Tutorial Problems 3

The TA will cover a subset of these problems during the tutorial. Solutions will not be provided outside of the tutorial.

1. For a binary string s, define its weight $w(s)$ to be the number of 1's in the string plus the length of the string itself. For example, $w(110100001) = 13$.

   (a) Let S_n be the set of all binary strings of length n. Use the product lemma to determine $\Phi_{S_n}(x)$.

   (b) Let T be the set of all binary strings (regardless of length). Determine $\Phi_T(x)$.

2. Let n be a non-negative integer. How many compositions of n are there where the i-th part has the same parity as i? For example, compositions of 7 that satisfy this condition are

$$(7), (5, 2), (3, 4), (1, 6), (1, 2, 1, 2, 1).$$

3. How many k-tuples $(a_1, \ldots, a_k)$ of positive integers satisfy the inequality $a_1 + \cdots + a_k < n$?

4. Let a_n denote the number of compositions of n. In class, we found out that for $n \geq 1$, $a_n = 2^{n-1}$. This tells us that for $n \geq 2$, a_n satisfies the recurrence $a_n = 2a_{n-1}$. Give a combinatorial proof of this recurrence.

Practice Problems for Assignment 3

1. At an intergalactic yard sale, there are three distinct planets costing 5, 7 and 9 gold coins respectively, one comet costing 12 gold coins, 120 identical stars selling for 3 gold coins each, and an unlimited supply of star bits selling for 2 gold coins each. For a positive integer n, how many ways can one spend exactly n gold coins in this sale?

2. How many compositions of n are there where each part is at most m? (The number of parts is not restricted.)

3. Let k be a fixed integer. How many compositions of n with k parts are there where each part is congruent to 1 modulo 5? Determine an explicit formula.

4. Let n, k be positive integers. Let $S_{n,k}$ be the set of all compositions of n with exactly k parts. Give a combinatorial proof that

$$|S_{n+1,k+1}| = \sum_{i=1}^{n-k+1} |S_{n+1-i,k}|.$$

Which algebraic identity does this also prove?

5. Let n, k be positive integers. Let $S_{n,k}$ be the set of all compositions of n with exactly k parts. Prove that

$$\sum_{(a_1,\ldots,a_k)\in S_{n,k}} a_1 \cdots a_k = \binom{n+k-1}{2k-1}.$$

For example, $S_{4,3} = \{(1,1,2),(1,2,1),(2,1,1)\}$. So the sum is $1\cdot 1\cdot 2 + 1\cdot 2\cdot 1 + 2\cdot 1\cdot 1 = 6$. This is equal to $\binom{6}{5}$.

6. From class, you have learned that for an integer $n \geq 1$, the total number of compositions of n is 2^{n-1}. This is also the number of subsets of $[n-1]$. Find a bijection f_n between the set S_n of all compositions of n and the set T_{n-1} of all subsets of $[n-1]$. Provide the inverse of your mapping.

7. For any $n \in \mathbb{N}_0$, let E_n be the set of all compositions of n with even number of parts, and let O_n be the set of all compositions of n with odd number of parts. Prove that for $n \geq 2$, $|E_n| = |O_n|$.

8. Prove that for $n \geq 2$, the number of compositions of n with even number of even parts is equal to the number of compositions of n with odd number of even parts.

9. Let f_n be the n-th term in the Fibonacci sequence. Let S_n be the set of all compositions of n where each part is odd. From class, you know that $|S_n| = f_n$. Let T_n be the set of all compositions of n where each part is at least 1. From the assignment, you can derive that $|T_n| = f_{n-1}$. This means that $|S_{n+1}| = |T_n|$ for all integers $n \geq 2$. Find a bijection between S_{n+1} and T_n.