Solution of linear differential equations with constant coefficients a <u>linear</u> differential equation has the form

$$\frac{d^n}{dt^n} y(t) + a_{n-1}(t) \frac{d^{n-1}}{dt^{n-1}} y(t) + \ldots + a_0(t) y(t)$$

$$= b_n(t) \frac{d^n}{dt^n} f(t) + \ldots + b_0(t) f(t)$$

One of the few cases that admit analytical solution is that in which the $a_i(\cdot)$ and the $b_j(\cdot)$ are constant.

We'll restrict attention to such linear ODEs with <u>constant coefficients</u>:

$$\frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \ldots + a_0 y(t)$$

$$= b_n \frac{d^n}{dt^n} f(t) + \ldots + b_0 f(t)$$

For notational convenience, we'll think of $\frac{d}{dt}$ as a differentiation operator denoted by D. Hence,

$$D\, y(t) = \frac{d}{dt}\, y(t) \,,$$

$$D^2\, y(t) = \frac{d^2}{dt^2}\, y(t) \,,$$

etc.

In this way, we can think of the two sides of the equation as containing polynomials in the differentiation operator:

$$Q(D)\, y(t) = P(D)\, f(t)$$

In this first part of the course, we're taking a traditional approach to differential equations, which assumes that the "forcing term" on the right-hand side is given — and we need to solve for $y(t)$.

We'll do so by first solving for $\tilde{y}(t)$ in the diff. eq.

$$Q(D)\,\tilde{y}(t) = f(t)$$

Given such a $\tilde{y}(t)$, we have

$$\begin{aligned} &Q(D)\,P(D)\,\tilde{y}(t) \\ =\ &P(D)\,Q(D)\,\tilde{y}(t) \\ =\ &P(D)\,f(t), \end{aligned}$$

so $y(t) = P(D)\,\tilde{y}(t)$ solves the original equation.

FACT:

If $f(t)$ is continuous on an interval $a \le t \le b$ then there exists a solution $y(\cdot)$ satisfying the above differential equation and also the "initial conditions" for $a \le t_0 \le b$:

$$y(t_0) = p_0,\ \dot{y}(t_0) = p_1,\ \ldots,\ y^{(n-1)}(t_0) = p_{n-1},$$

where $p_0, p_1, \ldots, p_{n-1} \in \mathbb{C}$ are constants. Moreover, this solution is unique.

<u>Terminology</u>

The <u>general solution</u> of the equation is an expression for $y(t)$ that satisfies the equation and contains n "arbitrary constants" ...

... a set of n initial conditions determines values of these arbitrary constants.

Example:

$\boxed{m}$ $\downarrow y$

$$\ddot{y}(t) = g$$

Integrating,

$$\dot{y}(t) = \int g\,dt$$

$$= gt + c_1$$

$$\Longrightarrow \quad y(t) = \int (gt + c_1)\,dt$$

$$= \tfrac{1}{2}gt^2 + c_1 t + c_2$$

c_1, c_2 — "arbitrary constants"

— If $y(0) = 0$, then $c_2 = 0$...

... and if $\dot{y}(0) = 0$, then $c_1 = 0$.

We'll find the general solution of the equation, and plug in initial conditions to evaluate the "arbitrary constants."

Q: How?

A: 1. We'll first find the general solution of the <u>auxiliary equation</u>

$$\frac{d^n}{dt^n} y(t) + a_n \frac{d^{(n-1)}}{dt^{(n-1)}} y(t) + \ldots + a_0 y(t) = 0$$

Call it $y_c(t)$ — the "complementary solution". It contains n arbitrary constants.

2. Then we'll find any solution $y_p(t)$ of the original equation — this will be called a <u>particular solution</u>.

To see why this suffices, consider

$$\tilde{y}(t) = y_c(t) + y_p(t)$$

This contains n arbitrary constants.

Moreover, if we substitute it into the differential equation, we get

$$\begin{aligned} & Q(D)\,(y_c(t) + y_p(t)) \\ = \;& Q(D)\, y_c(t) + Q(D)\, y_p(t) \\ = \;& 0 + f(t) \end{aligned}$$

— so $y(t)$ is a solution, and therefore (by the above FACT), the general solution.

Example

Applying this procedure to the above example, we first consider the auxiliary equation

$$\ddot{y}(t) = 0$$

Integrating twice, we get

$$y_c(t) = c_1 t + c_2$$

– the "complementary solution."

Now, it can easily be checked that

$$y_p(t) = \frac{1}{2} g t^2$$

is a "particular solution." (We'll study systematic ways of finding complementary and particular solutions later.)

Adding, we get the general solution

$$\tilde{y}(t) = \frac{1}{2} g t^2 + c_1 t + c_2$$

Note that it doesn't matter what particular solution we choose. For example, if we had instead written

$$y_p(t) = \frac{1}{2} g t^2 + d$$

(where d is a constant), we would have obtained the general solution

$$\tilde{y}(t) = \frac{1}{2} g t^2 + c_1 t + (c_2 + d)$$

— which is effectively the same.

- The complementary solution

How to find the general solution of

$$Q(D)\, y(t) = 0 \quad ?$$

Consider the simple example

$$(D - 3)\, y(t) = 0$$

Its general solution is $y(t) = c e^{3t}$.

This could be found by assuming $y(t) = c e^{mt}$, in which case

$$\begin{aligned} Q(D)\, y(t) &= (D - 3)\, y(t) \\ &= (m - 3)\, c e^{m(t)} \\ &= Q(m)\, c e^{m(t)} \end{aligned}$$

which solves the auxiliary equation iff

$$Q(m) = 0,$$

i.e., $m = 3$.

We'll call

$$Q(m) = 0$$

the <u>characteristic equation</u>.

- Its solutions, or <u>roots</u>, will determine the form of the complementary solution.

Let's try another example:

$$(D^2 + 8D + 15)\, y = 0$$

If we suppose $y = c e^{mt}$, and substitute, we get

$$(m^2 + 8m + 15)\, c e^{mt} = 0.$$

which holds iff m is a root of the characteristic equation

$$m^2 + 8m + 15 = 0.$$

$$\Longleftrightarrow \quad m = -3 \text{ or } m = -5$$

- This means that $c_1 e^{-3t}$ and $c_2 e^{-5t}$ are both solutions.

- By linearity, so is their sum

$$y(t) = c_1 e^{-3t} + c_2 e^{-3t},$$

which, since it has two arbitrary constants, is the complementary solution.

Generally, the n roots of the characteristic equation give rise in this way to a solution with n arbitrary constants.

Q 1 : What if there are repeated roots, rather than n distinct ones?

Q 2 : What if there are complex roots?

Repeated roots:

Consider

$$(D^2 - 6D + 9)y = 0$$

The characteristic equation

$$m^2 - 6m + 9 = 0$$

has repeated roots at $m = 3, 3$.

Obviously,

$$y(t) = c_1 e^{3t} + c_2 e^{3t}$$

can't be considered to have two arbitrary constants, so it's not the complementary solution.

For this, bring in the following

FACT:

If $y = y_1$ is a solution of $Q(D)y(t) = f(t)$, then substituting $y = y_1 v$ yields an equation of order $n-1$ in $\dot{v}$

Try it:

$$y = e^{3t} v$$

$$\Rightarrow Dy = 3e^{3t} v + e^{3t} \dot{v}$$

$$\Rightarrow D^2 y = 9e^{3t} v + 6e^{3t} \dot{v} + e^{3t} \ddot{v}$$

Substituting into $Q(D)\, y = 0$ and simplifying yields

$$e^{3t} \ddot{v} = 0$$

$$\Longleftrightarrow \quad \ddot{v} = 0$$

Now the general solution of this last equation is

$$v(t) = c_1 + c_2 t,$$

so we get a solution

$$y(t) = c_1 e^{3t} + c_2 t e^{3t}$$

that has two arbitrary constants and is the general solution.

In general, if the repeated root λ is of order k, then the complementary solution contains a corresponding term

$$\left(c_1 + c_2 t + \ldots + c_m t^{(m-1)} \right) \cdot e^{\lambda t}$$

Example

Returning again to

$$\ddot{y}(t) = 0,$$

we find the auxiliary equation

$$m^2 = 0$$

has a repeated root of order 2 at $m = 0$. Since it has no other roots, the complementary solution is

$$y_c(t) = c_1 + c_2 t$$

– which is effectively the same as our earlier solution.

Recap

- We'll solve diff. eqs. of the form

$$Q(D)\, y(t) = P(D)\, f(t)$$

by first solving

$$Q(D)\, \tilde{y}(t) = f(t)$$

(and then setting $y(t) = P(D)\, \tilde{y}(t)$).

- We'll solve

$$Q(D)\, \tilde{y}(t) = f(t)$$

by finding the general solution $y_c(t)$ of the "auxiliary equation"

$$Q(D)\, \tilde{y}(t) = 0$$

and a "particular solution" $y_p(t)$ of

$$Q(D)\, \tilde{y}(t) = f(t)$$

and then setting

$$\tilde{y}(t) = y_c(t) + y_p(t).$$

- The key to finding the "complementary solution" $y_c(t)$ of the auxiliary equation

$$Q(D)\,\tilde{y}(t) = 0$$

lies in the roots of the "characteristic equation"

$$Q(m) = 0$$

($Q(m)$ is often called the "characteristic polynomial".)

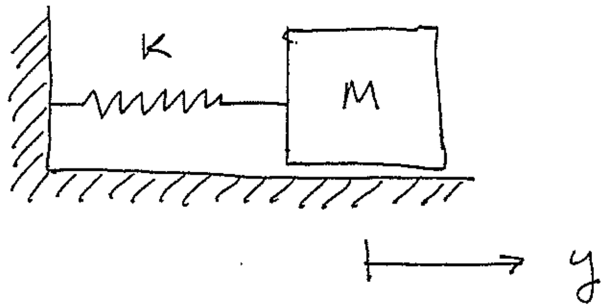
- A real root λ of order k gives rise to a term

$$\left(c_1 + c_2 t + \ldots + c_k t^{(k-1)} \right) e^{\lambda t}$$

- The complementary solution $y_c(t)$ is obtained by adding all such terms for all distinct roots.

Now let's consider <u>complex roots</u>.

<u>Example</u> :



$$M \ddot{y} = -Ky \; ;$$

i.e.,

$$\ddot{y} + \frac{K}{M} y = 0$$

- characteristic equation :

$$m^2 + K/m = 0 \; ;$$

i.e.

$$m = \pm j \sqrt{K/m}$$

- imaginary !

Q: What does the physics tell us the form of the solution should be?

Perhaps surprisingly, the form of the solution is <u>mathematically</u> rather similar to that of the case of real roots — it involves exponentials.

Bring in the <u>complex exponential</u> function e^z, where $z \in \mathbb{C}$, defined by the absolutely convergent series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$$

Note:

1. - if z is real, this is the familiar real exponential function;

2. - it can be shown (by multiplication of absolutely convergent series) that

$$e^{z_1} e^{z_2} = e^{(z_1 + z_2)} \quad (\forall z_1, z_2 \in \mathbb{C})$$

— this is called the <u>addition property</u>, familiar from the real exponential.

3. - Furthermore, functions of complex variables like e^z can be differentiated using the same rules that apply to functions of real variables. So, for example,

$$\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t},$$

even if λ is complex.

The addition property shows how e^z depends on the real and imaginary parts of z: let $z = \alpha + j\theta$, where $\alpha, \theta \in \mathbb{R}$; then

$$e^z = e^{\alpha + j\theta}$$

$$= e^{\alpha} e^{j\theta}$$

Now consider the form of $e^{j\theta}$:

$$e^{j\theta} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)$$

$$+ j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

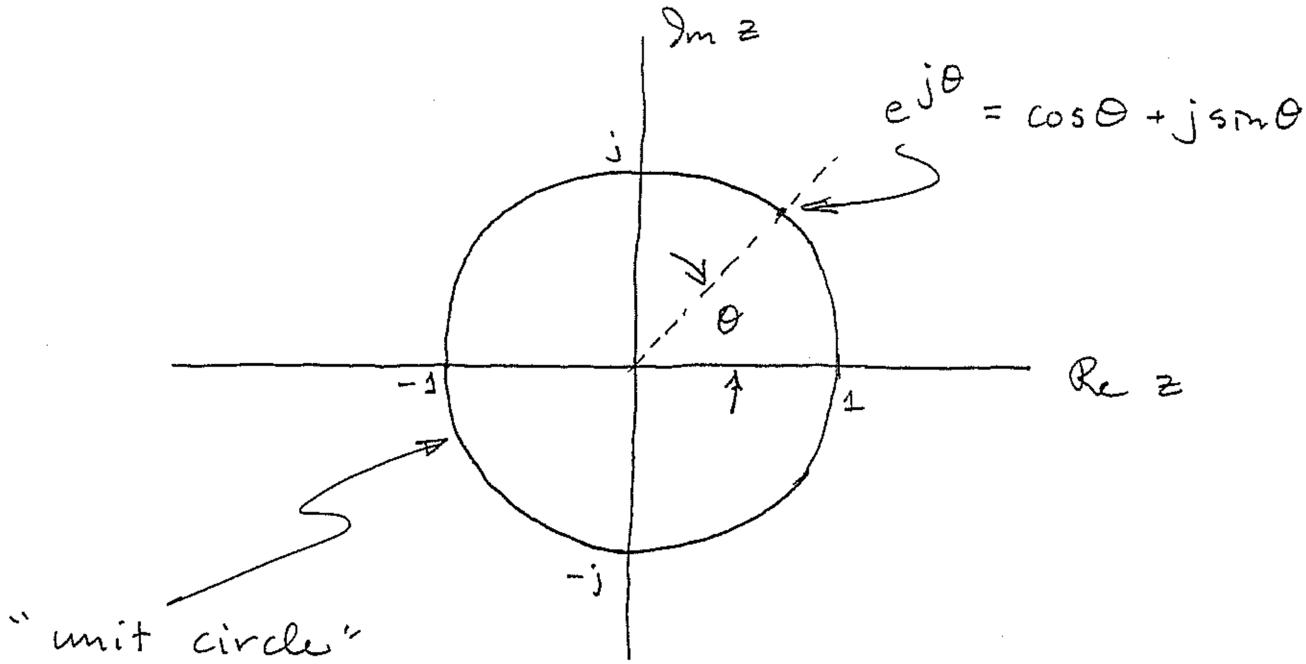
($\because$ the latter two series are convergent)

$$= \cos\theta + j\sin\theta$$

This yields the <u>Euler identity</u>:

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

Here's the <u>picture</u>, on the complex plane:



Let's go back to our example.

$$\ddot{y} + \frac{K}{M} y = 0$$

The roots of the characteristic equation are

$$m = \pm j \sqrt{\frac{K}{M}}$$

Following the example of the case of real roots, we might guess the complementary solution to be

$$y_c(t) = c_1 e^{j\sqrt{\frac{K}{M}}\,t} + c_2 e^{-j\sqrt{\frac{K}{M}}\,t}$$

— and it is! (Check this.)

This yields a real-valued solution provided the initial conditions are real-valued:

$$y_c(t_0) = c_1 e^{j\sqrt{\frac{K}{M}}\, t_0} + c_2 e^{-j\sqrt{\frac{K}{M}}\, t_0}, \ \&$$

$$\dot{y}_c(t_0) = j\sqrt{\frac{K}{M}} \left[c_1 e^{j\sqrt{\frac{K}{M}}\, t_0} - c_2 e^{-j\sqrt{\frac{K}{M}}\, t_0} \right];$$

so,

$$c_1 = \frac{j\sqrt{\frac{K}{M}}\, y(t_0) + \dot{y}(t_0)}{2j\sqrt{\frac{K}{M}}\, e^{j\sqrt{\frac{K}{M}}\, t_0}}, \ \&$$

$$c_2 = \frac{j\sqrt{\frac{K}{M}}\, y(t_0) - \dot{y}(t_0)}{2j\sqrt{\frac{K}{M}}\, e^{-j\sqrt{\frac{K}{M}}\, t_0}};$$

therefore, $c_2 = c_1^*$.

Hence,

$$y_c(t) = c_1 e^{j\sqrt{\frac{K}{M}}\, t} + c_2 e^{j\sqrt{\frac{K}{M}}\, t}$$

$$= \left[\mathrm{Re}\, c_1 + j\,\mathrm{Im}\, c_1\right]\left(\cos\sqrt{\tfrac{K}{M}}\, t + j\sin\sqrt{\tfrac{K}{M}}\, t\right)$$

$$+ \left[\mathrm{Re}\, c_1 - j\,\mathrm{Im}\, c_1\right]\left(\cos\sqrt{\tfrac{K}{M}}\, t - j\sin\sqrt{\tfrac{K}{M}}\, t\right)$$

$$= 2\,\mathrm{Re}\, c_1 \cos\sqrt{\tfrac{K}{M}}\, t - 2\,\mathrm{Im}\, c_1 \sin\sqrt{\tfrac{K}{M}}\, t$$

$$= 2\,|c_1|\left[\cos(\angle c_1)\cos\sqrt{\tfrac{K}{M}}\, t - \sin(\angle c_1)\sin\sqrt{\tfrac{K}{M}}\, t\right],$$

$$= 2\,|c_1|\cos\left(\sqrt{\tfrac{K}{M}}\, t + \angle c_1\right)$$

- so the solution is sinusoidal, as the physics suggests.

- Note: the roots of the characteristic polynomial give the angular frequency

at which the mass oscillates.
This is commonly called the <u>natural frequency</u> of the system.

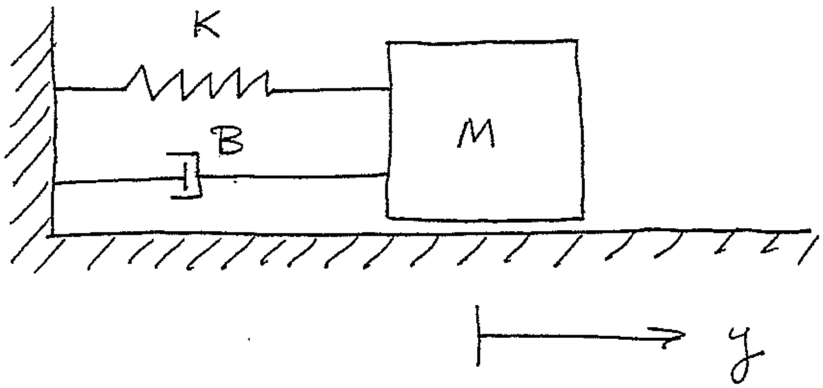
Exercise:

Use the addition property and Euler's identity to prove the "double-angle" formulas of trigonometry,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

used in the last step of the above development.

Example:



$$M\ddot{y} + B\dot{y} + Ky = 0 \;;$$

i.e. $$\ddot{y} + \frac{B}{M}\dot{y} + \frac{K}{M}y = 0$$

→ characteristic equation:

$$m^2 + \frac{B}{M}m + \frac{K}{M} = 0$$

roots:

$$m = \frac{-B}{2M} \pm \sqrt{\frac{B^2}{4M^2} - \frac{K}{M}}$$

We've looked at the cases of real and repeated roots, so let's suppose

$$\frac{B^2}{4M^2} < \frac{K}{M}$$

Then the (complex) roots are

$$m = -\frac{B}{2M} \pm j\sqrt{\frac{K}{M} - \frac{B^2}{4M^2}}$$

If we let

$$\omega_n = \sqrt{\frac{K}{M}}$$ (the "natural frequency" of the undamped system)

and $\zeta = \frac{1}{\omega_n}\left(\frac{B}{2M}\right) < 1$ (the "damping ratio").

then we can rewrite this as

$$m = -\zeta\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2}$$

Following the usual pattern, we guess (correctly) that the complementary solution is

$$y_c(t) = c_1 e^{(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})t} + c_2 e^{(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})t}$$

$$= e^{-\zeta\omega_n t}\left[c_1 e^{j\omega_n\sqrt{1-\zeta^2}\,t} + c_2 e^{-j\omega_n\sqrt{1-\zeta^2}\,t}\right]$$

Again, if the initial conditions are real-valued, then $c_2 = c_1{}^*$, so

$$y_c(t) = 2|c_1|\, e^{-\zeta\omega_n t} \cos\left(\omega_n\sqrt{1-\zeta^2}\, t + \phi\right),$$

where $\phi = \operatorname{Tan}^{-1}\left(\frac{\operatorname{Re} c_1}{\operatorname{Im} c_1}\right)$

- so the effect of the dashpot (which makes ζ nonzero) is:

a) to reduce the angular frequency of oscillation, and

b) to "damp" the oscillations, so that their amplitude decays exponentially.

- <u>Note</u> that the coefficient $-\zeta\omega_n$ of t in the real exponential is the real part of the roots, while the angular frequency of oscillation $\omega_n\sqrt{1-\zeta^2}$ corresponds to the imaginary part.

Finding the complementary solution: the general case.

Our result on repeated roots generalizes to the case of complex roots, so a root λ of order $k \geq 1$ gives rise to a term in the complementary solution of the form

$$(c_1 + c_2 t + \ldots + c_k t^{(k-1)}) e^{\lambda t}$$

... these are the only terms in the complementary solution.

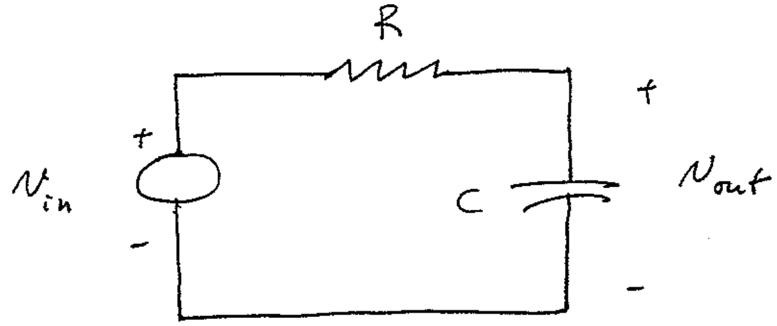
If the coefficients in the differential equation are real, then for every complex root λ, the complex conjugate λ^* is also a root, and if the initial conditions are real then the terms $e^{\lambda t}$ and $e^{\lambda^* t}$ give rise to sinusoids multiplied by real exponentials.

- Finding particular solutions

To find particular solutions, let's look at the method of undetermined coefficients, which works for polynomials and exponentials (and their sums).

- The main idea is to "guess" at a particular solution of the same form as the input signal.

Example: RC circuit



$$\left(D + \frac{1}{RC}\right) v_{out} = \frac{1}{RC} v_{in}$$

- Supposing the time constant RC to be 20ms, we write

$$(D + 50)\, v_{out} = 50\, v_{in}$$

- If $v_{in}(t) \equiv 1$ V, we "guess" that

that $v_{out}(t) = k, \quad t \geq 0$, and evaluate the 'undetermined constant' k by substitution

$$(D + 50)\, v_{out}(t) = 50k = 50 \cdot 1\ V$$

$$\longrightarrow \quad k = 1\ V.$$

Q: What if $v_{in}(\cdot)$ is a "unit ramp,"

$v_{in}(t) = t$?

- 1^{st} try $v_{out}(t) = kt$.

- Substituting, we get

$$(D+50)\, v_{out}(t) = k + 50kt = 50t$$

. . . but this has no constant solution

- Instead, try $v_{out}(t) = k_1 + k_2 t$:

$$(D+50)\, v_{out}(t) = k_2 + 50k_1 + 50k_2 t$$

- equating terms, we find

$$k_2 = 1,$$

$$k_1 = -0.02$$

This gives the particular solution

$$v_{out}(t) = -0.02 + t$$

- The general method is the following:

a) for a polynomial input signal of degree d, assume a particular solution that is a polynomial of degree d with undetermined coefficients — i.e.,

$$k_1 + k_2 t + \ldots + k_{d+1} t^d$$

b) for an exponential input signal $e^{\lambda t}$, assume a particular solution $k e^{\lambda t}$

- Obviously, for sums of input signals of the two kinds, assume sums of the corresponding candidate solutions.

<u>Example</u>: Response of RC circuit to exponentials

- If $v_{in}(t) = e^{\lambda t}$, seek a particular solution

$$v_{out}(t) = k e^{\lambda t}.$$

→ Substituting,

$$(D + 50) v_{out}(t) = (k\lambda + 50k) e^{\lambda t},$$

so $$k = \frac{50}{50 + \lambda} \qquad (\text{if } \lambda \neq -50)$$

- So, for example, if $\lambda = -100$, a particular solution is

$$v_{out}(t) = - e^{-100t};$$

and the general solution is

$$v_{out}(t) = c e^{-50t} - e^{-100t},$$

where the value of c is determined by an initial condition.

- What if $\lambda = -50$?

Then
$$(D+50)e^{\lambda t} = (-50+50)e^{\lambda t}$$
$$= 0$$
$$\neq 50e^{\lambda t}$$

— this doesn't work, because -50 is a root of the auxiliary polynomial . . .

. . . so $e^{\lambda t}$ solves the complementary equation, and makes the left-hand side 0.

- We need a 3^{rd} rule for this:

$$\boxed{\text{c) if any of the terms in a) or b) occurs in the complementary solution, multiply it by the smallest power of } t \text{ that gives a term that \underline{doesn't} appear in the complementary solution}}$$

... then evaluate the undetermined coefficients via substitution into the equation.

RC circuit example:

$$\text{If} \quad v_{in}(t) = e^{-50t},$$

assume

$$v_{out}(t) = k\,t\,e^{-50t}.$$

Substituting;

$$(D+50)\,v_{out}(t) = (k - 50kt + 50kt)e^{-50t}$$

$$= k\,e^{-50t}.$$

— so $k = 50$.

- This yields the general solution

$$v_{out}(t) = c\,e^{-50t} + 50\,t\,e^{-50t}.$$

- Note that this method allows for sinusoidal inputs, since by Euler's identity

$$\cos \omega t = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right],$$

$$\& \quad \sin \omega t = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$$

Example : RC circuit

- If $v_{in}(t) = e^{j\omega t}$, then we get the particular solution

$$v_{out}(t) = \frac{50}{50+j\omega} e^{j\omega t}$$

... so if $v_{in}(t) = \cos \omega t$, we get

$$v_{out}(t) = \frac{1}{2}\left[\frac{50}{50+j\omega} e^{j\omega t} + \frac{50}{50-j\omega} e^{-j\omega t}\right]$$

$$= \frac{50}{2}\left[\frac{50-j\omega}{50^2+\omega^2} e^{j\omega t} + \frac{50+j\omega}{50^2+\omega^2} e^{-j\omega t}\right]$$

$$= \frac{50}{\sqrt{50^2+\omega^2}}\left[\frac{50}{\sqrt{50^2+\omega^2}} \cos \omega t + \frac{\omega}{\sqrt{50^2+\omega^2}} \sin \omega t\right]$$

So, if $\phi = \tan^{-1} \omega/50$,

$$v_{out}(t) = \frac{50}{\sqrt{50^2+\omega^2}} \left[\cos\phi \cos\omega t + \sin\phi \sin\omega t \right]$$

$$= \frac{50}{\sqrt{50^2+\omega^2}} \cos(\omega t - \phi)$$

$$= \left|\frac{50}{50+j\omega}\right| \cos\left(\omega t + \angle\left(\frac{50}{50+j\omega}\right)\right)$$

- This circuit is a "low-pass filter":
  - as $\omega \to 0$, $\left|\frac{50}{50+j\omega}\right| \to 1$,
  - as $\omega \to \infty$, $\left|\frac{50}{50+j\omega}\right| \to 0$.

- Initial condition

Suppose again that $v_{in}(t) = t$, and that $v_{out}(0) = 0$.

The general solution of the diff. eq. is

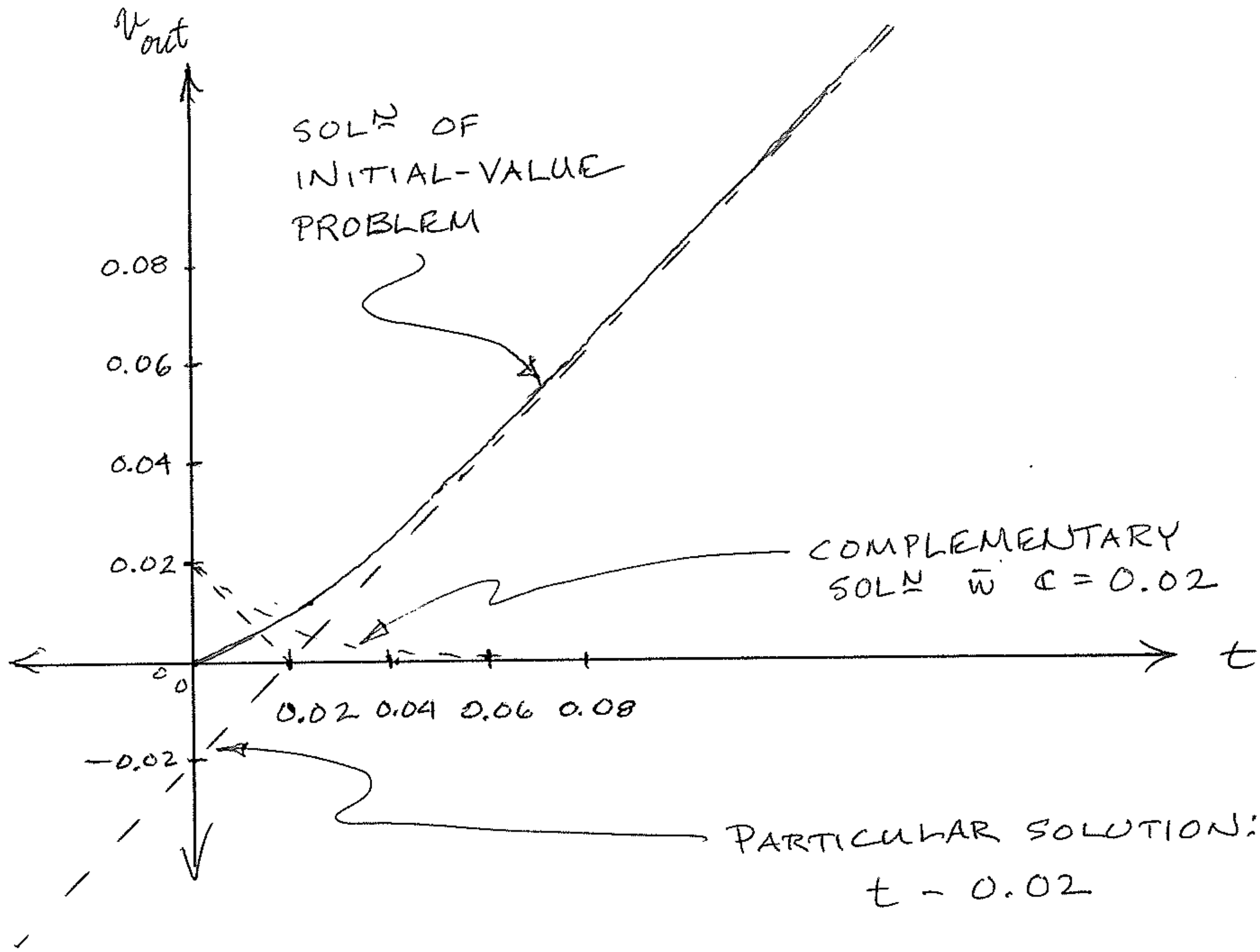
$$v_{out}(t) = c\, e^{-50t} - 0.02 + t$$

so

$$v_{out}(0) = c - 0.02$$

The unique solution of the initial-value problem is therefore

$$v_{out}(t) = 0.02\, e^{-50t} - 0.02 + t$$


v_{out}
SOL$^{\underline{N}}$ OF INITIAL-VALUE PROBLEM
0.08
0.06
0.04
0.02
0
-0.02
0.02 0.04 0.06 0.08
t
COMPLEMENTARY SOL$^{\underline{N}}$ $\bar{w}$ C = 0.02
PARTICULAR SOLUTION: t - 0.02