Part II:

A Signals-and-Systems Approach

The first part of the course looked at differential equations from a traditional perspective, such as a mathematician or physicist might usually take.

We assumed a given "forcing function," $f(t)$, and found the corresponding solution $y(t)$.

In this part of the course we'll think of $f(t)$ as an input and $y(t)$ as an output of a "system." We'll be interested in understanding how the system responds to a broad range of inputs, not just a particular $f(t)$.

This approach is more typical of engineering — for example, control, signal-processing or communications engineering.

Intro (ctd):

To model systems mathematically, we need to make a few definitions:

- Signal: a real- or complex-valued function of a real variable t

  - t will usually represent time, though sometimes a different variable is preferable

    - e.g., crankshaft angle in engine control

  - if the domain of the signal is $\mathbb{R}$ or one of its intervals, the signal is continuous-time (CT)

    - e.g., most "physical" signals

  - if the domain is a discrete set like $\mathbb{Z}$ or $\mathbb{N}$ the signal is discrete-time (DT)

    - e.g.,

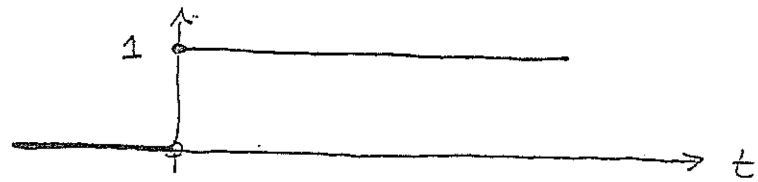
      - monthly bank balance

      - value of a variable in a computer program.

      - sampled version of a continuous-time signal

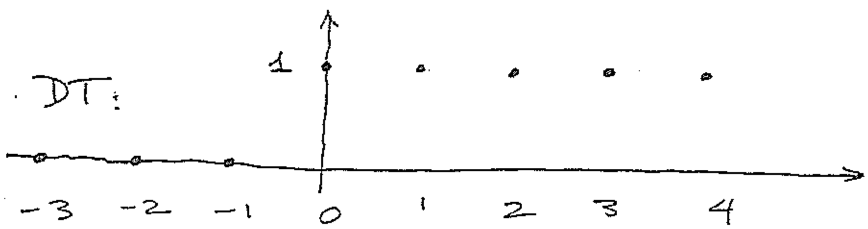
A simple signal that is often used as a reference in control systems is a <u>unit step</u> :

CT:



$u(t)$

DT:



$u[k]$

- <u>system</u> :

  - informally, a device or process whereby certain "input" signals determine certain "output" signals.

  - mathematically, a mapping (function) from a class $\mathcal{F}$ of input signals to a class $\mathcal{Y}$ of output signals

Notation:

$$\mathcal{F} \xrightarrow{S} \mathcal{Y}$$

$$y(t) = (Sf)(t)$$

$$y = Sf$$

Output $y(\cdot) = (Sf)(\cdot)$ is called the system's <u>response</u> to input $f(\cdot)$.

Properties of systems:

1. CT, DT & hybrid
   - if the input and output classes are of CT signals then the system is continuous - time (CT)
     - e.g. most models of physical systems
   - if the input and output classes are of DT signals then the system is discrete - time (DT).
     - e.g. digital hardware
   - in a hybrid system, the signals classes are of different kinds

     e.g. A/D converter

     D/A "

A differential equation may represent a CT system, provided that for any signal in the input class, there is a unique signal in the output class that satisfies the equation.

DT systems are often represented by <u>difference equations</u>. Technically, these are equations involving DT signals, say $y[\cdot]$ and $f[\cdot]$, and their <u>differences</u>, e.g.

$$\begin{aligned}
\nabla y[k] &= y[k] - y[k-1] & &(1^{st} \text{ diff.}) \\
\nabla^2 y[k] &= \nabla y[k] - \nabla y[k-1] & &(2^{nd} \text{ ..}) \\
&\vdots \\
\nabla^n y[k] &= \nabla^{n-1} y[k] - \nabla^{n-1} y[k-1] & &(n^{th} \text{ ..})
\end{aligned}$$

<u>initial conditions</u> give the values of the differences of $y[\,]$ at some "starting time"

It is more common to write a <u>recurrence equation</u>, e.g.

$$\begin{aligned}
y[k] + a_1 y[k-1] + a_2 y[k-2] + \ldots + a_n y[k-n] \\
= b_0 f[k] + b_1 f[k-1] + \ldots + b_m f[k-m]
\end{aligned}$$

and to specify values of $y[\,]$ at a number of different time points. We still commonly use the terms "difference equation" and "initial conditions" in this case.

Properties of systems

2. Memoryless vs. dynamic

- In a <u>memoryless</u> system, the instantaneous output value $y(t)$ depends only on the input value $f(t)$

  - e.g. ideal amplifier:

$$V_{out}(t) = K\, V_{in}(t)$$

- A system that is not memoryless is <u>dynamic</u>.

  - e.g. mechanical system:

$$M\ddot{y}(t) = f(t), \qquad f(t) = 0,\ \forall t \le \bar{t}, \quad \dot{y}(\bar{t}) = y(\bar{t}) = 0$$

$$\Rightarrow \quad y(t) = \frac{1}{M}\int_{-\infty}^{t}\left[\int_{-\infty}^{\tau} f(\theta)\,d\theta\right]d\tau$$

This system is dynamic because of mechanical inertia.

Most interesting control problems involve dynamic "plants."

3. Causality

S is <u>causal</u> if $y(t) = (Sf)(t)$ depends only on

$$\{ f(\tau) : \tau \le t \}$$

— i.e. only on prior (& present) values of the input

In other words, if $f_1(\tau) = f_2(\tau)$, $\forall \tau \le t$, and $y_1 = Sf_1$ & $y_2 = Sf_2$, then

$$y_1(\tau) = y_2(\tau), \quad \forall \tau \le t$$

examples:

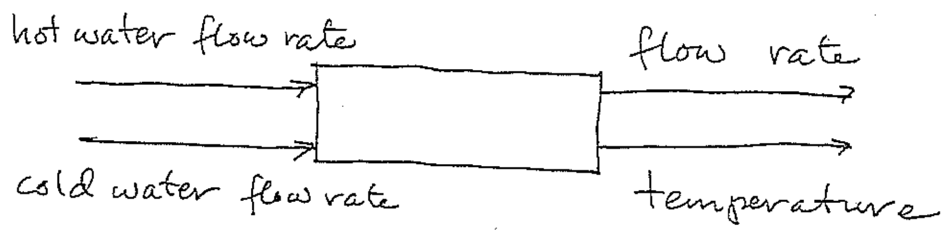
- memoryless systems — causal
- $y[k] = f[k+2]$ — noncausal

- Real-time controllers are causal, but much (off-line) signal processing involves noncausal systems.

4. Multivariable / scalar

- multivariable - system with multiple inputs & outputs

  - e.g., shower



- scalar, or single-input, single-output

  - as the name suggests.

Multivariable systems pose special problems for control.

5. <u>Linearity</u>

- if the input is a linear combination of input signals, then the output is a linear combination (of the same form) of their respective responses.

- more precisely, $\forall c_1, c_2 \in \mathbb{R}$, $\forall f_1, f_2 \in \mathcal{F}$,

$$S(c_1 f_1 + c_2 f_2) = c_1 S(f_1) + c_2 S(f_2)$$

<u>examples</u>:

$$M\ddot{y}(t) = f(t) \quad [f(t)=0, \forall t \le \bar{t}, \dot{y}(\bar{t}) = y(\bar{t}) = 0]$$

- yes

$$y(t) = f(t) + 1$$

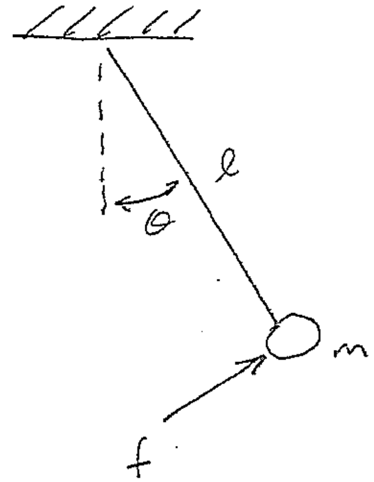
- No

- Linearity greatly simplifies mathematical analysis.

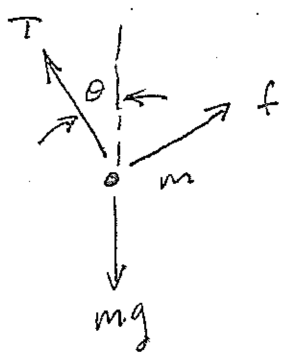
- Among physical systems, nonlinearity is common

→ We often approximate the operation of a nonlinear system about an "operating point" with a "linearized" model.

example:



- free-body diagram



- Newton's law:

$$m l^2 \ddot{\theta} = fl - mgl\underbrace{\sin\theta}_{\text{nonlinear}}$$

- for sufficiently small θ,

$$m l^2 \qquad = fl - mgl\theta$$

- linear

6. Time - invariance

Roughly speaking, a system is <u>time - invariant</u> if its behaviour doesn't change with time ...

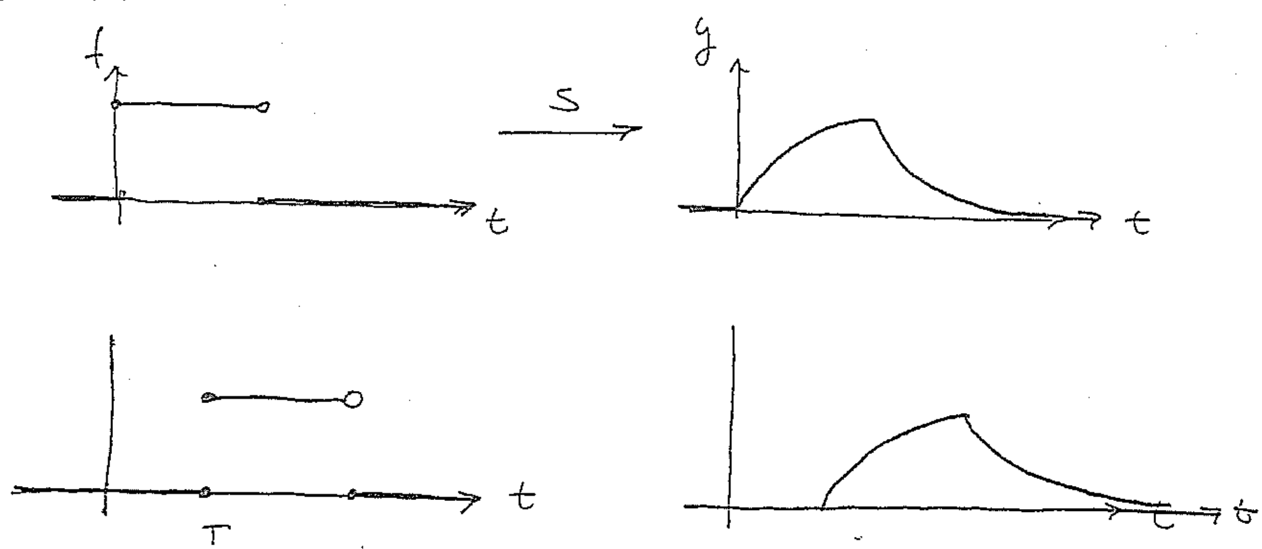
... mathematically, if

$$f(t) \xrightarrow{S} y(t),$$

then

$$f(t-T) \xrightarrow{S} y(t-T)$$

Picture:



Examples

a. $M\ddot{y}(t) = f(t)$, $\quad f(t) = 0, \ \forall t \le t_0$,
$\dot{y}(t_0^-) = y(t_0^-) = 0$

$$\Rightarrow \quad y(t) = \frac{1}{M} \int_{-\infty}^{t} \int_{-\infty}^{\tau} f(\theta)\, d\theta$$

Now replace $f(t)$ with $\tilde{f}(t) = f(t-T)$.
The corresponding response is

$$\tilde{y}(t) = \frac{1}{M} \int_{-\infty}^{t} \left[\int_{-\infty}^{\tau} f(\theta - T)\, d\theta \right] d\tau$$

$$= \frac{1}{M} \int_{-\infty}^{t} \left[\int_{-\infty}^{\tau - T} f(\theta)\, d\theta \right] d\tau \quad \text{(change of variable)}$$

$$= \frac{1}{M} \int_{-\infty}^{t-T} \left[\int_{-\infty}^{\tau} f(\theta)\, d\theta \right] d\tau \quad (\quad " \quad)$$

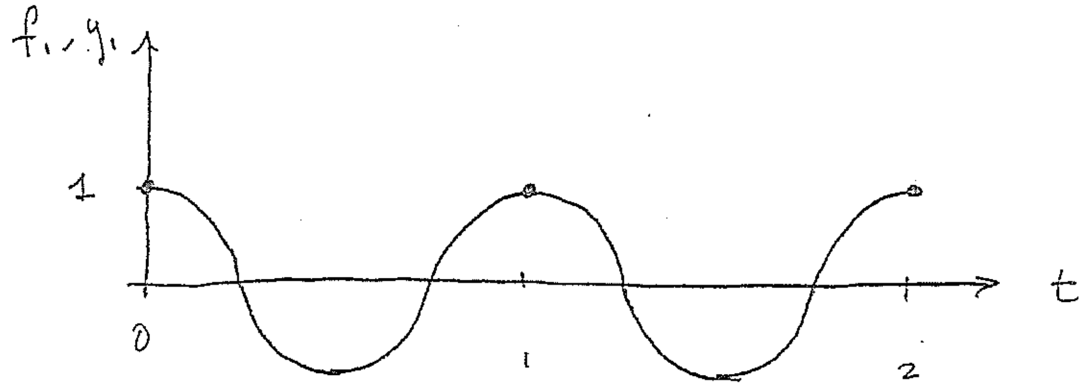
$$= y(t-T)$$

$\rightarrow$ time-invariant

b. "Ideal sampler"

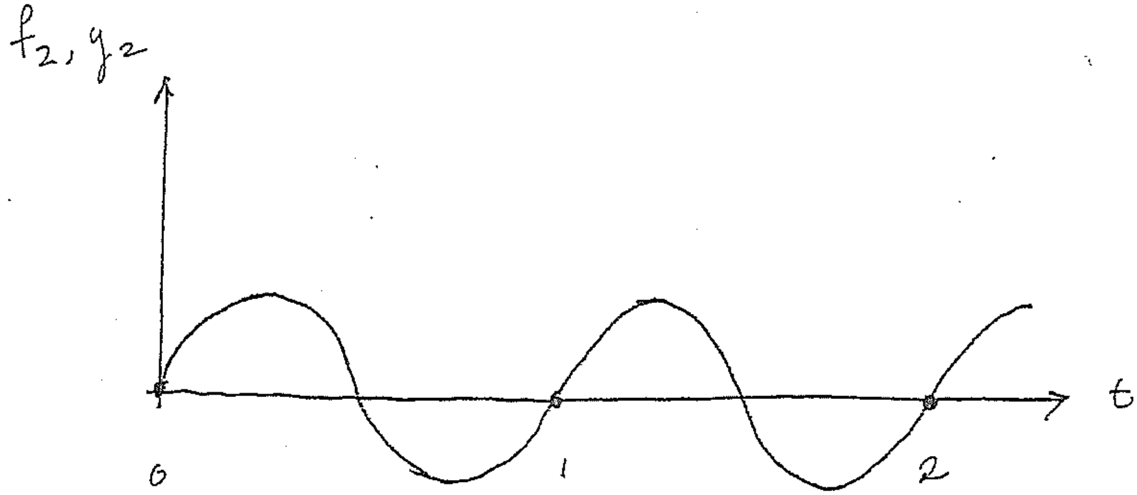
$$f(t) \xrightarrow[\;1\;]{\quad} y[k] = f(k),\ k \in \mathbb{Z}$$

$$f_1(t) = \cos 2\pi t \xrightarrow{S} y_1[k] = 1,\ \forall k \in \mathbb{Z}$$



$$\begin{aligned} f_2(t) &= f_1(t - \tfrac{1}{4}) \\ &= \cos(2\pi(t - \tfrac{1}{4})) \\ &= \cos(2\pi t - \tfrac{\pi}{2}) \\ &= \sin(2\pi t) \end{aligned}$$

$$\xrightarrow{S} \quad y_2[k] = 0,\ \forall k \in \mathbb{Z}$$



$\longrightarrow$ NOT time-invariant.

Lumped- vs. distributed-parameter systems

- A lumped-parameter system is one in which signals depend only on the variable t ...

... a distributed-parameter system is one in which signal values depend not only on t but also on other independent variables such as spatial variables.

example: In the introductory part of the course we modelled a vibrating string, whose displacement varied not only in time, but also along the length of the string — this is a distributed-parameter system.

- On the other hand, the motion of a rigid body can be described by a finite number of variables – e.g., 3 position variables and 3 orientation variables – that depend only on time — this is a <u>lumped-parameter</u> system.

To understand the terminology, compare two different views of an electrical circuit.

Engineers often consider the signals in a circuit to be confined to the conductors. They also assume that wires have negligible impedance: resistance, inductance and capacitance are limited to a finite number of discrete circuit components.

On the other hand, a theoretical physicist who knew nothing of circuit theory might analyze the circuit using the PDEs of Maxwell's equations.

The engineer sees the state of the circuit as consisting of a finite number of currents and voltages that vary with time;

to the physicist, it consists of electrical and magnetic fields and current densities, all of which vary in space as well as in time.

The physicist's view is more accurate, and if signals are changing rapidly in time, it may be necessary to use his or her model . . .

. . . but for many applications, the engineer's approximation works very well. Because, in the engineer's view the only spatial variation in the key quantities occurs at a finite number of points in the circuit, his or her model is called a <u>lumped - parameter</u> approximation of the physicist's <u>distributed - parameter</u> model.

Piecewise - continuous functions

A function $f : \mathbb{R} \to \mathbb{C}$ is piecewise - continuous on a given interval if it has only a finite number of discontinuities in that interval, and at each such discontinuity, both its right - and left - hand limits exist.

That is, for any point of discontinuity $t_0 \in \mathbb{R}$,

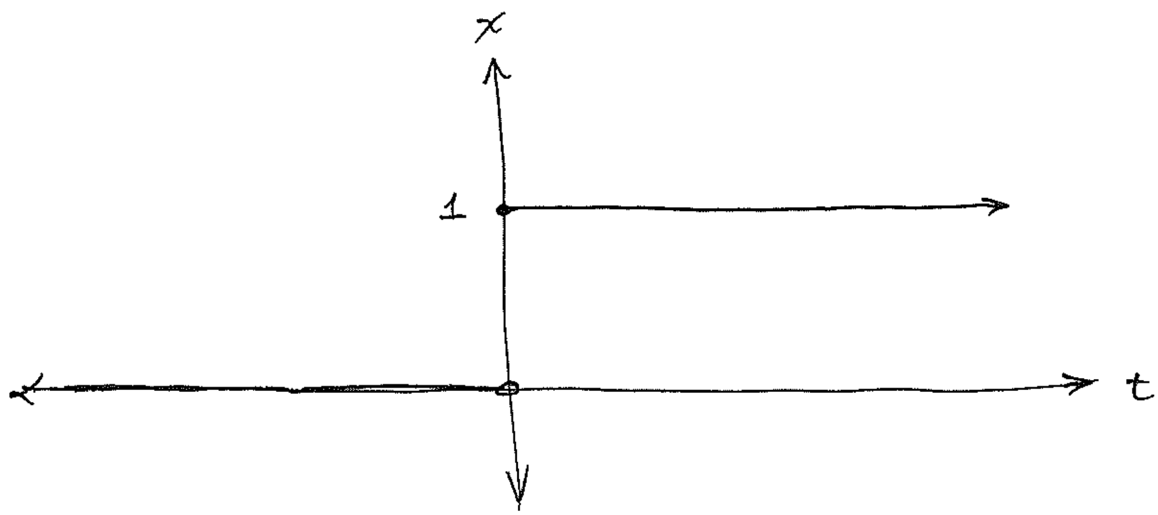
$$\lim_{t \uparrow t_0} f(t) \quad \text{and} \quad \lim_{t \downarrow t_0} f(t)$$

both exist.

If f is piecewise - continuous on any interval of finite length, we'll simply say that f is piecewise - continuous.

We'll restrict attention to signals that are piecewise-continuous.

example: unit step



$$x(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases}$$

$$\lim_{t \uparrow 0} x(t) = 0, \quad \lim_{t \downarrow 0} x(t) = 1$$

- We'll respectively denote the left-hand sides of the above two equations $x(0^-)$ and $x(0^+)$.

Zero-state and zero-input responses

Suppose that a causal, linear, time-invariant, lumped-parameter system, with input $f(t)$ and output $y(t)$, is modelled by a linear ODE with constant coefficients

$$Q(D)\, y(t) = P(D)\, f(t)$$

of order n, and satisfies n initial conditions

$$y(0^-) = p_0$$

$$\dot{y}(0^-) = p_1$$

$$\vdots$$

$$y^{(n-1)}(0^-) = p_{n-1}$$

We'll typically be interested in the response $y(t)$ for $t \geq 0$.

If the system is dynamic, then that part of the response will depend on values of $f(t)$ for <u>both</u> negative and nonnegative values of t.

Let's decompose $f(t)$ into the sum of a function $f_-(t)$, whose value is zero for all $t \geq 0$, and a function $f_+(t)$ whose value is zero for all $t < 0$:

$$f(t) = f_-(t) + f_+(t)$$

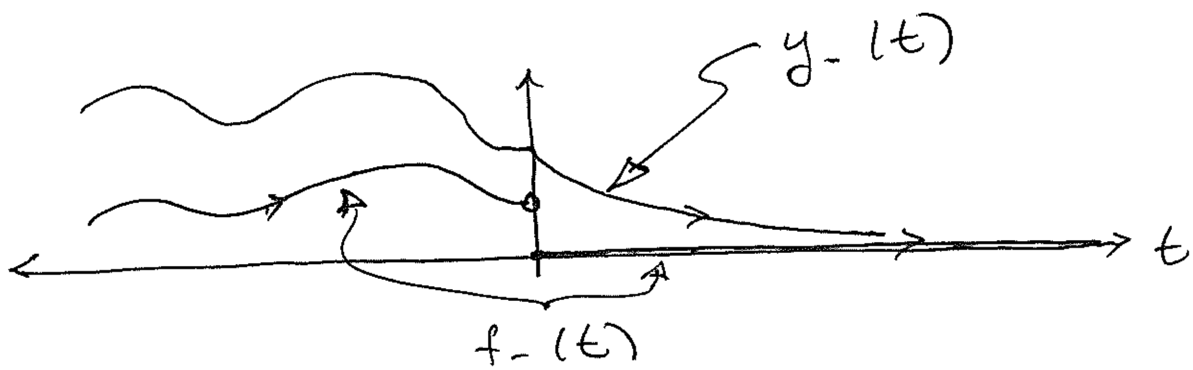
Now suppose that $f(t)$ and all of its derivatives are piecewise-continuous. Then the equation

$$Q(D)\, y(t) = P(D)\, f_-(t),$$

together with the above initial conditions, has a unique solution — call it $y_-(t)$.

This signal is called the zero-input response, because it's the response that we get if the input $f(t)$ is 'turned off' at $t=0$.

Note that $y_-(t)$ need not be zero for $t \geq 0$:



- Indeed, $y_-(t)$ solves the auxiliary equation, and the initial conditions.

Now let $y(t)$ be the unique solution of

$$Q(D)\, y(t) = P(D)\, f(t),$$

together with the initial conditions.

Note that, for all $t < 0$,

$$f(t) = f_-(t)$$

It follows from causality that

$$y(t) = y_-(t),$$

for all $t < 0$.

Let

$$y_+(t) := y(t) - y_-(t).$$

Then, for all $t < 0$, $y_+(t) = 0$.

Moreover, because both y and y_- satisfy the above initial conditions, we must have

$$y_+(0^-) = 0$$
$$\dot{y}_+(0^-) = 0$$
$$\vdots$$
$$y_+^{(n-1)}(0^-) = 0$$

So $y_+(t)$ is the response to the input $f_+(t)$, with the system initially 'at rest' – with all initial conditions equal to zero.

The response $y_+(t)$ is called the <u>zero-state response</u>.

<u>Note</u>: Even for $t \geq 0$, the zero-input and zero-state responses are generally <u>not</u> the parts of the solution $y(t)$ corresponding to the complementary solution and the particular solution: the zero-input response must itself satisfy the initial conditions, and the zero-state response must satisfy initial conditions that are all zero-valued.

For example, the zero-state response will generally contain terms from the complementary solution...

... these will be necessary for the zero-state solution to satisfy the zero-valued initial conditions.

For our purposes, it will suffice to study the zero-state response:

- it can often be considered to subsume the zero-input response, in the sense that a set of non-zero initial conditions can often be satisfied (at a time $t_0 > 0$) by suitable choice of an $f_+(t)$;
- and in any case, the zero-state response will tell us all we need to know about the system's dynamics.

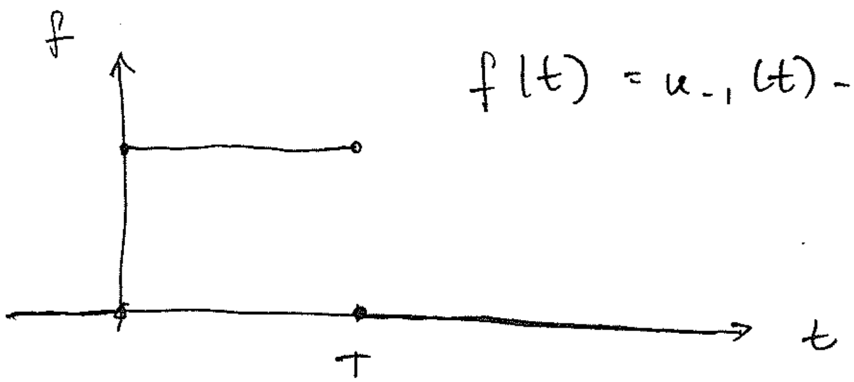
Mathematically, we can picture the convolution operation as follows.

Suppose for simplicity that $f(t) = 0$ for negative t, and likewise for $h(t)$.

Then $$\int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

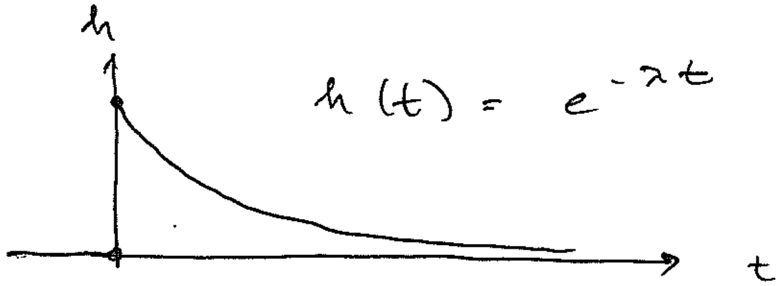
$$= \int_{0^-}^{\tau} f(\tau)\, h(t-\tau)\, d\tau$$

Let's suppose that $f(t)$ is a square pulse ...



$$f(t) = u_{-1}(t) - u_{-1}(t-T)$$

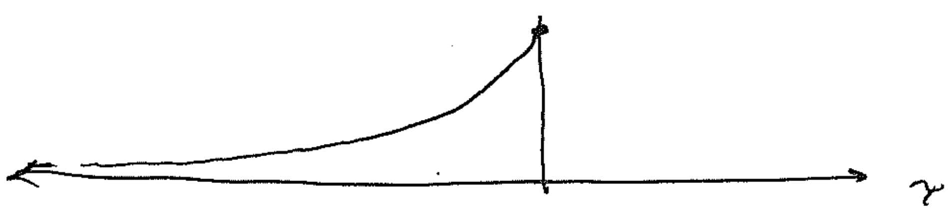
... and that $h(t)$ is a decaying exponential



Let's plot $f(\tau)$ and $h(t-\tau)$ against τ.

For this, rewrite $h(t-\tau)$ as $h(-(\tau-t))$.

First, consider the plot of $h(-\tau)$:



- This key property of LTI systems reduces the analysis of their time-domain responses to convolution . . .

. . . for another key property, let's consider the response of an LTI system to an <u>exponential</u> input.

Suppose that S is LTI, and that, for some $s \in \mathbb{C}$,

$$e^{st} \xrightarrow{S} y(t).$$

Then, by time-invariance, for any $T \in \mathbb{R}$,

$$e^{s(t-T)} \xrightarrow{S} y(t-T).$$

... but $e^{s(t-T)} = e^{-sT} e^{st}$, so by linearity,

$$e^{s(t-T)} = e^{-sT} e^{st} \xrightarrow{S} e^{-sT} y(t) = y(t-T)$$

Since this holds for any $T \in \mathbb{R}$, we can, in particular, set $T = t$, for any given $t \in \mathbb{R}$: then

$$e^{-st} \, y(t) = y(0)$$

$$\Longleftrightarrow \qquad y(t) = y(0) \, e^{st}$$

... so the response is just the input e^{st}, multiplied by a constant, $y(0)$.

What's the value of this constant?

By the convolution integral,

$$y(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, f(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

We'll call the function $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$ the <u>transfer function</u> of the system...

... so to find its response to an exponential input e^{st}, we simply multiply this input by $H(s)$:

$$y(t) = H(s)\, e^{st}$$

Discrete-time convolution

$$y[k] = \sum_{j=-\infty}^{\infty} f[j]\,h[k-j]$$

(sum of all products of values of f and h whose indices sum to k)

- fundamental to discrete-time LTI systems
- examples:
  - math:

    multiplication of polynomials

    $$\sum_{j=0}^{m} f[j]\,x^j \quad , \quad \sum_{j=0}^{n} h[j]\,x^j$$

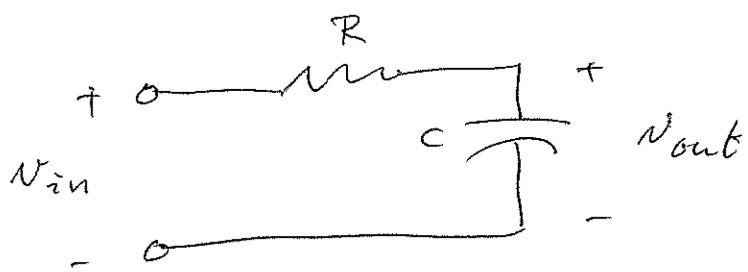
  - engineering

    convolutional codes for error-correction in digital communications

. . . but it will be helpful if we first study the responses of LTI systems in the time domain.

It's particularly important to relate the forms of these responses to the positions of the poles of the transfer function.

Example: RC circuit



What is the transfer function relating the input to the output?

Differential equation:

$$RC\frac{dv_{out}}{dt} + v_{out} = v_{in}$$

Taking Laplace transforms:

$$RC\left[sV_{out}(s) - v_{out}(0^-)\right] + V_{out}(s) = V_{in}(s)$$

Now, the transfer function is the transform of the impulse response, and the impulse response is a zero-state response — so set the initial condition to zero.

$$[sRC + 1]\, V_{out}(s) = V_{in}(s)$$

$$\Longleftrightarrow \quad V_{out}(s) = \frac{1}{sRC + 1}\, V_{in}(s)$$

So the transfer function is

$$H(s) = \frac{1}{sRC + 1}$$

- This is called a <u>first-order</u> transfer function, because it has only one pole.

Standard 1st-order system

$$H(s) = \frac{K}{s\tau + 1} \quad , \quad K, \tau > 0$$

(e.g., dc motor with negligible armature inductance, RC circuits)

- impulse response:

$$y(t) = \mathcal{L}^{-1}\{H(s) \cdot 1\}$$

$$= K/\tau \; e^{-t/\tau} \cdot u_{-1}(t)$$

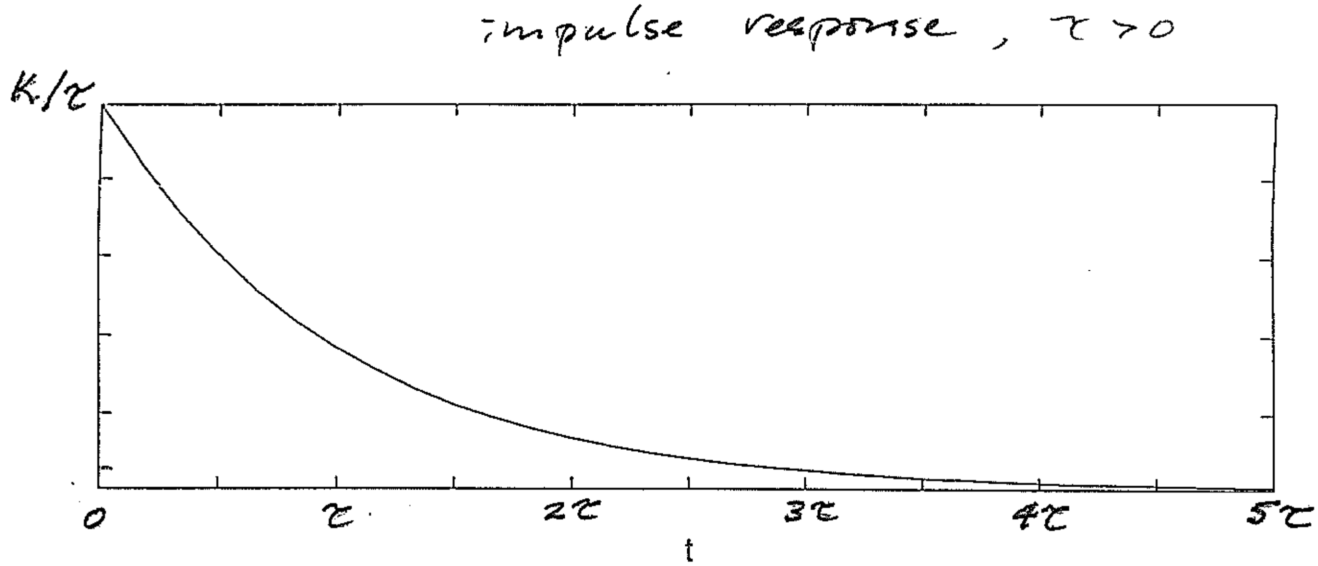
(unit step)

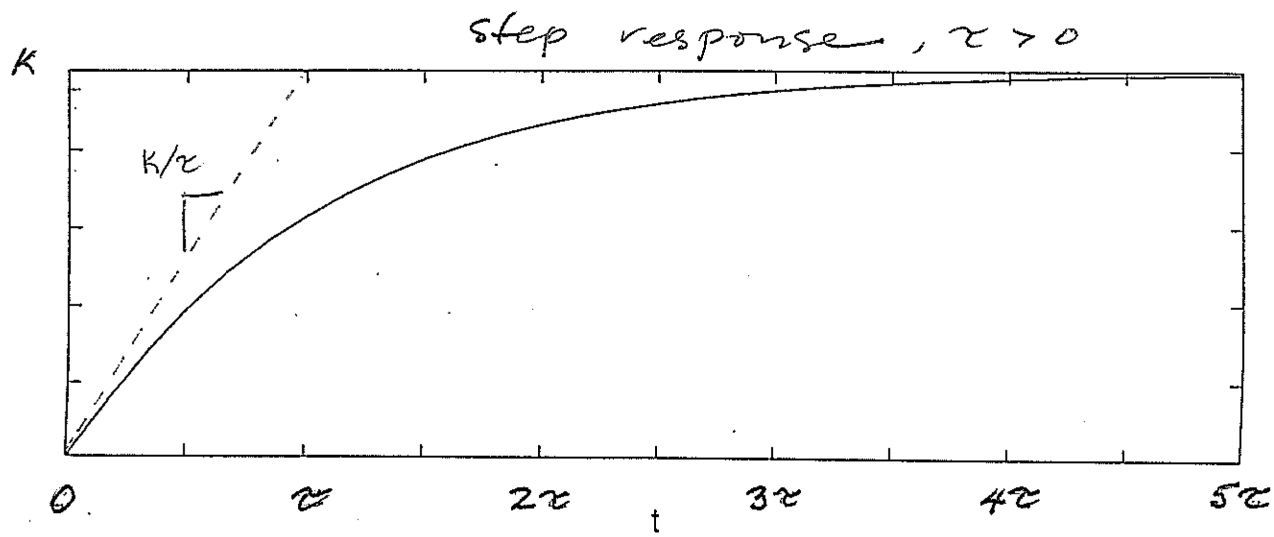
- step response:

$$y(t) = \mathcal{L}^{-1}\{H(s) \cdot \tfrac{1}{s}\}$$ (antiderivative of impulse response)

$$= K\,[1 - e^{-t/\tau}]\, u_{-1}(t)$$

> Note that the steady-state value of the step response is K — the "dc gain" of the transfer function — provided $\tau > 0$.


impulse response , τ > 0
K/τ
0
τ
2τ
3τ
4τ
5τ
t


step response , τ > 0
K
K/τ
0
τ
2τ
3τ
4τ
5τ
t

Note that the other transfer-function parameter, τ, determines the rate of decay of the transient term $e^{-t/\tau}$ — for this reason, τ is called the <u>time constant</u>.

If $t = \tau$, $e^{-t/\tau} = e^{-1}$;

if $t = 3\tau$, $e^{-t/\tau} = e^{-3} \simeq 0.05$

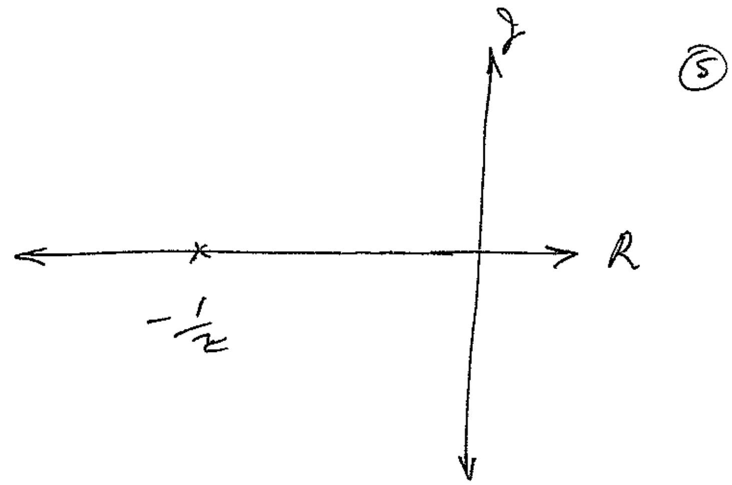
(i.e. transient has decayed to 5% of its initial value after three time constants) ;

if $t = \tau$, $e^{-t/\tau} = e^{-4} \simeq 0.02$

(i.e., transient has decayed to 2% after four time constants).

Position of pole on s-plane:

- 1 real pole: $s = -\frac{1}{\tau}$

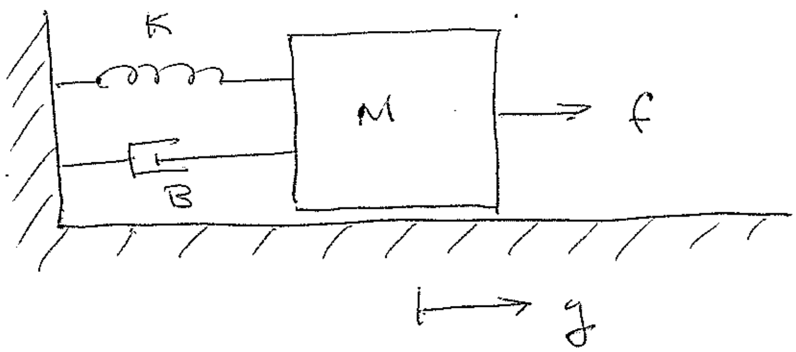


- as pole $\to 0$, $\tau \to \infty$, so response slows (transient decays more slowly)

- as pole $\downarrow -\infty$, $\tau \to 0$, so response speeds up.

Note in particular that if the pole lies to the right of the imaginary axis, τ is negative, and the exponentials are growing, rather than decaying exponentials.

Example: Mass - spring - damper



$$M\ddot{y} + B\dot{y} + Ky = f$$

Taking Laplace transforms with initial conditions set to zero:

$$\left[Ms^2 + Bs + K\right]Y(s) = F(s)$$

$$\Longleftrightarrow \quad Y(s) = \frac{1}{Ms^2 + Bs + K}F(s)$$

So the transfer function is

$$H(s) = \frac{1}{Ms^2 + Bs + K}$$

$$= \frac{1/M}{s^2 + \frac{B}{M}s + \frac{K}{M}}$$

– a <u>second-order</u> transfer function.

- Standard 2^{nd}-order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \omega_n > 0$$

- e.g., spring-mass-damper systems, RLC circuits

- We'll consider only the "underdamped case", $0 < \zeta < 1$, since we've already looked at the case of real poles.

- impulse response:

$$y(t) = \mathcal{L}^{-1}\{H(s) \cdot 1\}$$

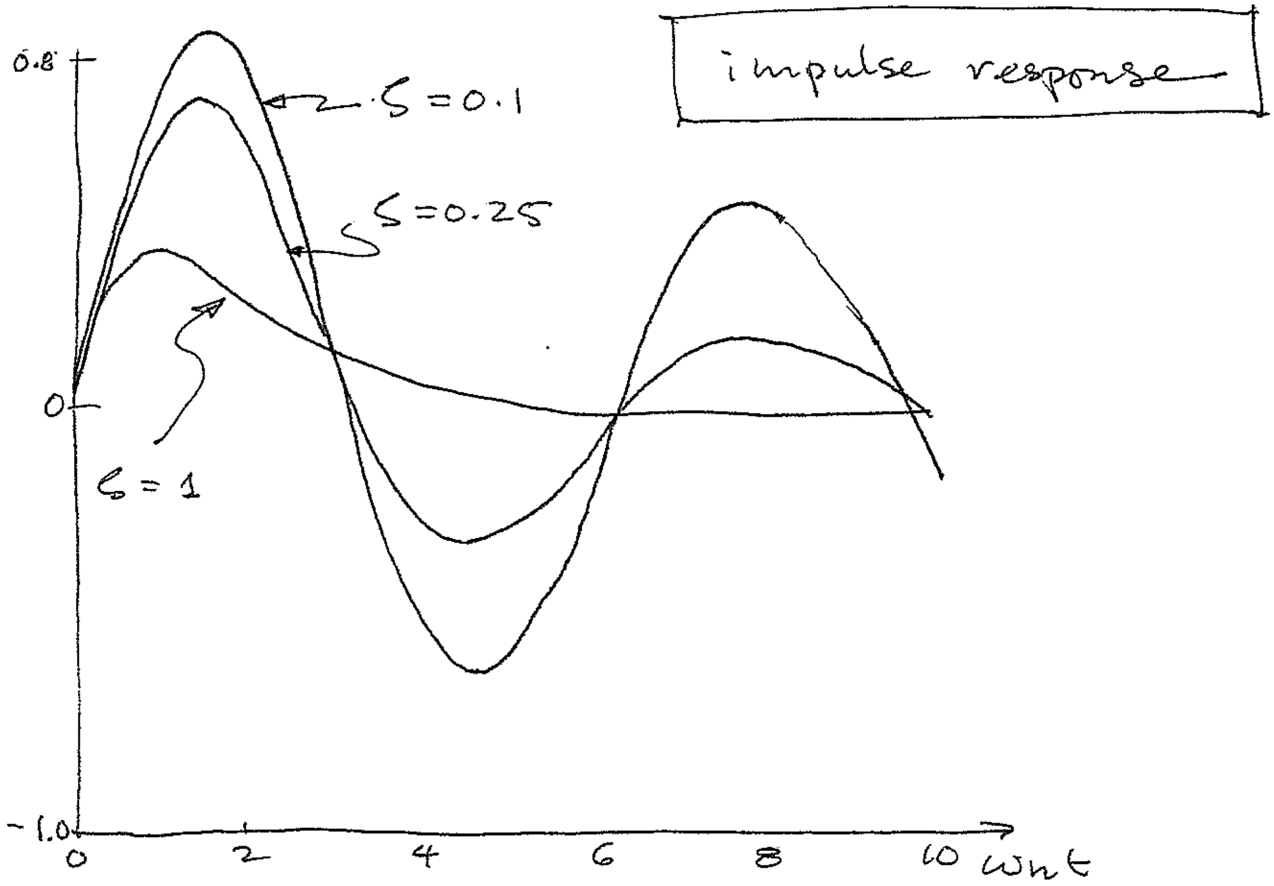
$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

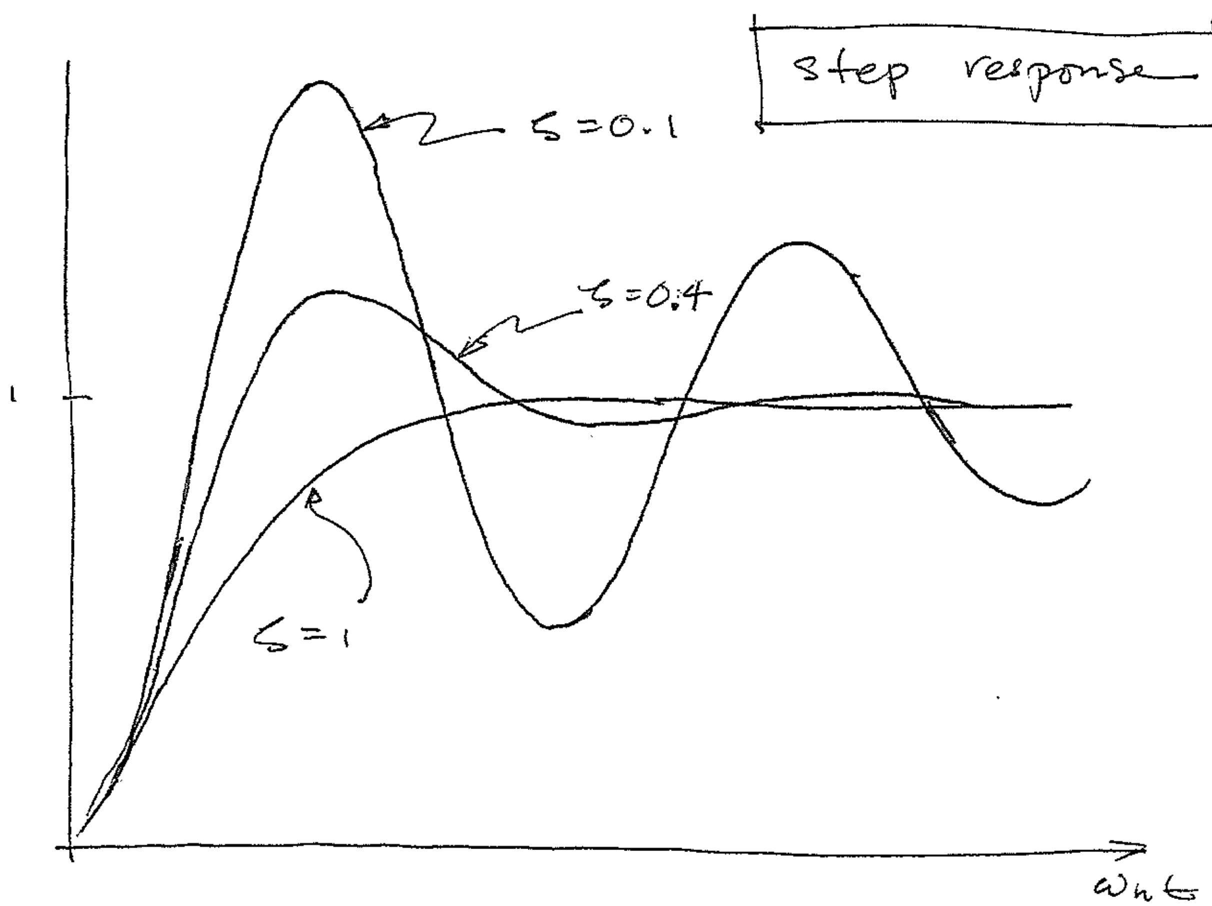
- step response:

$$y(t) = \mathcal{L}^{-1}\{H(s) \cdot \tfrac{1}{s}\}$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \theta),$$

where $\theta = \cos^{-1}\zeta$


impulse response
0.8
0
-1.0
0
2
4
6
8
10
$\omega_n t$
$\zeta = 0.1$
$\zeta = 0.25$
$\zeta = 1$


step response
1
$\omega_n t$
$\zeta = 0.1$
$\zeta = 0.4$
$\zeta = 1$

It's convenient to plot the responses as functions of $\omega_n t$; because wherever t appears in the expressions for $y(t)$, it's multiplied by ω_n

$\Longrightarrow$ ω_n acts as a time-scale factor:

$\omega_n \uparrow \Longrightarrow$ response speeds up

$\omega_n \downarrow \Longrightarrow$ response slows

(for a fixed value of ζ).

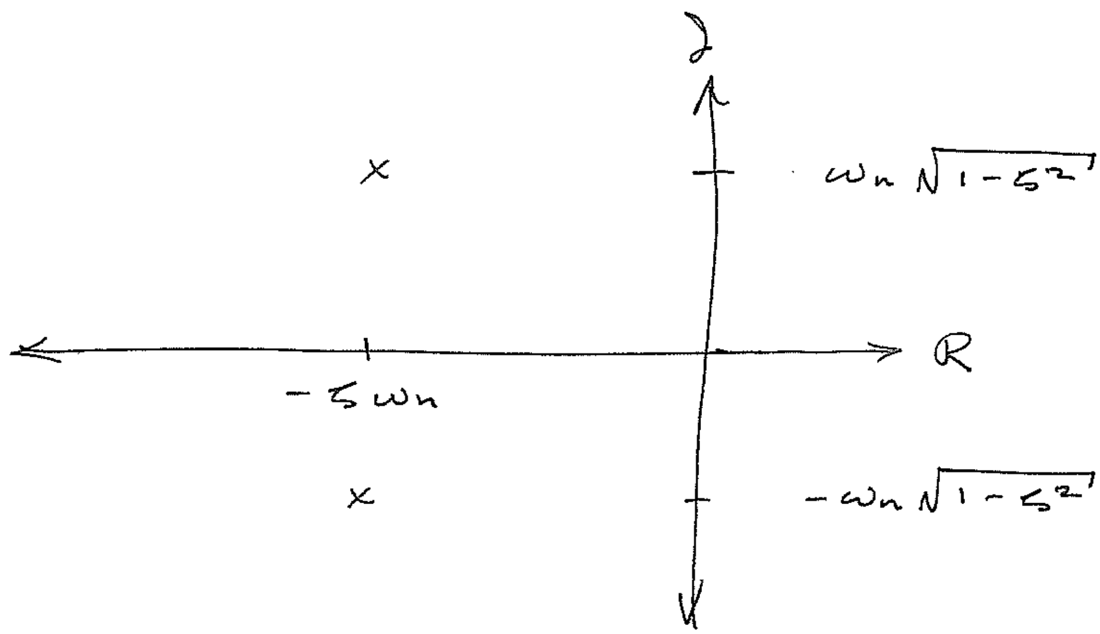
The plots show that (for a fixed ω_n)

$\zeta \uparrow \Longrightarrow$ response is less oscillatory

We call ω_n the <u>natural frequency</u> (the frequency of oscillations if ζ were 0), and ζ the <u>damping ratio</u>.

Positions of pole on s-plane:

$0 < \zeta < 1 \implies s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$



- So, the real part of the poles, $-\zeta\omega_n$, determines the rate of decay of the amplitude of the oscillations $e^{-\zeta\omega_n t}$ . . .

. . . and the imaginary part, $\pm \omega_n \sqrt{1-\zeta^2}$, determines the angular frequency of the oscillations $\sin \omega_n \sqrt{1-\zeta^2}\, t \; (+\theta)$.

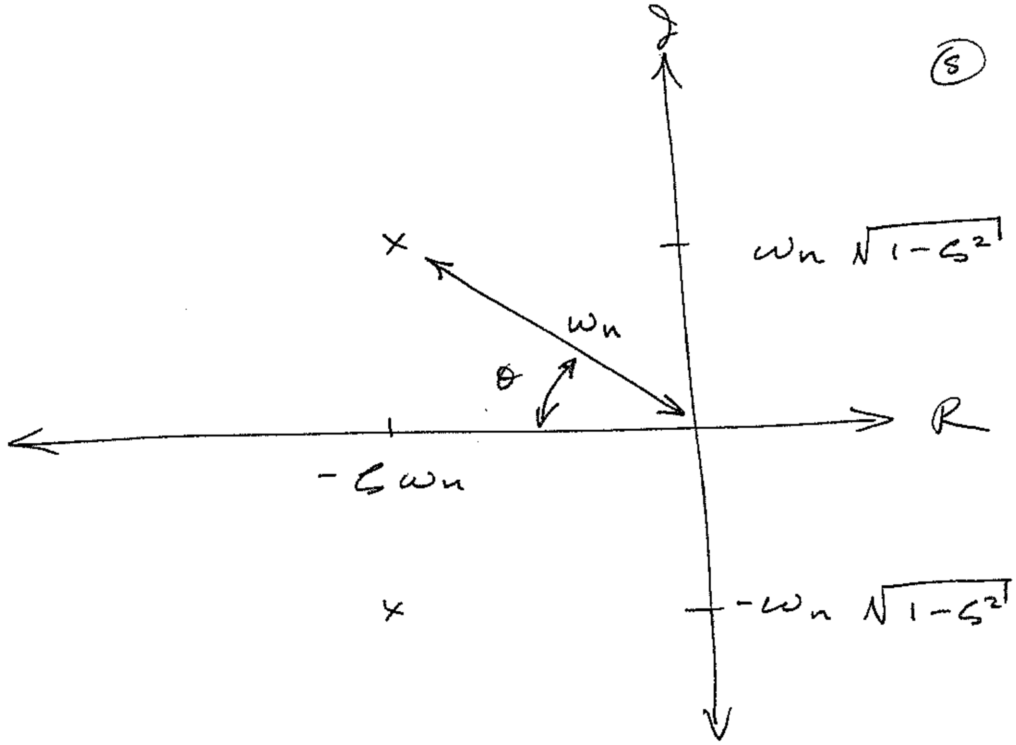
But there's also another simple way of relating the pole positions to the response:

- the <u>modulus</u> of the poles is

$$\sqrt{(\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2} = \sqrt{\omega_n^2}$$
$$= \omega_n$$

. . . and the angle that the poles make with the real axis is therefore $\cos^{-1}\zeta = \theta$

(see the next page).



So, we also have the following relationship:

- modulus $\uparrow \iff \omega_n \uparrow \implies$ response speeds up

- $\theta \downarrow \iff \zeta \uparrow \implies$ response less oscillatory

Again, we find that the further the pole(s) from the origin, the faster the response; but also, the closer they are to the real axis (for fixed modulus), the less oscillatory the response.

This analysis can easily be extended to more complex transfer functions

- To see how, consider adding a zero to the 2nd-order transfer function:

$$H_z(s) = \frac{\omega_n^2 \left(\frac{s}{\alpha \zeta \omega_n} + 1 \right)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (\alpha > 0)$$

- step response:

$$y(t) = \mathcal{L}^{-1} \left\{ H_z(s) \cdot \frac{1}{s} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \right.$$

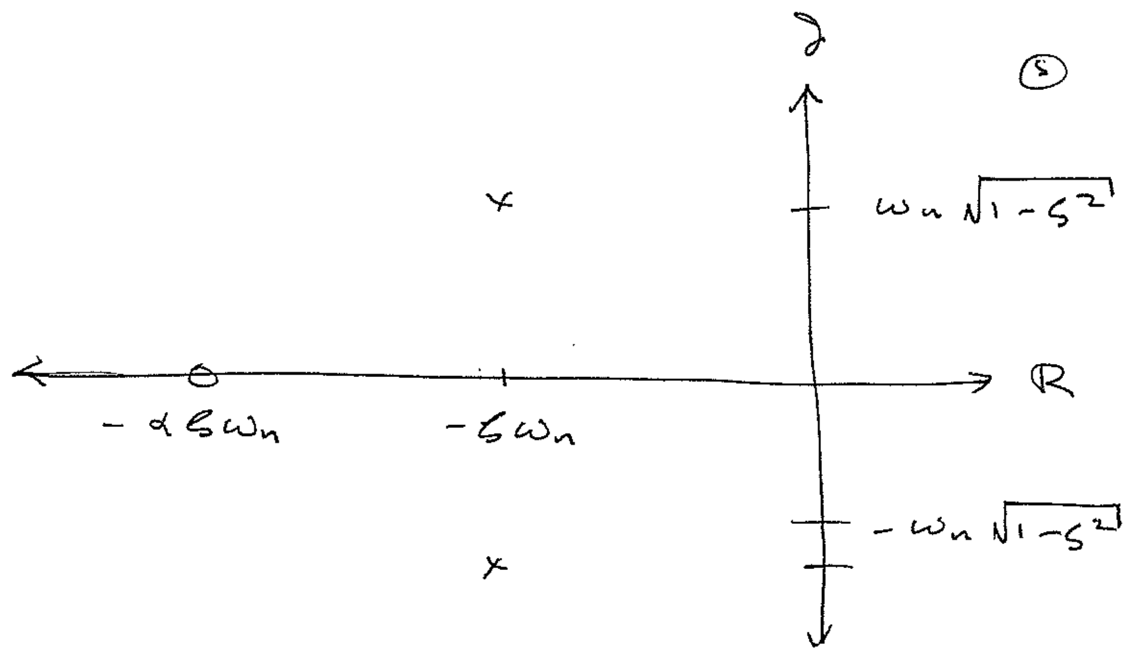
$$\left. + \frac{s}{\alpha \zeta \omega_n} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \right\}$$

But note that this means that

$y(t)$ = step response of standard 2nd-order system

$+ \frac{1}{\alpha \zeta \omega_n} \times$ impulse response of standard 2nd-order system

Consider the pole-zero diagram:

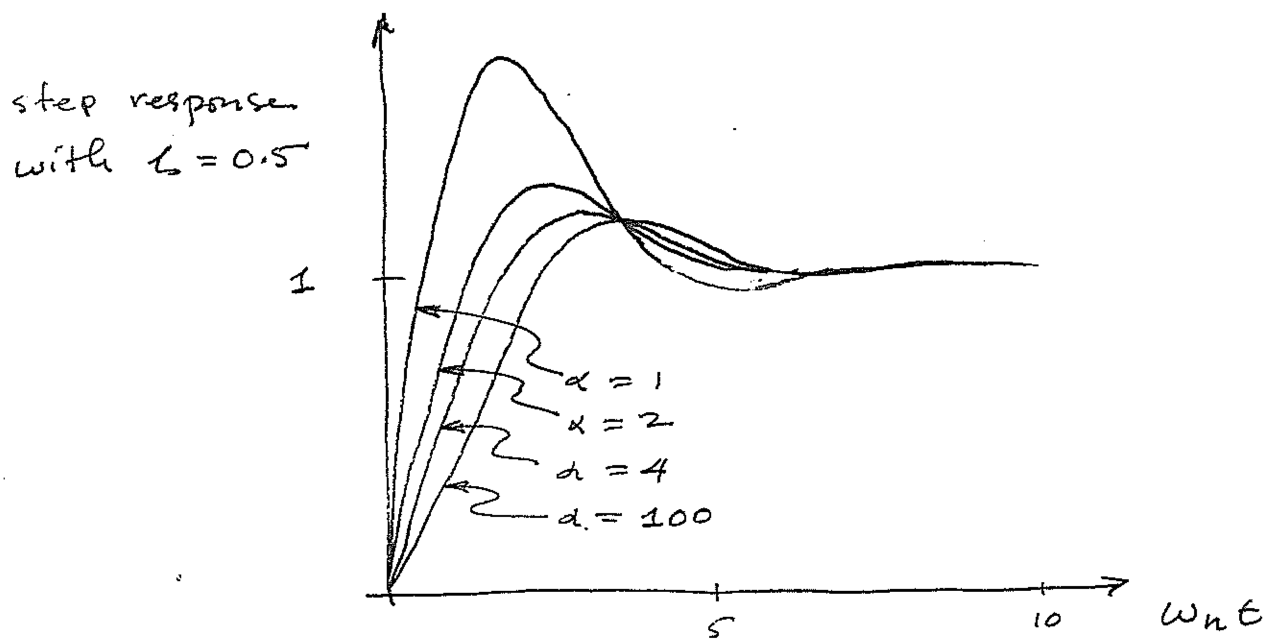


- By the expression for $y(t)$

zero $\downarrow -\infty \iff \alpha \to \infty \implies y(t) \to$ standard 2nd-order step response

... and by the following plot,

zero $\to 0 \iff \alpha \to 0 \Rightarrow$ response more oscillatory.



To see the effect of an added pole, consider the transfer function

$$H_p(s) = \frac{\omega_n^2}{\left(\frac{s}{\alpha\zeta\omega_n}+1\right)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

$(\alpha > 0)$

$$= \frac{a}{\frac{s}{\alpha\zeta\omega_n}+1} + \frac{bs+c}{s^2+2\zeta\omega_n s+\omega_n^2}$$

(by an appropriate partial fractions decomposition)

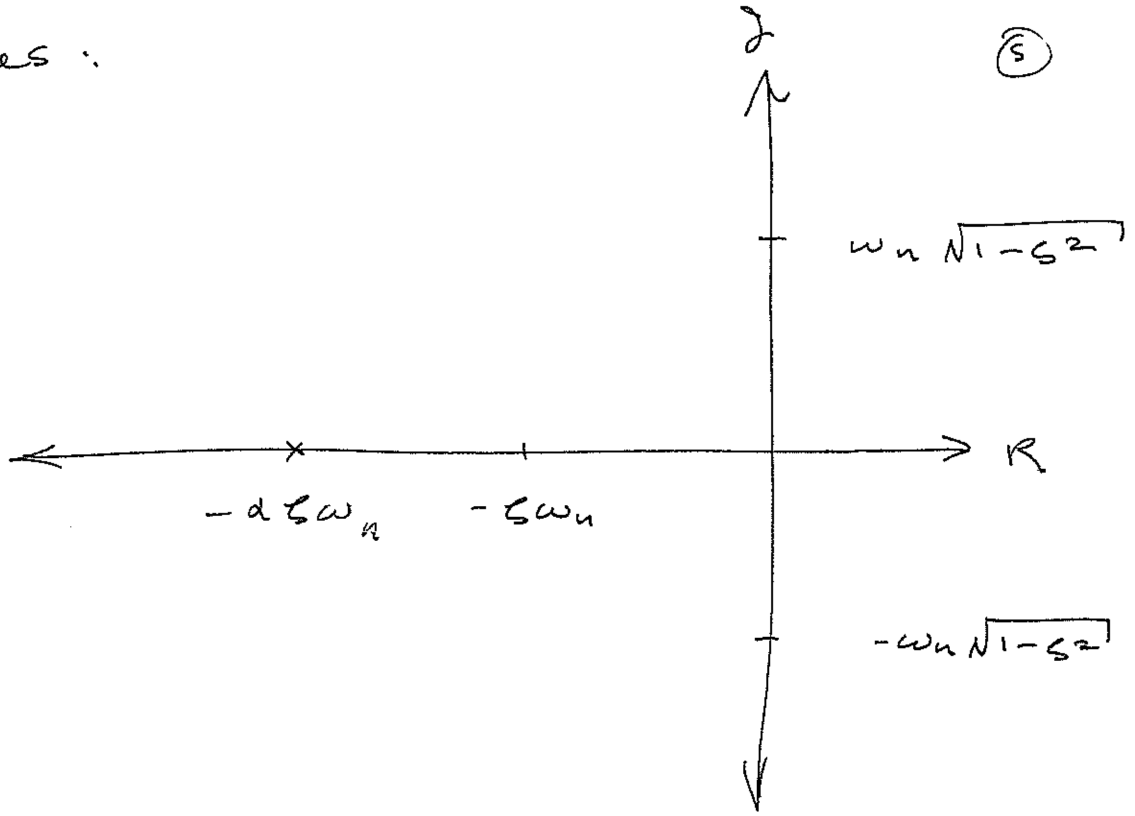
where

$$a = \frac{1}{1-2\alpha\zeta^2+\alpha^2\zeta^2}$$

$$b = -\alpha\zeta\omega_n a$$

$$\& \quad c = \alpha(\alpha-2)\zeta^2\omega_n^2 a$$

Poles:



Note that,

real pole $\to 0 \iff \alpha \to 0 \iff a \to 1,$

$b, c \to 0$

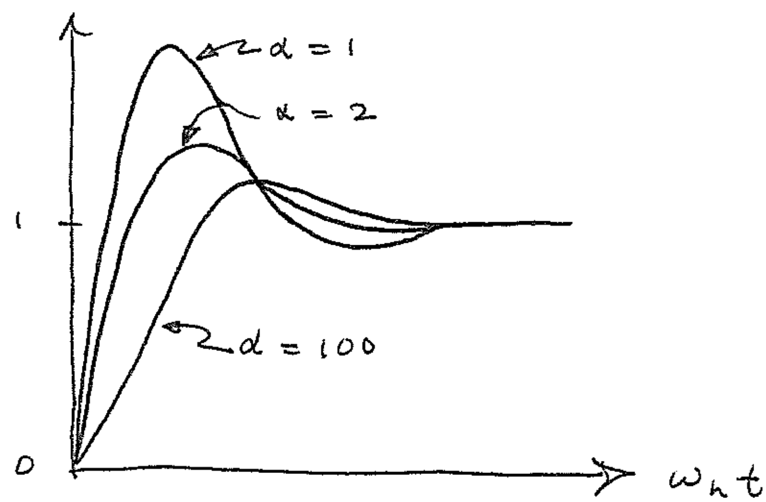
$\iff$ response approaches that of 1^{st}-order system

real pole $\to -\infty \iff \alpha \to \infty \iff a \to \dfrac{1}{\alpha^2\zeta^2} \to 0$

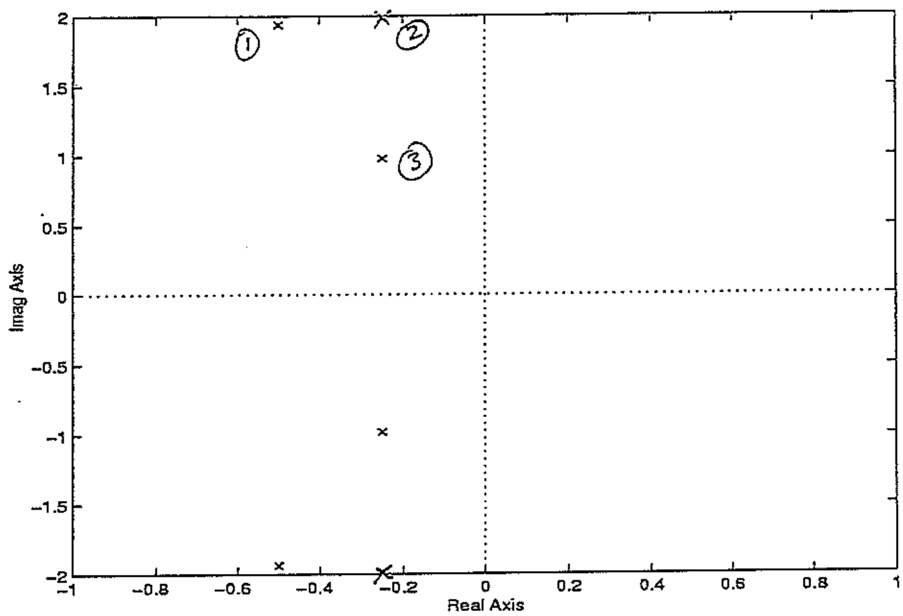
$b \to -\dfrac{\alpha\zeta\omega_n}{\alpha^2\zeta^2} \to 0$

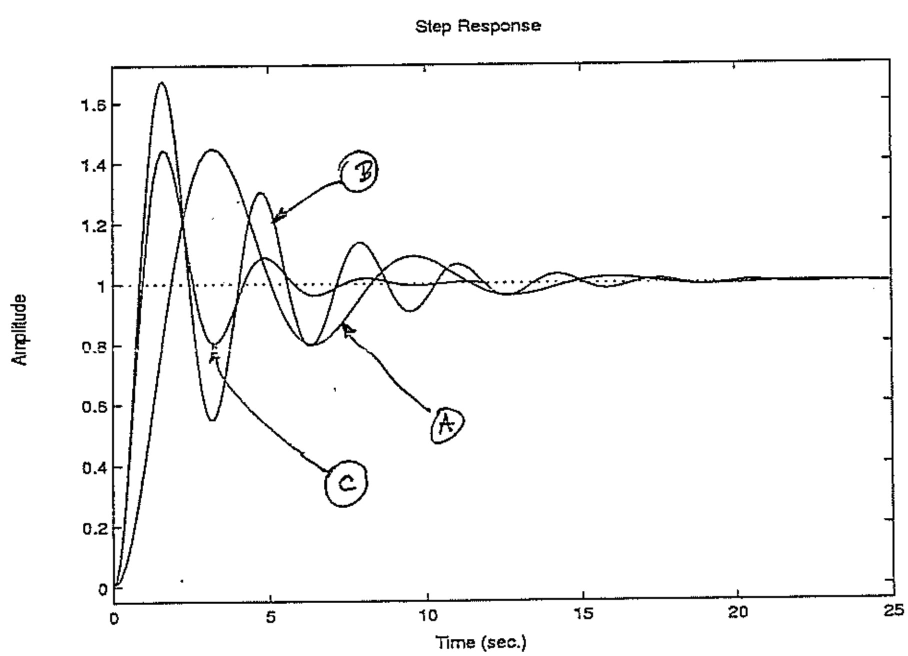
$c \to \dfrac{\alpha^2\zeta^2\omega_n^2}{\alpha^2\zeta^2} = \omega_n^2$

$\iff$ response approaches that of std. 2^{nd}-order sys.


α = 1
α = 2
α = 100
1
0
$\omega_n t$

The main message is that you can tell a lot about a system's transient response from its poles:





A ⟷ 3

B ⟷ 2

C ⟷ 1

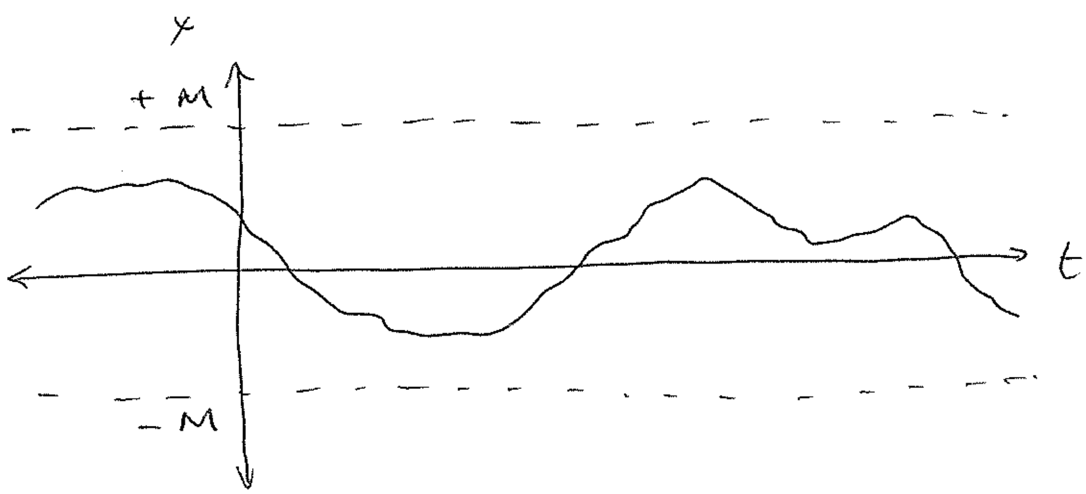
In all cases, poles that lie to the right of the imaginary axis give rise to growing exponentials in the time domain . . .

For this reason, we say that a rational transfer function is <u>stable</u> if all of its poles lie strictly to the left of the imaginary axis — that is, if they all have real parts that are negative.

A useful notion of stability of an LTI system rests on the notion of <u>boundedness</u> of a signal:

A signal $x(t)$ is <u>bounded</u> if there exists a real number M such that

$$|x(t)| \le M, \quad \forall t$$

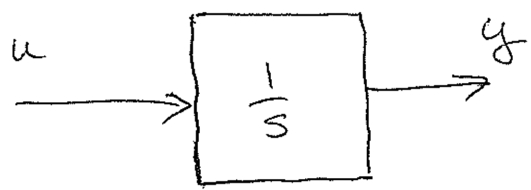


A SISO LTI system with a rational transfer function is <u>bounded-input, bounded-output (BIBO) stable</u> if its zero-state response is bounded whenever its input is.

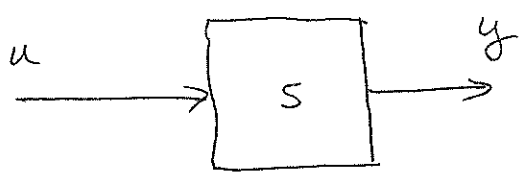
- *example*: if their transfer functions are stable, then our standard first- and second-order systems are BIBO stable.

- *non-examples*:

  - integrator:



  - differentiator:



(Consider their step responses.)

Theorem: A SISO LTI system with rational transfer function is BIBO stable if and only if its transfer function is both stable and proper.

Proof:

(If) If the transfer function $H(s)$ is stable and proper, it can be decomposed into a constant a_0 plus terms of the form

$$\frac{a_{ik}}{(s-p_i)^k}, \quad Re(p_i) < 0$$

Such a term has an inverse Laplace transform of the form

$$t^k e^{p_i t} u_{-1}(t)$$

If the input is a bounded signal $u(t)$, with $|u(t)| \leq M$, then the output is a sum of convolutions of the form

$$\int_{0^-}^{\infty} u(t-\tau)\, a_0\, \delta(\tau)\, d\tau$$

$$\leq M a_0$$

and

$$\int_{0^-}^{\infty} u(t-\tau)\, \tau^k e^{p_i \tau}\, d\tau$$

$$\leq M \int_{0^-}^{\infty} \tau^k e^{p_i \tau}\, d\tau$$

Each of the convolutions, and therefore their sum, is bounded.

(Only if):

Suppose that the transfer function $H(s)$ is unstable. Then $H(s)$ has a pole p with $Re(p) \geq 0$.

If $Re(p) > 0$, then the step response includes an increasing exponential term.

If $p = 0$, then the step response includes a ramp.

If $p = j\omega$, for some $\omega \in \mathbb{R}$, then let the input $u(t) = e^{j\omega t}$, then the output contains a term

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-j\omega)^2}\right\}$$

$$= t\, e^{j\omega t}$$

It follows that if $H(s)$ is unstable, the system is not BIBO stable.

Suppose now that $H(s)$ is stable but improper. Then

$$H(s) = Q(s) + \frac{N(s)}{D(s)}$$

where Q, N and D are polynomials (Q nonconstant and D nonzero) and $N(s)/D(s)$ is strictly proper.

If $u(t) = u_{-1}(t)$, then the output $y(t)$ contains a term

$$\mathcal{L}^{-1}\left\{ Q(s) \frac{1}{s} \right\}$$

which includes a unit impulse $\delta(t)$ (and possibly "derivatives" of unit impulses). It follows that $y(t)$ is unbounded.

$\square$