MATH239 Intro to Combinatorics

Brandon Yeh

MWF 2:30-3:30 MC4021

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1 Preface

These lecture notes will be compiled using Konstantinos Georgiou's lecture material supplemented by the course notes.

2 Lecture 1 - September 9 2013

2.1 Contact Information

Konstantinos Georgiou
MC6316
email: k2georgiou@math
Please add MATH239 in subject

2.2 Weekly Schedules

Weekly Lectures
MWF 2:30-3:20pm, MC4021

Weekly Office Hours
MW 5:30-6:30, MC6316 or by appointment

Weekly Tutorials
Starting week of Sept 16 W 4:30-5:20, MC4059

2.3 Grading Scheme

Best 8/9 Assignments - 10%
Midterm - 30 %
Final exam - 60 %

2.4 Course Overview

Combinatorics - The study of finite or countable discrete objects

- Combinatorial Analysis
    - Compositions and Strings
    - Recurrences
- Graph Theory

2.5 Questions

What problems are we solving in combinatorial analysis?

Problem 1 How many binary strings are there of length n, that avoi blocks of two 0's and two 1's?

$$b_n = [x^n]\frac{1-x^2+x^3}{1-2x+x^2-x^3}, n/geq0$$

What problems are we solving in graph theory?

Problem 3 The following pictures represent people and their relations. Are these configurations different?

Problem 4 How many colours do we need to colour the map of Spain?

2.5.1 Other Problems

- How do we systematically traverse a graph?
- Can we draw a circuit on a plane so that no two wires overlap?
- Two grops of men and women declare their partner-preferences. Is there a matching satisfying them all?

2.6 Section 1.1

Main question we study: Count how many objects are there of certain type
Two objects to study: Compositions and Binary Strings

2.6.1 Composition

Definition
Fix integers $n \geq 0$. A sequence of positive integers. $(m_1, m_2, m_3, ..., m_r)$, for which $m_1 + m_2 + m_3 + \cdots + m_r = n$ is called a composition of m.

$m_i \rightarrow$ is called a part of the composition
$m \rightarrow$ is called the weight of the composition

Example of Composition
$(2, 3, 3, 4, 1)$ is a composition of weight 13

Questions to Study

- How many compositions of m are there?
- How many compositions are there with k parts?
- How many compositions of n are there where all parts are all integers?

2.6.2 Binary String

Definition A binary string of length n is sequence $a_1, a_2, ..., a_n$ where $a_i \in \{0, 1\}, i = 1, ..., n$

Example of Binary String
0001 is a binary string of length 4.

Questions to Study

- How many binary strings of length n are there? 2^n
- How many binary strings of length n, that avoid 1010?

2.7 Section 1.2

2.7.1 Review

Let A,B be two sets. We define $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Example
$A_0 = \{1, 2, 3\}$ $\quad$ $B_0 = \{3, 7, 8, 4\}$ $\quad$ $A_0 \cup B_0 = \{1, 2, 3, 7, 8, 4\}$

$|A|$ denotes the size of set A
$\emptyset$ denotes the empty set, the set with no elements

Example
$|\emptyset| = 0$ $\quad$ $|\{\emptyset\}| = 1$

Question When is it true that $|A \cup B| = |A| + |B|$?

Answer Exactly when the intersection of A, B is the empty set
i.e. $A \cap B = \{x : x \in A \text{ AND } x \in B\}$ is empty

In such cases, we call A, B disjoint.

Example: $A_0 \cap B_0 = \{3\}$

2.7.2 Cartesian Product

Definition Let A, B be two sets. Their cartesian product is $A \times B = \{(a, b) : a \in A, b \in B\}$.

i.e. all ordered pairs (a, b), where $a \in A, b \in B$.

Example $A_1\{1, 2\}$, $\quad$ $B_1 = \{a, b, c\}$ $\quad$ $A_1 \times B_1 = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

Lemma $|A \times B| = |A| \cdot |B|$

2.7.3 Cartesian Power

Definition For a positive integer k and a set A. Define the Cartesian Power A^k as follows:

$$A^1 := A$$
$$A^{k+1} := A \times A^k, k \geq 1$$

Example
$B = \{0,1\} \qquad B^4 = \{(b_1, b_2, b_3, b_4) : b_i \in \{0,1\}, c = 1,2,3,4\}$
B^4 Binary sring of length 4

3 Lecture 2 - September 11 2013

3.1 Last Lecture

$A \times B$

$|A \times B| = |A| \cdot |B|$

$A^k = A \times A \times A \times \cdots \times A \ \}k$

$B = 0,1 \qquad B^4 = \{(b_1, b_2, b_3, b_4) : b_i \in B, i = 1,2,3,4\}$

3.2 Chapter 1.2 Continued

3.2.1 Cartesian Power

Lemma $|A^k| = |A|^k$

Example
for $B = \{0,1\}$ as before, $B^4 = 2^4$

3.2.2 Sets of Compositions

Definition Let A,B be two sets of compositions. For every $(a,b) \in A \times B$, where a,b are compositions. We define the weight of (a,b) as weight of a plus the weight of b. We denote it by $w(a,b)$.

Reminder that $(2,3,3,4,1)$ is a composition of weight 13.

Example
$a = (3,3,2)$ a composition of 8
$b = (1,2)$ a composition of 3
$w(a,b) = w(a) + w(b) = 8 + 3 = 11$

Lemma $w_k(A \times B) = \sum_{i=0}^{k} w_i(A) \cdot w_{k-1}(B)$

This is the number of elements of $A \times B$ that have weight k.

In order for $w(a,b) = k$ We need $w(a) = i$ and $w(b) = k - i$ since $w(a,b) = w(a) + w(b) = k$

Proof Let A_i denote the subset of A of elements of weight i. Let B_j denote the subset of B of elements of weight j.

- All elements of $A_i \times B_j$ have weight $i + j$ and weight is exactly k if $j = k - i(i = 0, 1, ..., k)$
- $\forall i, j |A_i \times B_j| = |A_i| \times |B_j| = w_i(A) \cdot w_j(B)$
- $A_0 \times B_k, A_1 \times B_{k-1}, ..., A_k \times B_0$ are pairwise disjoint.

Hence, $w_k(A \times B) = \sum_{i=0}^{k} |A_i \times B_{k-i}| = \sum_{i=0}^{k} w_i(A) \cdot w_{k-1}(B)$

□

3.3 Section 1.3 - Binomial Coefficients

Example/Lemma

In how many ways can we order n distinct elements? (n-permutations)

Solution

Position	1	2	3	4	$\cdots$	n
# of Options	n	$n-1$	$n-2$	$n-3$	$\cdots$	1

$n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$

Example/Lemma

in how many ways can you permute r elements chosen out of n elements. r-permutations of n elements.

Solution

Position	1	2	3	$\cdots$	n
# of Options	n	$n-1$	$n-2$	$\cdots$	$n-(r-1)\star$

Overall, $n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$

□

Comment Previous formula stays true when $r > n$

Definition We define $0! = 1$ in which case we write $\star$

$$\frac{n!}{(n-r)!} \star\star \qquad n \geq r$$

Please verify that $\star$ and $\star\star$ agree.

Theorem For non negative integers n, k the number of k-element subsets of an n-element set equals:

$$\frac{n \cdot (n-1) \cdots (n-(k-1))}{k!}$$

The above counts the different ways we can choose k elements out of n.

Proof - Combinatorial Argument We do not know that the number of k-permutations of n objects are $n \cdot (n-1) \cdots (n-(k-1)$.

Equivalently, we can first choose k many elements out of n and suppose that this can be done in x many ways. We know that this k elements can be permuted in $k!$ many ways.

$x \cdot k!$ is the number of different ways we can permute k objects out of n.

$x \cdot k! = n \cdot (n-1) \cdots (n-(k-1)$
$x = \dfrac{n \cdot (n-1) \cdots (n-(k-1))}{k!}$
Solve for x.

□

Notation "n choose k"

$$\binom{n}{k} := \frac{n \cdot (n-1) \cdots (n-(k-1))}{k!}$$

Example $\binom{n}{k} = 1$

Lemma $\binom{n}{k} = \binom{n}{n-k}$

4 Lecture 3 - September 13 2013

4.1 Last Lecture

Binomial Coefficient: $\binom{n}{k} := \dfrac{n \cdot (n-1) \cdots (n-(k-1))}{k!}$

Count number of different ways we can choose k elements out of n many. Ordering dos not matter repetition is not allowed.

4.2 Section 1.3 Continued

Observation
When $0 \le k \le n$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Theorem $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}_{\ge 0}, (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

Example
$(1+x)^3 = 1 + 3x + 3x^2 + x^3 = \binom{3}{0} + \binom{3}{1} x + \binom{3}{2} x^2 + \binom{3}{3} x^3$

Proof
$(1+x)^n = (1+x)\cdot(1+x)\cdot(1+x)\cdots(1+x)$
n many factors (brackets)

- If we expand and collect common monomials x^k, what would be the coefficient of x^k?
- Every term is a product of 1's and x's each chose from each of the factors

$(1+x)^2 = (1+x)(1+x) = 1(1+x) + x(1+x) = 1\cdot 1 + 1\cdot x + x\cdot 1 + x\cdot x$

We obtain x^k for each different we choose k many x's. (from equally many factors) out of the n factors. We know taht there are $\left(\begin{array}{c} n \\ k \end{array}\right)$ different ways we can choose k brackets out of n.

Hence, $(1+x)^m = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) x^k \quad \square$

Binomial Theorem $\quad (1+x)^m = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) x^k$

Corollary $\quad 2^n = \sum_{k=0}^{n} \binom{n}{k}$

Proof 1
In Binomial Theorem, set $x = 1$

Proof 2
Consider the set $\{1, 2, ..., n\} = [n]$. LHS counts the different subsets $[n]$.

Let A_i denote the subsets of $[n]$ with size exactly i. All different subsets of n are exactly:

$$A_0 \cup A_1 \cup \cdots \cup A_n$$

Observe that all A_i are pairwise disjoint.

$|A_0 \cup A_1 \cup \cdots \cup A_n| = \sum_{i=0}^{k} |A_i| = \sum_{i=0}^{n} \binom{n}{i} \quad \square$

Example
$\forall n, k \geq 0 \qquad$ (LHS) $\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$ (RHS)

$\boxed{\binom{n}{k} = \binom{n}{n-k}}$

Proofs (Combinatorial Argument) LHS: counts the different n-sized subsets of $A = \{1, 2, ..., n, n+1, ..., k\}$

Let $\mathcal{L}$ denote all size-n subsets of A

Ovservation: Any subsets of A of size m, has largest eleemnt either n or n+1 or,...,or n+k

How many such subsets of $\mathcal{L}$ have largest elements equal to n?

$\binom{n-1}{n-1}$ since we need to choose $n-1$ elements out of $\{1, 2, ..., n-1\}$

How many subsets of A of size n, have largest elements $n+1$?

$\binom{n}{n-1}$

What about element $n+i$

$\binom{n-1+i}{n-i}$
i ranges from 0 up to k

$\mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_k$ are positive disjoint and $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_k$

$\binom{n+k}{n} = |\mathcal{L}| \sum_{i=0}^{k} |\mathcal{L}_i| = \sum_{i=0}^{k} \binom{n-1+i}{n-1}$ $\square$

4.3 Section 1.4 - Generating Series

High level Description
Let S ne a set of "configurations" (objects). For each $\sigma \in S$ we will have a non-negative weight $w(\sigma)$. Given an integer k, how many objects are there in S of weight k?

Example
Let S denote all subsets of $\{1, 2, ..., n\} \forall \sigma \in S$, we define $w(\sigma)$ as the size of σ.
Let α_k denote the number of configurations in S with weight k.

Question $\alpha_k = ?$

Answer $\alpha_k = \binom{n}{k}$

Definition
Let S be a set of configurations with weight function w.

4.3.1 Generating Series

The **Generating Series** for S with respect to w is $\Phi_s(x) = \sum_{\sigma \in S} x^{w(\sigma)}$

Informal

$\Phi_s(x)$ is a polynomial (in x) with one summond x^k, for each configuration of weight k. summond is a term that you sum up.

Corollary

If α_k denotes the number of elements of S with weight k, then:
$\Phi_s(x) = \sum\limits_{k\geq 0} \alpha_k x^k$

Example Let S denote all subsets of $\{1, 2, ..., n\} \forall \sigma \in S$, we define $w(\sigma)$ as the size of σ. What is the configuration?

$$\Phi_s(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \quad \square$$

5 Lecture 4

5.1 Last Lecture

Let S be a set of configurations.
$w(\sigma)$ is the weight of $\sigma \in S$.
Generating series of S $\Phi_s(x) = \sum\limits_{\sigma \in S} x^{w(\sigma)}$
$= \sum\limits_{k\geq 0}$(number of configurations in S with weight k)$\cdot x^k$

Main Idea We need to count number of configurations with weight k. It will be easy to find generating series.

5.2 Section 1.4 Continued

Example

$A = \{1, 3, 5\} \qquad \forall a \in A \; w(a) = a$
$B = \{2, 4\} \qquad \forall b \in B \; w(b) = b$

$\Phi_A(x) = x^1 + x^3 + x^5$
$\Phi_B(x) = x^2 + x^4$

$A \times B = \{(1,2), (1,4), (3,2), (3,4), (5,2), (5,4)\}$

Define $w(a, b) = w(a) + w(b)$

$$\begin{aligned} \Phi_{A\times B}(x) &= x^3 + x^5 + x^5 + x^7 + x^7 + x^9 \\ &= x^3 + 2x^5 + 2x^7 + x^9 \end{aligned}$$

This shows that we have 1 configuration of weight 3, 2 configurations of weight 5, 2 for weight 7 and 1 for weight 9.

5.2.1 Theorem 1.4.3

Let $\Phi_S(x)$ be a generating series for a finite set S with respect to some weight fuction w. Then:

(i) $\Phi_S(1) = |S|$

(ii) Sum of weights of configuration in S is $\Phi'_S(1)$

First derivative at x=1

(iii) Average weight of a configuration in S is $\dfrac{\Phi'_S(1)}{\Phi_S(1)}$

Proof

(i) $\Phi_S(x) = \sum\limits_{\sigma \in S} x^{w(\sigma)}$

$\Phi_S(1) = \sum\limits_{\sigma \in S} 1^{w(\sigma)} = |S|$

(ii) $\Phi'_S(x) = \sum\limits_{\sigma \in S} w(\sigma) \cdot x^{w(\sigma)-1}$

$\Phi'_S(1) = \sum\limits_{\sigma \in S} w(\sigma) \cdot 1^{w(\sigma)-1}$

$\Phi'_S(1) = \sum\limits_{\sigma \in S} 1w(\sigma)$

(iii) follows from (i),(ii) $\square$

Example S is composed by all subsets of $\{1, 2, ..., n\}$
$\forall \sigma \in S, w(\sigma) =$the size of subset σ. Where σ is a subset of $\{1, 2, ..., n\}$

We have shown that $\Phi_S(x) = (1 + x)^n$. Previous theorem says $|S| = \Phi_S(i) = 2^n$

Theorem (ii) says sum of weight of configuration in S is $\Phi'_S(1)$

$\Phi'_S(x) = n(1 + x)^{n-1}$, so sum of weights of σ in S is $n \cdot 2^{n-1}$

According to previous theorem(iii), average weight of $\sigma \in S$ is:

$$\frac{n \cdot 2^{n-1}}{2^n} = \frac{n}{2}$$

5.3 Section 1.5 - Formal Power Series

5.3.1 Discussion

S is a set of configurations

If S is finite, $\Phi_S(x)$ is a polynomial
If S is infinite, $\Phi_S(x)$ is an infinite sum

5.3.2 Formal Power Series

Definition An expression:

$$\begin{aligned} A(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots \\ &\sum_{k\geq 0} \alpha_k \cdot x^k \end{aligned}$$

where $\{\alpha_k\}_k$ sequence of rational numbers is called a formal power series (FPS)

α_k is called the coefficient of x^k in $A(x)$

We also introduce notation $[x^k]A(x) := \alpha_k$

Example: $[X^1]A(x) = \alpha_1$

Comments
Every polynomial is a FPS.

Example
$P(x) = 1 + x + 3x^234$ $P(x) = \sum_{k\geq 0} \alpha_k x^k$ $\alpha_0 = 1, \alpha_1 = 1, \alpha_{234} = 3, \alpha_i = 0\ \forall i \neq 0, 1, 234$
$[X^5]P(x) = 0$

$A(x) = \sum_{k\geq 0} \alpha_k x^k$, $B(x) = \sum_{k\geq 0} \alpha_k x^k$ are equal $(A(x) = B(x))$ if $\forall k \geq 0, \alpha_k - \beta_k$

Definition
$A(x) + B(x) = \sum_{k\geq 0} (\alpha_k + \beta_k) x^k$ (sum of the two fps)
$A(x) \cdot B(x) = \sum_{n\geq 0} \left(\sum_{k00}^{n} \alpha_k \beta_{n-k} \right) x^n$ (product of two fps)

Example
$A(x) = \sum_{k\geq 0} kx^k$, $B(x) = \sum_{k\geq 0} x^k$
$A(x) \cdot B(x) = \sum_{n\geq 0} \left(\sum_{k-0}^{n} k \cdot 1 \right) x^n = \sum_{n\geq 0} \frac{n(n+1)}{2} \cdot x^n$
$[X^234]A(x) \cdot B(x) = \frac{239 \cdot 240}{2}$

Example
Does there exists $A(x) = \sum_{k\geq 0} \alpha_k x^k$ satisfying $(1 - x - x^2)A(x) = 1 + x$?

Let $1 + x = \sum_{k\geq 0} B_k x^k$

Answer

$$\begin{aligned}
\text{LHS} &= 1 - x - x^2) \cdot A(x) \\
&= (1 - x - x^2) \sum_{k \geq 0} \alpha_k x^k \\
&= \sum_{k \geq 0} \alpha_k x^k - x \sum_{k \geq 0} \alpha_k x^k = x^2 \sum_{k \geq 0} \alpha_k x^k \\
&= \sum_{k \geq 0} \alpha_k x^k - \sum_{k \geq 0} \alpha_k x^{k+1} - \sum_{k \geq 0} \alpha_k x^{k+2} \\
&= \sum_{k \geq 0} \alpha_k x^k - \sum_{k \geq 1} \alpha_{k-1} x^k - \sum_{k \geq 2} \alpha_{k-2} x^k \\
&= \alpha_0 + (\alpha_1 + \alpha_0) x' + \sum_{k \geq 2} (\alpha_k - \alpha_{k-1} - \alpha_{k-2}) x^k
\end{aligned}$$

Take $\alpha_0 = \beta_0 = 1, \alpha_1 = \beta_1 = 1, \alpha_k - \alpha_{k-1} - \alpha_{k-2} = B_k = 0 \ \forall k \geq 2 \quad \square$

6 Definitions

6.1 Chapter 1

6.1.1 Section 1.1

Composition

A composition of a non-negative integer n is a sequence: $m_1, ..., m_r$ of positive integers such that

$$m_1 + \cdots m_r = n$$

The numbers $m_1, ..., m_r$ are called the **parts** of the composition.
The **weight** of the composition is the sum of its parts.

6.1.2 Binary Strings

A binary string of length n is a sequence $a_1, ..., a_n$ where each a_i is 0 or 1.

6.1.3 Union

The **Union** $A \cup B$ is defined by:

$$A \cup B := \{x : x \in A or x \in B\}$$

If A and B are **Disjoint**, that is, $A \cap B = \emptyset$ then,

$$|A \cup B| = |A| + |B|$$

We define define $|S|$ as the number of elements in set S.

6.1.4 Cartesian Product

The **Cartesian Product** $A \times B$ of sets A and B is the set of all ordered pairs whose first element is an element of A and second elemenet is an element of B, that is:

$$A \times B := \{(a, b) : a \in A, b \in B\}$$

Then

$$|A \times B = |A||B|$$

6.1.5 Cartesian Power

We define the **Cartesian Power** A^k inductively by setting $A^1 = A$ and:

$$A^{k+1} := A \times A^k$$

We have that $|A^k| = |A|^k$. The elements of A^k are the ordered k-tuples of elemnts from A.