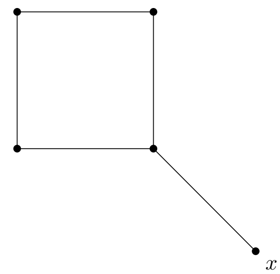
Math 239 Spring 2014 Assignment 7 Solutions

1. {6 marks} For each of the following statements about any graph G and three of its vertices u, v, w, determine whether it is true or false. If it is true, give a proof. If it is false, give a counterexample with appropriate explanations.

   (a) If G contains a cycle, then every vertex has degree at least 2.

   Solution. False. The following graph has a cycle, but vertex x has degree 1.



   (b) If there is a walk containing u, v, w, then there is a path containing u, v, w.

   Solution. False. Consider the following graph. There is a walk containing u, v, w, namely u, x, v, x, w. But there is no path containing u, v, w, since each vertex has degree 1 and if there's a path containing all of them, one of them has to have degree at least 2.

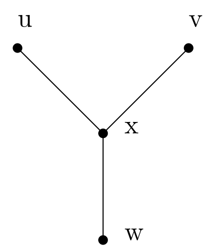
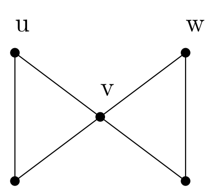


Figure 1: Flux capacitor.

   (c) If there exist a cycle containing vertices u, v and a cycle containing vertices v, w, then there exists a cycle containing vertices u, w.

   Solution. False.



2. Consider the following proposition.

   Proposition. *Let G be a non-empty graph where every vertex has degree at least k. Then G contains a path of length at least k.*

   (a) {2 marks} The following is an incorrect proof of the proposition using induction on k. Determine the main flaw of this proof.

   Bad proof. When $k = 1$, G has at least 1 vertex v with degree at least 1, so it has at least one neighbour w. Then v, w is a path of length at least 1 in G.
   Assume that the statement is true for $k-1$. Let G be a graph where every vertex has degree at least $k-1$. By induction hypothesis, there exists a path $P = v_1, v_2, \ldots, v_k$ of length $k-1$. Construct a new graph G' by adding a new vertex x and join x with every vertex in G. Then every vertex has degree at least k in G'. In particular, $v_k x$ is an edge, so $v_1, v_2, \ldots, v_k, x$ is a path of length k in G'. Therefore, G' contains a path of length at least k. $\square$

Solution. The problem is that this does not prove the statement for ALL possible graphs where every vertex has degree at least k. In particular, not every such graph has a vertex that is joined to all the other vertices. (A correct proof by induction is provided below.)

(b) {4 marks} Give a correct proof of this proposition. (You do not need to use induction.)

Solution. Let $v_0, v_1, \ldots, v_m$ be a path of the longest length in G. Note that neighbours of v_m must all be on the path, for otherwise we may extend the path to get a longer one, contradicting our original choice. Since v_m has degree at least k, it is adjacent to at least k distinct vertices on the path. This implies that $m \geq k$, hence this path has length at least k.

Alternative proof (by induction). When $k = 1$, there is at least one edge, so there is a path of length at least 1. Assume that every graph where each vertex has degree at least $k-1$ has a path of length at least $k-1$. Let G be a graph where each vertex has degree at least k. Then every vertex in G has degree at least $k-1$, so by induction hypothesis, there exists a path of length $k-1$, say it is $v_0, v_1, \ldots, v_{k-1}$. The vertex v_{k-1} has degree at least k, and there are only $k-1$ other vertices on the path. So at least one neighbour of v_{k-1} is not on the path. Let v_k be this neighbour. Therefore, $v_0, v_1, \ldots, v_{k-1}, v_k$ is a path of length k in G.

3. {4 marks} Let $k \geq 2$ be an integer. Let G be a graph where every vertex has degree at least k. Prove that G contains at least $\lfloor k/2 \rfloor$ edge-disjoint cycles (i.e. no two of these cycles share any edges, but they may share some vertices).

Solution. We prove this by induction on k. For $k = 2, 3$, $\lfloor k/2 \rfloor = 1$. Since $k \geq 2$, there exists a cycle in G. This proves the base cases.

We now assume that the statement is true for $2, 3, \ldots, k-1$. Let G be a graph where every vertex has degree at least k. Since $k \geq 2$, there exists a cycle C in G. Remove the edges of C to obtain a graph G'. Notice that every vertex that is in C has degree 2 in C, so by removing the edges of C from G, the degree of each vertex decreases by either 2 or 0. This means that ever vertex in G' has degree at least $k-2$. By induction hypothesis, G' contains $\lfloor \frac{k-2}{2} \rfloor = \lfloor k/2 \rfloor - 1$ edge-disjoint cycles. These do not share any edges with C since edges in C are not present in G'. So together with C, we have $\lfloor k/2 \rfloor$ edge-disjoint cycles in G.

Alternate proof. Consider a longest path $P = v_1, v_2, \ldots, v_m$. All the neighbours of v_1 must be on the path, for otherwise we can extend P to get a longer path. Since every vertex has degree at least k, there are at least k neighbours of v_1 on P, let them be $v_{a_1}, v_{a_2}, \ldots, v_{a_k}$ where $a_1 < a_2 < \cdots < a_k$. Consider the cycles C_i where $C_i = v_1, v_{a_{2i}}, v_{a_{2i}+1}, \ldots, v_{a_{2i+1}}, v_1$. These cycles are edge-disjoint since they are using different parts of P and different edges incident with v_1. The maximum possible value of i is $i = \lfloor k/2 \rfloor$, hence there are $\lfloor k/2 \rfloor$ edge-disjoint cycles in G.

4. {4 marks} Let G be a bipartite graph with n vertices. Prove that if every vertex has degree at least $\frac{n}{4} + 1$, then G is connected.

Solution. Suppose by way of contradiction that G is disconnected. So there are at least two components in G. Let H be one component. We know that H is bipartite, so H contains a bipartition (A, B) where each vertex in A is adjacent to at least $\frac{n}{4} + 1$ vertices in B, and vice versa. Since there is at least one vertex in H, say wlog it is in A, it implies that $|B| \geq \frac{n}{4} + 1$. Since there is at least 1 vertex in B and all of its neighbours are in A, it implies that $|A| \geq \frac{n}{4} + 1$. Therefore, $|V(H)| \geq |A| + |B| \geq \frac{n}{2} + 2$. But there are at least 2 components, so the total number of vertices in G is at least $2|V(H)| \geq n + 4$. This contradicts the fact that G has n vertices. Therefore, G must be connected.

Alternate solution. Suppose $V(G)$ has bipartition (A, B). Wlog assume $|A| \geq |B|$. In particular, $|B| \leq \frac{n}{2}$. Let $v \in A$. We will prove that G is connected by showing that there is a v, x-path for all $x \in V(G)$. If $x \in A$, then consider the set of neighbours $N(v)$ and $N(x)$ of v and x respectively. Since each vertex has degree at least $\frac{n}{4} + 1$, $|N(v)| \geq \frac{n}{4} + 1$ and $|N(x)| \geq \frac{n}{4} + 1$. In addition, both $N(v)$ and $N(x)$ are subsets of B, whose size is at most $\frac{n}{2}$. So by pigeonhole principle, there exists $y \in B$ such that $y \in N(v) \cap N(x)$. Then v, y, x form a v, x-path.

Now suppose $x \in B$. Let $N(x)$ be the set of neighbours of x. If $v \in N(x)$, then vx is an edge, so v, x is a v, x-path. Otherwise, let $y \in N(x)$. From the argument above, we see that v and y must have a common neighbour in B, say z. We know that $z \neq x$, therefore v, z, y, x form a v, x-path.

Since a v, x-path exists for all $x \in V(G)$, G is connected.

5. For some $k \in \mathbb{N}$, let G be a connected graph with $2k$ odd-degree vertices, and any number of even-degree vertices.

   (a) {4 marks} Prove that there exist k walks such that each edge in G is used in exactly one walk exactly once. What is so special about the end vertices of the k walks? (Hint: Add some edges to create an Eulerian circuit, and then remove them.)

   Solution. Let $v_1, v_2, \ldots, v_{2k}$ be the set of all odd-degree vertices in G. We obtain a new graph G' by adding k edges $v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}$ to G. Since we added one to each of these odd-degree vertices, G' is a connected graph where every vertex has even degree. Therefore, G' contains an Eulerian circuit, i.e. a closed walk containing each edge exactly once. By removing the k edges from the Eulerian circuit, we break it down to k walks where each edge in G is in exactly one of them.

   The end vertices have to be odd-degree vertices in G.

   (b) {2 marks} Prove that it is not possible that a set of $k-1$ walks in G uses each edge exactly once. (This shows that to cover G with walks containing no repeated edges, you need at least k walks.)

   Solution. Let $W_1, \ldots, W_{k-1}$ be edge-disjoint walks in G. For each W_i, if it is a closed walk, then the edges contribute an even degree to every vertex. If it is not a closed walk, then the edges contribute an even degree to every vertex except the two endpoints, which have odd degrees. Over all $k-1$ walks, we have at most $2(k-1)$ vertices of odd degrees, which is not possible since there are $2k$ vertices of odd degrees.

   (c) {2 marks} Partition the edges of the leftmost graph below into as few walks as possible.

   Solution. One possible solution is the following. Note that there are 8 odd-degree vertices, so we must use exactly 4 walks, each starting and ending at distinct odd-degree vertices.