MATH 213, ASSIGNMENT #1

— SOLUTIONS —

1. Let t be the time, in years, elapsed since the death of the organism, and let x be the percentage of carbon atoms that are C14. Then

$$x(t) = x(0)\, e^{-\lambda t},$$

where

$$e^{-\lambda 5730} = 0.5$$

$$\Longleftrightarrow \quad \lambda = \frac{-1}{5730} \ln 0.5$$

$$= 1.21 \times 10^{-4}$$

So

$$x(t) = 1.0 \times 10^{-10}\, e^{-1.21 \times 10^{-4} t} \, \%.$$

2.

$$i_c(t) + i_R(t) = i(t) = 0$$

$$\Longrightarrow \quad C \frac{d}{dt} v(t) + \frac{1}{R} v(t) = 0$$

$$\Longleftrightarrow \quad \frac{d}{dt} v(t) + \frac{1}{RC} v(t) = 0$$

$$\Longleftrightarrow \quad v(t) = v(0)\, e^{-\frac{1}{RC} t}$$

Therefore

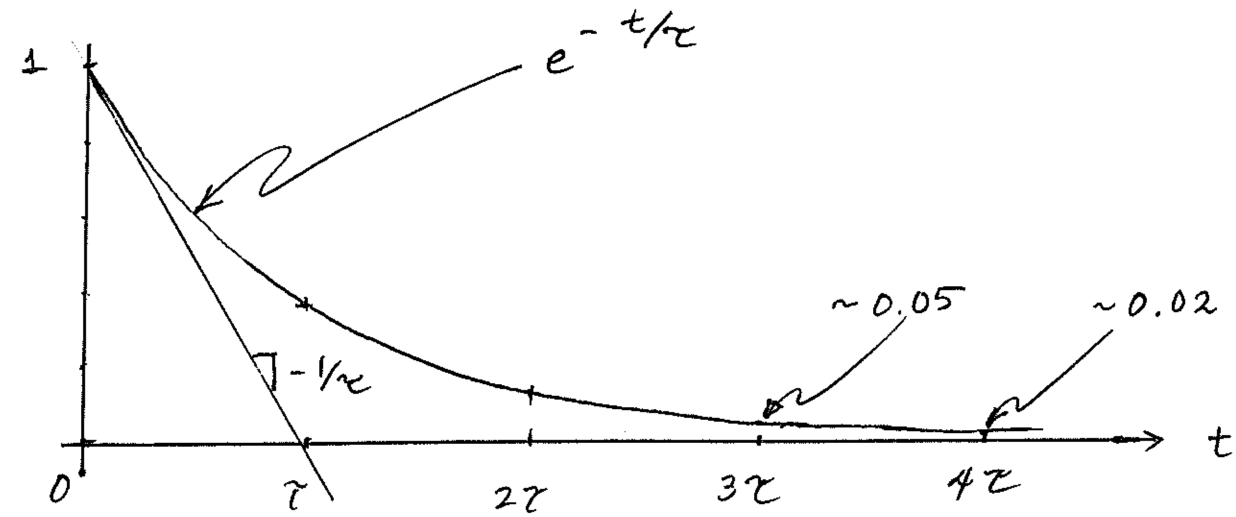
$$v(t) = 5\, e^{-10t} \quad \text{Volts}$$

(where t is in seconds).

3. $\ln(0.05) = -2.996 \sim -3$

& $\ln(0.02) = -3.91 \sim -4$

So, it takes approximately 3 time constants for an exponential to decay to 5% of its initial value, and approximately 4 time constants to decay to 2%.



The derivative of $e^{-t/\tau}$ is $-\frac{1}{\tau} e^{-t/\tau}$.

at $t = 0$, this is $-1/\tau$; it follows that the tangent intersects the t axis at $t = \tau$. See the above sketch.

4a) characteristic equation:

$$m^2 - 7m + 12 = 0$$

roots: $m = 3, 4$

The general solution is therefore

$$y = c_1 e^{3t} + c_2 e^{4t}$$

$$\Rightarrow \frac{dy}{dt} = 3c_1 e^{3t} + 4c_2 e^{4t}$$

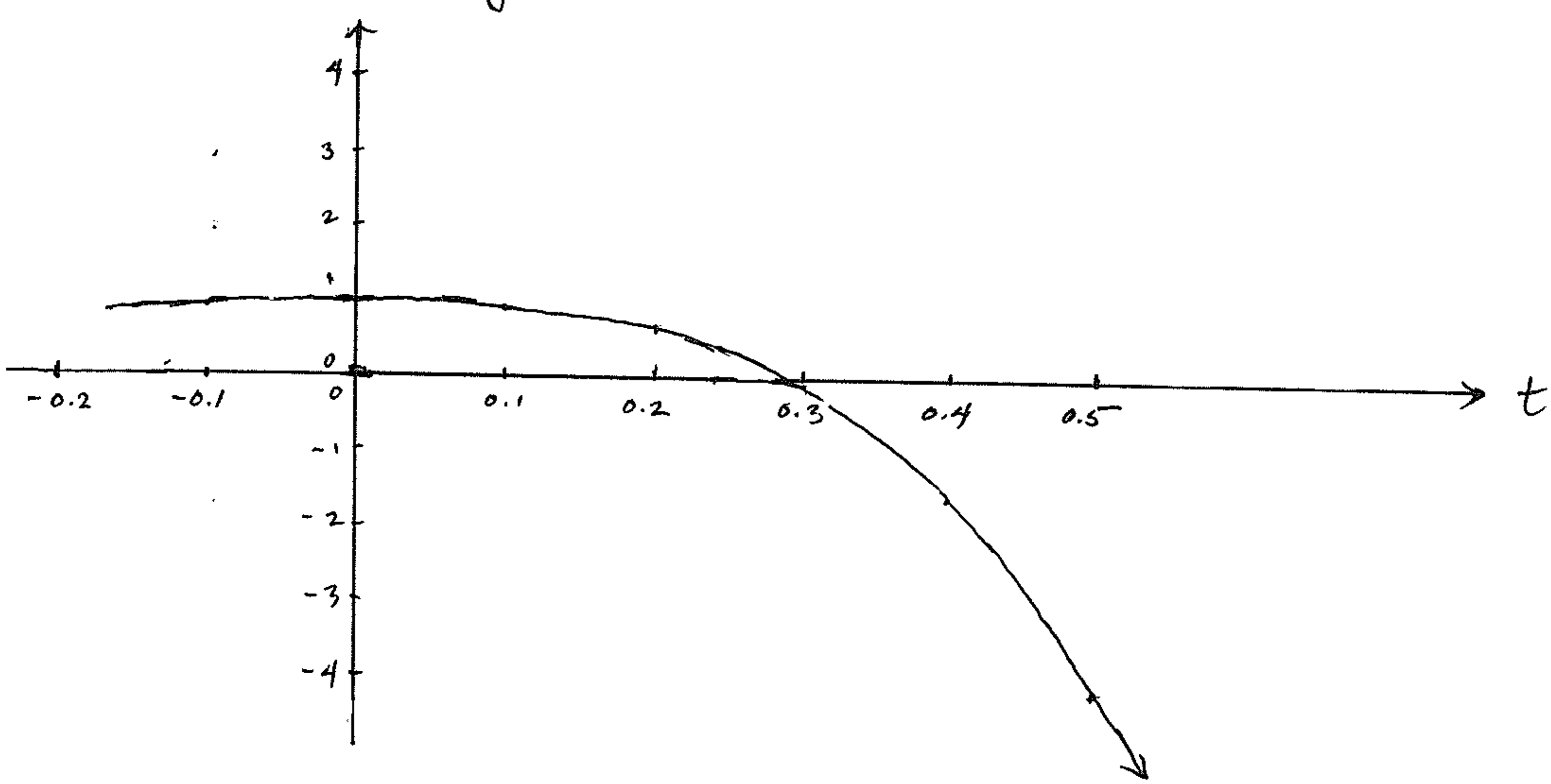
The initial conditions therefore mean that

$$c_1 + c_2 = 1$$
$$3c_1 + 4c_2 = 0$$

Solving, we find $c_1 = 4$ and $c_2 = -3$,

so

$$y(t) = 4e^{3t} - 3e^{4t}$$



4 b) Characteristic equation:

$$m^2 - 3m + 2 = 0$$

roots: $m = 1, 2$

general solution:

$$y = c_1 e^{t} + c_2 e^{2t}$$

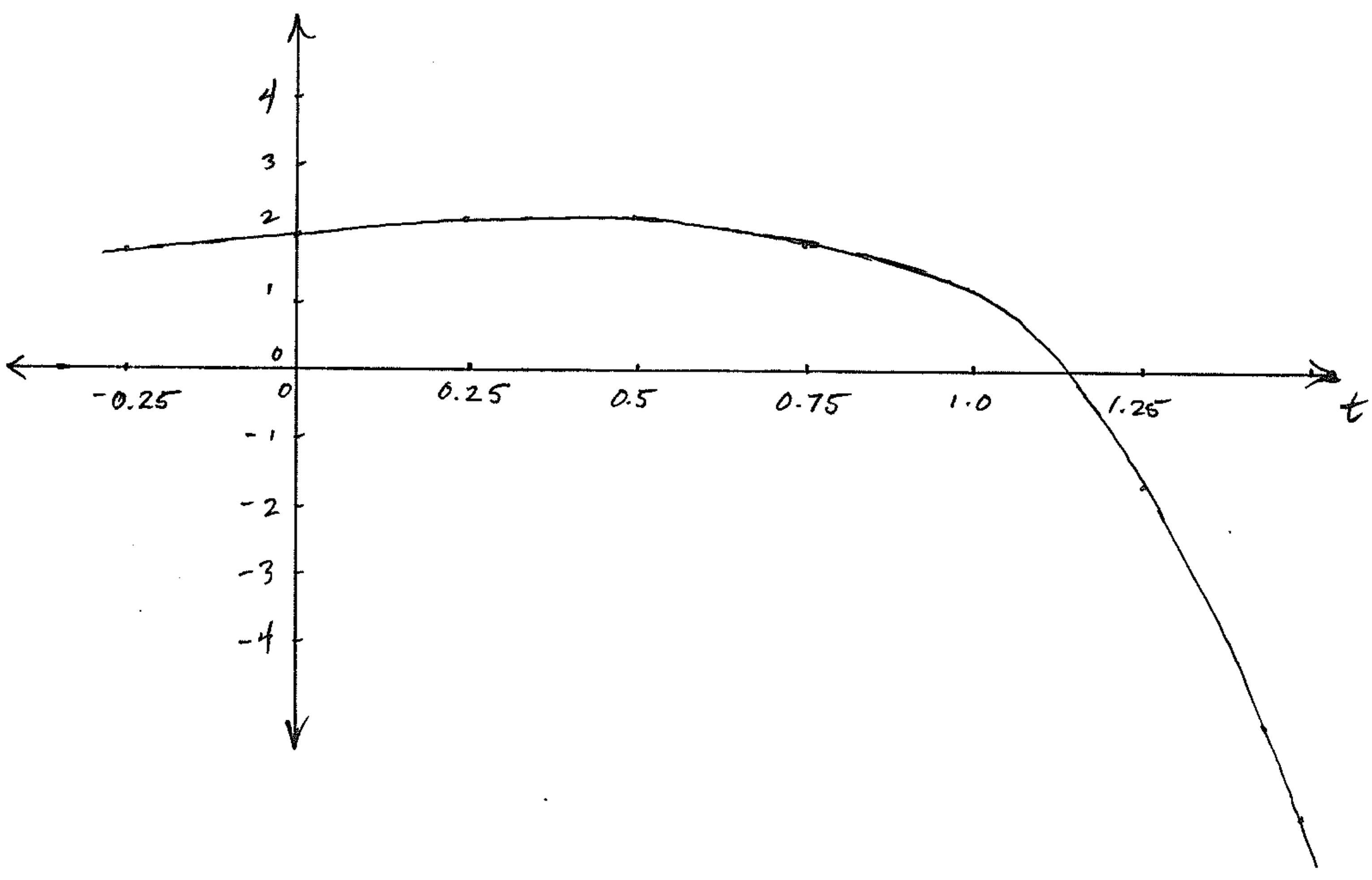
$$\Rightarrow \dot{y} = c_1 e^{t} + 2c_2 e^{2t}$$

initial conditions:

$$c_1 + c_2 = 2$$

$$c_1 + 2c_2 = 1$$

$$\Rightarrow c_1 = 3, \quad c_2 = -1, \quad y(t) = 3e^{t} - e^{2t}$$



5. To solve this prior to considering the case of complex roots of the characteristic equation in class, one might simply ask oneself what functions are the negatives of their second derivatives? Why, sin and cos! In particular, sin satisfies the initial conditions, so a solution is

$$y(t) = \sin t .$$