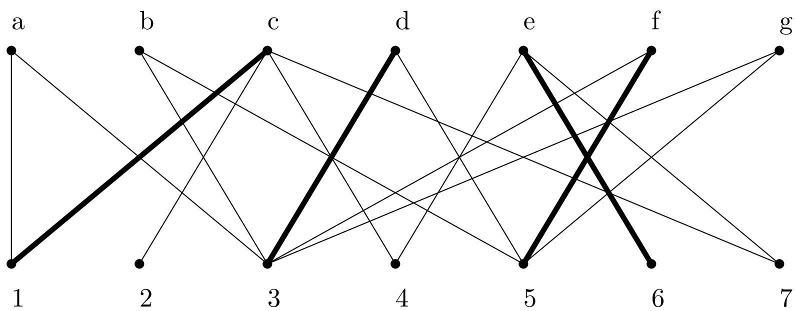
Math 239 Spring 2014 Assignment 11 Solutions

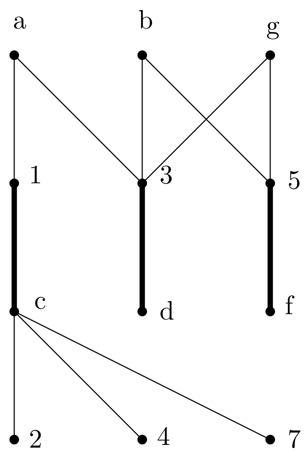
1. Prove that every tree has at most 1 perfect matching.

   Solution. Suppose M and M' are distinct perfect matchings of a tree T. Consider the set of edges $M \Delta M'$ that consist of edges that are in M or in M', but not both. For any vertex v, suppose vw is in M and vw' is in M'. If $w = w'$, then the degree of v is 0 in $M \Delta M'$. If $w \neq w'$, then the degree of v is 2 in $M \Delta M'$. Since M and M' are different matchings, there exists at least 1 vertex of degree 2 in $M \Delta M'$. Therefore, there is a component where every vertex has degree 2. This is a cycle, which is not possible in T. Therefore, there is at most 1 perfect matching.

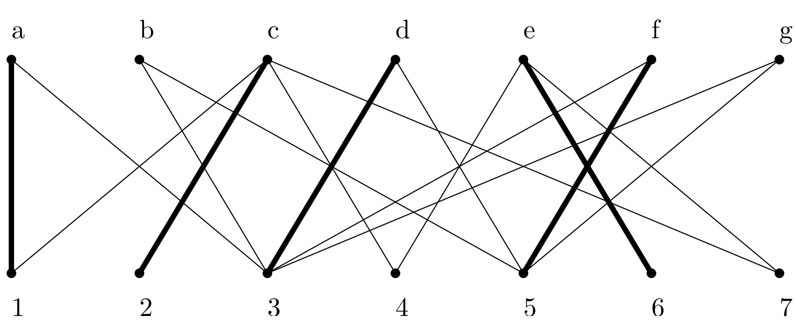
2. For the following bipartite graph with bipartition $A = \{a, b, c, d, e, f, g\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$, perform the maximum matching algorithm using XY-construction. At the end of the algorithm, produce a maximum matching, a minimum cover, and the sets X and Y from the algorithm. Prove that there is no matching that saturates every vertex in A by giving a set $D \subseteq A$ such that $|N(D)| < |D|$.



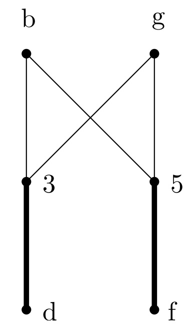
   Solution. In the first iteration, we have $X_0 = \{a, b, g\}$, and construct $X = \{a, b, g, c, d, f\}$, $Y = \{1, 3, 5, 2, 4, 7\}$. We also find several augmenting paths, one of which is $a, 1, c, 2$.



   Augmenting on this path, we get the following new matching.

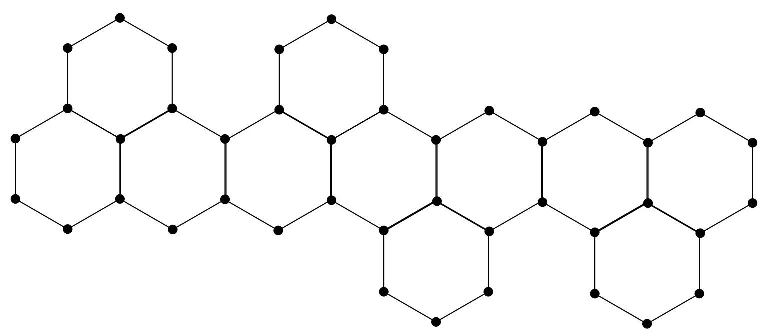


   In the second iteration, we have $X_0 = \{b, g\}$, and construct $X = \{b, g, d, f\}$ and $Y = \{3, 5\}$. We cannot find any augmenting paths, so this is a maximum matching with a minimum cover $Y \cup (A \setminus X) = \{3, 5, a, c, e\}$.

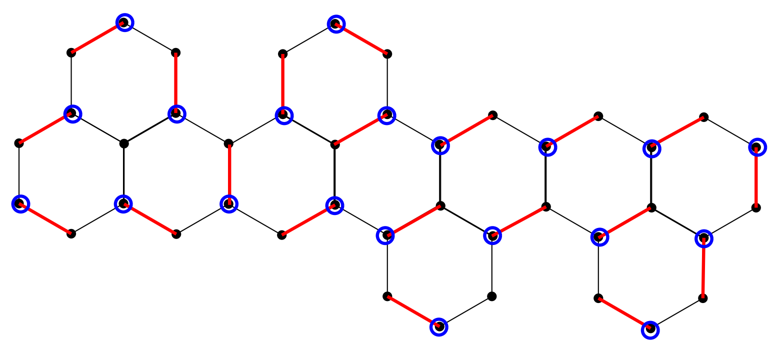


To use Hall's Theorem, we can look at the set $X = \{b, g, d, f\}$ and notice that $N(X) = Y$. Here $|X| = 4 > 2 = |N(X)|$, hence X is a set that violates Hall's condition.

3. Find a maximum matching of the following graph. Prove that your matching is maximum using a vertex cover.



Solution. The following shows a matching of size 20 and a cover of size 20.



4. An *independent set* of a graph G is a subset of the vertices $S \subseteq V(G)$ such that no two vertices in S are adjacent. Prove that C is a vertex cover of G if and only if $V(G) \setminus C$ is an independent set. If x is the size of a maximum independent set and y is the size of a minimum vertex cover, determine $x + y$.

   Solution. C is a vertex cover if and only if each edge of G has at least one end in C, if and only if no edge joins two vertices of $V(G) \setminus C$, if and only if $V(G) \setminus C$ is an independent set.

   If C is a minimum vertex cover with size x, then $V(G) \setminus C$ is an independent set with size $|V(G)| - x$. If there is any larger independent set I, then $V(G) \setminus I$ is a vertex cover whose size is smaller than C, which is not possible. So the size of a largest independent set is $y = |V(G)| - x$, and so $x + y = |V(G)|$.

5. Suppose that a connected graph G has exactly one maximum matching. Prove that G has a perfect matching.

   Solution. Let M be the only maximum matching in G. Suppose M is not a perfect matching. Then there exists an unsaturated vertex v in G. Since G is connected, v has degree at least 1. Let u be a neighbour of v. Now u must be saturated, for otherwise we could add the edge uv and get a larger matching. Suppose uw is a matching edge in M. Then $M - uw + uv$ is another matching of G, which has the same size as M, so it is another maximum matching in G. This is a contradiction.

6. Prove that the edges of a k-regular bipartite graph can be partitioned into k perfect matchings.

   Solution. We prove our statement by induction on k.

   Base case: When $k = 0$, there are no edges, this is trivially true.

   Induction hypothesis: Assume that the edges of any $(k-1)$-regular bipartite graph can be partitioned into $k-1$ perfect matchings.

   Induction step: Let G be a k-regular bipartite graph. From class, we know that G has a perfect matching, let M be one of them. Now M is a 1-regular graph, so $G - M$ is a $(k-1)$-regular bipartite graph. By induction hypothesis, the edges of $G-M$ can be partitioned into $k-1$ perfect matchings. Together with M, we partitioned $E(G)$ into k perfect matchings for G.

7. Let G be a bipartite graph with bipartition (A, B) where $|A| = |B| = 2n$. Suppose for each $X \subseteq A$ where $|X| \leq n$, $|N(X)| \geq |X|$, and for each $Y \subseteq B$ where $|Y| \leq n$, $|N(Y)| \geq |Y|$ (i.e. Hall's condition holds for subsets of A and B of size at most n). Prove that G has a perfect matching.

   Solution. To use Hall's Theorem, it suffices to show that for any set $X \subseteq A$ where $|X| > n$, $|N(X)| \geq |X|$. Suppose by way of contradiction that there exists one set $X \subseteq A$ and $|X| > n$ where $|N(X)| < |X|$. Let X' be a subset of X of size exactly n. Then $|N(X')| \geq |X'| = n$ by assumption. But $N(X') \subseteq N(X)$, so $|N(X)| \geq n$. Let $Y = B \setminus N(X)$. So $|Y| = |B| - |N(X)| \leq n$. By assumption, $|N(Y)| \geq |Y|$. Since there is no edge between X and Y, $N(Y) \subseteq A \setminus X$. Therefore,

   $$|A| \geq |X| + |N(Y)| > |N(X)| + |Y| = |B|.$$

   This is a contradiction since $|A| = |B|$.