MATH 239 Spring 2014: Assignment 1

Due: 3:00 PM, Tuesday May 20, 2014 in the dropboxes outside MC 4066

Last Name: First Name:

I.D. Number: Section:

Mark (For the marker only): /25

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1. {4 marks} Let $n \geq 2$ be an integer. The n children of the Von Trapp family all have different ages. Whenever they sing, they will all line up so that the oldest child is to the right of the youngest child (it does not have to be directly to the right, for example, the oldest child could be 3 to the right of the youngest child). How many ways can they line up? Use a bijection to prove your answer.

   (For notation, you may label the n children in order of age from the youngest as $1, 2, \ldots, n$. Represent one possible line up as a permutation $\sigma : [n] \to [n]$ where $\sigma(i)$ represents the position of the i-th youngest child in the line up, counting from the left.)

2. {3 marks} Let $n \geq 3$ be an integer. How many subsets of $[2n]$ have exactly 3 odd integers and any number of even integers? Describe two sets S and T such that the answer to our question is $|S \times T|$, then determine what is this answer.

3. {4 marks} Let a, b, c, n be positive integers such that $a \leq b \leq c \leq n$. Consider the following set.

$$S = \{(A, B, C) \mid A \subseteq B \subseteq C \subseteq [n], |A| = a, |B| = b, |C| = c\}.$$

By counting S in two different ways, prove that

$$\binom{n}{a}\binom{n-a}{b-a}\binom{n-b}{c-b} = \binom{n}{c}\binom{c}{b}\binom{b}{a}.$$

4. Consider the following identity.

$$3^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}.$$

(a) {4 marks} Give a combinatorial proof of this identity.

(b) {3 marks} Give an algebraic proof of this identity. (You may assume the binomial theorem.)

5. Let S be the set of all subsets of $[3]$.

(a) {2 marks} Let w be the weight function on S such that $w(\emptyset) = 0$, and for any nonempty set $A \in S$, $w(A)$ is the sum of all elements of A. Determine the generating series $\Phi_S(x)$ with respect to w.

(b) {2 marks} Let w^* be the weight function on S such that for any $A \in S$, $w^*(A) = 3w(A)$. Determine the generating series $\Phi^*_S(x)$ with respect to w^*.

(c) {2 marks} In general, let T be a set and let w be a weight function on T. Let $\Phi_T(x)$ be the generating series with respect to w. For a positive integer k, define w^* to be the weight function on T where for any $a \in T$, $w^*(a) = k \cdot w(a)$. Let $\Phi^*_T(x)$ be the generating series with respect to w^*. Use the definition of generating series to determine a relationship between $\Phi_T(x)$ and $\Phi^*_T(x)$.