Module 5: Hashing

CS 240 - Data Structures and Data Management

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Lower bound for search

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Theorem: In the comparison model (on the keys), $\Omega(\log n)$ comparisons are required to search a size-n dictionary.

Proof: Similar to lower bound for sorting.

Any algorithm defines a binary decision tree with comparisons at the nodes and actions at the leaves.

There are at least $n + 1$ different actions (return an item, or "not found").

So there are $\Omega(n)$ leaves, and therefore the height is $\Omega(\log n)$.

□

Direct Addressing

Requirement: For a given $M \in \mathbb{N}$,
every key k is an integer with $0 \leq k < M$.

Data structure : An array of *values* A with size M

search(k) : Check whether $A[k]$ is empty

insert(k, v) : $A[k] \leftarrow v$

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Total storage is $\Theta(M)$.

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What sorting algorithm does this remind you of? *Counting Sort*

Hashing

Direct addressing isn't possible if keys are not integers.
And the storage is very wasteful if $n \ll M$.

Say keys come from some *universe* U.
Use a *hash function* $h : U \to \{0, 1, \ldots, M-1\}$.
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Generally, h is not injective, so many keys can map to the same integer.

Hash table Dictionary: Array T of size M (the *hash table*).
An item with key k is stored in $T[h(k)]$.
search, *insert*, and *delete* should all cost $O(1)$.

Challenges:

- Choosing a good hash function
- Dealing with *collisions* (when $h(k_1) = h(k_2)$)

Choosing a good hash function

Uniform Hashing Assumption: Each hash function value is equally likely.

Proving is usually impossible, as it requires knowledge of the input distribution and the hash function distribution.

We can get good performance by following a few rules.

A good hash function should:

- be very efficient to compute
- be unrelated to any possible patterns in the data
- depend on all parts of the key

Basic hash functions

If all keys are integers (or can be mapped to integers), the following two approaches tend to work well:

Division method: $h(k) = k \bmod M$.
We should choose M to be a prime not close to a power of 2.

Multiplication method: $h(k) = \lfloor M(kA - \lfloor kA \rfloor) \rfloor$,
for some constant floating-point number A with $0 < A < 1$.
Knuth suggests $A = \varphi = \frac{\sqrt{5}-1}{2} \approx 0.618$.

Collision Resolution

Even the best hash function may have *collisions*:
when we want to insert (k, v) into the table,
but $T[h(k)]$ is already occupied.

Two basic strategies:

- Allow multiple items at each table location (buckets)
- Allow each item to go into multiple locations (open addressing)

We will examine the average cost of *search*, *insert*, *delete*,
in terms of n, M, and/or the *load factor* $\alpha = n/M$.

We probably want to rebuild the whole hash table and change
the value of M when the load factor gets too large or too small.
This is called *rehashing*, and should cost roughly $\Theta(M + n)$.

Chaining

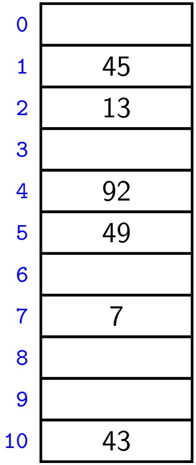
Each table entry is a *bucket* containing 0 or more KVPs.
This could be implemented by any dictionary (even another hash table!).

The simplest approach is to use an unsorted linked list in each bucket.
This is called collision resolution by *chaining*.

- *search*(k): Look for key k in the list at $T[h(k)]$.
- *insert*(k, v): Add (k, v) to the front of the list at $T[h(k)]$.
- *delete*(k): Perform a search, then delete from the linked list.

Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

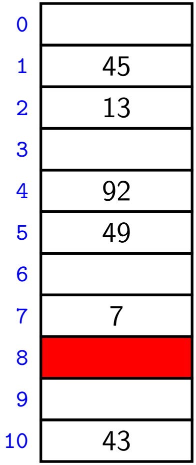


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(41)

$h(41) = 8$



Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(41)

$h(41) = 8$

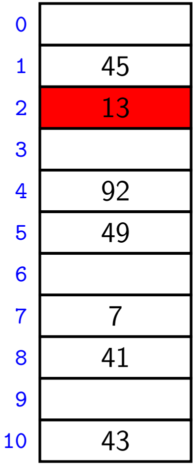
0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(46)

$h(46) = 2$

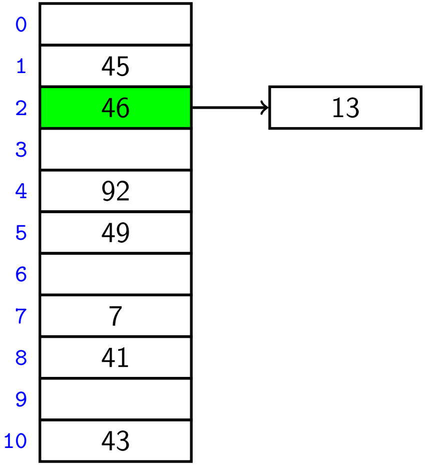


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(46)

$h(46) = 2$

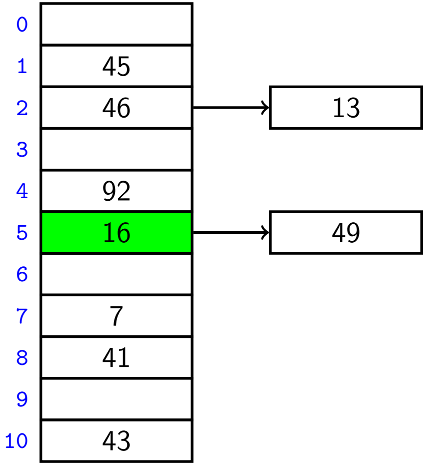


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(16)

$h(16) = 5$

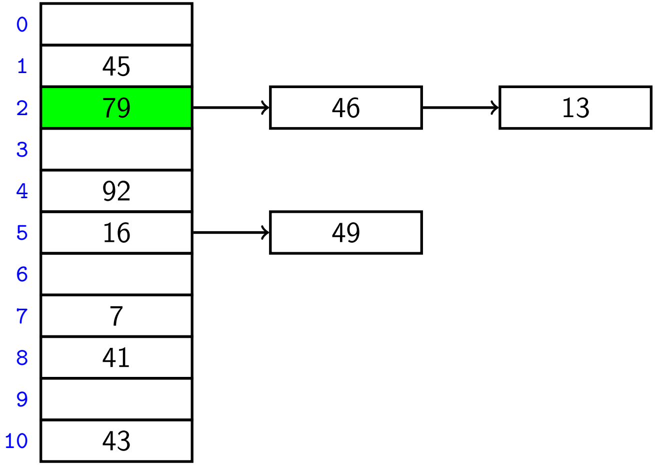


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(79)

$h(79) = 2$



Complexity of chaining

Recall the load balance $\alpha = n/M$.

Assuming uniform hashing, average bucket size is exactly α.

Analysis of operations:

search $\Theta(1 + \alpha)$ average-case, $\Theta(n)$ worst-case

insert $O(1)$ worst-case, since we always insert in front.

delete Same cost as *search*: $\Theta(1 + \alpha)$ average, $\Theta(n)$ worst-case

If we maintain $M \in \Theta(n)$, then average costs are all $O(1)$.
This is typically accomplished by rehashing whenever $n < c_1 M$ or $n > c_2 M$, for some constants c_1, c_2 with $0 < c_1 < c_2$.

Open addressing

Main idea: Each hash table entry holds only one item,
but any key k can go in multiple locations.

search and *insert* follow a *probe sequence* of possible locations for key k:
$\langle h(k,0), h(k,1), h(k,2), \ldots\rangle$.

delete becomes problematic; we must distinguish between
empty and *deleted* locations.

Open addressing

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delete becomes problematic; we must distinguish between
empty and *deleted* locations.

Simplest idea: *linear probing*
$h(k,i) = (h(k) + i) \bmod M$, for some hash function h.

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11$

insert(41)

$h(41, 0) = 8$

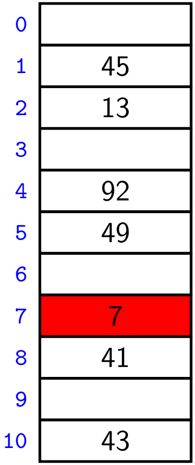
0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11$

insert(84)

$h(84, 0) = 7$

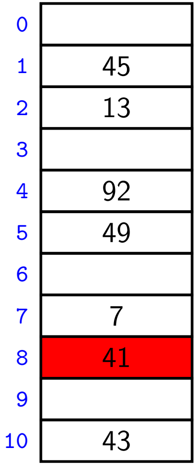


Linear probing example

$M = 11, \qquad h(k) = k \bmod 11$

insert(84)

$h(84, 1) = 8$



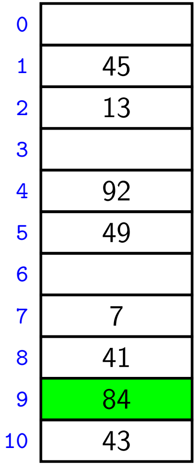
0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	🟥 41
9	
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11$

insert(84)

$h(84, 2) = 9$

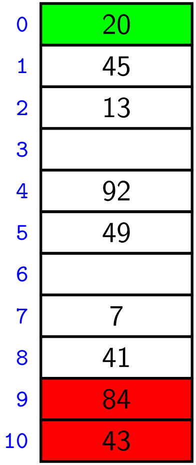


Linear probing example

$M = 11, \qquad h(k) = k \bmod 11$

insert(20)

$h(20, 2) = 0$



0	20 🟩
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84 🟥
10	43 🟥

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11$

delete(43)

$h(43, 0) = 10$

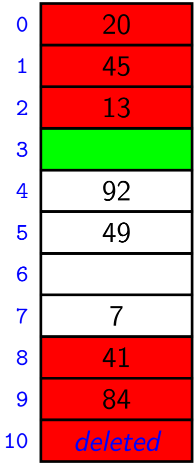
0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11$

search(63)

$h(63, 6) = 3$



Double Hashing

Say we have **two** hash functions h_1, h_2 that are **independent**.

So, under uniform hashing, we assume the probability that a key k has $h_1(k) = a$ and $h_2(k) = b$, for any particular a and b, is

$$\frac{1}{M^2}.$$

For *double hashing*, define $h(k, i) = h_1(k) + i \cdot h_2(k) \bmod M$.

search, *insert*, *delete* work just like for linear probing, but with this different probe sequence.

Cuckoo hashing

This is a relatively new idea from Pagh and Rodler in 2001.

Again, we use two independent hash functions h_1, h_2.
The idea is to *always* insert a new item into $h_1(k)$.
This might "kick out" another item, which we then attempt to re-insert into its alternate position.

Insertion might not be possible if there is a loop.
In this case, we have to rehash with a larger M.

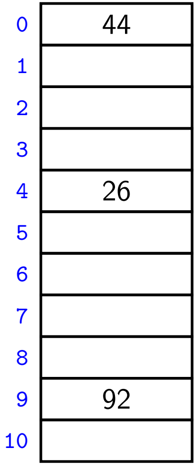
The big advantage is that an element with key k
can only be in $T[h_1(k)]$ or $T[h_2(k)]$.

Cuckoo hashing insertion

```
cuckoo-insert(T,x)
T: hash table,   x: new item to insert
1.    $y \leftarrow x$,   $i \leftarrow h_1(x.key)$
2.    do at most $n$ times:
3.        swap($y$, $T[i]$)
4.        if $y$ is "empty" then return "success"
5.        if $i = h_1(y.key)$ then $i \leftarrow h_2(y.key)$
6.        else $i \leftarrow h_1(y.key)$
7.    return "failure"
```

Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$



0	44
1	
2	
3	
4	26
5	
6	
7	
8	
9	92
10	

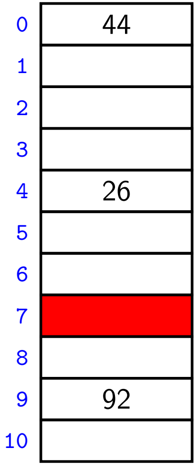
Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(51)

$y.key = 51$
$i = 7$

$h_1(y.key) = 7$
$h_2(y.key) = 5$



Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(51)

$y.key =$

$i =$

$h_1(y.key) =$

$h_2(y.key) =$

0	44
1	
2	
3	
4	26
5	
6	
7	51
8	
9	92
10	

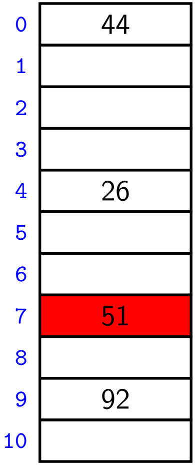
Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$y.key = 95$
$i = 7$

$h_1(y.key) = 7$
$h_2(y.key) = 7$



0	44
1	
2	
3	
4	26
5	
6	
7	🟥 51
8	
9	92
10	

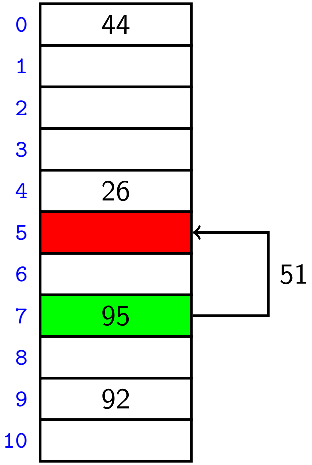
Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$y.key = 51$
$i = 5$

$h_1(y.key) = 7$
$h_2(y.key) = 5$



Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$y.key =$
$i =$

$h_1(y.key) =$
$h_2(y.key) =$

0	44
1	
2	
3	
4	26
5	51
6	
7	95
8	
9	92
10	

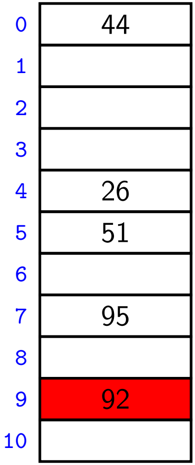
Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(97)

$y.key = 97$
$i = 9$

$h_1(y.key) = 9$
$h_2(y.key) = 10$



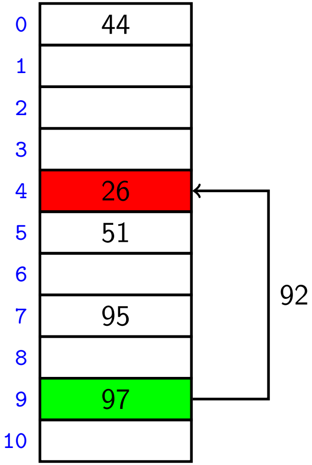
Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(97)

$y.key = 92$
$i = 4$

$h_1(y.key) = 4$
$h_2(y.key) = 9$



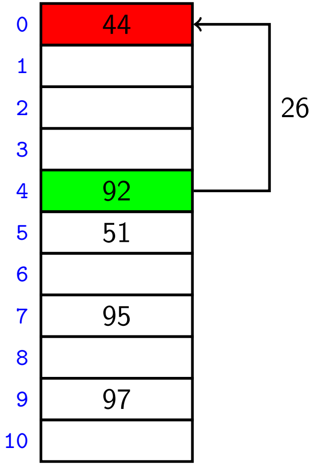
Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(97)

$y.key = 26$
$i = 0$

$h_1(y.key) = 4$
$h_2(y.key) = 0$



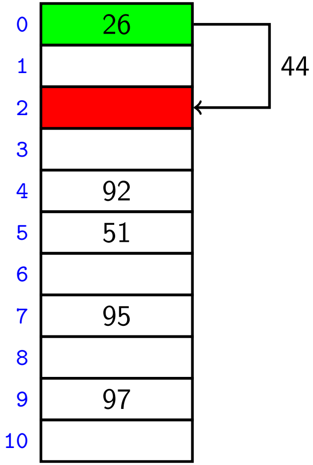
Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(97)

$y.key = 44$
$i = 2$

$h_1(y.key) = 0$
$h_2(y.key) = 2$



Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(97)

$y.key =$
$i =$

$h_1(y.key) =$
$h_2(y.key) =$

0	26
1	
2	44
3	
4	92
5	51
6	
7	95
8	
9	97
10	

Cuckoo hashing example

$M = 11, \qquad h_1(k) = k \bmod 11, \qquad h_2(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

search(26)

$h_1(26) = 4$
$h_2(26) = 0$

0	26
1	
2	44
3	
4	92
5	51
6	
7	95
8	
9	97
10	

Complexity of open addressing strategies

We won't do the analysis, but just state the costs.

For any open addressing scheme, we *must* have $\alpha < 1$ (why?).
Cuckoo hashing requires $\alpha < 1/2$.

The following gives the *big-Theta* cost of each operation for each strategy:

	search	*insert*	*delete*
Linear Probing	$\frac{1}{(1-\alpha)^2}$	$\frac{1}{(1-\alpha)^2}$	$\frac{1}{1-\alpha}$
Double Hashing	$\frac{1}{1-\alpha}$	$\frac{1}{1-\alpha}$	$\frac{1}{\alpha}\log\left(\frac{1}{1-\alpha}\right)$
Cuckoo Hashing	1	$\frac{\alpha}{(1-2\alpha)^2}$	1

Hashing in External Memory

If we have a *very large* dictionary that must be stored externally, how can we hash and minimize page faults?

Most hash strategies covered have scattered data access patterns.

Linear Probing: All hash table accesses will usually be in the same page.
But α must be kept small to avoid "clustering",
so there is a lot of wasted space.
Also, there is a need for frequent rehashing of the *entire table*.

Extendible Hashing

Say external memory is stored in blocks (or “pages”) of size S.
The goal: Use very few blocks, access only 1 page per operation.

Basic idea: Similar to a B-tree with height 1 and max size S at the leaves

The *directory* (similar to root node) is stored in *internal memory*.
Contains a hashtable of size 2^d, where d is called the *order*.

Each directory entry points to a *block* (similar to leaves)
stored in *external memory*.
Each block contains at most S items, sorted by hash value.

Parameters: an integer $L > 0$ and a hash function
$h : U \rightarrow \{0, 1, 2, \ldots, 2^L - 1\}$
(thought of as binary sequences of length L.)

Extendible Hashing Directory

Properties of the directory:

- Directory has *order* $d \le L$,
- Directory contains a hashtable with indices $0, 1, \ldots, 2^d - 1$.
- To look up a key k in the directory, use the high-order d bits of $h(k)$, that is
$$\left\lfloor \frac{h(k)}{2^{L-d}} \right\rfloor .$$
- Each directory entry points to a single block.
(Many entries can point to the same block.)

Extendible Hashing Blocks

Properties of a block B:

- B has a *local depth* $k_B \leq d$ and size $n_B \leq S$.
- B stores n_B KVPs, sorted by the hash values of the keys.
- Hash values in B agree on the high-order k_B bits.
  Call this the *block index* i_B, where $0 \leq i_B < 2^{k_B}$.
- Every key *key* in B satisfies

$$i_B \cdot 2^{L-k_B} \leq h(key) < (i_B + 1) \cdot 2^{L-k_B}.$$

- Exactly 2^{d-k_B} directory entries point to block B.

Searching in extendible hashing

Searching is done in the directory, then in a block:

- Given a key k, compute $h(k)$.
- Lookup a block B in the directory with index first d bits of $h(k)$: $\lfloor h(k)/2^{L-d} \rfloor$.
- Perform a binary search in B for all items with hash value $h(k)$.

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- Perform a binary search in B for all items with hash value $h(k)$.

Cost:

CPU time: $\Theta(\log S)$

Page faults: 1 (directory resides in internal memory)

Insertion in Extendible Hashing

insert(k, v) is done as follows:

- Search for $h(k)$ to find the proper block B for insertion
- If the B has space ($n_B < S$), then put (k, v) there.

Insertion in Extendible Hashing

insert(k, v) is done as follows:

- Search for $h(k)$ to find the proper block B for insertion
- If the B has space ($n_B < S$), then put (k, v) there.
- ElseIf the block is full and $k_B < d$, perform a *block split*:
  - Split B into B_0 and B_1.
  - Separate items according to the $(k_B + 1)$-th bit:
    if $h(k) \bmod 2^{L-k_B} < 2^{L-k_B-1}$, then k goes in B_0, else k goes in B_1.
  - Set local depth in B_0 and B_1 to $k_B + 1$
  - Update references in the directory

Insertion in Extendible Hashing

insert(k, v) is done as follows:

- Search for $h(k)$ to find the proper block B for insertion
- If the B has space ($n_B < S$), then put (k, v) there.
- ElseIf the block is full and $k_B < d$, perform a *block split*:
  - Split B into B_0 and B_1.
  - Separate items according to the $(k_B + 1)$-th bit:
    if $h(k) \bmod 2^{L-k_B} < 2^{L-k_B-1}$, then k goes in B_0, else k goes in B_1.
  - Set local depth in B_0 and B_1 to $k_B + 1$
  - Update references in the directory
- ElseIf the block is full and $k_B = d$, perform a *directory grow*:
  - Double the size of the directory ($d \leftarrow d + 1$)
  - Update references appropriately.
  - Then split block B (which is now possible).

Extendible hashing conclusion

delete(k) is performed in a reverse manner to *insert*:

- Search for block B and remove k from it
- If n_B is too small, then we perform a *block merge*
- If every block B has local depth $k_B \leq d - 1$, perform a *directory shrink*

Extendible hashing conclusion

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- Search for block B and remove k from it
- If n_B is too small, then we perform a *block merge*
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Cost of *insert* and *delete*:

CPU time: $\Theta(S)$ without a directory grow/shrink (which rarely happen)
Directory grow/shrink costs $\Theta(2^d)$.

Page faults: 1 or 2, depending on whether there is a block split/merge.

Summary of extendible hashing

- Directory is much smaller than total number of stored keys and should fit in main memory.
- Only 1 or 2 external blocks are accessed by *any* operation.
- To make more space, we only add a block.
  Rarely do we have to change the size of the directory.
  Never do we have to move all items in the dictionary
  (in constrast to normal hashing).
- Space usage is not too inefficient: can be shown that
  under uniform hashing, each block is expected to be 69% full.
- Main disadvantage:

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- Main disadvantage: extra CPU cost of $O(\log S)$ or $O(S)$

Hashing vs. Balanced Search Trees

Advantages of Balanced Search Trees

- $O(\log n)$ worst-case operation cost
- Does not require any assumptions, special functions, or known properties of input distribution
- No wasted space
- Never need to rebuild the entire structure

Advantages of Hash Tables

- $O(1)$ cost, but only on average
- Flexible load factor parameters
- Cuckoo hashing achieves $O(1)$ worst-case for search & delete

External memory:
Both approaches can be adopted to minimize page faults.

Multi-dimensional Data

What if the keys are multi-dimensional, such as strings?

Say $U = \{0, 1, \ldots, s-1\}^d$, length-$d$ strings from size-s alphabet

Standard approach is to *flatten* to integers using a function $f : U \to \mathbb{N}$.
We combine this with a standard hash function
$h : \mathbb{N} \to \{0, 1, 2, \ldots, M-1\}$.

With $h(f(k))$ as the hash values, we then use any standard hash table.

Note: computing each $h(f(k))$ takes $\Omega(d)$ time.