Fourier Series

We've seen that sinusoidal signals are easy to analyze — they only require the use of the frequency response, and its graphical representation, the Bode plot.

It's natural to ask how far we can push this analysis, perhaps by decomposing signals into sinusoids.

It turns out that, given a sinusoidal signal, we can often express it as an infinite sum of sinusoids. Surprisingly, this can work even when the periodic signal is discontinuous.

Such infinite sums are called <u>Fourier series</u>, and they were first used in the study of partial differential equations.

Recall our vibrating-string PDE. We found that it had an infinite number of sinusoidal solutions, with frequencies that were integer multiples of a fundamental frequency (because of the boundary conditions on the ends of the string). A more comprehensive treatment of that example might have attempted to satisfy an initial condition on the shape of the string at $t=0$ by means of a superposition of all these sinusoids.

This approach makes use of some of the concepts of linear algebra. It turns out that the infinite set of sinusoids constitutes an orthogonal basis of a vector space. In order to represent a signal as a sum of these sinusoids, we'll take its projection onto each of them.

Let's get more concrete. First, we say that a function is <u>periodic</u> with period T if, for all t,

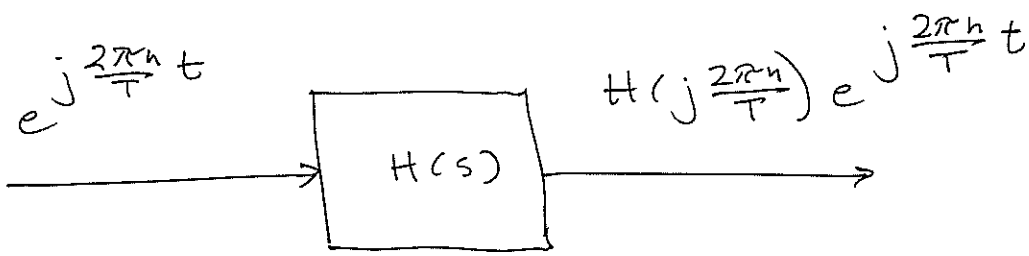
$$f(t+T) = f(t).$$

Of course, this implies that f is also periodic with period $2T$, $3T$, $4T$, etc., so we usually choose T to be the smallest for which this property holds.

Given such a periodic function, we'll try to represent it as a superposition of sinusoidal functions

$$e^{j\frac{2\pi n}{T}t}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Why? We've just seen that such functions are "eigenvectors" of linear systems:



A function f is piecewise-smooth on the interval $[-\frac{T}{2}, T/2]$ if there exists a finite set of points

$$-T/2 = t_0 < t_1 < \dots < t_k = T/2$$

such that

- f and $\dot{f}$ are bounded and continuous on each interval (t_i, t_{i+1}); and
- the limits $f(t_i^-)$, $f(t_i^+)$, $\dot{f}(t_i^-)$ and $\dot{f}(t_i^+)$ all exist.

The set of all piecewise-smooth complex-valued functions on $[-\frac{T}{2}, \frac{T}{2}]$ forms a vector space (under multiplication by complex-valued scalars).

We can make it into an <u>inner-product space</u> by defining

$$\langle f, g \rangle = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \left(g(t) \right)^* dt$$

The complex exponentials $e^{j\frac{2\pi n}{T}t}$ $(n \in \mathbb{Z})$ then form an orthonormal basis of a subspace of this inner-product space.

The <u>Fourier series</u> is obtained by projecting a function onto this subspace to yield

$$\sum_{n=-\infty}^{\infty} c_n \, e^{j\frac{2\pi n}{T}t},$$

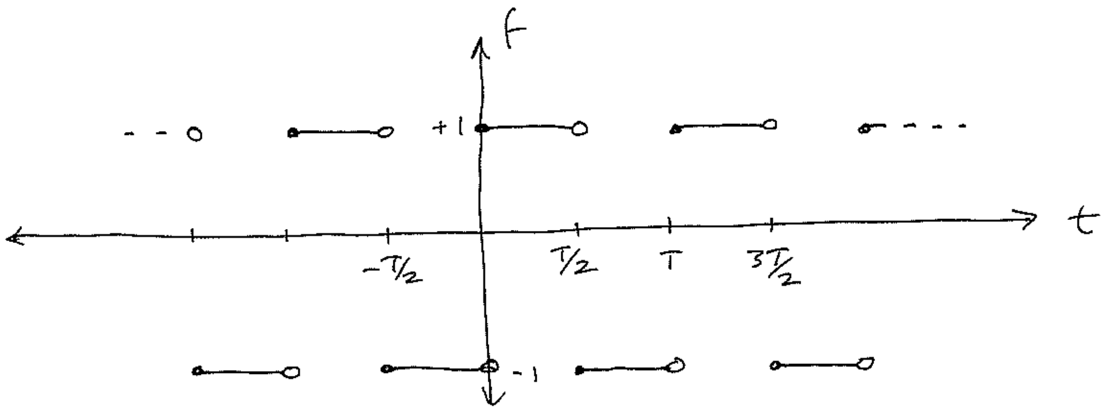
where

$$c_n = \langle f, e^{j\frac{2\pi n}{T}t} \rangle$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-j\frac{2\pi n}{T}t}\, dt$$

[Note that if $f(t)$ is real-valued, $c_{-n} = c_n^*$.]

Example: square wave



$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} f(t)\, e^{-j\frac{2\pi n}{T}t}\, dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{0} -e^{-j\frac{2\pi n}{T}t}\, dt + \int_{0}^{T/2} e^{-j\frac{2\pi n}{T}t}\, dt\right]$$

$$= \frac{1}{j2\pi n}\left[e^{-j\frac{2\pi n}{T}t}\Big|_{-T/2}^{0} - e^{-j\frac{2\pi n}{T}t}\Big|_{0}^{T/2}\right] \quad (n \neq 0)$$

$$= \frac{1}{j2\pi n}\left[1 - e^{j\pi n} - \left(e^{-j\pi n} - 1\right)\right] \quad (n \neq 0)$$

$$= \frac{1}{j\pi n}\left[1 - (-1)^n\right] \quad (n \neq 0)$$

For $n = 0$,

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt = 0$$

So

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} = \frac{4}{\pi} \sum_{m=1,3,5} \frac{1}{n} \sin \frac{2\pi n}{T} t$$

For every partial sum, we can define an error function,

$$e_N(t) = f(t) - \sum_{n=-N}^{N} c_n e^{j\frac{2\pi n}{T}t}$$

By the orthogonality of the sinusoids and the definition of the c_n

$$\langle e_N, e_N \rangle = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| f(t) - \sum_{n=-N}^{N} c_n e^{j\frac{2\pi n}{T}t} \right|^2 dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{T/2} |f(t)|^2 dt - 2\sum_{n=-N}^{N} |c_n|^2 T + \sum_{n=-N}^{N} |c_n|^2 T\right]$$

$$= \frac{1}{T}\left[\int_{-T/2}^{T/2} |f(t)|^2 dt\right] - \sum_{n=-N}^{N} |c_n|^2$$

Of the two terms on the right-hand side, the first represents the "average power" of $f(t)$, and the second that of the partial sum.

Dirichlet convergence theorem

For any piecewise-smooth function $f(t)$ on $[-\frac{T}{2}, \frac{T}{2}]$,

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} = \begin{cases} f(t), & \text{if } f \text{ is cont. @ } t \\ \frac{f(t^-)+f(t^+)}{2}, & \text{otherwise.} \end{cases}$$

This implies that

$$\lim_{N \to \infty} \langle e_N , e_N \rangle = 0$$

As a consequence, we have
Parseval's Theorem:

$$\frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

- the average power of $f(t)$ is the sum of the average power of each of its "frequency components."

For the square wave, Parseval implies that

$$1 = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left| \frac{1}{j\pi n} \left[1 - (-1)^n \right] \right|^2$$

$$= \frac{4}{\pi^2} \sum_{\substack{n=-\infty \\ n \neq 0, \\ n \text{ odd}}}^{\infty} \frac{1}{n^2}$$

$$= \frac{8}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2}$$

N	Avg. power in partial sum
1	81%
3	90%
5	93.3%
7	94%
⋮	

Note that this also means that

$$\pi = 2\sqrt{2} \cdot \sqrt{\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2}}$$

- An algorithm for computing π !

- Similar to Euler's Theorem:

$$\pi = \sqrt{6 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}}$$