Math 239 Spring 2014 Assignment 1 Solutions

1. {4 marks} Let $n \geq 2$ be an integer. The n children of the Von Trapp family all have different ages. Whenever they sing, they will all line up so that the oldest child is to the right of the youngest child (it does not have to be directly to the right, for example, the oldest child could be 3 to the right of the youngest child). How many ways can they line up? Use a bijection to prove your answer.

   (For notation, you may label the n children in order of age from the youngest as $1, 2, \ldots, n$. Represent one possible line up as a permutation $\sigma : [n] \to [n]$ where $\sigma(i)$ represents the position of the i-th youngest child in the line up, counting from the left.)

   Solution. We could formulate the problem as the set S of all permutations σ of $[n]$ where $\sigma(1) < \sigma(n)$. Let T be the set of all permutations where $\sigma(1) > \sigma(n)$. We define the function $f : S \to T$ where for each $\sigma \in S$, $f(\sigma) = \sigma'$ where

   $$\sigma'(i) = \begin{cases} \sigma(n) & i = 1 \\ \sigma(1) & i = n \\ \sigma(i) & \text{otherwise} \end{cases}$$

   In other words, we are switching $\sigma(1)$ and $\sigma(n)$ to produce σ'. Notice that since $\sigma(1) < \sigma(n)$, we must have $\sigma'(1) > \sigma'(n)$, so $f(\sigma) \in T$. Also, the mapping $f^{-1} : T \to S$ with the same operation as f is the inverse of f. So f is a bijection. This implies that $|S| = |T|$. Since $S \cup T$ is the set of all permutations of $[n]$ and $S \cap T = \emptyset$, $|S| + |T| = n!$, hence $|S| = n!/2$.

2. {3 marks} Let $n \geq 3$ be an integer. How many subsets of $[2n]$ have exactly 3 odd integers and any number of even integers? Describe two sets S and T such that the answer to our question is $|S \times T|$, then determine what is this answer.

   Solution. Let S be the set of all subsets of $\{1, 3, \ldots, 2n-1\}$ of size 3. Let T be the set of all subsets of $\{2, 4, \ldots, 2n\}$. For any set X that is a subset of $[2n]$ with exactly 3 odd integers, $X = A \cup B$ where $A \in S$ and $B \in T$. So the total number of subsets X that we want is exactly $|S \times T|$. Since $|S| = \binom{n}{3}$ and $|T| = 2^n$, the total number is $\binom{n}{3}2^n$.

3. {4 marks} Let a, b, c, n be positive integers such that $a \leq b \leq c \leq n$. Consider the following set.

   $$S = \{(A, B, C) \mid A \subseteq B \subseteq C \subseteq [n], |A| = a, |B| = b, |C| = c\}.$$

   By counting S in two different ways, prove that

   $$\binom{n}{a}\binom{n-a}{b-a}\binom{n-b}{c-b} = \binom{n}{c}\binom{c}{b}\binom{b}{a}.$$

   Solution. We will count the set of all triples in S in two ways.

   In the first method, we will choose A first. Since A is an a-subset of $[n]$, there are $\binom{n}{a}$ ways to pick A. After we have picked A, we note that B must include A, so out of the b elements of B, a of them are chosen. So we need to choose $b - a$ elements out of $[n] \setminus A$, which has size $n - a$. So for every A, there are $\binom{n-a}{b-a}$ possible B. After we have picked A, B, we note that C must include B, so out of the c elements of C, b of them are chosen. So we need to choose $c - b$ elements out of $[n] \setminus B$, which has size $n - b$. So for every choice of A and B, there are $\binom{n-b}{c-b}$ ways to choose C. In total, $|S| = \binom{n}{a}\binom{n-a}{b-a}\binom{n-b}{c-b}$.

   In the second method, we will choose C first. Since C is a c-subset of $[n]$, there are $\binom{n}{c}$ ways to pick C. After we have picked C, we see that B is a b-subset of C, so there are $\binom{c}{b}$ ways to choose B. After we have picked B, we see that A is an a-subset of B, so there are $\binom{b}{a}$ ways to choose A. In total, $|S| = \binom{n}{c}\binom{c}{b}\binom{b}{a}$, hence the identity holds.

4. Consider the following identity.

   $$3^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}.$$

(a) {4 marks} Give a combinatorial proof of this identity.

Solution. Consider the set $S = \{1, 2, 3\}^n$, which consist of all n-tuples $(a_1, \ldots, a_n)$ where each $a_i \in \{1, 2, 3\}$. Clearly $|S| = 3^n$.

We partition S into $n+1$ sets $S_0, \ldots, S_n$ where for each $i = 0, \ldots, n$, S_i is the set of elements of S that contains exactly i 1's. We can count S_i by first deciding which i of the n spots are 1's, then fill in the remaining $n - i$ spots with either 2 or 3. There are $\binom{n}{i}$ ways to choose the i spots, and 2^{n-i} ways to fill in the remaining spots. So $|S_i| = \binom{n}{i} 2^{n-i}$.

Since $S = S_0 \cup \cdots \cup S_n$ is a disjoint union,

$$3^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}.$$

(b) {3 marks} Give an algebraic proof of this identity. (You may assume the binomial theorem.)

Solution. By the binomial theorem,

$$(1+2x)^n = \sum_{j=0}^{n} \binom{n}{j} (2x)^j = \sum_{j=0}^{n} \binom{n}{n-j} (2x)^j.$$

Putting $i = n - j$, we get

$$(1+2x)^n = \sum_{i=0}^{n} \binom{n}{i} (2x)^{n-i}.$$

Substitute $x = 1$ to get the identity.

5. Let S be the set of all subsets of $[3]$.

(a) {2 marks} Let w be the weight function on S such that $w(\emptyset) = 0$, and for any nonempty set $A \in S$, $w(A)$ is the sum of all elements of A. Determine the generating series $\Phi_S(x)$ with respect to w.

Solution. We can check the weight of each subset of $[3]$:

$$w(\emptyset) = 0, \quad w(\{1\}) = 1, \quad w(\{2\}) = 2, \quad w(\{3\}) = 3,$$
$$w(\{1,2\}) = 3, \quad w(\{1,3\}) = 4, \quad w(\{2,3\}) = 5, \quad w(\{1,2,3\}) = 6.$$

Using these weights, we see that

$$\Phi_S(x) = 1 + x + x^2 + 2x^3 + x^4 + x^5 + x^6.$$

(b) {2 marks} Let w^* be the weight function on S such that for any $A \in S$, $w^*(A) = 3w(A)$. Determine the generating series $\Phi_S^*(x)$ with respect to w^*.

Solution. The weights from part (a) are all tripled, so the new generating series is

$$\Phi_S^*(x) = 1 + x^3 + x^6 + 2x^9 + x^{12} + x^{15} + x^{18}.$$

(c) {2 marks} In general, let T be a set and let w be a weight function on T. Let $\Phi_T(x)$ be the generating series with respect to w. For a positive integer k, define w^* to be the weight function on T where for any $a \in T$, $w^*(a) = k \cdot w(a)$. Let $\Phi_T^*(x)$ be the generating series with respect to w^*. Use the definition of generating series to determine a relationship between $\Phi_T(x)$ and $\Phi_T^*(x)$.

Solution. Using the definition of generating functions, we have

$$\begin{aligned}
\Phi_T^*(x) &= \sum_{\sigma \in T} x^{w^*(\sigma)} \\
&= \sum_{\sigma \in S} x^{k \cdot w(\sigma)} \\
&= \sum_{\sigma \in S} (x^k)^{w(\sigma)} \\
&= \Phi_S(x^k).
\end{aligned}$$