MATH 239 Spring 2014: Assignment 3

Due: 3:00 PM, Monday June 2, 2014 in the dropboxes outside MC 4066

Last Name: First Name:

I.D. Number: Section:

Mark (For the marker only): /26

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1. {4 marks} Let $n \in \mathbb{N}$. Suppose we have an unlimited supply of Canadian nickels, dimes, quarters, loonies, and toonies (they are worth $5, 10, 25, 100, 200$ cents each, respectively). How many ways can we make n cents using these coins such that the number of nickels is at least the number of quarters? Note that coins with the same denomination are considered to be identical, so we only care about the number of coins of each denomination. You may say that your answer is equal to a certain coefficient of a power series.

2. {4 marks} Let $n \in \mathbb{N}$. How many compositions of n consists of either 5 or 6 parts, and each part is even? Determine a generating series for the set of all such compositions (for all n), and then determine an explicit formula for the answer.

3. Let n be a non-negative integer, and let S_n be the set of all compositions of n where each part is greater than 1. (The number of parts is not restricted.) Let $a_n = |S_n|$.

   (a) {4 marks} Prove that
$$a_n = [x^n]\frac{1-x}{1-x-x^2}.$$

(b) {4 marks} The generating series from part (a) gives us the recurrence

$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2.$$

Give a combinatorial proof of this recurrence by finding a bijection between S_n and $S_{n-1} \cup S_{n-2}$, and finding its inverse.

(c) {2 marks} Illustrate your bijection by matching up compositions in S_7 with compositions in S_6 and S_5.

$S_5 \cup S_6$:	(5)	$(2,3)$	$(3,2)$	(6)	$(2,4)$	$(3,3)$	$(4,2)$	$(2,2,2)$
S_7 :	(7)	$(2,5)$	$(3,4)$	$(4,3)$	$(5,2)$	$(2,2,3)$	$(2,3,2)$	$(3,2,2)$

4. Let $n \in \mathbb{N}$. Consider the problem of finding the number of compositions of n with exactly 3 parts (a_1, a_2, a_3) such that $1 \leq a_1 < a_2 < a_3$. For example, when $n = 9$, there are three such compositions: $(1, 2, 6), (1, 3, 5), (2, 3, 4)$. Let S be the set of all such compositions, i.e. $S = \{(a_1, a_2, a_3) \mid 1 \leq a_1 < a_2 < a_3\}$. We will determine the generating series of S with the help of another set $T = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, which is the set of all compositions with exactly 3 parts.

   (a) {3 marks} Define a bijection $f : S \to T$, and write down its inverse f^{-1}. Illustrate your bijection by determining $f(2, 3, 9)$ and $f^{-1}(3, 1, 4)$.

   (b) {2 marks} Let w be the weight function on S where $w(a_1, a_2, a_3) = a_1 + a_2 + a_3$. For each $(b_1, b_2, b_3) \in T$, define a weight $w^*(b_1, b_2, b_3)$ such that $w^*(f(a_1, a_2, a_3)) = w(a_1, a_2, a_3)$ for all $(a_1, a_2, a_3) \in S$. (You need to prove that this property holds.)

   (c) {3 marks} Determine the generating series of T with respect to w^*, and explain why this is the same as the generating series of S with respect to w. (Hint: You may use question 5(c) from assignment 1.)