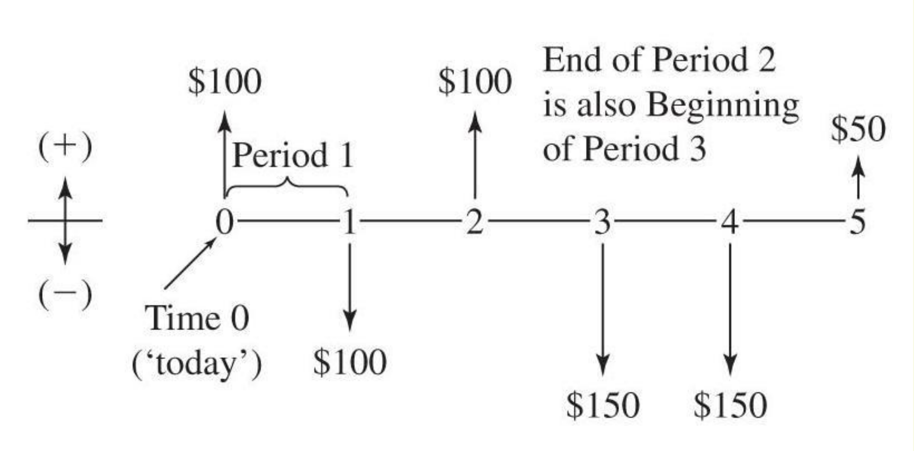
Chapter 3.
Interest and equivalence

Money has value over time. The same $100 has different value depending on when you have it. Money's value over time is expressed by an *interest rate*. This chapter describes two introductory concepts involving the time value of money: *interest* and *cash flow equivalence*.

1. CASH FLOW DIAGRAM

- Cash flow comes at different points in time with different sizes.
  - Investment cost; annual tax; monthly insurance; weekly wage, etc.
- We can summarize it with a *cash flow diagram*.
  - It illustrates the size, sign, and timing of individual cash flows.
  - *Receipts* with a positive value vs. *disbursements* with a negative value
  - We evaluate the value of money at date zero. ‘Today’ is t=0.
  - Most payments are made at the end of the year.
  - End of period t = beginning of period $t + 1$



EXAMPLE 1-1. CASH FLOW DIAGRAM

- A man borrowed $1,000 from a bank. He agreed to repay the loan in two end-of-year payments. At the end of the first year, he pays $580 and at the end of the second year he pays $540. Draw a cash flow diagram.

2. Time value of money

- "Money has different value over time."
- Would you receive $100 today, or $100 a year from now?
  - Of course today! Why?
  - You could invest $100 today (e.g., bank, stock, bond, real estate, etc.) to make more money in a year.
  - You could buy some stuff (e.g., music player, CD, watch, etc.) today with $100, and enjoy them for a year.
  - Who knows what will happen during the next year? The "B" word.
- Now, would you receive $100 today, or $120 a year from now?
  - Well, it depends!
  - Some people may still want to take $100 today.
  - Others may wait and get $120 next year.
  - Preference types?

2.1. Interest

- Suppose you want to borrow \$100 from your friend for a year.
  - Will she lend you \$100 if you pay back \$100 a year from now?
  - Probably not!
  - To borrow \$100, you need to pay *compensation* to her.
- Interest: The *compensation* for giving up the present use of money for the duration of a loan
  - To persuade your friend to give up the use of \$100 for a year, you should give her compensation, *interest*.
  - When you rent an apartment, you pay monthly *rent*. At the end of the contract, you return the apartment to the owner.
  - When you borrow money, you pay *interest* (monthly or yearly). At the end of the contract, you return the *principal* to the lender.
  - This compensation is needed –even- without *inflation*.

2.1. Interest rate

- Interest rate: The *rate* to calculate the size of compensation
  - *Interest period*: The base unit of time over which an interest rate is calculated.
  - The stated interest rate is an annual rate, unless otherwise stated. (e.g., APR of a credit card of 18%; mortgage rate of 4%)
  - An interest rate is considered the *price of (borrowing) money*!
  - Who determines the interest rate? [Bank of Canada – overnight rate]
  - Check the "*prime interest rate*" in any bank's webpage.
- Why does a (positive) interest rate exist?
  - *Time preference*: People prefer goods now to later.
  - Alternative *investment* opportunity
  - *Risk premium* for bankruptcy, default, etc.
  - *Liquidity preference*: cash vs. long-term saving

2.1. Case: APR of a credit card

Terms and Conditions

	Travelocity Rewards MasterCard
Annual Percentage Rate (APR) for Purchases	**12.24%** variable, **15.24%** variable, or **18.24%** variable, depending on our review of your application and credit history.
Other APRs	Balance Transfers and Convenience Checks APR: 0 % for the first nine billing cycles after you open your account.* After that, 12.24% variable, 15.24% variable, or 18.24% variable, depending on our review of your application and credit history. Cash Advance APR: 20.24% variable. Penalty APR: Up to 30.24% variable.†
Variable Rate Information	Your APR may vary. The rate is determined monthly by adding the Prime Rate and: • 8.99%, 11.99%, or 14.99% for Purchases • 8.99%, 11.99%, or 14.99% for Balance Transfers and Convenience Checks after the introductory period • 16.99% for Cash Advances , and **this rate will not be lower than** 19.99% . • Up to 26.99% for the Penalty APR.† See explanation below.**
Grace Period	At least 21-days Grace Period for Purchases from the Statement Closing Date on your periodic statement (provided you pay your previous balance in full by the due date). The Grace Period does not apply to Cash Advances, Balance Transfers or Convenience Checks.
Annual Fee	**$69 for Annual Fee Card** **$0 for No Annual Fee Card**
Minimum Finance Charge	$1.00 / For Residents of Iowa at the time of Account opening $0.50
Method of Computing the Balance for Purchases	Average Daily Balance (including new purchases)
Transaction Charges	• Balance Transfer and Convenience Check Charge: During the first nine billing cycles: 3% of the amount of each transfer or check, $5 minimum, $50 maximum. After the first nine billing cycles: 3% of the amount of each transfer or check with a minimum fee of $5. • Cash Advance Charge: 3% of the amount advanced, $10 minimum, no maximum.

2.1. Case: Canada prime interest rate

2.2. Simple interest

- Simple interest
  - The interest is computed only on the *original sum* (principal), and not on accrued interest.
  - Total interest earned = P (principal) × i (interest rate) × n (years)
- If you lend $1,000 for 4 years at a *simple interest rate* of 10% per year, what is the total amount of interest earned?
  - The total amount of interest is **$400** (= $1,000 × 0.10 × 4 years).
- From now on, however, you can forget the simple interest.
  - No lenders (banks, financial institutions) use this method in calculating the interest owed. Why?
  - It is very rare to observe the use of this method in real life.
  - If the interest rate is low and the number of periods is short, however, this method may be an approximation of a compounding.

2.3. Compound interest

- Compound interest
  - Interest is computed on *accumulated amount* (both on the principal and any unpaid interest).
  - It is the “interest on top of interest.”
  - In most cases, you should assume a compound interest rate (for any loans, investment, or other financial transactions).
- If you lend $1,000 for 4 years at a *compound interest rate* of 10% per year, what is the total amount of interest earned?

EXAMPLE 2-1. PRICE OF THE MANHATTAN ISLAND

- *Peter Minuit* was the director general of the Dutch colony of New Netherland (now New York) during 1626 – 1633.
- He purchased the island of Manhattan from Native Americans on May 24, 1626 for $24 worth of beads.

- Manhattan island was sold for $24 in 1626. At 6% interest rate compounded annually, what is it worth in 2014 (388 years later)?

Example 2-2. Budget buster

- In April 1993, a couple in Nevada, USA, presented the state government with a $1,000 bond issued by the state in 1865. The bond carried an annual interest rate of 24%. The couple claimed the bond was now worth several trillion dollars (*Newsweek*, 1993, "Budget Buster").
- What would be the present worth of the bond in 1993 (127 years later)?
  -
  -
- The Wilsons, the owner of the bond, were willing to settle for a lesser amount, a paltry $54 million…
- A state judge ruled that the bond should have been cashed by 1872. ☹

2.3. Compound interest: Definitional issues

- This loan comes with a "6% interest rate."
  - It is an *annual* interest rate of 6%, because nothing is mentioned.
  - It is compounded *annually*, same as the interest rate period.
  - This is the same as "6% annual interest rate, compounded annually."
- This loan comes with a "6% interest rate, compounded monthly."
  - It is an *annual* interest rate of 6%.
  - However, it is compounded *monthly*, by 0.5% (= 6%/12) each month.
- This loan comes with a "6% monthly interest rate."
  - It is a *monthly* interest rate of 6%.
  - It is compounded *monthly*, same as the interest rate period.
  - It is equivalent to "6% monthly interest rate, compounded monthly."
- If nothing is mentioned, the *compounding period* is the same as the *interest rate period*.

3. EQUIVALENCE

- If you owe some money, how do you pay back the debt?
  - You can pay back in different structures (lump-sum, equal annual payment, interest only till the last period, etc.).
  - If you are indifferent to each type of payment structures, they are called equivalent.
- Equivalence
  - We can convert different cash flows at different times to an equivalent value at a *common reference point*.
  - This concept is very important in finance.
    - If you invest $100 now, how much will you have 5 years later?
    - If I buy a Honda Civic for $20,000, what is the monthly payment?
    - If I save $200 every month, how much will I have 40 years later?
  - Equivalence depends on the given *interest rate*.

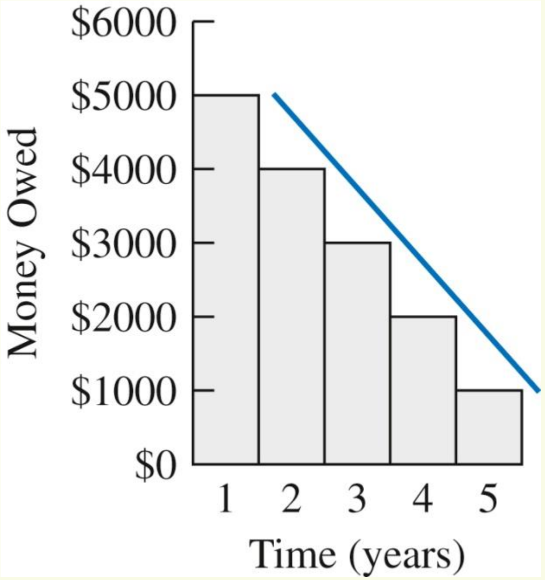
3.1. Equivalence: Repaying a debt

- Suppose you borrow \$5,000 now and have to pay the debt within 5 years, with 8% interest rate. How do you pay?
- We can consider four plans.
  1. Plan 1: Pay (\$1,000 + interest) at the end of each year.
  2. Plan 2: Pay only *interest* each year, and principal at the end of five year.
  3. Plan 3: Pay in five *equal payments* each year.
  4. Plan 4: Pay principal and interest, all at the end of *five year*.
- They all achieve the same purpose of repaying the loan.
  - The total end-of-year payment of each plan will be different.
  - But which plan is the "best"?

3.2. Repaying a debt: Plan 1

- Pay ($1,000 + interest) at the end of each year.
  - Interest at the end of the 1st year: $400 (= $5,000 × 0.08)
  - Interest at the end of the 2nd year: $320 (= $4,000 × 0.08)
  - Sum of end-of-year payment: $6,200

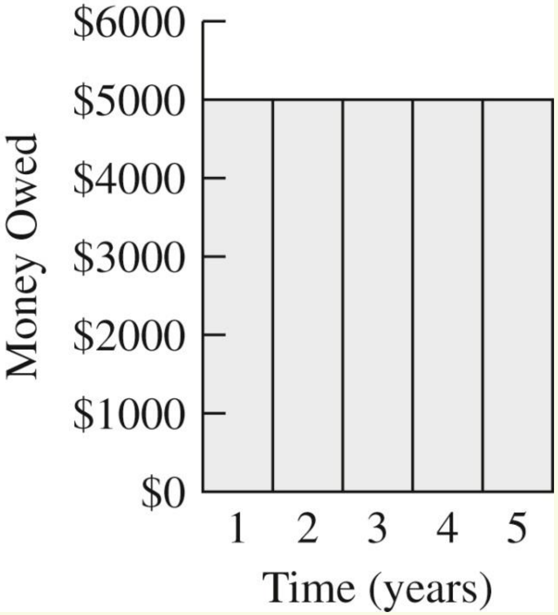
Year	Total at the beginning	Interest	Total at the end of year	Principal payment	Total end-of-year payment
1	5,000	400	5,400	1,000	1,400
2	4,000	320	4,320	1,000	1,320
3	3,000	240	3,240	1,000	1,240
4	2,000	160	2,160	1,000	1,160
5	1,000	80	1,080	1,000	1,080
		1,200		5,000	6,200



3.2. Repaying a debt: Plan 2

- Pay only interest each year, and principal at the end of 5 year.
  - Interest at the end of each year : \$400 (= \$5,000 × 0.08)
  - Payment at the end of five years: \$5,400 (= \$5,000 + \$400)
  - Sum of end-of-year payment: \$7,000

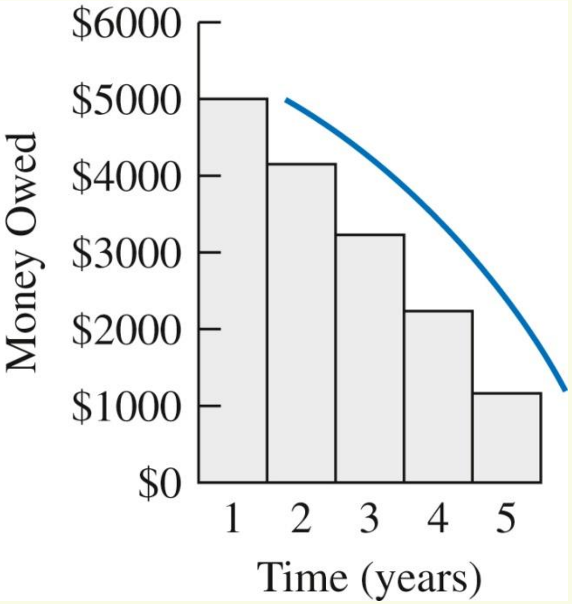
Year	Total at the beginning	Interest	Total at the end of year	Principal payment	Total end-of-year payment
1	5,000	400	5,400	0	400
2	5,000	400	5,400	0	400
3	5,000	400	5,400	0	400
4	5,000	400	5,400	0	400
5	5,000	400	5,400	5,000	5,400
		2,000		5,000	7,000



3.2. Repaying a debt: Plan 3

- Pay in five *equal payments* each year.
  - The way to calculate the annual payment will be covered later.
  - Each year's payment includes some of principal and interest.
  - Sum of end-of-year payment: $6,260

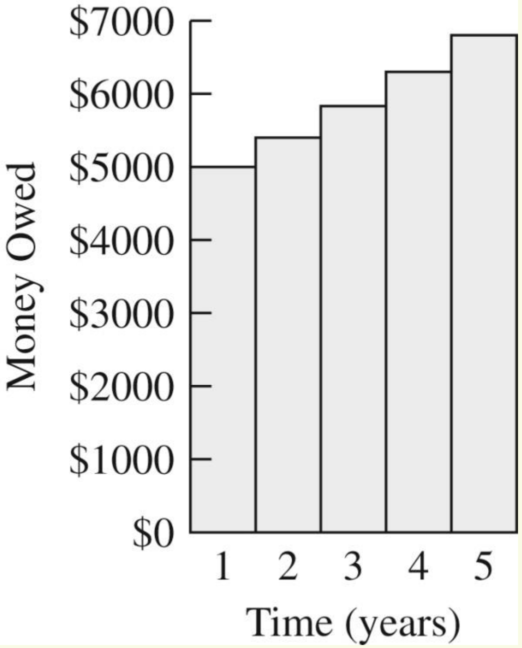
Year	Total at the beginning	Interest	Total at the end of year	Principal payment	Total end-of-year payment
1	5,000	400	5,400	852	1,252
2	4,148	331	4,479	921	1,252
3	3,227	258	3,485	994	1,252
4	2,233	178	2,411	1,074	1,252
5	1,159	93	1,252	1,159	1,252
		1,260		5,000	6,260



3.2. Repaying a debt: Plan 4

- Pay principal and interest, all at the end of five years.
  - Debt at the start of the 2nd year: \$5,400 (= 5,000 + 5,000 × 0.08)
  - Debt at the start of the 3rd year: \$5,832 (= 5,400 + 5,400 × 0.08)
  - Sum of end-of-year payment: \$7,347

Year	Total at the beginning	Interest	Total at the end of year	Principal payment	Total end-of-year payment
1	5,000	400	5,400	0	0
2	5,400	432	5,832	0	0
3	5,832	467	6,299	0	0
4	6,299	504	6,803	0	0
5	6,803	544	7,347	5,000	7,347
		2,347		5,000	7,347



3.3. Equivalence: Comparison of plans

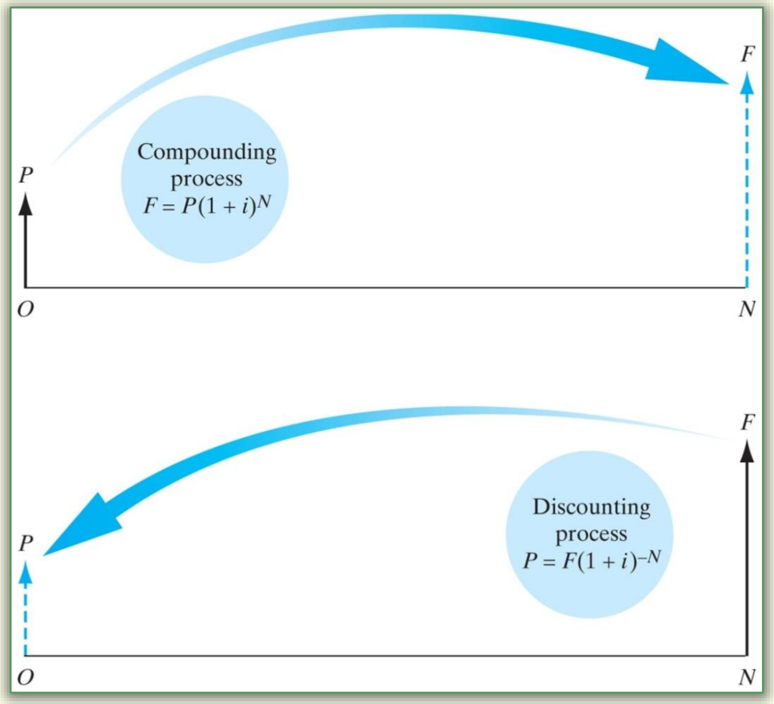
- Which plan is the best?

Plan	Total end-of-year payment	Total interest paid	Dollar-year (area under curve)	Total interest/ dollar-year
1	$6,200	$1,200	$15,000	0.08
2	$7,000	$2,000	$25,000	0.08
3	$6,260	$1,260	$15,767	0.08
4	$7,347	$2,347	$29,334	0.08

- The four repayment plans are equivalent to one another.

4. Single payment compound interest

- We will study a series of interest formulas which convert cash flows to different point of time.
- Compounding process
  - If the present worth (P) is known, what is the future worth (F), given interest rate i and number of periods n?
  - $F = P(1 + i)^n$ or $F = P(F/P, i, n)$
    where $(F/P, i, n) = (1 + i)^n$
- Discounting process
  - If the future worth (F) is known, what is the present worth (P), given i and n?
  - $P = F/(1 + i)^n$ or $P = F(P/F, i, n)$
    where $(P/F, i, n) = 1/(1 + i)^n$



EXAMPLE 4-1. *F* GIVEN *P*

- If \$500 were deposited in a bank now, how much would it be worth 3 years later, if the interest rate is 6%?

Example 4-2. *P* given *F*

- If you wish to have \$800 at the end of 4 years, how much should you put into the savings bank now, given a 5% interest rate?

4.2. How to use the factor table?

- Calculating the *P*/*F* or *F*/*P* factors using the formula can be tedious.
- Instead, we can use the factor table (right).
- To find the *F*/*P* factor for i = 12% and n = 10, we could evaluate the following equation using the factor table.

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor $(F/P,i,N)$	Present Worth Factor $(P/F,i,N)$	Compound Amount Factor $(F/A,i,N)$	Sinking Fund Factor $(A/F,i,N)$	Present Worth Factor $(P/A,i,N)$	Capital Recovery Factor $(A/P,i,N)$	Gradient Uniform Series $(A/G,i,N)$	Gradient Present Worth $(P/G,i,N)$	N
1	1.1200	0.8929	1.0000	1.0000	0.8929	1.1200	0.0000	0.0000	1
2	1.2544	0.7972	2.1200	0.4717	1.6901	0.5917	0.4717	0.7972	2
3	1.4049	0.7118	3.3744	0.2963	2.4018	0.4163	0.9246	2.2208	3
4	1.5735	0.6355	4.7793	0.2092	3.0373	0.3292	1.3589	4.1273	4
5	1.7623	0.5674	6.3528	0.1574	3.6048	0.2774	1.7746	6.3970	5
6	1.9738	0.5066	8.1152	0.1232	4.1114	0.2432	2.1720	8.9302	6
7	2.2107	0.4523	10.0890	0.0991	4.5638	0.2191	2.5515	11.6443	7
8	2.4760	0.4039	12.2997	0.0813	4.9676	0.2013	2.9131	14.4714	8
9	2.7731	0.3606	14.7757	0.0677	5.3282	0.1877	3.2574	17.3563	9
10	$\boxed{3.1058}$	0.3220	17.5487	0.0570	5.6502	0.1770	3.5847	20.2541	10

$$F = \$20{,}000\underbrace{(1 + 0.12)^{10}}_{3.1058} = \$62{,}116$$

Example 4-3. Compounding period

- If you deposit $500 and the bank pays "6% interest rate, *compounded quarterly*," how much would you have at the end of three years?

Example 4-4. Uneven payment

- Suppose you borrowed $\$P$ now and will pay it in two payments: \$400 at the end of three years and \$600 at the end of five years. If the interest rate is 12%, what is the value of P?

Example 4-5. Picasso painting

- In 1995, someone purchased a painting by Picasso, *Angel Fernandez de Soto*, for $29 million. The painting was done in 1903 and was valued then at $600. What *rate of return* did the owner receive on the $600 investment?

Example 4-6. Berkshire Hathaway stock price

- Warren Buffett’s Berkshire Hathaway Company went public in 1965. The public offering price was \$18 per share. The stock was trading at \$172,220 on December 2, 2013. What is the firm’s annual rate of return over the last 48 years?

5. Nominal and effective interest

- Nominal interest rate (r)
  - Annual interest rate without considering the effect of any compounding during the year.
  - If the bank pays 3% interest every *quarter*, the nominal interest rate per year is r = 12% (= 3% × 4 quarters).
  - *Annual percentage rate* (APR) for any loan
- Effective interest rate (i_e)
  - Annual interest rate taking into account the effect of any *compounding* during the year.
  - If the bank pays 3% interest every *quarter*, the effective interest rate per year is i_e = 12.55%.
  - *Annual percentage yield* (APY): The effective annual interest rate earned on savings account, certificates of deposit, bonds, etc.

5.1. Calculation of effective interest rate

- Notation
  - r: Nominal interest rate per year
  - i: Interest rate per compounding period ($i = r/m$)
  - i_e: Effective interest rate per year
  - m: Number of compounding periods per year
- If a \$1 deposit is compounded m times per year with a nominal interest rate r, the interest rate per compounding period is r/m.
  - Nominal interest rate r is often given for a one-year period.
- The effective interest rate (per year) is

$$i_e = (1 + r/m)^m - 1 = (1 + i)^m - 1$$

- When a nominal interest rate is compounded annually, the nominal interest rate equals the effective interest rate.

Example 5-1. Effective interest rate

- A credit card company charges a nominal rate of 24% on overdue accounts, compounded *daily*. What is the effective interest rate?

Example 5-2. Credit card rates

- Three credit card companies charge different interests on an overdue account.
  - The Vic Visa card charges 20% compounded *daily*.
  - The Mag Master card charges 21% compounded *semiannually*.
  - The Am Ex card charges 22% compounded *quarterly*.
- Which card has the best deal?

EXAMPLE 5-3. PAYDAY LOAN RATE (1)

- What is a "payday" loan?
  - A small, short term loan until the borrower's next payday.
  - People usually don't (or can't) borrow a few hundred dollars from a bank.
  - Typical loan amount is less than $500, due in two weeks.
  - Most people who use payday loan service receive checks bi-weekly.
  - The fee is usually $15 – 20 for each $100: that is, 15% - 20% of interest over two weeks.
  - http://www.parl.gc.ca/content/lop/researchpublications/prb0581-e.html

Example 5-3. Payday loan rate (2)

- Suppose Sam writes a check of $550 to Money Mart to borrow $500 for two weeks. What is the nominal annual interest rate?

- What is the effective (annual) interest rate of this loan?

- If the fee is $100 for two weeks (which is more common), what is the effective interest rate?

Example 5-4. Nominal vs. effect interest rate

Nominal Rate	Semi-Annual	Quarterly	Monthly	Daily	Continuous
1%	1.00%	1.00%	1.01%	1.01%	1.01%
5%	5.06%	5.10%	5.12%	5.13%	5.13%
10%	10.25%	10.38%	10.47%	10.52%	10.52%
15%	15.56%	15.87%	16.08%	16.18%	16.18%
20%	21.00%	21.55%	21.94%	22.13%	22.14%
30%	32.25%	33.55%	34.49%	34.97%	34.99%
40%	44.00%	46.41%	48.21%	49.15%	49.18%
50%	56.25%	60.18%	63.21%	64.82%	64.87%

5.2. Continuous compounding

- Continuous compounding

$$i_e = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m - 1 = e^r - 1$$

  - This is a theoretical concept, indicating the most (possible) frequent compounding.
  - In some financial markets (stock, mutual funds, etc.), the transaction occurs several times a day. Nearly continuous!

- If a new investment returns a nominal interest of 30%, compounded *continuously*, what is the effective interest rate?

Summary of Chapter 3

- Time value of money
  - Definition of interest and interest rate
  - Compound vs. simple interest
- Equivalence of cash flows
- Relation between P and F
  - Compounding process: $F = P(1 + i)^n$
  - Discounting process: $P = F/(1 + i)^n$
- Effective interest rate and continuous compounding

- End-of-chapter problems you should try…
  - 16, 17, 19, 21, 29, 31, 33, 37, 41, 45, 49