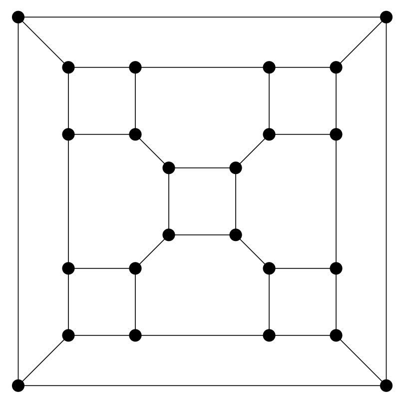
Math 239 Spring 2014 Assignment 9 Solutions

1. {5 marks} Let G be a 3-regular connected planar graph with a planar embedding where each face has degree either 4 or 6. In addition, each vertex is incident with exactly one face of degree 4. Determine the number of vertices, edges, faces of degree 4, and faces of degree 6 in this embedding. Draw a planar embedding of G.

   Solution. Suppose G has n vertices, m edges, f_4 faces of degree 4, and f_6 faces of degree 6. By the handshaking lemma, $2m = 3n$. By the handshaking lemma for faces, $2m = 4f_4 + 6f_6$. By Euler's formula, $n - m + f_4 + f_6 = 2$. Since each vertex is incident with exactly one face of degree 4, the number of vertices is 4 times the number of faces of degree 4, hence $n = 4f_4$. By solving the four equations, we get

$$n = 24, m = 36, f_4 = 6, f_6 = 8.$$

   One possible embedding is the following. (This is the truncated octahedron.)



2. {4 marks} Prove that any planar embedding of a simple connected planar graph contains a vertex of degree at most 3 or a face of degree at most 3.

   Solution. Suppose by way of contradiction that there is a planar embedding of G where every vertex has degree at least 4 and every face has degree at least 4. Suppose G has n vertices, m edges and s faces. By the handshaking lemma, $2m \geq 4n$, so $n \leq m/2$. By the handshaking lemma for faces, $2m \geq 4s$, so $s \leq m/2$. By Euler's formula,

$$2 = n - m + s \leq m/2 - m + m/2 = 0.$$

   This is a contradiction.

3. {4 marks} Let G be a connected planar graph with a planar embedding where every face boundary is a cycle of even length. Prove that G is bipartite. (Hint: Consider any cycle. Count the degrees of the faces inside this cycle.)

   Solution. Let C be any cycle in G. Let $F_1, \ldots, F_k$ be the faces inside C in the planar embedding. Consider the sum of the face degrees of these k faces. Each edge of C is being counted once, the side that is inside C. Let D be the set of edges that are in some boundary of some F_i, but are not on the cycle C itself. The edges in D are being counted twice, once on each side. So

$$\sum_{i=1}^{k} \deg(F_i) = |E(C)| + 2|D|.$$

   By assumption, every face boundary is a cycle of even length, so $\deg(F_i)$ is even for each i. Since $2|D|$ is also even, it must be the case that $|E(C)|$ is even. So C is an even cycle. Since every cycle in G has even length, G is bipartite.

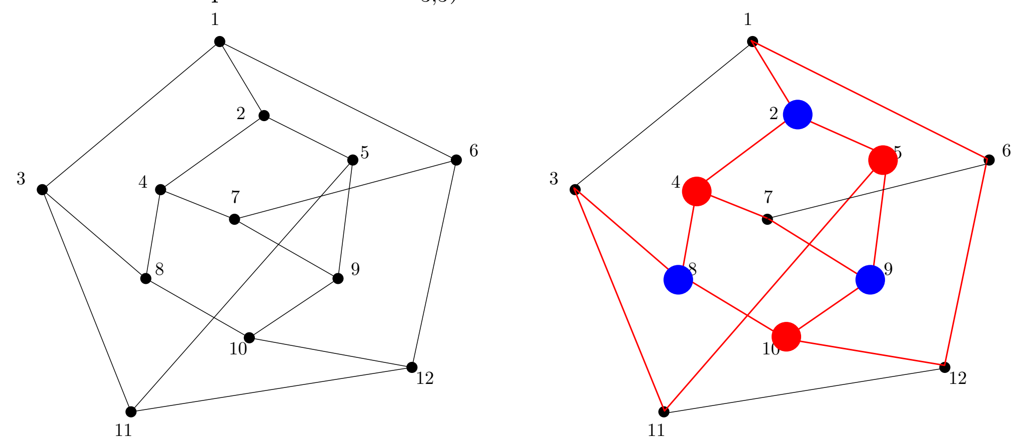
   Alternate solution. Suppose by way of contradiction that there exists an odd cycle C. We pick one with the fewest number of faces inside C. Since every face boundary is a cycle of even length, there must be at least 2 faces inside C. This means that in order to separate one face from another, there must exist a path P between two vertices of C that is in the interior of C. Let u, v be the two end vertices of P, which are on C. Then u and

v partitions the cycle into two paths Q_1 and Q_2. Since $P + Q_1$ and $P + Q_2$ are cycles that contain fewer faces than C, so they must be of even length. So $|E(P)| + |E(Q_1)|$ and $|E(P)| + |E(Q_2)|$ are both even, and their sum $2|E(P)| + |E(Q_1)| + |E(Q_2)|$ is also even. Then $|E(Q_1)| + |E(Q_2)|$ is even. But this is precisely the length of C, which is odd. This is a contradiction, hence no odd cycle exists in G, and G is bipartite.

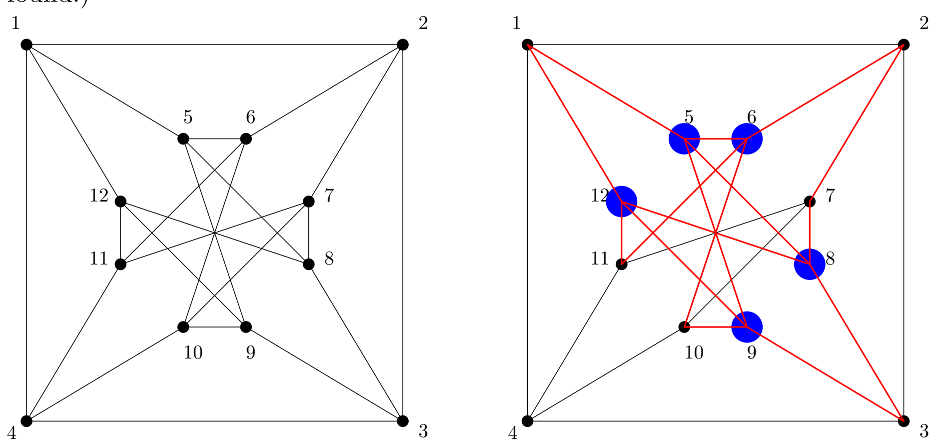
4. {9 marks} For each of the following graphs, determine whether or not it is planar. Prove your assertions.

Solution.

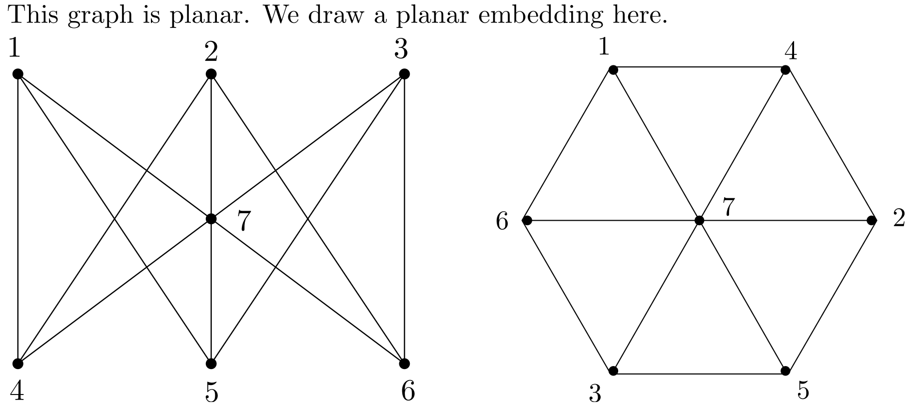
(a) This graph is not planar. We find an edge subdivision of $K_{3,3}$ here (the red and blue vertices represent the vertices in the bipartition of the $K_{3,3}$).



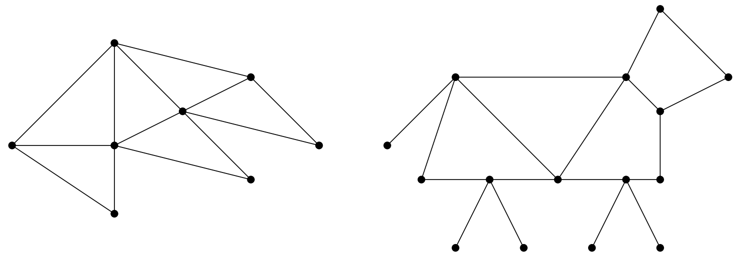
(b) This graph is not planar. We find an edge subdivision of K_5 here. (Edge subdivisions of $K_{3,3}$ can also be found.)



(c) This graph is planar. We draw a planar embedding here.



5. A graph is *outerplanar* if it has a planar embedding where every vertex lies on the unbounded face. Two examples of outerplanar graphs are drawn below. An example of a planar graph that is not outerplanar is K_4.



(a) {3 marks} Prove that a connected outerplanar graph on $n \geq 2$ vertices has at most $2n - 3$ edges. (Hint: Use Euler's formula. What is the minimum degree of the unbounded face?)

Solution. Let G be a connected outerplanar graph with n vertices, m edges and s faces. If G has no cycles, then $m = n - 1$, and $n - 1 \leq 2n - 3$ when $n \geq 2$. Suppose G has some cycles. Then every face has degree at least 3. Since the unbounded face contains all vertices, its boundary walk must contain all n vertices, hence it has degree at least n. So the sum of the face degrees is at least $n + 3(s - 1)$. By the handshaking lemma for faces and Euler's formula, we get

$$2m \geq n + 3(s - 1) = n + 3(2 - n + m - 1) = 3 - 2n + 3m.$$

Therefore, $m \leq 2n - 3$.

(b) {3 marks} Use Kuratowski's Theorem to prove that a graph is outerplanar if and only if it does not have an edge subdivision of K_4 or $K_{2,3}$ as a subgraph. (Hint: Modify the graph in a way so that you can apply Kuratowski's Theorem.)

Solution. Let G be any graph, and let G' be the graph obtained from G by adding a new vertex v^* and add an edge between v^* and every vertex in G. First we claim tha G is outerplanar if and only if G' is planar. If G is outerplanar, then it has an embedding with every vertex on the unbounded face. Place v^* on the unbounded face, and we can draw the extra edges without crossing. So G' is planar. On the other hand, if G' is planar, then there is a drawing of G' with v^* on the unbounded face. By removing v^* and its incident edges, the unbounded face now contains all vertices in G, hence G is outerplanar.

By Kuratowski's theorem, G' is planar if and only if G' does not have an edge subdivision of K_5 or $K_{3,3}$. We now claim that G' has an edge subdivision of K_5 or $K_{3,3}$ if and only if G has an edge subdivision of K_4 or $K_{2,3}$. If G' has an edge subdivision of K_5 or $K_{3,3}$, then this edge subdivision may (or may not) include v^*. By removing v^*, we get an edge subdivision of K_4 or $K_{2,3}$ in G. On the other hand, if G contains an edge subdivision of K_4 or $K_{2,3}$, then by adding v^* to the edge subdivision, we create an edge subdivision of K_5 or $K_{3,3}$.

By combining all the claims, we get G is outerplanar if and only if G does not have an edges subdivision of K_4 or $K_{2,3}$.