INTEGRATION TECHNIQUES

1. Review of formulas and techniques

Summary of known integral formulas.

$\displaystyle\int x^r\,dx = \frac{x^{r+1}}{r+1} + c$, for $r \neq -1$ (power rule)	$\displaystyle\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$ for $x \neq 0$
$\displaystyle\int \sin x\,dx = -\cos x + c$	$\displaystyle\int \cos x\,dx = \sin x + c$
$\displaystyle\int \sec^2 x\,dx = \tan x + c$	$\displaystyle\int \sec x \tan x\,dx = \sec x + c$
$\displaystyle\int \csc^2 x\,dx = -\cot x + c$	$\displaystyle\int \csc x \cot x\,dx = -\csc x + c$
$\displaystyle\int e^x\,dx = e^x + c$	$\displaystyle\int e^{-x}\,dx = -e^{-x} + c$
$\displaystyle\int \tan x\,dx = -\ln\lvert\cos x\rvert + c$	$\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$
$\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$	$\displaystyle\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$

Example 1.1. (A Simple Substitution)
Evaluate $\displaystyle\int \sin(ax)dx$, for $a \neq 0$.

Example 1.2. (Generalizing a Basic Integration Rule)
Evaluate $\displaystyle\int \frac{1}{a^2+x^2}\,dx$, for $a \neq 0$.

Example 1.3. (An Integrand That Must Be Expanded)
Evaluate $\displaystyle\int (x^2-5)^2\,dx$.

Example 1.4. (An Integral Where We Must Complete the Square)
Evaluate $\displaystyle\int \frac{1}{\sqrt{-5+6x-x^2}}\,dx$.

Example 1.5. (An Integral Requiring Some Imagination)
Evaluate $\displaystyle\int \frac{4x+1}{2x^2+4x+10}\,dx$.

2. Integration by Parts

Remember that if u and v are differentiable functions of x, then $u \cdot v$ is also differentiable and

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{d}{dx}v + v \cdot \frac{d}{dx}u.$$

Using differential forms this may be written as

$$d(uv) = udv + vdu. \tag{1}$$

Taking the integral of the both sides of (1), we obtain

$$\int d(uv) = \int udv + \int vdu. \tag{2}$$

Note that $\int d(uv) = uv$. Therefore (2) can be written as

$$uv = \int udv + \int vdu,$$

or, equivalently,

$$\int udv = uv - \int vdu.$$

Remark. This technique requires the separation of the integral into two parts as u and dv (which includes the differential of integral, say dx) such that

- u gets simpler when differentiated.
- dv is easy to integrate.

Example 2.1. Evaluate the following integrals.

(1) $\int x \sin x \, dx$

(2) $\int x^2 e^x \, dx$

(3) $\int e^x \cos x \, dx$

(4) $\int x \ln x \, dx$

(5) $\int \ln x \, dx$

(6) $\int \sin^{-1} x \, dx$

(7) $\int x^3 e^{x^2} \, dx$

(8) $\int x \sec^2 x \, dx$

(9) $\int x 3^x \, dx$

(10) $\int \cos(\ln x) \, dx$

(11) $\int x^4 e^x \, dx$

(12) $\int \ln \sqrt{x}\, dx$

(13) $\int e^{\sqrt{x}}\, dx$

(14) $\int x^2 \sin x\, dx$

(15) $\int xe^{2x}\, dx$

(16) $\int xe^{-x}\, dx$

(17) $\int e^{2x} \cos 3x\, dx$

(18) $\int \tan^{-1} x\, dx$

(19) $\int x^5 e^{-x^2}\, dx$

(20) $\int e^{\sin^{-1} x}\, dx$

(21) $\int x\sqrt{1-x}\, dx$

(22) $\int_1^2 x^3 \ln x\, dx$

3. Trigonometric Integrals

3.1. **Products of Powers of Sines and Cosines.** If we are given an integral of the form

$$\int \sin^m x \cos^n x\, dx, \quad \text{where} \quad m, n \geq 0,$$

There are three possible cases:
Case 1. If m is odd, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x,$$

and, then

$$\int \sin^m x \cos^n x\, dx = \int (1-\cos^2 x)^k \cos^n x \sin x\, dx.$$

Using the substitution $u = \cos x$, we get

$$\int \sin^m x \cos^n x\, dx = \int (1-\cos^2 x)^k \cos^n x \sin x\, dx = -\int (1-u^2)^k u^n\, du.$$

Now, the integral is easy to evaluate.
Case 2. If m is even and n is odd, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ and repeat what we have done in Case 1 changing the places of sine and cosine.
Case 3. If both m and n are even, we substitute

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

3.2. **Integrals of Powers of $\tan x$ and $\sec x$.** We know how to integrate the tangent and secant and their squares. To integrate higher powers we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

3.3. **Products of Sines and Cosines.** To evaluate the integrals of the form

$$\int \sin mx \sin nx \, dx, \quad \int \sin mx \cos nx \, dx, \quad \text{and} \quad \int \cos mx \cos nx \, dx,$$

we may apply the method of integration by parts, but two such integrations are required in each case. So, it is better to use the identities

$$\sin mx \sin nx = \frac{1}{2}(\cos(m-n)x - \cos(m+n)x)$$
$$\sin mx \cos nx = \frac{1}{2}(\sin(m-n)x + \cos(m+n)x)$$
$$\cos mx \cos nx = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)$$

Example 3.1. Evaluate the following integrals.

(1) $\int \cos^4 x \sin x \, dx$

(2) $\int \cos^4 x \sin^3 x \, dx$

(3) $\int \sqrt{\sin x} \cos^5 x \, dx$

(4) $\int \cos^3 x \, dx$

(5) $\int \sin^3 x \cos^7 x \, dx$

(6) $\int \sin^2 x \, dx$

(7) $\int \cos^4 x \, dx$

(8) $\int \sin^2 x \cos^2 x \, dx$

(9) $\int \tan^3 x \sec^3 x \, dx$

(10) $\int \tan^2 x \, dx$

(11) $\int \tan^2 x \sec^4 x \, dx$

(12) $\int \tan^3 x \, dx$

(13) $\int \tan^3 x \sec^5 x \, dx$

(14) $\int \sec x \, dx$

(15) $\int \sqrt{\tan x} \sec^4 x \, dx$

(16) $\int \cot^4 x \, dx$

(17) $\int \csc x \, dx$

(18) $\int \cot^3 x \csc^3 x \, dx$

(19) $\int \frac{\sin^5 x}{\sqrt{\cos x}} \, dx$

3.4. **Trigonometric Substitutions.** Trigonometric substitutions can be effective in transforming complicated integrals involving $a^2 - x^2$, $a^2 + x^2$ and $x^2 - a^2$ into integrals we can evaluate directly.
The following table gives the suitable trigonometric substitution in each case and the trigonometric identity to be used.

Expression	Trigonometric Substitution	Related Trigonometric Identity
$a^2 + x^2$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$a^2 - x^2$	$x = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$
$x^2 - a^2$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$

Remark. We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a \tan \theta$, we want to be able to set $\theta = \tan^{-1}\left(\frac{x}{a}\right)$ after the integration takes place. As we know, the functions in these substitution have inverses only for selected values of θ. For reversibility,

$$x = a \tan \theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a \sin \theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

$$x = a \sec \theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & \text{if} \quad \frac{x}{a} \geq 1, \\ \frac{\pi}{2} < \theta \leq \pi & \text{if} \quad \frac{x}{a} \leq -1, \end{cases}$$

To simplify calculations with the substitution $x = a \sec \theta$, we will restrict its use to integrals in which $\frac{x}{a} \geq 1$. This will place θ in $\left[0, \frac{\pi}{2}\right)$ and make $\tan \theta \geq 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2 \tan^2 \theta} = |a \tan \theta| = a \tan \theta$, free of absolute values, provided that $a > 0$.

Example 3.2. (1) Evaluate the following integrals.

(a) $\int \frac{dx}{x^2 \sqrt{4 - x^2}}$

(b) $\int \frac{dx}{\sqrt{9 + x^2}}$

(c) $\int \frac{\sqrt{x^2 - 25}}{x} \, dx$, for $x \geq 5$.

(d) $\displaystyle\int \frac{dx}{(1+x^2)^2}$

(e) $\displaystyle\int \sqrt{4-x^2}\,dx$

(f) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$

(g) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-2}}$

(h) $\displaystyle\int \frac{\sqrt{9x^2-4}}{x}\,dx$

(i) $\displaystyle\int \frac{dx}{x^2+2x+10}$

(j) $\displaystyle\int e^x\sqrt{1-e^{2x}}\,dx$

(k) $\displaystyle\int \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$

(l) $\displaystyle\int \frac{dx}{(4x^2+4x+5)^2}$

(m) $\displaystyle\int \frac{2x+2}{x^2-4x+8}\,dx$

(n) $\displaystyle\int \frac{5x+3}{x^2-4x+8}\,dx$

(o) $\displaystyle\int \frac{x^2}{(x^2-1)^{\frac{5}{2}}}\,dx$

(2) Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = \dfrac{2}{1+x^2}$, and the line $x = 1$.

4. Integration of Rational Functions Using Partial Fractions

The method based on the fact that "every rational function can be written as a sum of simpler fractions that we can integrate with the techniques we already know".

General Description of the Method of Partial Fractions. To write a rational function $\dfrac{P(x)}{Q(x)}$ as a sum of partial fractions, do the following:

(1) The degree of $P(x)$ must be less than the degree of $Q(x)$. (Otherwise, divide $P(x)$ by $Q(x)$ and work on the remainder.)

(2) Factor out the polynomial $Q(x)$.

- For each linear factor, assign the sum of m partial fractions to this factor where m is the exponent of that linear factor as follows:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

- For each irreducible quadratic factor (a polynomial of second degree that cannot be written as a product of linear factors), assign the sum of n partial fractions to this factor where n is the exponent of that quadratic factor as:

$$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\cdots+\frac{B_nx+C_n}{(x^2+px+q)^n}$$

Several Examples. We assume that the degree of $P(x)$ is less than the degree of $Q(x)$.

(1) Distinct Linear Factors:

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x-r_1)(x-r_2)(x-r_3)}=\frac{A}{x-r_1}+\frac{B}{x-r_2}+\frac{C}{x-r_3}.$$

(2) Repeated Linear Factors:

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x-r)^4}=\frac{A}{x-r}+\frac{B}{(x-r)^2}+\frac{C}{(x-r)^3}+\frac{D}{(x-r)^4}.$$

(3) Some distinct, some repeated linear factors:

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{x^2(x-r_1)(x-r_2)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-r_1}+\frac{D}{x-r_2}+\frac{E}{(x-r_2)^2}+\frac{F}{(x-r_3)^3}.$$

(4) Distinct irreducible quadratic factors:

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x^2+p_1x+q_1)(x^2+p_2x+q_2)}=\frac{Ax+B}{x^2+p_1x+q_1}+\frac{Cx+D}{x^2+p_2x+q_2}.$$

(5) Repeated irreducible quadratic factors:

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x^2+p_1x+q_1)^3}=\frac{Ax+B}{x^2+p_1x+q_1}+\frac{Cx+D}{(x^2+p_1x+q_1)^2}+\frac{Ex+F}{(x^2+p_1x+q_1)^3}.$$

(6) General Case:

$$\begin{aligned}\frac{P(x)}{Q(x)}&=\frac{P(x)}{x(x-r)^2(x^2+p_1x+q_1)^2(x^2+p_2x+q_2)}\\&=\frac{A}{x}+\frac{B}{x-r}+\frac{C}{(x-r)^2}+\frac{Dx+E}{x^2+p_1x+q_1}+\frac{Fx+G}{(x^2+p_1x+q_1)^2}+\frac{Hx+I}{x^2+p_2x+q_2}.\end{aligned}$$

Example 4.1. Evaluate the following integrals.

(1) $\displaystyle\int\frac{1}{x^2+x-2}\,dx$

(2) $\displaystyle\int\frac{3x^2-7x-2}{x^3-x}\,dx$

(3) $\displaystyle\int\frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx$

(4) $\displaystyle\int\frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

(5) $\displaystyle\int\frac{2x^2-5x+2}{x^3+x}\,dx$

(6) $\displaystyle\int\frac{5x^2+6x+2}{(x+2)(x^2+2x+5)}\,dx$

(7) $\displaystyle\int\frac{3x-1}{(x+2)(x-3)}\,dx$

(8) $\displaystyle\int \frac{x^2+1}{x(x-1)(x-2)(x-3)}\,dx$

(9) $\displaystyle\int \frac{x}{(x-3)^2}\,dx$

(10) $\displaystyle\int \frac{x-1}{(x+1)^3}\,dx$

(11) $\displaystyle\int \frac{x^2+1}{(x-1)^2(x+1)}\,dx$

(12) $\displaystyle\int \frac{x+1}{(x^2+1)(x-2)}\,dx$

(13) $\displaystyle\int \frac{6x^2-15x+22}{(x+3)(x^2+2)^2}\,dx$

(14) $\displaystyle\int \frac{2x+1}{3x+1}\,dx$

(15) $\displaystyle\int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}\,dx$

(16) $\displaystyle\int \frac{x^3+1}{x^2-x}\,dx$

(17) $\displaystyle\int \sec x\,dx$

(18) $\displaystyle\int \frac{x^3}{x^2+6x+5}\,dx$

(19) $\displaystyle\int \frac{x^4}{x^4-1}\,dx$

(20) $\displaystyle\int \frac{dx}{x^3-1}$

(21) $\displaystyle\int \frac{dx}{e^x-e^{2x}}$

(22) $\displaystyle\int \frac{\sin x}{\cos x+\cos 2x}\,dx$

(23) $\displaystyle\int \frac{\sin^4 x}{\cos^3 x}\,dx$

(24) $\displaystyle\int \frac{dx}{x(1-x^2)^{\frac{3}{2}}}$

(25) $\displaystyle\int \frac{2x^3+5x^2+8x+4}{(x^2+2x+2)^2}\,dx$

Brief Summary of Integration Techniques.

- **Integration by Substitution:** $\int f(u(x))u'(x)\,dx = \int f(u)\,du$

  What to look for

  – Compositions of the form $f(u(x))$, where the integrand also contains $u'(x)$; for example,

  $$\int 2x\cos(x^2)\,dx = \int \cos(x^2)2x\,dx = \int \cos u du.$$

  – Compositions of the form $f(ax+b)$; for example,

  $$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du.$$

- **Integration by Parts:** $\int u\,dv = uv - \int v\,du$

  What to look for: products of different types of functions: x^n, $\cos x$, e^x; for example,

  $$\int 2x\cos x\,dx = x\sin x - \int \sin x\,dx.$$

- **Trigonometric Substitutions:**

  What to look for:

  – Terms like $\sqrt{a^2-x^2}$: Let $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so that $dx = a\cos\theta\,d\theta$ and $\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$; for example,

  $$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \sin^2\theta\,d\theta.$$

  – Terms like $\sqrt{x^2+a^2}$: Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $dx = a\sec^2\theta\,d\theta$ and $\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = a\sec\theta$; for example,

  $$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = 27\int \tan^3\theta\sec\theta\,d\theta.$$

  – Terms like $\sqrt{x^2-a^2}$: Let $x = a\sec\theta$, for $\theta \in [0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]$, so that $dx = a\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\tan\theta$; for example,

  $$\int x^3\sqrt{x^2-4}\,dx = 32\int \sec^4\theta\tan^2\theta\,d\theta.$$

- **Partial Fractions:**

  What to look for: rational functions, for example,

  $$\int \frac{x+2}{x^2-4x+3}\,dx = \int \frac{x+2}{(x-1)(x-3)}\,dx = \int \left(\frac{A}{x-1} + \frac{B}{x-3}\right)\,dx.$$

5. Improper Integrals

Example 5.1. Evaluate $\displaystyle\int_{-1}^{2} \frac{1}{x^2}\, dx$.

When studying the definite integrals, we required two things. First, the domain of integration (from a to b) $[a, b]$, be finite. Second, function is finite on the domain of integration.
Question: What happens if the domain of integration is infinite? What if function becomes infinite in the domain of integration?
Answer: Improper integration!
The integrals of type described above are called *improper integrals.* If the limits exist, we evaluate them with the following definitions

(1) If f is continuous on $[a, \infty)$, then
$$\int_a^{\infty} f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx.$$
(2) If f is continuous on $(-\infty, b]$, then
$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a\to-\infty} \int_a^b f(x)\, dx.$$
(3) If f is continuous on $[a, b)$ then
$$\int_a^b f(x)\, dx = \lim_{c\to b^-} \int_a^c f(x)\, dx.$$
(4) If f is continuous on $(a, b]$ then
$$\int_a^b f(x)\, dx = \lim_{c\to a^+} \int_c^b f(x)\, dx.$$

In each case, if the limit exists and is finite we say that the improper integral *converges* and the limit is the value of the improper integral. Otherwise the improper integral *diverges.*
Similarly, if f becomes infinite at an interior point $d \in [a, b]$, then
$$\int_a^b f(x)\, dx = \int_a^d f(x)\, dx + \int_d^b f(x)\, dx.$$
This integral (on $[a, b]$) converges if both integrals (on $[a, d]$ and on $[d, b]$) converges. Otherwise, the integral from a to b diverges.
Finally, if f is continuous on $(-\infty, \infty)$ and if $\displaystyle\int_{-\infty}^{a} f(x)\, dx$ and $\displaystyle\int_a^{\infty} f(x)\, dx$ both converge, then $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx$ converges and its value is
$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{a} f(x)\, dx + \int_a^{\infty} f(x)\, dx.$$
If either one or both of the integrals on the right-hand side of this equation diverge, the integral diverges.

Example 5.2. In each part, determine whether the improper integral converges or diverges, and find its value if it converges.

(1) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}.$

(2) $\displaystyle\int_{-1}^0 \frac{1}{x^2}\,dx.$

(3) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx.$

(4) $\displaystyle\int_1^2 \frac{1}{x-1}\,dx.$

(5) $\displaystyle\int_{-1}^2 \frac{1}{x^2}\,dx.$

(6) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx.$

(7) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx.$

(8) $\displaystyle\int_1^\infty e^{-x}\,dx.$

(9) $\displaystyle\int_0^\infty \sin x\,dx.$

(10) $\displaystyle\int_0^\infty xe^{-x}\,dx.$

(11) $\displaystyle\int_{-1}^1 \frac{1}{x^{\frac{1}{3}}}\,dx.$

(12) $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx.$

(13) $\displaystyle\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}.$

(14) $\displaystyle\int_{-1}^1 \frac{dx}{x}.$

(15) $\displaystyle\int_0^1 \frac{dx}{x^p}.$

(16) $\displaystyle\int_{-\infty}^{-1} \frac{1}{x}\,dx.$

(17) $\displaystyle\int_{-\infty}^0 \frac{1}{(x-1)^2}\,dx.$

(18) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx.$

(19) $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx.$

(20) $\displaystyle\int_{-\infty}^\infty e^{-x}\,dx.$

(21) $\displaystyle\int_0^\infty \frac{1}{(x-1)^2}\,dx.$

(22) $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx.$

(23) $\displaystyle\int_{-\infty}^\infty e^{-|x|}\,dx.$

(24) $\displaystyle\int_{-\infty}^\infty \cos x\,dx.$

Exercises 5.3. In each part, determine whether the improper integral converges or diverges, and find its value if it converges.

(1) $\displaystyle\int_{-1}^1 \frac{dx}{x^2}.$

(2) $\displaystyle\int_0^{16} \frac{dx}{\sqrt[4]{x}}.$

(3) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$

(4) $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx.$

(5) $\displaystyle\int_1^\infty \frac{dx}{x^{1.001}}.$

(6) $\displaystyle\int_{-8}^1 \frac{dx}{x^{\frac{1}{3}}}.$

(7) $\displaystyle\int_0^1 \frac{dx}{x^{0.999}}.$

(8) $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{\frac{3}{2}}}.$

(9) $\displaystyle\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx.$

(10) $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta.$

(11) $\displaystyle\int_0^1 -\ln x\,dx.$

(12) $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x^2+1)}.$

(13) $\displaystyle\int_{-1}^1 -x\ln|x|\,dx.$

(14) $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}.$

(15) $\displaystyle\int_{e^e}^\infty \ln(\ln x).$

A comparison test.

Theorem 5.1 (Direct Comparison Test)**.** *Let f and g be continuous on $[a,\infty)$ and suppose that $0 \le f(x) \le g(x)$ for all $x \ge a$.*

(i) *If $\displaystyle\int_a^\infty g(x)\,dx$ converges then $\displaystyle\int_a^\infty f(x)\,dx$ converges.*

(ii) *If* $\int_a^\infty f(x)\,dx$ *diverges then* $\int_a^\infty g(x)\,dx$ *diverges.*

Example 5.4. In each part, determine whether the improper integral converges or diverges.

(1) $\displaystyle\int_0^\infty \frac{1}{x+e^x}\,dx.$

(2) $\displaystyle\int_0^\infty e^{-x^2}\,dx.$

(3) $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x}}\,dx.$

(4) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx.$

(5) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^4+5}}.$

(6) $\displaystyle\int_1^\infty \frac{\ln x}{\sqrt{x}}\,dx.$

(7) $\displaystyle\int_1^\infty \frac{dx}{5^x+2^x}.$

(8) $\displaystyle\int_0^\infty \frac{dx}{1+e^x}.$

(9) $\displaystyle\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx.$

(10) $\displaystyle\int_2^\infty \frac{dx}{\ln x}.$

(11) $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx.$

Answers

Answers 1.1. $-\dfrac{\cos(ax)}{a}+c.$

Answers 1.2. $\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+c.$

Answers 1.3. $\dfrac{x^5}{5}-\dfrac{10x^3}{3}+25x+c.$

Answers 1.4. $\sin^{-1}\left(\dfrac{x-3}{2}\right)+c.$

Answers 1.5. $\ln\left(x^2+2\,x+5\right)-\dfrac{3}{4}\,\tan^{-1}\left(\dfrac{x+1}{2}\right)+c.$

Answers 2.1.
(1) $\sin x - x\cos x + c.$
(2) $x^2e^x-2xe^x+2e^x+c.$
(3) $\dfrac{e^x(\cos x+\sin x)}{2}+c.$
(4) $\dfrac{x^2\ln x}{2}-\dfrac{x^2}{4}+c.$
(5) $x\ln x - x + c.$
(6) $x\sin^{-1}x+\sqrt{1-x^2}+c.$
(7) $\dfrac{e^{x^2}(x^2-1)}{2}+c.$
(8) $x\tan x+\ln(\cos x)+c.$
(9) $\dfrac{x3^x}{\ln 3}-\dfrac{3^x}{\ln^2 3}+c.$
(10) $\dfrac{x(\cos(\ln x)+\sin(\ln x))}{2}+c.$
(11) $x^4e^x-4x^3e^x+12x^2e^x-24xe^x+24e^x+c.$
(12) $\dfrac{x\ln x}{2}-\dfrac{x}{2}+c.$
(13) $2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+c.$
(14) $-x^2\cos x+2x\sin x+2\cos x+c.$
(15) $\dfrac{xe^{2x}}{2}-\dfrac{e^{2x}}{4}+c.$
(16) $-xe^{-x}-e^{-x}+c.$
(17) $\dfrac{2e^{2x}\cos(3x)}{13}+\dfrac{3e^{2x}\sin(3x)}{13}+c.$
(18) $x\tan^{-1}x-\dfrac{1}{2}\ln(1+x^2)+c.$
(19) $-\dfrac{x^4e^{-x^2}}{2}-x^2e^{-x^2}-e^{-x^2}+c.$
(20) $\dfrac{e^{\sin^{-1}x}(\sqrt{1-x^2}+x)}{2}+c.$
(21) $\dfrac{2}{5}(1-x)^{\frac{5}{2}}-\dfrac{2}{3}(1-x)^{\frac{3}{2}}+c.$
(22) $4\ln 2-\dfrac{15}{16}+c.$

Answers 3.1. (1) $-\frac{\cos^5 x}{5} + c.$

(2) $\frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + c.$

(3) $\frac{2}{3}\sin^{\frac{3}{2}} x - \frac{4}{7}\sin^{\frac{3}{2}} x + \frac{2}{11}\sin^{\frac{11}{2}} x + c.$

(4) $\sin x - \frac{\sin^3 x}{3} + c.$

(5) $\frac{\cos^{10} x}{10} - \frac{\cos^8}{8} + c.$

(6) $\frac{x}{2} - \frac{\sin(2x)}{4} + c.$

(7) $\frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} c.$

(8) $\frac{x}{8} - \frac{\sin(4x)}{32} + c.$

(9) $\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c.$

(10) $\tan x - x + c.$

(11) $\frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + c.$

(12) $\frac{\tan^2 x}{2} + \ln|\cos x| + c.$

(13) $\frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + c.$

(14) $\ln|\sec x + \tan x| + c.$

(15) $\frac{2}{7}\tan^{\frac{7}{2}} + \frac{2}{3}\tan^{\frac{3}{2}} + c.$

(16) $-\frac{1}{3}\cot^3 x + \cot x + x + c.$

(17) $-\ln|\csc x + \cot x| + c.$

(18) $\frac{\csc^3 x}{3} - \frac{\csc^5 x}{5} + c.$

(19) $-\sqrt{\cos x} + \frac{4}{5}\cos^{\frac{5}{2}} x - \frac{2}{9}\cos^{\frac{9}{2}} x + c.$

Answers 3.2. (1) (a) $-\frac{\sqrt{4-x^2}}{4x} + c.$

(b) $\ln\left(\frac{x+\sqrt{9+x^2}}{3}\right) + c.$

(c) $\sqrt{x^2-25} - 5\sec^{-1}\frac{x}{5} + c.$

(d) $\frac{1}{2}\tan^{-1} x + \frac{x}{2(1+x^2)} + c.$

(e) $2\sin^{-1}\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + c.$

(f) $\sqrt{x^2-1} - \sec^{-1} x + c.$

(g) $\frac{\sqrt{x^2-2}}{2x} + c.$

(h) $\sqrt{9x^2-4} - 2\sec^{-1}\dfrac{3x}{2} + c.$

(i) $\dfrac{1}{3}\tan^{-1}\dfrac{x+1}{3} + c.$

(j) $\dfrac{1}{2}\sin^{-1}(e^x) + \dfrac{e^x\sqrt{1-e^{2x}}}{2} + c.$

(k) $\ln\left|\sqrt{1+\sin^2 x} + \sin x\right| + c.$

(l) $\dfrac{1}{32}\tan^{-1}\dfrac{2x+1}{2} + \dfrac{2x+1}{16(4x^2+4x+5)} + c.$

(m) $\ln\left(x^2-4\,x+8\right) + 3\,\tan^{-1}\left(\dfrac{x-2}{2}\right) + c.$

(n) $\dfrac{5}{2}\,\ln\left(x^2-4\,x+8\right) + \dfrac{13}{2}\,\tan^{-1}\left(\dfrac{x-2}{2}\right) + c.$

(o) $-\dfrac{x^3}{3(x^2-1)^{\frac{3}{2}}} + c.$

(2) $\dfrac{\pi(\pi+2)}{2}.$

Answers 4.1. (1) $\dfrac{1}{3}\,\ln|x-1| - \dfrac{1}{3}\,\ln|x+2| + c.$

(2) $2\,\ln|x| + 4\,\ln|x+1| - 3\,\ln|x-1| + c.$

(3) $x^2 + \dfrac{3}{2}\,\ln|x-4| - \dfrac{1}{2}\,\ln|x+2| + c.$

(4) $-\dfrac{9}{x+1} + 6\,\ln|x| - \ln|x+1| + c.$

(5) $-5\,\tan^{-1}x + 2\,\ln|x| + c.$

(6) $\dfrac{3}{2}\,\ln\left(x^2+2\,x+5\right) - \dfrac{7}{2}\,\tan^{-1}\left(\dfrac{x+1}{2}\right) + 2\,\ln|x+2| + c.$

(7) $\dfrac{8}{5}\,\ln|x-3| + \dfrac{7}{5}\,\ln|x+2| + c.$

(8) $-\dfrac{5}{2}\,\ln|x-2| - \dfrac{1}{6}\,\ln|x| + \ln|x-1| + \dfrac{5}{3}\,\ln|x-3| + c.$

(9) $-\dfrac{3}{x-3} + \ln|x-3| + c.$

(10) $-\dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} + c.$

(11) $\dfrac{1}{2}\,\ln|x+1| + \dfrac{1}{2}\,\ln|x-1| - \dfrac{1}{x-1} + c.$

(12) $-\dfrac{3}{10}\,\ln\left(x^2+1\right) - \dfrac{1}{5}\,\tan^{-1}x + \dfrac{3}{5}\,\ln|x-2| + c.$

(13) $\ln|3+x| + \dfrac{5}{2(x^2+2)} - \dfrac{1}{2}\,\ln\left(x^2+2\right) + \dfrac{3\sqrt{2}}{2}\,\tan^{-1}\left(\dfrac{x\sqrt{2}}{2}\right) + c.$

(14) $\dfrac{2}{3}\,x + \dfrac{1}{9}\,\ln|3\,x+1| + c.$

(15) $x + \dfrac{x^2}{2} + \ln\left(x^2+1\right) + \tan^{-1}x - 2\,\ln|x-1| - \dfrac{1}{x-1} + c.$

(16) $x + \dfrac{x^2}{2} - \ln|x| + 2\,\ln|x-1| + c.$

(17) $\ln|\sec x + \tan x| + c$.

(18) $\dfrac{x^2}{2} - 6\,x - \dfrac{1}{4}\,\ln|x+1| + \dfrac{125}{4}\,\ln|5+x| + c$.

(19) $x + \dfrac{1}{4}\,\ln|x-1| - \dfrac{1}{4}\,\ln|x+1| - \dfrac{1}{2}\,\tan^{-1} x + c$.

(20) $-\dfrac{1}{6}\,\ln\left(x^2+x+1\right) - \dfrac{\sqrt{3}}{3}\,\tan^{-1}\left(\dfrac{(2\,x+1)\,\sqrt{3}}{3}\right) + \dfrac{1}{3}\,\ln|x-1| + c$.

(21) $-e^{-x} + x - \ln|e^x - 1| + c$.

(22) $-\dfrac{1}{3}\,\ln|2\,\cos x - 1| + \dfrac{1}{3}\,\ln|1+\cos x| + c$.

(23) $\sin x - \dfrac{1}{4(\sin x + 1)} - \dfrac{3}{4}\,\ln|\sin x + 1| + \dfrac{3}{4}\,\ln|\sin x - 1| - \dfrac{1}{4(\sin x - 1)} + c$.

(24) $\dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{2}\,\ln\left|\sqrt{1-x^2}+1\right| + \dfrac{1}{2}\,\ln\left|\sqrt{1-x^2}-1\right| + c$.

(25) $\ln\left(x^2+2\,x+2\right) - \tan^{-1}(x+1) - \dfrac{1}{x^2+2\,x+2} + c$.

Answers 5.1. See, Answers 5.2, (5).

Answers 5.2. (1) Convergent, 2.

(2) Divergent.

(3) Convergent, 2.

(4) Divergent.

(5) Divergent.

(6) Convergent, 1.

(7) Divergent.

(8) Convergent, $\dfrac{1}{e}$.

(9) Divergent.

(10) Convergent, 1.

(11) Convergent, 0.

(12) Convergent, $\dfrac{\pi}{2}$.

(13) Convergent, $3 + 3\sqrt[3]{2}$.

(14) Divergent.

(15) Divergent if $p \geq 1$. Convergent if $p < 1$, $\dfrac{1}{1-p}$.

(16) Divergent.

(17) Convergent, 1.

(18) Convergent, π.

(19) Convergent, 0.

(20) Divergent.

(21) Divergent.

(22) Divergent if $p \leq 1$. Convergent if $p > 1$, $\dfrac{1}{p-1}$.

(23) Convergent, 2.

(24) Divergent.

Answers 5.3. (1) Divergent.

(2) Convergent, $\dfrac{32}{3}$.
(3) Convergent, $\dfrac{\pi}{2}$.
(4) Divergent.
(5) Convergent, 1000.
(6) Convergent, $-\dfrac{9}{2}$.
(7) Convergent, 1000.
(8) Convergent, 0.
(9) Convergent, $2\pi^2$.
(10) Convergent, 1.
(11) Convergent, 1.
(12) Convergent, $\dfrac{\pi}{4}$.
(13) Convergent, 0.
(14) Convergent, $\dfrac{\pi}{2}$.

Answers 5.4. (1) Convergent.
(2) Convergent.
(3) Divergent.
(4) Convergent.
(5) Convergent.
(6) Divergent.
(7) Convergent.
(8) Convergent.
(9) Convergent.
(10) Divergent.
(11) Convergent.
(12) Divergent.