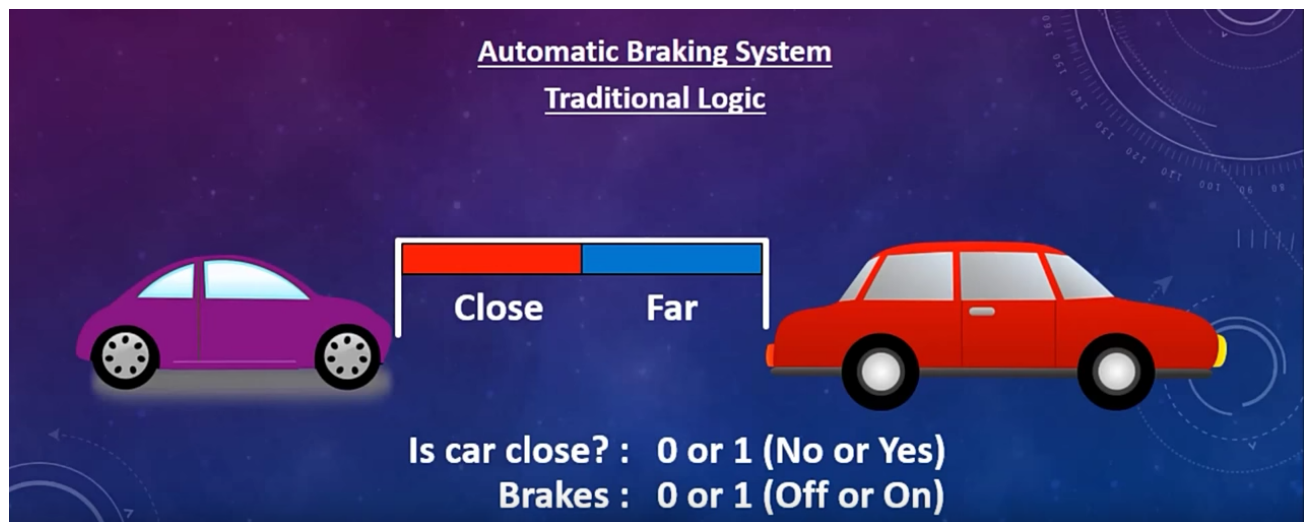


# Origins

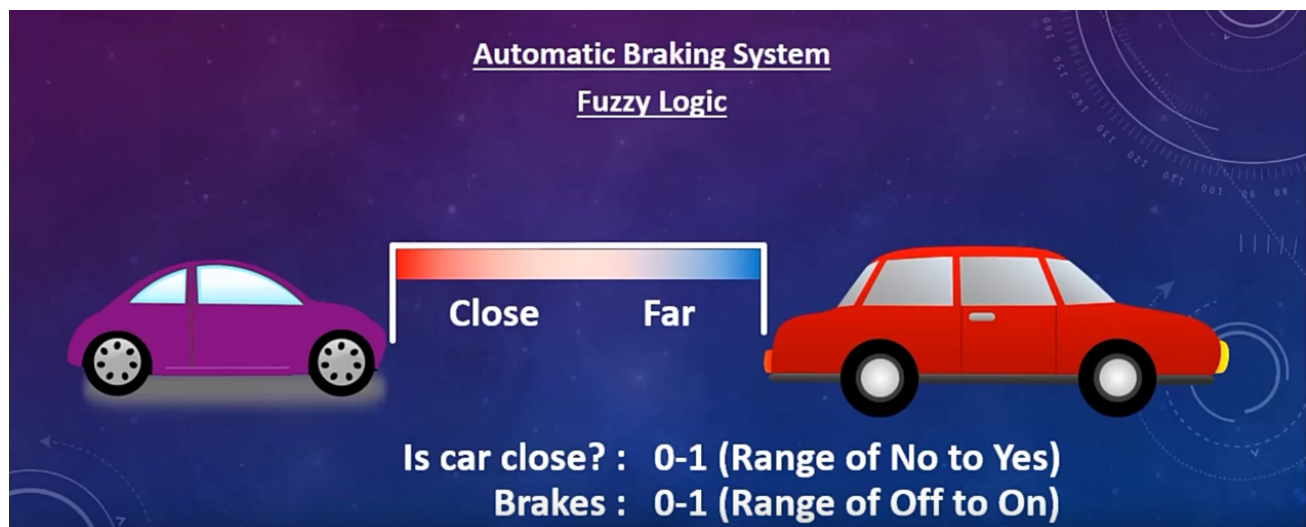
- Fuzzy logic was first developed by L.A. Zadeh in 1960's to extend conventional (binary) crisp logic to make it suitable to incorporate knowledge and mimic human-like approximate reasoning.
- Two-state (bivalent) crisp logic uses two quantities: true (T) and false (F) as truth values.
- Real-life situations are usually characterized by ambiguity and partial truth.  
⇒ cannot always assume two crisp states T and F with a crisp line dividing them.

# Illustration 1



Usually we would say that when the car is close to another car we start breaking. We want to pass our ability to determine "too close" to the car so that it can detect it. We can't just have a binary value of break, no break since everything is continuous.

## Illustration 1 (cont.)



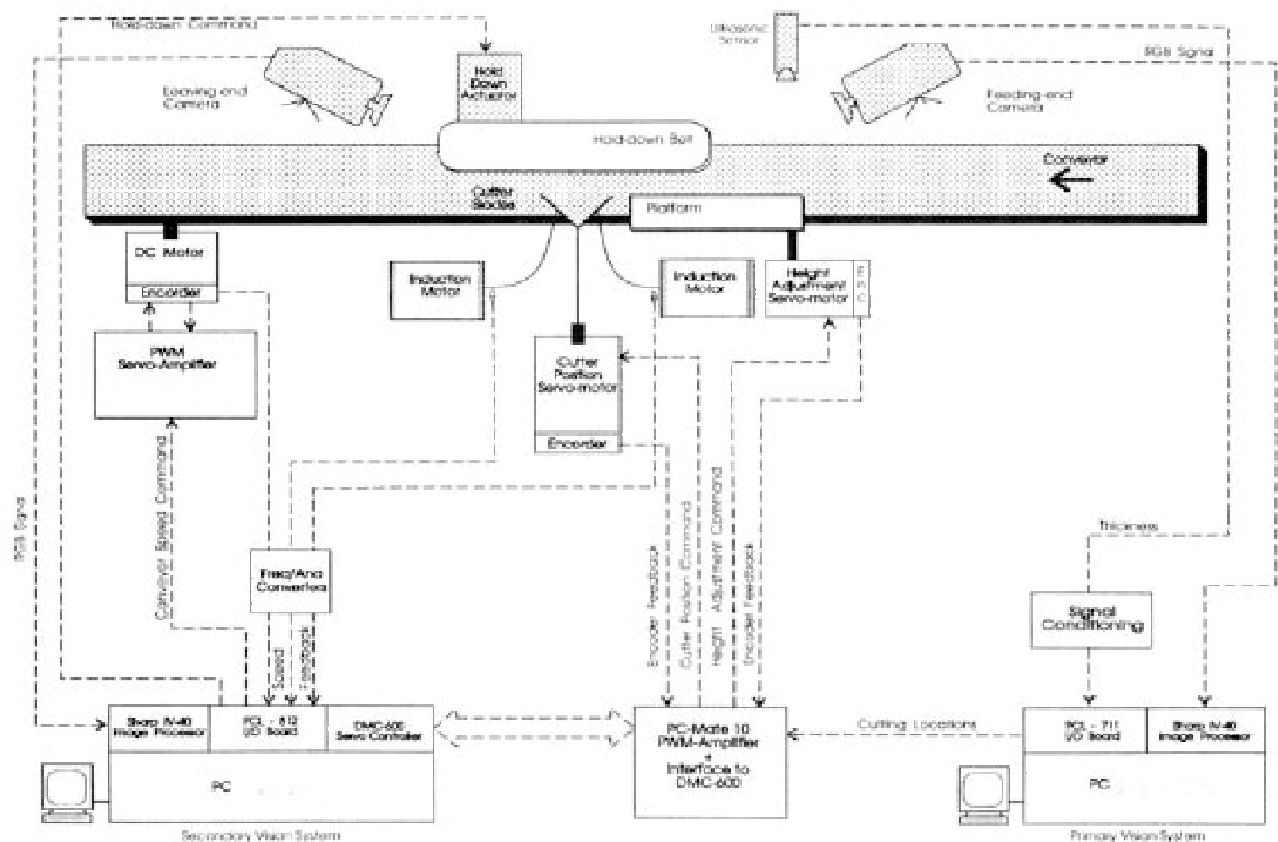
In real life we drive by evaluating on a spectrum rather than binary values.

## Illustration 2

- The linguistic descriptors “fast”, “warm”, and “large” are not crisp quantities and tend to be quite **subjective**, **approximate**, and **qualitative**.
- The statement “the water is warm” may have **varying** degrees of truth values (not always T or F).

English is not very good at crisp descriptors.

## Illustration 3: Fish Cutting (System Architecture)





## Illustration 3: Fish Cutting (cont.)

- Machine adjustments (control system's outputs) may include descriptive changes (e.g., small, medium, large) in the cutter blade speed, conveyor speed, and holding force.
- Human knowledge is represented in the form of a set of if-then rules:  
**If** the “quality” of the processed fish is “not acceptable”, **and**  
**if** the cutting load appears to be “high”,  
**then** “moderately” decrease the conveyor belt speed.

## Illustration 3: Fish Cutting (cont.)

- These rules contain qualitative, descriptive, and linguistic terms such as “quality”, “high”, and “slight”, which reflects human knowledge/expertise in operating such systems.
- It is clear that these fuzzy descriptors cannot be directly incorporated by conventional bivalent logic theories.

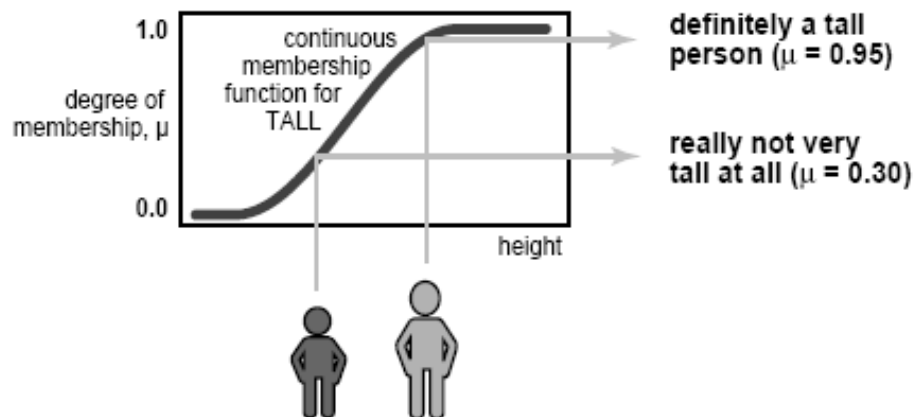
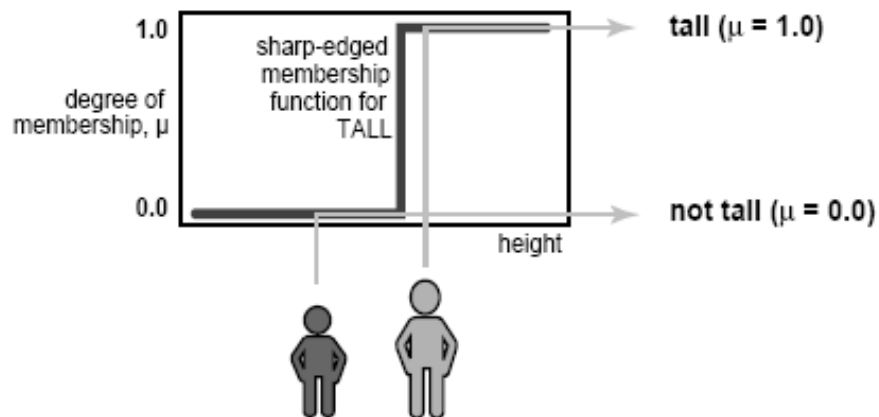
# Applications of Fuzzy Logic

Consumer	Automotive	Industrial	Decision support
VCRs Microwave ovens Camcorders Washing machines Vacuum cleaners Clothes dryers	Air conditioning Emission control Engine control Fuel control Suspension control Cruise control	Extruding Printing Painting Food processing Injection molding Packaging Conveyor control Temperature control AGV Control Mixing Furnace control Plating	Building HVAC Transportation Medical diagnostics Stock market analysis Database processing

# Fuzzy Sets

- Fuzzy sets represent the corner stone of fuzzy logic theory.
- Unlike a crisp set, where an element either belongs to it or not, **partial** membership in a fuzzy set is possible.
- An example of a fuzzy set is “the set of narrow streets in a city.”
- How to quantify the term “narrow”?  $\Rightarrow$  **Gray** zone.

# Definition of Tall: Crisp and Fuzzy Representation

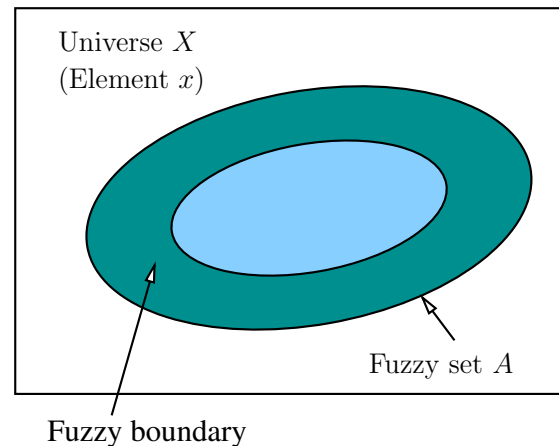


# Universe of Discourse and a Fuzzy Set

## Definition

Let  $X$  be a set that contains every set of interest in the context of a given class of problems (e.g., set of all students). Then,  $X$  is called the **universe of discourse** (or simply the **universe**), whose elements are denoted by  $x$ .

- A fuzzy set  $A$  in the universe of discourse  $X$  may be represented by a Venn diagram.



- Generally, the elements of a fuzzy set are not numerical quantities.
- However, for analytical convenience, they are assigned real values.

We have the set of all things we are considering for the given context, called the **universe of discourse**. Within that set we get a fuzzy set. That fuzzy set has an inner part that is known to belong to the set and an outer boundry that is fuzzy.

# Membership Function

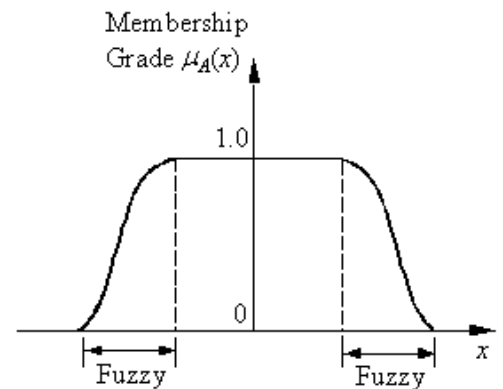
- A fuzzy set may be represented by a membership function, which gives the grade (or degree) of membership within the set, of every element of the universe of discourse.
- The membership function maps the elements of the universe of discourse onto numerical values in the interval  $[0, 1]$ .

## Membership function

A fuzzy set  $A$  may be represented as a set of ordered pairs:

$$A = \{(x, \mu_A(x)); x \in X, \mu_A(x) \in [0, 1]\},$$

where  $\mu_A(x)$  is the membership function of the fuzzy set  $A$ .





# Membership Function

- The membership function  $\mu_A(x)$  represents the grade of possibility (not probability) that an element  $x$  belongs to the fuzzy set  $A$ .

$\mu_A(x) = 1 \quad \Rightarrow x$  is definitely an element of  $A$ .

$\mu_A(x) = 0 \quad \Rightarrow x$  is definitely not an element of  $A$ .

$\mu_A(x) = 0.2 \quad \Rightarrow$  the possibility that  $x$  is an element of  $A$  is 0.2.

## Remark

A crisp set is a special case of a fuzzy set where the membership function can take only two values, 0 and 1.

We define a **membership function** that helps us determine if an item belongs in the set as a degree. Its important to note that the membership function does not output probability but the grade of possibility.

# Symbolic Representation

- If the universe of discourse is discrete with elements  $x_i$ , then a fuzzy set  $A$  may be symbolically represented with a “formal series”

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n = \sum_{x_i \in X} \mu_A(x_i)/x_i$$

- If the universe of discourse is continuous, an equivalent form notation is given in terms of a “symbolic” integration

$$A = \int_{x_i \in X} \mu_A(x_i)/x_i$$

- $\sum$  and  $\int$  do not represent summation or integration but rather the collection of members in discrete or continuous domain.

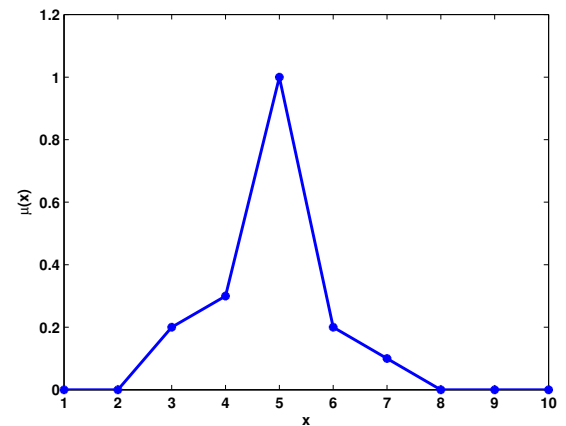
# Fuzzy Representation of 5

## Discrete Universe

Suppose that the universe of discourse  $X$  is the set of positive integers (or natural numbers). Consider the fuzzy set  $A$  in this discrete universe, given by the Zadeh notation:

$$A = 0.2/3 + 0.3/4 + 1.0/5 + 0.2/6 + 0.1/7$$

This set may be interpreted as a fuzzy representation of the integer 5.



# Fuzzy Representation of “ $x = a$ ”

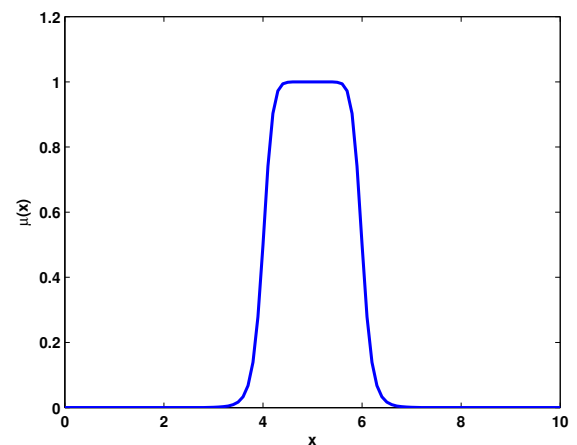
## Continuous Universe

Consider the continuous universe of discourse  $X$  representing the set of real number. The membership function

$$\mu_A(x) = \frac{1}{1 + (x - a)^{10}}$$

Defines a fuzzy set  $A$  whose elements  $x$  vaguely represent those satisfying the crisp relation  $x = a$ . This fuzzy set corresponds to a *fuzzy relation*.

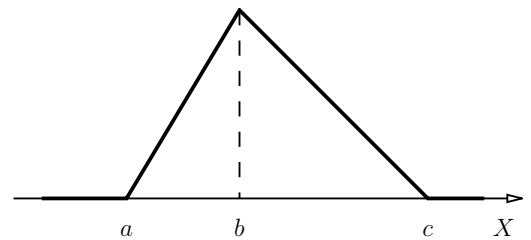
$a = 5$



We can represent the membership function as a discrete function or as an continuous function. This will be determined by the universe in which we are working

# Triangular Membership Function

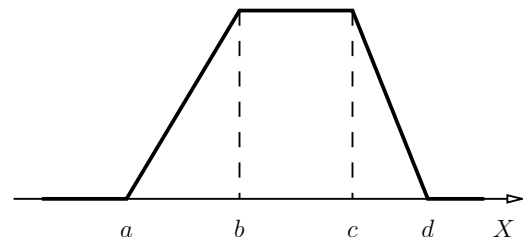
A triangular membership function is characterized by 3 parameters  $\{a, b, c\}$ ,  $a < b < c$ .



$$\begin{aligned} \text{triangle}(x|a, b, c) &= \begin{cases} 0 & , x \leq a \\ \frac{x-a}{b-a} & , a \leq x \leq b \\ \frac{c-x}{c-b} & , b \leq x \leq c \\ 0 & , x \geq c \end{cases} \\ &= \max \left[ \min \left( \frac{x-a}{b-a}, \frac{c-x}{c-b} \right), 0 \right] \end{aligned}$$

# Trapezoidal Membership Function

A trapezoidal membership function is characterized by 4 parameters  $\{a, b, c, d\}$ ,  $a < b < c < d$ .

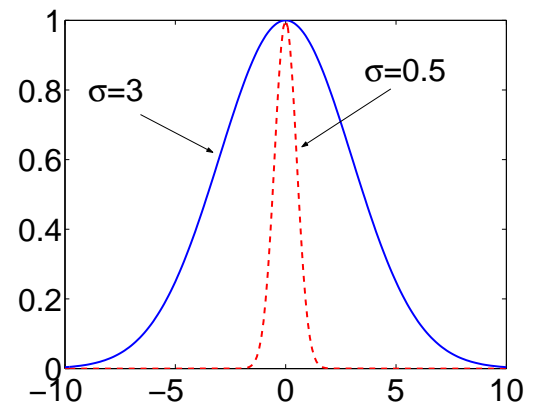


$$\text{trapezoid}(x|a, b, c, d) = \begin{cases} 0 & , x \leq a \\ \frac{x-a}{b-a} & , a \leq x \leq b \\ 1 & , b \leq x \leq c \\ \frac{d-x}{d-c} & , c \leq x \leq d \\ 0 & , x \geq d \end{cases}$$
$$= \max \left[ \min \left( \frac{x-a}{b-a}, 1, \frac{d-x}{d-c} \right), 0 \right]$$



# Gaussian Membership Function

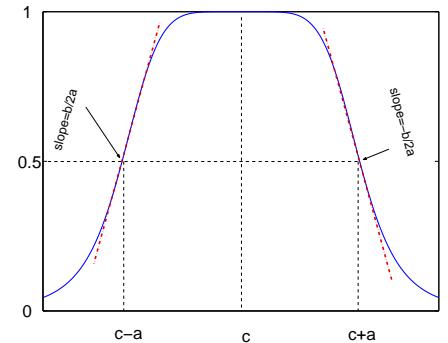
A Gaussian membership function is characterized by 2 parameters  $\{\sigma, c\}$ , with  $\sigma$  indicating the width, and  $c$  being the center of the membership function, respectively.



$$\text{gauss}(x|\sigma, c) = e^{-0.5\left(\frac{x-c}{\sigma}\right)^2}$$

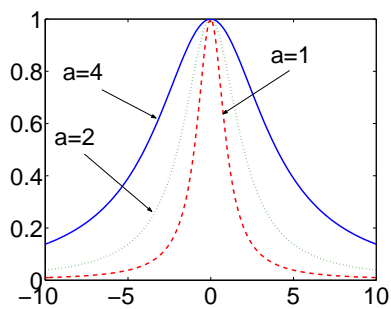
# Generalized Bell Membership Function

A Generalized Bell (or simply Bell) membership function is characterized by 3 parameters  $\{a, b, c\}$ , with  $a$  and  $b$  defining the width and the slope, and  $c$  being the center of the membership function, respectively.

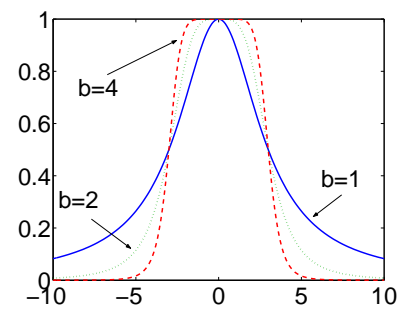


$$gbell(x|a, b, c) = \frac{1}{1 + \left| \frac{x-c}{a} \right|^{2b}}$$

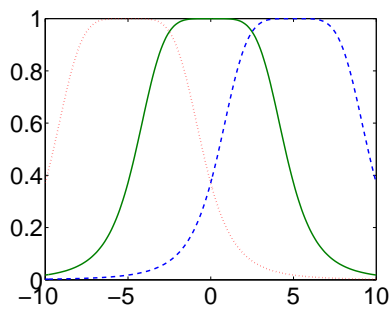
# Bell MF: Parameters Significance



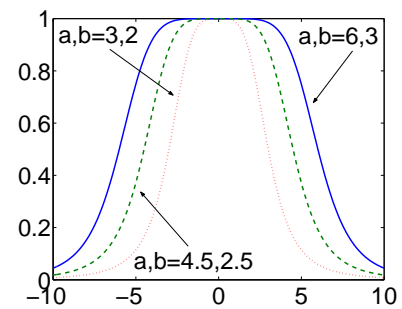
Varying  $a$



Varying  $b$



Varying  $c$



Varying  $a$  and  $b$

## Remark

- Triangular and trapezoidal membership functions:
  - + Linear  $\Rightarrow$  Computationally inexpensive.
  - Non-differentiable  $\Rightarrow$  May not be suitable to be used with gradient-descent optimization algorithms.
- Gaussian and Bell membership functions:
  - + Differentiable  $\Rightarrow$  Suitable to be used with gradient-descent optimization algorithms.
  - Nonlinear  $\Rightarrow$  Computationally expensive.

Membership functions are bitchin.

Triangular and trapezoidal membership functions are the cheapest to calculate, but they are non-differentiable which turns out to be very important later on.

# Fuzzy Logic Operations

- Like in crisp sets, several operations can be performed on fuzzy sets.
- A fuzzy logic operation is an operation applied to “one” or more membership functions (or membership grades) at a time, resulting in one membership function (or grade).
- The associated mapping is indicated by:

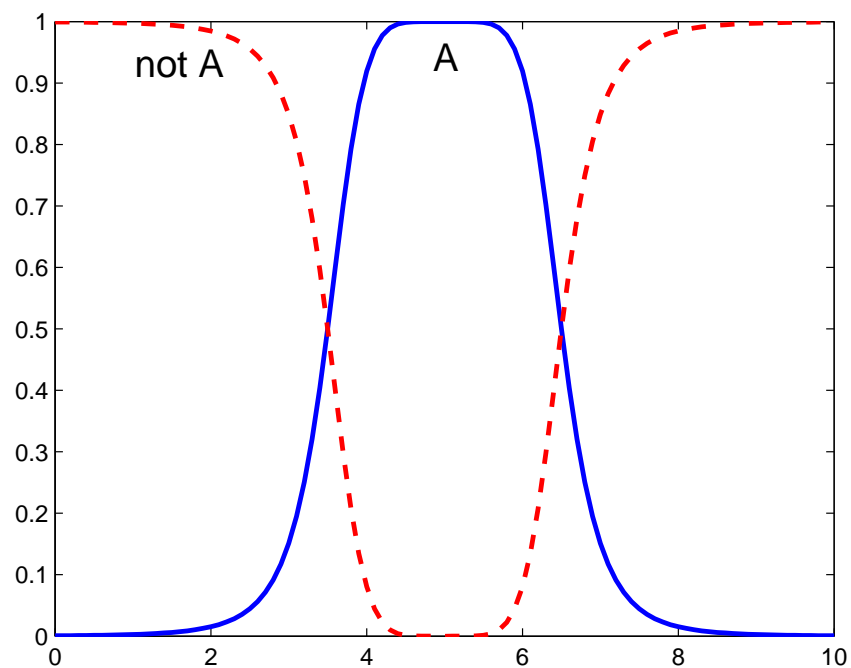
$$f : [0, 1] \times [0, 1] \times \dots \times [0, 1] \rightarrow [0, 1]$$

- Among the most useful operations are:
  - the complement
  - the union
  - the intersection

# Fuzzy Complement (Negation, Not)

- Consider a fuzzy set  $A$  in a universe of discourse  $X$ . Its complement  $A'$  is a fuzzy set with membership function:

$$\mu_{A'}(x) = 1 - \mu_A(x), \quad x \in X$$



# Fuzzy Union and Intersection

Assume two fuzzy sets  $A$  and  $B$  are given in universe  $X$ .

- The fuzzy union of  $A$  and  $B$ :

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)), \forall x \in X.$$

- The fuzzy intersection of  $A$  and  $B$ :

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)), \forall x \in X.$$



Union and intersection of fuzzy sets work a bit weirdly. For now we are just going to use these equations but a better definition will be provided.

## Example: Fuzzy Complement

### Discrete Universe

- Suppose that the universe of discourse  $X$  is defined as  $X = \{0, 7\}$ .
- Consider the fuzzy set  $A$  in this discrete universe given by:

$$A = 0.2/3 + 0.3/4 + 1.0/5 + 0.2/6 + 0.1/7$$

- The complement of fuzzy set  $A$  is as follows:

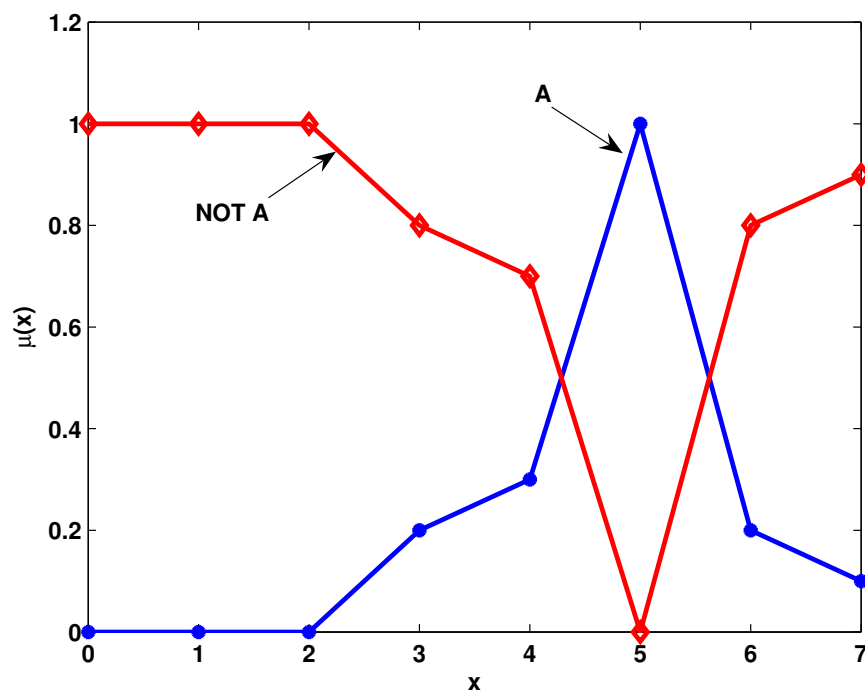
$$A' = 1.0/0 + 1.0/1 + 1.0/2 + 0.8/3 \\ + 0.7/4 + 0.0/5 + 0.8/6 + 0.9/7$$

Lets consider this discrete fuzzy set A. To find the compliment we go through each value in the universe of discourse and subtract each value in the set A from 1 (if there is no value in the set A for the entry its "value" is 0).

**Important note:** A/B notation is not division, but equivalent to (B,A) coordinates. Basically given input B we get output A from the given function. The pluses denote set addition not mathematical addition.

## Example: Fuzzy Complement (cont.)

### Graphical Representation



## Example: Fuzzy Union and Intersection

### Discrete Universe

Suppose that the universe of discourse  $X$  denotes the driving speeds in a highway. Consider the fuzzy sets  $A$  and  $B$  are given by:

- Fuzzy state **Slow** ( $A$ )

$$A = 1.0/40 + 1.0/50 + 0.8/60 + 0.4/80 + 0.1/100$$

- Fuzzy state **Moderate** ( $B$ )

$$B = 0.6/50 + 0.8/60 + 1.0/70 + 1.0/80 + 1.0/90 + 1.0/100$$

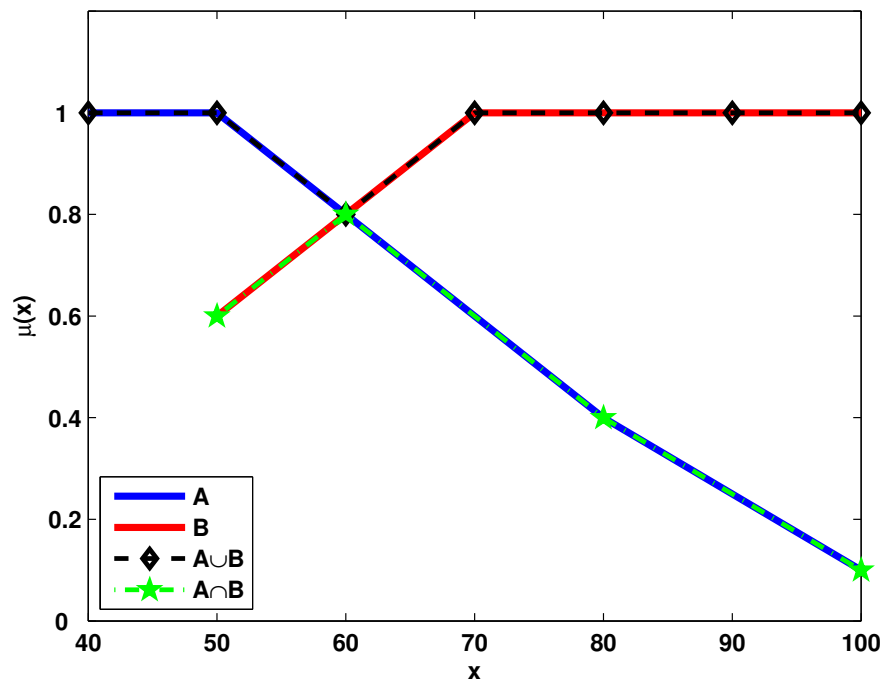
The fuzzy union and intersection of  $A$  and  $B$  are as follows:

$$A \cup B = 1.0/40 + 1.0/50 + 0.8/60 + 1.0/70 + 1.0/80 + 1.0/90 + 1.0/100$$

$$A \cap B = 0.6/50 + 0.8/60 + 0.4/80 + 0.1/100$$

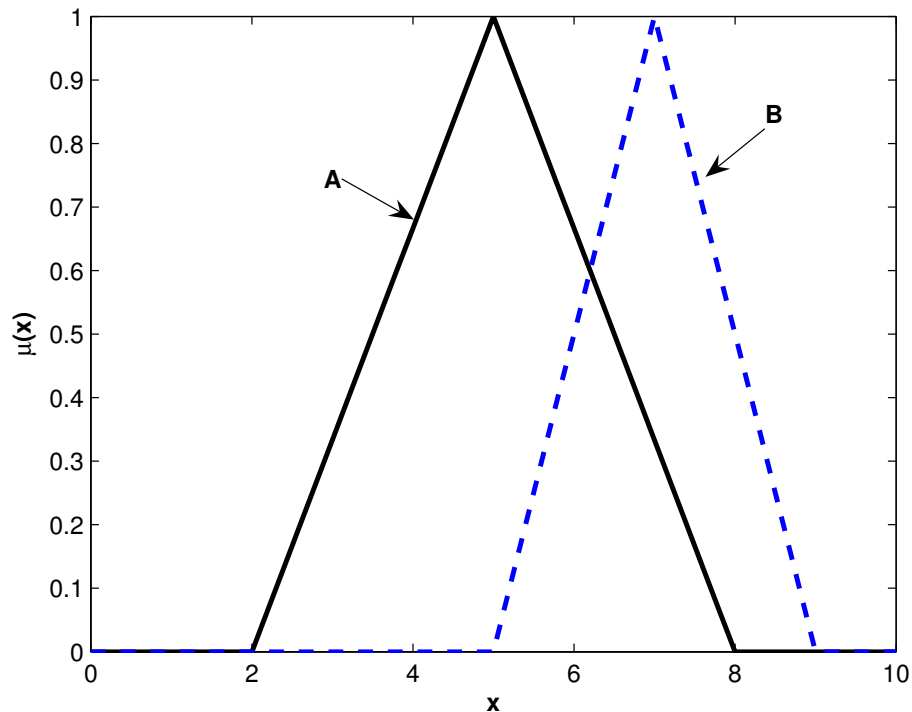
## Example: Fuzzy Union and Intersection (cont.)

### Graphical Representation



# Example: Continuous Universe

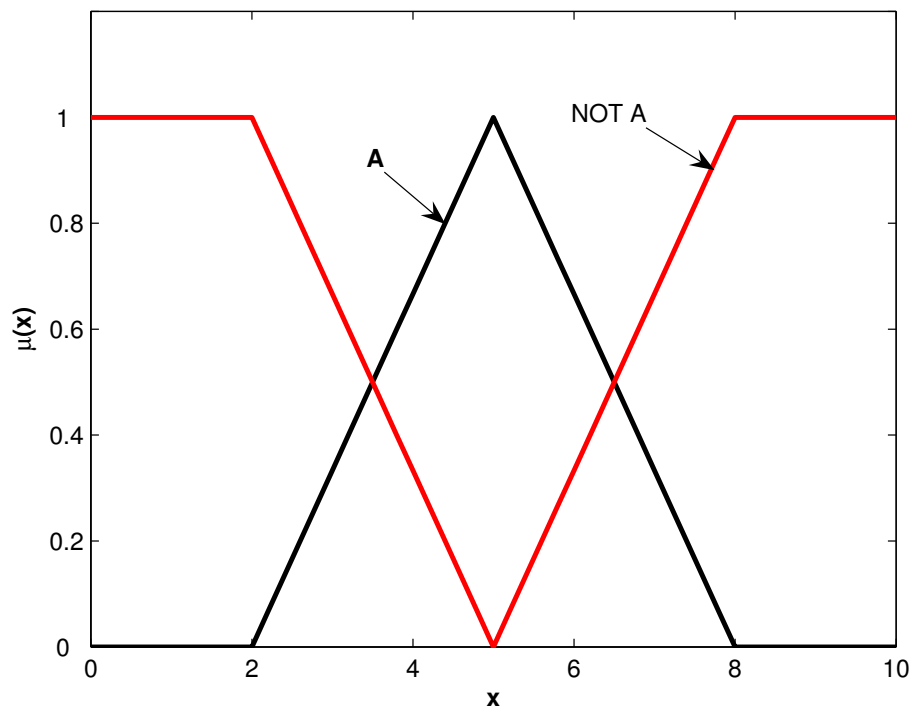
## Consider Two Fuzzy Sets $A$ and $B$



We are interested in finding the fuzzy complement of  $A$  as well as fuzzy union and intersection of  $A$  and  $B$ .

## Example (cont.)

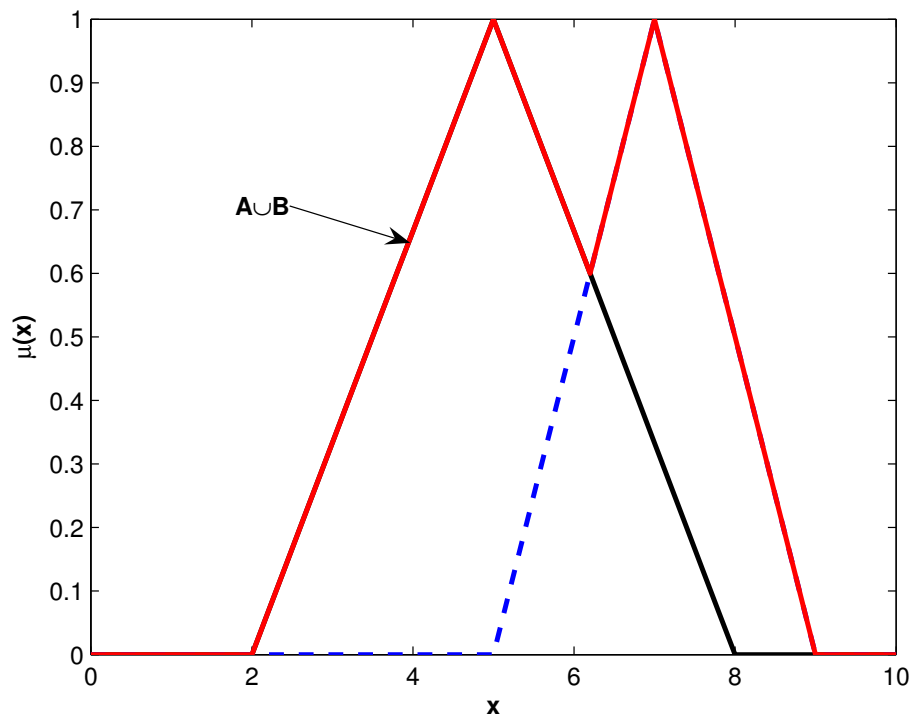
### Fuzzy Complement of $A$





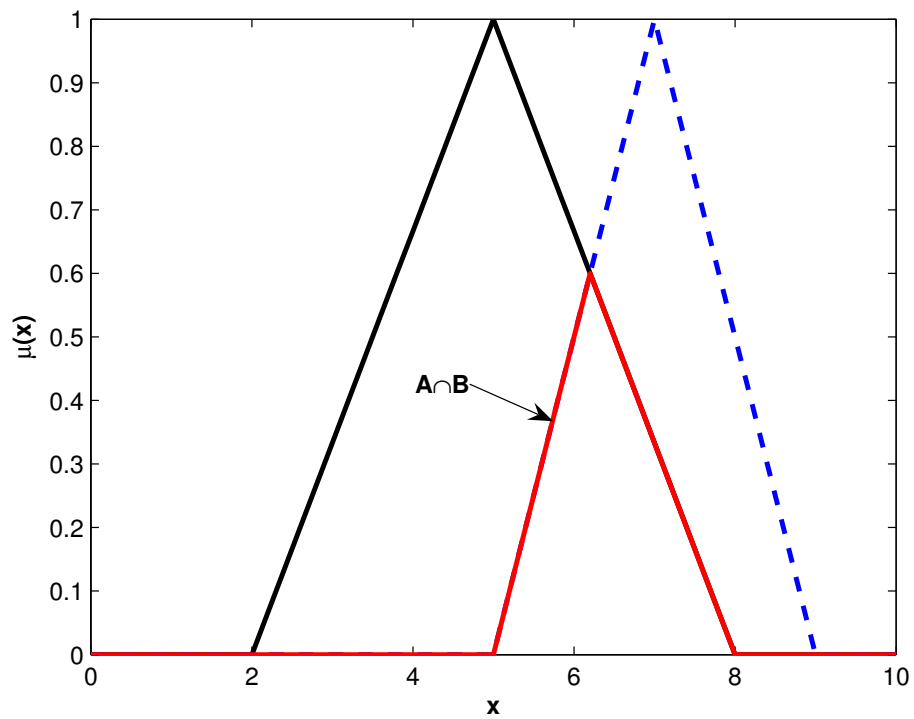
## Example (cont.)

### Fuzzy Union of $A$ and $B$



## Example (cont.)

### Fuzzy Intersection of $A$ and $B$



# Generalized Fuzzy Complement

A generalized complement operation, denoted by  $C : [0, 1] \rightarrow [0, 1]$ , should satisfy the following axioms:

- Boundary conditions:  $C(\emptyset) = X$  and  $C(X) = \emptyset$ , where  $X$  is the universe of discourse and  $\emptyset$  is the null set.
- Non-increasing: if  $\mu_A(x) < \mu_B(x)$  then  $C(\mu_A(x)) \geq C(\mu_B(x))$ , and vice versa.
- Involution:  $C(C(A)) = A$ .

## Two of the Well Known Fuzzy Complement Operators

- 1 Sugeno's complement:  $C(a) = \frac{1 - a}{1 + pa}$  ,  $p \in (-1, \infty)$
- 2 Yager's complement:  $C(a) = (1 - a^p)^{1/p}$  ,  $p \in (0, \infty)$

Here we have three axioms that we must satisfy inorder to have a fuzzy compliment. There are infintely many operatators that could provide a compliment.

**KNOW THIS!!!!**

# Fuzzy Intersection and T-norm

- The intersection of two fuzzy sets  $A$  and  $B$  is given by an operation  $T$  which maps two membership functions to:

$$\mu_{A \cap B}(x) = T(\mu_A(x), \mu_B(x)).$$

- $T(\cdot)$  is known as the T-norm operator.

## T-norm (Generalized Intersection)

- Consider two membership functions that are given by  $a = \mu_A(x)$  and  $b = \mu_B(x)$ .
- The t-norm operation or generalized intersection may be represented by  $T(a, b)$  or more commonly  $aTb$ .

### T-norm Properties

- 1 It is non-decreasing in each argument. i.e., if  $a \leq b$  and  $c \leq d$  then  $aTc \leq bTd$ .
- 2 It satisfies commutativity. i.e.,  $aTb = bTa$ .
- 3 It satisfies associativity. i.e.,  $(aTb)Tc = aT(bTc)$ .
- 4 It satisfies the boundary conditions, i.e.,  $aT1 = a$  and  $aT0 = 0$ .

Here is the criteria required for a intersection operator.

**KNOW THIS!!!!**

# Well Known T-norm Operators

- Two general forms of T-norms are:

$$1 - \min[1, ((1 - a)^p + (1 - b)^p)^{1/p}], \quad p \geq 1$$
$$\max[0, (\lambda + 1)(a + b - 1) - \lambda ab], \quad \lambda \geq -1$$

- Four of the well known T-norm operators are:

Min:  $T(a, b) = \min(a, b) = a \wedge b$

Algebraic product:  $T(a, b) = a \times b$

Bounded product:  $T(a, b) = 0 \vee (a + b - 1)$

Basic product:  $T(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{if } a, b < 1 \end{cases}$

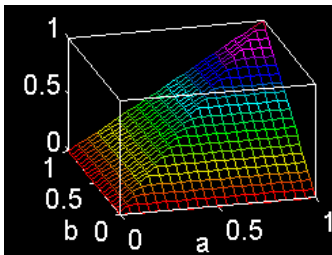


I don't really understand the first part of this slide.

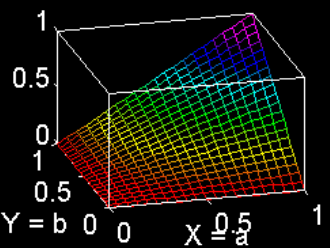
Here are four basic forms of intersection functions. We want to think of the intersection as a sort of minimum. There are infinite number of T-norm operators here are just basic ones.

# Well Known T-norm Operators: Graphical Example

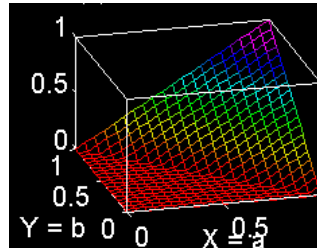
Minimum



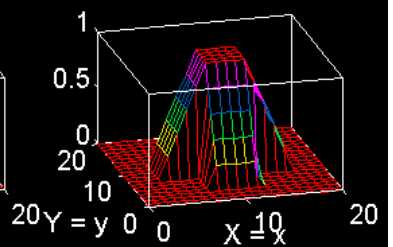
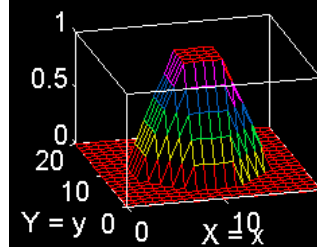
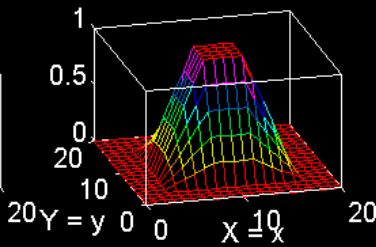
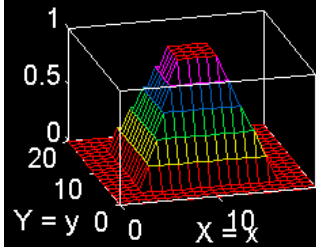
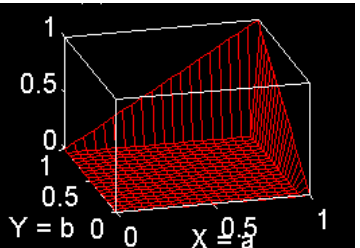
Algebraic product



Bounded product



Basic product



# Fuzzy Union and S-norm

- The union of two fuzzy sets  $A$  and  $B$  is given by an operation  $S$  which maps two membership functions to:

$$\mu_{A \cup B}(x) = S(\mu_A(x), \mu_B(x)).$$

- $S(\cdot)$  is known as the S-norm operator, or T-conorm operator.

## Theorem: DeMorgan's Laws

According to DeMorgan's Law, there exists a complementary S-norm associated to every T-norm, and vice versa.

$$aSb = 1 - (1 - a)T(1 - b)$$

$$aTb = 1 - (1 - a)S(1 - b)$$

We call the union operator the S-norm or the T-conorm operator.

S-norm and T-norm must follow DeMorgan's law.

The compliment of (a times S-norm times b) is equal to the compliment of a times T-norm times the compliment of b

## Example

Prove that the S-norm associated to the T-norm *min* is *max*.

### Proof

- Use DeMorgan's Law:  $aSb = 1 - (1 - a)T(1 - b)$

- Direct substitution of *min* for T gives:

$$\begin{aligned} aSb &= 1 - \min[(1 - a), (1 - b)] = 1 - (1 - a) = a && \text{if } a \geq b \\ &= 1 - (1 - b) = b && \text{if } b < a \end{aligned}$$

- Hence:  $aSb = \max(a, b)$

We defined the T norm as the minimum function so when we sub it into DeMorgan's equation we can prove that the S-norm is the max function.

## S-norm (Generalized Union)

- Consider two membership functions that are given by  $a = \mu_A(x)$  and  $b = \mu_B(x)$ .
- The s-norm operation or generalized union may be represented by  $S(a, b)$  or more commonly  $aSb$ .

### S-norm Properties

- 1 It is non-decreasing in each argument. i.e., if  $a \leq b$  and  $c \leq d$  then  $aSc \leq bSd$ .
- 2 It satisfies commutativity. i.e.,  $aSb = bSa$ .
- 3 It satisfies associativity. i.e.,  $(aSb)Sc = aS(bSc)$ .
- 4 It satisfies the boundary conditions, i.e.,  $aS1 = 1$  and  $aS0 = a$ .

More properties

**KNOW THIS!!!**



# Well Known S-norm Operators

- Two general forms of S-norms are:

$$\min[1, (a^p + b^p)^{1/p}], \quad p \geq 1$$
$$\min[1, a + b + \lambda ab], \quad \lambda \geq -1$$

- Four of the well known S-norm operators are:

Max:  $S(a, b) = \max(a, b) = a \vee b$

Algebraic sum:  $T(a, b) = a + b - ab$

Bounded sum:  $T(a, b) = 1 \wedge (a + b)$

Basic sum:  $S(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{if } a, b < 1 \end{cases}$

The algebraic sum is  $a + b - ab$  because we always want the sum to be in  $[0,1)$ . WATCH THIS, its a common mistake and with fuck up everything.

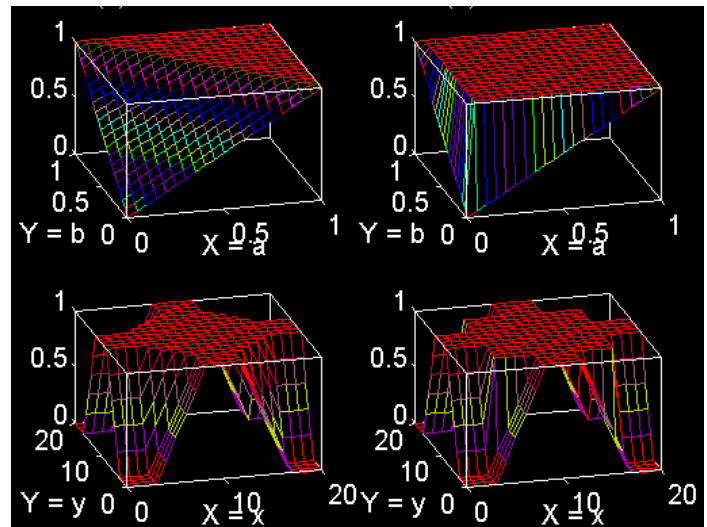
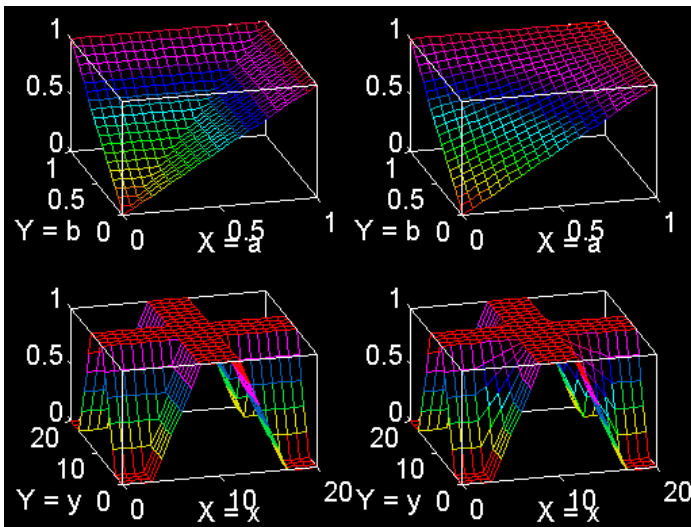
# Well Known S-norm Operators: Graphical Example

Maximum

Algebraic sum

Bounded sum

Basic sum



## Set Inclusion ( $A \subseteq B$ )

- The concept of subset (or, a set “included” in another set) in crisp sets may be conveniently extended to the case of fuzzy sets.
- Specifically, a fuzzy set  $A$  is considered a subset of another fuzzy set  $B$ , in universe  $X$ , if and only if  $\mu_A(x) \leq \mu_B(x)$ , for all  $x \in X$ .
  - This is denoted by  $A \subseteq B$ .

### Set Inclusion

$$A \subset B \Leftrightarrow \mu_A(x) < \mu_B(x), \forall x \in X$$

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \forall x \in X$$

### Definition: Proper Subset

In the case where  $A \subset B$ ,  $A$  is called a **proper subset** of  $B$ .

A set  $A$  is fully included in a set  $B$  if the membership function of  $A$  is strictly less than the membership function of  $B$ . We can limit this over a set interval.

# Grade of Inclusion

In the context of fuzzy logic, a set may be partially included in another set.

- Then, it is convenient to define a grade of inclusion of a fuzzy set  $A$  in another fuzzy set  $B$ .
- This may be interpreted as the membership function of the *fuzzy relation*  $A \subset B$  (or  $A \subseteq B$ ).

## Grade of Inclusion

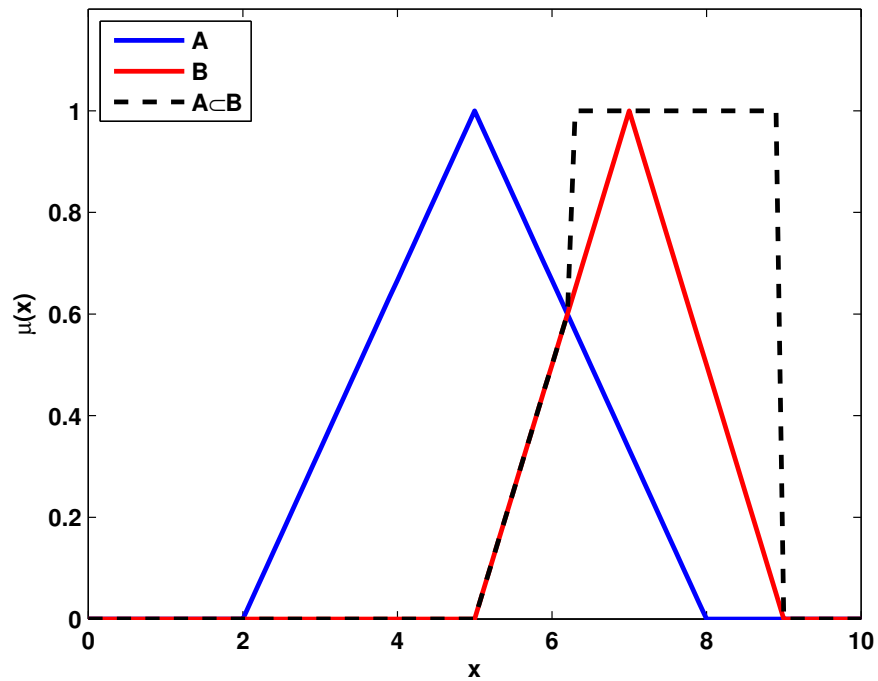
$$\mu_{A \subset B}(x) = \begin{cases} 1 & , \text{ if } \mu_A(x) < \mu_B(x) \\ \mu_A(x) \bar{T} \mu_B(x) & , \text{ otherwise} \end{cases}$$

$$\mu_{A \subseteq B}(x) = \begin{cases} 1 & , \text{ if } \mu_A(x) \leq \mu_B(x) \\ \mu_A(x) \bar{T} \mu_B(x) & , \text{ otherwise} \end{cases}$$

With fuzzy sets we can have fuzzy inclusion. Basically the grade of inclusion is given to sets.

# Example

## Graphical Representation





## Set Equality ( $A = B$ )

- The equality of two fuzzy sets is a special case of set inclusion.
- A fuzzy set  $A$  is equal to another fuzzy set  $B$ , in universe  $X$ , if and only if  $\mu_A(x) = \mu_B(x)$ , for all  $x \in X$ .
  - This is denoted by  $A = B$ .

### Set Equality

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \forall x \in X$$

## Grade of Equality

- A grade of equality for two fuzzy sets may be defined similar to the grade of inclusion.
- This may be interpreted as the membership function of the *fuzzy relation*  $A = B$ .

### Grade of Equality

$$\mu_{A=B} = \begin{cases} 1 & , \text{ if } \mu_A(x) = \mu_B(x) \\ \mu_A(x) T \mu_B(x) & , \text{ otherwise} \end{cases}$$

# Dilation and Contraction

- Let  $A$  be a fuzzy set in the universe  $X$  with membership function  $\mu_A$ .
  - Its  $K$ th dilation is a fuzzy set  $A'$  in the universe  $X$  with membership function  $\mu_{A'}(x) = \mu_A^{1/K}(x)$ .
  - Its  $K$ th contraction is a fuzzy set  $A''$  in the universe  $X$  with membership function  $\mu_{A''}(x) = \mu_A^K(x)$ .

## Dilation

$$\text{dil}(A) = A^{1/2} = \int \frac{\sqrt{\mu_A(x)}}{x} \equiv \{\text{more or less}\} \equiv \{\text{somehow}\}$$

## Contraction

$$\text{con}(A) = A^2 = \int \frac{(\mu_A(x))^2}{x} \equiv \{\text{very}\} \equiv \{\text{too}\}$$

Dilation and contraction are other operations applied to the fuzzy set.

Dilation is denoted as  $A^{\frac{1}{2}}$  gives a more elongated membership function over the universe. The integral in the equation is only used in the continuous case, it might not always apply. This gives us richness of the values. We use it to say its more or less "adjective". It increases the membership grade of elements that have small grades.

Contraction is basically the opposite of dilation.

Helpful slides: Helpful slides

# Example 1

## Discrete Case

- Consider the following fuzzy set:

$$A = 0.0/1 + 0.5/2 + 1.0/3 + 0.5/4 + 0.0/5$$

- The second dilation of fuzzy set  $A$  is given as follows:

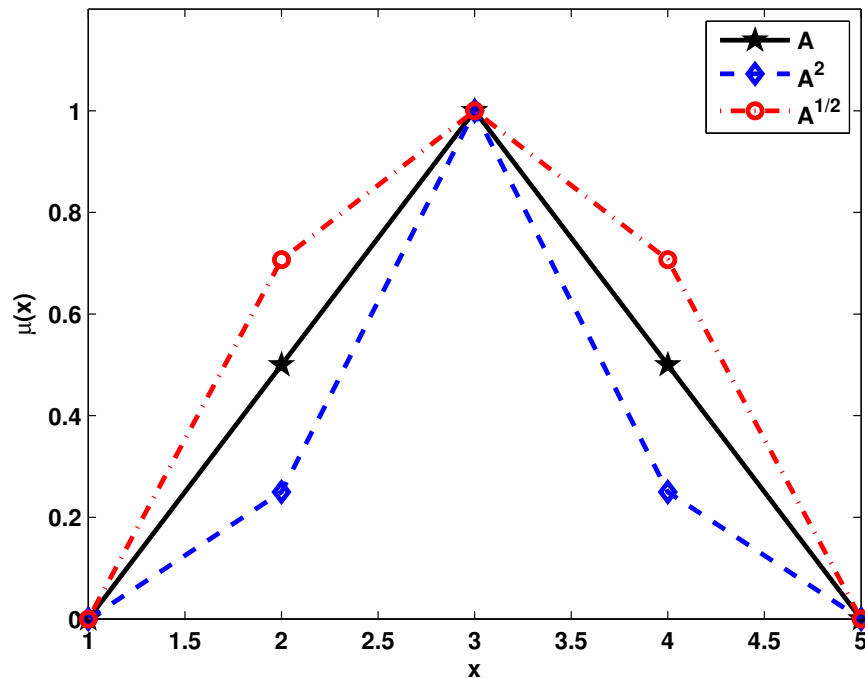
$$\begin{aligned} A^{1/2} &= 0.0^{1/2}/1 + 0.5^{1/2}/2 + 1.0^{1/2}/3 + 0.5^{1/2}/4 + 0.0^{1/2}/5 \\ &= 0.0/1 + 0.7071/2 + 1.0/3 + 0.7071/4 + 0.0/5 \end{aligned}$$

- Contraction of fuzzy set  $A$  is given as follows:

$$\begin{aligned} A^2 &= 0.0^2/1 + 0.5^2/2 + 1.0^2/3 + 0.5^2/4 + 0.0^2/5 \\ &= 0.0/1 + 0.25/2 + 1.0/3 + 0.25/4 + 0.0/5 \end{aligned}$$

## Example 1 (cont.)

### Graphical Illustration

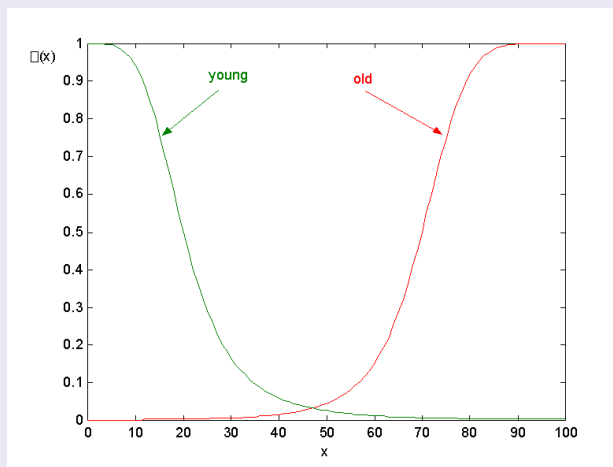


## Example 2

### Continuous Case

$$\mu_{\text{young}}(x) = \text{gbell}(x|20, 2, 0) = \frac{1}{1 + \left|\frac{x}{20}\right|^4}$$

$$\mu_{\text{old}}(x) = \text{gbell}(x|30, 3, 100) = \frac{1}{1 + \left|\frac{x-100}{30}\right|^6}$$



## Example 2 (cont.)

### Construction of linguistic Representation

$$\text{More or less old} \equiv \int_x \left[ \sqrt{\frac{1}{1 + \left| \frac{x-100}{30} \right|^6}} \right] / x$$

$$\text{Not young and not old} \equiv \overline{\text{young}} \cap \overline{\text{old}}$$

$$\equiv \int_x \left[ 1 - \frac{1}{1 + \left| \frac{x}{20} \right|^4} \right] \wedge \left[ 1 - \frac{1}{1 + \left| \frac{x-100}{30} \right|^6} \right] / x$$

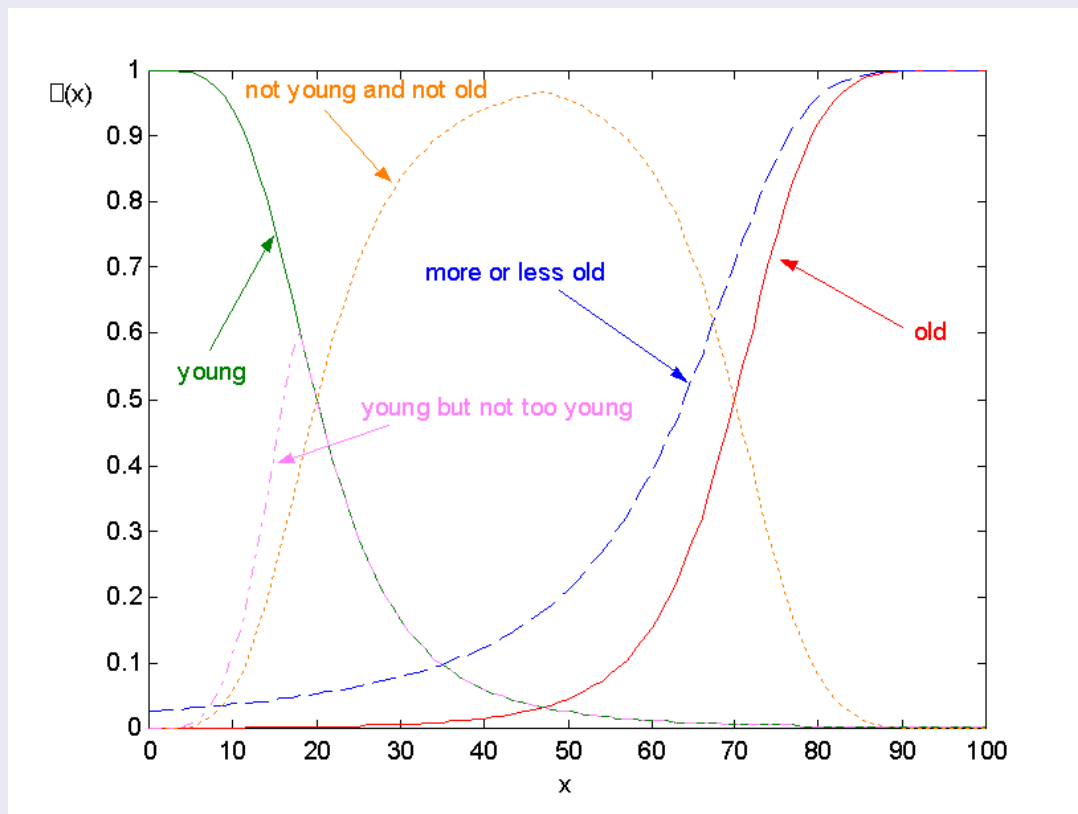
$$\text{Young but not too old} \equiv \text{young} \cap \overline{\text{too old}}$$

$$\equiv \int_x \left[ \frac{1}{1 + \left| \frac{x}{20} \right|^4} \right] \wedge \left[ 1 - \left\{ \frac{1}{1 + \left| \frac{x-100}{30} \right|^6} \right\}^2 \right] / x$$



## Example 2 (cont.)

### Graphical Representation



In an assignment if not specified:

- T-norm = min
- S-norm = max
- Compliment = 1-value

# Implication (if-then)

- Consider two fuzzy sets:  $A$  in  $X$  and  $B$  in  $Y$ .
- The fuzzy implication  $A \rightarrow B$  is a fuzzy relation in the Cartesian product  $X \times Y$  defined by:

## Implication

- Larsen implication:  $\mu_{A \rightarrow B}(x, y) = \mu_A(x) \mu_B(y)$   
 $\forall (x, y) \in (X \times Y)$
- Mamdai implication:  $\mu_{A \rightarrow B}(x, y) = \min [\mu_A(x), \mu_B(y)]$   
 $\forall (x, y) \in (X \times Y)$
- Zadeh implication:  $\mu_{A \rightarrow B}(x, y) = \max [\min \{\mu_A(x), \mu_B(y)\}, 1 - \mu_A(x)]$   
 $\forall (x, y) \in (X \times Y)$
- Dienes-Rascher implication:  $\mu_{A \rightarrow B}(x, y) = \max [1 - \mu_A(x), \mu_B(y)]$   
 $\forall (x, y) \in (X \times Y)$
- Lukasiewicz implication:  $\mu_{A \rightarrow B}(x, y) = \min [1, 1 - \mu_A(x) + \mu_B(y)]$   
 $\forall (x, y) \in (X \times Y)$

There are many different types of implication operators. They are very similar to the intersection operator in terms of input and output but they mean very different things.

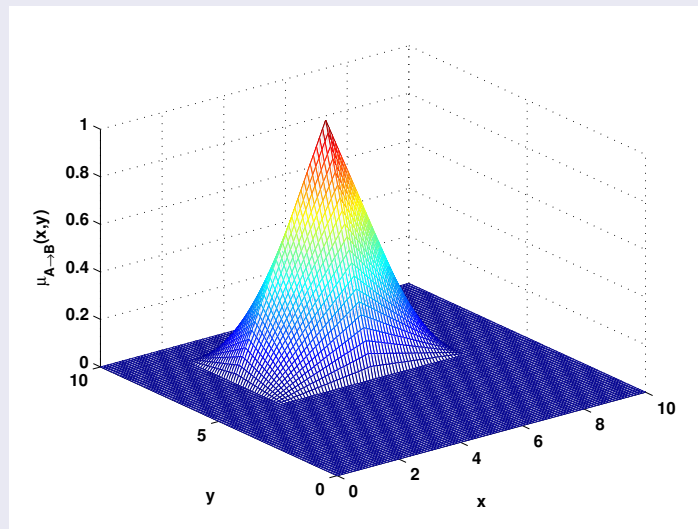
He kind of defined it as a transition between fuzzy sets. It was a bit unclear.

helpful slides: [Helpful slides](#)

# Example

- Assume two fuzzy sets  $A$  and  $B$  are defined on universes  $X$ ,  $Y$ , respectively: triangular membership functions.
- Both sets  $A$  and  $B$  have triangular membership functions given by  $A = \text{triangle}(x|2, 5, 8)$ ,  $B = \text{triangle}(y|5, 7, 9)$ .

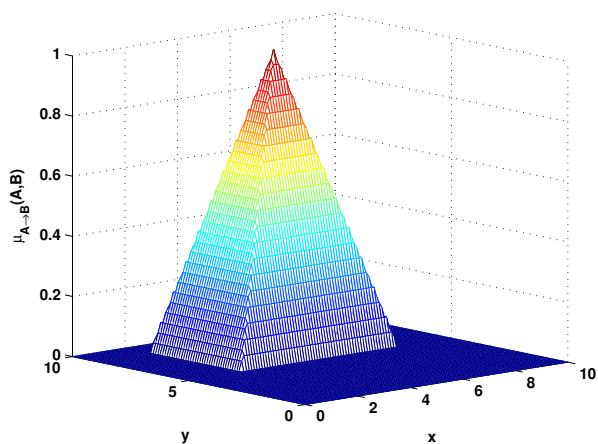
## Larsen Implication



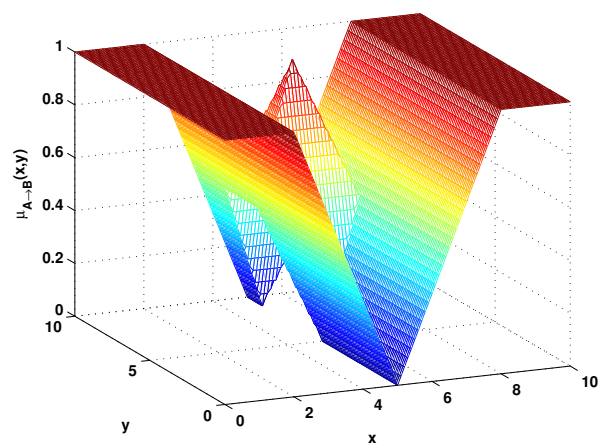
This is an example of the larsen implication operator where both fuzzy sets have a triangle membership function.

## Example (cont.)

### Mamdani Implication

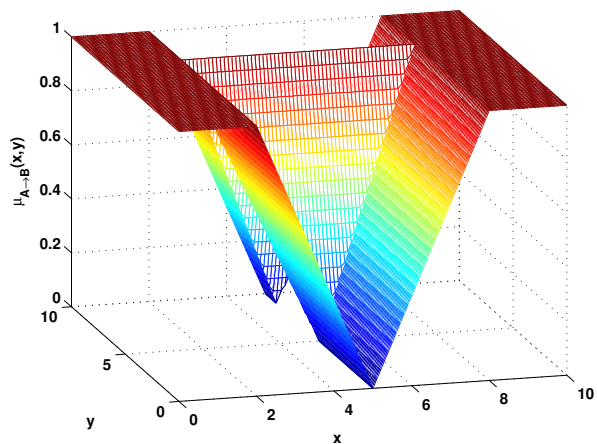


### Zadeh Implication

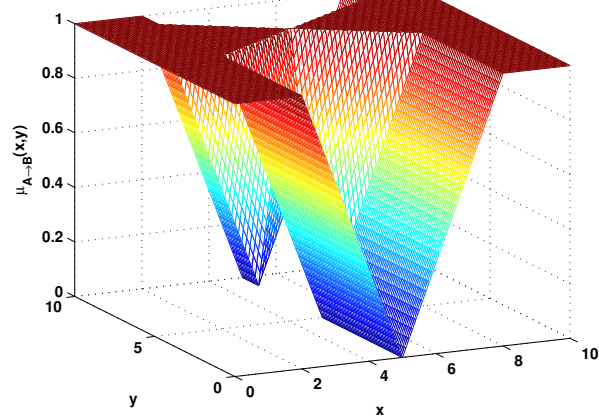


## Example (cont.)

### Deines-Rescher Implication



### Lukasiewicz Implication





# Height of a Fuzzy Set

- The **height** (also called **modal grade**) of a fuzzy set is the **maximum** value of its membership function.

## Height of a Fuzzy Set

For a fuzzy set  $A$  in the universe  $X$ , with membership function  $\mu_A$ , the height of  $A$  is defined as

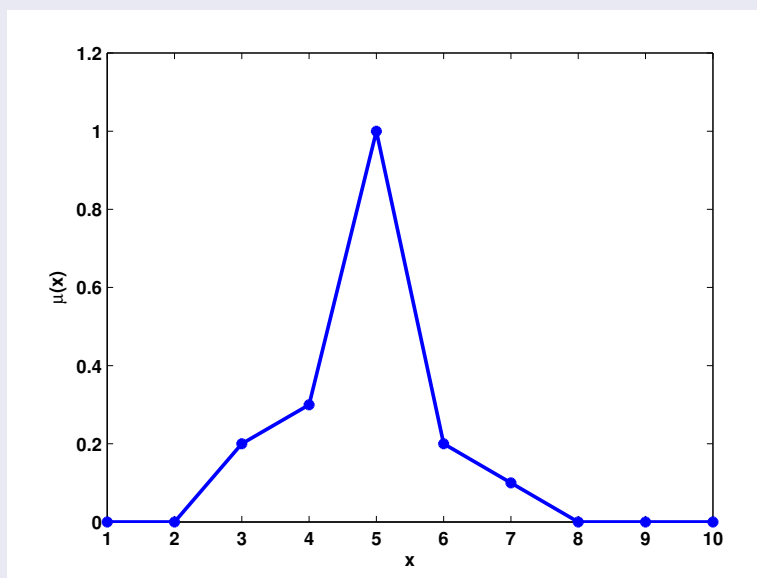
$$\text{hgt}(A) = \sup_{x \in X} \mu_A(x)$$

- The element  $x^* \in X$  corresponding to the modal grade of the fuzzy set (i.e.,  $\mu_A(x^*) = \text{hgt}(A)$ ) is called the **modal element value**, or simply **modal point**.

The height of a fuzzy set is the highest value of its membership function. The modal point is the value at which the max occurs.

# Example

## Fuzzy Set A



- Height of fuzzy set A:  $\text{hgt}(A) = 1$
- Modal point:  $x = 5$

# Support Set

- The **support set** of a fuzzy set is a **crisp** set containing all the elements (in the universe) whose membership grades are **greater** than zero.

## Support Set

The support set  $S$  of a fuzzy set  $A$  in the universe  $X$ , with membership function  $\mu_A$ , is defined by

$$S = \{x \in X \mid \mu_A(x) > 0\}$$

The crisp set (aka support set) is the set of all points where the membership function has a nonzero value.

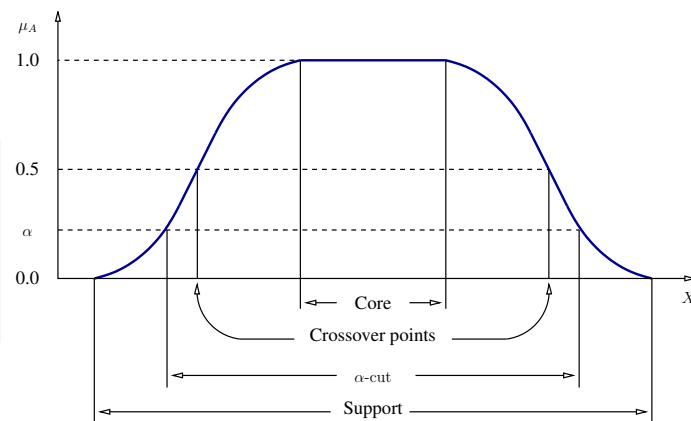
## $\alpha$ -cut of a Fuzzy Set

The  $\alpha$ -**cut** of a fuzzy set  $A$  is the **crisp** set denoted by  $A_\alpha$  formed by the elements of  $A$  whose membership function grades are **greater than or equal** to a specified threshold value  $\alpha \in [0, 1]$ .

### $\alpha$ -cut

Mathematically,  $A_\alpha$  is defined by

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}, \alpha \in [0, 1]$$



This is the generalized bell membership function. This is considered the richest membership function. You can see from this graph that it has a crisp set, but also that its modal point is actually a set of points since the graph has a plateau of highest points.

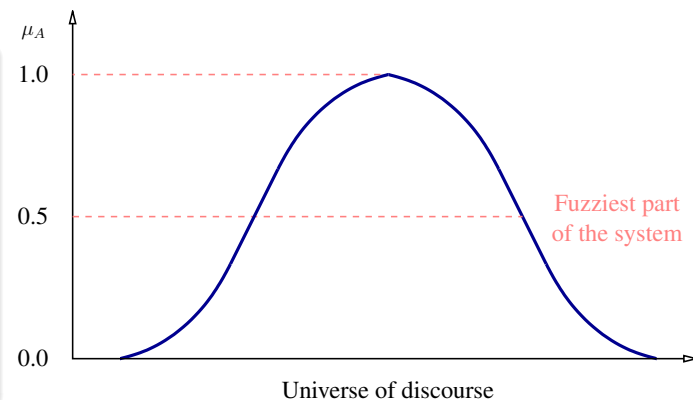
The **core** is the set of points at which the function has its highest values (aka the set of modal points). The **cross over points** are the points where the membership function has the value of 0.5. The **alpha cut** is the crisp set at which the membership function has values higher than some given  $\alpha$ .

# Measure of Fuzziness

A measure of fuzziness of a fuzzy set  $A$  in the universe  $X$  defines the closeness of its membership function  $\mu_A$  to the most fuzzy grade (0.5).

## Different Measures of Fuzziness

- 1 Closeness to grade 0.5 ( $A_1$ ),
- 2 Distance from 1/2-cut ( $A_2$ ),
- 3 Inverse of distance from the complement ( $A_3$ ).





The closeness to grade 0.5 - the fuzziest part of the membership function is at the cross over point. If we go lower than 0.5 we have values that are "low" so we are almost exiting the fuzzy set and if we go up then we are reaching values that are close to being in the fuzzy set, but at 0.5 we aren't sure what to do.

Distance from  $1/2$  cut - this is a specific alpha cut at 0.5 which is very similar to the above one.

# Measures of Fuzziness

## Closeness to grade 0.5

$$A_1 = \int_{x \in S} f(x) dx, \quad S \text{ denotes the support set.}$$
$$f(x) = \begin{cases} \mu_A(x) & \text{for } \mu_A(x) \leq 0.5 \\ 1 - \mu_A(x) & \text{otherwise} \end{cases}$$

## Distance from 1/2-cut

$$A_2 = \int_{x \in X} \left| \mu_A(x) - \mu_{A_{1/2}}(x) \right| dx$$

# Measures of Fuzziness

## Inverse of distance from the complement

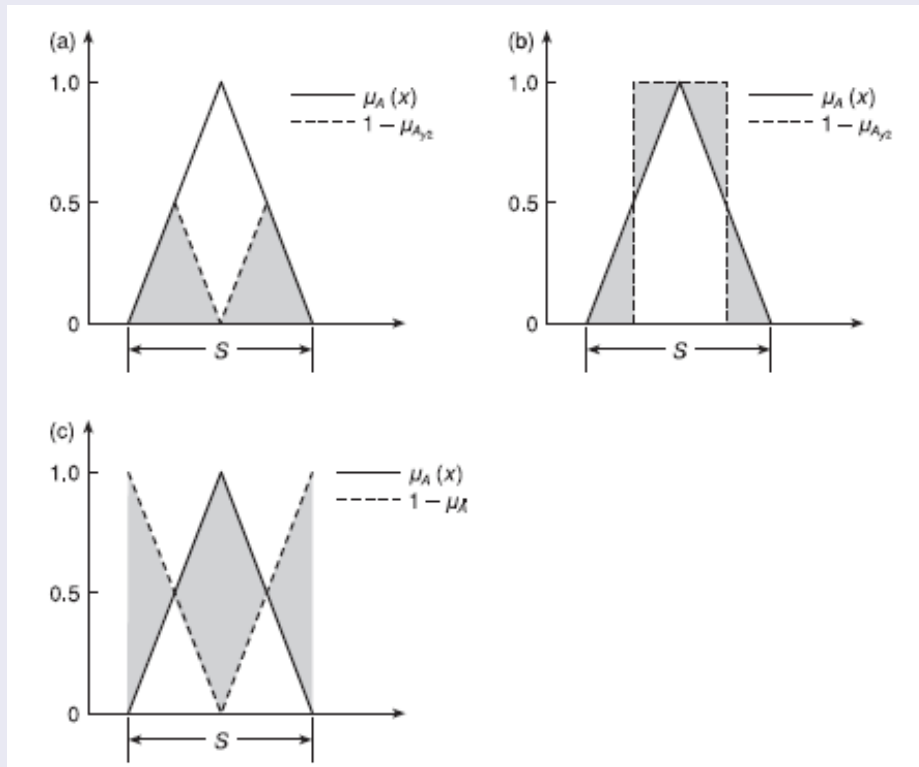
$$\begin{aligned} A_3 &= 2 \int_{x \in X} |\mu_A(x) - 0.5| dx = \int_{x \in X} |2\mu_A(x) - 1| dx \\ &= \int_{x \in X} |\mu_A(x) - (1 - \mu_A(x))| dx = \int_{x \in X} |\mu_A(x) - \mu_{\bar{A}}(x)| dx \end{aligned}$$

## Relationship Between Different Measures of Fuzziness

$$A_1 = A_2 = \frac{1}{2}(S * 1 - A_3)$$

# Illustration of Three Measures of Fuzziness

(a)  $A_1$ , (b)  $A_2$ , (c)  $A_3$



The measure of fuzziness is that total amount of fuzziness in your system. It is represented as an integral of a fuzziness measurement function.

The closeness to 0.5 is super simple, we just integrate across a function for the difference.

The closeness to  $1/2$  cut is found by integrating across the absolute value of the difference between the membership value and its corresponding alpha cut at 0.5

The inverse distance of the complement works out to be just literally the integral of the absolute value of the difference between the membership value and its complement.

We have a relationship between these

# Relations

- Consider the following boolean relation as an example in crisp sets and binary logic.

$$a = x \cdot \bar{y} + z$$

- Then, the relation  $a$  is equivalent to the logical expression (proposition)

“ $x$  AND NOT  $y$  OR  $z$ ”

and is denoted by  $a$ .

- An equivalent expression may be written using three sets and the operations: Complement, Intersection, and Union, but we will not do so here.

## Truth Table of the Relation $a$

$x$	$y$	$z$	$\bar{y}$	$x \cdot \bar{y}$	$a$
0	0	0	1	0	0
0	0	1	1	0	1
0	1	0	0	0	0
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	0	0	1

## Example of Fuzzy Relation

Consider a fuzzy logic proposition  $A$  with membership function  $\mu_A$ .

- Then,  $\mu_A$  may be taken to represent the degree of validity of the relation “ $A$  is True”.
- Similarly, the complement membership function  $\mu_{\bar{A}} = 1 - \mu_A$  represents the degree of validity of the fuzzy relation “ $\bar{A}$  is True”, or equivalently “ $A$  is NOT True”.



## Example of Fuzzy Relation (cont.)

Consider a fuzzy logic proposition “A AND B”.

- If  $\mu_A$  is the membership function of  $A$  and  $\mu_B$  is the membership function of  $B$ , then using the min T-norm to represent AND, the membership function of “A AND B” is given by

$$R(x, y) = \min [\mu_A(x), \mu_B(y)]$$

- This membership function represents the degree of validity of the fuzzy relation “A AND B is True”

# Analytical Representation of a Fuzzy Relation

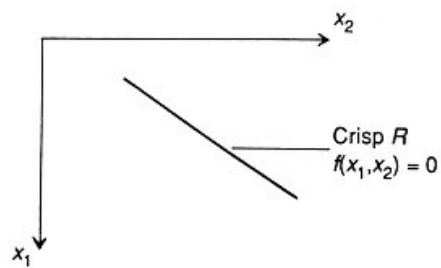
- Any membership function represents a fuzzy relation in the universe (domain, or space) of definition of the particular membership function.
- The space can be one dimensional; say the real line  $\mathbb{R}$  as in the case of  $\mu_R(x)$ , and this gives a 1-D fuzzy relation.
- This idea can be extended to higher dimensions.

## Analytical Representation of a Fuzzy Relation (cont.)

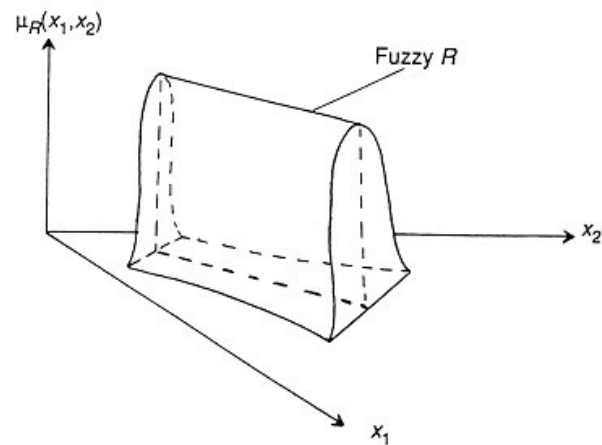
- Consider two universes  $X_1 = \{x_1\}$  and  $X_2 = \{x_2\}$ .
- A crisp set consisting of a subset of ordered pairs  $(x_1, x_2)$  is a crisp relation  $R$  in the two-dimensional Cartesian product space  $X_1 \times X_2$ .
- We may imagine that a truth value of 1 is associated to each of these ordered pairs, giving the characteristic function (special case of membership function) of the crisp relation.

# Relation Representation in Crisp and Fuzzy Cases

(a)



(b)



# Cartesian Products

## Recall for the Crisp Case

- Consider a crisp set  $A_1$  defined on the universe  $X_1$  and a second crisp set  $A_2$  defined on a different (independent, orthogonal) universe  $X_2$ .
- The Cartesian product  $A_1 \times A_2$  is the rectangular area which is a subset of the Cartesian product space  $X_1 \times X_2$  defined on the usual manner, which is the entire 2-D space (plane) containing the two axes  $x_1$  and  $x_2$ .

# Cartesian Product of Fuzzy Sets

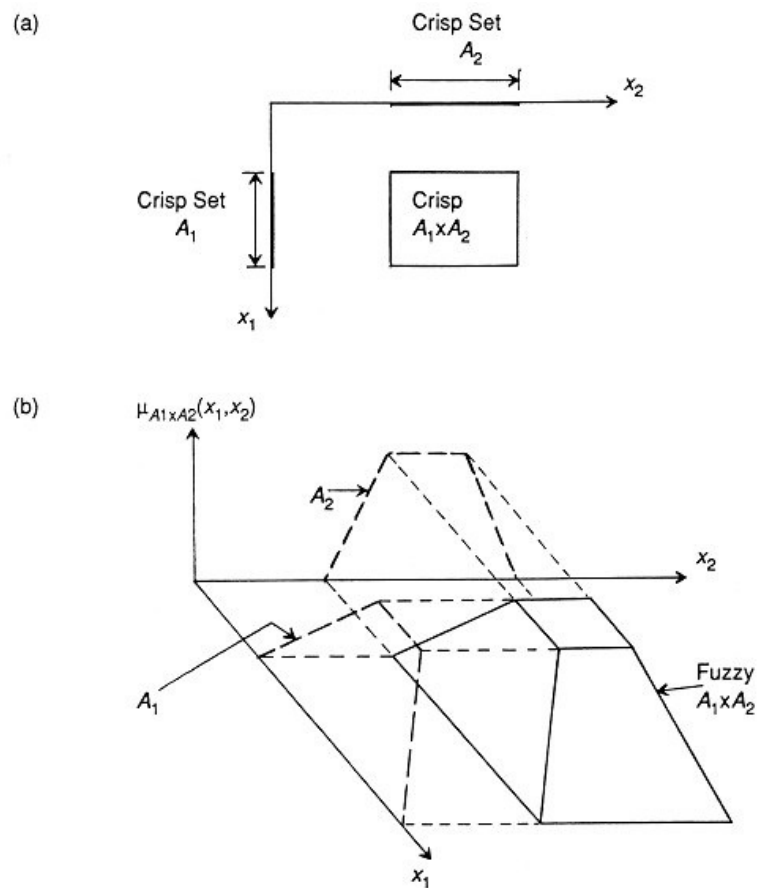
- Consider a fuzzy set  $A_1$  defined on the universe  $X_1$  and a second fuzzy set  $A_2$  defined on a different (independent, orthogonal) universe  $X_2$ .
- The Cartesian product  $A_1 \times A_2$  is the fuzzy subset of the Cartesian space  $X_1 \times X_2$ . Its membership function is given by

$$\mu_{(A_1 \times A_2)}(x_1, x_2) = \min [\mu_{A_1}(x_1), \mu_{A_2}(x_2)] , \quad \forall (x_1, x_2) \in (X_1 \times X_2)$$

## Cartesian Product of Fuzzy Sets (cont.)

- The min combination applies here because, each element  $(x_1, x_2)$ , in the Cartesian product is formed by taking both elements  $x_1$  “and”  $x_2$  together (an “AND” operation), not just one or the other.
- The Cartesian product  $A \times B$  provides the region of definition of the two fuzzy sets  $A$  and  $B$ .
- It is a subset of the Cartesian product of their support sets, which in turn is a subset of the Cartesian product of their universes  $(X \times Y)$ .

# Cartesian Product: Graphical Representation





# Extension Principle

- Consider a mapping function  $f : X \rightarrow Y$
- Let  $A$  be a fuzzy set on  $X$

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \dots + \frac{\mu_A(x_n)}{x_n}$$

- The Extension Principle states that the image of  $A$  under mapping  $f(\cdot)$  is expressed on  $Y$

$$B = f(A) = \frac{\mu_A(x_1)}{y_1} + \frac{\mu_A(x_2)}{y_2} + \dots + \frac{\mu_A(x_n)}{y_n}, \text{ where}$$
$$y_1 = f(x_1), y_2 = f(x_2), \dots, y_n = f(x_n)$$

presuming that the function  $f$  is a one-to-one mapping.

- If  $f$  is a many-to-one mapping, i.e.,  $\exists x_1 \neq x_2$  such that  $y_1 = f(x_1) = f(x_2) = y_2$ , Then  $\mu_B(y) = \max[\mu_A(x_1), \mu_A(x_2)]$ .

important note: remember that we are using different functions. there is no division or addition going on here,

Here we have a discrete fuzzy set A. We apply to it some mapping  $f()$ . The output of this mapping is a discrete fuzzy set B. You can see that the y values of the pairs is not changing, just the x values, thus the height is the same.

Where we have multiple x's for the same y we take the maximum of them.

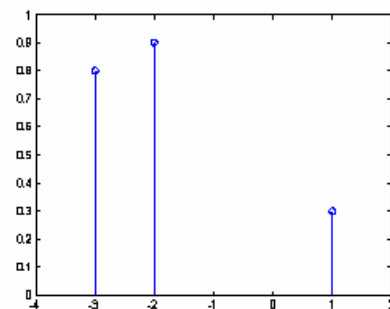
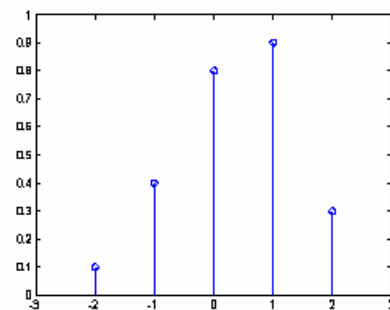
# Example 1

## Discrete Case

$$A = \frac{0.1}{-2} + \frac{0.4}{-1} + \frac{0.8}{0} + \frac{0.9}{1} + \frac{0.3}{2}$$

$$f : x \mapsto x^2 - 3$$

$$\begin{aligned} B &= \frac{0.1}{1} + \frac{0.4}{-2} + \frac{0.8}{-3} + \frac{0.9}{-2} + \frac{0.3}{1} \\ &= \frac{(0.1 \vee 0.3)}{1} + \frac{(0.4 \vee 0.9)}{-2} + \frac{(0.8)}{-3} \\ &= \frac{0.3}{1} + \frac{0.9}{-2} + \frac{0.8}{-3} \end{aligned}$$



## Example 2

### Continuous Case

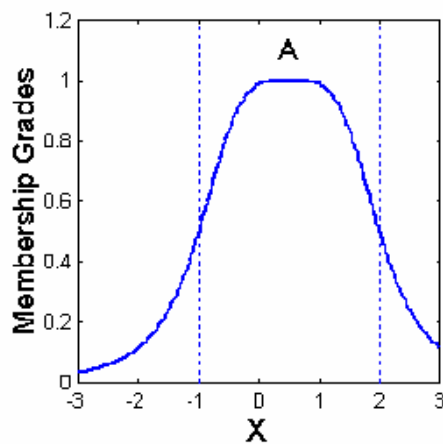
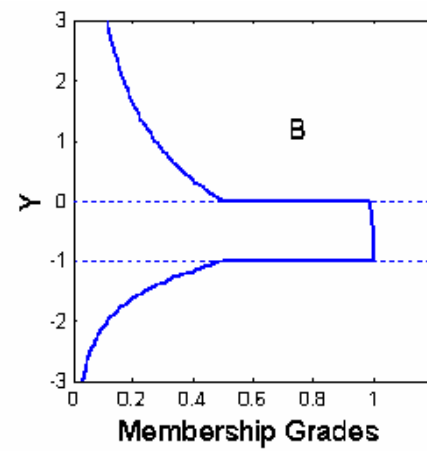
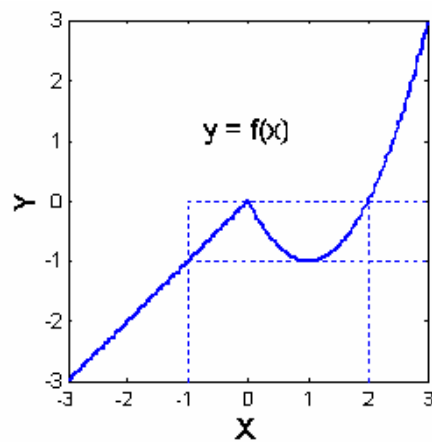
$$\mu_A(x) = \text{gbell}(x|1.5, 2, 0.5)$$

$$f(x) = \begin{cases} (x-1)^2 - 1 & , \text{ if } x \geq 0 \\ x & , \text{ if } x < 0 \end{cases}$$

Define  $B$  and  $\mu_B(y)$ .

Here part of the mapping is bijective and part is not. Weird.

## Example 2: Graphical Solution

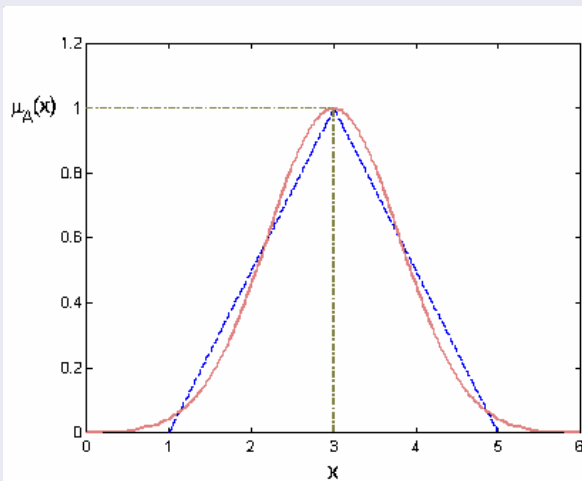


## Example 3

### Continuous Case

Let  $f$  be a crisp mapping  $f : x \mapsto y = \sqrt{x}$ , with  $x$  being a fuzzy number around 3.

$$\mu_A(x) = \begin{cases} (x-1)/2 & , \text{ if } x \leq 3 \text{ monotonic } (f(x)) \\ (5-x)/2 & , \text{ if } 3 \leq x \leq 5 \text{ monotonic } (f(x)) \end{cases}$$



Find out  $\mu_B(y)$ .

## Example 3: Solution

### Solution

$$B = f(A) = \int_y \frac{\mu_A(x)}{f(x)} = \int_y \frac{\mu_A(f^{-1}(y))}{f(x)} = \int_y \frac{\mu_A(f^{-1}(y))}{y}$$

Case 1:  $1 \leq x \leq 3$

$$x = f^{-1}(y) = y^2$$

$$\mu_B(y) = (y^2 - 1)/2$$

$$1 \leq y \leq \sqrt{3}$$

Case 2:  $3 \leq x \leq 5$

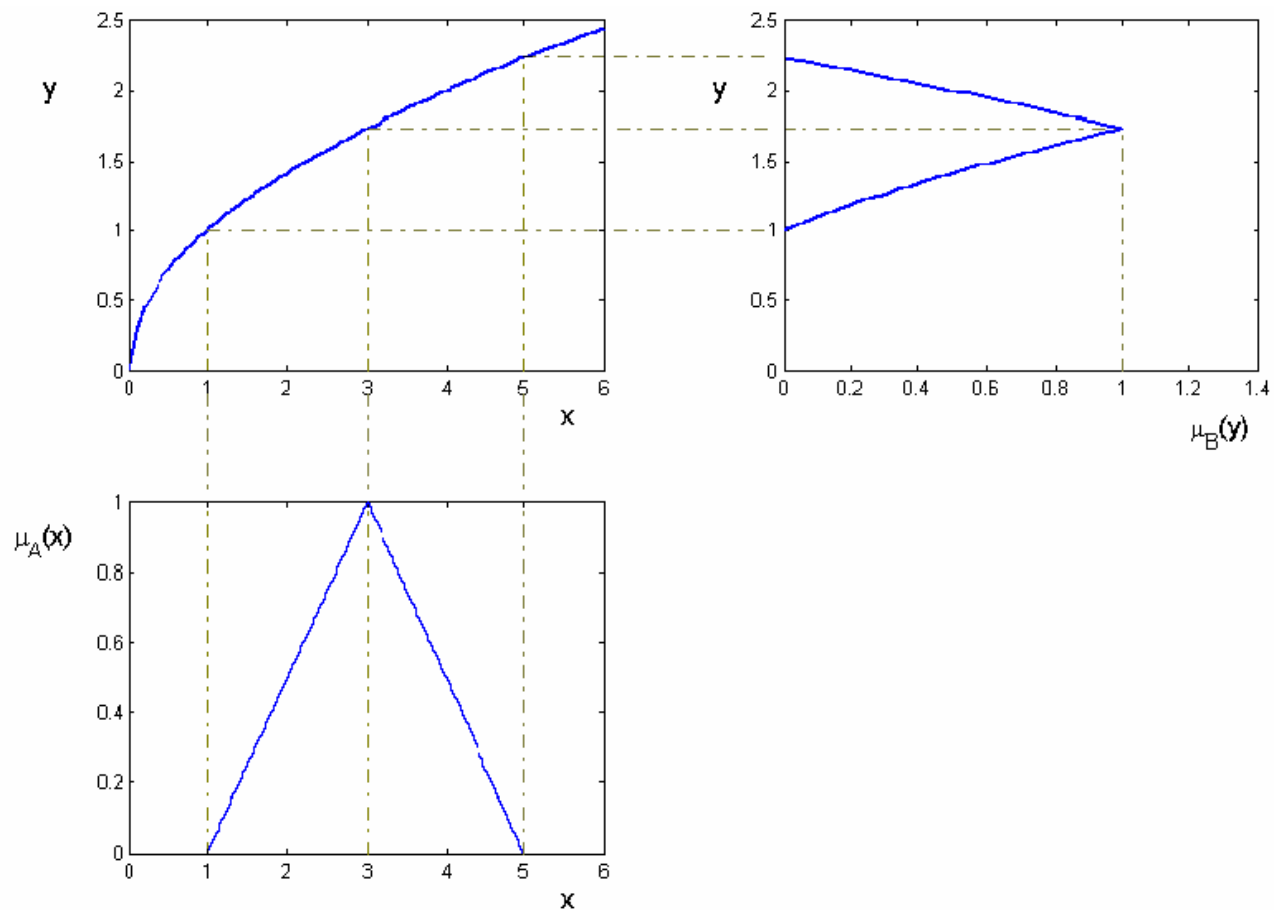
$$\mu_B(y) = (5 - y^2)/2$$

$$\sqrt{3} \leq y \leq \sqrt{5}$$

$$\mu_B(y) = \bigvee_y \mu_A[f^{-1}(y)]$$



## Example 3: Graphical Solution



# Projection

- Given the relation  $R$  defined by  $R = \int_{X \times Y} \frac{\mu_R(x, y)}{(x, y)}$
- The first projection is a fuzzy set that results by eliminating the second set  $Y$  of  $X \times Y$  by projecting the relation on  $X$ .

$$R_1 = \int_X \frac{\mu_{R_1}(x)}{x}, \quad \mu_{R_1}(x) = \bigvee_y \max [\mu_R(x, y)]$$

- The second projection is a fuzzy set that results by eliminating the first set  $X$  of  $X \times Y$  by projecting the relation on  $Y$ .

$$R_2 = \int_Y \frac{\mu_{R_2}(y)}{y}, \quad \mu_{R_2}(y) = \bigvee_x \max [\mu_R(x, y)]$$

- The **total projection** is a combined projection over the spaces  $X$  and  $Y$  and is defined by

$$\mu_{R_T}(x, y) = \bigvee_x \bigvee_y \max [\mu_R(x, y)] = \bigvee_y \bigvee_x \max [\mu_R(x, y)]$$

Given a relation R. We use a projection to eliminate the set Y by projecting it onto X. We go across the whole spectrum of y and project its max value onto x. The next two slides show such projections.

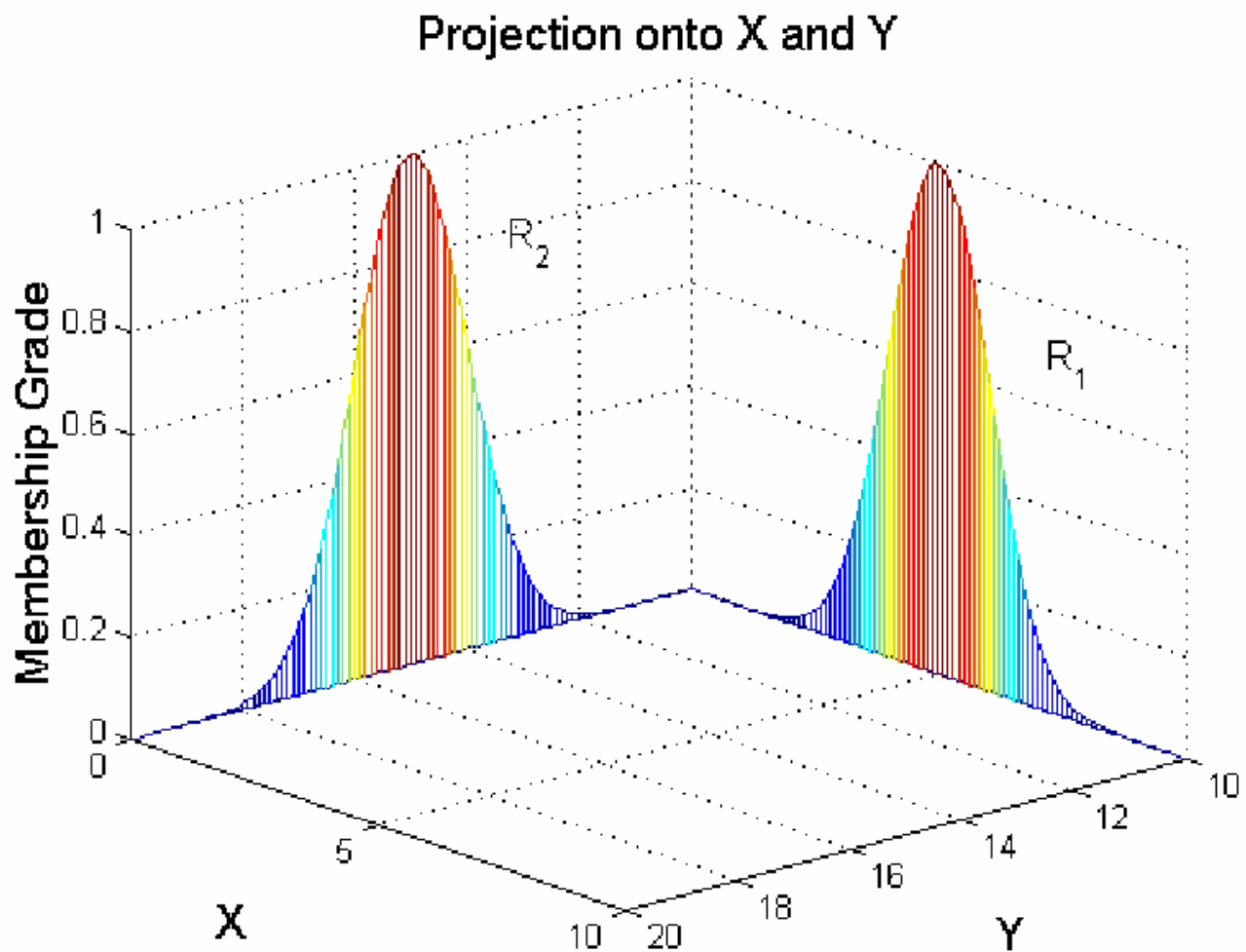
Usually the total projection is a constant.

We can expand this out to the nth dimension. In this case the first projection would be

$$\mu_R(x_1 \dots x_n) = \vee_{x_n} \max[\mu_R(x_1 \dots x_{n-1})] \quad (1)$$

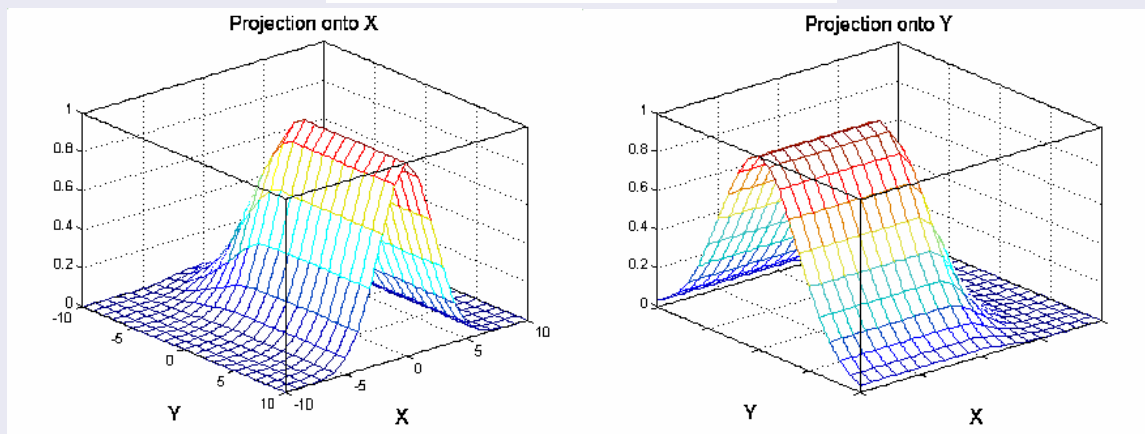
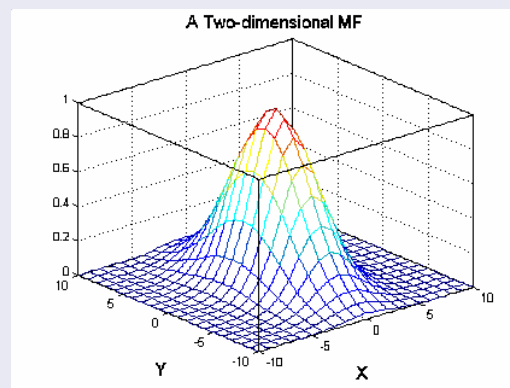
We keep going for the second, third, and so on until you get the total projection.

# First and Second Projections of Relation $R$



# Example 1

## Continuous Case



## Example 2

### Discrete Case

$$R = \begin{matrix} & \begin{matrix} x \\ y \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{bmatrix} 0.1 & 0.2 & 0.4 & 0.8 & 1.0 & 0.6 \\ 0.2 & 0.4 & 0.8 & 0.9 & 0.8 & 0.6 \\ 0.5 & 0.9 & 1.0 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{matrix}$$

$$R^1 = \sum_i \frac{\mu_{R^1}(x_i)}{x_i} = \frac{1.0}{x_1} + \frac{0.9}{x_2} + \frac{1.0}{x_3} = \begin{bmatrix} 1.0 \\ 0.9 \\ 1.0 \end{bmatrix}$$

$$R^2 = \sum_j \frac{\mu_{R^2}(y_j)}{y_j} = \frac{0.5}{y_1} + \frac{0.9}{y_2} + \frac{1.0}{y_3} + \frac{0.9}{y_4} + \frac{1.0}{y_5} + \frac{0.6}{y_6} \\ = \begin{bmatrix} 0.5 & 0.9 & 1.0 & 0.9 & 1.0 & 0.6 \end{bmatrix}^T$$

$$R^T = 1$$

Say we take the value 0.2 at  $y=6$   $x=3$ , this value is like a a three dimensional modal representing a system with two control variable. When we set them to 6 and 3 we got the value 0.2. So the matrix in this example  $R$  represents a three dimensional graph.

So we take the maximum  $y$  value for instances where  $x = 1$ . Basically going across the top row, so in this case 1.0. Repeat for each row. This is where  $R^1$  comes from.

The second projection works the same way but sideways. We just take the max of each column.

These values get projected onto the 3d graph by having  $R^1$  on the  $x$  axis and  $R^2$  on the  $y$  axis.

The total projection we take the max of the values in  $R^1$  or  $R^2$  which in this case is 1.0. This is a single point, one dimensional (well technically 0 dimensional but its easier to think of it as one).

So a projection is a reduction of order in the fuzzy dimensions.

# Cylindrical Extension

$$C(R) = \int_{X_1 \times X_2 \times \dots \times X_n} \frac{\mu_R(X_1, X_2, \dots, X_n)}{(X_1, X_2, \dots, X_n)}$$

## Example: Discrete Case

$$C(R^1) = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$C(R^2) = \begin{bmatrix} 0.5 & 0.9 & 1.0 & 0.9 & 1.0 & 0.6 \\ 0.5 & 0.9 & 1.0 & 0.9 & 1.0 & 0.6 \\ 0.5 & 0.9 & 1.0 & 0.9 & 1.0 & 0.6 \end{bmatrix}$$



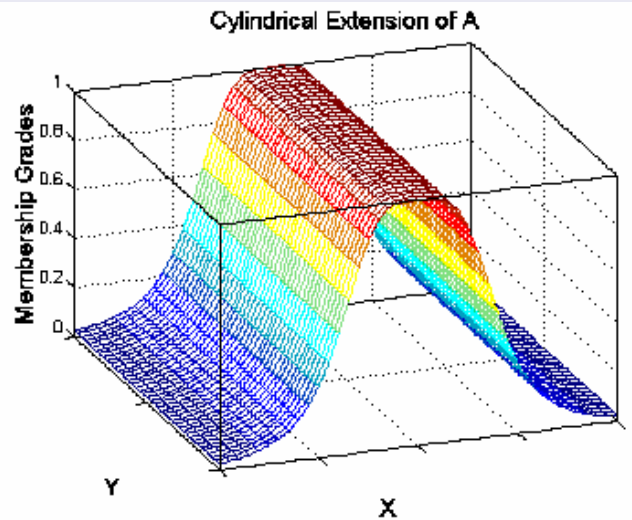
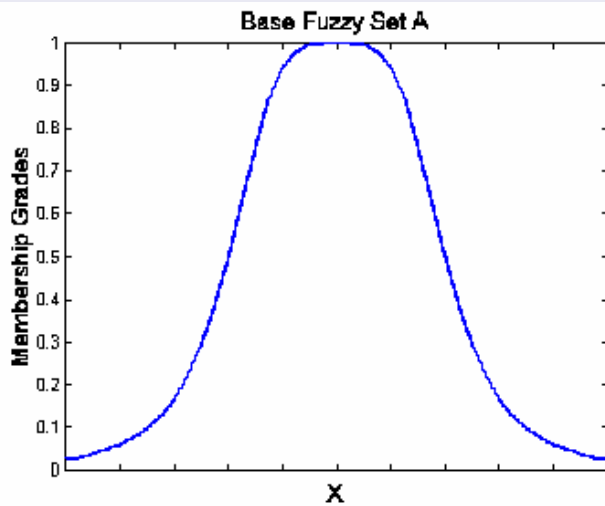
The cylindrical expansion is the opposite of projections, it increases order.

We take the elements from the initial projections. In this example  $R^1$  and  $R^2$  are the same as in the previous example. These are expanded out to match the initial dimensions of the relation by replicating across columns/rows.

The cylindrical expansion does not undo a projection. Once you projection you cannot get back to the initial relation.

# Cylindrical Extension (cont.)

## Example: Continuous Case



# Compositional Rule of Inference

- A typical fuzzy logic system's knowledge base is composed of a series of if-then rules.
- Each rule is a fuzzy relation employing the (if-then) fuzzy implication.
- When all individual rules (relations) are aggregated (executed), they result in one fuzzy relation represented by a fuzzy set, say  $K$ , and a multi-variable membership function.

The knowledge base is just the mathematical representation of a system composed of rules. We create a function or descriptive value for each rule our system gives us. Say we have 3 potential temperatures, 3 pressure, and 3 humidities we will have  $3 \times 3 \times 3 = 27$  rules for our system. Each of these rules are fuzzy relations. If the temperature is high then the air conditioner is on, and so on. This is an implication operator (T-norm). We want a model that encapsulates all potential input types because we don't know what we will get.

# Compositional Rule of Inference (cont.)

## Fuzzy Inference

- 1 In a fuzzy decision making process, the rule base  $K$  is first collectively matched with the available data.
  - 2 Next, an inference is made on another fuzzy variable (consequent part) that is represented in the knowledge base.
- The above process is conducted based on the **compositional rule of inference** (CRI).

## Compositional Rule of Inference (cont.)

Suppose that the available data (context) is denoted by the fuzzy set (or relation)  $D$  and the inference (action) is denoted by a fuzzy set (or relation)  $I$ .

- Then, the compositional rule of inference states that

$$I = D \circ K, \quad \circ : \text{“composed with”}$$

- The membership function of the inference  $I$  (also called decision, or action) is determined as

$$\mu_I = \sup_y \{ \min[\mu_D, \mu_K] \}$$

where the inference  $I$  is the output of the knowledge base decision making system.

- This is known as the **sup-min** (or max-min) composition.
- Another method to compute the inference  $I$  is through the **sup-product** composition.

$$\mu_I = \sup_y \{ \mu_D \cdot \mu_K \}$$

The context is the current set of input data. The system processes the context and decides what action it needs to do. It keeps repeating this cycle.

- Context = D (for data)
- Rules = K (for knowledge)
- Relation = I (for inference)

The membership is of the inference is

$$\mu_I = \sup_y \{\min[\mu_D, \mu_K]\} \quad (2)$$

The super min (or max) is the min value tending to the absolute min.

*Look up what the sup min composition operator is*

Another way to compose thins is the sup produce.

*look this up too*

# Example

## Rule Base

- Fuzzy set  $A$  of universe  $Y$  represents the output of a process.
- Fuzzy set  $C$  of universe  $Z$  represents the control input to the process.

$$A = 0.0/y_1 + 0.2/y_2 + 1.0/y_3 + 0.8/y_4 + 0.1/y_5$$

$$C = 0.1/z_1 + 0.7/z_2 + 1.0/z_3 + 0.4/z_4$$

- A fuzzy relation  $R : A \rightarrow C$  is defined by applying the **min** operation to  $A$  and  $C$ .

$$\mu_R(y_i, z_j) = \begin{matrix} & \begin{matrix} z_j \end{matrix} \\ \begin{matrix} y_i \end{matrix} & \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.7 & 1.0 & 0.4 \\ 0.1 & 0.7 & 0.8 & 0.4 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \end{matrix}$$



## Example (cont.)

### Context

A process measurement  $y_0$  is made and found to be closest to  $y_4$ . This crisp set may be represented by a fuzzy singleton  $A_0$  defined by

$$A_0 = 0.0/y_1 + 0.0/y_2 + 0.0/y_3 + 0.8/y_4 + 0.0/y_5$$

### Inference

The membership function of the corresponding fuzzy control inference  $C'$  is obtained using the compositional rule of inference.

$$C' = A_0 \circ R$$

Define  $C'$ .

### Remark

Note how the matrix  $R$  represents the rule base in the fuzzy decision making.

A process takes input  $C$  and outputs  $A$ . The output  $A$  feeds into a controller that has to make decisions, it takes input  $A$  and outputs  $C$ . We then define a relation  $R = A \implies C$ .

We take some measurements and get a context of  $A_0$ . The controller maps this to  $R$  using a composition rule, doing so it decides what to do.

## Example (cont.)

### Solution: max-min Composition Rule

$$\begin{aligned}
 \mu_{C'}(z_i) &= \max_{\text{rows}} \left( \min_{\text{columns}} \left( \left( \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.8 \\ 0.0 \end{bmatrix}, \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.7 & 1.0 & 0.4 \\ 0.1 & 0.7 & 0.8 & 0.4 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \right) \right) \\
 &= \max_{\text{rows}} \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.7 & 0.8 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \\
 &= [0.1, 0.7, 0.8, 0.4]
 \end{aligned}$$

Hence,  $C' = 0.1/z_1 + 0.7/z_2 + 0.8/z_3 + 0.4/z_4$ .

## Example (cont.)

### Solution: max-product Composition Rule

$$\begin{aligned}\mu_{C'}(z_i) &= \max_{\text{rows}} \left( \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.8 & 0.0 \end{bmatrix} \cdot \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.7 & 1.0 & 0.4 \\ 0.1 & 0.7 & 0.8 & 0.4 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \right) \\ &= \max_{\text{rows}} \begin{bmatrix} 0.08 & 0.56 & 0.64 & 0.32 \end{bmatrix}\end{aligned}$$

Hence,  $C' = 0.08/z_1 + 0.56/z_2 + 0.64/z_3 + 0.32/z_4$ .

# Properties of Composition

- In general, any **sup-T** operator can be used to perform the compositional rule of inference, where T is a T-norm operator.
  - The sup-min and sup-product compositions are special cases of the sup-T composition where the min and product operations are used as T-norms, respectively.
- Let  $P$  and  $R$  be two fuzzy sets represented by two membership functions  $\mu_P(x, z)$  and  $\mu_R(z, y)$ .

## Definition: Sup-T and Inf-S Compositions

The Sup-T composition is characterized by

$$\mu_{P \circ R}(x, y) = \sup_{z \in Z} [\mu_P(x, z) \ T \ \mu_R(z, y)] \quad T : \text{T-norm}$$

The Inf-S composition is characterized by

$$\mu_{P \otimes R}(x, y) = \inf_{z \in Z} [\mu_P(x, z) \ S \ \mu_R(z, y)] \quad S : \text{S-norm}$$

Now we are trying to compose relations, a relation of relations if you will.

# Properties of Composition

**Commutativity:**  $P \circ R = R \circ P$   
 $P \otimes R = R \otimes P$

**Associativity:**  $P \circ (Q \circ R) = (P \circ Q) \circ R$   
 $P \otimes (Q \otimes R) = (P \otimes Q) \otimes R$

**Distributivity:**  $(P \cup Q) \circ R = (P \circ R) \cup (Q \circ R)$

**DeMorgan's Laws:**  $\overline{P \circ R} = \overline{P} \otimes \overline{R}$   
 $\overline{P \otimes R} = \overline{P} \circ \overline{R}$

**Inclusion:** if  $R_1 \subset R_2$ , then  $P \circ R_1 \subset P \circ R_2$

# Case Study



## Problem Statement

- A discrete fuzzy relation  $R(x_i, y_j)$  is given by the following membership function matrix, defined in  $X \times Y$  where  $X = \{0, 1, 2, 3, 4\}$  and  $Y = \{0, 1, 2, 3, 4\}$ .

### Fuzzy Relation $R$

	$y_0 = 0$	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	$y_4 = 4$
$x_0 = 0$	0.0	0.4	0.7	0.3	0.0
$x_1 = 1$	0.1	0.5	0.8	0.4	0.1
$x_2 = 2$	0.6	0.7	1.0	0.5	0.2
$x_3 = 3$	0.3	0.4	0.9	0.7	0.4
$x_4 = 4$	0.0	0.1	0.5	0.3	0.1

# Problem Statement

- The following discrete fuzzy sets are derived from  $R(x_i, y_j)$ :
  - $A(x_i) = \text{Projection}_{x_i} R(x_i, y_j)$
  - $B(y_j) = \text{Projection}_{y_j} R(x_i, y_j)$
  - $A_1(x_i) = R(x_i, 1)$
  - $A_2(x_i) = R(x_i, 2)$

# Problem Statement

## Part (a)

Determine and sketch the membership functions of the following fuzzy sets and fuzzy relations:

### Required Membership Functions

- |                   |  |
|-------------------|--|
| 1. $A(x_j)$       | 7. $\alpha$ -cut of $A_1$ for $\alpha = 0.2, 0.42$ , and $0.5$ |
| 2. $A_1(x_i)$     | 8. $A \longrightarrow B$                                       |
| 3. $A_2(x_i)$     | 9. $A \times B$  |
| 4. $B(y_j)$       | 10. Cylindrical extension of $A$ in $X \times Y$               |
| 5. $A_1 \cup A_2$ | 11. Cylindrical extension of $B$ in $X \times Y$               |
| 6. $A_1 \cap A_2$ |  |

# Problem Statement

## Part (b)

### Crisp Mapping of $R(x_i, y_j)$ in $X \times Y$ to $C(z_z)$ in $Z$

- The fuzzy relation  $R(x_i, y_j)$  in  $X \times Y$  is to be mapped to a fuzzy set  $C(z_k)$  in  $Z$ , using the crisp function  $z = x + y$ .
- Determine and sketch C.

## Solution (a).1

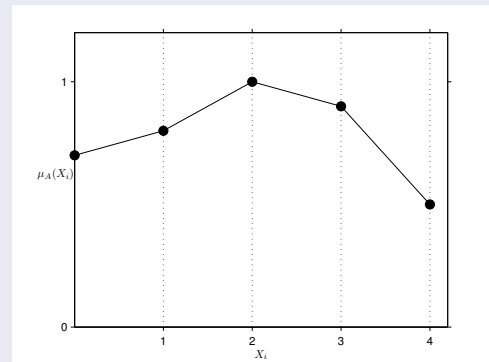
### Membership Function

$$A(x_i) = \text{Projection}_{x_i} R(x_i, y_j)$$

$$\mu_A(x_i) = \sup_{y_j} [\mu_R(x_i, y_j)]$$

$$A(x_i) = \left[ \frac{0.7}{0}, \frac{0.8}{1}, \frac{1.0}{2}, \frac{0.9}{3}, \frac{0.5}{4} \right]$$

### Graphical Representation



Remember the projection onto  $x$  is the max of all  $y$  values for a given  $x$  in the relation (max of rows).

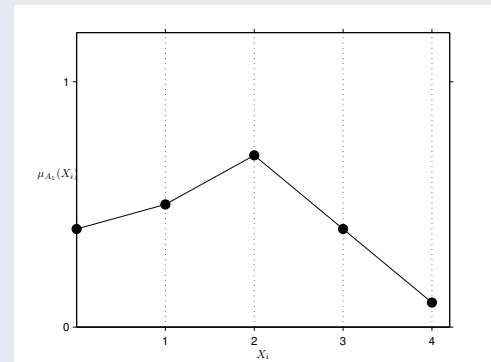
## Solution (a).2

### Membership Function

$$A_1(x_i) = R(x_i, 1)$$

$$A_1(x_i) = \left[ \frac{0.4}{0}, \frac{0.5}{1}, \frac{0.7}{2}, \frac{0.4}{3}, \frac{0.1}{4} \right]$$

### Graphical Representation



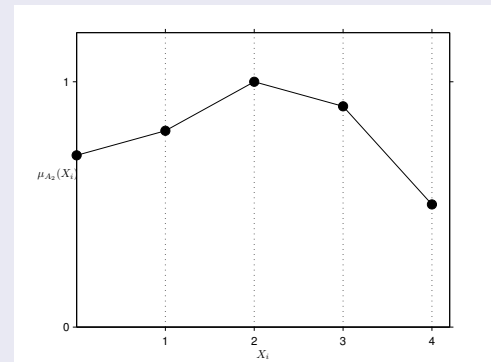
## Solution (a).3

### Membership Function

$$A_2(x_i) = R(x_i, 2)$$

$$A_2(x_i) = \left[ \frac{0.7}{0}, \frac{0.8}{1}, \frac{1.0}{2}, \frac{0.9}{3}, \frac{0.5}{4} \right]$$

### Graphical Representation





## Solution (a).4

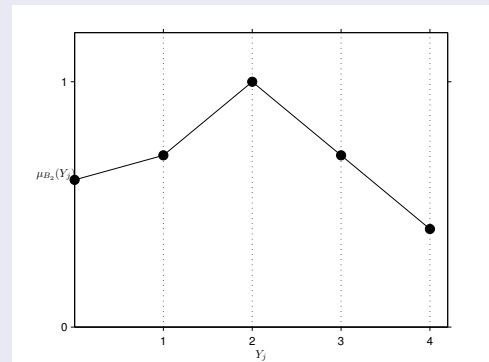
### Membership Function

$$B(y_j) = \text{Projection}_{y_j} R(x_i, y_j)$$

$$\mu_B(y_j) = \sup_{x_i} [\mu_R(x_i, y_j)]$$

$$B(y_j) = \left[ \frac{0.6}{0}, \frac{0.7}{1}, \frac{1.0}{2}, \frac{0.7}{3}, \frac{0.4}{4} \right]$$

### Graphical Representation



## Solution (a).5

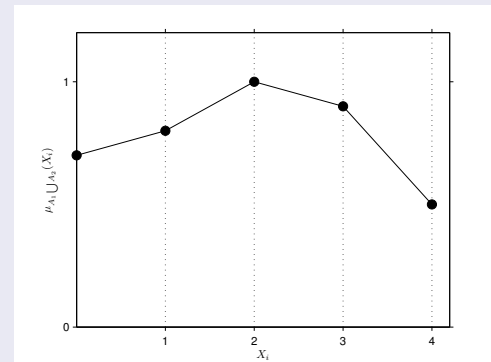
### Membership Function

$$A_1 \cup A_2$$

$$\mu_{A_1 \cup A_2}(x_i) = \max\{\mu_{A_1}(x_i), \mu_{A_2}(x_i)\}$$

$$A_1 \cup A_2 = \left[ \frac{0.7}{0}, \frac{0.8}{1}, \frac{1.0}{2}, \frac{0.9}{3}, \frac{0.5}{4} \right]$$

### Graphical Representation



## Solution (a).6

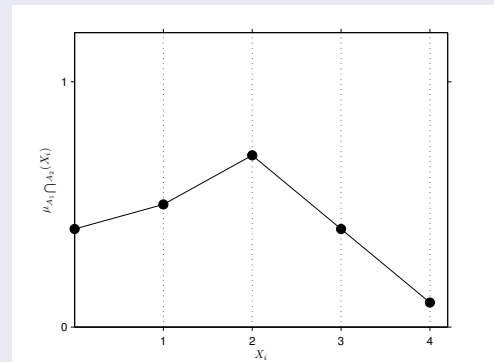
### Membership Function

$$A_1 \cap A_2$$

$$\mu_{A_1 \cap A_2}(x_i) = \min\{\mu_{A_1}(x_i), \mu_{A_2}(x_i)\}$$

$$A_1 \cap A_2 = \left[ \frac{0.4}{0}, \frac{0.5}{1}, \frac{0.7}{2}, \frac{0.4}{3}, \frac{0.1}{4} \right]$$

### Graphical Representation



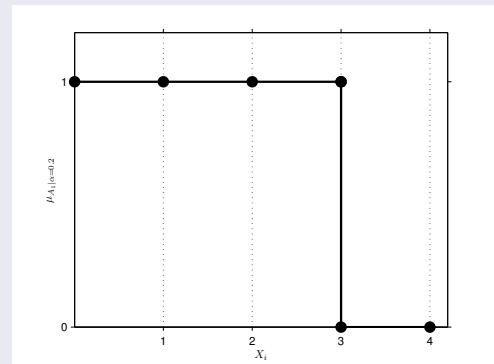
## Solution (a).7 ( $\alpha = 0.2$ )

### Membership Function

$$\begin{aligned}\mu_{F_\alpha}(x) &= 1 \quad \text{for } \mu_F(x) \geq \alpha \\ &= 0 \quad \text{for } \mu_F(x) < \alpha\end{aligned}$$

$$A_{1\alpha|\alpha=0.2} = \left[ \frac{1.0}{0}, \frac{1.0}{1}, \frac{1.0}{2}, \frac{1.0}{3}, \frac{0.0}{4} \right]$$

### Graphical Representation

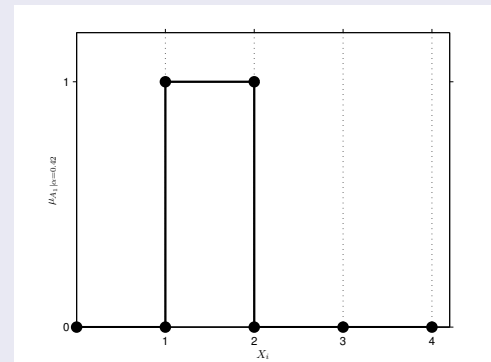


## Solution (a).7 ( $\alpha = 0.42$ )

### Membership Function

$$A_{1\alpha|\alpha=0.42} = \left[ \frac{0.0}{0}, \frac{1.0}{1}, \frac{1.0}{2}, \frac{0.0}{3}, \frac{0.0}{4} \right]$$

### Graphical Representation

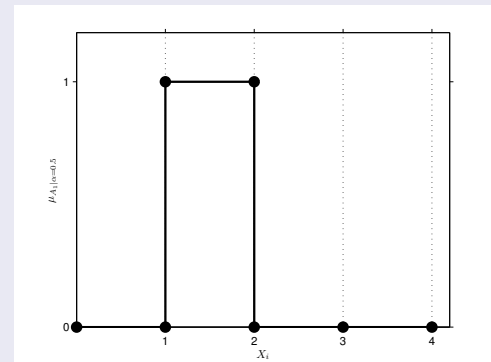


## Solution (a).7 ( $\alpha = 0.5$ )

### Membership Function

$$A_{1\alpha|\alpha=0.5} = \left[ \frac{0.0}{0}, \frac{1.0}{1}, \frac{1.0}{2}, \frac{0.0}{3}, \frac{0.0}{4} \right]$$

### Graphical Representation

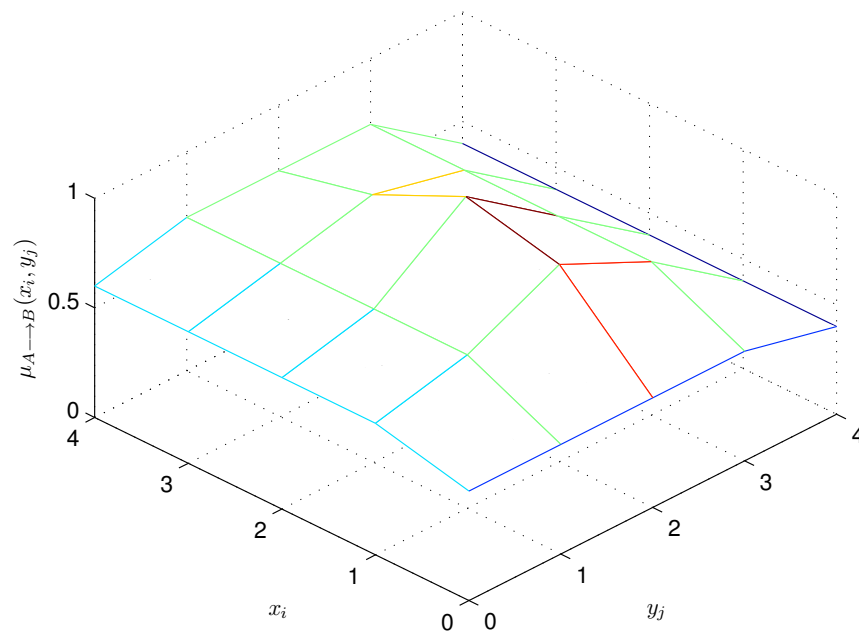


## Solution (a).8 : Membership Function

$$\mu_{A \rightarrow B} = \min\{\mu_A(x_i), \mu_B(y_j)\}$$

$$A \rightarrow B = x_i \begin{matrix} y_j \\ \begin{bmatrix} 0.6 & 0.7 & 0.7 & 0.7 & 0.4 \\ 0.6 & 0.7 & 0.8 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 0.9 & 0.7 & 0.4 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.4 \end{bmatrix} \end{matrix}$$

## Solution (a).8 : Graphical Representation



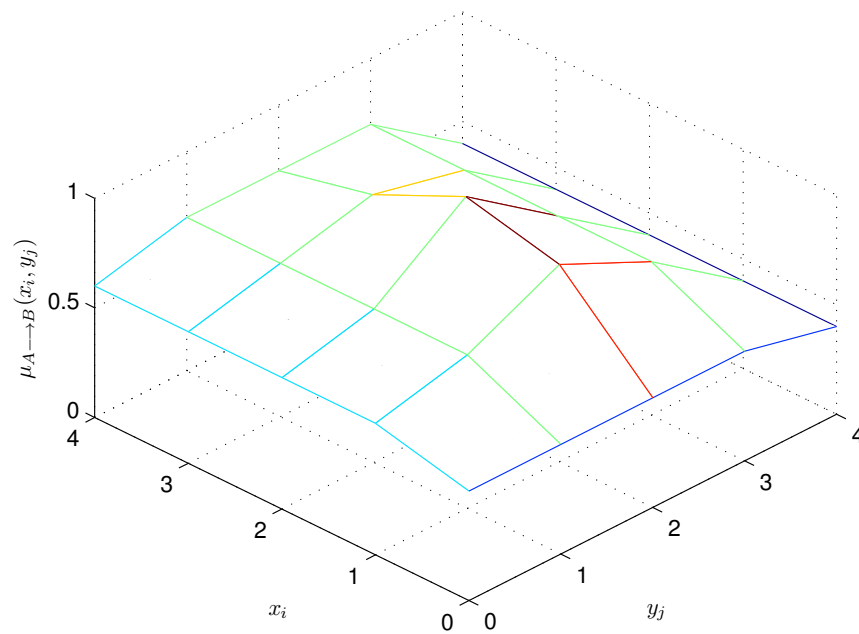


## Solution (a).9 : Membership Function

$$\mu_{A \times B} = \min\{\mu_A(x_i), \mu_B(y_j)\}$$

$$A \times B = \begin{matrix} & y_j \\ x_i & \begin{bmatrix} 0.6 & 0.7 & 0.7 & 0.7 & 0.4 \\ 0.6 & 0.7 & 0.8 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 0.9 & 0.7 & 0.4 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.4 \end{bmatrix} \end{matrix}$$

## Solution (a).9 : Graphical Representation

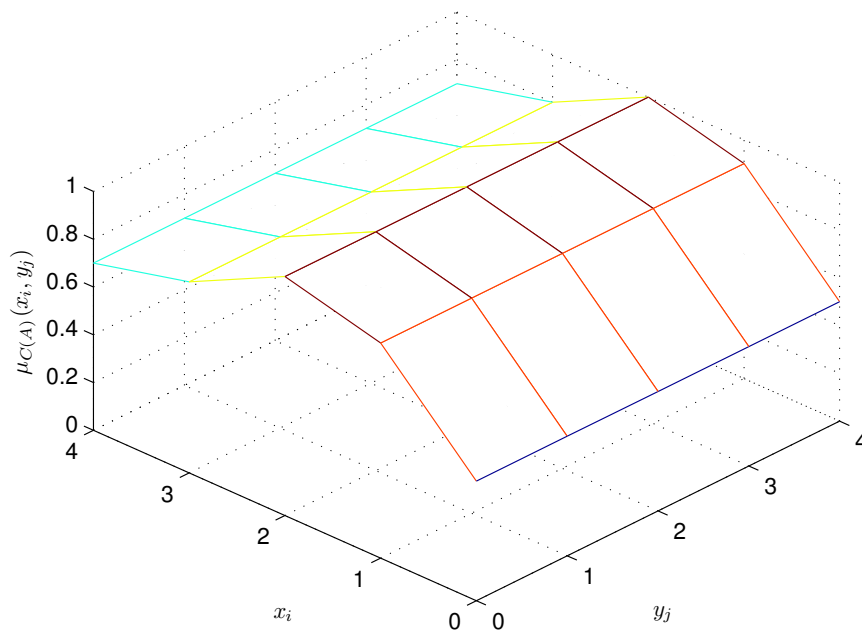


## Solution (a).10 : Membership Function

$$C_{X \times Y}(A) = \sum_{x_i, y_j} \frac{\mu_A(x_i)}{x_i, y_j}$$

$$C_{X \times Y}(A) = x_i \begin{matrix} & y_j \\ \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 0.9 & 0.9 & 0.9 & 0.9 & 0.9 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \end{matrix}$$

## Solution (a).10 : Graphical Representation

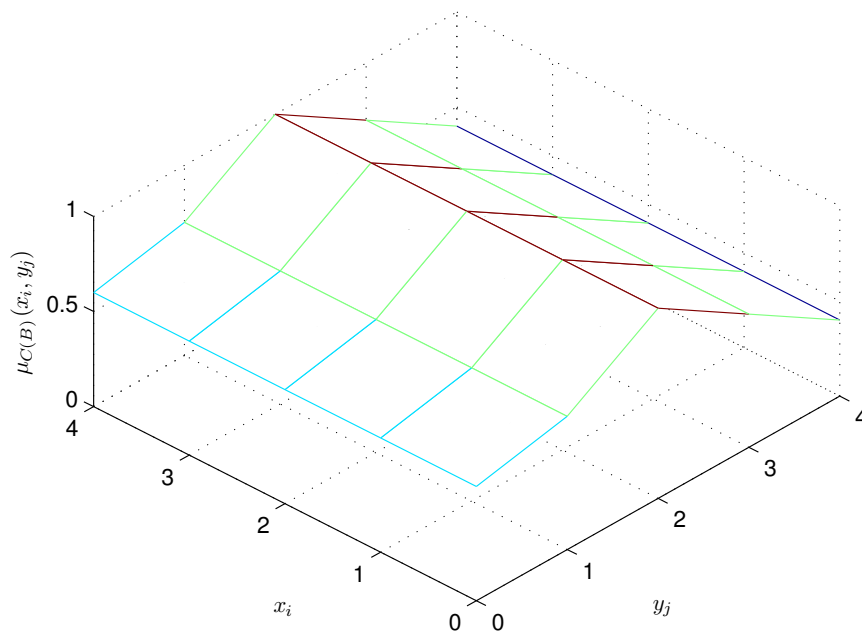


## Solution (a).11 : Membership Function

$$C_{X \times Y}(B) = \sum_{x_i, y_j} \frac{\mu_B(y_j)}{x_i, y_j}$$

$$C_{X \times Y}(B) = x_i \begin{matrix} & y_j \\ \begin{bmatrix} 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \end{bmatrix} \end{matrix}$$

## Solution (a).11 : Graphical Representation



## Solution (b)

- The extension principle is applied, where the crisp relation  $z = x + y$  is used.

$$\mu_C(z_k) = \sup_{z=x+y}(\mu_R(x_i, y_j))$$

- The fuzzy relation  $R(x_i, y_j)$  in the  $X \times Y$  domain is mapped to  $C(z_k)$  in the  $Z$  domain.
- Note that the diagonal broken lines are the  $x + y = z$  Constant lines, for different feasible values of  $z$ .

$$C(z_k) = \left\{ \frac{0.0}{0}, \frac{0.4}{1}, \frac{0.7}{2}, \frac{0.8}{3}, \frac{1.0}{4}, \frac{0.9}{5}, \frac{0.7}{6}, \frac{0.4}{7}, \frac{0.1}{8} \right\}$$

# Solution (b)

## Graphical Representation

