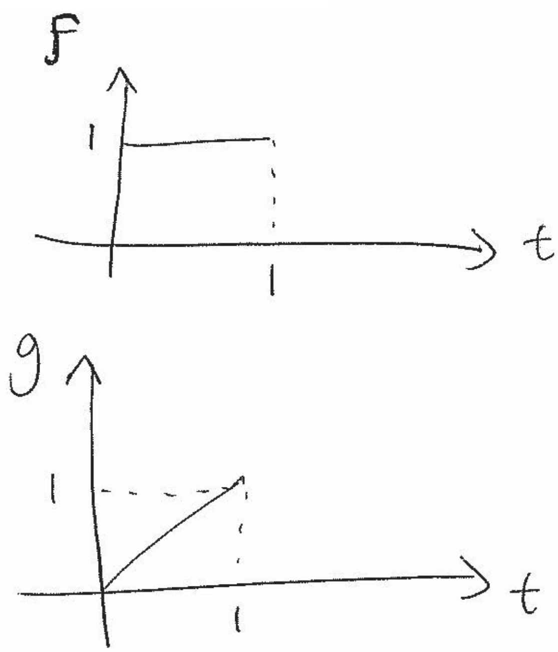
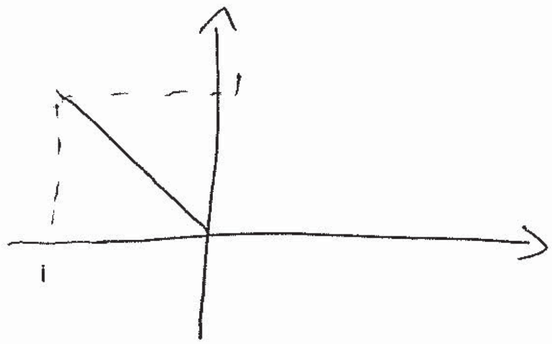
Solution of assignment # 5



$$f*g_{(t)} = ?$$

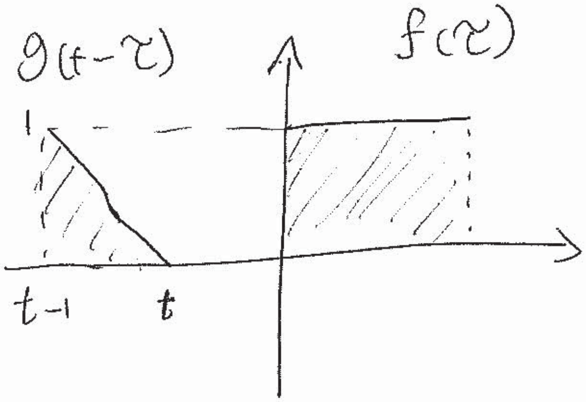
$$f*g_{(t)} = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

$g(-\tau)$:



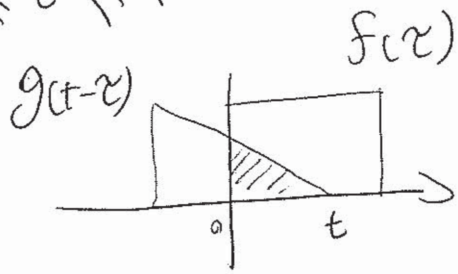
we calculate $\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$ for different values of "t"

for $-\infty \leq t \leq 0$:



$f(\tau), g(t-\tau)$ have no intersection, therefore convallution is zero.

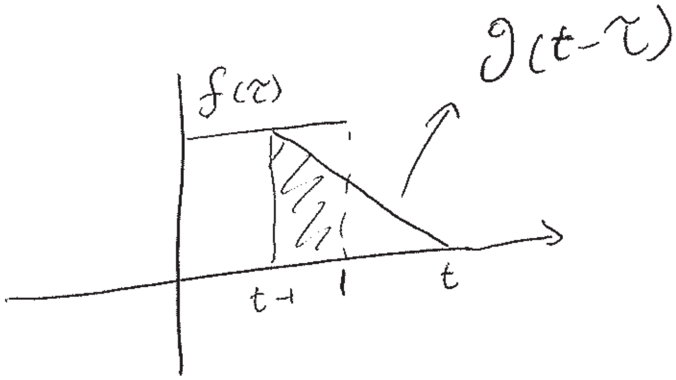
for $0 \leq t \leq 1$:



$$\int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t (t-\tau)\, d\tau = t^2 - \frac{t^2}{2} = \frac{t^2}{2}$$

for $1 \leq t \leq 2$:



$$\int_{t-1}^{1} (t-\tau)\, d\tau$$

$$= t\tau - \frac{\tau^2}{2}\Big]_{t-1}^{1}$$

$$= t - \frac{1}{2} - \left((t(t-1)) - \frac{(t-1)^2}{2}\right)$$

$$= -\frac{t^2}{2} + t$$

Note that the limit $0 \leq t \leq 1$ can be represented by $(u(t) - u(t-1))$ and the period $1 \leq t \leq 2$ can be represented by $u(t-1) - u(t-2)$.

therefore

$$\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau = \frac{t^2}{2}(u(t) - u(t-1)) + \left(t - \frac{t^2}{2}\right)(u(t-1) - u(t-2))$$

$$= \frac{t^2}{2}u(t) + (-t^2 + t)u(t-1) - \left(t - \frac{t^2}{2}\right)u(t-2)$$

b) $f(t) = u(t) - u(t-1)$

$$\Rightarrow F(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s}$$

$$g(t) = t\, f(t)$$

$$\Rightarrow G(s) = -\frac{d}{ds} F(s)$$

$$\Rightarrow G(s) = \frac{+1}{s^2} - \frac{e^{-s}(s+1)}{s^2}$$

c) $$F(s)\, G(s) = \frac{-(1-e^{-s})(s+1)e^{-s}}{s^3} + \frac{1-e^{-s}}{s^3}$$

$$= \frac{1 - se^{-s} - 2e^{-s} + se^{-2s} + e^{-2s}}{s^3}$$

$$= \frac{1}{s^3} - \frac{e^{-s}}{s^2} - \frac{2e^{-s}}{s^3} + \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s^3}$$

d) final form of solution in part (a) is

$$\frac{t^2 u(t)}{2} - t(t-1)\,u(t-1) + \frac{t}{2}(t-2)\,u(t-2).$$

it is easy to check that laplace transform of above equation is the same as part (b):

$$\mathcal{L}\left(\frac{t^2 u(t)}{2}\right) = \frac{1}{s^3},$$

$$\mathcal{L}\left(-t(t-1)\,u(t-1)\right) = -\mathcal{L}\left(t(t-1)\,u(t-1)\right)$$

$$= -\left[-\frac{d}{ds}\left(\mathcal{L}\left((t-1)\,u(t-1)\right)\right)\right]$$

$$= \frac{d}{ds}\left(\frac{e^{-s}}{s^2}\right) = \frac{-e^{-s}}{s^2} - \frac{2e^{-s}}{s^3}$$

Finally:

$$\mathcal{L}\left(\frac{t}{2}(t-2)\,u(t-2)\right) = \frac{1}{2}\mathcal{L}\left(t(t-2)\,u(t-2)\right)$$

$$= -\frac{1}{2}\times\frac{d}{ds}\left(\frac{e^{-2s}}{s^2}\right)$$

$$= \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s^3}$$