MATH 213

ASSIGNMENT NO. 5

SOLUTIONS

1.

$$\frac{25s + 50}{2s^2 + 10\sqrt{2}\,s + 50}$$

$$= \frac{1}{2}\,\frac{25s}{s^2 + 5\sqrt{2}\,s + 25} + \frac{25}{s^2 + 5\sqrt{2}\,s + 25}$$

Set $\omega_n = 5$, $\zeta = \frac{1}{\sqrt{2}}$

Then the inverse transform is the impulse response of the standard second-order system

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$+ \ \frac{1}{2}$ the derivative of that impulse response.

The impulse response is

$$\frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

$$= 5\sqrt{2}\, e^{-\frac{5}{\sqrt{2}}t} \sin \frac{5}{\sqrt{2}} t$$

Its derivative is

$$5\sqrt{2}\left[-\frac{5}{\sqrt{2}} e^{-\frac{5}{\sqrt{2}}t} \sin \frac{5}{\sqrt{2}} t + \frac{5}{\sqrt{2}} e^{-\frac{5}{\sqrt{2}}t} \cos \frac{5}{\sqrt{2}} t\right]$$

So the inverse transform is

$$5\sqrt{2}\, e^{-\frac{5}{\sqrt{2}}t}\left[\left(1-\frac{5}{2\sqrt{2}}\right) \sin \frac{5}{\sqrt{2}} t + \frac{5}{2\sqrt{2}} \cos \frac{5}{\sqrt{2}} t\right]$$

$$= e^{-\frac{5}{\sqrt{2}}t}\left[\frac{5}{\sqrt{2}}\left(2-\frac{5}{\sqrt{2}}\right) \sin \frac{5}{\sqrt{2}} t + \frac{25}{2} \cos \frac{5}{\sqrt{2}} t\right]$$

alternatively,

$$\frac{25s + 50}{25^2 + 10\sqrt{2}\,s + 50}$$

$$= \frac{\frac{25}{2}s + 25}{\left(s + \frac{5}{\sqrt{2}}\right)^2 + \frac{25}{2}}$$

$$= \frac{\frac{25}{2}\left(s + \frac{5}{\sqrt{2}}\right) + 25\left(1 - \frac{5}{2\sqrt{2}}\right)}{\left(s + \frac{5}{\sqrt{2}}\right)^2 + \frac{25}{2}}$$

$$= \mathcal{L}\Big\{ 5\sqrt{2}\left(1 - \frac{5}{2\sqrt{2}}\right) e^{-\frac{5}{\sqrt{2}}t} \sin\frac{5}{\sqrt{2}}t$$

$$+ \frac{25}{2} e^{-\frac{5}{\sqrt{2}}t} \cos\frac{5}{\sqrt{2}}t \Big]$$

$$= \mathcal{L}\left\{ e^{-\frac{5}{\sqrt{2}}t} \left[ \frac{5}{\sqrt{2}}\left(2 - \frac{5}{\sqrt{2}}\right) \sin\frac{5}{\sqrt{2}}t \right.\right.$$

$$\left.\left. + \frac{25}{2}\cos\frac{5}{\sqrt{2}}t \right]\right\}$$

2. $$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$$

So $$\mathcal{L}\{t \sin \omega t\} = -\frac{d}{ds}\frac{\omega}{s^2+\omega^2}$$

$$= +\frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\mathcal{L}\{\cos \omega t\} = \mathcal{L}\left\{\frac{1}{\omega}\frac{d}{dt}\sin \omega t\right\}$$

$$= \frac{1}{\omega}\left[s\mathcal{L}\{\sin \omega t\} - \sin 0\right]$$

$$= \frac{s}{s^2+\omega^2}$$

So $$\mathcal{L}\{t \cos \omega t\} = -\frac{d}{ds}\frac{s}{s^2+\omega^2}$$

$$= -\left[\frac{s^2+\omega^2}{(s^2+\omega^2)^2} - \frac{2s^2}{(s^2+\omega^2)^2}\right]$$

$$= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Alternatively,

$$\mathcal{L}\{t e^{j\omega t}\} = \frac{1}{(s-j\omega)^2}$$

$$= \frac{(s+j\omega)^2}{(s^2+\omega^2)^2}$$

$$= \frac{(s^2-\omega^2) + j\,2s\omega}{(s^2+\omega^2)^2}$$

So $\mathcal{L}\{t\cos\omega t\} = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

& $\mathcal{L}\{t\sin\omega t\} = \dfrac{2s\omega}{(s^2+\omega^2)^2}$

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( Also

$$\mathcal{L}\{e^{j\omega t}\} = \frac{1}{s-j\omega}$$

$$= \frac{s+j\omega}{s^2+\omega^2}$$

So $\mathcal{L}\{\cos\omega t\} = \dfrac{s}{s^2+\omega^2}$

& $\mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2}$ )

3. $g(t) = u_{-1}(t) - u_{-1}(t-1)$

so $G(s) = \dfrac{1 - e^{-s}}{s}$

4. a) $h(t) = g(t) + h(t-2)$

b) $H(s) = G(s) + e^{-2s} H(s)$

$$\iff H(s) = \frac{G(s)}{1 - e^{-2s}}$$

$$= \frac{1 - e^{-s}}{s(1 - e^{-2s})}$$

5. Take Laplace transforms:

$$10\left[s V_o(s) - v_o(0^-)\right] + V_o(s) = V_i(s) = 5\,\frac{0.1}{s^2 + 0.01}$$

$$V_o(s) = \frac{5\,\dfrac{0.1}{s^2+0.01} + 100}{10s + 1}$$

$$= \frac{0.05 + 10(s^2 + 0.01)}{(s^2 + 0.01)(s + 0.1)}$$

$$= \frac{10s^2 + 0.15}{(s^2 + 0.01)(s + 0.1)}$$

- partial fractions

$$V_o(s) = \frac{As + B}{s^2 + 0.01} + \frac{C}{s + 0.1}$$

where $C = \dfrac{0.25}{0.02} = 12.5$

$$\frac{As + B}{s^2 + 0.01} = \frac{10s^2 + 0.15}{(s^2 + 0.01)(s + 0.1)} - \frac{12.5}{s + 0.1}$$

$$= \frac{10s^2 + 0.15 - (12.5s^2 + 0.125)}{(s^2 + 0.01)(s + 0.1)}$$

$$= \frac{-2.5s^2 + 0.025}{(s^2 + 0.01)(s + 0.1)}$$

$$= -2.5 \frac{s^2 - 0.01}{(s^2 + 0.01)(s + 0.1)}$$

$$= -2.5 \frac{(s - 0.1)\cancel{(s + 0.1)}}{(s^2 + 0.01)\cancel{(s + 0.1)}}$$

So

$$V_o(s) = -2.5 \frac{s - 0.1}{s^2 + 0.01} + \frac{12.5}{s + 0.1}$$

$$\Rightarrow v_o(t) = -2.5 \cos 0.1t + 2.5 \sin 0.1t + 12.5 e^{-0.1t}$$

6. Note that

$$\mathcal{L}\{\ddot{y}_1\} = s\,\mathcal{L}\{\dot{y}\} - \dot{y}(0^-)$$
$$= s\left[sY(s) - y(0^-)\right] - \dot{y}(0^-)$$
$$= s^2 Y(s) - s\,y(0^-) - \dot{y}(0^-)$$

So, taking Laplace transforms,

$$2s^2 Y_1(s) - 2s\,y_1(0^-) + 40s\,Y_1(s) + 15\,Y_1(s)$$
$$- 15s\,Y_2(s) - 5\,Y_2(s) = 0$$

&

$$s^2 Y_2(s) - s\,y_2(0^-) + 15s\,Y_2(s) + 5\,Y_2(s)$$
$$- 15s\,Y_1(s) - 5\,Y_1(s) = 0$$

(in SI units).

Simplifying,

$$\left[2s^2 + 40s + 15\right] Y_1(s) - \left[15s + 5\right] Y_2(s) = -2s$$
$$\left[s^2 + 15s + 5\right] Y_2(s) - \left[15s + 5\right] Y_1(s) = s$$

That is

$$\begin{bmatrix} [2s^2+40s+15] & -[15s+5] \\ -[15s+5] & [s^2+15s+5] \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} -2s \\ s \end{bmatrix}$$

Solving for $Y_1(s)$ (e.g., by Cramer's Rule),

$$Y_1(s) = \frac{\begin{vmatrix} -2s & -[15s+5] \\ s & [s^2+15s+5] \end{vmatrix}}{\begin{vmatrix} [2s^2+40s+15] & -[15s+5] \\ -[15s+5] & [s^2+15s+5] \end{vmatrix}}$$

$$= \frac{s\left[-2\left[s^2+15s+5\right]+\left[15s+5\right]\right]}{(2s^2+40s+15)(s^2+15s+5)-(15s+5)^2}$$

$$= \frac{s\left[-2s^2-15s-5\right]}{2(s+28)(s+6.2)(s+0.42)(s+0.34)}$$

(by numerical computation)

By partial fractions,

$$Y_1(s) = \frac{-0.97}{s+28} + \frac{-0.048}{s+6.2}$$

$$+ \frac{0.016}{s+0.42} + \frac{0.0017}{s+0.34}$$

So

$$y_1(t) = -0.97\,e^{-28t} - 0.048\,e^{-6.2t}$$

$$+ 0.016\,e^{-0.42t} + 0.0017\,e^{-0.34t},$$

$$t \geq 0$$