Math 239 Spring 2014 Assignment 5 Solutions

1. {5 marks} Let $\{a_n\}$ be the sequence which satisfies

$$a_n - 7a_{n-1} + 15a_{n-2} - 9a_{n-3} = 0$$

for $n \geq 3$ with initial conditions $a_0 = 7, a_1 = 10, a_2 = 13$. Determine an explicit formula for a_n.

Solution. The characteristic polynomial is

$$x^3 - 7x^2 + 15x - 9 = (x-3)^2(x-1).$$

The root 3 has multiplicity 2, and the root 1 has multiplicity 1. So

$$a_n = (A + Bn) \cdot 3^n + C \cdot 1^n$$

for some constants A, B, C. Plugging in the initial conditions, we get

$$\begin{aligned} 7 &= A + C \\ 10 &= 3A + 3B + C \\ 13 &= 9A + 18B + C \end{aligned}$$

Solving this gives us $A = 3, B = -1, C = 4$. So an explicit formula for a_n is

$$a_n = (3 - n) \cdot 3^n + 4.$$

2. {5 marks} Let $\{b_n\}$ be the sequence which satisfies

$$b_n - 7b_{n-1} + 15b_{n-2} - 9b_{n-3} = 16 \cdot (-3)^n$$

for $n \geq 3$ with initial conditions $b_0 = 10, b_1 = -13, b_2 = 14$. Determine an explicit formula for b_n. (Note: This recurrence is similar to the one in question 1.)

Solution. We suppose that $c_n = \alpha(-3)^n$ is a specific solution to the recurrence. Then

$$\begin{aligned} c_n - 7c_{n-1} + 15c_{n-2} - 9c_{n-3} &= \alpha(-3)^n - 7\alpha(-3)^{n-1} + 15\alpha(-3)^{n-2} - 9\alpha(-3)^{n-3} \\ &= \alpha(-3)^{n-3}((-3)^3 - 7(-3)^2 + 15(-3) - 9) \\ &= -144\alpha(-3)^{n-3} = -16\alpha(-3)^{n-1}. \end{aligned}$$

This equals $16 \cdot (-3)^n$, so $\alpha = 3$. A specific solution is then $c_n = 3 \cdot (-3)^n$.
The characteristic polynomial is the same as part (a), so

$$b_n = 3 \cdot (-3)^n + (A + Bn) \cdot 3^n + C \cdot 1^n$$

for some constants A, B, C. Plugging in the initial conditions, we get

$$\begin{aligned} 10 &= 3 + A + C \\ -13 &= -9 + 3A + 3B + C \\ 14 &= 27 + 9A + 18B + C \end{aligned}$$

Solving this gives us $A = -\frac{23}{2}, B = 4, C = \frac{37}{2}$. So an explicit formula for b_n is

$$b_n = 3 \cdot (-3)^n + (-\tfrac{23}{2} + 4n) \cdot 3^n + \tfrac{37}{2}.$$

3. Consider the sequence $\{a_n\}$ where for each integer $n \geq 0$,

$$a_n = \frac{(1+\sqrt{3})^n - (1-\sqrt{3})^n}{2\sqrt{3}}.$$

(a) {3 marks} Derive a simplified rational expression for $A(x) = \sum_{n\geq 0} a_n x^n$.

Solution.

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} \frac{(1+\sqrt{3})^n - (1-\sqrt{3})^n}{2\sqrt{3}} x^n \\
&= \frac{1}{2\sqrt{3}} \left(\sum_{n\geq 0} (1+\sqrt{3})^n x^n - \sum_{n\geq 0} (1-\sqrt{3})^n x^n \right) \\
&= \frac{1}{2\sqrt{3}} \left(\frac{1}{1-(1+\sqrt{3})x} - \frac{1}{1-(1-\sqrt{3})x} \right) \\
&= \frac{(1-(1-\sqrt{3})x) - (1-(1+\sqrt{3}x)}{2\sqrt{3}(1-(1+\sqrt{3})x)(1-(1-\sqrt{3})x)} \\
&= \frac{2\sqrt{3}x}{2\sqrt{3}(1-2x-2x^2)} \\
&= \frac{x}{1-2x-2x^2}.
\end{aligned}$$

(b) {3 marks} Use part (a) to prove that a_n is an integer for all $n \geq 0$.

Solution. From the power series in part (a), we see that a_n satisfies the recurrence $a_n - 2a_{n-1} - 2a_{n-2} = 0$ for $n \geq 2$ with initial conditions $a_0 = 0$ and $a_1 = 1$. Using strong induction, we see that for each $n \geq 2$, $a_n = 2a_{n-1} + 2a_{n-2}$, which is a sum of two integers. So a_n is an integer for all $n \geq 0$.

4. {4 marks} Consider the sequence $\{a_n\}$ defined by $a_0 = -2$, $a_1 = 3$, and for $n \geq 2$,

$$a_n + 6a_{n-1} + 12a_{n-2} = 0.$$

The roots of the characteristic polynomial are complex. Convert them into polar form (with sines and cosines), and then derive an explicit formula for a_n that does not involve any imaginary parts. (Hint: $\sin(\theta) = -\sin(-\theta)$ and $\cos(\theta) = \cos(-\theta)$. Your final answer should look like $\ldots(\ldots\cos\ldots + \ldots\sin\ldots)$.)

Solution. The characteristic polynomial is $x^2 + 6x + 12$, which has roots $-3 \pm \sqrt{3}i$. In polar form, they are $2\sqrt{3}(\cos \pm \frac{5\pi}{6} + i \sin \pm \frac{5\pi}{6})$. Then a_n has the form

$$a_n = A \cdot (2\sqrt{3}(\cos \tfrac{5\pi}{6} + i \sin \tfrac{5\pi}{6}))^n + B \cdot ((2\sqrt{3}(\cos \tfrac{-5\pi}{6} + i \sin \tfrac{-5\pi}{6}))^n.$$

Using de Moivre's Theorem and trigonometric properties, we see that

$$\begin{aligned}
a_n &= (2\sqrt{3})^n (A \cos \tfrac{5\pi}{6} n + Ai \sin \tfrac{5\pi}{6} n + B \cos \tfrac{-5\pi}{6} n + Bi \sin \tfrac{-5\pi}{6} n) \\
&= (2\sqrt{3})^n (A \cos \tfrac{5\pi}{6} n + Ai \sin \tfrac{5\pi}{6} n + B \cos \tfrac{5\pi}{6} n - Bi \sin \tfrac{5\pi}{6} n) \\
&= (2\sqrt{3})^n ((A+B) \cos \tfrac{5\pi}{6} n + (A-B)i \sin \tfrac{5\pi}{6} n).
\end{aligned}$$

Using initial conditions, we see that

$$\begin{aligned}
a_0 &= -2 = A + B \\
a_1 &= 3 = 2\sqrt{3}((A+B)\cos \tfrac{5\pi}{6} + (A-B)i \sin \tfrac{5\pi}{6} \\
&= 2\sqrt{3}(-2 \cdot \tfrac{-\sqrt{3}}{2} + (A-B)i\tfrac{1}{2}) \\
&= 6 + (A-B)i\sqrt{3}.
\end{aligned}$$

This gives us $A - B = \frac{-3}{\sqrt{3}i} = -\sqrt{3}i$. So an explicit formula for a_n is

$$a_n = (2\sqrt{3})^n (-2\cos \tfrac{5\pi}{6} n - \sqrt{3} \sin \tfrac{5\pi}{6} n).$$

5. For each $n \in \mathbb{N}$, let a_n be the total number of blocks among all 2^n binary strings of length n. For example, $a_1 = 2$, and $a_2 = 6$ (each of the strings $00, 11$ has 1 block, and each of the strings $01, 10$ has 2 blocks, for a total of 6 blocks).

(a) {2 marks} Let S_n be the set of all binary strings of length n. For $n \geq 2$, we split S_n into two sets A_n and B_n in the following way: let A_n be strings of length n where the last block has length 1; let B_n be strings of length n where the last block has length at least 2. We define two functions $f : A_n \to S_{n-1}$ and $g : B_n \to S_{n-1}$ where both functions take the input string and remove the last bit. Prove that these are bijections by determining the inverses for both functions.

Solution. For $f^{-1} : S_{n-1} \to A_n$, for each string $s \in S_{n-1}$, $f^{-1}(s)$ adds one bit at the end that is different from the last bit of s. For $g^{-1} : S_{n-1} \to B_n$, $g^{-1}(s)$ adds one bit at the end that is the same as the last bit of s.

(b) {2 marks} Use part (a) to derive the following recurrence for $n \geq 2$:

$$a_n = 2a_{n-1} + 2^{n-1}.$$

Solution. For each $s \in A_n$, the corresponding string $f(s)$ has one fewer block than s. So the sum over all the blocks over all strings in A_n is a_{n-1} plus the number of strings in A_n, which is 2^{n-1}. For each $s \in B_n$, the corresponding string $g(s)$ has the same number of block as s. So the sum over all the blocks over all strings in B_n is a_{n-1}. In total, the number of blocks in S_n is $a_n = 2a_{n-1} + 2^{n-1}$.

(c) {3 marks} Solve for an explicit formula for a_n.

Solution. The characteristic polynomial is $x - 2$, so it has one root 2. The homogeneous part of the recurrence has the form $A \cdot 2^n$.

To find a specific solution to the nonhomogeneous recurrence, we cannot use $b_n = \alpha 2^n$, as 2 is a root of the characteristic polynomial. So we try $b_n = \alpha n 2^n$. Then

$$b_n - 2b_{n-1} = \alpha(n2^n - 2(n-1)2^{n-1}) = \alpha(n2^n(n-1)2^n) = \alpha 2^n.$$

So $\alpha = 1/2$. So a specific solution is then $b_n = n2^{n-1}$. For a_n, an explicit formula has the form

$$a_n = n2^{n-1} + A \cdot 2^n$$

Using $a_1 = 2$, we get $A = 1/2$. So an explicit formula is

$$a_n = (n+1)2^{n-1}.$$