Module 6: Dictionary Tricks

CS 240 - Data Structures and Data Management

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Dictionary ADT: Review

A *dictionary* is a collection of *key-value pairs* (KVPs), supporting operations *search*, *insert*, and *delete*.

Realizations

- **Unordered array or linked list**: $\Theta(1)$ insert, $\Theta(n)$ search and delete
- **Ordered array**: $\Theta(\log n)$ search, $\Theta(n)$ insert and delete
- **Balanced search trees** (AVL trees, 2-3 trees):
  $\Theta(\log n)$ search, insert, and delete
- **Hash tables** (on average, under UHA):
  $\Theta(1)$ search, insert, and delete

Interpolation Search

Ordered array

- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$

binary search$(A[\ell, r], k)$: Check index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r-\ell) \rfloor$

Question: What if the keys are numbers?

Interpolation Search

Ordered array

- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$

binary search$(A[\ell, r], k)$: Check index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r-\ell) \rfloor$

Question: What if the keys are numbers?

Idea: Use the value of the key to *guess* its location

Interpolation Search$(A[\ell, r], k)$: Check index $\ell + \lfloor \frac{k-A[\ell]}{A[r]-A[\ell]}(r-\ell) \rfloor$

Works well if keys are uniformly distributed: $O(\log \log n)$ on average.
Bad worst case performance: $O(n)$

Gallop Search

Problem in Binary-Search: Sometimes we cannot see the end of the array (data streams, a huge file, etc.)

Gallop-Search(A, k)
A: An ordered array, k: a key
1. $i \leftarrow 0$
2. **while** $i <$ size(A) **and** $k > A[i]$ **do**
3. $\quad i \leftarrow 2i + 1$
4. **return** Binary-Search$(A\big[\lceil i/2 \rceil, \min(i, \text{size}(A) - 1)\big], k)$

$O(\log m)$ comparisons (m: location of k in A)

Self-Organizing Search

- Unordered linked list
  search: $\Theta(n)$, *insert*: $\Theta(1)$, *delete*: $\Theta(1)$ (after a search)
- Linear search to find an item in the list
- Is there a more useful ordering?

Self-Organizing Search

- Unordered linked list
  search: $\Theta(n)$, *insert*: $\Theta(1)$, *delete*: $\Theta(1)$ (after a search)
- Linear search to find an item in the list
- Is there a more useful ordering?

- No: if items are accessed equally likely
- Yes: otherwise (we have a probability distribution for items)

- Optimal static ordering: sorting items by their probabilities of access in non-increasing order minimizes the expected cost of Search.
- Proof Idea: For any other ordering, exchanging two items that are out-of-order according to their access probabilities makes the total cost decrease.

Dynamic Ordering

- What if we do not know the access probabilities ahead of time?
- Move-To-Front(MTF): Upon a successful search, move the accessed item to the front of the list
- Transpose: Upon a successful search, swap the accessed item with the item immediately preceding it

Dynamic Ordering

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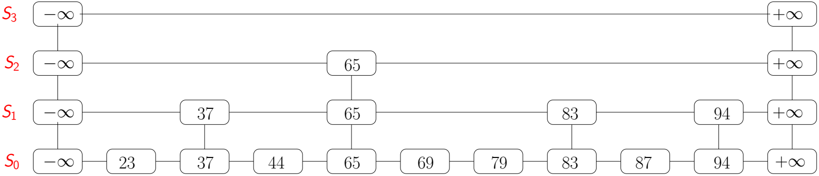
Performamce of dynamic ordering:

- Both can be implemented in arrays or linked lists.
- Transpose can perform very badly on certain input.
- MTF Works well in practice.
- Theoretically MTF is "competitive":
  No more than twice as bad as the optimal "offline" ordering.

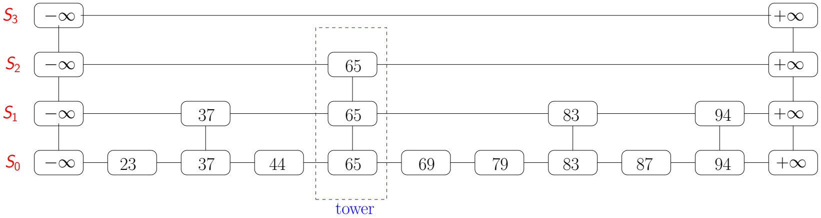
Skip Lists

- Randomized data structure for dictionary ADT
- A hierarchy of ordered linked lists
- A skip list for a set S of items is a series of lists $S_0, S_1, \cdots, S_h$ such that:
  - Each list S_i contains the special keys $-\infty$ and $+\infty$
  - List S_0 contains the keys of S in nondecreasing order
  - Each list is a subsequence of the previous one, i.e., $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$
  - List S_h contains only the two special keys



Skip Lists

- A skip list for a set S of items is a series of lists $S_0, S_1, \cdots, S_h$
- A two-dimensional collection of positions: levels and towers
- Traversing the skip list: after(p), below(p)



Search in Skip Lists

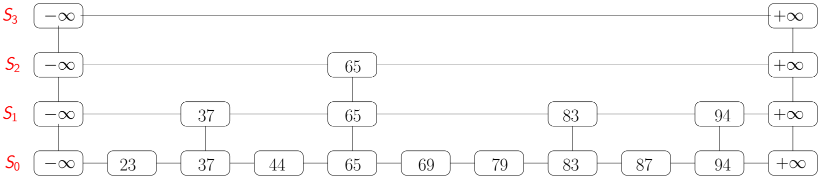
skip-search(L, k)
L: A skip list, k: a key

```
p ← topmost left position of L
S ← stack of positions, initially containing p
while below(p) ≠ null do
    p ← below(p)
    while key(after(p)) < k do
        p ← after(p)
    push p onto S
return S
```

- S contains positions of the largest key **less than** k at each level.
- $after(top(S))$ will have key k, iff k is in L.
- drop down: $p \leftarrow below(p)$
- scan forward: $p \leftarrow after(p)$

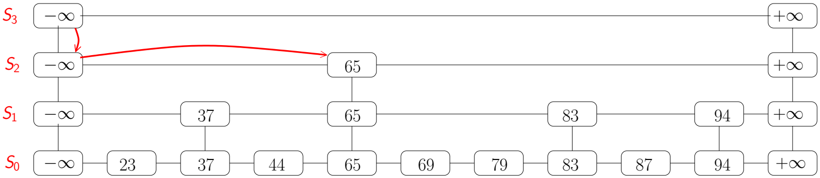
Search in Skip Lists

Example: Skip-Search$(S, 87)$



Search in Skip Lists

Example: Skip-Search(S, 87)



Search in Skip Lists

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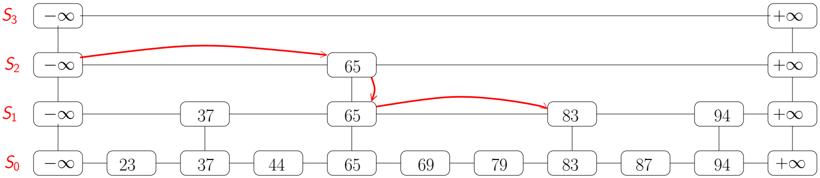
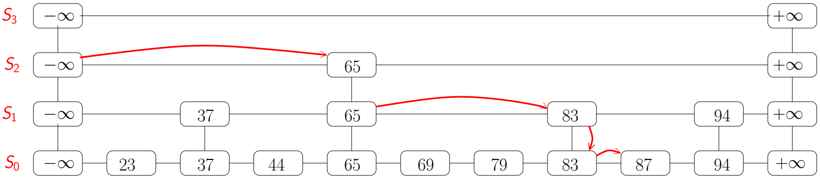


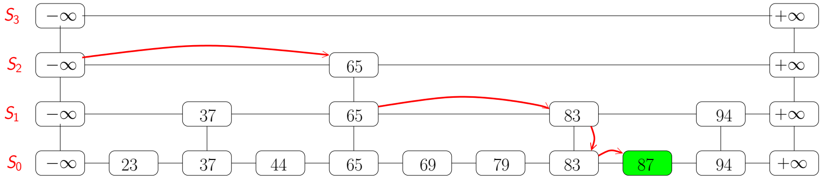
Search in Skip Lists

Example: Skip-Search$(S, 87)$



Search in Skip Lists

Example: Skip-Search(S, 87)

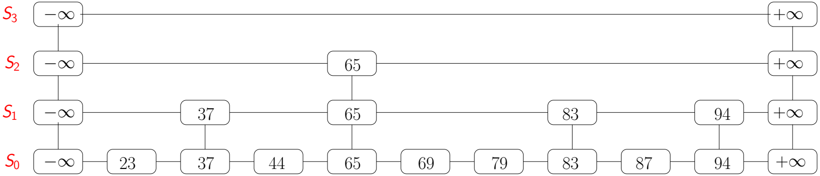


Insert in Skip Lists

- *Skip-Insert*(S, k, v)
  - Randomly compute the height of new item: repeatedly toss a coin until you get tails, let i the number of times the coin came up heads
  - Search for k in the skip list and find the positions $p_0, p_1, \cdots, p_i$ of the items with largest key less than k in each list $S_0, S_1, \cdots, S_i$ (by performing *Skip-Search*(S, k))
  - Insert item (k, v) into list S_j after position p_j for $0 \leq j \leq i$ (a tower of height i)

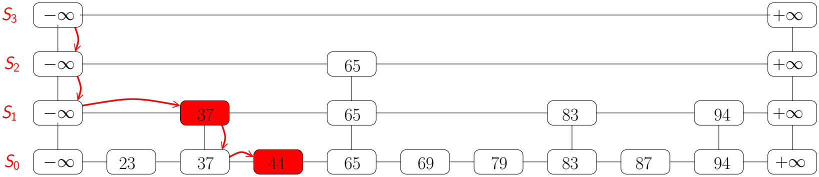
Insert in Skip Lists

Example: Skip-Insert$(S, 52, v)$
Coin tosses: H,T $\Rightarrow i = 1$



Insert in Skip Lists

Example: Skip-Insert(S, 52, v)
Coin tosses: H,T $\Rightarrow i = 1$
Skip-Search(S, 52)



Insert in Skip Lists

Example: Skip-Insert($S, 52, v$)
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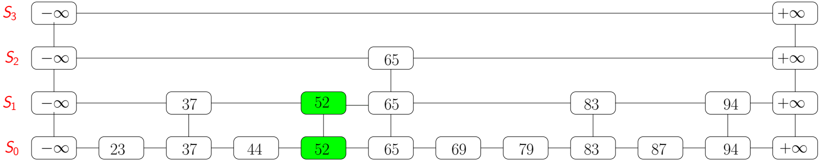
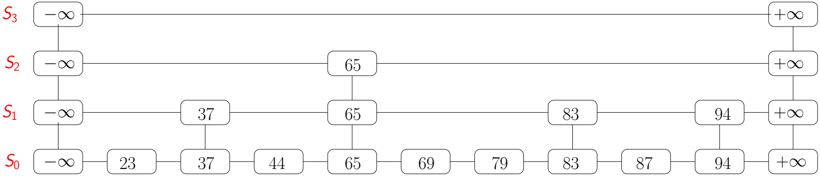


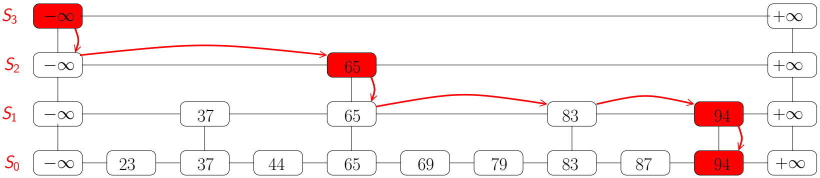
Insert in Skip Lists

Example: Skip-Insert(S, 100, v)
Coin tosses: H,H,H,T $\Rightarrow i = 3$



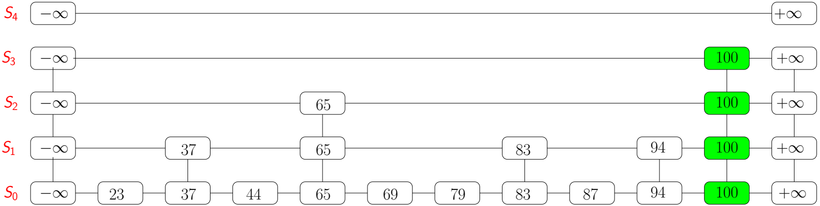
Insert in Skip Lists

Example: Skip-Insert(S, 100, v)
Coin tosses: H,H,H,T $\Rightarrow i = 3$
Skip-Search(S, 100)



Insert in Skip Lists

Example: Skip-Insert(S, 100, v)
Coin tosses: H,H,H,T $\Rightarrow i = 3$
Height increase

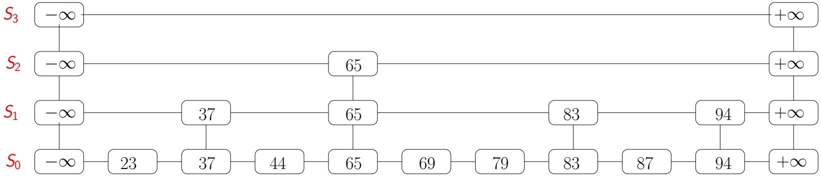


Delete in Skip Lists

- Skip-Delete(S, k)
  - Search for k in the skip list and find all the positions $p_0, p_1, \ldots, p_i$ of the items with the largest key smaller than k, where p_j is in list S_j. (this is the same as Skip-Search)
  - For each i, if $key(after(p_i)) == k$, then remove $after(p_i)$ from list S_i
  - Remove all but one of the lists S_i that contain only the two special keys

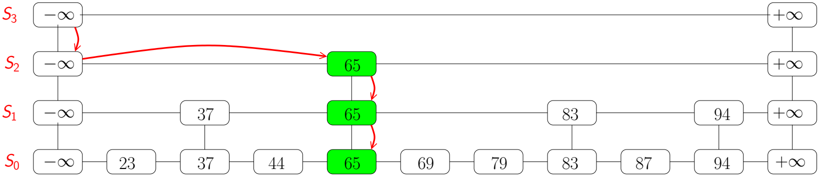
Delete in Skip Lists

Example: Skip-Delete(S, 65)



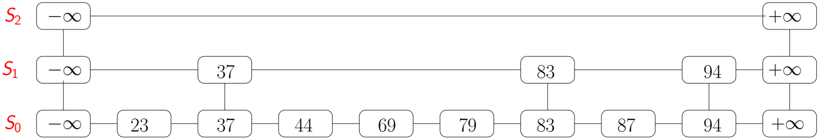
Delete in Skip Lists

Example: Skip-Delete(S, 65)
Skip-Search(S, 65)



Delete in Skip Lists

Example: Skip-Delete(S, 65)



Summary of Skip Lists

- Expected space usage: $O(n)$
- Expected height: $O(\log n)$
  A skip list with n items has height at most $3 \log n$ with probability at least $1 - 1/n^2$
- *Skip-Search*: $O(\log n)$ expected time
- *Skip-Insert*: $O(\log n)$ expected time
- *Skip-Delete*: $O(\log n)$ expected time
- Skip lists are fast and simple to implement in practice