MATH 239 Spring 2014: Assignment 0

Due: 3:00 PM, Monday May 12, 2014 in the dropboxes outside MC 4066

Last Name: First Name:

I.D. Number: Section:

Mark (For the marker only): /22

Note: This assignment is a review of some materials you may have learned before that will be used in this course, and a preview of some materials that you will see later on in the course. This is a bonus assignment.

1. {6 marks} Counting

   This is a review of some counting problems. We only require the final answer, and you do not need to expand large powers, factorials and binomial coefficients.

   (a) A binary string is a sequence of digits where each digit is either 0 or 1. A string is a palindrome if it reads the same forwards and backwards (for example, 101101 and 00100 are palindromes). How many binary strings of length 13 are palindromes?

   (b) There are 7 seats around a round table available for the 7 participants of a discussion panel. Among the 7 participants, 2 of them are married and hence cannot sit next to each other. How many ways can these 7 people be seated?

   (c) For a hypothetical group project, there are 15 combinatorial students that need to be divided into three groups of 5 students each. How many groupings are possible? (Note that the groups are indistinguishable, so if we had only 3 students divided into three groups of 1 student each, there is only 1 way to do it.)

2. {4 marks} Power series

We will be using the coefficients of power series to solve some counting problems. Consider the following two power series:

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i \geq 0} x^i, \qquad g(x) = 1 - x + x^2 - x^3 + x^4 - \cdots = \sum_{i \geq 0} (-x)^i$$

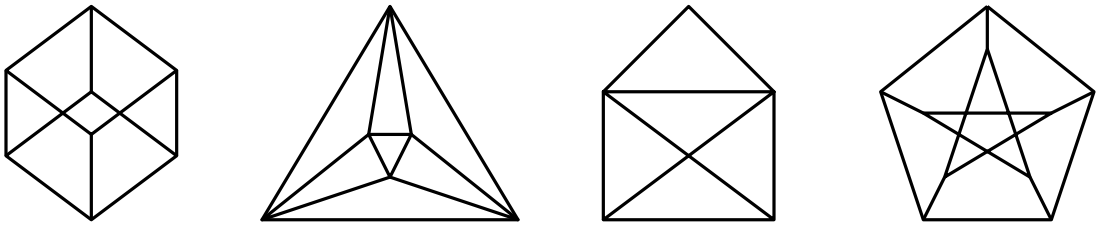
Determine the coefficient of x^{2014} in $(f(x))^2$ and $f(x)g(x)$.

3. {3 marks} Recurrence relation

Let $\{a_n\}_{n \geq 0}$ be the sequence defined by $a_0 = 6, a_1 = 12$, and for any integer $n \geq 2$, $a_n - 4a_{n-1} + 4a_{n-2} = n - 1$. You will learn how to find an explicit formula for recurrences like this. For now, we will give the answer to you and ask you to prove that for any integer $n \geq 0$, $a_n = (n + 3)(2^n + 1)$.

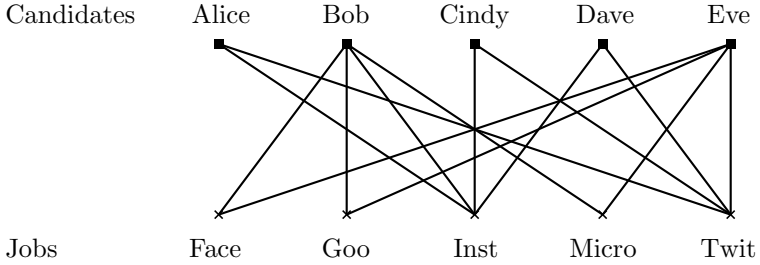
4. {2 marks} Graph theory

For each of the following figures, decide whether or not it can be drawn in one stroke without repeating any line segments (a point of intersection can be visited multiple times). This may sound like child's play, but it relates to one of the first problems that was studied in graph theory. Circle all figures for which only one stroke is needed (and do not circle others), no explanations required.



5. {3 marks} Matching

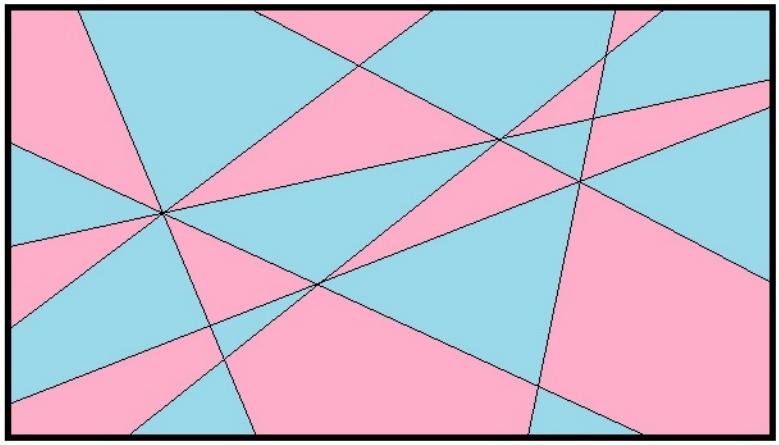
There are 5 jobs available to 5 possible candidates. In the following diagram, there is a straight line joining a candidate and a job if and only if that candidate had an interview for that job. Assume that each candidate can hold at most one job, each job can be filled by at most one candidate, and only those candidates that have been interviewed for a job are eligible for it. If you attempt to match the candidates to the jobs, you may find this to be an impossible task in this instance. Give a reasonable explanation as to why that is the case.



6. {4 marks} Colouring

A famous result in graph theory is the 4-colour theorem, which tells us that in any map, it is possible to use just 4 colours to paint the regions so that bordering regions receive different colours. Here is a simplified problem on colouring, which will also give you a taste of induction in perhaps an unfamiliar situation.

We are given a rectangle with several line segments, each joining border to border inside of the rectangle. This divides the rectangle into several regions. We wish to colour these regions with red and blue so that any two regions that border each other through some line segment receive different colours (two regions that touch each other at only a point may have the same colour). An example is given below.



Prove (using induction on the number of line segments) that regardless of how we divide the rectangle, such a colouring is always possible.