Math 239 Spring 2014 Assignment 2 Solutions

1. {6 marks} Consider the following power series.

$$f(x) = \sum_{i=2}^{157} (-3)^{i-2} x^{2i} = x^4 - 3x^6 + 9x^8 - \cdots - 3^{155} x^{314} \qquad g(x) = \sum_{i \geq 3} f(x)^i.$$

(a) Express both $f(x)$ and $g(x)$ as rational functions, i.e. $\frac{p(x)}{q(x)}$ where $p(x), q(x)$ are explicit polynomials (you should be able to write them out without resorting to sums). Simplify your expressions as much as possible.

Solution. We first factor out x^4 from $f(x)$. Re-indexing gives us

$$f(x) = x^4 \sum_{i=0}^{155} (-3)^i x^{2i} = x^4 \sum_{i=0}^{155} (-3x^2)^i.$$

Using geometric series,

$$f(x) = x^4 \frac{1 - (-3x^2)^{156}}{1 - (-3x^2)} = \frac{x^4 - 3^{156} x^{316}}{1 + 3x^2}.$$

For $g(x)$, we can factor out $f(x)^3$ first, and then use geometric series (this is possible since the constant term in $f(x)$ is 0).

$$g(x) = f(x)^3 \sum_{i \geq 0} f(x)^i = \frac{f(x)^3}{1 - f(x)} = \frac{\left(\frac{x^4 - 3^{156} x^{316}}{1+3x^2}\right)^3}{1 - \frac{x^4 - 3^{156} x^{316}}{1+3x^2}} = \frac{(x^4 - 3^{156} x^{316})^3}{(1+3x^2)^3 - (1+3x^2)^2 (x^4 - 3^{156} x^{316})}.$$

(b) Does $g(x)$ have an inverse? If so, determine a rational function for it. If not, explain why not.

Solution. We see that the constant term in $f(x)$ is 0, and $g(x)$ is equal to $f(x)^3$ multiplied by some series. So the constant term in $g(x)$ is 0, therefore $g(x)$ does not have an inverse.

2. {4 marks} Using mathematical induction on k, prove that for any integer $k \geq 1$,

$$(1-x)^{-k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n.$$

Solution. When $k = 1$, $\binom{n+k-1}{k-1} = \binom{n}{0} = 1$. So $(1-x)^{-1} = \sum_{n \geq 0} x^n = \sum_{n \geq 0} \binom{n+1-1}{k-1} x^n$. So the base case holds.

Assume that for some positive integer m, $(1-x)^{-m} = \sum_{n \geq 0} \binom{n+m-1}{m-1} x^n$.

We need to prove the equation for $m+1$. We see that

$$(1-x)^{-(m+1)} = (1-x)^{-m} (1-x)^{-1}.$$

By induction hypothesis, $[x^i](1-x)^{-m} = \binom{i+m-1}{m-1}$. Also, we know that $[x^i](1-x)^{-1} = 1$. Using rules of multiplication of power series, we get

$$[x^n](1-x)^{-(m+1)} = \sum_{i=0}^{n} ([x^i](1-x)^{-m})([x^{n-i}](1-x)^{-1}) = \sum_{i=0}^{n} \binom{i+m-1}{m-1} = \binom{n+m}{m}$$

where the final step uses an identity from class. Therefore,

$$(1-x)^{-(m+1)} = \sum_{n \geq 0} \binom{n+m}{m} x^n.$$

Therefore, by induction, the result holds.

3. {4 marks} Determine the value of the following coefficient.

$$[x^{26}](3+x^2)(1-2x^6)^{-31}(1+x^9)^{-41}.$$

Solution. We see that

$$[x^{26}](3+x^2)(1-2x^6)^{-31}(1+x^9)^{-41} = 3[x^{26}](1-2x^6)^{-31}(1+x^9)^{-41} + [x^{24}](1-2x^6)^{-31}(1+x^9)^{-41}.$$

Note that in the expansion of $(1-2x^6)^{-31}(1+x^9)^{-41}$, the exponents of x are integer combinations of 6's and 9's. Such exponents are multiples of 3, so the coefficient of x^{26} is 0. The required coefficient is then equal to the coefficient of x^{24} in $(1-2x^6)^{-31}(1+x^9)^{-41}$.

There are 2 ways to get x^{24} in this multiplication: $[x^{24}](1-2x^6)^{-31}[x^0](1+x^9)^{-41}$ and $[x^6](1-2x^6)^{-31}[x^{18}](1+x^9)^{-41}$. These correspond to the numbers $2^4\binom{4+31-1}{31-1}\cdot(-1)^0$ and $2^1\binom{1+31-1}{31-1}\cdot(-1)^2\binom{2+41-1}{41-1}$. So the required coefficient is $2^4\binom{34}{30}+2\binom{31}{30}\binom{42}{40}=795398$.

4. {4 marks} Let $\{a_n\}_{n\geq 0}$ be a sequence whose corresponding power series $A(x)=\sum_{i\geq 0} a_i x^i$ satisfies

$$A(x)=\frac{-6-34x}{1+2x-3x^2}.$$

Determine a recurrence relation that $\{a_n\}$ satisfies, with sufficient initial conditions to uniquely specify $\{a_n\}$. Use this recurrence relation to find a_4.

Solution. We see that

$$(1+2x-3x^2)A(x) = -6-34x.$$

So

$$\begin{aligned}
-6-34x &= (1+2x-3x^2)(a_0+a_1x+a_2x^2+a_3x^3+\cdots)\\
&= a_0+(a_1+2a_0)x+\sum_{n\geq 2}(a_n+2a_{n-1}-3a_{n-2})x^n
\end{aligned}$$

By comparing the coefficients, we see that $a_0=-6$; $a_1+2a_0=-34$, so $a_1=-22$; and $a_n+2a_{n-1}-3a_{n-2}=0$ for $n\geq 2$. These are the initial conditions and the recurrence that $\{a_n\}$ satisfies. To get a_4, we apply the recurrence relation.

$$\begin{aligned}
a_2 &= -2a_1+3a_0 = 26\\
a_3 &= -2a_2+3a_1 = -118\\
a_4 &= -2a_3+3a_2 = 314
\end{aligned}$$

5. Let $n\in\mathbb{N}$. For a permutation $\sigma:[n]\to[n]$, we use the notation $(\sigma(1)\sigma(2)\cdots\sigma(n))$ to describe the mapping. A pair of integers (i,j) is called an *inversion* of σ if $i<j$ and $\sigma(i)>\sigma(j)$. For example, the permutation (32415) on [5] has 4 inversions: $(1,2),(1,4),(2,4),(3,4)$. Define the weight function w on a permutation σ to be the number of inversions in σ. Let S_n be the set of all permutations of $[n]$.

(a) {2 marks} Determine the generating series for S_1, S_2, S_3 with respect to w. (No work required.)

Solution. $S_1=\{(1)\}$, which has 0 inversions. So $\Phi_{S_1}(x)=1$.
For S_2, (12) has no inversions, but (21) has one inversion. So $\Phi_{S_2}(x)=1+x$.
For S_3, (123) has no inversions, (132), (213) have one inversion, (231), (312) have two inversions, and (321) has three inversions. So $\Phi_{S_3}(x)=1+2x+2x^2+x^3$.

(b) {4 marks} Prove that for $n\geq 2$,

$$\Phi_{S_n}(x)=(1+x+\cdots+x^{n-1})\Phi_{S_{n-1}}(x).$$

You may use the following (non-standard) notation: If σ is a permutation of $[n]$, denote σ' to be the permutation of $[n-1]$ obtained from σ by removing the element n. For example, if $\sigma=(31524)$, then $\sigma'=(3124)$.

Solution. We split S_n into n sets according to the location of the element n in the permutation. For $i = 1, \ldots, n$, let T_i be the set of all permutations $\sigma \in S_n$ where $\sigma(i) = n$. Then

$$S_n = T_1 \cup T_2 \cup \cdots \cup T_n.$$

For each T_i, we can form a bijection between T_i and S_{n-1} as follows: $f : T_i \to S_{n-1}$ where $f(\sigma) = \sigma'$. We now compare the number of inversions between σ and σ'. Each inversion in σ' is still an inversion of σ. However, there are additional inversions introduced by n in σ. Since n is the largest possible element in σ, it creates an inversion with any element after it. Since $\sigma(i) = n$, there are $n - i$ additional inversions, namely $(i, i+1), (i, i+2), \ldots, (i, n)$. Therefore, $w(\sigma) = w(\sigma') + (n - i)$. Since we have a bijection between T_i and S_{n-1}, we can say that

$$\Phi_{T_i}(x) = x^{n-i}\Phi_{S_{n-1}}.$$

Using the sum lemma, we get

$$\Phi_{S_n}(x) = \sum_{i=1}^{n} \Phi_{T_i}(x) = \sum_{i=1}^{n} x^{n-i}\Phi_{S_{n-1}}(x) = (1 + x + x^2 + \cdots + x^{n-1})\Phi_{S_{n-1}}(x).$$

(c) {2 marks} Prove that the number of permutations of $[n]$ with k inversions is

$$[x^k]\frac{\prod_{i=1}^{n}(1 - x^i)}{(1 - x)^n}.$$

Solution. We see that $\Phi_{S_1}(x) = 1$ from part (a), so this is satisfied for $n = 1$. Using induction, we see that

$$\begin{aligned}
\Phi_{S_n}(x) &= (1 + x + \cdots + x^{n-1})\Phi_{S_{n-1}}(x) \text{ by part (b)} \\
&= \frac{1 - x^n}{1 - x}\Phi_{S_{n-1}}(x) \\
&= \frac{1 - x^n}{1 - x}\,\frac{\prod_{i=1}^{n-1}(1 - x^i)}{(1 - x)^{n-1}} \text{ by ind hyp} \\
&= \frac{\prod_{i=1}^{n}(1 - x^i)}{(1 - x)^n}
\end{aligned}$$