Module 7: Dictionaries for Multi-Dimensional Data

CS 240 - Data Structures and Data Management

Romain Lebreton
Lectures notes by Arne Storjohann
Based on lecture notes by R. Dorrigiv and D. Roche

David R. Cheriton School of Computer Science, University of Waterloo

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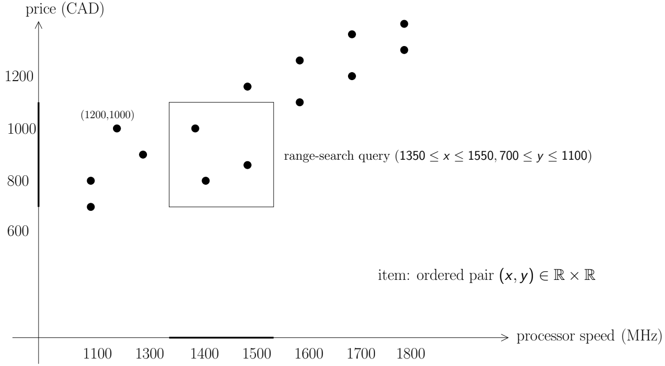
Multi-Dimensional Data

- Various applications
  - Attributes of a product (laptop: price, screen size, processor speed, RAM, hard drive,$\cdots$)
  - Attributes of an employee (name, age, salary,$\cdots$)
- Dictionary for multi-dimensional data
  A collection of d-dimensional items
  Each item has d aspects (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
  Operations: insert, delete, range-search query
- (Orthogonal) Range-search query: specify a range (interval) for certain aspects, and find all the items whose aspects fall within given ranges.
  Example: laptops with screen size between 11 and 13 inches, RAM between 8 and 16 GB, price between 1,500 and 2,000 CAD

Multi-Dimensional Data

- Each item has d aspects (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
- Aspect values (x_i) are numbers
- Each item corresponds to a point in d-dimensional space
- We concentrate on $d = 2$, i.e., points in Euclidean plane



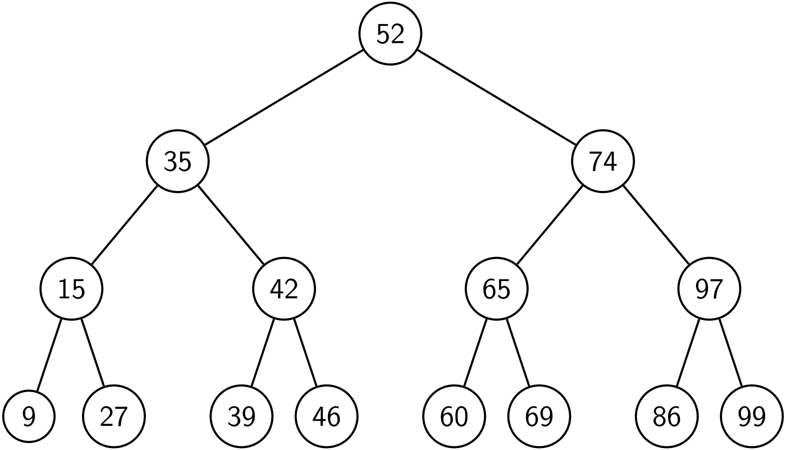
One-Dimensional Range Search

- First solution: ordered arrays
  - Running time: $O(\log n + k)$, k: number of reported items
  - Problem: does not generalize to higher dimensions
- Second solution: balanced BST (e.g., AVL tree)

```
BST-RangeSearch(T, k1, k2)
T: A balanced search tree, k1, k2: search keys
Report keys in T that are in range [k1, k2]
    if T = nil then return
    if key(T) < k1 then
        BST-RangeSearch(T.right, k1, k2)
    if key(T) > k2 then
        BST-RangeSearch(T.left, k1, k2)
    if k1 ≤ key(T) ≤ k2 then
        BST-RangeSearch(T.left, k1, k2)
        report key(T)
        BST-RangeSearch(T.right, k1, k2)
```

Range Search example

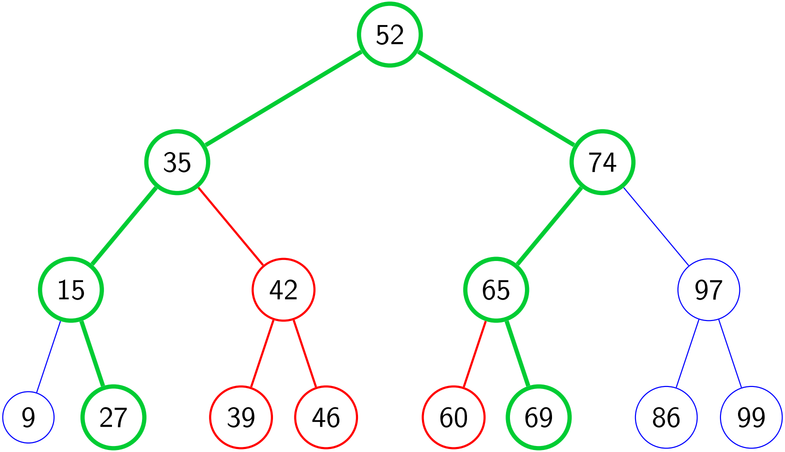
BST-RangeSearch$(T, 30, 65)$



Range Search example

BST-RangeSearch$(T, 30, 65)$

Nodes either on boundary, inside, or outside.



Range Search example

BST-RangeSearch$(T, 30, 65)$

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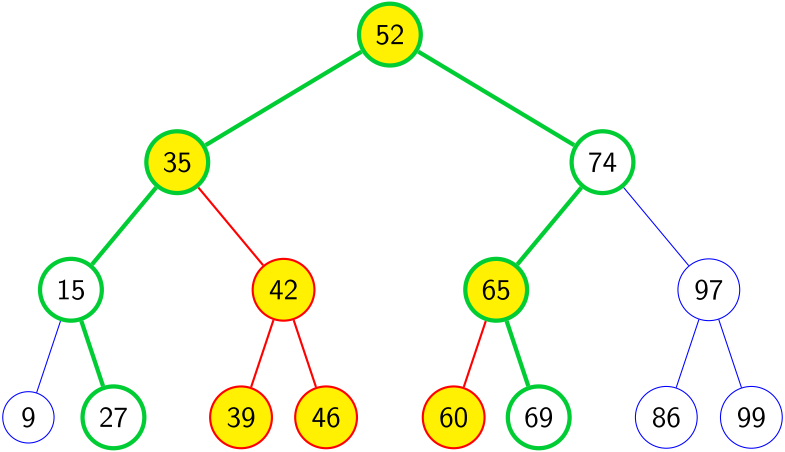


Note: Not every boundary node is returned.

One-Dimensional Range Search

- P_1: path from the root to a leaf that goes right if $k < k_1$ and left otherwise
- P_2: path from the root to a leaf that goes left if $k > k_2$ and right otherwise
- Partition nodes of T into three groups:
  1. boundary nodes: nodes in P_1 or P_2
  2. inside nodes: non-boundary nodes that belong to either (a subtree rooted at a right child of a node of P_1) or (a subtree rooted at a left child of a node of P_2)
  3. outside nodes: non-boundary nodes that belong to either (a subtree rooted at a left child of a node of P_1) or (a subtree rooted at a right child of a node of P_2)

One-Dimensional Range Search

- k: number of reported items
- Nodes visited during the search:
  - $O(\log n)$ boundary nodes
  - $O(k)$ inside nodes
  - No outside nodes
- Running time $O(\log n + k)$

2-Dimensional Range Search

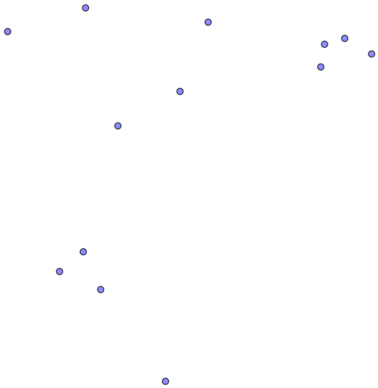
- Each item has 2 aspects (coordinates): (x_i, y_i)
- Each item corresponds to a point in Euclidean plane
- Options for implementing d-dimensional dictionaries:
  - Reduce to one-dimensional dictionary: combine the d-dimensional key into one key
    Problem: Range search on one aspect is not straightforward
  - Use several dictionaries: one for each dimension
    Problem: inefficient, wastes space
  - Partition trees
    - ★ A tree with n leaves, each leaf corresponds to an item
    - ★ Each internal node corresponds to a region
    - ★ quadtrees, kd-trees
  - multi-dimensional range trees

Quadtrees

- We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane
- How to build a quadtree on P:
  - Find a square R that contains all the points of P (We can compute minimum and maximum x and y values among n points)
  - Root of the quadtree corresponds to R
  - Split: Partition R into four equal subsquares (quadrants), each correspond to a child of R
  - Recursively repeat this process for any node that contains more than one point
  - Points on split lines belong to left/bottom side
  - Each leaf stores (at most) one point
  - We can delete a leaf that does not contain any point

Quadtrees

- Example: We have 13 points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{12}, y_{12})\}$ in the plane



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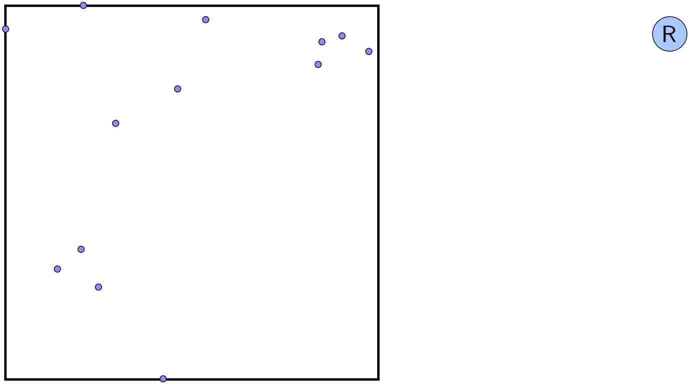

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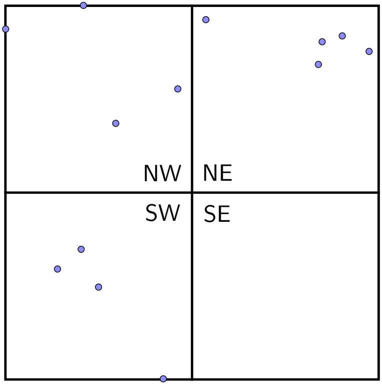

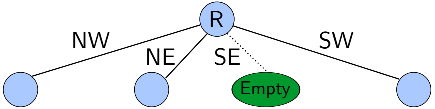

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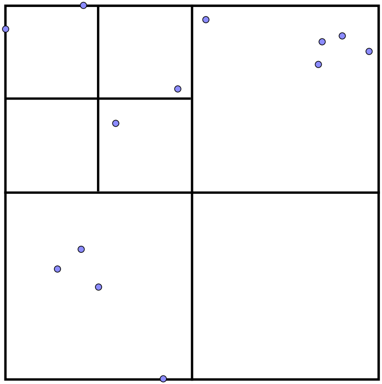

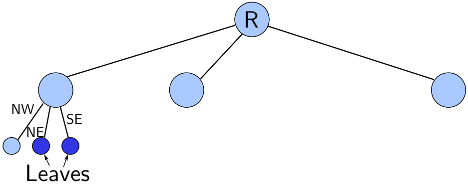

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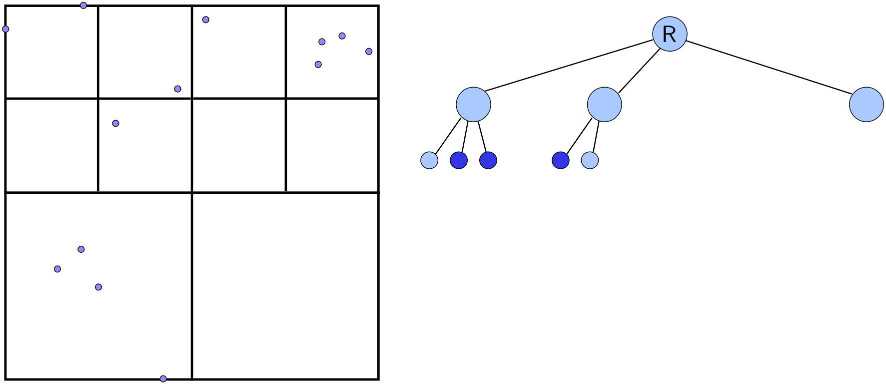

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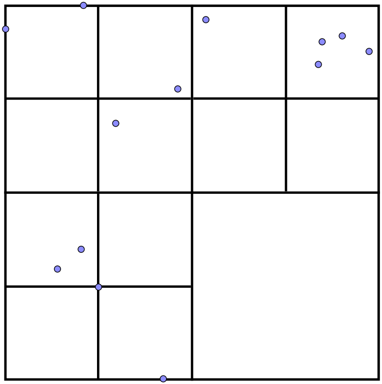

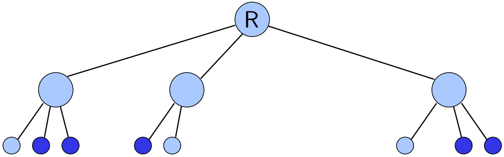

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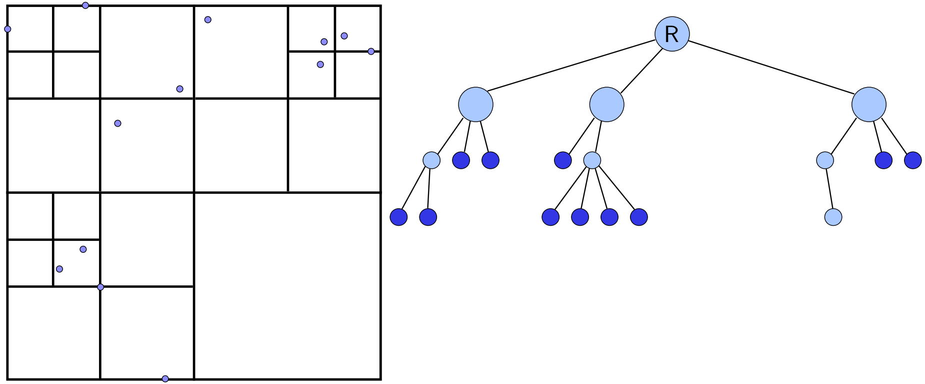

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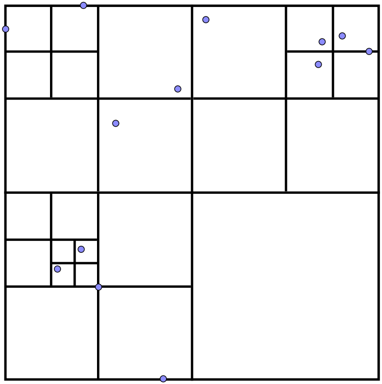
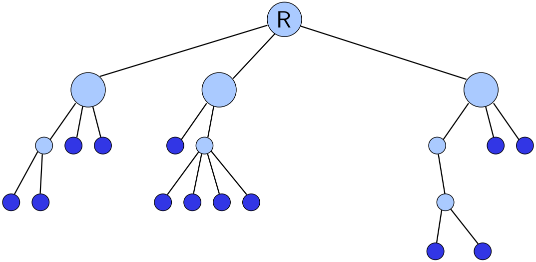




Quadtree Operations

- Search: Analogous to binary search trees
- Insert:
  - Search for the point
  - Split the leaf if there are two points
- Delete:
  - Search for the point
  - Remove the point
  - Walk back up in the tree to discard unnecessary splits

Quadtree: Range Search

QTree-RangeSearch(T, R)
T: A quadtree node, R: Query rectangle

```
if (T is a leaf) then
    if (T.point ∈ R) then
        report T.point
for each child C of T do
    if C.region ∩ R ≠ ∅ then
        QTree-RangeSearch(C, R)
```

- spread factor of points P : $\beta(P) = d_{max}/d_{min}$
- $d_{max}(d_{min})$: maximum (minimum) distance between two points in P
- height of quadtree: $h \in \Theta(\log_2 \frac{d_{max}}{d_{min}})$
- Worst-case complexity to build initial tree $= \Theta(\#nodes) = \Theta(nh)$
- Worst-case complexity of range search $= O(\#nodes) = O(nh)$ even if the answer is $\emptyset$

Quadtree Conclusion

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (usually the boundary box is padded to get a power of two).
- Space wasteful
- Major drawback: can have very large height for certain nonuniform distributions of points
- Easily generalizes to higher dimensions (octrees, *etc.*).

kd-trees

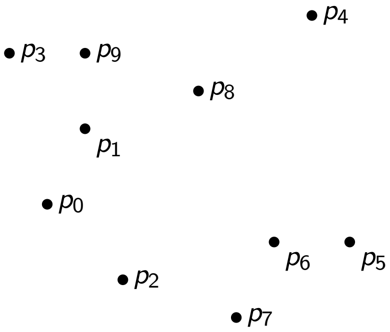
- We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane
- Quadtrees split square into quadrants regardless of where points actually lie
- kd-tree idea: Split the points into two (roughly) equal subsets
- How to build a kd-tree on P:
  - Split P into two equal subsets using a vertical line
  - Split each of the two subsets into two equal pieces using horizontal lines
  - Continue splitting, alternating vertical and horizontal lines, until every point is in a separate region

kd-trees

- We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane
- Quadtrees split square into quadrants regardless of where points actually lie
- kd-tree idea: Split the points into two (roughly) equal subsets
- More details:
  - Initially, we sort the n points according to their x-coordinates.
  - The root of the tree is the point with median x coordinate (index $\lfloor n/2 \rfloor$ in the sorted list)
  - All other points with x coordinate less than or equal to this go into the left subtree; points with larger x-coordinate go in the right subtree.
  - At alternating levels, we sort and split according to y-coordinates instead.
- Complexity: $\Theta(n \log n)$, height of the tree: $\Theta(\log n)$

kd-trees

- kd-tree idea: Split the points into two (roughly) equal subsets
- A balanced binary tree



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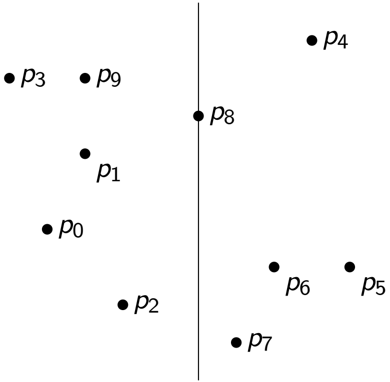

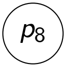

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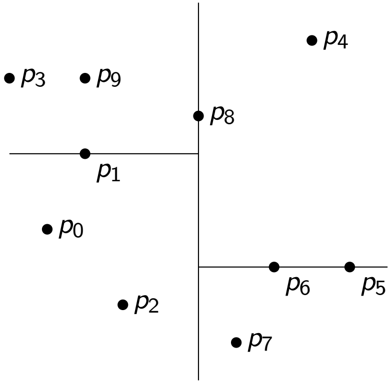

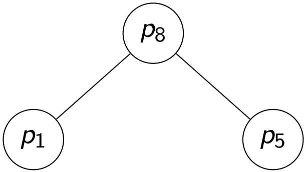

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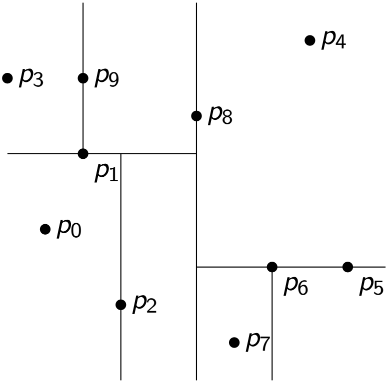

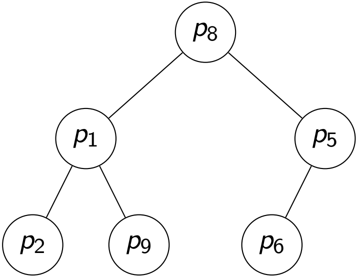

kd-trees

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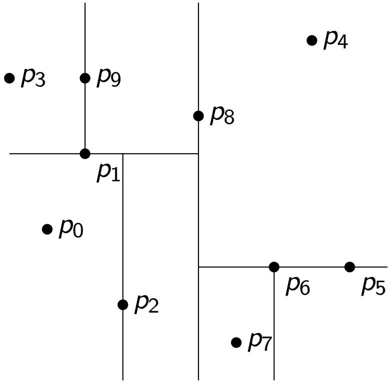
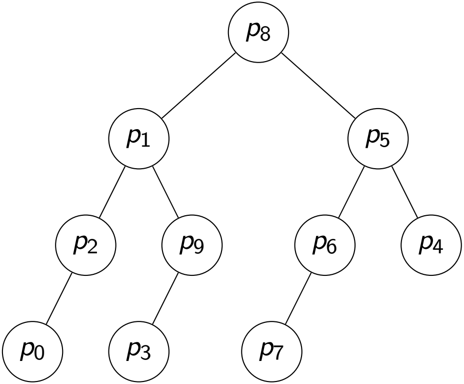




kd-tree: Range Search

```
kd-rangeSearch(T, R)
T: A kd-tree node, R: Query rectangle
    if T is empty then return
    if T.point ∈ R then
        report T.point
    for each child C of T do
        if C.region ∩ R ≠ ∅ then
            kd-rangeSearch(C, R)
```

kd-tree: Range Search

```
kd-rangeSearch(T, R, split[← 'x'])
T: A kd-tree node, R: Query rectangle
    if T is empty then return
    if T.point ∈ R then
        report T.point
    if split = 'x' then
        if T.point.x ≥ R.leftSide then
            kd-rangeSearch(T.left, R, 'y')
        if T.point.x < R.rightSide then
            kd-rangeSearch(T.right, R, 'y')
    if split = 'y' then
        if T.point.y ≥ R.bottomSide then
            kd-rangeSearch(T.left, R, 'x')
        if T.point.y < R.topSide then
            kd-rangeSearch(T.right, R, 'x')
```

kd-tree: Range Search Complexity

- The complexity is $O(k + U)$ where k is the number of keys reported and U is the number of regions we go to but unsuccessfully
- U corresponds to the number of regions which intersect but are not fully in R
- Those regions have to intersect one of the four sides of R
- $Q(n)$: Maximum number of regions in a kd-tree with n points that intersect a vertical (horizontal) line
- $Q(n)$ satisfies the following recurrence relation:

$$Q(n) = 2Q(n/4) + O(1)$$

- It solves to $Q(n) = O(\sqrt{n})$
- Therefore, the complexity of range search in kd-trees is $O(k + \sqrt{n})$

kd-tree: Higher Dimensions

- kd-trees for d-dimensional space
  - At the root the point set is partitioned based on the first coordinate
  - At the children of the root the partition is based on the second coordinate
  - At depth $d-1$ the partition is based on the last coordinate
  - At depth d we start all over again, partitioning on first coordinate
- Storage: $O(n)$
- Construction time: $O(n \log n)$
- Range query time: $O(n^{1-1/d} + k)$

(Note: d is considered to be a constant.)

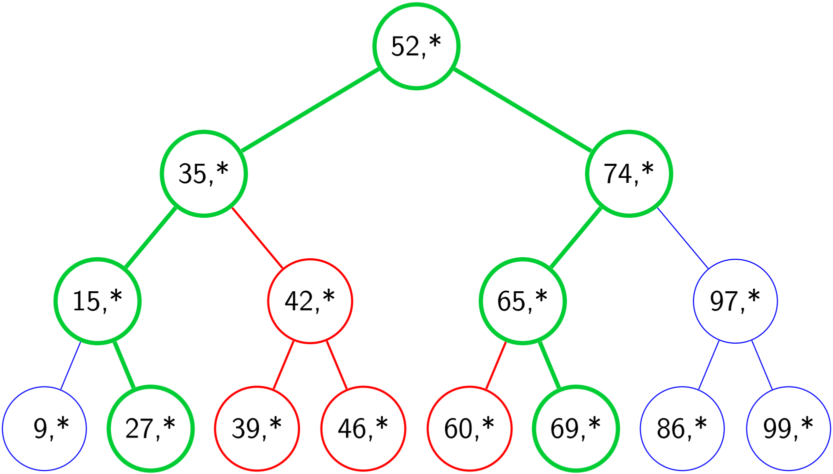
Range Trees

- We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane
- A range tree is a tree of trees (a *multi-level* data structure)
- How to build a range tree on P:
  - Build a balanced binary search tree τ determined by the x-coordinates of the n points
  - For every node $v \in \tau$, build a balanced binary search tree $\tau_{assoc}(v)$ (associated structure of τ) determined by the y-coordinates of the nodes in the subtree of τ with root node v

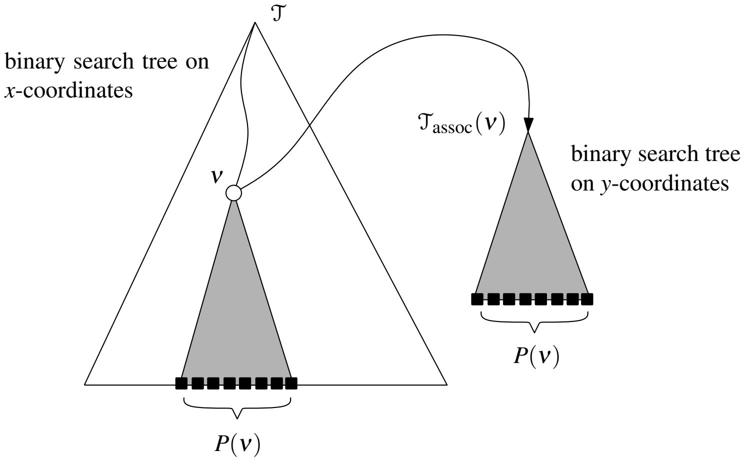
Range Tree Structure

Range search in $[28; 69] \times [y_1; y_2]$

Nodes either on boundary, inside, or outside.



Range Tree Structure



Range Trees: Operations

- Search: trivially as in a binary search tree
- Insert: insert point in τ by x-coordinate
- From inserted leaf, walk back up to the root and insert the point in all associated trees $\tau_{assoc}(v)$ of nodes v on path to the root
- Delete: analogous to insertion
- Note: re-balancing is a problem!

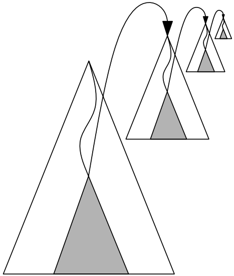
Range Trees: Range Search

- A two stage process
- To perform a range search query $R = [x_1, x_2] \times [y_1, y_2]$:
  - Perform a range search (on the x-coordinates) for the interval $[x_1, x_2]$ in τ (*BST-RangeSearch*(τ, x_1, x_2))
  - For every outside node, do nothing.
  - For every "top" inside node v, perform a range search (on the y-coordinates) for the interval $[y_1, y_2]$ in $\tau_{assoc}(v)$. During the range search of $\tau_{assoc}(v)$, do not check any x-coordinates (they are all within range).
  - For every boundary node, test to see if the corresponding point is within the region R.
- Running time: $O(k + \log^2 n)$
- Range tree construction time: $O(n \log n)$
- Range tree space usage: $O(n \log n)$

Range Trees: Higher Dimensions

- Range trees for d-dimensional space
  - Storage: $O(n\,(\log n)^{d-1})$
  - Construction time: $O(n\,(\log n)^{d-1})$
  - Range query time: $O((\log n)^{d} + k)$

(Note: d is considered to be a constant.)



Range Trees: Higher Dimensions

- Space/time trade-off
  - Storage: $O(n\,(\log n)^{d-1})$ kd-trees: $O(n)$
  - Construction time: $O(n\,(\log n)^{d-1})$ kd-trees: $O(n \log n)$
  - Range query time: $O((\log n)^d + k)$ kd-trees: $O(n^{1-1/d} + k)$

(Note: d is considered to be a constant.)