MATH 213

ASSIGNMENT No. 3

- SOLUTIONS -

1 a) Try $y(t) = k e^{-0.10t}$

$\Rightarrow \dot{y}(t) = -0.10 k e^{-0.10t}$

Substituting,

$$9.90 k e^{-0.10t} = 10 e^{-0.10t}$$

$$\Leftrightarrow k = 1.01$$

So a particular solution is

$$y(t) = 1.01 e^{-0.10t}$$

b) Here we might try $y(t) = k e^{-3t}$, but it wouldn't work, as that function solves the auxiliary equation.

$\longrightarrow$ Multiply by the smallest power of t that gives a function that doesn't solve the auxiliary equation:

$$y(t) = k t e^{-3t}$$
$$\dot{y}(t) = k(1-3t)e^{-3t}$$

$$\ddot{y}(t) = k(-3)(2-3t)e^{-3t}$$

$$y^{(3)}(t) = k(9)(3-3t)e^{-3t}$$

LHS of equation is

$$(D+3)(D^2+9D+20)\,y(t)$$

$$= (D^3 + 12D^2 + 47D + 60)\,y(t)$$

So substitution of our candidate solution into the diff. eqn. yields

$$9k(3-3t) - 36k(2-3t) + 47k(1-3t) + 60kt = 5$$

$$\Longleftrightarrow \quad 2k = 5$$

$$\Longleftrightarrow \quad k = 5/2$$

So $y(t) = \frac{5}{2} t e^{-3t}$

is a particular solution.

So a particular solution is

$$y(t) = \frac{5}{6} e^{-3t}$$

d) First look for a particular sol^{n} of

$$\ddot{y} + 4y = e^{-2t}$$

$$\Rightarrow \quad y = k e^{-2t}$$
$$\dot{y} = -2 k e^{-2t}$$
$$\ddot{y} = 4 k e^{-2t}$$

- substituting,

$$8k = 1$$
$$\Leftrightarrow \quad k = 1/8$$

So $y(t) = \frac{1}{8} e^{-2t}$ will do.

Now find a particular sol^{n} of

$$\ddot{y} + 4y = t$$

$\Rightarrow$ try $y = k_1 + k_2 t$

$\Rightarrow \dot{y} = k_2, \quad \ddot{y} = 0$

Substituting, we get

$$4(k_1 + k_2 t) = t$$

$$\Leftrightarrow \quad k_1 = 0 \ \& \ k_2 = \frac{1}{4}$$

So $y(t) = \frac{1}{4} t$ is a particular sol$^{\underline{n}}$.

Now simply add the above particular solutions to find a particular solution of the original linear equation:

$$y(t) = \frac{1}{8} e^{-2t} + \frac{1}{4} t$$

e) Here we proceed as in d), but we need to note that e^{-2t} solves the auxiliary equation.

→ Try substituting $y = k t e^{-2t}$

into

$$(D+2)(D^2+9)\, y(t) = e^{-2t}$$

$$\Leftrightarrow \quad (D^3 + 2D^2 + 9D + 18)\, y(t) = e^{-2t}$$

$$\dot{y} = k\,(1-2t)\, e^{-2t}$$
$$\ddot{y} = k\,(-4+4t)\, e^{-2t}$$
$$y^{(3)} = k\,(12-8t)\, e^{-2t}$$

Substituting,

$$k\left[\begin{array}{l} 12 - 8t \\ -8 + 8t \\ +9 - 18t \\ \quad +18t \end{array}\right] e^{-2t} = e^{-2t}$$

$$\Leftrightarrow \quad 13k = 1$$

$$\Leftrightarrow \quad k = \frac{1}{13}, \quad \text{i.e. } y(t) = \frac{1}{13} t e^{-2t}$$

Now substitute $y = k_1 + k_2 t$

into

$$(D^3 + 2D^2 + 9D + 18)\,y(t) = t$$

$$\Rightarrow \quad (18k_1 + 9k_2) + 18k_2 t = t$$

$$\Leftrightarrow \quad k_2 = \frac{1}{18}, \quad k_1 = -\frac{1}{36}$$

$$\Rightarrow \quad y(t) = -\frac{1}{36} + \frac{1}{18}t$$

Now add these two particular solutions to get a particular solution of the original linear equation

$$y(t) = \frac{1}{13} t e^{-2t} - \frac{1}{36} + \frac{1}{18} t .$$

f) First solve

$$(D^4 + 4D^3 + 13D^2 + 36D + 36)(kt^2 e^{-2t}) = e^{-2t}$$

$$y = kt^2 e^{-2t}$$

$$\Rightarrow \dot{y} = k(2t - 2t^2)e^{-2t},$$

$$\ddot{y} = k(2 - 8t + 4t^2)e^{-2t},$$

$$y^{(3)} = k(-12 + 24t - 8t^2)e^{-2t},$$

$$y^{(4)} = k(48 - 64t + 16t^2)e^{-2t}.$$

Substituting,

$$\begin{aligned}[48 &- 64t + 16t^2 \\ -48 &+ 96t - 32t^2 \\ +26 &- 104t + 52t^2 \\ &+ 72t - 72t^2]ke^{-2t} = e^{-2t}\end{aligned}$$

$$\Longleftrightarrow \quad 26k = 1$$

$$\Longleftrightarrow \quad k = \frac{1}{26}$$

$$\therefore \quad y(t) = \frac{1}{26} t^2 e^{-2t}$$

is a particular solution.

then solve

$$(D^4 + 4D^3 + 13D^2 + 36D + 36)\, y = t$$

Try $\quad y = k_1 + k_2 t$

$$\Rightarrow \dot{y} = k_2,$$
$$\ddot{y} = 0$$

Substituting,

$$36k_2 + 36k_1 + 36k_2 t = 1$$

$$\Longleftrightarrow \quad k_2 = \frac{1}{36}, \quad k_2 = \frac{-1}{36}$$

So $\quad y = \frac{1}{36}(t-1)$

is a particular sol$\underline{n}$s.

Now add the above two sol$\underline{n}$s to get a particular solution of the original linear ODE:

$$y(t) = \frac{1}{26} t^2 e^{-2t} + \frac{1}{36}(t-1).$$

g) Try $y(t) = k_1 + k_2 t + k_3 t^2$

$$\Rightarrow \dot{y}(t) = k_2 + 2k_3 t$$

$$\ddot{y}(t) = 2k_3$$

Substitute

$$(D^4 + 18D^2 + 81)\, y(t) = 1 - 2t^2$$

$$\Longleftrightarrow \quad 36k_3 + 81k_1 + 81k_2 t + 81k_3 t^2 = 1 - 2t^2$$

$$\Longleftrightarrow \quad k_3 = \frac{-2}{81},$$

$$k_2 = 0$$

$$81k_1 = 1 - 36k_3 = 1 + \frac{72}{81} = \frac{153}{81}$$

$$\Rightarrow k_1 = \frac{17}{729}$$

So a particular solution is

$$y(t) = \frac{17}{729}(1 - 18t^2)$$

2. Characteristic equation:

$$m^2 + 2\zeta\omega_n m + \omega_n^2 = 0$$

roots:

$$m = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Now the 'forcing term' is

$$\frac{B}{M}\dot{x} + \frac{K}{M}x = \left(j\frac{10}{\sqrt{2}}\frac{B}{M} + \frac{K}{M}0.1\right)e^{j\frac{100}{\sqrt{2}}t}$$

- This doesn't solve the auxiliary equation in any of the three cases.
- Therefore, seek a particular solution of the form

$$y(t) = k e^{j\frac{100}{\sqrt{2}}t}$$

$$\Rightarrow \dot{y}(t) = j\frac{100}{\sqrt{2}} k e^{j\frac{100}{\sqrt{2}}t}$$

$$\Rightarrow \ddot{y}(t) = -5000 k e^{j\frac{100}{\sqrt{2}}t}$$

b) $\frac{K}{M} = \omega_n^2 = 10000 \quad (\text{rad}/s)^2$

$\frac{B}{M} = 2\zeta\omega_n = \sqrt{2} \times 100 \quad \text{rad}/s$

So
$$k = \frac{1}{10} \frac{(10000 + j10000)}{(5000 + j\ 10000)}$$

$$= \frac{1}{10} \frac{10000\sqrt{2}\ e^{j45°}}{11180\ e^{j63.4°}}$$

$$= 0.089\ e^{-j18.4°}$$

So in this case, a particular solution is

$$y(t) = 0.089\ e^{j(70.7t - 18.4°)}$$

c) $\frac{B}{M} = 2\zeta\omega_n = 100$ rad/s

$$\Rightarrow \quad k = \frac{1}{10} \frac{\left(10000 + j\frac{10000}{\sqrt{2}}\right)}{5000 + j\frac{10000}{\sqrt{2}}}$$

$$= \frac{1}{10} \frac{12250\, e^{j35.3^\circ}}{8660\, e^{j54.7^\circ}}$$

$$= 0.141\, e^{-j19.4^\circ}$$

So, in this case, a particular solution is

$$y(t) = 0.141\, e^{j(70.7t - 19.4^\circ)}$$

3. $$\frac{dy}{dx} = 2y + 3x^2$$

$$\Longleftrightarrow \quad \frac{dy}{dx} - 2y = 3x^2$$

$\Longrightarrow$ complementary solution is

$$y = c\, e^{2x}$$

For a particular solution, set

$$y = k_1 + k_2 x + k_3 x^2$$

$$\Longrightarrow \frac{dy}{dx} = k_2 + 2k_3 x$$

Substituting,

$$k_2 + 2k_3 x - 2(k_1 + k_2 x + k_3 x^2) = 3x^2$$

$$\Longleftrightarrow \quad (k_2 - 2k_1) + (2k_3 - 2k_2)x - 2k_3 x^2 = 3x^2$$

$$\Longleftrightarrow \quad k_3 = -\frac{3}{2}$$

$$\& \quad k_2 = \frac{3}{2}$$

$$\& \quad k_1 = \frac{3}{4}$$

So a particular solution is

$$y = \frac{3}{4} + \frac{3}{2}x - \frac{3}{2}x^2$$

The general solution is

$$y = C e^{2x} + \frac{3}{4} + \frac{3}{2}x - \frac{3}{2}x^2$$

substituting the initial condition

$y(0) = 0$:

$$0 = C + \frac{3}{4} \iff C = -\frac{3}{4}$$

Therefore we must have

$$y = \frac{3}{4}(1 + e^{2x}) + \frac{3}{2}x(1-x)$$

4.

$$\frac{dy}{dx} = 2y + 2e^{2x}$$

$$\iff \frac{dy}{dx} - 2y = 2e^{2x}$$

The complementary solution is the same:

$$y = C e^{2x}$$

To find a particular solution, try

$$y = k x e^{2x}$$

(because e^{2x} solves the auxiliary equation). Then,

$$\frac{dy}{dx} = k(1+2x)e^{2x}$$

Substituting,

$$k(1+2x)e^{2x} - 2kxe^{2x} = 2e^{2x}$$

$$\iff k = 2,$$

So the general solution is

$$y = C e^{2x} + 2x e^{2x}$$

"Initial" condⁿ:

$$y(2) = 0$$

$$\Rightarrow \quad 0 = C e^{4} + 4 e^{4}$$

$$\Leftrightarrow \quad C = -4$$

So

$$y = (2x - 4) e^{2x}$$