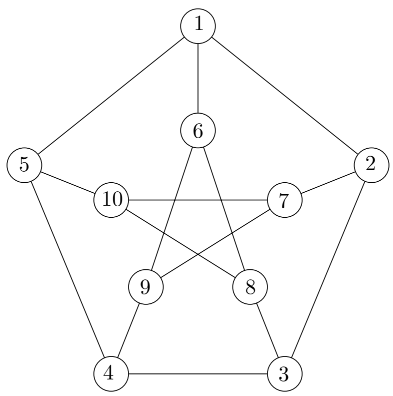
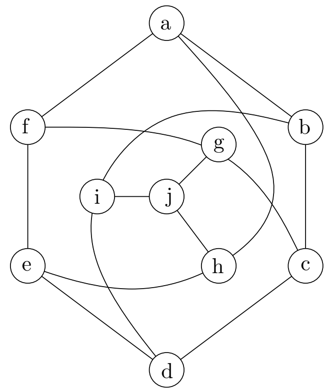
# Tutorial Problems 6

The TA will cover a subset of these problems during the tutorial. Solutions will not be provided outside of the tutorial.

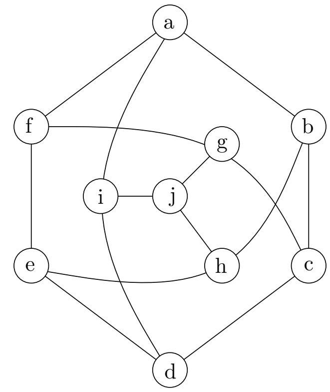
1. Graph $G$ is the Petersen graph. Among the graphs $H_1$ and $H_2$, one of them is isomorphic to the Petersen graph, and the other is not. For the one that is isomorphic, give an isomorphism. For the one that is not isomorphic, explain why it is not isomorphic to $G$.




Graph $G$




Graph $H_1$




Graph $H_2$

2. Let $G$ be a graph. The complement of $G$, denoted $\overline{G}$, is the graph where $V(\overline{G}) = V(G)$, and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

   (a) Draw a graph $G$ on 5 vertices such that $G$ is isomorphic to $\overline{G}$.

   (b) Suppose a graph $G$ with $n$ vertices is isomorphic to $\overline{G}$. Prove that $n \equiv 0, 1 \pmod 4$.

   (c) Prove that if $G$ has at least 5 vertices, then $G$ and $\overline{G}$ cannot both be bipartite.

3. There are $n$ participants at a certain meeting with no purpose whatsoever ($n \geq 2$). During this gathering, certain participants exchanged phone numbers with each other. Prove that at least two participants received the same number of phone numbers.

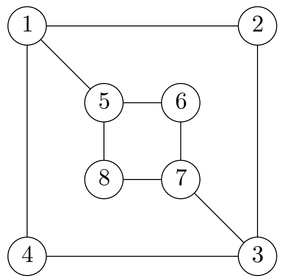
4. An *automorphism* of a graph $G$ is an isomorphism of $G$ onto itself. Consider the $n$-cube $C_n$ where the set of vertices $V(C_n)$ consists of all binary strings of length $n$, and two strings are adjacent if they differ in exactly one position. Define $f : V(C_n) \to V(C_n)$ such that for any string $a_1 a_2 \cdots a_n$, $f(a_1 a_2 \cdots a_n) = b_1 b_2 \cdots b_n$ where

$$b_i = \begin{cases} 0 & \text{if } a_i = 1 \\ 1 & \text{if } a_i = 0 \end{cases}$$

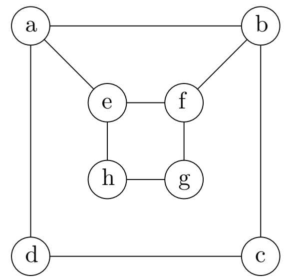
   (In other words, we are switching every bit of the string.) Prove that $f$ is an automorphism.

# Practice Problems for Assignment 6

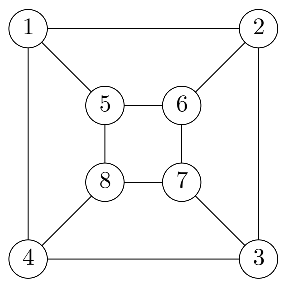
1. Determine if $G_1$ and $G_2$ are isomorphic, and if $G_3$ and $G_4$ are isomorphic. Prove your claims.



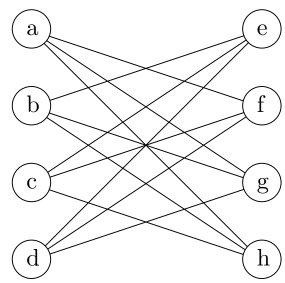
Graph $G_1$



Graph $G_2$

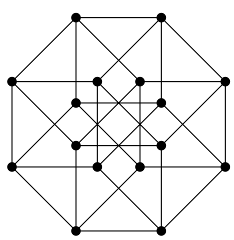


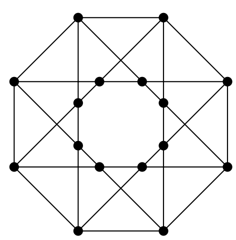
Graph $G_3$



Graph $G_4$

2. Are these two graphs isomorphic?





3. The *degree sequence* of a graph is the list of vertex degrees, usually written in nonincreasing order, as $d_1 \geq \cdots \geq d_p$. Determine whether or not each of the following is the degree sequence of a graph. If so, draw the graph. If not, explain why.

   (a) 7, 6, 5, 4, 4, 3, 2, 2, 2, 2

   (b) 3, 3, 3, 2, 2, 1

   (c) 5, 5, 1, 1, 1, 1

4. Assume that for any pair of people, they are either acquaintances or strangers. Prove that among any given group of 6 people, at least 3 of them are mutual acquaintances or mutual strangers.

5. A graph is *self-complementary* if it is isomorphic to its complement. From the tutorial problems, we see that any self-complementary graph with $n$ vertices satisfies $n \equiv 0, 1 \pmod 4$. For each such $n$, construct a self-complementary graph on $n$ vertices.

6. Prove that the 3-cube has 48 automorphisms.