Some Examples of Fourier Series

Example:

It should be obvious what we'll get if we find the Fourier series of a sinusoid, but let's make the calculation anyway

a) a complex sinusoid $f(t) = e^{j\omega t}$

- periodic, with period $T = \frac{2\pi}{\omega}$

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-j\frac{2\pi}{T}nt}\, dt$$

$$= \frac{\omega}{2\pi} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} e^{j\omega t}\, e^{-j\omega n t}\, dt$$

So $$c_n = \begin{cases} 1 & , \text{ if } n = 1 \\ 0 & , \text{ otherwise} \end{cases}$$

and the Fourier series is

$$\sum_{n=-\infty}^{\infty} c_n e^{j\omega n t} = e^{j\omega t}$$

b) a real sinusoid $\sin \omega t$:

$$f(t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} ,$$

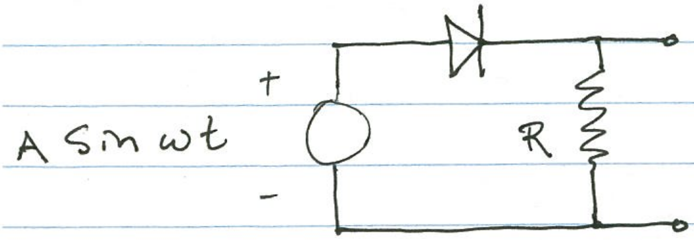
so a similar calculation gives

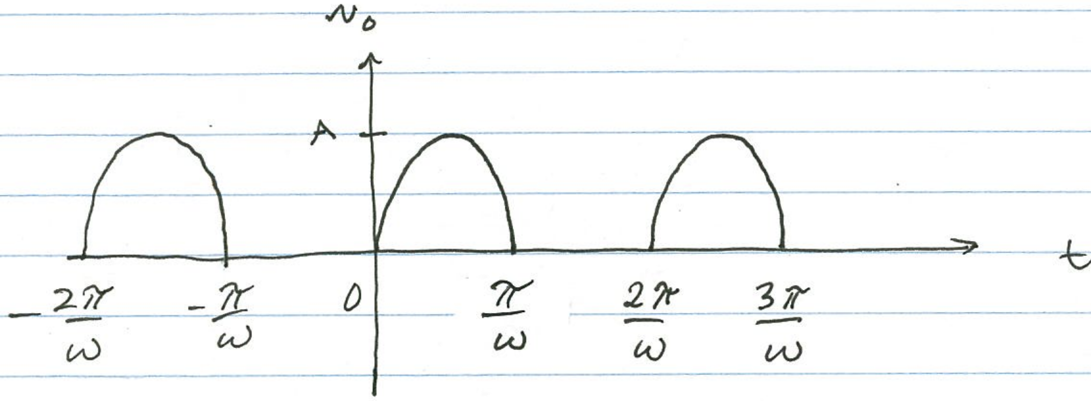
$$c_n = \begin{cases} \frac{1}{2j} & , \quad n = 1 \\ -\frac{1}{2j} & , \quad n = -1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

and the Fourier series is

$$\sum_{n=-\infty}^{\infty} c_n e^{j\omega n t} = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} = \sin \omega t$$

Example: Half-wave rectifier:





The Fourier coefficients of $v_o(t)$ are given by

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} v_o(t)\, e^{-j\frac{2\pi n}{T}t}\, dt$$

$$= \frac{\omega}{2\pi} \int_{0}^{\pi/\omega} A \sin \omega t \; e^{-j\omega n t}\, dt$$

So, for $n \neq \pm 1$

$$c_n = \frac{\omega A}{4\pi j}\left[\frac{1-(-1)^{n-1}}{j\omega(n-1)} - \frac{1-(-1)^{n+1}}{j\omega(n+1)}\right]$$

$$= \frac{\cancel{\omega} A}{4\pi \cancel{j}}\left(1+(-1)^n\right)\left[\frac{2\cancel{j}\cancel{\omega}}{-\cancel{\omega^2}(n^2-1)}\right]$$

$$= \frac{A}{2\pi}\ \frac{1+(-1)^n}{(1-n^2)}$$

For $n = 1$, we set

$$c_n = \frac{\cancel{\omega}}{2\cancel{\pi}}\ \frac{A}{2j}\ \frac{\cancel{\pi}}{\cancel{\omega}} = \frac{A}{4j},$$

and for $n = -1$,

$$c_n = \frac{\cancel{\omega}}{2\cancel{\pi}}\left(\frac{-A}{2j}\right)\left[\frac{\cancel{\pi}}{\cancel{\omega}}\right] = -\frac{A}{4j}$$

What does this tell us about the effectiveness of our rectifier?

- The dc component of the output (which is normally what we're after) is

$$c_0 = \frac{A}{\pi}$$

- The other terms in the Fourier series are sinusoidal – we'd normally want to filter them out.

- The component at the frequency of the input is given by the terms for $n = \pm 1$:

$$c_1 e^{j\omega t} + c_{-1} e^{-j\omega t}$$

$$= \frac{A}{4j} \left[e^{j\omega t} - e^{-j\omega t} \right]$$

$$= \frac{A}{4j} \, 2j \sin \omega t$$

$$= \frac{A}{2} \sin \omega t$$

- this is exactly one half of the input sinusoid.

The remaining terms (for $n \neq 0, \pm 1$)

give us sinusoids at multiples of ω:

$$c_n e^{j\omega n t} + c_{-n} e^{-j\omega n t}$$

$$= \frac{A}{2\pi}\left[\frac{1+(-1)^n}{1-n^2} e^{j\omega n t} + \frac{1+(-1)^{-n}}{1-(-n)^2} e^{-j\omega n t}\right]$$

$(n \neq \pm 1)$

$$= \frac{A}{2\pi} \frac{1+(-1)^n}{1-n^2}\left[e^{j\omega n t} + e^{-j\omega n t}\right]$$

$(n \neq \pm 1)$

$$= \frac{A}{\cancel{2}\pi} \frac{1+(-1)^n}{1-n^2} \cancel{2} \cos \omega n t$$

$$= \begin{cases} \dfrac{-2A}{\pi(n^2-1)} \cos \omega n t, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

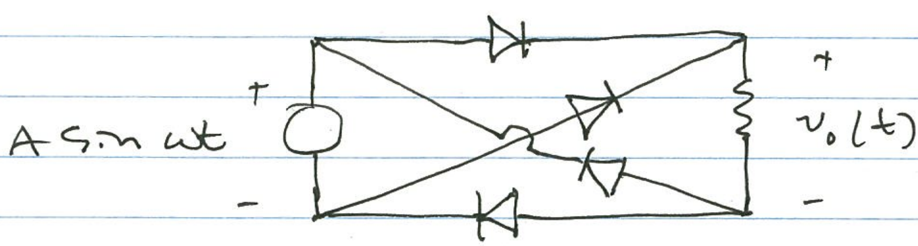
So the entire Fourier series is

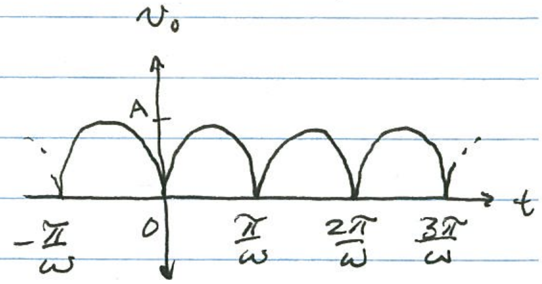
$$\sum_{n=-\infty}^{\infty} c_n e^{j\omega n t}$$

$$= \underset{(n=0)}{\frac{A}{\pi}} + \underset{(n=\pm 1)}{\frac{A}{2}\sin\omega t} - \frac{2A}{\pi} \sum_{\substack{n=2 \\ n \text{ even} \\ (n \neq 0, \pm 1)}}^{\infty} \frac{1}{n^2-1} \cos\omega n t$$

In order to use the half-wave rectifier in a dc power supply, we'd need a low-pass filter that would provide lots of attenuation at the input frequency ω — and we'd still only get a dc component of $\frac{A}{\pi}$.

Full-wave rectifier:





$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} v_0(t)\, e^{-j\frac{2\pi n}{T}t}\, dt$$

$$= \frac{\omega}{2\pi}\left[\int_{-\frac{\pi}{\omega}}^{0} (-A\sin\omega t)\, e^{-j\omega n t}\, dt\right.$$

$$\left. + \int_{0}^{\pi/\omega} (+A\sin\omega t)\, e^{-j\omega n t}\, dt\right]$$

Now $\int \sin \omega t \, e^{-j\omega n t} \, dt$

$$= \int \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \cdot e^{-j\omega n t} \, dt$$

$$= \frac{1}{2j} \int \left[e^{-j\omega(n-1)t} - e^{-j\omega(n+1)t} \right] dt$$

$$= \frac{1}{2j} \left[\frac{-1}{j\omega(n-1)} e^{-j\omega(n-1)t} - \frac{1}{-j\omega(n+1)} e^{-j\omega(n+1)t} \right] \quad (n \neq \pm 1)$$

$$= \frac{1}{2\omega} \left[\frac{1}{n-1} e^{-j\omega(n-1)t} - \frac{1}{n+1} e^{-j\omega(n+1)t} \right] \quad (n \neq \pm 1)$$

So

$$c_n = \frac{A}{4\pi} \cdot \left[\frac{1}{n-1} e^{-j\omega(n-1)t} \Big|_{-\frac{\pi}{\omega}}^{0} - \frac{1}{n+1} e^{-j\omega(n+1)t} \Big|_{-\frac{\pi}{\omega}}^{0} \right.$$

$$\left. - \frac{1}{n-1} e^{-j\omega(n-1)t} \Big|_{0}^{\frac{\pi}{\omega}} + \frac{1}{n+1} e^{j\omega(n+1)t} \Big|_{0}^{\pi/\omega} \right]$$

So for $n \neq \pm 1$

$$c_n = \frac{A}{4\pi}\left[\frac{1}{n-1}\left[1 - e^{-j\pi(n-1)} - e^{-j\pi(n-1)} + 1\right]\right.$$

$$\left. + \frac{1}{n+1}\left[-1 + e^{+j\pi(n+1)} + e^{j\pi(n+1)} - 1\right]\right]$$

$$= \frac{A}{2\pi}\left[1 + (-1)^n\right]\left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$

$$= \frac{-A}{\pi}\left[1 + (-1)^n\right]\frac{1}{n^2 - 1}$$

In particular,

$$c_0 = \frac{-A}{\pi} \, 2 \, \frac{1}{-1} = \frac{2A}{\pi}$$

– as one would expect, twice the value for the half-wave rectifier output.

Now, for $n = 1$,

$$c_1 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} v_o(t)\, e^{-j\frac{2\pi}{T}t}\, dt$$

$$= \frac{A\omega}{2\pi} \left[\int_{-\frac{\pi}{\omega}}^{0} (-\sin\omega t)\, e^{-j\omega t}\, dt \right.$$

$$\left. + \int_{0}^{\frac{\pi}{\omega}} \sin\omega t\; e^{-j\omega t}\, dt \right]$$

But

$$\int \sin\omega t\; e^{-j\omega t}\, dt$$

$$= \int \frac{e^{j\omega t} - e^{-j\omega t}}{2j}\, e^{-j\omega t}\, dt$$

$$= \int \frac{1 - e^{-2j\omega t}}{2j}\, dt$$

So

$$c_1 = \frac{A\omega}{2\pi}\left[\left(\frac{-\pi}{2j\omega}\right) + \frac{\pi}{2j\omega}\right] = 0$$

and similarly,

$$c_{-1} = 0$$

The Fourier series is therefore

$$\frac{2A}{\pi} + \sum_{n=2}^{\infty}\left(c_n e^{j\omega n t} + c_{-n} e^{-j\omega n t}\right)$$

But, for $n \geq 2$,

$$c_n e^{j\omega nt} + c_{-n} e^{-j\omega nt}$$

$$= -\frac{A}{\pi}\left[1+(-1)^n\right]\frac{1}{n^2-1}\left[e^{j\omega nt} + e^{-j\omega nt}\right]$$

$$= -\frac{2A}{\pi}\frac{\left[1+(-1)^n\right]}{n^2-1}\cos \omega n t$$

$$= \begin{cases} -\dfrac{4A}{\pi}\dfrac{1}{n^2-1}\cos \omega n t & , n \text{ even} \\ 0 & , \text{otherwise} \end{cases}$$

So the series is

$$\frac{2A}{\pi} - \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{4A}{\pi}\frac{1}{n^2-1}\cos \omega n t$$

Not only does the full-wave rectifier give twice the dc component, it doesn't produce any output signal at the frequency of the input sinusoid.

Parseval's Theorem tells how the power is distributed along the spectrum:

Average Power	total	Output signal power – by frequency: dc	ω	$> \omega$
½-wave	$\frac{A^2}{4}$	$\frac{A^2}{\pi^2}$	$\frac{A^2}{8}$	$\frac{A^2}{8} - \frac{A^2}{\pi^2}$
full-wave	$\frac{A^2}{2}$	$\frac{4A^2}{\pi^2}$	0	$\underbrace{\frac{A^2}{2} - \frac{4A^2}{\pi^2}}_{\text{(by Parseval)}}$