CHAPTER 4. EQUIVALENCE FOR REPEATED CASH FLOWS

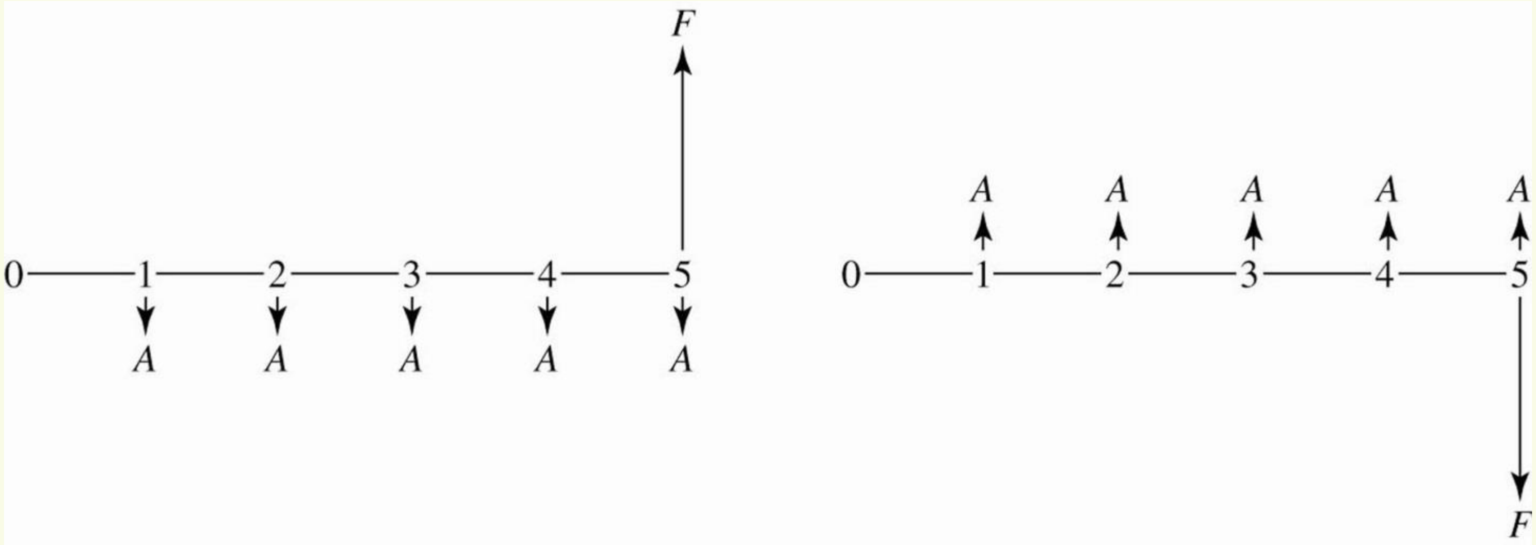
This chapter further develops formulas for cash flows that are a uniform series or are changing on an arithmetic or geometric series. We learn how to convert cash flow into different series. We also discuss the difference between nominal and effective interest. This chapter is the basis of the whole course and you are expected to completely understand the materials.

ANNUITY

- In chapter 3, we examined the conversion of a *one-time payment* between present (P) and future (F).
  - Compound factor: $(F/P, i, n) = (1 + i)^n$
  - Discount factor: $(P/F, i, n) = 1/(1 + i)^n$
- However, pattern and timing of cash flows are more complicated.
  - Cash flows → paid weekly (wage), monthly (electricity bill, mortgage), quarterly (tuition), or yearly (property tax), etc.
  - Some flows → constant each period (mortgage), while others fluctuate (car repair bills or piece work pay).
- This chapter considers a *series* of cash flows, such as car or mortgage payment, instead of a one-time payment.
  - Annuity (A): A series of cash flows during n periods.
  - Annuity can be *uniform* or *changing* in each period.

1. Uniform series: *F* and *A*

- Consider the following questions.
  - If I save an amount \$*A* every year for *n* years, how much do I have at the end of *n* years?
  - If I want to have \$*F* at the end of *n* years, how much do I have to save \$*A* each year?
  - We evaluate cash flows at date zero.



1.1. Conversion of uniform series: *F* and *A*

- Uniform series *compound amount* factor, (*F/A, i, n*)

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

- Uniform series *sinking fund* factor, (*A/F, i, n*)

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

  - These two factors are inverse to each other.
  - You don't need to memorize it, but it is useful to know how to derive the formula.
  - We will use the factor tables in class, but you can use Excel.

1.2. Derivation of factors between *F* and *A*

0	1	2	…	n – 1		n
	A				$\Rightarrow$	$A(1+i)^{n-1}$
		A			$\Rightarrow$	$A(1+i)^{n-2}$
			…			…
				A	$\Rightarrow$	$A(1+i)^{1}$
						$A(1+i)^{0}$

$$F = A(1+i)^{n-1} + A(1+i)^{n-2} + \ldots + A(1+i)^{1} + A(1+i)^{0}$$

$$(1+i)F = A(1+i)^{n} + A(1+i)^{n-1} + \ldots + A(1+i)^{2} + A(1+i)^{1}$$

$$iF = A[(1+i)^{n} - 1]$$

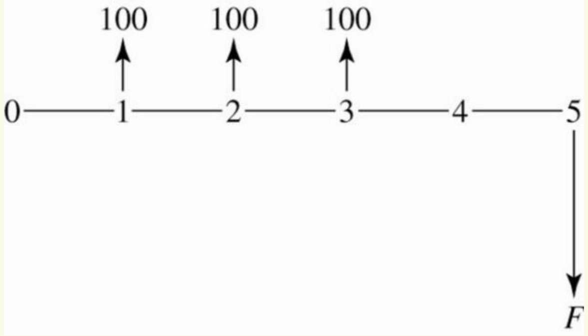
EXAMPLE 1-1. *F* AND *A*

- If you deposit \$500 in a bank each year with an interest rate of 5%, how much will you have at the end of five years?

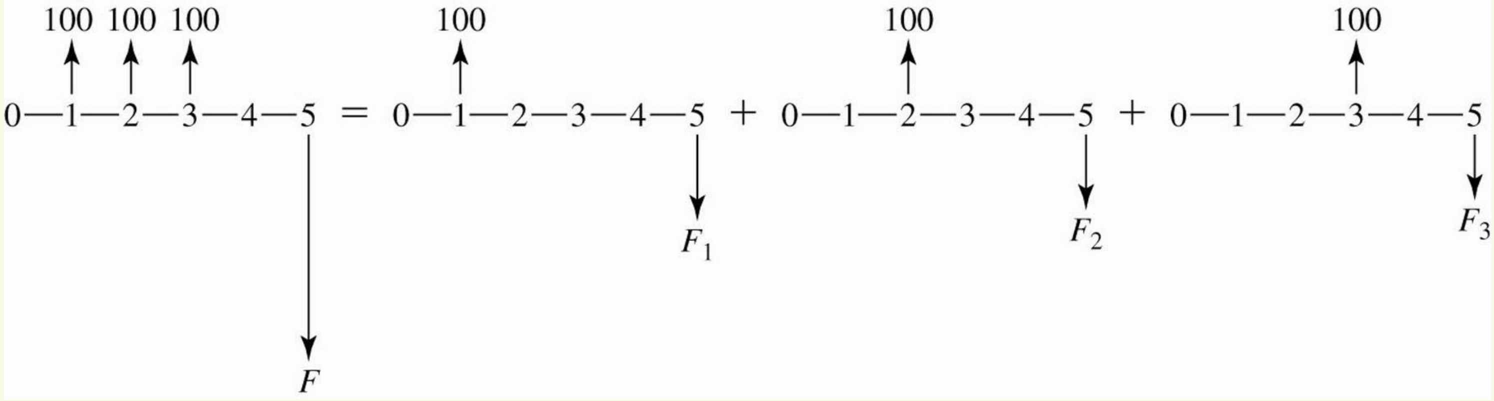
- Jim wants to save \$*A* each month to buy a notebook of \$1,000 at the end of the year. If bank pays 6% interest, compounded *monthly*, how much does Jim have to deposit each month?

EXAMPLE 1-2. *F* AND *A* (1)

- A student is borrowing \$100 per year for 3 years, starting from year 1. The loan will be repaid 2 years later at a 15% interest rate. How much should he pay 5 years from now?

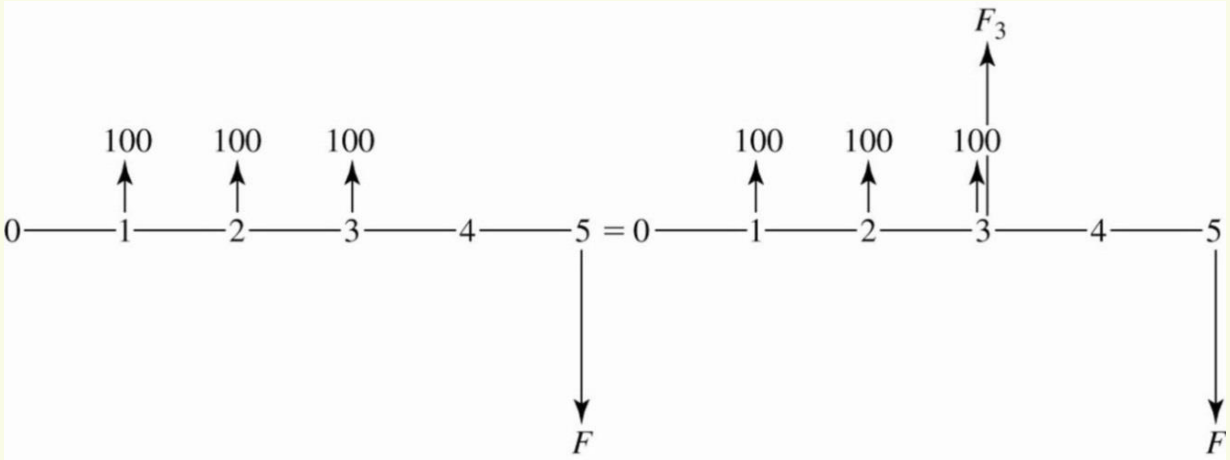


- Method 1



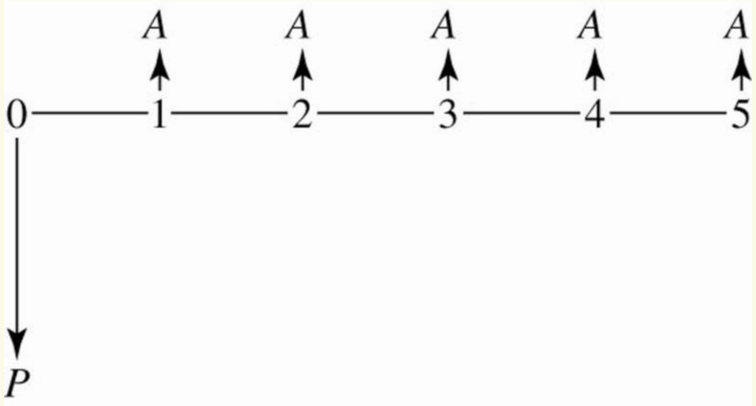
EXAMPLE 1-2. *F* AND *A* (2)

- Method 2



2. Uniform series: *P* and *A*

- We now consider the following questions.
  - How much must be saved over *n* periods to recover an investment of \$*P* made today?
  - If I pay \$*A* for a computer over *n* periods, how much do I pay for the product now?
  - Again, be careful on the timing of each cash flow!
    - For *F* and *A*, *F* was evaluated at year 5.
    - For *P* and *A*, *P* is evaluated at year zero, not year 1.



2.1. Conversion of uniform series: *P* and *A*

- Uniform series *present worth* factor, (*P*/*A*, *i*, *n*)

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

- Uniform series *capital recovery* factor, (*A*/*P*, *i*, *n*)

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

  - These two factors are inverse to each other.
  - (*A*/*P*, *i*, *n*) is called "*capital recovery factor*": How large does the annual return *A* have to be to recover the capital investment of *P*?

2.2. Derivation of factors between *P* and *A*

0		1	2	…	$n-1$	n
$A/(1+i)^1$	$\Leftarrow$	A				
$A/(1+i)^2$	$\Leftarrow$		A			
				…		
$A/(1+i)^{n-1}$	$\Leftarrow$				A	
$A/(1+i)^n$	$\Leftarrow$					A

$$P=\frac{A}{(1+i)}+\frac{A}{(1+i)^2}+\ldots+\frac{A}{(1+i)^{n-1}}+\frac{A}{(1+i)^n}$$

$$(1+i)P=A+\frac{A}{(1+i)}+\ldots+\frac{A}{(1+i)^{n-2}}+\frac{A}{(1+i)^{n-1}}$$

$$iP=A\left[1-\frac{1}{(1+i)^n}\right]$$

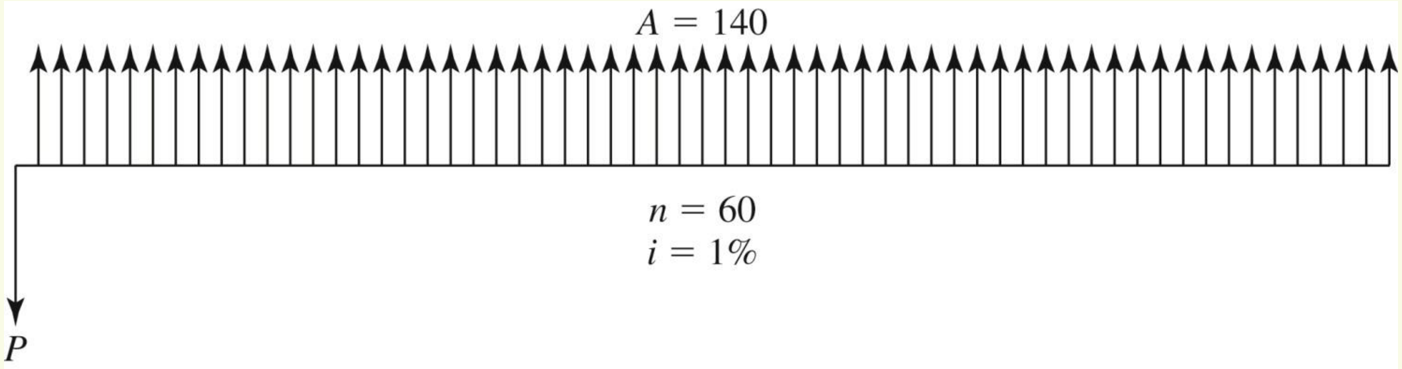
EXAMPLE 2-1. *P* AND *A*

- A machine costs $5,000 and has a service life of 5 years. If interest rate is 8%, how much must be saved every year to recover the investment?

- Should you spend $6,800 to buy a contract that pays $140 at the end of each month for 5 years if the desired return is 1% per month?

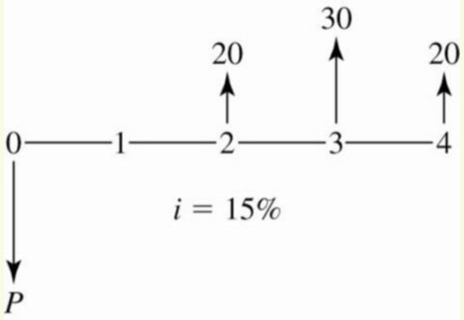
EXAMPLE 2-2. *P* AND *A*

- Continuing from the previous example, if \$6,800 is paid for a project that pays \$140 at the end of each month for 5 years, what is the monthly rate of return?

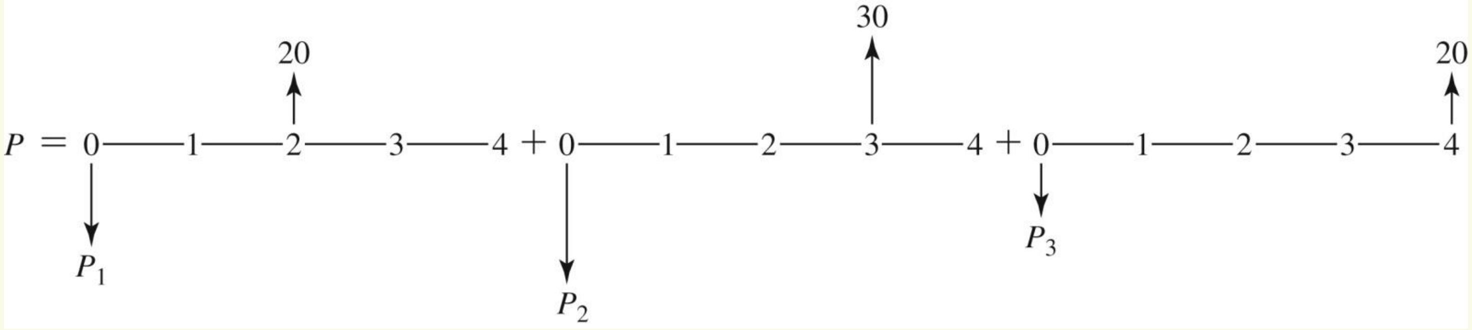


Example 2-3. Irregular payment (1)

- What amount, *P*, needs to be deposited in a saving account that pays 15% interest, to support 3 later withdraws as follows?



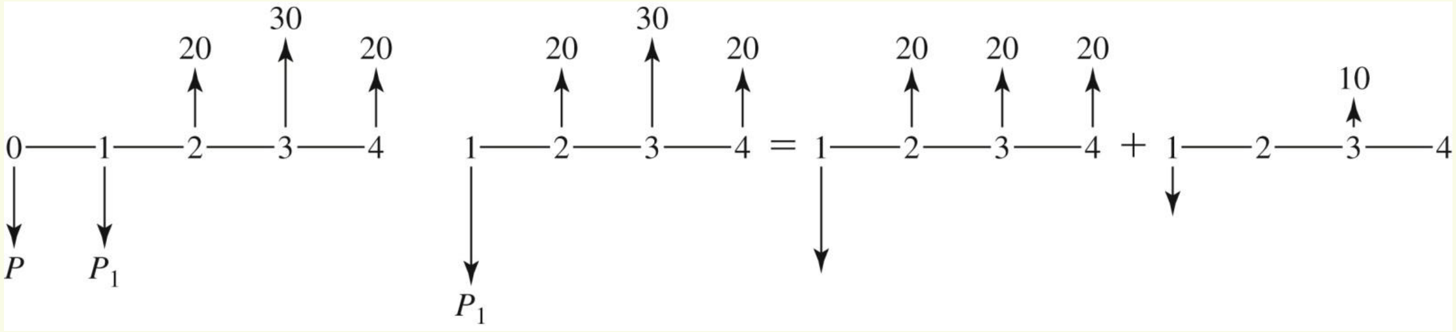
- Method 1



Example 2-3. Irregular payment (2)

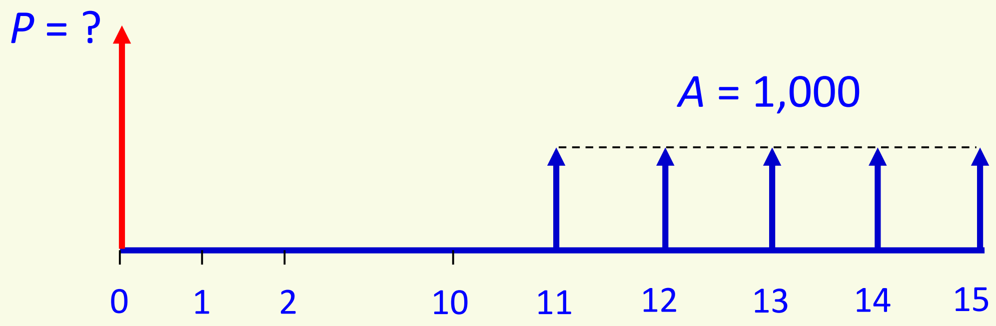
- Method 2
  - Convert each annuity to future worth and then to present worth.

- Method 3



EXAMPLE 2-4. DEFERRED ANNUITY

- My father promises to give me \$1,000 per year for 5 years starting 11th year from now. With i = 5%, what is the present worth of that promise?

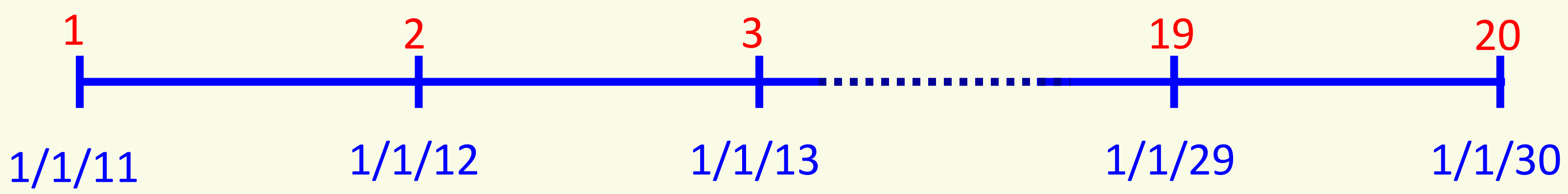


Example 2-5. Value of "Jackpot" (1)

- Are you dreaming of winning a Jackpot?
- If you won a $20 million Jackpot, are you richer by $20 million?
  - In Canada, you would be richer by $20 million.
  - They pay you $20 million immediately, with no income tax.
- Winning a Jackpot in the United States
  - However, the rule is different in the United States.
  - They pay you an equal *annuity* over 10 – 40 years, with an income tax.
  - Alternatively, you can opt for an immediate lump sum payment, which is much less than $20 million due to discounting.
  - "Gloria MacKenzie won a $590.5 million jackpot in June, and donated $2 million of the $370.8 million lump-sum payment to a local school."
  - Why do they have this rule?
  - Marketing! The Jackpot size looks larger in this way.

Example 2-5. Value of "Jackpot" (2)

- Suppose I won a $20 million jackpot in the United States, which pays an equal amount for 20 years from next year. If I choose an immediate lump-sum payment, how much do I get today (Jan 1, 2010), with *i* = 10%? (assuming no tax)



- $P = A(P/A, 10\%, 20) = 1{,}000{,}000(8.5136) =$ **$8,513,600**

PW of $1 million received in 2011: $\$1\text{ mil}/(1 + 0.1)^1 = \$909{,}091$

PW of $1 million received in 2012: $\$1\text{ mil}/(1 + 0.1)^2 = \$826{,}446$

PW of $1 million received in 2030: $\$1\text{ mil}/(1 + 0.1)^{20} = \$148{,}644$

Example 2-5. Value of "Jackpot" (3)

- More realistically, suppose the prize is paid from today (Jan, 1. 2010). What is the present worth of the jackpot?

$$P = \frac{A}{(1+i)^0} + \frac{A}{(1+i)^1} + \ldots + \frac{A}{(1+i)^{18}} + \frac{A}{(1+i)^{19}}$$

P = 1,000,000 + A(P/A, 0.1, 19)

= 1,000,000 + 1,000,000(8.3649) = **$9,364,900**.

  - If the interest rate is 20%, the present worth is $5,843,496.
  - If the prize is paid over 30 years, the present worth is $6,913,071.

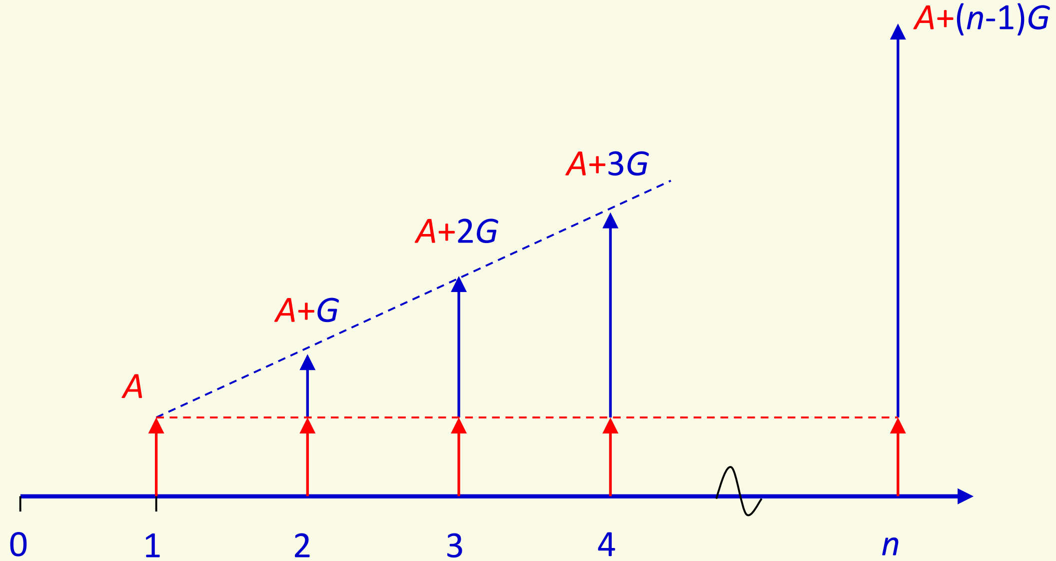
- Question: If the PW of the earned prize today is $20 million, what would be the "*stated*" lottery prize?

Example 2-6. Saving of Jackpot

- You won a Jackpot of $20 million which pays you $1 million per year for 20 years. Like others, you want spend your winning money to improve your lifestyle. You are considering two options.
  - Option 1: You save all of your winnings for the first 6 years, and then spend all the winnings which come in the remaining 14 years.
  - Option 2: You do the reverse, spending for the first 6 years and then saving for 14 years.
- If you can save winnings at 8% interest, how much would you have at the end of 20 years under each option?

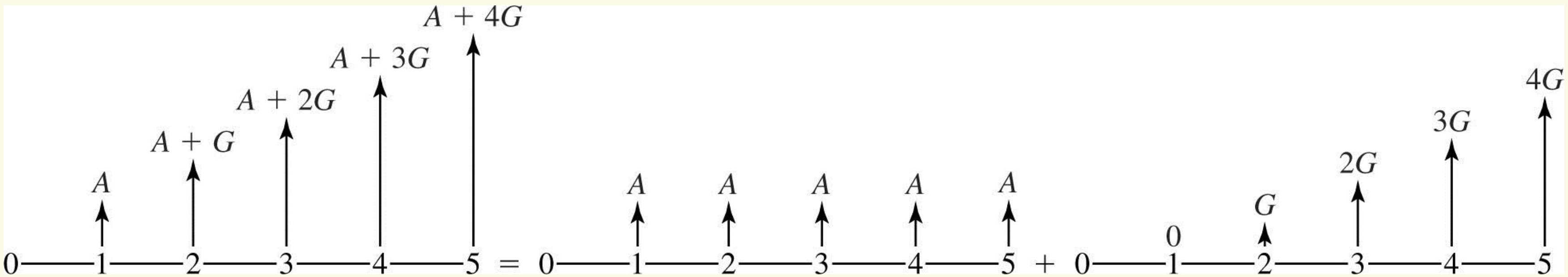
3. Arithmetic gradient (Non-uniform)

- Arithmetic gradient series
  - A series that increases by a constant amount per period
  - Examples: Increasing operating cost of a machine; salary package
  - Be careful about the timing of each cash flow: You are evaluating at year zero, and the first increase *G* starts at year 2.



3.1. Arithmetic gradient

- Cash flows of the arithmetic gradient series can be resolved into *two components*.



  - The first component is simply a *uniform series*.
  - The second component increases by a fixed increment G per period.
  - Note that the second gradient series starts with zero at year 1.
- The present worth of the above series is

$$P = A(P/A, i, n) + G(P/G, i, n)$$

- Then, what is the value of $(P/G, i, n)$?

3.1. Arithmetic gradient factors

- Arithmetic gradient *present worth* factor

$$(P/G, i, n) = \frac{(1+i)^n - in - 1}{i^2(1+i)^n}$$

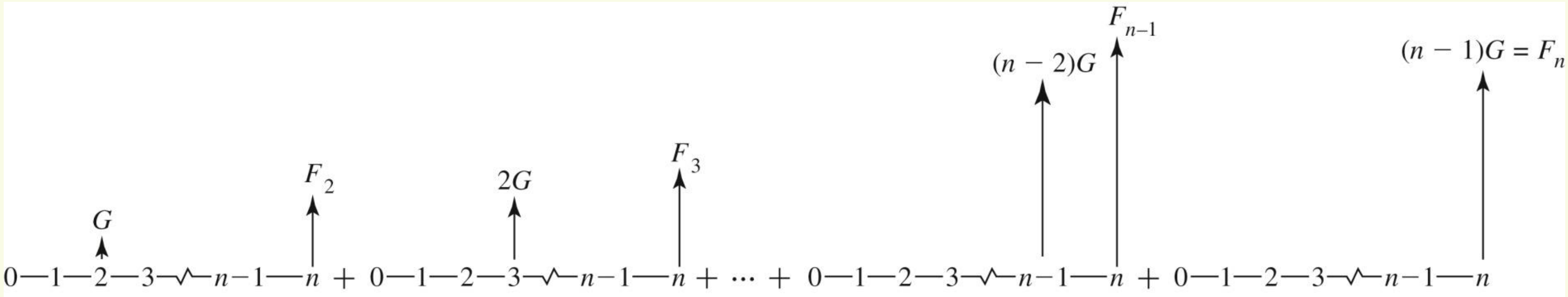
- Arithmetic gradient *uniform series* factor

$$(A/G, i, n) = \frac{1}{i} - \frac{n}{(1+i)^n - 1}$$

  - Again, you don’t need to memorize these factors.
  - You better use the factor tables to find the value, or use Excel if possible.

3.2. Derivation of arithmetic series (1)

- The arithmetic gradient series is a series of individual cash flows.



- The future worth of each cash flow is

$$F = G(1+i)^{n-2} + 2G(1+i)^{n-3} + \ldots + (n-2)G(1+i)^1 + (n-1)G$$

- Multiply by $(1 + i)$.

$$(1+i)F = G(1+i)^{n-1} + 2G(1+i)^{n-2} + \ldots + (n-2)G(1+i)^2 + (n-1)G(1+i)^1$$

- Subtract the first equation from the second equation.

$$iF = G[(1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i)^2 + (1+i) + 1] - nG$$

3.2. Derivation of arithmetic series (2)

- Using the series compound amount factor, we get

$$F=\frac{G}{i}\left[\frac{(1+i)^n-1}{i}-n\right]$$

- Convert the future worth into present worth.

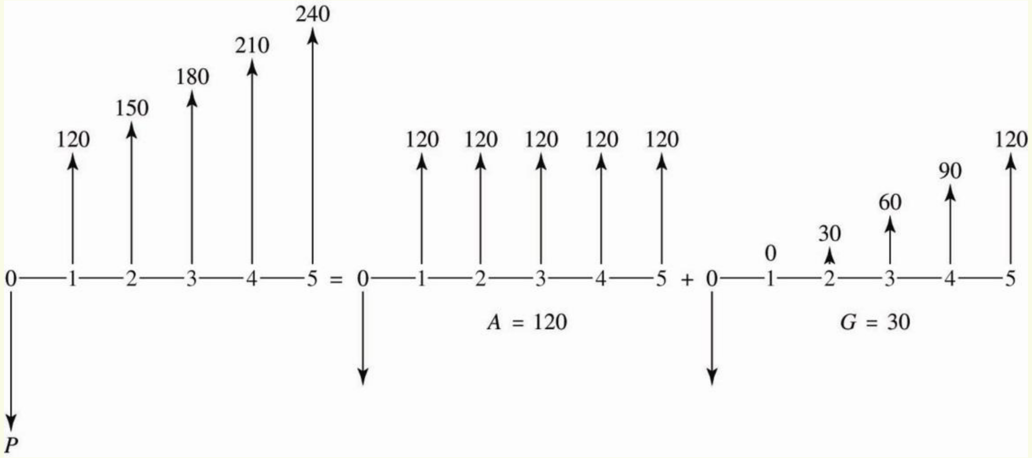
$$P=\frac{G}{i}\left[\frac{(1+i)^n-1}{i}-n\right]\left[\frac{1}{(1+i)^n}\right]=G\left[\frac{(1+i)^n-in-1}{i^2(1+i)^n}\right]=G(P/G,i,n)$$

- Or, convert the future worth into uniform series.

$$A=\frac{G}{i}\left[\frac{(1+i)^n-1}{i}-n\right]\left[\frac{i}{(1+i)^n-1}\right]=G\left[\frac{(1+i)^n-in-1}{i(1+i)^n-1}\right]=G(A/G,i,n)$$

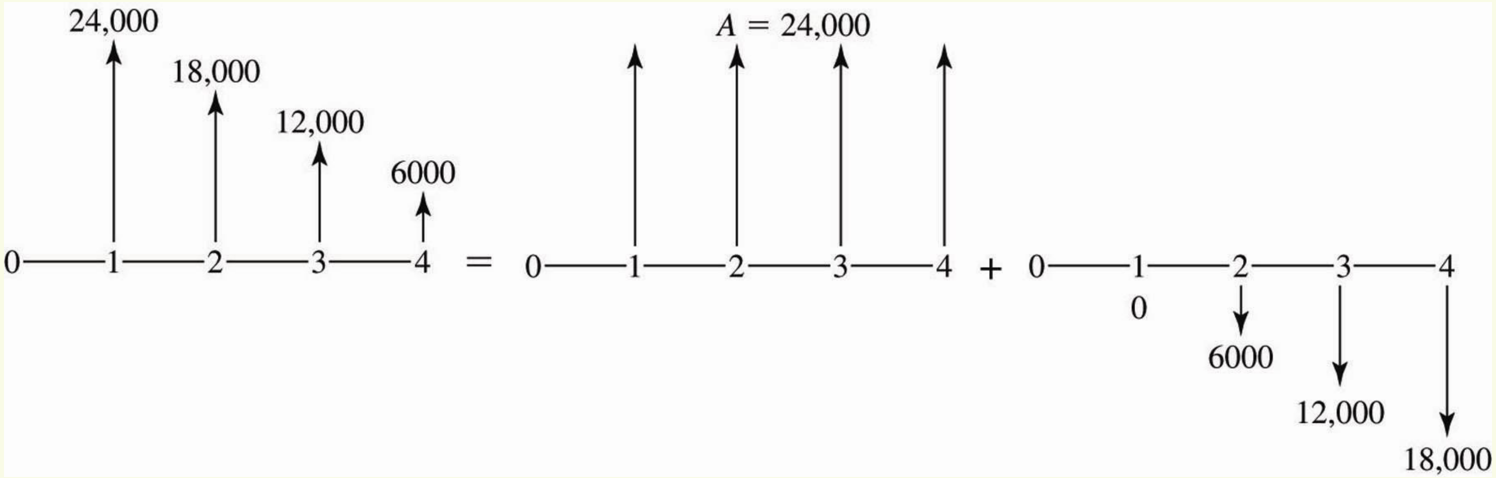
EXAMPLE 3-1. ARITHMETIC GRADIENT PW

- A man who purchased a car wants to set aside enough money for its maintenance for the next five years. If he needs \$120 in the first year with an annual increase of \$30, how much should he deposit now, with i = 5%?



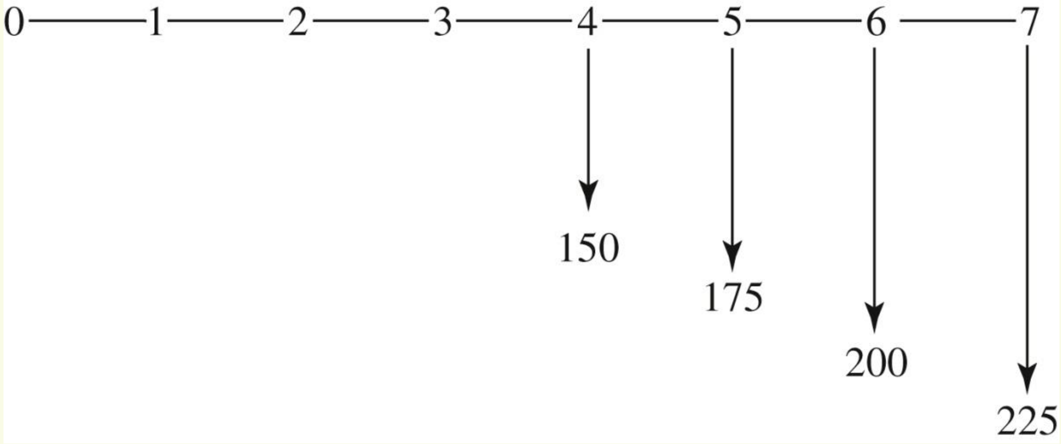
EXAMPLE 3-2. ARITHMETIC GRADIENT AW

- A textile mill has a maintenance cost, which starts at $24,000 in the first year, reducing by $6,000 each year for 4 years. What is the *equivalent annual* maintenance cost if interest rate is 10%?

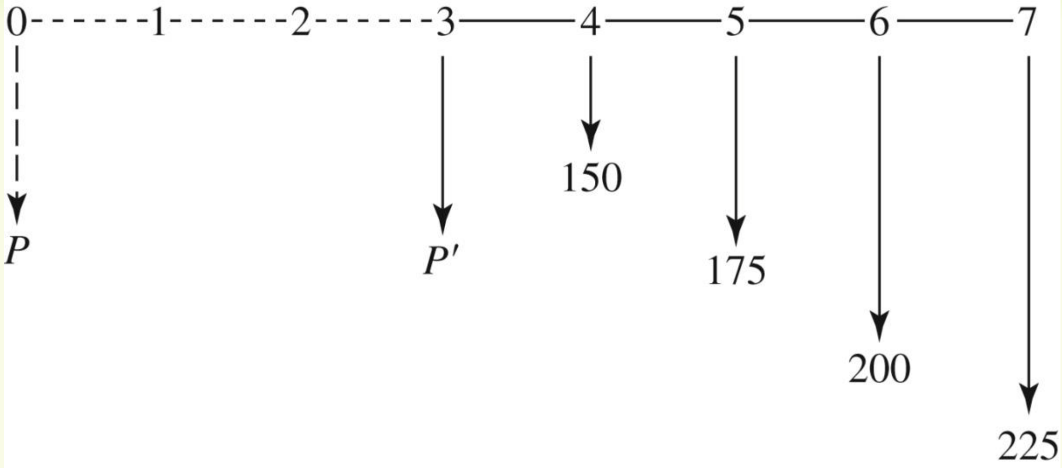


Example 3-3. Arithmetic gradient PW (1)

- The warranty on a car is three years. Once it expires, annual maintenance starts at $150 and then increases $25 per year until the car is sold at the end of year 7. Using a 10% interest rate, find the present worth of these expenses.

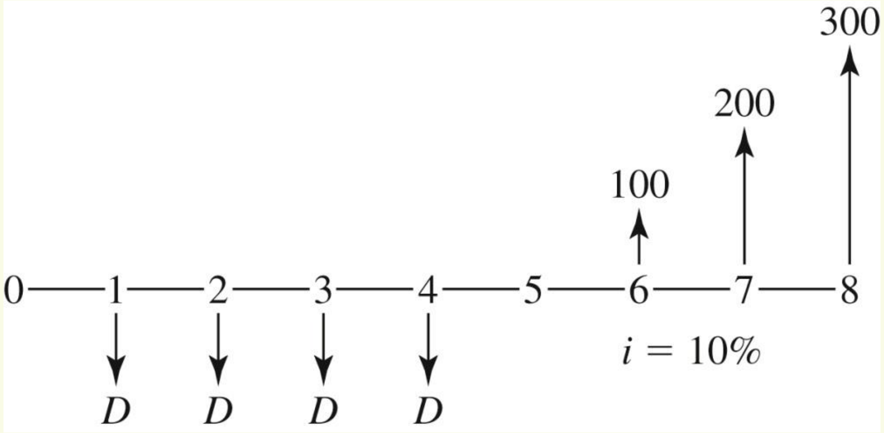


EXAMPLE 3-3. ARITHMETIC GRADIENT PW (2)



EXAMPLE 3-4. ARITHMETIC GRADIENT PW

- Compute the value of *D* in the diagram for i = 10%.



Example 3-5. Multi-million dollar contract

- An NHL player has signed a 10-year, $130 million contract extension, which comes with three categories of payments.
  - The base salary starts at $1 million, increasing by $2 million every year.
  - The initial signing bonus ($20 million) is prorated over the first 5 years.
  - The second roster bonus ($10 million) is prorated over the final 5 years.
- If all the payments are paid at the end of each year, what is the contract's annual worth, at an interest rate of 8%?

3.3. Case: RIM's patent dispute in 2006 (1)

- 1984: RIM was founded at Waterloo, Canada, initially producing two-way paging machines.
- 1998: RIM introduced BlackBerry 950, the first wireless email device.

- 2001.11: NTP (a Virginia-based patent holding company) filed a lawsuit against RIM on patent infringement.
  - In 2000, NTP had asked RIM to license its patents, but RIM rejected.
- 2002.11: The jury found RIM's infringement on the NTP patents.
  - NTP was awarded a total of $53 million, but RIM appealed.
- 2003: RIM requested US PTO to re-examine NTP's patents .
  - A few of the NTP's patents were later ruled invalid.
  - Some argued that all the disputed patents might *eventually* be ruled invalid. However, it would take several years to resolve the validity.

3.3. Case: RIM's patent dispute in 2006 (2)

- 2003.08: The district court ruled a permanent "injunction" against RIM, prohibiting RIM from selling the device.
  - RIM appealed the decision, and the federal court lifted this injunction.
- 2006.03: RIM finally agreed to pay NTP $612 million.
  - RIM's annual revenue in 2006 was $2 billion.

- Why did RIM settle the lawsuit in 2006?
  - Even if the NTP's patents were ruled invalid, the process would take several years and RIM would suffer during the period.
- How did RIM come up with the amount of $612 million?
  - During the legal process, sales of BlackBerry service have dropped.
  - "In March 2006, RIM revised its revenue forecast from $590 million to $550 million."

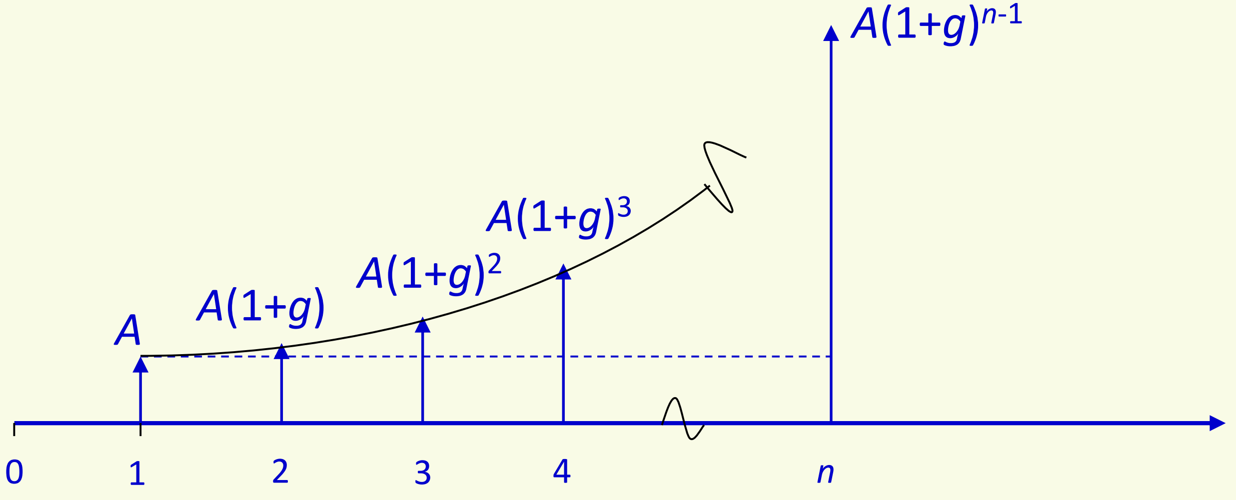
3.3. Case: RIM's patent dispute in 2006 (3)

- What would be the present worth of RIM's loss in sales if the loss of sales is $40 million per quarter? Assume the legal dispute would last for 3 more years, and the interest rate is 6%, compounded quarterly.

- More realistically, suppose the $40 million quarterly loss would increase by $4 million every quarter. What would be the present worth of the loss for 3 years?

4. GEOMETRIC GRADIENT (NON-UNIFORM)

- Geometric gradient series
  - A series that increases by a constant percentage per period
  - Examples: inflation, productivity growth, college tuition
  - g: constant growth rate per period
  - Again, be careful of the timings of each cash flow.



4.1. Geometric gradient

- Geometric gradient series
  - The derivation is omitted here (see the textbook).
  - The present worth of geometric gradient series is

$$P = A(P/A, g, i, n)$$

- Two cases of the geometric series

$$(P/A, g, i, n) = \left[1 - \left(\frac{1+g}{1+i}\right)^n\right] / [i-g] \qquad \text{where } i \neq g$$

$$(P/A, g, i, n) = \frac{n}{1+i} \qquad \text{where } i = g$$

Example 4-1. Geometric gradient series: $i \neq g$

- The first-year maintenance cost for a new car is estimated to be \$100, and it increases at a rate of 10% per year. Using an 8% interest rate, calculate the present worth of the maintenance cost of the first 5 years.

Example 4-2. Geometric gradient series: $i = g$

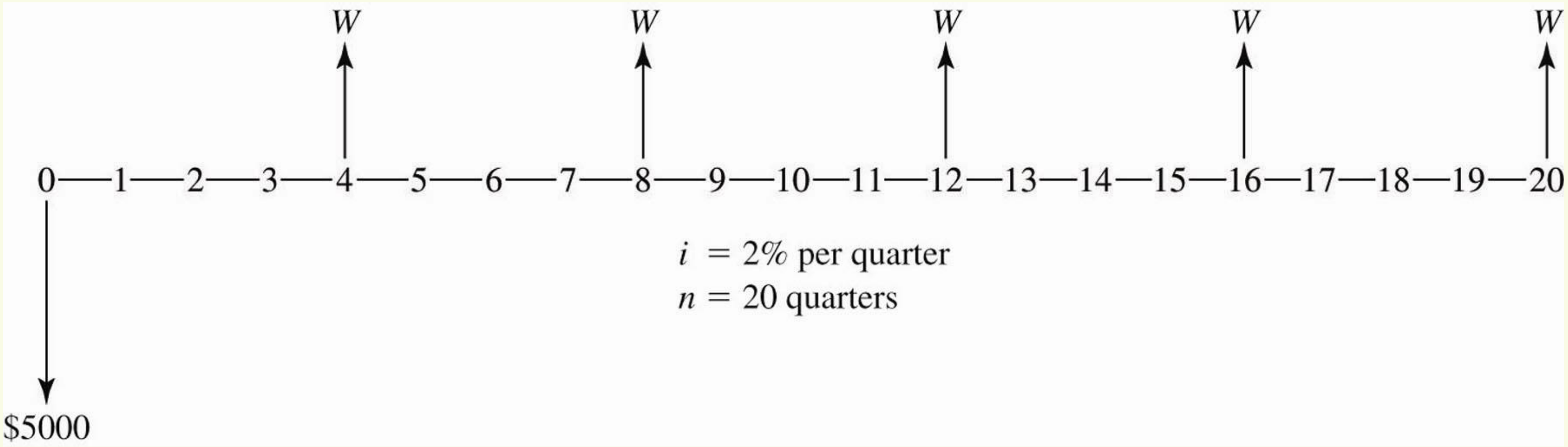
- Emery is expected to receive dividends for the next 10 years, starting $110,000 at the end of this year with a 10% increase each year. If all dividends are invested at 10% interest, how much will Emery accumulate in 10 years?

EXAMPLE 4-3. RIM CASE REVISITED

- In the RIM case discussed above, suppose the $40 million quarterly loss is increased by 5% every quarter due to the switching of existing customers to other service providers. What would be the present worth of the loss during the 3 year period, if the interest rate is 6%, compounded quarterly?

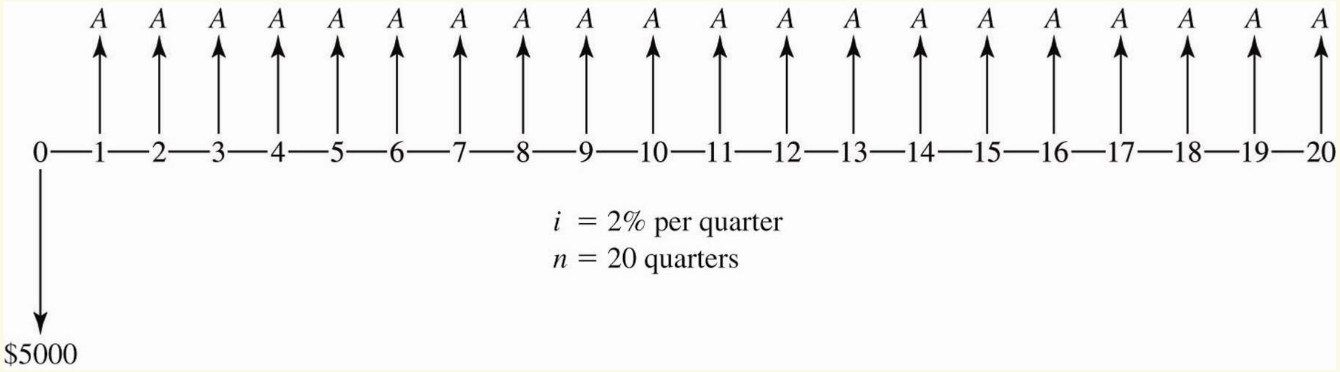
EXAMPLE 5-1. SPORADIC PAYMENT (1)

- Sometimes, compounding interest periods and payment periods differ, and an adjustment is required.
- On January 1, a woman deposits $5,000 in a credit union that pays 8% nominal annual interest, compounded *quarterly*. She wishes to withdraw the money in five equal yearly sums beginning December 31 of the first year. How much should she withdraw each year?

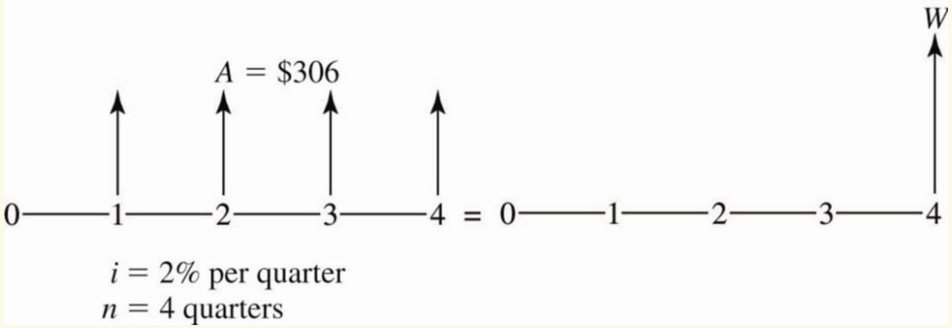


EXAMPLE 5-1. SPORADIC PAYMENT (2)

- Method 1: Compute an equivalent *A* for each quarter.

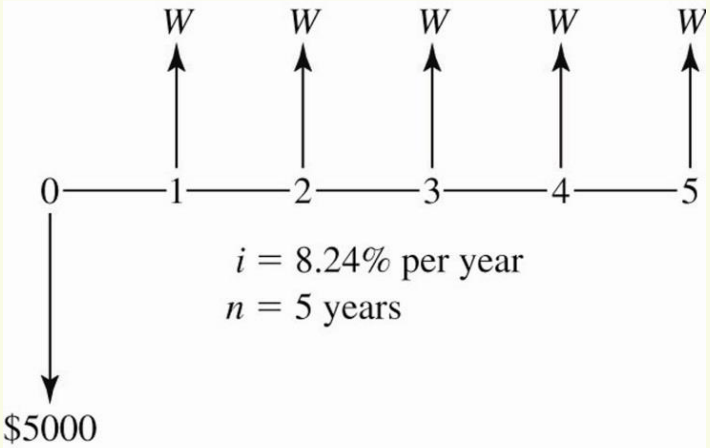


  - We now convert every quarter annuity *A* into a yearly annuity.



EXAMPLE 5-1. SPORADIC PAYMENT (3)

- Method 2: Compute an effective *i* for the 4-quarter period.



Example 5-2. Frequent compounding period

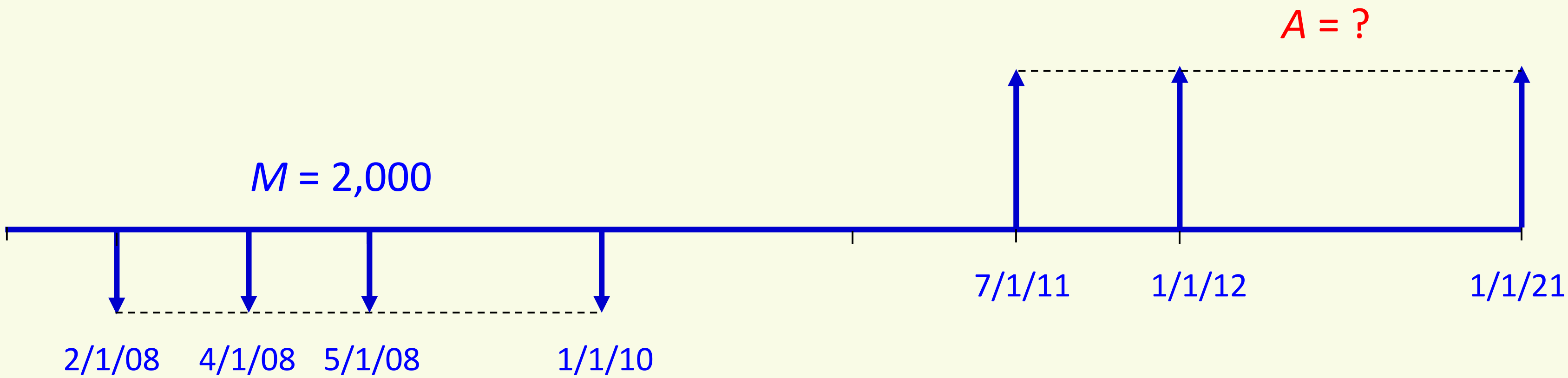
- Suppose you make equal *quarterly* deposits of $1,000 into a fund that pays interest rate of 6% compounded *monthly*. How much will you have at the end of 5 years?

Example 5-3. Less compounding period

- Suppose you make \$500 *monthly* deposits to a registered retirement savings plan (RRSP) that pays interest at a rate of 10% compounded *quarterly*. How much will you have at the end of 10 years?

Example 5-4. Compounding period (1)

- Kathy deposited $2,000 per *month* from February 1, 2008 till January 1, 2010. She then withdrew money *semi-annually* from July 1, 2011 till January 1, 2021. What is the amount of each semi-annual cash flow, given the interest rate of 12% with *monthly compounding* on all accounts?



- We should make the value of monthly cash flows equal to the value of semi-annual cash flows at the *same point* of time (Jan 1, 2011).

Example 5-2. Compounding period (2)

- Value of monthly cash flow

- Value of semi-annual cash flow

Excel commands of annuity

- Find the equivalent of *P*
  - (P/F) factor: F*PV(i, N, 0, -1)
  - (P/A) factor: A*PV(i, N, -1)
- Find the equivalent of *F*
  - (F/P) factor: P*FV(i, N, 0, -1)
  - (F/A) factor: A*FV(i, N, -1)
- Find the equivalent of *A*
  - (A/F) factor: F*PMT(i, N, 0, -1)
  - (A/P) factor: P*PMT(i, N, -1)
- Find n: NPER(i, A, P, F, Type)
- Find i: RATE(n, A, P, F, Type, guess)
- The gradient series factors are too complicated.

Summary of chapter 4

- Uniform series formula
  - *F* vs. *A*
  - *P* vs. *A*
- Arithmetic gradient formulas
- Geometric gradient formulas
- Check Excel commands for each case

- Selective end-of-chapter problems
  - 2, 4, 9, 16, 22, 25, 36, 45, 56, 60, 63, 68, 74, 84, 92, 104, 115