MATH 239 Spring 2014: Assignment 7

Due: 3:00 PM, Tuesday July 8, 2014 in the dropboxes outside MC 4066

Last Name: First Name:

I.D. Number: Section:

Mark (For the marker only): /28

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1. {6 marks} For each of the following statements about any graph G and three of its vertices u, v, w, determine whether it is true or false. If it is true, give a proof. If it is false, give a counterexample with appropriate explanations.

   (a) If G contains a cycle, then every vertex has degree at least 2.

   (b) If there is a walk containing u, v, w, then there is a path containing u, v, w.

   (c) If there exist a cycle containing vertices u, v and a cycle containing vertices v, w, then there exists a cycle containing vertices u, w.

2. Consider the following proposition.

Proposition. *Let G be a non-empty graph where every vertex has degree at least k. Then G contains a path of length at least k.*

(a) {2 marks} The following is an incorrect proof of the proposition using induction on k. Determine the main flaw of this proof.

Bad proof. When $k = 1$, G has at least 1 vertex v with degree at least 1, so it has at least one neighbour w. Then v, w is a path of length at least 1 in G.

Assume that the statement is true for $k - 1$. Let G be a graph where every vertex has degree at least $k - 1$. By induction hypothesis, there exists a path $P = v_1, v_2, \ldots, v_k$ of length $k - 1$. Construct a new graph G' by adding a new vertex x and join x with every vertex in G. Then every vertex has degree at least k in G'. In particular, $v_k x$ is an edge, so $v_1, v_2, \ldots, v_k, x$ is a path of length k in G'. Therefore, G' contains a path of length at least k. $\square$

(b) {4 marks} Give a correct proof of this proposition. (You do not need to use induction.)

3. {4 marks} Let $k \geq 2$ be an integer. Let G be a graph where every vertex has degree at least k. Prove that G contains at least $\lfloor k/2 \rfloor$ edge-disjoint cycles (i.e. no two of these cycles share any edges, but they may share some vertices).

4. {4 marks} Let G be a bipartite graph with n vertices. Prove that if every vertex has degree at least $\frac{n}{4} + 1$, then G is connected.

5. For some $k \in \mathbb{N}$, let G be a connected graph with $2k$ odd-degree vertices, and any number of even-degree vertices.

(a) {4 marks} Prove that there exist k walks such that each edge in G is used in exactly one walk exactly once. What is so special about the end vertices of the k walks? (Hint: Add some edges to create an Eulerian circuit, and then remove them.)

(b) {2 marks} Prove that it is not possible that a set of $k-1$ walks in G uses each edge exactly once. (This shows that to cover G with walks containing no repeated edges, you need at least k walks.)

(c) {2 marks} Partition the edges of the leftmost graph below into as few walks as possible.