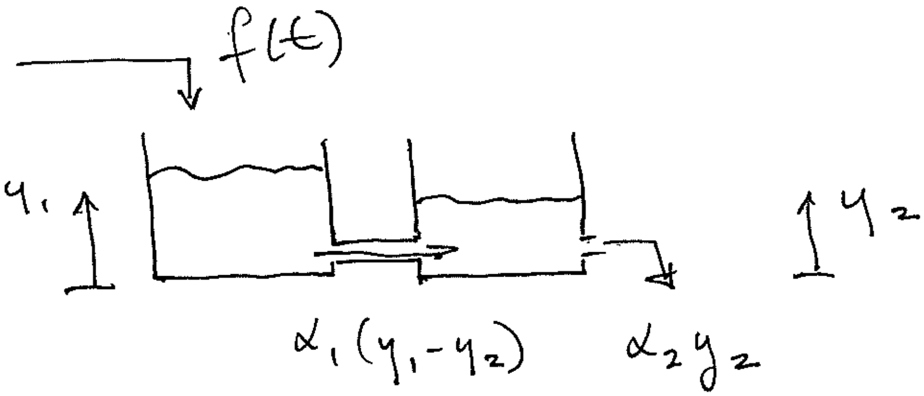
# Additional ODE example:

- to illustrate all of the steps of our approach.

- Fluid flow example




$f(t)$ - volume flow rate into system

$\alpha_1(y_1-y_2)$ - " " " from tank 1 to tank 2

$\alpha_2 y_2$ - volume flow rate out of tank 2

If the respective cross-sectional areas of the tanks are constants $A_1$ & $A_2$, then the levels $y_1$, $y_2$ in the tanks obey the following eq$^{\underline{n}}$s.

$$A_1 \dot{y}_1 = f - \alpha_1 (y_1 - y_2)$$

$$A_2 \dot{y}_2 = \alpha_1 (y_1 - y_2) - \alpha_2 y_2$$

– or, in our "operator" notation,

$$(A_1 D + \alpha_1) y_1 - \alpha_1 y_2 = f$$

$$-\alpha_1 y_1 + (A_2 D + \alpha_1 + \alpha_2) y_2 = 0$$

Let's eliminate $y_2$ by "multiplying" the first equation by $(A_2 D + \alpha_1 + \alpha_2)$ and the second by $\alpha_1$, and adding:

$$\left[ (A_1 D + \alpha_1)(A_2 D + \alpha_1 + \alpha_2) - \alpha_1^2 \right] y_1 = (A_2 D + \alpha_1 + \alpha_2) f$$

This is a linear ODE with constant coefficients.

$$Q(D)\, y = P(D)\, f .$$

We know that $y = P(D)\, \tilde{y}$ solves this equation provided $\tilde{y}$ solves the simpler equation

$$Q(D)\, \tilde{y} = f$$

To solve the latter equation, we add the complementary solution and a particular solution (for a given function $f(t)$).

Suppose that (in, say, SI units) the numerical values of the constants are given by

$$A_1 = A_2 = 1$$

$$\alpha_1 = \alpha_2 = 1/2$$

Then the equation becomes

$$\left(D^2 + \frac{3}{2}D + \frac{1}{4}\right)\tilde{y} = f$$

and the auxiliary equation is

$$\left(D^2 + \frac{3}{2}D + \frac{1}{4}\right)\tilde{y} = 0$$

$\Rightarrow$ find the roots of the characteristic equation

$$m^2 + \frac{3}{2}m + \frac{1}{4} = 0$$

$$\Longleftrightarrow \quad m = -\frac{3}{4} \pm \frac{\sqrt{5}}{4}$$

The <u>complementary solution</u> is therefore

$$y_c = c_1 e^{\left(-\frac{3}{4} + \frac{\sqrt{5}}{4}\right)t} + c_2 e^{\left(-\frac{3}{4} - \frac{\sqrt{5}}{4}\right)t}$$

(Recall that if any of the roots of the char. eq. is of multiplicity k, the corresponding exponential is multiplied by a general polynomial of degree k-1; in this case, k = 1 for both roots.)

Let's suppose that $f(t)$ is given as

$$f(t) = \sin 2t$$

and find a <u>particular solution</u>.

We can apply our method for exponential functions, because by Euler's identity,

$$\sin 2t = \frac{e^{j2t} - e^{-j2t}}{2j}$$

Specifically, we can find particular solutions for the cases $f(t) = e^{\pm j2t}$, and then apply superposition.

Now if $f(t) = e^{j2t}$,

then the method of undetermined coefficients is to look for a solution of the form

$$y_p(t) = k e^{j2t}$$

[ Of course, if $2j$ were a root of the characteristic equation, we'd have to multiply this by the smallest power of $t$ that didn't yield a solution of the auxiliary equation. ]

- We substitute this "candidate" solution into the diff. eq. to see what values of k yield solutions:

$$\left(D^2 + \frac{3}{2}D + \frac{1}{4}\right) y_p = e^{j2t}$$

$$\left(-4 + 3j + \frac{1}{4}\right) k e^{j2t} = e^{j2t}$$

$$\left(-\frac{15}{4} + 3j\right) k = 1$$

$$k = \frac{1}{-\frac{15}{4} + 3j}$$

$$= \frac{1}{-\frac{15}{4} + 3j} \frac{-\frac{15}{4} - 3j}{-\frac{15}{4} - 3j}$$

$$= \frac{-\frac{15}{4} - 3j}{\left(\frac{15}{4}\right)^2 + 3^2} = \frac{-60 - 48j}{369}$$

$$= \frac{-20 - 16j}{123}$$

So a particular solution is

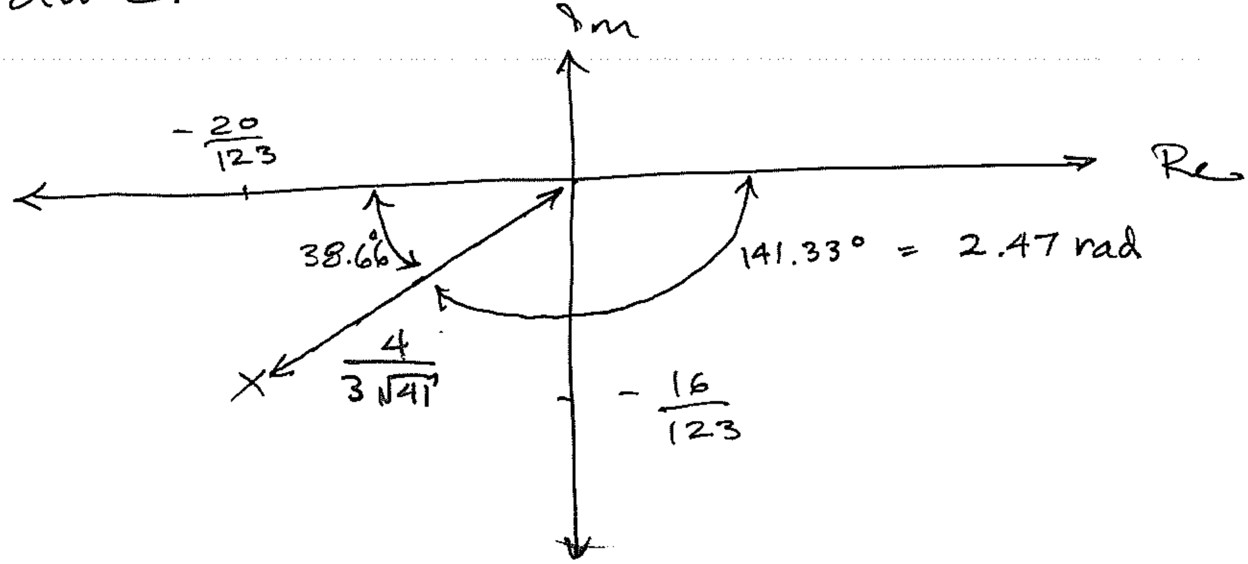
$$y_p(t) = \frac{-20 - 16j}{123} e^{j2t}$$

It may be easier to make sense of this result if we rewrite the coefficient in terms of its modulus and its angle:

$$k = \frac{-20 - 16j}{123} = \frac{\sqrt{656}}{123} e^{-j\,2.47}$$

$$= \frac{4}{3\sqrt{41}} e^{-j\,2.47}$$

Picture:



In other words,

$$y_p(t) = \frac{4}{3\sqrt{41}} e^{j(2t - 2.47)}$$

$$= |k| e^{j(2t + \angle k)}$$

If we go through the same exercise for $f(t) = e^{-j2t}$, we'll simply find

$$y_p(t) = \frac{4}{3\sqrt{41}} e^{-j(2t - 2.47)}$$

$$= |k^*| e^{j(2t + \angle k^*)}$$

(Check this.)

So, by superposition, if

$$f(t) = \sin 2t = \frac{e^{j2t} - e^{-j2t}}{2j},$$

a particular solution is

$$y_p(t) = \frac{4}{3\sqrt{41}} \frac{e^{j(2t-2.47)} - e^{-j(2t-2.4}}{2j}$$

$$= \frac{4}{3\sqrt{41}} \sin(2t - 2.47)$$

$$= |k| \sin(2t + \angle k).$$

So $|k|$ and $\angle k$ respectively relate the amplitude and phase of $y_p$ to those of $f$.

We now have the general solution of

$$Q(D)\,\tilde{y} = f \;;$$

namely,

$$\tilde{y}(t) = c_1 e^{\left(-\frac{3}{4} + \frac{\sqrt{5}}{4}\right)t} + c_2 e^{\left(-\frac{3}{4} - \frac{\sqrt{5}}{4}\right)t}$$

$$+ \frac{4}{3\sqrt{41}} \sin(2t - 2.47)$$

To find the general solution of

$$Q(D)\,y = P(D)\,f \,,$$

we need only set

$$y = P(D)\,\tilde{y}\,.$$

Recall that

$$P(D) = (A_2 D + \alpha_1 + \alpha_2)$$

$$= D + 1$$

Differentiating $\tilde{y}$, we get

$$D\tilde{y} = \left(-\tfrac{3}{4} + \tfrac{\sqrt{5}}{4}\right) c_1 e^{\left(-\frac{3}{4} + \frac{\sqrt{5}}{4}\right)t}$$
$$+ \left(-\tfrac{3}{4} - \tfrac{\sqrt{5}}{4}\right) c_2 e^{\left(-\frac{3}{4} - \frac{\sqrt{5}}{4}\right)t}$$
$$+ \frac{8}{3\sqrt{41}} \cos(2t - 2.47)$$

So $y = (D+1)\tilde{y}$ is

$$\left(\tfrac{1}{4} + \tfrac{\sqrt{5}}{4}\right) c_1 e^{\left(-\frac{3}{4} + \frac{\sqrt{5}}{4}\right)t}$$
$$+ \left(\tfrac{1}{4} - \tfrac{\sqrt{5}}{4}\right) c_2 e^{\left(-3/4 - \frac{\sqrt{5}}{4}\right)t}$$
$$+ \frac{4}{3}\sqrt{\frac{5}{41}} \left(\frac{1}{\sqrt{5}} \sin(2t - 2.47) + \frac{2}{\sqrt{5}} \cos(2t - 2.47)\right)$$

$$\Rightarrow \quad y = c_3\, e^{\left(-\frac{3}{4} + \frac{\sqrt{5}}{4}\right)t}$$

$$+ \; c_4\, e^{\left(-\frac{3}{4} - \frac{\sqrt{5}}{4}\right)t}$$

$$+ \; \frac{4}{3}\sqrt{\frac{5}{41}}\, \sin(2t - 2)$$

(because $\frac{1}{\sqrt{5}} \sim \sin(0.47)$).

Given initial conditions (values of $y(0)$, $\dot{y}(0)$), we could now evaluate the arbitrary constants $c_3$, $c_4$.

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Note: we could simply have applied $P(D)$ to the particular solution of $Q(D)\tilde{y} = f$ to find a particular solution of $Q(D)\,y = P(D)f$ — the complementary solutions of both equations are the same.