

## Tutorial Problems 8

The TA will cover a subset of these problems during the tutorial. Solutions will not be provided outside of the tutorial.

1. Let  $G$  be a  $k$ -regular graph where  $k$  is even. Prove that  $G$  does not have a bridge.
2. Let  $G$  be a connected graph, and let  $T$  be a spanning tree of  $G$ . Suppose in addition to  $E(T)$ , there are  $k$  edges in  $G$ ,  $e_1, \dots, e_k$ . Let  $C_i$  be the unique cycle in  $G + e_i$ . Prove that if  $C_i$  has even length for all  $i$ , then  $G$  is bipartite.
3. Let  $G$  be a connected graph, and let  $T$  and  $T'$  be distinct spanning trees of  $G$ . Prove that there exist edges  $e \in E(T)$  and  $e' \in E(T')$  such that both  $T - e + e'$  and  $T' + e - e'$  are spanning trees of  $G$ .
4. Let  $G$  be a connected graph with given edge weights, and let  $T$  be the minimum spanning tree produced by Prim's algorithm starting at vertex  $v$ . For any  $x \in V(G)$ , is it true that the weight of a minimum weight  $v, x$ -path is the same as the weight of the unique  $v, x$ -path in  $T$ ?
5. Let  $G$  be a connected graph with edge weights  $w$ .
  - (a) Let  $T^*$  be a minimum spanning tree, and let  $e$  be an edge in  $G$  but not in  $T^*$ . We know that  $T^* + e$  contains exactly one cycle  $C$ . Prove that for all  $e' \in E(C)$ ,  $w(e') \leq w(e)$ .
  - (b) Suppose no two edges in  $G$  have the same weight. Prove that  $G$  has a unique minimum spanning tree.