MATH 239 Spring 2014: Assignment 2

Due: 3:00 PM, Monday May 26, 2014 in the dropboxes outside MC 4066

Last Name: First Name:

I.D. Number: Section:

Mark (For the marker only): /26

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1. {6 marks} Consider the following power series.

$$f(x) = \sum_{i=2}^{157} (-3)^{i-2} x^{2i} = x^4 - 3x^6 + 9x^8 - \cdots - 3^{155} x^{314} \qquad g(x) = \sum_{i \geq 3} f(x)^i.$$

   (a) Express both $f(x)$ and $g(x)$ as rational functions, i.e. $\frac{p(x)}{q(x)}$ where $p(x), q(x)$ are explicit polynomials (you should be able to write them out without resorting to sums). Simplify your expressions as much as possible.

   (b) Does $g(x)$ have an inverse? If so, determine a rational function for it. If not, explain why not.

2. {4 marks} Using mathematical induction on k, prove that for any integer $k \geq 1$,

$$(1-x)^{-k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n.$$

3. {4 marks} Determine the value of the following coefficient.

$$[x^{26}](3+x^2)(1-2x^6)^{-31}(1+x^9)^{-41}.$$

4. {4 marks} Let $\{a_n\}_{n\geq 0}$ be a sequence whose corresponding power series $A(x) = \sum_{i\geq 0} a_i x^i$ satisfies

$$A(x) = \frac{-6 - 34x}{1 + 2x - 3x^2}.$$

Determine a recurrence relation that $\{a_n\}$ satisfies, with sufficient initial conditions to uniquely specify $\{a_n\}$. Use this recurrence relation to find a_4.

5. Let $n \in \mathbb{N}$. For a permutation $\sigma : [n] \to [n]$, we use the notation $(\sigma(1)\sigma(2)\cdots\sigma(n))$ to describe the mapping. A pair of integers (i, j) is called an *inversion* of σ if $i < j$ and $\sigma(i) > \sigma(j)$. For example, the permutation (32415) on $[5]$ has 4 inversions: $(1, 2), (1, 4), (2, 4), (3, 4)$. Define the weight function w on a permutation σ to be the number of inversions in σ. Let S_n be the set of all permutations of $[n]$.

   (a) {2 marks} Determine the generating series for S_1, S_2, S_3 with respect to w. (No work required.)

(b) {4 marks} Prove that for $n \geq 2$,

$$\Phi_{S_n}(x) = (1 + x + \cdots + x^{n-1})\Phi_{S_{n-1}}(x).$$

You may use the following (non-standard) notation: If σ is a permutation of $[n]$, denote σ' to be the permutation of $[n-1]$ obtained from σ by removing the element n. For example, if $\sigma = (31524)$, then $\sigma' = (3124)$.

(c) {2 marks} Prove that the number of permutations of $[n]$ with k inversions is

$$[x^k]\frac{\prod_{i=1}^{n}(1-x^i)}{(1-x)^n}.$$