# MATH 239 Past Midterm

**Note:** This is the midterm from last Spring with some modifications. This is good for practice purposes, however it is not an indication of the content nor the difficulty of the midterm you will be writing. Solutions will not be provided, but you are free to discuss them with TAs and instructors.

1. Counting and power series

   (a) {3 marks} Among the 7 children of the von Trapp family, 2 of them are boys and 5 of them are girls. Whenever they sing, all 7 of them will line up so that the 2 boys are standing next to each other. How many ways can they line up?

   (b) {4 marks} Determine the following coefficient. Simplify your answer as much as possible.

$$[x^n]\frac{(1+x)^n}{1-x}.$$

2. Bijections

   Let $k, m, n$ be positive integers such that $k \le m \le n$.

   (a) {2 marks} Define a bijection between the following two sets

$$\begin{aligned}\mathcal{S}_k &= \{(A,B) \mid A,B \subseteq [n], |A| = |B| = m, |A \cap B| = k\},\\ \mathcal{T}_k &= \{(X,Y,Z) \mid X,Y,Z \subseteq [n], |X| = |Y| = m-k, |Z| = k,\\ &\quad X \cap Y = X \cap Z = Y \cap Z = \emptyset\}.\end{aligned}$$

   Provide the inverse of your bijection. You do not need to prove that either function is a bijection.

   (b) {3 marks} Give a combinatorial proof of the following identity:

$$\binom{n}{m}^2 = \sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k}\binom{n-m}{m-k}.$$

   You may assume that there exists a bijection between $\mathcal{S}_k$ and $\mathcal{T}_k$ from part (a).

3. Generating series

   {4 marks} Let $\mathbb{N}$ be the set $\{1, 2, 3, \dots\}$ of positive integers. Note that every positive integer $n$ has a unique factorization as $2^l m$, where $l$ is a non-negative integer and $m$ is a positive odd integer. Define a weight function $w$ on $\mathbb{N}$ as follows:

$$w(n) = l + m,$$

   where $(l, m)$ is the unique pair of integers as given by the factorization above. Find the generating series $\Phi(x)$ of $\mathbb{N}$ with respect to this weight function, and express it as a rational function of $x$.

4. Compositions

   Let $A_n$ be the set of all compositions of $n$ where each part is either 1 or 3. Note that the number of parts is not restricted. For example, compositions in $A_4$ include $(1,3)$ and $(1,1,1,1)$, but not $(2,1,1)$ nor $(4)$.

   (a) {4 marks} Prove that

$$|A_n| = [x^n]\frac{1}{1-x-x^3}.$$

   (b) {4 marks} Let $a_n = |A_n|$. Derive a homogeneous recurrence that the sequence $\{a_n\}$ satisfies, along with sufficient initial conditions to uniquely specify the sequence.

5. Binary strings

   (a) {2 marks} Describe in words the set of all binary strings that are in the following decomposition:

$$S = \{\varepsilon, 0, 00\}(\{11\}\{\varepsilon, 0, 00\})^*.$$

   (b) {3 marks} Given that the decomposition for $S$ from part (a) is unambiguous, determine the generating series for $S$ where the weight of a string is its length. Express your answer as a simplified rational expression.

(c) {3 marks} Write an unambiguous expression for the set of all binary strings that do not contain 1100 as a substring. (No justification required.)

6. Recurrences

{5 marks} Let $\{a_n\}$ be a sequence where $a_0 = 0, a_1 = 2$, and for $n \geq 2$,

$$a_n - 8a_{n-1} + 15a_{n-2} = 0.$$

Determine an explicit formula for $a_n$.

7. Graph theory

(a) For $n \in \mathbb{N}$, let $G_n$ be the graph whose vertices are all the compositions of $n$ and there is an edge between two compositions if we can add two consecutive terms in the longer composition to get the shorter composition. For example, in $G_{11}$, $(1,4,3,2,1)$ is adjacent to $(1,7,2,1)$ by combining the second and third part (4 and 3) to get 7. Furthermore, $(1,7,2,1)$ is adjacent to $(8,2,1)$.

i. {2 marks} Draw $G_4$.

$(3,1)$ $(2,1,1)$

$(4)$ $(1,1,1,1)$

$(2,2)$ $(1,2,1)$

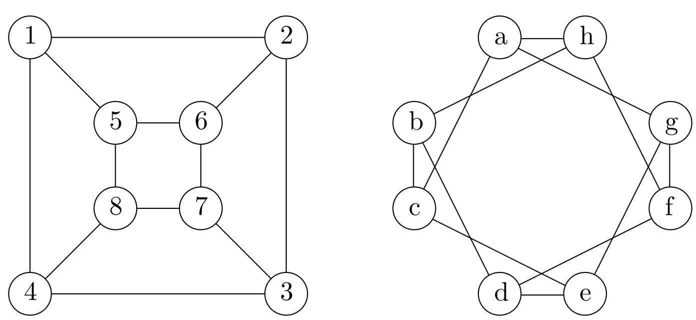
$(1,1,2)$ $(1,3)$

ii. {3 marks} Prove that $G_n$ is bipartite.

iii. {Extra credit: 2 marks} Prove that $G_n$ is $(n-1)$-regular.

iv. {2 marks} Recall that for $n \geq 1$, the total number of compositions of $n$ is $2^{n-1}$. Determine the number of edges in $G_n$. (You may assume that $G_n$ is $(n-1)$-regular.)

(b) {3 marks} The following two graphs $G$ and $H$ are isomorphic. Find an isomorphism. (You do not need to prove that your mapping is an isomorphism.)




$G$ $H$