Tutorial Problems 1

The TA will cover a subset of these problems during the tutorial. Solutions will not be provided outside of the tutorial.

1. Let n be a positive integer. Let E be the set of all subsets of $[n]$ of even size, and let O be the set of all subsets of $[n]$ of odd size. Find a bijection between the elements of E and O, and illustrate your bijection by matching up the E and O for $[4]$. Use this result to prove the following identity:
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

2. For any integers n, k, r where $n \geq k \geq r \geq 0$, give a combinatorial proof of the following identity.
$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}.$$

3. Give a combinatorial proof and an algebraic proof of the following identity for any positive integer n.
$$\sum_{i=0}^{n}\binom{n}{i}i = n2^{n-1}.$$

4. Let S be the set of all permutations of $[3]$.
   (a) For each permutation $\sigma : [3] \to [3]$, define $w(\sigma) = \min\{i \mid \sigma(i) \leq i, i \in [3]\}|$. Determine the generating series $\Phi_S(x)$ for S with respect to w.
   (b) We now define $w^*(\sigma) = \min\{i + 2 \mid \sigma(i) \leq i, i \in [3]\}$. Determine the generating series $\Phi^*_S(x)$ for S with respect to w^*.
   (c) What is the relationship between $\Phi_S(x)$ and $\Phi^*_S(x)$?

Practice Problems for Assignment 1

1. Consider the k-tuples $(T_1, \ldots, T_k)$ where each $T_i \subseteq [n]$. In other words, if P is the set of all subsets of $[n]$, then such a k-tuple is in the cartesian product P^k. We define the following two subsets of P^k:

   (a) S is all such k-tuples where $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$.

   (b) T is all such k-tuples that are mutually disjoint, i.e. $T_i \cap T_j = \emptyset$ for any $i \neq j$.

   Find a bijection between S and T, which proves that $|S| = |T|$. What is this cardinality?

2. For $0 \leq r \leq n$, prove the following identity.

$$\binom{n}{r} 3^{n-r} = \sum_{k=r}^{n} \binom{n}{k} \binom{k}{r} 2^{n-k}.$$

3. Let n be a positive integer. Consider all possible ways of arranging n pairs of parentheses "(" and ")" in a row. We want to arrange them in a way that it is possible to pair up the left parentheses with the right parenthesis so that "(" is to the left of ")". Let A_n be the set of all such arrangements with n pairs of parentheses. There are 5 elements in A_3, as shown below.

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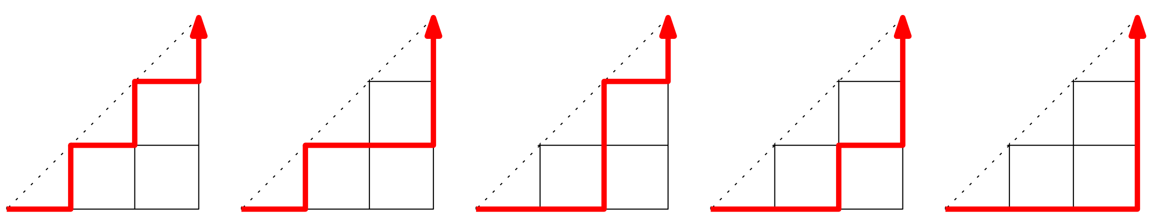
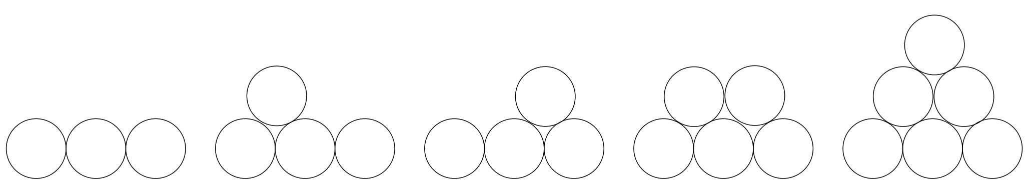
   On an $n \times n$ grid, we consider paths travelling along the grid lines from the SW corner to the NE corner of the grid with the conditions that we only travel in the N or E direction, and we never travel above the $x = y$ line. Let B_n be the set of all such paths in an $n \times n$ grid. There are 5 elements in B_3, as shown below.



   Let C_n be the set of all possible ways that we can stack identical coins with n coins at the bottom row. For example, the following are elements of C_3.



   Finding the exact size of A_n, B_n or C_n in general could be tricky. Even though these are completely different objects, they actually have the same size. Prove that $|A_n| = |B_n| = |C_n|$ by providing two bijections, one maps between A_n and B_n, the other maps between B_n and C_n. Provide inverses of your bijections. Illustrate your bijections by drawing the path in B_7 and a stack of coins in C_7 that corresponds to the following element in A_7.

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