Tutorial Problems 4

The TA will cover a subset of these problems during the tutorial. Solutions will not be provided outside of the tutorial.

1. Consider the following set of binary strings.

$$S = (1(0\{1\}^*0)^*1\{0\}^*)^*$$

   (a) List all the binary strings in S of length at most 4.

   (b) You are given that the expression for S is unambiguous. Using the length of a string as its weight, determine the generating series for the set S. Express it as a simplified rational expression.

2. Prove that $\{0,1\}^*(000111)\{0,1\}^*$ is an ambiguous expression for the set of all binary strings which contain 000111 as a substring.

3. Determine an unambiguous decomposition for each of the following sets of strings.

   (a) The set of binary strings that begin and end with the same bit.

   (b) The set of binary strings that does not contain 10000 as a substring.

   (c) The set of binary strings where no block has length divisible by 3.

   (d) The set of binary strings where any block of 1's of odd length must be followed by a block of 0's of the same length.

4. A binary string is a palindrome if it reads the same forwards and backwards. Examples of palindromes include 01100110, 11011, 1. Let P be the set of all binary strings that are palindromes. Determine a recursive equation for P, and use it to find the generating series for P with respect to lengths of the strings.

5. Let k be a fixed positive integer. Let S be the set of binary strings with no k consecutive 1's, and let b_n be the number of strings in S of length n. Prove that for $n \geq k$,

$$b_n = \sum_{i=1}^{k} b_{n-i}.$$

   Give a combinatorial proof of this recurrence.

Practice Problems for Assignment 4

1. Prove that $\{00, 101, 11\}^*$ is an unambiguous expression.

2. For some positive integer m, let $s_1, \ldots, s_k$ be distinct binary strings all of length m. Prove that $S = \{s_1, \ldots, s_k\}^*$ is an unambiguous expression.

3. Determine an unambiguous decomposition for the set of binary strings that does not have 10001 as a substring. To check that you might have the correct decomposition, the generating series should be $\frac{1+x^4}{1-2x+x^4-x^5}$.

4. Let S be the set of all binary strings where consecutive blocks have different parities. For example, things in S include 00011011111100000110, 1111111, 0011111, ε. Prove that the generating series for S is
$$\Phi_S(x) = \frac{1 + 2x + x^3 - x^4}{1 - 2x^2 - x^3 + x^4}.$$

5. Let S_n be the set of all subsets of $[n]$ which do not contain consecutive integers. For example, $S_3 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}\}$. The Fibonacci sequence $\{f_i\}$ is defined by $f_0 = 0, f_1 = 1$, and $f_i = f_{i-1} + f_{i-2}$ for $i \geq 2$. By converting to a binary string problem, prove that for each $n \in \mathbb{N}$, $|S_n| = f_{n+2}$.