MATH 213 ASSIGNMENT #6

— SOLUTIONS —

1 a) No – the transfer function is proper (in fact, strictly proper) but the transfer function is not stable: it has a pole at $s = +1$. Hence, the system is not BIBO stable.

b)
$$Y(s) = \frac{s+1}{s(s-1)} \; \frac{1}{s}$$

$$= \frac{A}{s-1} + \frac{B}{s} + \frac{C}{s^2}$$

By Heaviside cover-up,

$A = 2$, $C = -1$, so

$$\frac{B}{s} = \frac{s+1}{s^2(s-1)} - \frac{2}{s-1} + \frac{1}{s^2}$$

$$= \frac{s+1-2s^2+s-1}{s^2(s-1)}$$

$$= \frac{-2\cancel{s}\cancel{(s-1)}}{s^{\cancel{2}}\cancel{(s-1)}}$$

So $$Y(s) = \frac{2}{s-1} - \frac{2}{s} - \frac{1}{s^2}$$

$$\Longleftrightarrow \quad y(t) = \left[2e^{t} - 2 - t\right] u_{-1}(t)$$

(Growing exponential, owing to 'unstable' pole; step and ramp owing to step input and 'integrator' pole at $s=0$; negative signs owing to negative 'dc gain.')

c) $$Y(s) = \frac{s+1}{s(s-1)} \left[R(s) - K\,Y(s)\right]$$

$$\Longleftrightarrow \quad Y(s) = \frac{\frac{s+1}{s(s-1)}}{1 + K\frac{s+1}{s(s-1)}} R(s)$$

$$= \frac{s+1}{s(s-1) + K(s+1)} R(s)$$

$$= \frac{s+1}{s^2 + (K-1)s + K} R(s)$$

(Choice of controller gain K affects locations of poles of "closed-loop" system.)

d) If $K = 1$,

$$\frac{Y(s)}{R(s)} = \frac{s+1}{s^2 + 1} = \frac{s+1}{(s+j)(s-j)}$$

Transfer function is unstable, owing to poles on the imaginary axis

Step response:

$$Y(s) = \frac{s+1}{s^2+1} \frac{1}{s} = \frac{As+B}{s^2+1} + \frac{C}{s}$$

where $C = 1$.

$$\Longrightarrow \quad \frac{As+B}{s^2+1} = \frac{s+1}{s(s^2+1)} - \frac{s^2+1}{s(s^2+1)}$$

$$= \frac{-s^2+s}{s(s^2+1)} = -\frac{\cancel{s}(s-1)}{\cancel{s}(s^2+1)}$$

$$= \frac{-s+1}{s^2+1}$$

So $\quad Y(s) = \dfrac{-s+1}{s^2+1} + \dfrac{1}{s}$

$$\Longleftrightarrow \quad y(t) = [\sin t - \cos t + 1]\, u_{-1}(t)$$

(Poles on imaginary axis give rise to undamped sinusoids.)

e) With $K = 2$,

$$\frac{Y(s)}{R(s)} = \frac{s+1}{s^2 + s + 2}$$

poles: $s = -\frac{1}{2} \pm \frac{1}{2} j\sqrt{7}$

The transfer function is stable.

f) $\omega_n^2 = 2$, so $\omega_n = \sqrt{2}$

$2\zeta\omega_n = 1$, so $\zeta = \dfrac{1}{2\omega_n} = \dfrac{1}{2\sqrt{2}}$ (rather low!)

Step response:

$$Y(s) = \frac{s+1}{s^2+s+2} \cdot \frac{1}{s}$$

$$= \frac{1}{s^2+s+2} + \frac{1}{s^2+s+2} \cdot \frac{1}{s}$$

$$= \frac{1}{2}\left[\frac{2}{s^2+s+2} + \frac{2}{s^2+s+2}\,\frac{1}{s} \right]$$

$= \frac{1}{2}\Big[$ transforms of impulse response + step response of standard 2^{nd} order sys. $\dfrac{2}{s^2+2s+2}$ $\Big]$

So the step response is

$$y(t) = \frac{1}{2} \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

$$+ \frac{1}{2}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t + \cos^{-1}\zeta\right)\right]$$

$$= \frac{1}{2} \frac{4}{\sqrt{7}} e^{-t/2} \sin \frac{\sqrt{7}}{2} t$$

$$+ \frac{1}{2}\left[1 - \frac{2\sqrt{2}}{\sqrt{7}} e^{-t/2} \sin\left(\frac{\sqrt{7}}{2} t + \cos^{-1}\frac{1}{2\sqrt{2}}\right)\right]$$

$$= \frac{1}{2} + \frac{2}{\sqrt{7}} e^{-t/2} \sin \frac{\sqrt{7}}{2} t$$

$$- \frac{\sqrt{2}}{\sqrt{7}} e^{-t/2} \sin\left(\frac{\sqrt{7}}{2} t + \cos^{-1}\frac{1}{2\sqrt{2}}\right)$$

$$(t \geq 0)$$

g) Transform of step response:

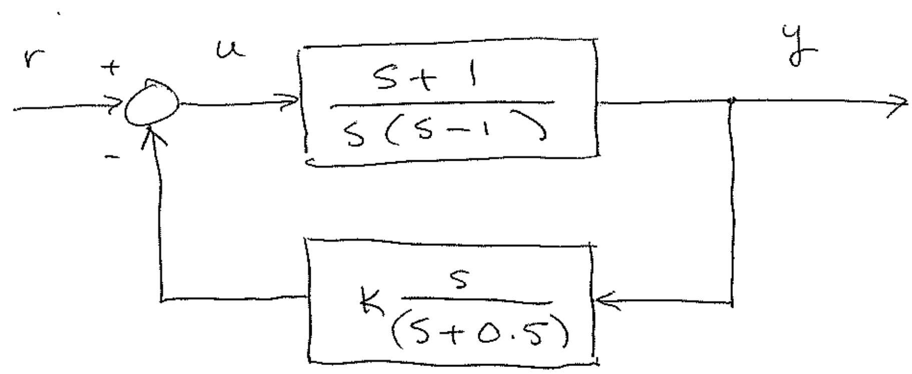
$$X(s) = K \frac{\cancel{8}}{s+0.5} \cdot \frac{1}{\cancel{8}}$$

The final-value theorem applies (only pole has negative real part)

so

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s X(s) = 0$$

h)



$$Y(s) = \frac{s+1}{s(s-1)} \, U(s)$$

$$U(s) = R(s) - K \frac{s}{s+0.5} Y(s)$$

So $$Y(s) = \frac{s+1}{s(s-1)} \left[R(s) - K \frac{s}{s+0.5} Y(s) \right]$$

$$Y(s) = \frac{\frac{s+1}{s(s-1)}}{1 + K \frac{\cancel{s}}{s+0.5} \frac{s+1}{\cancel{s}(s-1)}} R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{(s+1)(s+0.5)}{s\left((s+0.5)(s-1)+K(s+1)\right)}$$

$$= \frac{(s+1)(s+0.5)}{s\left(s^2+(K-0.5)s+K-0.5\right)}$$

- if $K=2$,

$$\frac{Y(s)}{R(s)} = \frac{(s+1)(s+0.5)}{s\left(s^2+1.5s+1.5\right)}$$

Poles: $s=0$, $s = -\frac{3}{4} \pm j\frac{\sqrt{15}}{4}$

Step response:

$$Y(s) = \frac{(s+1)(s+0.5)}{s\left(s^2+1.5s+1.5\right)}\frac{1}{s}$$

$$= \frac{As+B}{s^2+1.5s+1.5} + \frac{C}{s} + \frac{D}{s^2}$$

Where $D = \frac{1}{3}$, so

$$\frac{As+B}{s^2+1.5s+1.5} + \frac{C}{s} = \frac{(s+1)(s+0.5) - \frac{1}{3}(s^2+1.5s+1.5)}{s^2(s^2+1.5s+1.5)}$$

$$= \frac{\cancel{s}\left(\frac{2}{3}s+1\right)}{s^{\cancel{2}}\left(s^2+1.5s+1.5\right)}$$

So $C = \frac{2}{3}$, by Heaviside cover-up.

So

$$Y(s) = \frac{\frac{2}{3}s + 1}{s^2 + 1.5s + 1.5} \cdot \frac{1}{s} + \frac{1/3}{s^2}$$

$$\left(\omega_n = \sqrt{\frac{3}{2}}, \quad 2\zeta\omega_n = \frac{3}{2} \Rightarrow \zeta = \frac{3}{4\sqrt{3/2}} = \frac{1}{2}\sqrt{\frac{3}{2}} \right)$$

So

$$y(t) = \left(\frac{2}{3}\right)^2 \left[\frac{\sqrt{\frac{3}{2}}}{\frac{1}{2}\sqrt{5/2}} e^{-\frac{3}{4}t} \sin \sqrt{\frac{3}{2}}\, \frac{1}{2}\sqrt{\frac{5}{2}}\, t \right]$$

$$+ \frac{2}{3}\left[1 - \frac{1}{\frac{1}{2}\sqrt{\frac{5}{2}}} e^{-3/4\, t} \sin\left(\sqrt{\frac{3}{2}} \cdot \frac{1}{2} \cdot \sqrt{\frac{5}{2}}\, t + \cos^{-1} \frac{1}{2}\sqrt{\frac{3}{2}} \right) \right]$$

$$+ \frac{1}{3}t$$

$$= \frac{4}{9} \cdot 2\sqrt{\frac{3}{5}}\, e^{-3/4\, t} \sin \frac{\sqrt{15}}{4} t$$

$$- \frac{4}{3}\sqrt{\frac{2}{5}}\, e^{-\frac{3}{4}t} \sin\left(\frac{\sqrt{15}}{4} t + \cos^{-1} \frac{1}{2}\sqrt{\frac{3}{2}} \right)$$

$$+ \frac{2}{3} + \frac{1}{3}t$$

(So, for example, heading continues to change under a constant rudder input.)