University of Waterloo
Department of Electrical and Computer Engineering

MATH 213, Advanced Mathematics for Software Engineering
Midterm Examination
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Instructions:

- Time allowed: 90 minutes.
- No aids allowed.
- The exam comprises 4 questions with a total value of 100 points; answer all of them.

Question No. 1 (32 points)

Find the complementary solutions of the following differential equations:

(a) $\ddot{y} + 3\dot{y} + y = e^{\frac{3}{2}t}$

(b) $\ddot{y} + 2\dot{y} + y = e^{-t}$

(c) $\ddot{y} + 2\dot{y} + 1 = e^{-t} + t$

(d) $\ddot{y} + \dot{y} + y = \sin(\sqrt{3}/2)t$

(e) $\ddot{y} + 4\dot{y} = e^{j2t}$

(f) $(D+3)^3 y = t^2 e^{-3t}$

(g) $(D^2+4)^2 y = e^{-2t}$

(h) $(4D+5)y = (D+2)t^2$

Question No. 2 (32 points)

of parts (a) - (e)

Find particular solutions of each of the differential equations of the previous question.

Question No. 3 (16 points)

An electric circuit is modelled by the following differential equation

$$v_i(t) = RC\dot{v}_o(t) + v_o(t)$$

and the initial condition $v_o(t) = 5$ volts. Suppose that $R = 5\ \mathrm{M}\Omega$ and $C = 1\mu\mathrm{F}$ and that

$$v_i(t) = \begin{cases} 0 & , \ t < 0 \\ 10 \text{ volts} & , \ 0 \le t < 5 \text{ seconds} \\ 0 & , \ t > 5 \text{ seconds} . \end{cases}$$

(a) Does the initial-value problem have a unique solution?

(b) Find the most general solution.

Question No. 4 (20 points)

A mass-spring-damper system is modelled by the following differential equation

$$(D^2 + D + 4)y = (D + 5)x$$

and the initial condition $y(0) = 1$. Suppose that

$$x(t) = \begin{cases} 0 & , \ t < 1 \\ 1 & , \ t \ge 1 \end{cases}$$

(a) Does the initial-value problem have a unique solution?

(b) Find the most general solution. There's no need to simplify.

SOLUTIONS

1 a) $m^2 + 3m + 1 = 0$

$$\Leftrightarrow \quad m = \frac{-3 \pm \sqrt{9-4}}{2}$$

$$= -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

$$y_c(t) = c_1 e^{-\frac{3+\sqrt{5}}{2}t} + c_2 e^{-\frac{3-\sqrt{5}}{2}t}$$

b) $m^2 + 2m + 1 = 0$

$$m = \frac{-2 \pm \sqrt{4-4}}{2} = -1, -1$$

$$y_c(t) = (c_0 + c_1 t) e^{-t}$$

c) $\cancel{g}\, m^2 + 2m = 0$

$$m(m+2) = 0$$

$$m = 0, \quad m = -2$$

$$y_c(t) = c_1 + c_2 e^{-2t}$$

d) $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$y_c(t) = e^{-\frac{1}{2}} \left(c_1 e^{j\frac{\sqrt{3}}{2}t} + c_2 e^{-j\frac{\sqrt{3}}{2}t} \right)$$

e) $m^2 + 4m = 0$

$m = 0, -4$

$y_c(t) = c_0 + c_1 e^{-4t}$

f) $(m+3)^3 = 0 \qquad m = -3$

$y_c(t) = (c_0 + c_1 t + c_3 t^2) e^{-3t}$

g) $(m^2+4)^2 = 0 \Rightarrow m = \pm 2j, \pm 2j$

$y_c(t) = (c_0 + c_1 t) e^{2jt} + (c_2 + c_3 t) e^{-2jt}$

h) $4m + 5 = 0$

$m = -\frac{5}{4}$

$y_c(t) = c\, e^{-5/4\, t}$

2 a) $y_p(t) = k e^{3/2 t}$

$\Longrightarrow \dot{y}_p(t) = \frac{3}{2} k e^{3/2 t}$

$\ddot{y}_p(t) = \frac{9}{4} k e^{3/2 t}$

$$\left(\frac{9}{4} + \frac{9}{2} + 1\right) k e^{3/2 t} = e^{3/2 t}$$

$\Longleftrightarrow \frac{31}{4} k = 1$

$\Longleftrightarrow k = \frac{4}{31}$

$\therefore y_p(t) = \frac{4}{31} e^{3/2 t}$

b) e^{-t} is a sol^n of aux eq^n,

so try

$y_p(t) = kt^2 e^{-t}$

$\dot{y}_p(t) = 2kte^{-t} - kt^2 e^{-t}$

$\ddot{y}_p(t) = 2ke^{-t} - 2kte^{-t}$
$\quad - 2kte^{-t} + kt^2 e^{-t}$

$\quad = 2ke^{-t} - 4kte^{-t} + kt^2 e^{-t}$

Substituting,

$$2ke^{-t} - \cancel{4kte^{-t}} + \cancel{kt^2e^{-t}} + 2(\cancel{2kte^{-t}} - \cancel{kt^2e^{-t}})$$
$$+ \cancel{kt^2e^{-t}} = e^{-t}$$

$\Longleftrightarrow k = \frac{1}{2}$

$\therefore y_p(t) = \frac{1}{2} t^2 e^{-t}$

2c) RHS is actually

$$e^{-t} + t - 1$$

- find a particular sol^n for each term; add.

① - $y_p = k e^{-t}$

$\Rightarrow \dot{y}_p = -k e^{-t}$

$\ddot{y}_p = k e^{-t}$

Substituting

$(1 - 2) k e^{-t} = e^{-t}$

$\Rightarrow k = -1$

$\Rightarrow y_p(t) = -e^{-t}$

② ~~$y_p = \cancel{kt}\ k_0 + k_1 t$~~ ~~$\Rightarrow \dot{y}_p = k_1,\ \ddot{y}_p = 0$~~

Try $\dot{y}_p = k_1 + k_2 t \Rightarrow \ddot{y}_p = k_2$

$k_2 + 2(k_1 + k_2 t) = t$

$\Leftrightarrow (2k_2 = 1 \Leftrightarrow k_2 = \frac{1}{2})$

$\Rightarrow k_1 = -\frac{k_2}{2} = -\frac{1}{4}$

$\dot{y}_p = -\frac{1}{4} + \frac{1}{2} t$

$\Rightarrow y_p = \cancel{k_0} - \frac{1}{4} t + \frac{1}{4} t^2$

But $k_0 = 0$ yields a particular sol^n

(3)

$$\dot{y}_p = k_0$$

$$2k_0 = -1$$

$$k_0 = -\tfrac{1}{2}$$

So finally, a particular sol$^{\underline{n}}$ of the original equation is

$$y_p(t) = -e^{-t} - \tfrac{1}{2}t - \tfrac{1}{4}t + \tfrac{1}{4}t^2$$

$$= \underline{-e^{-t} - \tfrac{3}{4}t} + \tfrac{1}{4}t^2$$

d) 1^{st} try $y_p(t) = k\, e^{j\sqrt{3}/2\, t}$

$$\dot{y}_p(t) = \tfrac{\sqrt{3}}{2}kj\, e^{j\sqrt{3}/2\, t}$$

$$\ddot{y}_p(t) = -\tfrac{3}{2}k\, e^{j\sqrt{3}/2\, t}$$

So

$$\left(-\tfrac{3}{2}k + \tfrac{\sqrt{3}}{2}kj + k\right)e^{j\frac{\sqrt{3}}{2}t} = e^{j\frac{\sqrt{3}}{2}t}$$

$$\Rightarrow \quad -\tfrac{1}{2}k + \tfrac{\sqrt{3}}{2}kj = 1$$

$$\Rightarrow \quad k = \frac{1}{-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}j} = \frac{1}{e^{j2\pi/3}} = e^{-j\frac{2\pi}{3}}$$

So if we set $y_p(t) = k\, e^{-j\sqrt{3}/2\, t}$,

we get $k = e^{+j\frac{2\pi}{3}}$

Now $\sin\frac{\sqrt{3}}{2}t = \dfrac{e^{j\frac{\sqrt{3}}{2}t} - e^{-j\frac{\sqrt{3}}{2}t}}{2j}$

So a particular sol$^{\underline{n}}$ is

$$y_p(t) = \sin\left(\frac{\sqrt{3}}{2}t - \frac{2\pi}{3}\right)$$

e) $y_p(t) = k e^{2jt}$ $\quad \dot{y}_p(t) = 2j k e^{2jt}$

$\ddot{y}_p(t) = -4 k e^{2jt}$

$$(-4\cancel{k} + 4 \cdot 2j) k e^{2jt} = e^{2jt}$$

$$4(-1 + 2j) k = 1$$

$$k = \frac{1}{4(-1+2j)} = \frac{-1-2j}{4(5)}$$

$$= \frac{-1}{20}(1+2j)$$

So $y_p(t) = \frac{-1}{20}(1+2j) e^{2jt}$

(could be simplified further) to

$$y_p(t) = -\frac{\sqrt{5}}{20} e^{j(2t+\phi)}$$ where

$$\phi = \tan^{-1} 2$$

f)

Omit f – h in Q. 2

3. a) In any interval in which v_i is continuous, there exists a unique solution, provided a suitable initial condition is given.

Hence for $0 \le t < 5$, there is a unique solution for every initial condition $v(0^+)$, and for $t \ge 5$, there is a unique solution for every $v(5^+)$. On physical grounds, one can argue $v(0^-) = v(0^+)$ and $v(5^-) = v(5^+)$ — no instantaneous voltage change across a capacitor. Then there exists a unique solution

b) Characteristic equation:

$$RCm + 1 = 0$$

$$\Leftrightarrow \quad m + \frac{1}{5} = 0$$

$$\Leftrightarrow \quad m = -\frac{1}{5}$$

So the complementary solution is

$$y_c(t) = c\, e^{-t/5}$$

Particular solution

- for $t < 0$, $y_p(t) = 0$
- for $0 \leq t < 5$, $y_p(t) = k$

$$\Rightarrow \quad k = 10 \text{ volts}$$

- for $t \geq 5$, $y_p(t) = 0$

So for $t < 0$,

$$v_o(t) = v(0^-)\, e^{-t/5} = v(0^-) = 5 \cdot e^{-t/5} \text{ volts}$$

for $0 \leq t < 5$

$$v_o(t) = C\, e^{-t/5} + 10 \text{ volts}$$

$$\Rightarrow v_o(0^+) = C + 10 \text{ volts} = 5 \text{ volts}$$

So $v_o(t) = -5\, e^{-t/5} + 10$ volts

for $5 \leq t$

$$v_o(t) = C\, e^{-t/5}$$

$$\Rightarrow v_o(5^+) = C\, e^{-1}$$

Now according to the previous case, we should have

$$v_o(5^-) = 10 - 5e^{-1} \text{ volts}$$

Hence we should have

$$v_o(t) = (10e - 5)\, e^{-t/5} \text{ volts}$$

$$= 10\, e^{-(t-5)/5} - 5\, e^{-t/5} \text{ volts}$$

4\. a) No - the equation is second-order, but only one initial condition is given, so there does not exist a unique solution.

b) Characteristic equation:

$$m^2 + m + 4 = 0$$

$$\Rightarrow \quad m = -\frac{1}{2} \pm \frac{\sqrt{1-16}}{2}$$

$$= -\frac{1}{2} \pm j\frac{\sqrt{15}}{2}$$

So the complementary solution is

$$y_c(t) = e^{-t/2}\left(c_1 e^{j\frac{\sqrt{15}}{2}t} + c_2 e^{-j\frac{\sqrt{15}}{2}t}\right),$$

As for particular solutions, we know that we can first solve the modified equation

$$(D^2 + D + 4)\tilde{y} = x$$

and then write $y = (D+5)\tilde{y}$.

For $t < 1$, $x(t) = 0$,

So a particular solution is

$y_p(t) = 0$.

For $t \geq 1$, try $y_p(t) = k$

$\Longrightarrow \dot{y}_p(t) = 0, \quad \ddot{y}_p(t) = 0$

$\Longrightarrow k = 1/4$.

So for $t < 1$, the general solution is

$$\tilde{y}(t) = e^{-t/2}\left(\mathbb{C}_1 e^{j\frac{\sqrt{15}}{2}t} + \mathbb{C}_2 e^{-j\frac{\sqrt{15}}{2}t}\right)$$

and for $t > 1$,

$$\tilde{y}(t) = e^{-t/2}\left(\mathbb{C}_3 e^{j\frac{\sqrt{15}}{2}t} + \mathbb{C}_4 e^{-j\frac{\sqrt{15}}{2}t}\right) + 1/4$$

One might argue on physical grounds that $\tilde{y}(1^-) = \tilde{y}(1^+)$ (speed is bounded). In that case, we would have to have

$$\mathbb{C}_1 e^{j\frac{\sqrt{15}}{2}} + \mathbb{C}_2 e^{-j\frac{\sqrt{15}}{2}} = \mathbb{C}_3 e^{j\frac{\sqrt{15}}{2}} + \mathbb{C}_4 e^{-j\frac{\sqrt{15}}{2}} + 1/4$$

Similarly, one would expect

$$\dot{\tilde{y}}(1^-) = \dot{\tilde{y}}(1^+) \qquad (\text{force } x(t) \text{ is bounded})$$

which would lead to the additional constraint

$$\mathbb{C}_1 e^{+j\frac{\sqrt{15}}{2}} - \mathbb{C}_2 e^{-j\frac{\sqrt{15}}{2}}$$

$$= \mathbb{C}_3 e^{+j\frac{\sqrt{15}}{2}} - \mathbb{C}_4 e^{-j\frac{\sqrt{15}}{2}}$$

These two equations would let one solve for $\mathbb{C}_3$ and $\mathbb{C}_4$ in terms

$$\mathbb{C}_3 = \mathbb{C}_1 - \frac{1}{8} e^{-j\frac{\sqrt{15}}{2}}$$

$$\mathbb{C}_4 = \mathbb{C}_2 - \frac{1}{8} e^{j\frac{\sqrt{15}}{2}}$$

Finally, one applies

$$y(t) = (D+5)\,\tilde{y}(t) .$$