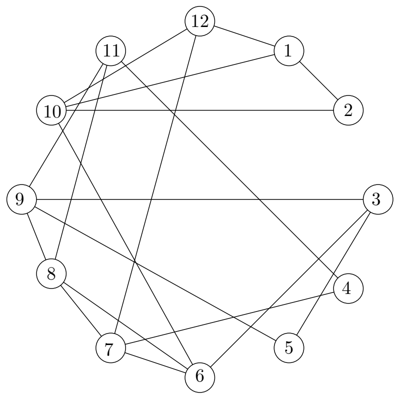
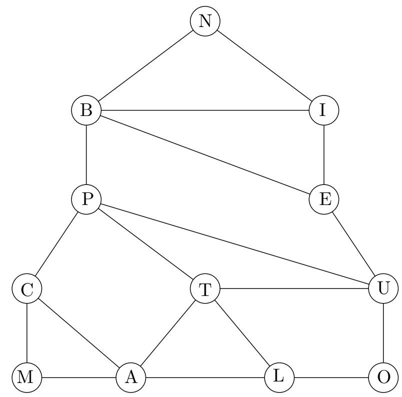
# Math 239 Spring 2014 Assignment 6 Solutions

1. {3 marks} The following two graphs $G$ and $H$ are isomorphic. Find an isomorphism. (You do not need to prove that your mapping is an isomorphism.)




Graph $G$



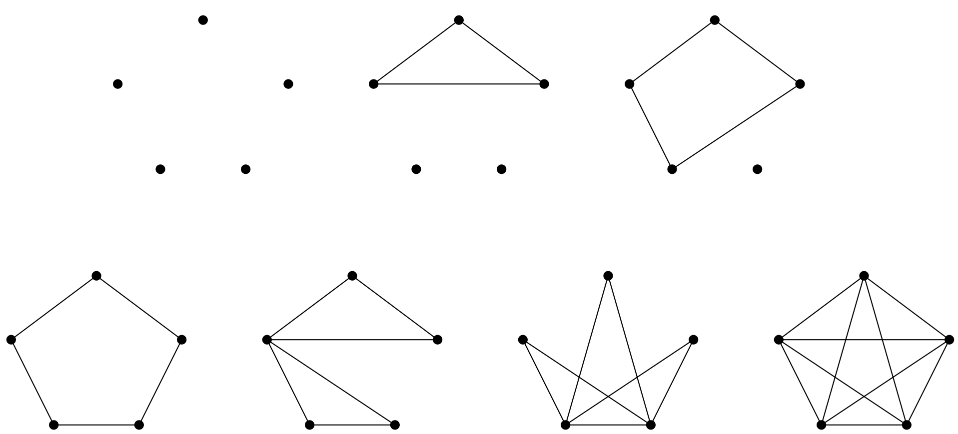

Graph $H$

**Solution.** There is only one possible isomorphism $f : V(G) \to V(H)$, where

| $v$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(v)$ | $I$ | $N$ | $C$ | $O$ | $M$ | $P$ | $U$ | $T$ | $A$ | $B$ | $L$ | $E$ |

2. {3 marks} Draw all non-isomorphic graphs with 5 vertices where the degree of each vertex is even.

**Solution.** There are 7 possibilities.



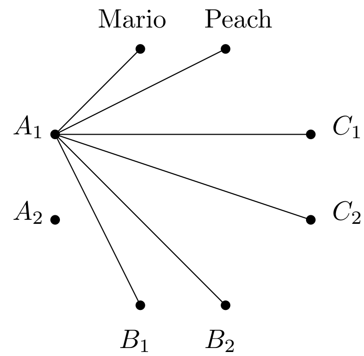
3. {3 marks} A group of 3141 software engineers gathered to change a lightbulb. Some of them have 13 friends within the group, some have 33 friends, and the rest have 37 friends. Using graph theory, prove that this group of software engineers does not exist (and it's not just because this is a hardware problem). Note that we assume that friendship is a symmetric relationship, i.e. if A is a friend of B, then B is a friend of A. In other words, we are assuming Facebook's model of friendship rather than Google+'s model of friendship. (Nobody uses Google+ anyway.)

**Solution.** We can model this using a graph with 3141 vertices (each representing an engineer), and two vertices are adjacent if their corresponding engineers are friends. This means that each vertex has one of three possible degrees: 13, 33 or 37. Notice that all of them are odd, and our graph has an odd number of vertices. This is impossible, since every graph has an even number of odd-degree vertices. Therefore, this group of software engineers does not exist.

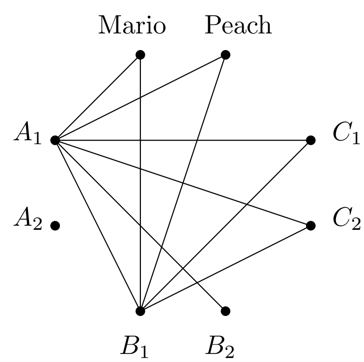
4. {4 marks} Married couple Mario and Peach invited 3 other couples to the castle on the mountain for a cake party (and it's no lie). During the party, some handshaking took place with the restriction that a person cannot shake hands with themselves nor with their own spouse. After all the handshaking was done, Peach went around

to ask the 7 others in the party how many people they shook hands with, and she received a different answer from everyone. How many hands did Mario shake? How many hands did Peach shake?

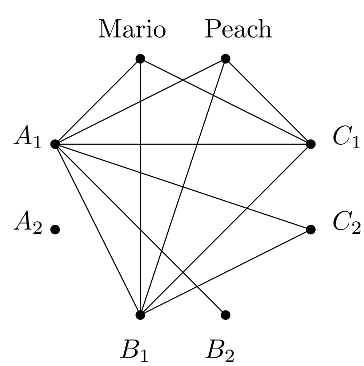
**Solution.** Let the 3 couples be $(A_1, A_2)$, $(B_1, B_2)$, and $(C_1, C_2)$. Each person can shake up to 6 people's hands. Since Peach asked 7 people and the possible number of handshakes is between 0 and 6, there is one person for each possibility. Mario cannot be the one who shook 6 hands, for he would shake hands with all 3 couples, and none of them can shake 0 hands. So WLOG suppose $A_1$ is the one who shook 6 hands. Then this means that $A_2$ is the only candidate for shaking 0 hands, resulting in the following situation.




Now Mario cannot be the one who shook 5 hands, for he would have to shake hands with $B_1, B_2, C_1, C_2$, meaning none of them could shake only 1 hand. So WLOG suppose $B_1$ is the one who shook 5 hands. Then this menas that $B_2$ is the only candidate for shaking 1 hand, resulting in the following situation.




Similarly, Mario cannot be the one who shook 4 hands, for he would have to shake hands with $C_1, C_2$, meaning none of them could shake only 2 hands. So WLOG suppose $C_1$ is the one who shook 4 hands. Then this means that $C_2$ is the only candidate for shaking 2 hands, resulting in the following final situation.




Therefore, both Mario and Peach shook 3 people's hands.

5. {3 marks} Let $k \in \mathbb{N}$. Prove that if $G$ is a $k$-regular bipartite graph with bipartition $(A, B)$, then $|A| = |B|$.

**Solution.** Each edge in a bipartite graph has exactly one end in $A$ and one end in $B$. So the total number of edges is equal to the sum of the degrees of vertices in $A$, which is also equal to the sum of the degrees of vertices in $B$, i.e.

$$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v).$$

Since $G$ is $k$-regular, $\deg(v) = k$ for all vertices $v$. Therefore,

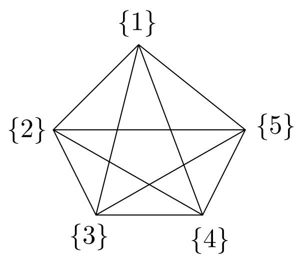
$$k|A| = k|B|.$$

Since $k > 0$, we can divide both sides by $k$ to get $|A| = |B|$.

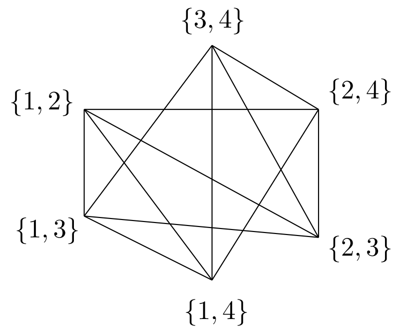
6. For integers $n \geq k \geq 1$, let $S_{n,k}$ be the graph whose vertices are all the $k$-subsets of $[n]$. There is an edge between two vertices $A, B$ in the graph if and only if the two sets intersect in $k-1$ elements, i.e. $|A \cap B| = k - 1$.

   (a) {2 marks} Draw $S_{5,1}$ and $S_{4,2}$.

   **Solution.**




$S_{5,1}$




$S_{4,2}$

   (b) {2 marks} Prove that $S_{n,k}$ is regular by determining the degree of each vertex.

   **Solution.** For each set $A$, any neighbour $B$ satisfies $|A \cap B| = k - 1$. So there are $\binom{k}{k-1} = k$ ways to choose the $k - 1$ elements in common. There is one element remaining in $B$ that can be chosen from the $n - k$ elements not in $A$. So there are $k(n - k)$ possible ways to choose $B$. This means that every vertex has degree $k(n - k)$, so $S_{n,k}$ is a regular graph.

   (c) {2 marks} Determine the total number of edges in $S_{n,k}$.

   **Solution.** There are $\binom{n}{k}$ vertices. Each vertex has degree $k(n - k)$. So the number of edges is half the sum of all vertex degrees, which is $\binom{n}{k}k(n - k)/2$.

   (d) {3 marks} Prove that when $n > k \geq 2$, $S_{n,k}$ is not bipartite.

   **Solution.** Consider the following three sets.

$$\begin{aligned} A &= \{1, 2, 4, \ldots, n\} \\ B &= \{1, 3, 4, \ldots, n\} \\ C &= \{2, 3, 4, \ldots, n\} \end{aligned}$$

   Notice that $|A \cap B| = |A \cap C| = |B \cap C| = k - 1$, so $AB, BC, CA$ are all edges in $S_{n,k}$. This forms a triangle, which is not bipartite. Hence $S_{n,k}$ is not bipartite.

   (e) {3 marks} Prove that for any $n \geq 2$, $S_{n,k}$ is isomorphic to $S_{n,n-k}$.

   **Solution.** Consider the mapping $f : V(S_{n,k}) \to V(S_{n,n-k})$ by $f(A) = [n] \setminus A$. From earlier in the course, we know that $f$ is a bijection. Now we need to prove that $f$ is an isomorphism. Suppose $AB$ is an edge in $S_{n,k}$. Then we know that $|A| = |B| = k$ and $|A \cap B| = k - 1$. In addition, we know that $|A \cup B| = k + 1$. Consider $f(A) = [n] \setminus A$ and $f(B) = [n] \setminus B$. The $n - k - 1$ elements of $[n] \setminus (A \cup B)$ are in both $f(A), f(B)$. There is one additional element in $f(A)$, which is $B \setminus A$, and similarly there is one additional element in $f(B)$, which is $A \setminus B$. These are two different elements. So $|f(A) \cap f(B)| = n - k - 1$, hence $f(A)f(B)$ is an edge in $S_{n,n-k}$. A similar argument proves that if $f(A)f(B)$ is an edge in $S_{n,n-k}$, then $AB$ is an edge in $S_{n,k}$. Hence $f$ is an isomorphism, and $S_{n,k}, S_{n,n-k}$ are isomorphic.