s^2 + \omega^2}$$

$$\lim_{s \to 0} sF(s) = 0$$

Check:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \cos \omega t$$

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \cos \omega t$$

— which doesn't exist

But the conditions of the FVT aren't satisfied — $F(s)$ has poles on the imaginary line.

Example: $F(s) = \dfrac{10}{5s + 1} \cdot \dfrac{1}{s}$

(satisfies the conditions).

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} \cancel{s} \frac{10}{5s+1} \cdot \frac{1}{\cancel{s}} = 10$$

Proof of FVT:

First note that an easy case analysis shows that if the conditions on $F(s)$ are not satisfied, then $\lim_{t\to\infty} f(t)$ does not exist.

Now suppose that the conditions are satisfied. Then $F(s)$ can be decomposed into a sum of terms

$\frac{A}{s}$ and $\frac{B_{ik}}{(s-p_i)^k}$, where the p_i are poles of $F(s)$ that lie to the left of the imaginary axis.

It follows that $\lim_{t\to\infty} f(t) = A$.

But what is the value of A?

- By the "Heaviside cover-up,"

$$A = \lim_{s\to 0} sF(s)$$

$\square$

Summary of Laplace-Transform Properties

Property	Time domain	Laplace domain
1. linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
2. time-scaling	$f(ct)$	$\frac{1}{c}F(\frac{s}{c})$
3. exponential modulation	$e^{\alpha t} f(t)$	$F(s-\alpha)$
4. time-shifting	$f(t-T)u_{-1}(t-T)$	$e^{-sT}F(s)$
5. t-multiplication	$tf(t)$	$-\frac{d}{ds}F(s)$
6. differentiation/integration	$f'(t)$	$sF(s) - f(0^-)$
7. convolution	$(f * g)(t)$	$F(s)G(s)$
8. initial-value theorem	$f(0^+)$	$\lim_{s\to\infty} sF(s)$ *
9. final-value theorem	$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$ **

* where the real part of s goes to infinity.

** provided that all poles of F(s) have negative real parts, with the possible exception of a single pole at the origin.