Height of Quadtrees

I made a mistake in the theorem and the proof of the height of quadtrees. Here is the correction and more details on the proof. The corrections are written in red.

Theorem. *Let* $d_{\max}$ *(resp.* $d_{\min}$*) be the maximal distance (resp. minimal) distance between two points.*

Then the height of the corresponding quadtree is at most $\log_2(d_{\max}/d_{\min}) + 3/2$*.*

Proof. Let s be the size of the edges of the minimal square bounding box R. Let us prove that

$$s \leqslant d_{\max}. \tag{1}$$

To prove this, we will first prove that we can include all the points in a square of edge $d_{\max}$. Then, since R is the minimal bounding box, we get $s \leqslant d_{\max}$. Let $\min_x$ (resp. $\max_x$) be the minimum (resp. maximum) of all x-coordinates of our points. Let $\min_y$ and $\max_y$ be the corresponding quantities for the y-coordinate. Then $\max_x - \min_x \leqslant d_{\max}$ (otherwise we would have points with distance $> d_{\max}$) and also $\max_y - \min_y \leqslant d_{\max}$. So the square of edge $d_{\max}$ whose bottom left corner is $(\min_x, \min_y)$ includes all the points.

Since s is the edge's size of R, the quadrants corresponding to nodes at depth 1 have size $s/2$, then the subquadrants at depth 2 have size $s/4$, ..., up to last level at depth h which have size $s/2^h$, where h is the height of the tree. Since all splits are necessary, it means that there exists a subquadrant at depth $h-1$ (of size $s/2^{h-1}$) that contains at least 2 points. These two points must have distance $\leqslant \sqrt{2}\,(s/2^{h-1})$ (diagonal of the subquadrant) and so

$$d_{\min} \leqslant \sqrt{2}\,(s/2^{h-1}). \tag{2}$$

Using Equations (1) and (2), we get $2^{h-1} \leqslant \sqrt{2}\, s/d_{\min} \leqslant \sqrt{2}\, d_{\max}/d_{\min}$. Taking the log in base-2, we get $h - 1 \leqslant \log_2(d_{\max}/d_{\min}) + 1/2$ and our result. □