MATH 239 Spring 2014: Assignment 5

Due: 3:00 PM, Monday June 16, 2014 in the dropboxes outside MC 4066

Last Name: First Name:

I.D. Number: Section:

Mark (For the marker only): /27

---

1. {5 marks} Let $\{a_n\}$ be the sequence which satisfies

$$a_n - 7a_{n-1} + 15a_{n-2} - 9a_{n-3} = 0$$

for $n \geq 3$ with initial conditions $a_0 = 7, a_1 = 10, a_2 = 13$. Determine an explicit formula for a_n.

2. {5 marks} Let $\{b_n\}$ be the sequence which satisfies

$$b_n - 7b_{n-1} + 15b_{n-2} - 9b_{n-3} = 16 \cdot (-3)^n$$

for $n \geq 3$ with initial conditions $b_0 = 10, b_1 = -13, b_2 = 14$. Determine an explicit formula for b_n. (Note: This recurrence is similar to the one in question 1.)

3. Consider the sequence $\{a_n\}$ where for each integer $n \geq 0$,

$$a_n = \frac{(1+\sqrt{3})^n - (1-\sqrt{3})^n}{2\sqrt{3}}.$$

(a) {3 marks} Derive a simplified rational expression for $A(x) = \sum_{n\geq 0} a_n x^n$.

(b) {3 marks} Use part (a) to prove that a_n is an integer for all $n \geq 0$.

4. {4 marks} Consider the sequence $\{a_n\}$ defined by $a_0 = -2$, $a_1 = 3$, and for $n \geq 2$,

$$a_n + 6a_{n-1} + 12a_{n-2} = 0.$$

The roots of the characteristic polynomial are complex. Convert them into polar form (with sines and cosines), and then derive an explicit formula for a_n that does not involve any imaginary parts. (Hint: $\sin(\theta) = -\sin(-\theta)$ and $\cos(\theta) = \cos(-\theta)$. Your final answer should look like $\ldots(\ldots\cos\ldots + \ldots\sin\ldots)$.)

5. For each $n \in \mathbb{N}$, let a_n be the total number of blocks among all 2^n binary strings of length n. For example, $a_1 = 2$, and $a_2 = 6$ (each of the strings $00, 11$ has 1 block, and each of the strings $01, 10$ has 2 blocks, for a total of 6 blocks).

   (a) {2 marks} Let S_n be the set of all binary strings of length n. For $n \geq 2$, we split S_n into two sets A_n and B_n in the following way: let A_n be strings of length n where the last block has length 1; let B_n be strings of length n where the last block has length at least 2. We define two functions $f : A_n \to S_{n-1}$ and $g : B_n \to S_{n-1}$ where both functions take the input string and remove the last bit. Prove that these are bijections by determining the inverses for both functions.

   (b) {2 marks} Use part (a) to derive the following recurrence for $n \geq 2$:

$$a_n = 2a_{n-1} + 2^{n-1}.$$

   (c) {3 marks} Solve for an explicit formula for a_n.