Math 239 Spring 2014 Assignment 3 Solutions

1. {4 marks} Let $n \in \mathbb{N}$. Suppose we have an unlimited supply of Canadian nickels, dimes, quarters, loonies, and toonies (they are worth $5, 10, 25, 100, 200$ cents each, respectively). How many ways can we make n cents using these coins such that the number of nickels is at least the number of quarters? Note that coins with the same denomination are considered to be identical, so we only care about the number of coins of each denomination. You may say that your answer is equal to a certain coefficient of a power series.

   Solution. Let $C_k = \{0, k, 2k, 3k, 4k, \ldots\}$. Each collection of coins can be represented by a 5-tuple $(n, d, q, l, t) \in C_5 \times C_{10} \times C_{25} \times C_{100} \times C_{200}$. We define the weight of such a 5-tuple to be $w(n, d, q, l, t) = n + d + q + l + t$, which represents the value of this collection of coins in cents.

   To ensure that the number of nickels is at least the number of quarters, we will consider the set S_n to be those collections with exactly n quarters. Then $S_n = \{5n, 5(n+1), 5(n+2), 5(n+3), \ldots\} \times C_{10} \times \{25n\} \times C_{100} \times C_{200}$. Using the weight function $\alpha(a) = a$ for each set in the cartesian product, by product lemma,

$$\Phi_{S_n}(x) = \frac{x^{5n}}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot x^{25n} \cdot \frac{1}{1-x^{100}} \cdot \frac{1}{1-x^{200}} = \frac{x^{30n}}{(1-x^5)(1-x^{10})(1-x^{100})(1-x^{200})}.$$

   Now the set of all valid collections is $S = \bigcup_{n \geq 0} S_n$. So by the sum lemma,

$$\begin{aligned}\Phi_S(x) &= \sum_{n\geq 0} \frac{x^{30n}}{(1-x^5)(1-x^{10})(1-x^{100})(1-x^{200})} \\ &= \frac{1}{(1-x^5)(1-x^{10})(1-x^{100})(1-x^{200})} \sum_{n\geq 0} x^{30n} \\ &= \frac{1}{(1-x^5)(1-x^{10})(1-x^{30})(1-x^{100})(1-x^{200})}.\end{aligned}$$

   The answer is then $[x^n]\Phi_S(x)$.

2. {4 marks} Let $n \in \mathbb{N}$. How many compositions of n consists of either 5 or 6 parts, and each part is even? Determine a generating series for the set of all such compositions (for all n), and then determine an explicit formula for the answer.

   Solution. Let $\mathbb{N}_e = \{2, 4, 6, 8, \ldots\}$ be the set of all positive even integers. The set of all compositions that we want is then $S = \mathbb{N}_e^5 \cup \mathbb{N}_e^6$. Using the weight $\alpha(a) = a$ for $\mathbb{N}_e$, we have

$$\Phi_{\mathbb{N}_e}(x) = x^2 + x^4 + x^6 + x^8 + \cdots = \frac{x^2}{1-x^2}.$$

   Define the weight of a composition as the sum of its parts. Then using the sum lemma and the product lemma, we get

$$\begin{aligned}\Phi_S(x) &= \Phi_{\mathbb{N}_e^5}(x) + \Phi_{\mathbb{N}_e^6}(x) \\ &= \left(\frac{x^2}{1-x^2}\right)^5 + \left(\frac{x^2}{1-x^2}\right)^6 \\ &= \left(\frac{x^2}{1-x^2}\right)^5 \left(1 + \frac{x^2}{1-x^2}\right) \\ &= \left(\frac{x^2}{1-x^2}\right)^5 \frac{1}{1-x^2} \\ &= \frac{x^{10}}{(1-x^2)^6}.\end{aligned}$$

   The answer to our question is then

$$[x^n]\frac{x^{10}}{(1-x^2)^6} = [x^{n-10}]\frac{1}{(1-x^2)^6} = \begin{cases} \binom{\frac{n-10}{2}+5}{5} & n \text{ is even and } n \geq 10 \\ 0 & \text{otherwise} \end{cases}$$

3. Let n be a non-negative integer, and let S_n be the set of all compositions of n where each part is greater than 1. (The number of parts is not restricted.) Let $a_n = |S_n|$.

(a) {4 marks} Prove that

$$a_n = [x^n]\frac{1-x}{1-x-x^2}.$$

Solution. Let $A = \{2, 3, 4, \ldots\}$. Then the set that we are counting is

$$S = \bigcup_{k\geq 0} A^k.$$

We use the weight function $\alpha(a) = a$ for A and the sum of the parts as the weight of a composition. Then

$$\Phi_A(x) = x^2 + x^3 + \cdots = \frac{x^2}{1-x}.$$

Using the sum lemma, we get

$$\begin{aligned}
\Phi_S(x) &= \sum_{k\geq 0} \Phi_{A^k}(x) \\
&= \sum_{k\geq 0} (\Phi_A(x))^k \text{ by product lemma} \\
&= \frac{1}{1-\Phi_A(x)} \text{ by geometric series} \\
&= \frac{1}{1-\frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2}.
\end{aligned}$$

So the answer is $[x^n]\frac{1-x}{1-x-x^2}$.

(b) {4 marks} The generating series from part (a) gives us the recurrence

$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2.$$

Give a combinatorial proof of this recurrence by finding a bijection between S_n and $S_{n-1} \cup S_{n-2}$, and finding its inverse.

Solution. We define a function $f : S_n \to S_{n-1} \cup S_{n-2}$ as follows: For any $(a_1, \ldots, a_k) \in S_n$,

$$f(a_1, \ldots, a_k) = \begin{cases} (a_1, \ldots, a_{k-1}, a_k - 1) & a_k \geq 3 \\ (a_1, \ldots, a_{k-1}) & a_k = 2 \end{cases}$$

For the first case, we have $a_k \geq 3$, so $a_k - 1 \geq 2$. So each part of the result $(a_1, \ldots, a_{k-1}, a_k - 1)$ is still greater than 1, and they sum up to $n-1$, hence it is in S_{n-1}. For the second case, by removing the last part, each part is still greater than 1, and all the parts add up to $n-2$, hence it is in S_{n-2}. Therefore, f is well-defined.

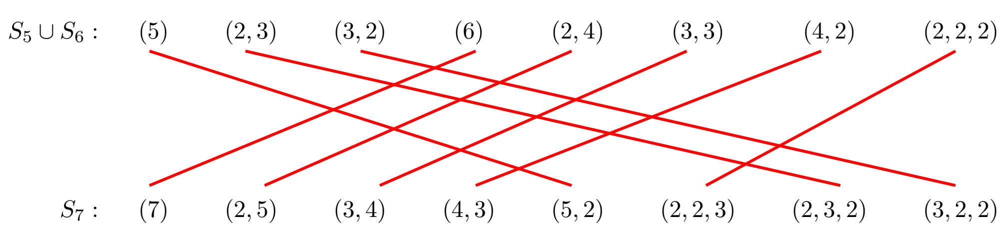
The inverse is the function $g : S_{n-1} \cup S_{n-2} \to S_n$ as follows: For any $(b_1, \ldots, b_l) \in S_{n-1} \cup S_{n-2}$,

$$g(b_1, \ldots, b_l) = \begin{cases} (b_1, \ldots, b_{l-1}, b_l + 1) & b_1 + \cdots + b_l = n-1 \\ (b_1, \ldots, b_l, 2) & b_1 + \cdots + b_l = n-2 \end{cases}$$

So f is a bijection

(c) {2 marks} Illustrate your bijection by matching up compositions in S_7 with compositions in S_6 and S_5.

Solution.



4. Let $n \in \mathbb{N}$. Consider the problem of finding the number of compositions of n with exactly 3 parts (a_1, a_2, a_3) such that $1 \leq a_1 < a_2 < a_3$. For example, when $n = 9$, there are three such compositions: $(1, 2, 6), (1, 3, 5), (2, 3, 4)$. Let S be the set of all such compositions, i.e. $S = \{(a_1, a_2, a_3) \mid 1 \leq a_1 < a_2 < a_3\}$. We will determine the generating series of S with the help of another set $T = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, which is the set of all compositions with exactly 3 parts.

(a) {3 marks} Define a bijection $f : S \to T$, and write down its inverse f^{-1}. Illustrate your bijection by determining $f(2, 3, 9)$ and $f^{-1}(3, 1, 4)$.

Solution. Define f where for each $(a_1, a_2, a_3) \in S$,

$$f(a_1, a_2, a_3) = (a_1, a_2 - a_1, a_3 - a_2).$$

Notice that since $a_3 > a_2 > a_1 \geq 1$, each part of $(a_1, a_2 - a_1, a_3 - a_2)$ is positive, hence $f(a_1, a_2, a_3) \in T$. For the inverse $f^{-1} : T \to S$, for each $(b_1, b_2, b_3) \in T$,

$$f^{-1}(b_1, b_2, b_3) = (b_1, b_1 + b_2, b_1 + b_2 + b_3).$$

We can check that this is valid: for each $(a_1, a_2, a_3) \in S$,

$$f^{-1}f(a_1, a_2, a_3) = f^{-1}(a_1, a_2 - a_1, a_3 - a_2) = (a_1, a_1 + (a_2 - a_1), a_1 + (a_2 - a_1) + (a_3 - a_2)) = (a_1, a_2, a_3).$$

And vice versa. So f is a bijection.
As illustrations, $f(2, 3, 9) = (2, 1, 6)$ and $f^{-1}(3, 1, 4) = (3, 4, 8)$.

(b) {2 marks} Let w be the weight function on S where $w(a_1, a_2, a_3) = a_1 + a_2 + a_3$. For each $(b_1, b_2, b_3) \in T$, define a weight $w^*(b_1, b_2, b_3)$ such that $w^*(f(a_1, a_2, a_3)) = w(a_1, a_2, a_3)$ for all $(a_1, a_2, a_3) \in S$. (You need to prove that this property holds.)

Solution. We define $w^*(b_1, b_2, b_3) = 3b_1 + 2b_2 + b_3$. Then for each $(a_1, a_2, a_3) \in S$,

$$\begin{aligned} w^*(f(a_1, a_2, a_3)) &= w^*(a_1, a_2 - a_1, a_3 - a_2) \\ &= 3a_1 + 2(a_2 - a_1) + (a_3 - a_2) \\ &= a_1 + a_2 + a_3 \\ &= w(a_1, a_2, a_3). \end{aligned}$$

(c) {3 marks} Determine the generating series of T with respect to w^*, and explain why this is the same as the generating series of S with respect to w. (Hint: You may use question 5(c) from assignment 1.)

Solution. For $T = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, we can think of the weight functions $\alpha(a) = 3a, \beta(b) = 2b, \gamma(c) = c$ for the three $\mathbb{N}$ in order. The generating series for $\mathbb{N}$ with respect to these 3 weight functions are $\frac{x^3}{1-x^3}, \frac{x^2}{1-x^2}, \frac{x}{1-x}$ respectively. Then using the product lemma, the generating series for T with respect to the weight function w^* is

$$\Phi_T^*(x) = \frac{x^6}{(1 - x^3)(1 - x^2)(1 - x)}.$$

The reason that this is equal to the generating series for S with respect to w can be shown using the definition of generating series:

$$\begin{aligned} \Phi_S(x) &= \sum_{(a_1,a_2,a_3) \in S} x^{w(a_1,a_2,a_3)} \\ &= \sum_{(a_1,a_2,a_3) \in S} x^{w^*(f(a_1,a_2,a_3))} \quad \text{by part (b)} \\ &= \sum_{(b_1,b_2,b_3) \in T} x^{w^*(b_1,b_2,b_3)} \quad \text{since } f \text{ is a bijection} \\ &= \Phi_T^*(x). \end{aligned}$$