Math 239 Spring 2014 Assignment 10 Solutions

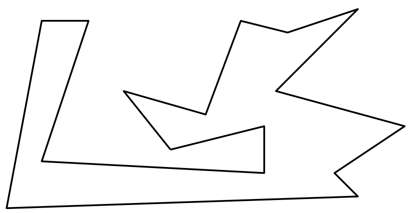
1. Recall from assignment 9 that a graph is *outerplanar* if it has a planar embedding where every vertex lies on the unbounded face.

   (a) {4 marks} Prove that every outerplanar graph is 3-colourable. For this question, you may assume (without proof) that every outerplanar graph has a vertex of degree at most 2.

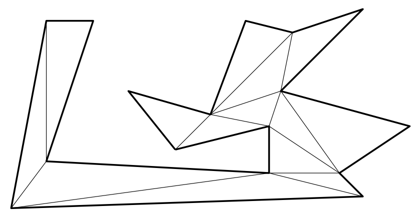
   Solution. We prove by induction on the number of vertices n. When $n = 1$, the only outerplanar graph is the isolated vertex, which is 3-colourable. We now assume that any outerplanar graph on $n - 1$ vertices is 3-colourable.

   Let G be an outerplanar graph with n vertices. Let v be a vertex of degree at most 2. Obtain $G - v$ by removing v and the incident edges. After removing v, every vertex is still on the unbounded face, so $G - v$ is outerplanar. In addition, $G - v$ has $n - 1$ vertices. By the induction hypothesis, $G - v$ is 3-colourable. We keep the same colours for G, and assign to v an unusued colour among its neighbours. This is possible since v has at most 2 neighbours and there are 3 colours available. Hence G is 3-colourable.

   (b) {3 marks} Martin is setting up a shop selling a combination of Peintings, Peigers and Peinkillers. Unlike some retailers, we accept PeiPal. The Peirimeter of the shop floor is shaped as a simple polygon with n sides. To catch non-Peiing customers, Martin decided to install some surveillance cameras in the shop. Prove that it is possible to place at most $\lfloor n/3 \rfloor$ cameras such that every point inside the shop can be tracked by some camera. (Assume that any camera has a 360-degree Peinoramic view. You may also assume that any simple polygon has a triangulation of its interior, as shown in an example below.)



A polygon



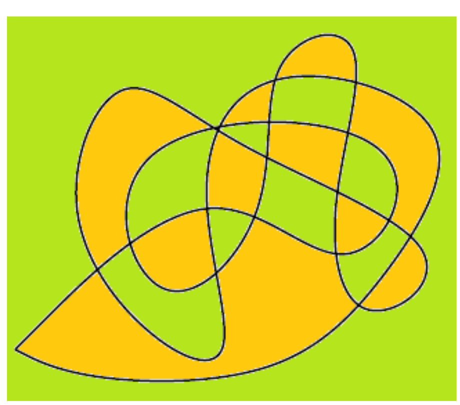
A triangulation of the polygon

   Solution. We may triangulate the polygon and obtain an outerplanar graph. By part (a), there is a 3-colouring of the vertices. Since there are n vertices each with one of 3 colours, there exists one colour (say red) that is used at most $\lfloor n/3 \rfloor$ times by pigeonhole principle. We have triangulated the shop floor, and each triangle must use all three colours for its three vertices. So each triangle contains a vertex coloured red. Since a triangle is convex, the red vertex can "see" the area inside the triangle. Therefore, by placing the cameras on the red vertices, we can see all the triangles, hence the entire shop can be tracked with at most $\lfloor n/3 \rfloor$ cameras.

2. {4 marks} Let G be a simple connected planar graph with at least 2 vertices, and let G^* be the dual of a planar embedding of G. Prove that if G is isomorphic to G^*, then G is not bipartite.

   Solution. Suppose G has n vertices, m edges and s faces. Then G^* has s vertices, m edges and n faces. Since G and G^* are isomorphic, they have the same number of vertices, so $n = s$. Using Euler's formula, $n - m + s = 2n - m = 2$, so $m = 2n - 2$. But any planar bipartite graph has at most $2n - 4$ edges, so G cannot be bipartite.

3. {4 marks} Consider any closed curve on the plane that does not repeat any segments, but possibly crossing itself at several points. Prove that the faces are 2-colourable. An example is shown below. (Hint: Use question 3 from assignment 9.)

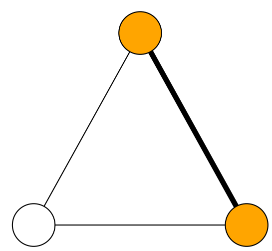


Solution. We may obtain a graph G from the embedding of the closed curve on the plane by having vertices at intersections of the curve. Since this is a closed curve, G is Eulerian. Therefore, every vertex has even degree. Consider the dual G^* of G. Each vertex of G corresponds to a face in G^* with the same degree. So every face boundary of G^* is an even cycle. Using question 3 from assignment 9, G^* must be bipartite, so the vertices are 2-colourable. Using the vertex colour of G^* as the face colour of G, we obtain a 2-colouring of the faces of G.

4. {4 marks} For each of the following parts, draw a non-bipartite graph that satisfy the conditions. Show the relevant matchings and covers.

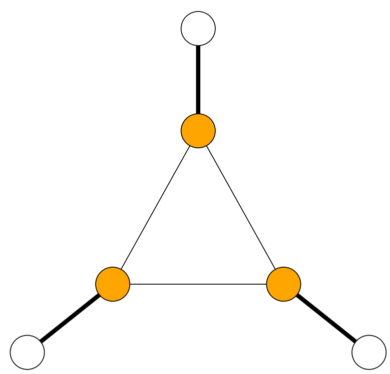
   (a) The size of a maximum matching is strictly less than the size of a minimum cover.

   Solution. In the following graph, a maximum matching has size 1 (since there are only 3 vertices), but a minimum cover cannot have size 1 since each vertex covers only 2 edges. A minimum cover of size 2 is illustrated.

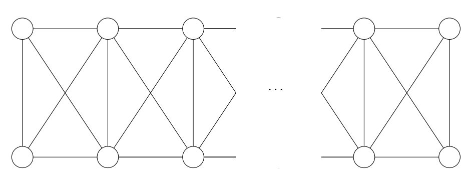


   (b) The size of a maximum matching is equal to the size of a minimum cover.

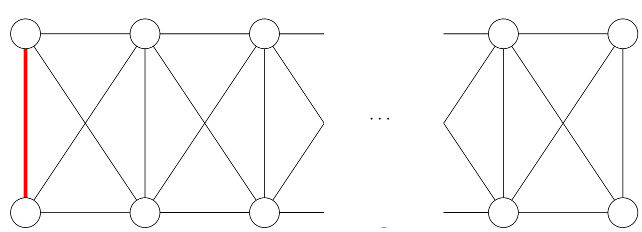
   Solution. In the following graph, the matching and the vertex cover have the same size.



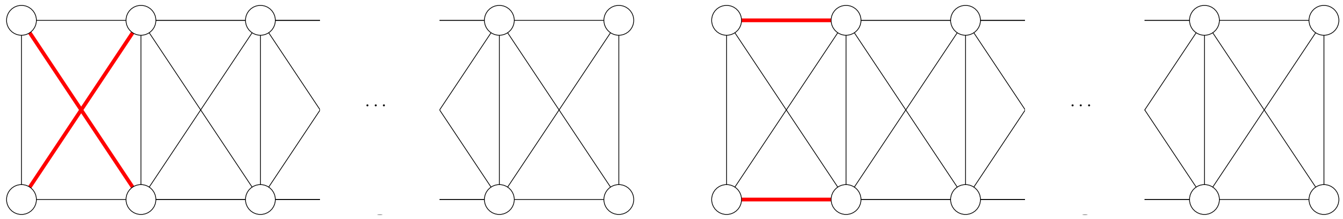
5. {4 marks} For each $n \in \mathbb{N}$, let L_n be the graph with $2n$ vertices shown below. Determine the number of perfect matchings in L_n. (Hint: Let a_n be this number, and derive a recurrence relation for a_n.)



Solution. In a perfect matching of L_n, there are 3 possible edges that the top left vertex is matched to. If it is matched to the vertex right below it (illustrated below), then the remainder of the matching is a perfect matching of L_{n-1}. So there are a_{n-1} possible perfect matchings in this case.



If the top left vertex is matched to one of the other two neighbours, then this gives only 1 choice for matched neighbour of the bottom left vertex (illustrated below). Then the remainder of the matching is a perfect matching of L_{n-2}. So there are $2a_{n-2}$ possible perfect matchings in this case.



The recurrence is then $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$, with initial conditions $a_1 = 1, a_2 = 3$. Solving this recurrence gives $a_n = \frac{2}{3} \cdot 2^n + \frac{1}{3}(-1)^n$.

6. $\{4$ marks$\}$ Two people play a game on a graph G by alternately selecting distinct vertices $v_1, v_2, \ldots$ forming a path. The last player who is able to select a vertex wins. Suppose G has no perfect matchings. Describe a winning strategy for the first player, and explain why this strategy works.

   Solution. Let M be a maximum matching. Suppose the vertices that the players select are $v_1, w_1, v_2, w_2, v_3, w_3, \ldots$ (where the v_i's are chosen by player 1 and w_i's are chosen by player 2). The aim of player 1 is to choose vertices so that this path is an alternating path with respect to M. Player 1 will select an unsaturated vertex for v_1 (one exists since G does not have a perfect matching). Regardless of player 2's choice for w_1, it must be saturated (for otherwise M is not maximum). So player 1 can now choose the other end of the matching edge containing w_1 to be v_2. And now regardless of player 2's choice for w_2, player 1 can choose the other end of the matching edge for v_3. This means that no matter what player 2 chooses, player 1 can always continue the game, so player 1 will win.