Integration: Summary

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Abstract

This handout summarizes the various integration techniques. It also give alternatives for finding definite integrals when an antiderivative cannot be found.

1   Strategy for Integration

We have studied the following techniques:

1. Fundamental theorem of Calculus. This changes the problem of finding an integral to the problem of finding antiderivatives.

2. Substitution.

3. Integration by parts.

4. Partial fraction decomposition.

5. Integrals involving trigonometric functions.

6. Trigonometric substitution.

7. Tables of integrals.

Besides knowing all the above techniques, students must also know the following antiderivatives:

1. $\int u^n du = \dfrac{u^{n+1}}{n+1} + C$ if $n \neq -1$

2. $\int \dfrac{1}{u} du = \ln |u| + C$

3. $\int e^u du = e^u + C$

4. $\int a^u du = \dfrac{a^u}{\ln a} + C$

5. $\int \sin u\, du = -\cos u + C$

6. $\int \cos u\, du = \sin u + C$

7. $\int \sec^2 u\, du = \tan u + C$

8. $\int \csc^2 u\, du = -\cot u + C$

9. $\int \sec u \tan u\, du = \sec u + C$

10. $\int \csc u \cot u\, du = -\csc u + C$

11. $\int \sec u\, du = \ln|\sec u + \tan u| + C$

12. $\int \tan u\, du = \ln|\sec u| + C$

13. $\int \dfrac{1}{u^2 + 1}\, du = \tan^{-1} u + C$

14. $\int \dfrac{1}{\sqrt{1 - u^2}}\, du = \sin^{-1} u + C$

The problem is: given an integral, which method should be used to find it. Unfortunately, there is not a magical recipe. A lot comes from experience. Also, when doing problems, look at which technique worked for a given problem, try to remember it and also try to understand why it worked. The following provides some guidelines regarding which methods to try depending on the type of integral given. Given an integral $\int f(x)\, dx$ ($f(x)$ is called the integrand), try the following steps:

1. **Simplify the integrand.** Before you try to integrate a function, make sure it is written in its simplest form. This can be achieved by using the various identities studied up to this point.

2. **Look for an obvious substitution.** Try to find some function $g(x)$ in the integrand whose differential $g'(x)\, dx$ also occurs (up to a constant). In this case, try the substitution $u = g(x)$. For example, given $\int \dfrac{x}{x^2 - 1}\, dx$, we note that if $u = x^2 - 1$, then $du = 2x\, dx$. So, we see that both u and du (except for $\dfrac{1}{2}$) then we would try $u = x^2 - 1$.

3. **Classify the integrand according to its form.** If steps 1 and 2 have failed, try the following:

   (a) *Trigonometric functions.* Try the techniques described in the handout on integrals involving trigonometric functions.

   (b) *Rational functions.* Try the techniques described in the handout on partial fractions.

(c) *Integration by parts.* If $f(x)$ is a product of a power of x (or a polynomial) and a transcendental function (trigonometric, exponential, logarithmic), try integration by parts.

(d) *Radicals.* If the integral contains $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 - a^2}$, use the appropriate trigonometric substitution. If $\sqrt[n]{ax + b}$ occurs, try the substitution $u = \sqrt[n]{ax + b}$.

4. **Try again.** If the first three steps have produced no answer, try harder.

(a) *Try substitution.* You may not have tried all the possibilities for the given integral.

(b) *Integration by parts.* Even if your integral does not have the form described in (3c), integration by parts may still work. Remember that it works sometimes on single functions. We used it to find $\int \sin^{-1} x dx$, $\int \tan^{-1} x dx$ and $\int \ln x dx$.

(c) *Manipulate the integrand.* Try to change the integrand using algebraic manipulations, rationalizing. If the integrand involve trigonometric functions, try to rewrite it in terms of other trigonometric functions using identities.

(d) *Relate the problem to other problems.* Think if you have done a similar problem in the part and how you did it. This is why it is important to remember the problems you do and analyze them.

(e) *Use several methods.* Sometimes, several methods may be used. Either you will repeat the same method several times, or you will mix the various methods studied.

We give a few examples to illustrate these guidelines. In each example, we indicate the method to use but do not carry out the integration.

Example 1 *Find* $\int \dfrac{\sin^4 x}{\sec x} dx$
The integrand contains trigonometric functions, but not of the forms studied. Remembering that $\sec x = \dfrac{1}{\cos x}$, it follows that $\cos x = \dfrac{1}{\sec x}$ and therefore $\int \dfrac{\sin^4 x}{\sec x} dx = \int \sin^4 x \cos x dx$ which we can integrate with the substitution $u = \sin x$.

Example 2 *Find* $\int \dfrac{x^5 + 1}{x^3 - 3x^2 - 10x} dx$
Since there does not seem to be an obvious simplification of the integrand, and it is a rational function, we decompose it into partial fractions. Remember first to perform long division since the degree of the numerator is larger.

Example 3 *Find* $\int \dfrac{dx}{x\sqrt{\ln x}}$
with $u = \ln x$, we see that both u and du are part of this integral. So, we try the substitution $u = \ln x$.

3

2 Approximation of Definite Integrals

By elementary functions, we mean polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric functions and functions which can be obtained from these by addition, subtraction, multiplication, division and composition. These are the functions students are more likely to encounter in traditional calculus classes. The question then becomes can we integrate all the elementary functions? The answer is no. A simple function such as e^{x^2} does not have an antiderivative which can be expressed in terms of elementary functions. Thus, we cannot evaluate $\int e^{x^2} dx$ in terms of functions we know. The same is true for the following:

- $\int \dfrac{e^x}{x} dx$

- $\int \sin\left(x^2\right) dx$

- $\int \cos\left(e^x\right) dx$

- $\int \dfrac{dx}{\ln x}$

- $\int \dfrac{\sin x}{x} dx$

What do we do when a problem arises in which we have to evaluate an integral containing one of these functions? We know the answer. We know we can approximate integrals using Riemann sums. In other words,

$$\int_a^b f\left(x\right) dx = \frac{b-a}{n} \sum_{i=1}^n f\left(x_i^*\right)$$

In the above formula, the following is assumed:

- The interval $[a, b]$ has been subdivided into n sub-intervals of equal length.

- x_i^* is a point selected in the ith sub-interval.

Though x_i^* can be selected anywhere, we usually consider the following five cases:

1. x_i^* is the left endpoint of the interval in which it is selected. We call this sum the left Riemann sum and denote it L_n.

2. x_i^* is the right endpoint of the interval in which it is selected. We call this sum the right Riemann sum and denote it R_n.

3. x_i^* is the midpoint of the interval in which it is selected. We denote this sum M_n.

4. x_i^* corresponds to the maximum value of f in the interval in which it is selected. This will always give a value larger than the actual value of the integral. We call this sum the upper Riemann sum, and denote it U_n.

5. x_i^* corresponds to the minimum value of f in the interval in which it is selected. This will always give a value smaller than the actual value of the integral. We call this sum the lower Riemann sum and denote it S_n.

For further details on how this is done, consult the handout on the definite integral given at the beginning of the semester.

We will always have

$$S_n \leq \int_a^b f(x)\, dx \leq U_n$$

This gives us an upper bound and a lower bound on the value of the integral. By taking n larger and larger, we will get a better and better approximation. The Riemann sum applet, which can be found at

http://science.kennesaw.edu/~plaval/applets/Riemann.html

can find these Riemann sums.

It should be noted that Riemann sums can be improved to give a better approximation. These techniques will not be discussed here. Some can be found in many calculus books. Others can be found in books on Numerical Analysis.

3   Additional Tools to Find Integrals

Advanced calculators such as the TI 82, TI 83, TI 86, TI 89, TI 92 as well as many computer software (CAS for Computer Algebra Systems) can evaluate integrals. They usually take one of two approaches. These are describes below.

3.1   Symbolic Integration

This refers to integration by finding an antiderivative first, then using the limits of integration if any are provided. Systems which can perform symbolic integration can find indefinite integrals such as $\int f(x)\, dx$ as well as definite integrals such as $\int_a^b f(x)\, dx$. They will give an exact answer. Unfortunately, this is very difficult to do, and few machines or programs have this capability. The TI 92 can perform symbolic integration. The following CAS, widely used in the scientific community can also perform symbolic integration:

- Mathematica

- Maple

- MuPad

- MATLab

Here, at KSU, we have Maple installed on all computers. The command to evaluate $\int f(x)\, dx$ is $int\,(f(x), x)\,;$. The command to evaluate $\int_a^b f(x)\, dx$ is $int\,(f(x), x = a..b)\,;$. Note that all commands in Maple must end with ;

These CAS do not always give the most simplified answer. You should practice with Maple.

3.2 Numerical Integration

This refers to techniques which approximate integrals using Riemann sums or similar techniques. This is much easier to implement. Calculators such as the TI 82/83/86 which cannot perform symbolic integration can perform numerical integration. The command to do this is $fnint\,(f(x), x, a, b)$. The $fnint$ command can be accessed through the MATH menu.

4 Problems

1. For each integral below, explain which method you would use and why, then evaluate the integral. Compare your answer with the answer Maple would give.

   (a) $\int \sin^2 x \cos^3 x\, dx$ (**answer**: $\dfrac{1}{3} \sin^3 x - \dfrac{1}{5} \sin^5 x$)

   (b) $\int_0^{\frac{1}{2}} \dfrac{x}{\sqrt{1-x^2}} dx$ (**answer**: $1 - \dfrac{\sqrt{3}}{2}$)

   (c) $\int \ln\left(1+x^2\right) dx$ (**answer**: $x \ln\left(1+x^2\right) - 2x + 2 \tan^{-1} x$)

   (d) $\int \dfrac{\sqrt{9-x^2}}{x} dx$ (**answer**: $3 \ln \left| \dfrac{3 - \sqrt{9-x^2}}{x} \right| + \sqrt{9-x^2}$)

   (e) $\int \dfrac{x}{x^2 - 2x + 2} dx$ (**answer**: $\dfrac{1}{2} \ln \left| x^2 - 2x + 2 \right| + \tan^{-1}(x-1)$)

   (f) $\int_0^1 \cos \pi x \tan \pi x\, dx$ (**answer**: $\dfrac{2}{\pi}$)

   (g) $\int \dfrac{e^x}{e^{2x} - 1} dx$ (**answer**: $\dfrac{1}{2} \ln \left| \dfrac{e^x - 1}{e^x + 1} \right|$)

2. Approximate to one decimal place $\int_0^1 e^{x^2}\, dx$. Compare your answer with the answer the TI 83 (or similar) gives. (**answer**: 1.4 and 1.462651746)