Tutorial Problems 2

The TA will cover a subset of these problems during the tutorial. Solutions will not be provided outside of the tutorial.

1. Find a simplified rational expression for each of the following power series.

   (a)
   $$f(x) = \sum_{i=3}^{141} (-3x)^i.$$

   (b)
   $$g(x) = \sum_{i \geq 0} \left(\frac{x}{1-x^2} \right)^i$$

2. Determine the following coefficient.
   $$[x^9](1-x^3)^{-5}(1-3x^2)^{-1}$$

3. Let $\{a_n\}$ be a sequence whose corresponding power series $A(x) = \sum_{i \geq 0} a_i x^i$ satisfies
   $$A(x) = \frac{6 - x + 5x^2}{1 - 3x^2 - 2x^3}.$$

   (a) Determine a recurrence relation that $\{a_n\}$ satisfies, with sufficient initial conditions to uniquely specify $\{a_n\}$. Use this recurrence relation to find a_5.

   (b) The denominator of $A(x)$ can be factored into $(1-2x)(1+x)^2$. Using results from partial fractions, it can be shown that there exist constants C_1, C_2, C_3 such that
   $$A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1+x} + \frac{C_3}{(1+x)^2}.$$
   Determine these three constants, and use this new expression for $A(x)$ to find a formula for a_n.

4. Let S_n be the set of all subsets of $[n]$, and for each $A \in S_n$, define $w(A)$ to be the sum of the elements in A. Give a combinatorial proof of the following:
   $$\Phi_{S_n}(x) = (1 + x^n)\Phi_{S_{n-1}}(x).$$
   Use this recursion to derive a formula for $\Phi_{S_n}(x)$.

Practice Problems for Assignment 2

1. More exercises on formal power series can be found on pages 22-23 in the course notes.

2. This question asks you to reverse engineer the process of finding a recurrence relation from a rational expression. Suppose a sequence $\{a_n\}$ satisfies $a_0 = 1$, $a_1 = 2$, $a_2 = 2$, and for $n \geq 3$, $a_n = 2a_{n-1} - a_{n-2} + a_{n-3}$. Find a rational expression whose power series representation is $\sum_{n \geq 0} a_n x^n$.